Calc-ulus

Volume 1

2 | LIMITS

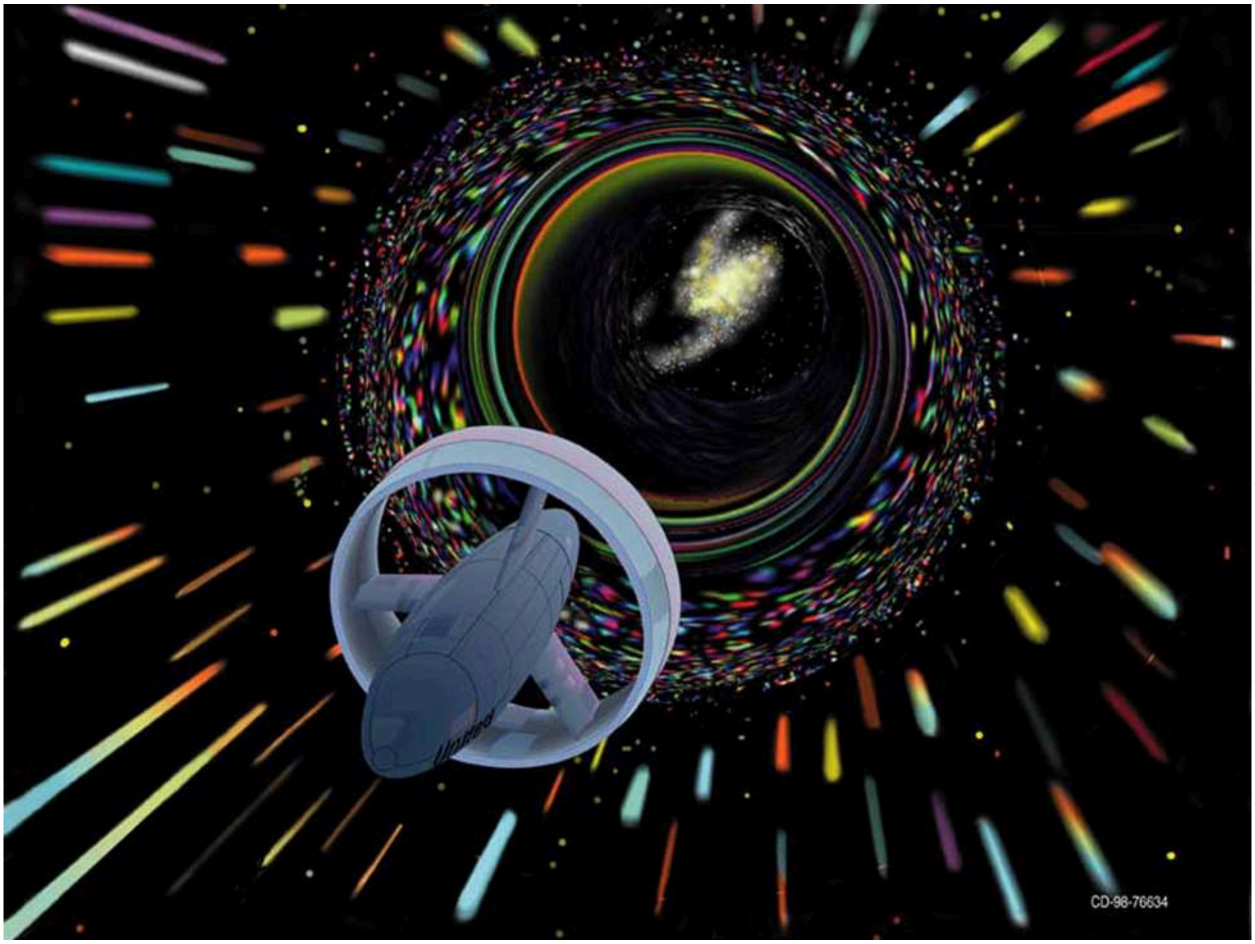


Figure 2.1 The vision of human exploration by the National Aeronautics and Space Administration (NASA) to distant parts of the universe illustrates the idea of space travel at high speeds. But, is there a limit to how fast a spacecraft can go? (credit: NASA)

Chapter Outline

Introduction

Science fiction writers often imagine spaceships that can travel to far-off planets in distant galaxies. However, back in 1905, Albert Einstein showed that a limit exists to how fast any object can travel. The problem is that the faster an object moves, the more mass it attains (in the form of energy), according to the equation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. What is this speed limit? (We explore this problem further in **Example 2.12**.)

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

2.1 | A Preview of Calculus

Learning Objectives

2.1.1 Describe the tangent problem and how it led to the idea of a derivative.
2.1.2 Explain how the idea of a limit is involved in solving the tangent problem.
2.1.3 Recognize a tangent to a curve at a point as the limit of secant lines.
2.1.4 Identify instantaneous velocity as the limit of average velocity over a small time interval.
2.1.5 Describe the area problem and how it was solved by the integral.
2.1.6 Explain how the idea of a limit is involved in solving the area problem.
2.1.7 Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3,\ g(x) = \frac{1}{2}x + 1,$ and $h(x) = 2,$ shown in **Figure 2.2**.

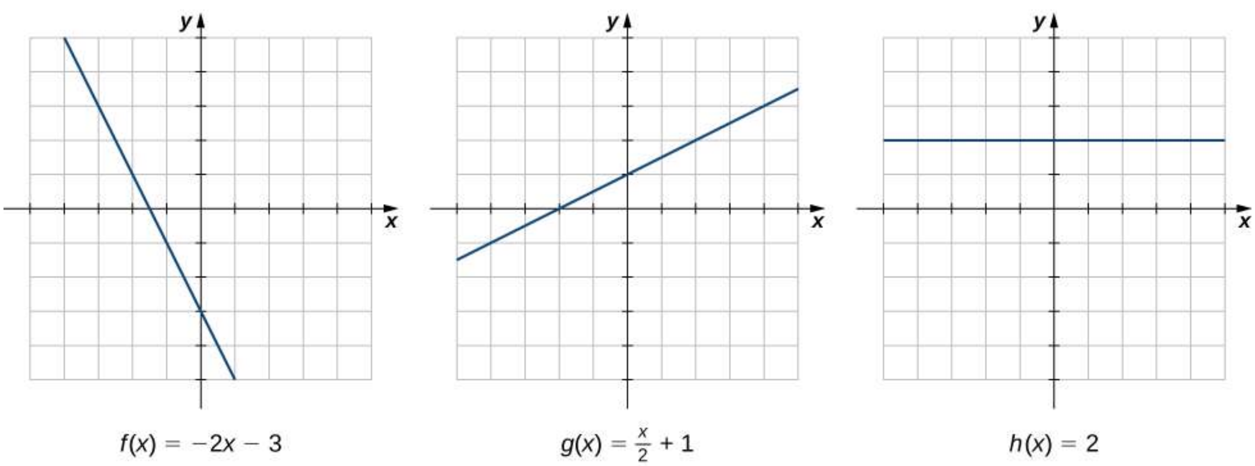


Figure 2.2 The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3,$ we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x-axis, the y-coordinate decreases by 2 units. This rate of change is determined by the slope (−2) of the line. Similarly, the slope of 1/2 in the function $g(x)$ tells us that for every change in x of 1 unit there is a corresponding change in y of 1/2 unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (**Figure 2.3**). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?

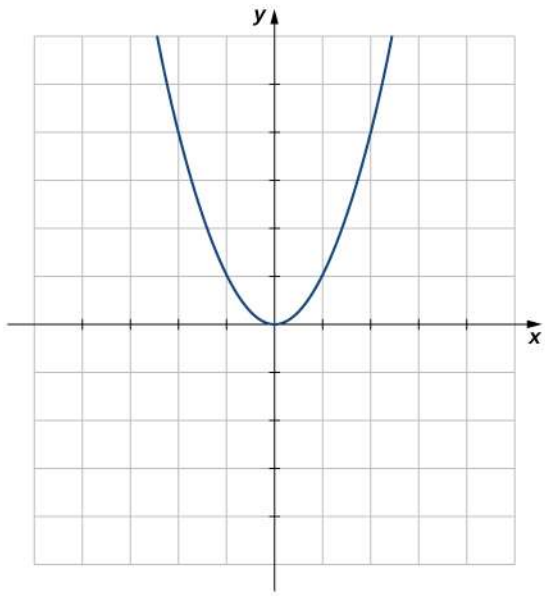


Figure 2.3 The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a **secant** line. **Figure 2.4** shows a secant line to a function $f(x)$ at a point $(a, f(a))$.

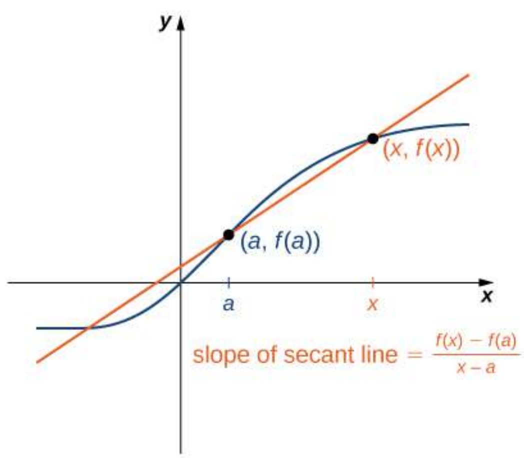


Figure 2.4 The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Definition

The **secant** to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{\sec} = \frac{f(x) - f(a)}{x - a}. \tag{2.1}$$

The accuracy of approximating the rate of change of the function with a secant line depends on how close x is to a. As we see in **Figure 2.5**, if x is closer to a, the slope of the secant line is a better measure of the rate of change of $f(x)$ at a.

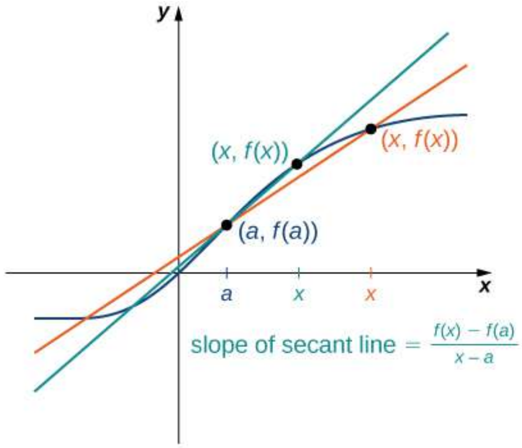


Figure 2.5 As x gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at a (**Figure 2.6**). The slope of the tangent line to the graph at a measures the rate of change of the function at a. This value also represents the derivative of the function $f(x)$ at a, or the rate of change of the function at a. This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

For an interactive demonstration of the slope of a secant line that you can manipulate yourself, visit this applet (*Note:* this site requires a Java browser plugin): **Math Insight (http://www.openstax.org/l/20_mathinsight)**.

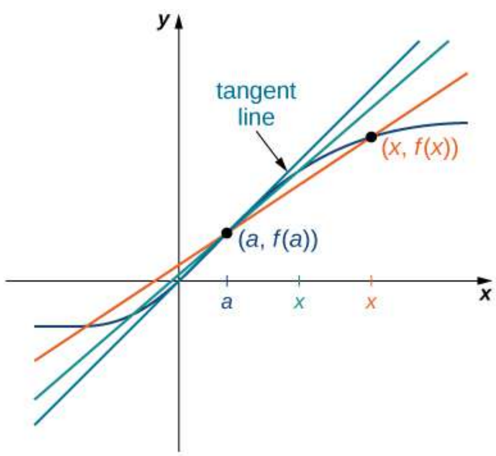


Figure 2.6 Solving the Tangent Problem: As x approaches a, the secant lines approach the tangent line.

Example 2.1 illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

Example 2.1

Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2, 4)$

b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

Solution

Use the formula for the slope of a secant line from the definition.

a. $m_{\text{sec}} = \frac{4 - 1}{2 - 1} = 3$

b. $m_{\text{sec}} = \frac{\frac{9}{4} - 1}{\frac{3}{2} - 1} = \frac{5}{2} = 2.5$

The point in part b. is closer to the point $(1, 1)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 (**Figure 2.7**).

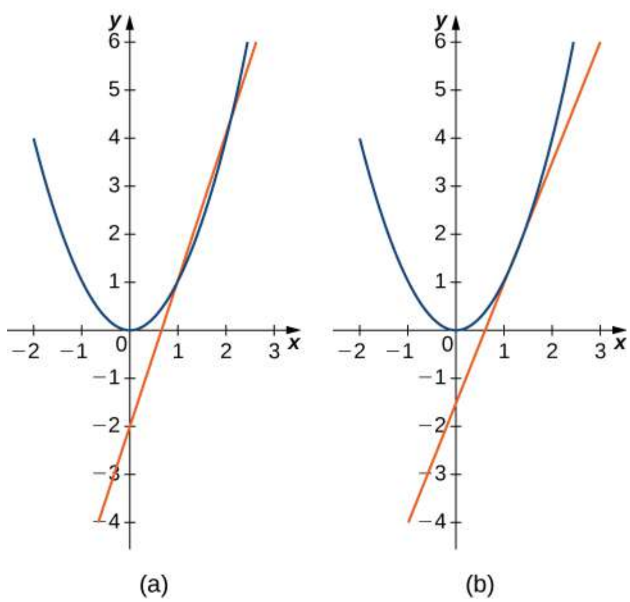


Figure 2.7 The secant lines to $f(x) = x^2$ at $(1, 1)$ through (a) $(2, 4)$ and (b) $\left(\frac{3}{2}, \frac{9}{4}\right)$ provide successively closer approximations to the tangent line to $f(x) = x^2$ at $(1, 1)$.

2.1 Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and the point $\left(\frac{5}{4}, \frac{25}{16}\right)$ on the graph of $f(x) = x^2$.

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time *t*. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t = a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the **average velocity** of an object over a time period to be the change in its position divided by the length of the time period.

Definition

Let $s(t)$ be the position of an object moving along a coordinate axis at time *t*. The **average velocity** of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}. \tag{2.2}$$

As *t* is chosen closer to *a*, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point *a*, we let the *x*-values approach *a* in the slope of the secant line. Similarly, to find the instantaneous velocity at time *a*, we let the *t*-values approach *a* in the average velocity. This process of letting *x* or *t* approach *a* in an expression is called taking a **limit**. Thus, we may define the **instantaneous velocity** as follows.

Definition

For a position function $s(t)$, the **instantaneous velocity** at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of *t* become closer to *a*, provided such a value exists.

Example 2.2 illustrates this concept of limits and average velocity.

Example 2.2

Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground *t* seconds later (for $0 \le t \le 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$

b. $[0.5, 0.51]$

Solution

Substitute the data into the formula for the definition of average velocity.

a. $v_{\text{ave}} = \dfrac{s(0.5) - s(0.49)}{0.5 - 0.49} = -15.84$

b. $v_{ave} = \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.16$

The instantaneous velocity is somewhere between −15.84 and −16.16 ft/sec. A good guess might be −16 ft/sec.

2.2 An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may be interpreted as the area under a curve. This leads us to ask the question: How can we find the area between the graph of a function and the *x*-axis over an interval (**Figure 2.8**)?

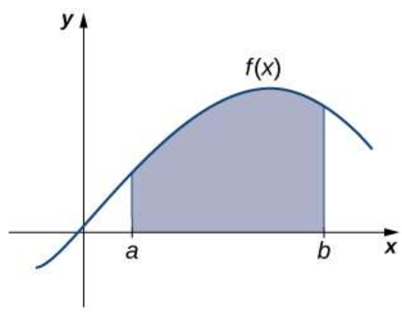


Figure 2.8 The Area Problem: How do we find the area of the shaded region?

As in the answer to our previous questions on velocity, we first try to approximate the solution. We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up the areas of these rectangles (**Figure 2.9**).

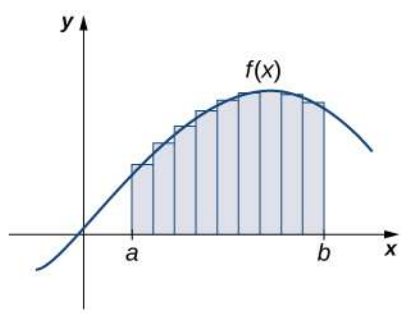


Figure 2.9 The area of the region under the curve is approximated by summing the areas of thin rectangles.

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the *x*-axis over the interval $[a, b]$. Once again, we find ourselves taking a limit. Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

Example 2.3

Estimation Using Rectangles

Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in **Figure 2.10**.

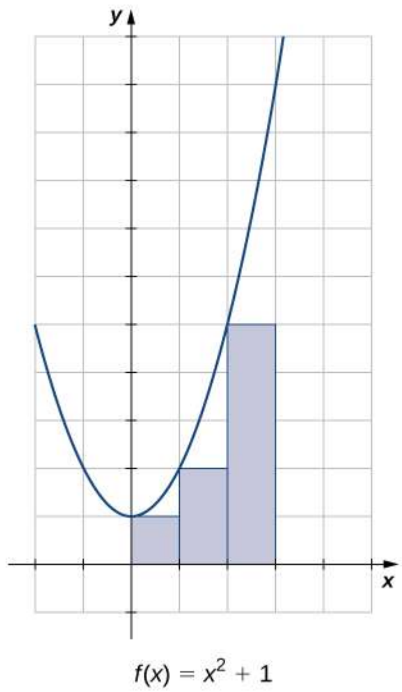
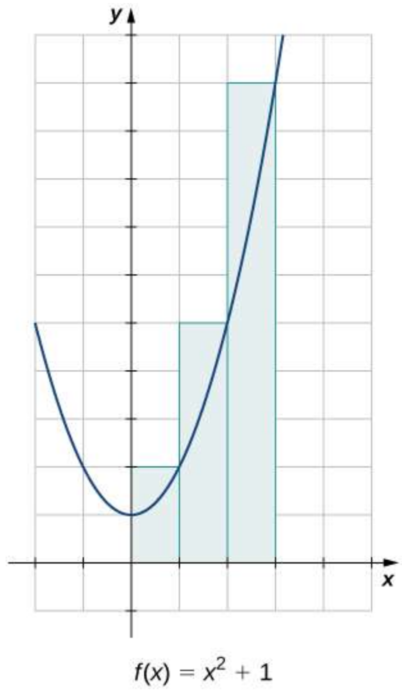


Figure 2.10 The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

Solution

The areas of the three rectangles are 1 unit^2, 2 unit^2, and 5 unit^2. Using these rectangles, our area estimate is 8 unit^2.

2.3 Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown here:



Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions. Suppose, for example, that instead of determining the velocity of an object moving along a coordinate axis, we want to determine the velocity of a rock fired from a catapult at a given time, or of an airplane moving in three dimensions. We might want to graph real-value functions of two variables or determine volumes of solids of the type shown in **Figure 2.11**. These are only a few of the types of questions that can be asked and answered using **multivariable calculus**. Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables. However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.

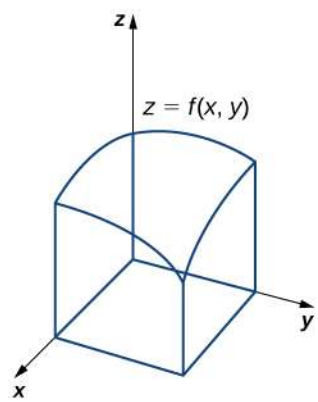


Figure 2.11 We can use multivariable calculus to find the volume between a surface defined by a function of two variables and a plane.

2.1 EXERCISES

For the following exercises, points $P(1, 2)$ and $Q(x, y)$ are on the graph of the function $f(x) = x^2 + 1$.

1. **[T]** Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q(x, y)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

2. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to *f* at $x = 1$.

3. Use the value in the preceding exercise to find the equation of the tangent line at point *P*. Graph $f(x)$ and the tangent line.

For the following exercises, points $P(1, 1)$ and $Q(x, y)$ are on the graph of the function $f(x) = x^3$.

4. **[T]** Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q(x, y)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

5. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to *f* at $x = 1$.

6. Use the value in the preceding exercise to find the equation of the tangent line at point *P*. Graph $f(x)$ and the tangent line.

For the following exercises, points $P(4, 2)$ and $Q(x, y)$ are on the graph of the function $f(x) = \sqrt{x}$.

7. **[T]** Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q(x, y)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
4.1	a.	e.	i.
4.01	b.	f.	j.
4.001	c.	g.	k.
4.0001	d.	h.	l.

8. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to *f* at $x = 4$.

9. Use the value in the preceding exercise to find the equation of the tangent line at point *P*.

For the following exercises, points $P(1.5, 0)$ and $Q(\phi, y)$ are on the graph of the function $f(\phi) = \cos(\pi\phi)$.

10. **[T]** Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(\varphi, y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(\phi, y)$	m_{sec}
1.4	a.	e.	i.
1.49	b.	f.	j.
1.499	c.	g.	k.
1.4999	d.	h.	l.

11. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $\varphi = 1.5$.

12. Use the value in the preceding exercise to find the equation of the tangent line at point P.

For the following exercises, points $P(-1, -1)$ and $Q(x, y)$ are on the graph of the function $f(x) = \frac{1}{x}$.

13. **[T]** Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x, y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
−1.05	a.	e.	i.
−1.01	b.	f.	j.
−1.005	c.	g.	k.
−1.001	d.	h.	l.

14. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x = -1$.

15. Use the value in the preceding exercise to find the equation of the tangent line at point P.

For the following exercises, the position function of a ball dropped from the top of a 200-meter tall building is given by $s(t) = 200 - 4.9t^2$, where position s is measured in meters and time t is measured in seconds. Round your answer to eight significant digits.

16. **[T]** Compute the average velocity of the ball over the given time intervals.

a. $[4.99, 5]$

b. $[5, 5.01]$

c. $[4.999, 5]$

d. $[5, 5.001]$

17. Use the preceding exercise to guess the instantaneous velocity of the ball at $t = 5$ sec.

For the following exercises, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h(t) = 15t - 4.9t^2$.

18. **[T]** Compute the average velocity of the stone over the given time intervals.

a. $[1, 1.05]$

b. $[1, 1.01]$

c. $[1, 1.005]$

d. $[1, 1.001]$

19. Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

For the following exercises, consider a rocket shot into the air that then returns to Earth. The height of the rocket in meters is given by $h(t) = 600 + 78.4t - 4.9t^2$, where t is measured in seconds.

20. **[T]** Compute the average velocity of the rocket over the given time intervals.

a. $[9, 9.01]$

b. $[8.99, 9]$

c. $[9, 9.001]$

d. $[8.999, 9]$

21. Use the preceding exercise to guess the instantaneous velocity of the rocket at $t = 9$ sec.

For the following exercises, consider an athlete running a 40-m dash. The position of the athlete is given by $d(t) = \frac{t^3}{6} + 4t$, where d is the position in meters and t is the time elapsed, measured in seconds.

22. **[T]** Compute the average velocity of the runner over the given time intervals.

a. $[1.95, 2.05]$

b. $[1.995, 2.005]$

c. $[1.9995, 2.0005]$

d. $[2, 2.00001]$

23. Use the preceding exercise to guess the instantaneous velocity of the runner at $t = 2$ sec.

For the following exercises, consider the function $f(x) = |x|$.

24. Sketch the graph of *f* over the interval $[-1, 2]$ and shade the region above the *x*-axis.

25. Use the preceding exercise to find the aproximate value of the area between the *x*-axis and the graph of *f* over the interval $[-1, 2]$ using rectangles. For the rectangles, use the square units, and approximate both above and below the lines. Use geometry to find the exact answer.

For the following exercises, consider the function $f(x) = \sqrt{1 - x^2}$. (*Hint*: This is the upper half of a circle of radius 1 positioned at $(0, 0)$.)

26. Sketch the graph of *f* over the interval $[-1, 1]$.

27. Use the preceding exercise to find the aproximate area between the *x*-axis and the graph of *f* over the interval $[-1, 1]$ using rectangles. For the rectangles, use squares 0.4 by 0.4 units, and approximate both above and below the lines. Use geometry to find the exact answer.

For the following exercises, consider the function $f(x) = -x^2 + 1$.

28. Sketch the graph of *f* over the interval $[-1, 1]$.

29. Approximate the area of the region between the *x*-axis and the graph of *f* over the interval $[-1, 1]$.

2.2 | The Limit of a Function

Learning Objectives

2.2.1 Using correct notation, describe the limit of a function.
2.2.2 Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
2.2.3 Use a graph to estimate the limit of a function or to identify when the limit does not exist.
2.2.4 Define one-sided limits and provide examples.
2.2.5 Explain the relationship between one-sided and two-sided limits.
2.2.6 Using correct notation, describe an infinite limit.
2.2.7 Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

$$f(x) = \frac{x^2 - 4}{x - 2}, \; g(x) = \frac{|x - 2|}{x - 2}, \text{ and } h(x) = \frac{1}{(x - 2)^2},$$

which are shown in **Figure 2.12**. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

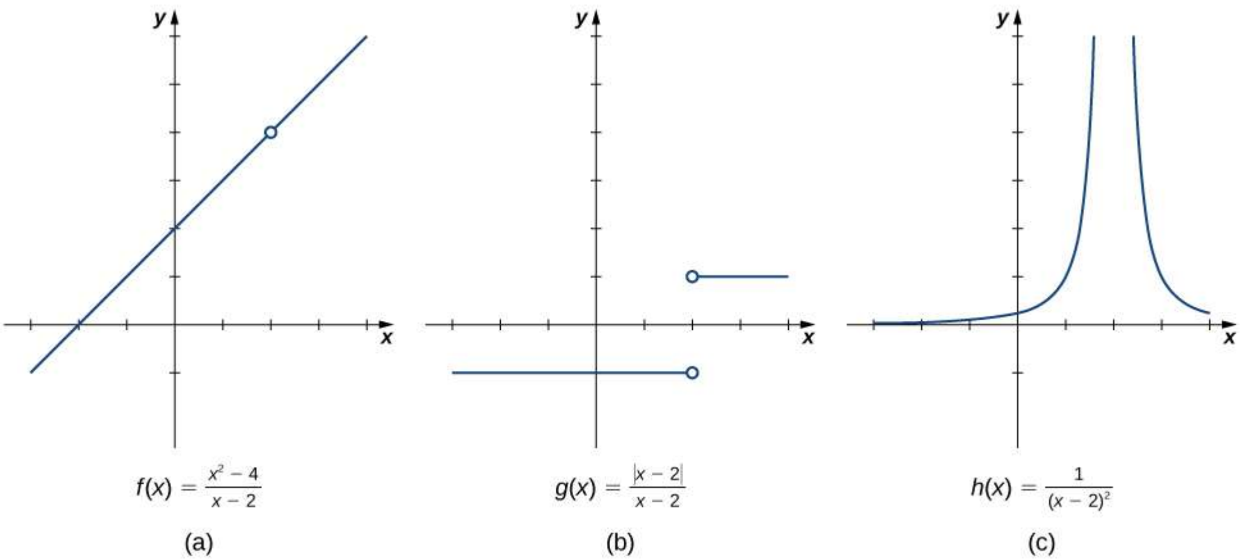


Figure 2.12 These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2,$ but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2.$ To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in **Figure 2.12**. As the values of *x* approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as *x* approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4.$$

From this very brief informal look at one limit, let's start to develop an **intuitive definition of the limit**. We can think of the limit of a function at a number a as being the one real number L that the functional values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following definition:

Definition

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If *all* values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. (More succinct, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L. \quad (2.3)$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x-values—one set of values approaching a and less than a, and another set of values approaching a and greater than a. Table 2.1 demonstrates what your tables might look like.

x	$f(x)$		x	$f(x)$
$a - 0.1$	$f(a - 0.1)$		$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$		$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$		$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$		$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.			Use additional values as necessary.	

Table 2.1 Table of Functional Values for $\lim_{x \to a} f(x)$

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a - 0.1), f(a - 0.01), f(a - 0.001)., f(a - 0.0001),$ and so on, and $f(a + 0.1), f(a + 0.01), f(a + 0.001), f(a + 0.0001),$ and so on. (*Note*: Although we have chosen the x-values $a \pm 0.1, a \pm 0.01, a \pm 0.001, a \pm 0.0001,$ and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)
3. If both columns approach a common y-value L, we state $\lim_{x \to a} f(x) = L.$ We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for *x*-values near *a* are in our window. We can use the trace feature to move along the graph of the function and watch the *y*-value readout as the *x*-values approach *a*. If the *y*-values approach *L* as our *x*-values approach *a* from both directions, then $\lim_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in **Example 2.4**.

Example 2.4

Evaluating a Limit Using a Table of Functional Values 1

Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$ using a table of functional values.

Solution

We have calculated the values of $f(x) = (\sin x)/x$ for the values of *x* listed in **Table 2.2**.

x	$\frac{\sin x}{x}$		x	$\frac{\sin x}{x}$
−0.1	0.998334166468		0.1	0.998334166468
−0.01	0.999983333417		0.01	0.999983333417
−0.001	0.999999833333		0.001	0.999999833333
−0.0001	0.999999998333		0.0001	0.999999998333

Table 2.2
Table of Functional Values for $\lim_{x \to 0} \frac{\sin x}{x}$

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\frac{(\sin x)}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x \to 0} \frac{\sin x}{x} = 1.$ A calculator or computer-generated graph of $f(x) = \frac{(\sin x)}{x}$ would be similar to that shown in **Figure 2.13**, and it confirms our estimate.

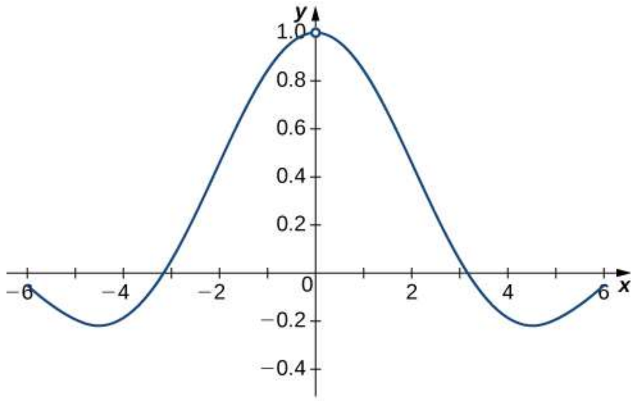


Figure 2.13 The graph of $f(x) = (\sin x)/x$ confirms the estimate from **Table 2.2**.

Example 2.5

Evaluating a Limit Using a Table of Functional Values 2

Evaluate $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$ using a table of functional values.

Solution

As before, we use a table—in this case, **Table 2.3**—to list the values of the function for the given values of *x*.

x	$\frac{\sqrt{x} - 2}{x - 4}$		x	$\frac{\sqrt{x} - 2}{x - 4}$
3.9	0.251582341869		4.1	0.248456731317
3.99	0.25015644562		4.01	0.24984394501
3.999	0.250015627		4.001	0.249984377
3.9999	0.250001563		4.0001	0.249998438
3.99999	0.25000016		4.00001	0.24999984

Table 2.3
Table of Functional Values for $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that

$\lim\limits_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = 0.25.$ We confirm this estimate using the graph of $f(x) = \frac{\sqrt{x} - 2}{x - 4}$ shown in **Figure 2.14**.

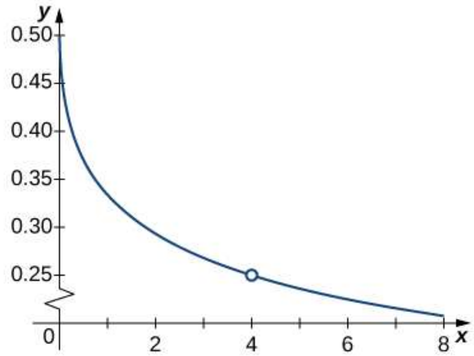


Figure 2.14 The graph of $f(x) = \frac{\sqrt{x} - 2}{x - 4}$ confirms the estimate from **Table 2.3**.

2.4 Estimate $\lim\limits_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1}$ using a table of functional values. Use a graph to confirm your estimate.

At this point, we see from **Example 2.4** and **Example 2.5** that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In **Example 2.6**, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

Example 2.6

Evaluating a Limit Using a Graph

For $g(x)$ shown in **Figure 2.15**, evaluate $\lim_{x \to -1} g(x)$.

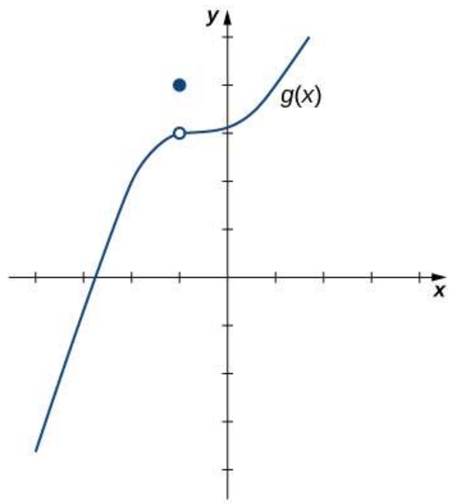


Figure 2.15 The graph of $g(x)$ includes one value not on a smooth curve.

Solution

Despite the fact that $g(-1) = 4,$ as the x-values approach -1 from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x \to -1} g(x) = 3.$ Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on **Example 2.6**, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

2.5 Use the graph of $h(x)$ in **Figure 2.16** to evaluate $\lim_{x \to 2} h(x),$ if possible.

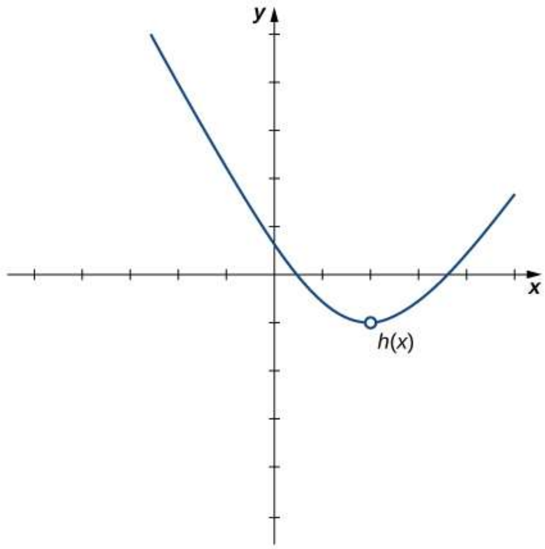


Figure 2.16

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Theorem 2.1: Two Important Limits

Let a be a real number and c be a constant.

i. $$\lim_{x \to a} x = a \quad (2.4)$$

ii. $$\lim_{x \to a} c = c \quad (2.5)$$

We can make the following observations about these two limits.

i. For the first limit, observe that as x approaches a, so does $f(x),$ because $f(x) = x.$ Consequently, $\lim_{x \to a} x = a.$

ii. For the second limit, consider **Table 2.4**.

x	$f(x) = c$		x	$f(x) = c$
$a - 0.1$	c		$a + 0.1$	c
$a - 0.01$	c		$a + 0.01$	c
$a - 0.001$	c		$a + 0.001$	c
$a - 0.0001$	c		$a + 0.0001$	c

Table 2.4 Table of Functional Values for $\lim_{x \to a} c = c$

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim_{x \to a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example 2.7

Evaluating a Limit That Fails to Exist

Evaluate $\lim_{x \to 0} \sin(1/x)$ using a table of values.

Solution

Table 2.5 lists values for the function $\sin(1/x)$ for the given values of x.

x	$\sin\left(\frac{1}{x}\right)$		x	$\sin\left(\frac{1}{x}\right)$
−0.1	0.544021110889		0.1	−0.544021110889
−0.01	0.50636564111		0.01	−0.50636564111
−0.001	−0.8268795405312		0.001	0.826879540532
−0.0001	0.305614388888		0.0001	−0.305614388888
−0.00001	−0.035748797987		0.00001	0.035748797987
−0.000001	0.349993504187		0.000001	−0.349993504187

Table 2.5
Table of Functional Values for $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let’s take a more systematic approach. Take the following sequence of x-values approaching 0:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding y-values are

$$1, -1, 1, -1, 1, -1, \ldots.$$

At this point we can indeed conclude that $\lim_{x \to 0} \sin(1/x)$ does not exist. (Mathematicians frequently abbreviate "does not exist" as DNE. Thus, we would write $\lim_{x \to 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in **Figure 2.17** and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between −1 and 1 as x approaches 0.

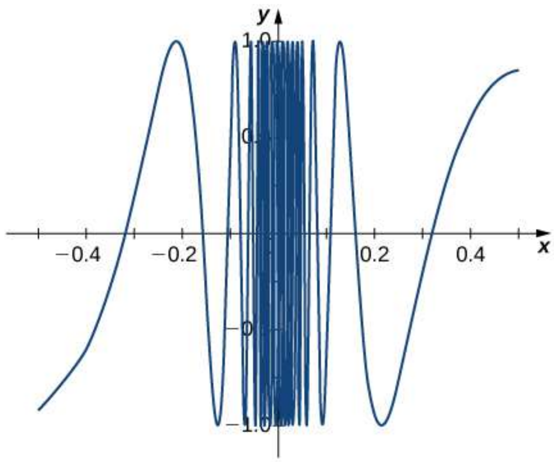


Figure 2.17 The graph of $f(x) = \sin(1/x)$ oscillates rapidly between −1 and 1 as x approaches 0.

2.6 Use a table of functional values to evaluate $\lim_{x \to 2} \frac{|x^2 - 4|}{x - 2}$, if possible.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x) = |x - 2|/(x - 2)$ introduced at the beginning of the section (see **Figure 2.12**(b)). As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist—that is, $\lim_{x \to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the x-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or *the negative side of* 2), $g(x) = -1$. Thus, as x approaches 2 from the left, $g(x)$ approaches −1. Mathematically, we say that the limit as x approaches 2 from the left is −1. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x) = -1.$$

Similarly, as x approaches 2 from the right (or *from the positive side*), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

Definition

We define two types of **one-sided limits**.

Limit from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (c, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L. \tag{2.6}$$

Limit from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L. \tag{2.7}$$

Example 2.8

Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x + 1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

Solution

We can use tables of functional values again **Table 2.6**. Observe that for values of x less than 2, we use $f(x) = x + 1$ and for values of x greater than 2, we use $f(x) = x^2 - 4$.

x	$f(x) = x + 1$		x	$f(x) = x^2 - 4$
1.9	2.9		2.1	0.41
1.99	2.99		2.01	0.0401
1.999	2.999		2.001	0.004001
1.9999	2.9999		2.0001	0.00040001
1.99999	2.99999		2.00001	0.0000400001

Table 2.6

Table of Functional Values for $f(x) = \begin{cases} x + 1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$. **Figure 2.18** shows a graph of $f(x)$ and reinforces our conclusion about these limits.

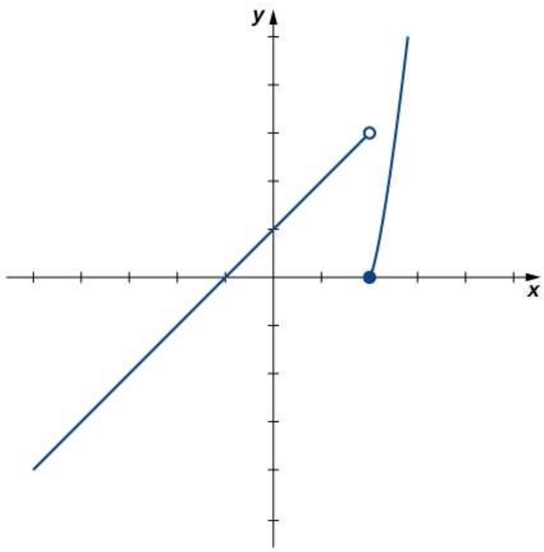


Figure 2.18 The graph of $f(x) = \begin{cases} x + 1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$ has a break at $x = 2$.

2.7 Use a table of functional values to estimate the following limits, if possible.

a. $\lim_{x \to 2^-} \frac{|x^2 - 4|}{x - 2}$

b. $\lim_{x \to 2^+} \frac{|x^2 - 4|}{x - 2}$

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in **Relating One-Sided and Two-Sided Limits**.

Theorem 2.2: Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing *a*, with the possible exception of *a* itself, and let *L* be a real number. Then,

$$\lim_{x \to a} f(x) = L. \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see **Figure 2.12**(c)). From its graph we see that as the values of x approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

$$\lim_{x \to 2} h(x) = +\infty.$$

More generally, we define **infinite limits** as follows:

Definition

We define three types of **infinite limits**.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty. \quad (2.8)$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty. \quad (2.9)$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty. \quad (2.10)$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty. \quad (2.11)$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a.

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty. \quad (2.12)$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty. \quad (2.13)$$

It is important to understand that when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim_{x \to a} f(x) = +\infty$, we always write $\lim_{x \to a} f(x) = +\infty$ rather than $\lim_{x \to a} f(x)$ DNE.

Example 2.9

Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \frac{1}{x}$

b. $\lim_{x \to 0^+} \frac{1}{x}$

c. $\lim_{x \to 0} \frac{1}{x}$

Solution

Begin by constructing a table of functional values.

x	$\frac{1}{x}$		x	$\frac{1}{x}$
−0.1	−10		0.1	10
−0.01	−100		0.01	100
−0.001	−1000		0.001	1000
−0.0001	−10,000		0.0001	10,000
−0.00001	−100,000		0.00001	100,000
−0.000001	−1,000,000		0.000001	1,000,000

Table 2.7
Table of Functional Values for $f(x) = \frac{1}{x}$

a. The values of $1/x$ decrease without bound as x approaches 0 from the left. We conclude that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as x approaches 0 from the right. We conclude that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x \to 0} \frac{1}{x} \text{ DNE}.$$

The graph of $f(x) = 1/x$ in **Figure 2.19** confirms these conclusions.

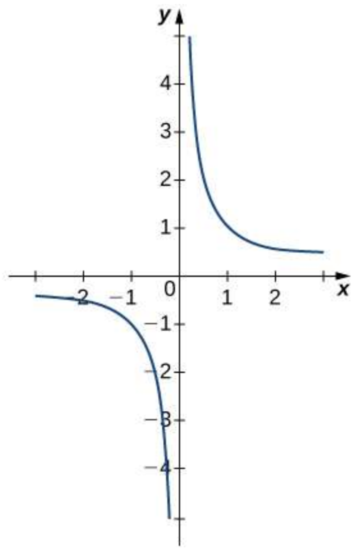


Figure 2.19 The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.

2.8 Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \frac{1}{x^2}$

b. $\lim_{x \to 0^+} \frac{1}{x^2}$

c. $\lim_{x \to 0} \frac{1}{x^2}$

It is useful to point out that functions of the form $f(x) = 1/(x - a)^n$, where n is a positive integer, have infinite limits as x approaches a from either the left or right (**Figure 2.20**). These limits are summarized in **Infinite Limits from Positive Integers**.

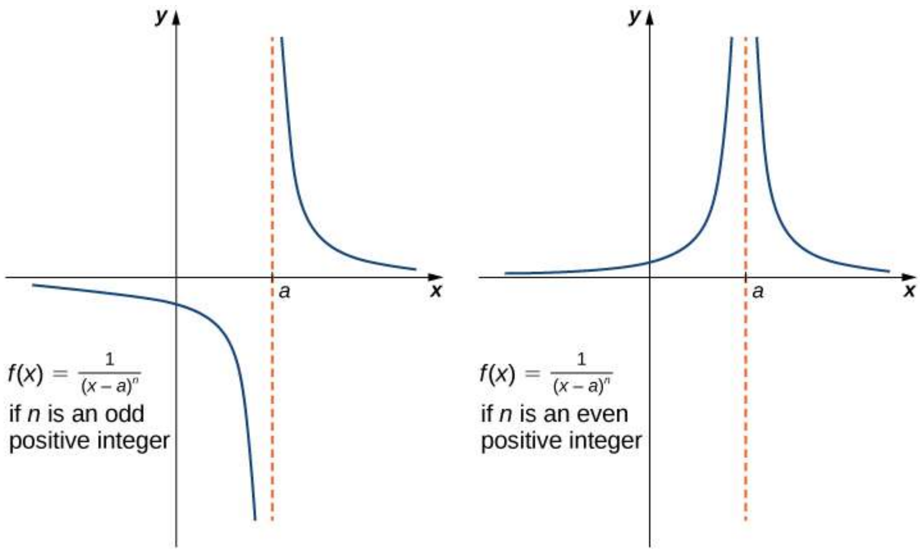


Figure 2.20 The function $f(x) = 1/(x - a)^n$ has infinite limits at a.

Theorem 2.3: Infinite Limits from Positive Integers

If n is a positive even integer, then

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If n is a positive odd integer, then

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having x-coordinates very near to a are very close to the vertical line $x = a$. That is, as x approaches a, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a **vertical asymptote** of the graph. We formally define a vertical asymptote as follows:

Definition

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a **vertical asymptote** of $f(x)$.

$$\lim_{x \to a^-} f(x) = +\infty \text{ or } -\infty$$
$$\lim_{x \to a^+} f(x) = +\infty \text{ or } -\infty$$
or
$$\lim_{x \to a} f(x) = +\infty \text{ or } -\infty$$

Example 2.10

Finding a Vertical Asymptote

Evaluate each of the following limits using **Infinite Limits from Positive Integers**. Identify any vertical asymptotes of the function $f(x) = 1/(x+3)^4$.

a. $\lim_{x \to -3^-} \frac{1}{(x+3)^4}$

b. $\lim_{x \to -3^+} \frac{1}{(x+3)^4}$

c. $\lim_{x \to -3} \frac{1}{(x+3)^4}$

Solution

We can use **Infinite Limits from Positive Integers** directly.

a. $\lim_{x \to -3^-} \frac{1}{(x+3)^4} = +\infty$

b. $\lim_{x \to -3^+} \frac{1}{(x+3)^4} = +\infty$

c. $\lim_{x \to -3} \frac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

2.9 Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \frac{1}{(x-2)^3}$.

a. $\lim_{x \to 2^-} \frac{1}{(x-2)^3}$

b. $\lim_{x \to 2^+} \frac{1}{(x-2)^3}$

c. $\lim_{x \to 2} \frac{1}{(x-2)^3}$

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example 2.11

Behavior of a Function at Different Points

Use the graph of $f(x)$ in **Figure 2.21** to determine each of the following values:

a. $\lim_{x \to -4^-} f(x);\ \lim_{x \to -4^+} f(x);\ \lim_{x \to -4} f(x);\ f(-4)$

b. $\lim_{x \to -2^-} f(x)$; $\lim_{x \to -2^+} f(x)$; $\lim_{x \to -2} f(x)$; $f(-2)$

c. $\lim_{x \to 1^-} f(x)$; $\lim_{x \to 1^+} f(x)$; $\lim_{x \to 1} f(x)$; $f(1)$

d. $\lim_{x \to 3^-} f(x)$; $\lim_{x \to 3^+} f(x)$; $\lim_{x \to 3} f(x)$; $f(3)$

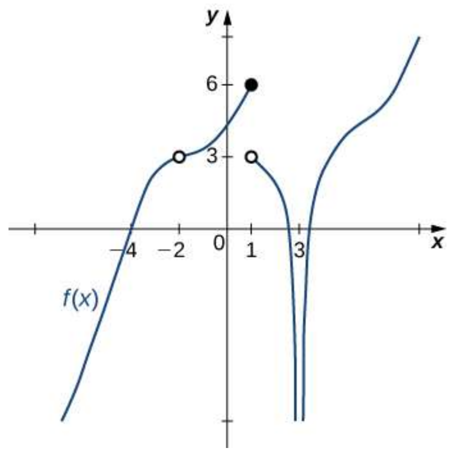


Figure 2.21 The graph shows $f(x)$.

Solution

Using **Infinite Limits from Positive Integers** and the graph for reference, we arrive at the following values:

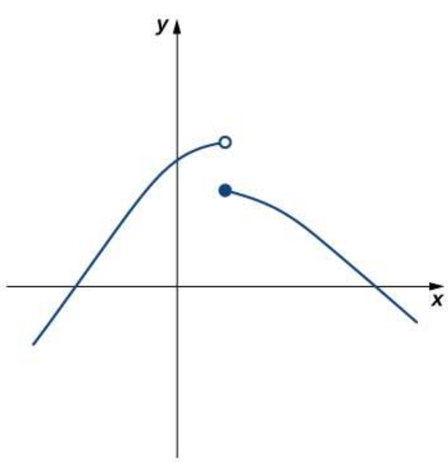
a. $\lim_{x \to -4^-} f(x) = 0$; $\lim_{x \to -4^+} f(x) = 0$; $\lim_{x \to -4} f(x) = 0$; $f(-4) = 0$

b. $\lim_{x \to -2^-} f(x) = 3.$; $\lim_{x \to -2^+} f(x) = 3$; $\lim_{x \to -2} f(x) = 3$; $f(-2)$ is undefined

c. $\lim_{x \to 1^-} f(x) = 6$; $\lim_{x \to 1^+} f(x) = 3$; $\lim_{x \to 1} f(x)$ DNE; $f(1) = 6$

d. $\lim_{x \to 3^-} f(x) = -\infty$; $\lim_{x \to 3^+} f(x) = -\infty$; $\lim_{x \to 3} f(x) = -\infty$; $f(3)$ is undefined

2.10 Evaluate $\lim_{x \to 1} f(x)$ for $f(x)$ shown here:



Example 2.12

Chapter Opener: Einstein's Equation

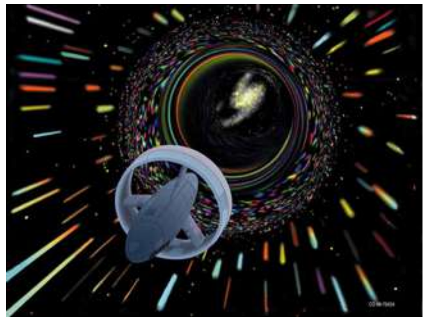


Figure 2.22 (credit: NASA)

In the chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object, what is the value of this bound?

Solution

Our starting point is Einstein's equation for the mass of a moving object,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we can graph the ratio of masses m/m_0 as a function of the ratio of speeds, v/c (Figure 2.23).

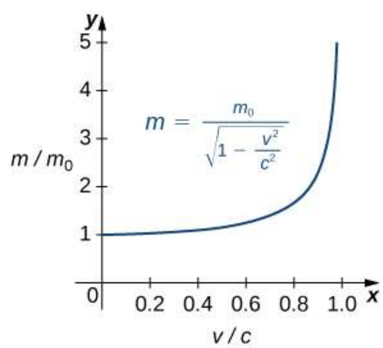


Figure 2.23 This graph shows the ratio of masses as a function of the ratio of speeds in Einstein's equation for the mass of a moving object.

We can see that as the ratio of speeds approaches 1—that is, as the speed of the object approaches the speed of light—the ratio of masses increases without bound. In other words, the function has a vertical asymptote at $v/c = 1.$ We can try a few values of this ratio to test this idea.

$\frac{v}{c}$	$\sqrt{1 - \frac{v^2}{c^2}}$	$\frac{m}{m_0}$
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.71

Table 2.8
Ratio of Masses and Speeds for a Moving Object

Thus, according to **Table 2.8**, if an object with mass 100 kg is traveling at 0.9999*c*, its mass becomes 7071 kg. Since no object can have an infinite mass, we conclude that no object can travel at or more than the speed of light.

2.2 EXERCISES

For the following exercises, consider the function $f(x) = \frac{x^2 - 1}{|x - 1|}$.

30. **[T]** Complete the following table for the function. Round your solutions to four decimal places.

x	$f(x)$		x	$f(x)$
0.9	a.		1.1	e.
0.99	b.		1.01	f.
0.999	c.		1.001	g.
0.9999	d.		1.0001	h.

31. What do your results in the preceding exercise indicate about the two-sided limit $\lim_{x \to 1} f(x)$? Explain your response.

For the following exercises, consider the function $f(x) = (1 + x)^{1/x}$.

32. **[T]** Make a table showing the values of f for $x = -0.01, -0.001, -0.0001, -0.00001$ and for $x = 0.01, 0.001, 0.0001, 0.00001$. Round your solutions to five decimal places.

x	$f(x)$		x	$f(x)$
−0.01	a.		0.01	e.
−0.001	b.		0.001	f.
−0.0001	c.		0.0001	g.
−0.00001	d.		0.00001	h.

33. What does the table of values in the preceding exercise indicate about the function $f(x) = (1 + x)^{1/x}$?

34. To which mathematical constant does the limit in the preceding exercise appear to be getting closer?

In the following exercises, use the given values to set up a table to evaluate the limits. Round your solutions to eight decimal places.

35. **[T]** $\lim_{x \to 0} \frac{\sin 2x}{x}$; $\pm 0.1, \pm 0.01, \pm 0.001, \pm .0001$

x	$\frac{\sin 2x}{x}$		x	$\frac{\sin 2x}{x}$
−0.1	a.		0.1	e.
−0.01	b.		0.01	f.
−0.001	c.		0.001	g.
−0.0001	d.		0.0001	h.

36. **[T]** $\lim_{x \to 0} \frac{\sin 3x}{x}$ $\pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

X	$\frac{\sin 3x}{x}$		x	$\frac{\sin 3x}{x}$
−0.1	a.		0.1	e.
−0.01	b.		0.01	f.
−0.001	c.		0.001	g.
−0.0001	d.		0.0001	h.

37. Use the preceding two exercises to conjecture (guess) the value of the following limit: $\lim_{x \to 0} \frac{\sin ax}{x}$ for a, a positive real value.

[T] In the following exercises, set up a table of values to find the indicated limit. Round to eight digits.

38. $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + x - 6}$

x	$\frac{x^2-4}{x^2+x-6}$		x	$\frac{x^2-4}{x^2+x-6}$
1.9	a.		2.1	e.
1.99	b.		2.01	f.
1.999	c.		2.001	g.
1.9999	d.		2.0001	h.

39. $\lim_{x \to 1} (1 - 2x)$

x	$1-2x$		x	$1-2x$
0.9	a.		1.1	e.
0.99	b.		1.01	f.
0.999	c.		1.001	g.
0.9999	d.		1.0001	h.

40. $\lim_{x \to 0} \frac{5}{1 - e^{1/x}}$

x	$\frac{5}{1-e^{1/x}}$		x	$\frac{5}{1-e^{1/x}}$
−0.1	a.		0.1	e.
−0.01	b.		0.01	f.
−0.001	c.		0.001	g.
−0.0001	d.		0.0001	h.

41. $\lim_{z \to 0} \frac{z - 1}{z^2 (z + 3)}$

z	$\frac{z-1}{z^2(z+3)}$		z	$\frac{z-1}{z^2(z+3)}$
−0.1	a.		0.1	e.
−0.01	b.		0.01	f.
−0.001	c.		0.001	g.
−0.0001	d.		0.0001	h.

42. $\lim_{t \to 0^+} \frac{\cos t}{t}$

t	$\frac{\cos t}{t}$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

43. $\lim_{x \to 2} \frac{1 - \frac{2}{x}}{x^2 - 4}$

x	$\frac{1-\frac{2}{x}}{x^2-4}$		x	$\frac{1-\frac{2}{x}}{x^2-4}$
1.9	a.		2.1	e.
1.99	b.		2.01	f.
1.999	c.		2.001	g.
1.9999	d.		2.0001	h.

[T] In the following exercises, set up a table of values and round to eight significant digits. Based on the table of values, make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?

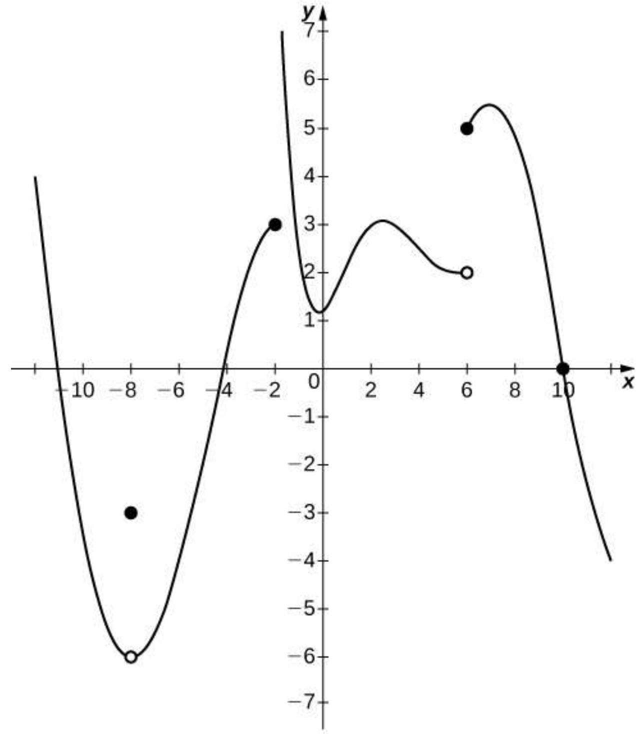
44. $\lim_{\theta \to 0} \sin\left(\frac{\pi}{\theta}\right)$

θ	$\sin\left(\frac{\pi}{\theta}\right)$		θ	$\sin\left(\frac{\pi}{\theta}\right)$
−0.1	a.		0.1	e.
−0.01	b.		0.01	f.
−0.001	c.		0.001	g.
−0.0001	d.		0.0001	h.

45. $\lim_{\alpha \to 0^+} \frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$

a	$\frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right)$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

In the following exercises, consider the graph of the function $y = f(x)$ shown here. Which of the statements about $y = f(x)$ are true and which are false? Explain why a statement is false.



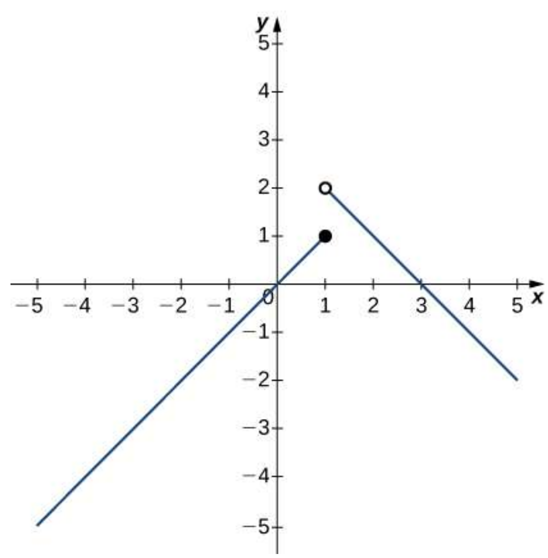
46. $\lim_{x \to 10} f(x) = 0$

47. $\lim_{x \to -2^+} f(x) = 3$

48. $\lim_{x \to -8} f(x) = f(-8)$

49. $\lim_{x \to 6} f(x) = 5$

In the following exercises, use the following graph of the function $y = f(x)$ to find the values, if possible. Estimate when necessary.



50. $\lim_{x \to 1^-} f(x)$

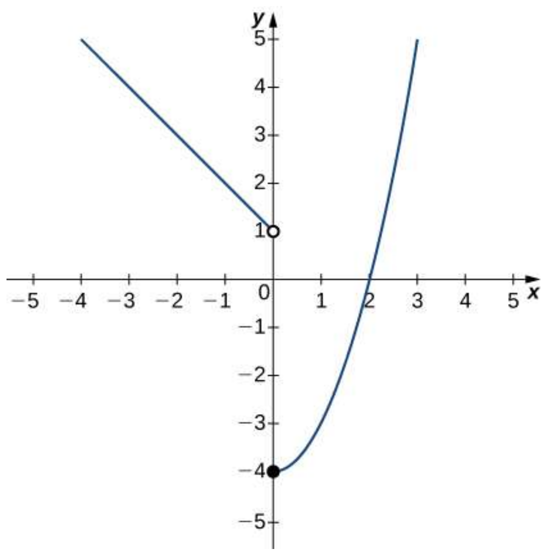
51. $\lim_{x \to 1^+} f(x)$

52. $\lim_{x \to 1} f(x)$

53. $\lim_{x \to 2} f(x)$

54. $f(1)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



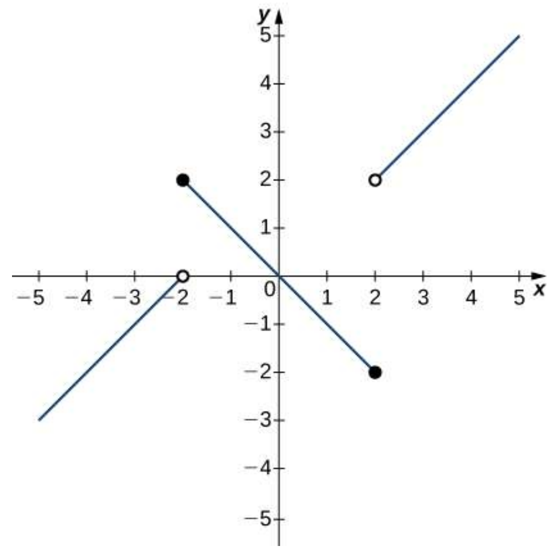
55. $\lim_{x \to 0^-} f(x)$

56. $\lim_{x \to 0^+} f(x)$

57. $\lim_{x \to 0} f(x)$

58. $\lim_{x \to 2} f(x)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



59. $\lim_{x \to -2^-} f(x)$

60. $\lim_{x \to -2^+} f(x)$

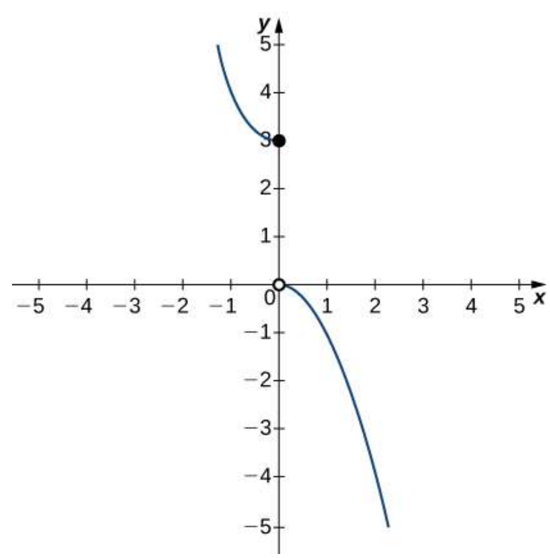
61. $\lim_{x \to -2} f(x)$

62. $\lim_{x \to 2^-} f(x)$

63. $\lim_{x \to 2^+} f(x)$

64. $\lim_{x \to 2} f(x)$

In the following exercises, use the graph of the function $y = g(x)$ shown here to find the values, if possible. Estimate when necessary.

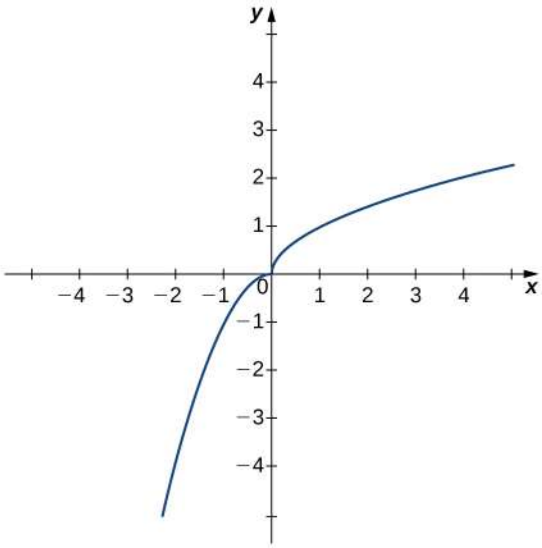


65. $\lim_{x \to 0^-} g(x)$

66. $\lim_{x \to 0^+} g(x)$

67. $\lim_{x \to 0} g(x)$

In the following exercises, use the graph of the function $y = h(x)$ shown here to find the values, if possible. Estimate when necessary.

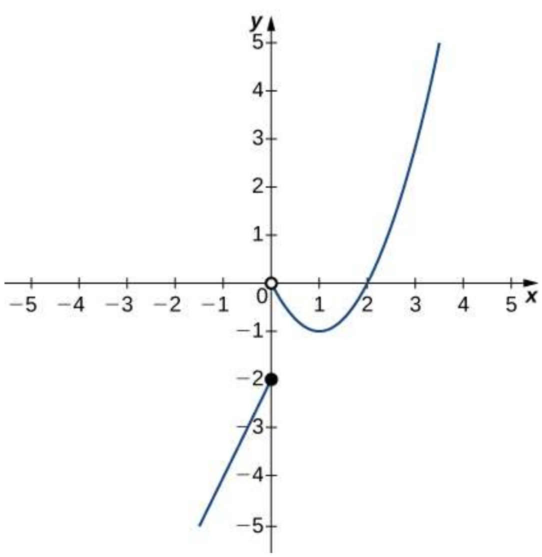


68. $\lim_{x \to 0^-} h(x)$

69. $\lim_{x \to 0^+} h(x)$

70. $\lim_{x \to 0} h(x)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



71. $\lim_{x \to 0^-} f(x)$

72. $\lim_{x \to 0^+} f(x)$

73. $\lim_{x \to 0} f(x)$

74. $\lim_{x \to 1} f(x)$

75. $\lim_{x \to 2} f(x)$

In the following exercises, sketch the graph of a function with the given properties.

76. $\lim_{x \to 2} f(x) = 1,\ \lim_{x \to 4^-} f(x) = 3,\ \lim_{x \to 4^+} f(x) = 6,\ f(4)$ is not defined.

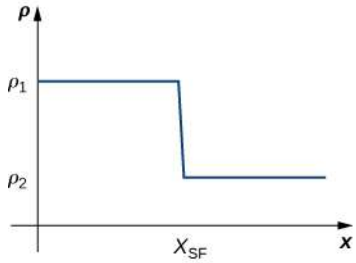
77. $\lim_{x \to -\infty} f(x) = 0,\ \lim_{x \to -1^-} f(x) = -\infty,\ \lim_{x \to -1^+} f(x) = \infty,\ \lim_{x \to 0} f(x) = f(0),\ f(0) = 1,\ \lim_{x \to \infty} f(x) = -\infty$

78. $\lim_{x \to -\infty} f(x) = 2,\ \lim_{x \to 3^-} f(x) = -\infty,\ \lim_{x \to 3^+} f(x) = \infty,\ \lim_{x \to \infty} f(x) = 2,\ f(0) = \frac{-1}{3}$

79. $\lim_{x \to -\infty} f(x) = 2,\ \lim_{x \to -2} f(x) = -\infty,\ \lim_{x \to \infty} f(x) = 2,\ f(0) = 0$

80. $\lim_{x \to -\infty} f(x) = 0,\ \lim_{x \to -1^-} f(x) = \infty,\ \lim_{x \to -1^+} f(x) = -\infty,\ f(0) = -1,\ \lim_{x \to 1^-} f(x) = -\infty,\ \lim_{x \to 1^+} f(x) = \infty,\ \lim_{x \to \infty} f(x) = 0$

81. Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a shock wave with respect to distance, x, is shown here. We are mainly interested in the location of the front of the shock, labeled x_{SF} in the diagram.



a. Evaluate $\lim_{x \to x_{SF}^+} \rho(x)$.

b. Evaluate $\lim_{x \to x_{SF}^-} \rho(x)$.

c. Evaluate $\lim_{x \to x_{SF}} \rho(x)$. Explain the physical meanings behind your answers.

82. A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where x is the position in meters of the runner and t is time in seconds. What is $\lim_{t \to 2} x(t)$? What does it mean physically?

t **(sec)**	***x*** **(m)**
1.75	4.5
1.95	6.1
1.99	6.42
2.01	6.58
2.05	6.9
2.25	8.5

2.3 | The Limit Laws

Learning Objectives

2.3.1 Recognize the basic limit laws.
2.3.2 Use the limit laws to evaluate the limit of a function.
2.3.3 Evaluate the limit of a function by factoring.
2.3.4 Use the limit laws to evaluate the limit of a polynomial or rational function.
2.3.5 Evaluate the limit of a function by factoring or by using conjugates.
2.3.6 Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated in **Two Important Limits** and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Theorem 2.4: Basic Limit Results

For any real number a and any constant c,

i. $$\lim_{x \to a} x = a \quad (2.14)$$

ii. $$\lim_{x \to a} c = c \quad (2.15)$$

Example 2.13

Evaluating a Basic Limit

Evaluate each of the following limits using **Basic Limit Results**.

a. $\lim_{x \to 2} x$

b. $\lim_{x \to 2} 5$

Solution

a. The limit of x as x approaches a is a: $\lim_{x \to 2} x = 2.$

b. The limit of a constant is that constant: $\lim_{x \to 2} 5 = 5.$

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Theorem 2.5: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

Sum law for limits: $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

Difference law for limits: $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$

Product law for limits: $\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

Quotient law for limits: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x \to a}(f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every positive integer n.

Root law for limits: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even and $f(x) \geq 0$.

We now practice applying these limit laws to evaluate a limit.

Example 2.14

Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate $\lim_{x \to -3}(4x + 2)$.

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \to -3}(4x + 2) &= \lim_{x \to -3} 4x + \lim_{x \to -3} 2 && \text{Apply the sum law.} \\ &= 4 \cdot \lim_{x \to -3} x + \lim_{x \to -3} 2 && \text{Apply the constant multiple law.} \\ &= 4 \cdot (-3) + 2 = -10. && \text{Apply the basic limit results and simplify.} \end{aligned}$$

Example 2.15

Using Limit Laws Repeatedly

Use the limit laws to evaluate $\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$.

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\begin{aligned}
\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4} &= \frac{\lim_{x \to 2}\left(2x^2 - 3x + 1\right)}{\lim_{x \to 2}\left(x^3 + 4\right)} && \text{Apply the quotient law, making sure that. } (2)^3 + 4 \neq 0 \\
&= \frac{2 \cdot \lim_{x \to 2} x^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{\lim_{x \to 2} x^3 + \lim_{x \to 2} 4} && \text{Apply the sum law and constant multiple law.} \\
&= \frac{2 \cdot \left(\lim_{x \to 2} x\right)^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{\left(\lim_{x \to 2} x\right)^3 + \lim_{x \to 2} 4} && \text{Apply the power law.} \\
&= \frac{2(4) - 3(2) + 1}{(2)^3 + 4} = \frac{1}{4}. && \text{Apply the basic limit laws and simplify.}
\end{aligned}$$

2.11 Use the limit laws to evaluate $\lim_{x \to 6} (2x - 1)\sqrt{x + 4}$. In each step, indicate the limit law applied.

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x \to a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Theorem 2.6: Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\lim_{x \to a} p(x) = p(a)$$

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

To see that this theorem holds, consider the polynomial $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$. By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned}
\lim_{x \to a} p(x) &= \lim_{x \to a}\left(c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0\right) \\
&= c_n\left(\lim_{x \to a} x\right)^n + c_{n-1}\left(\lim_{x \to a} x\right)^{n-1} + \cdots + c_1\left(\lim_{x \to a} x\right) + \lim_{x \to a} c_0 \\
&= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 \\
&= p(a).
\end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a) \neq 0$, then

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Example 2.16 applies this result.

Example 2.16

Evaluating a Limit of a Rational Function

Evaluate the $\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4}$.

Solution

Since 3 is in the domain of the rational function $f(x) = \frac{2x^2 - 3x + 1}{5x + 4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4} = \frac{10}{19}.$$

2.12 Evaluate $\lim_{x \to -2} (3x^3 - 2x + 7)$.

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x \to a} f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x \neq a$, $f(x) = g(x)$ over some open interval containing a, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

To understand this idea better, consider the limit $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

The function

$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x - 1} \\ &= \frac{(x - 1)(x + 1)}{x - 1} \end{aligned}$$

and the function $g(x) = x + 1$ are identical for all values of $x \neq 1$. The graphs of these two functions are shown in **Figure 2.24**.

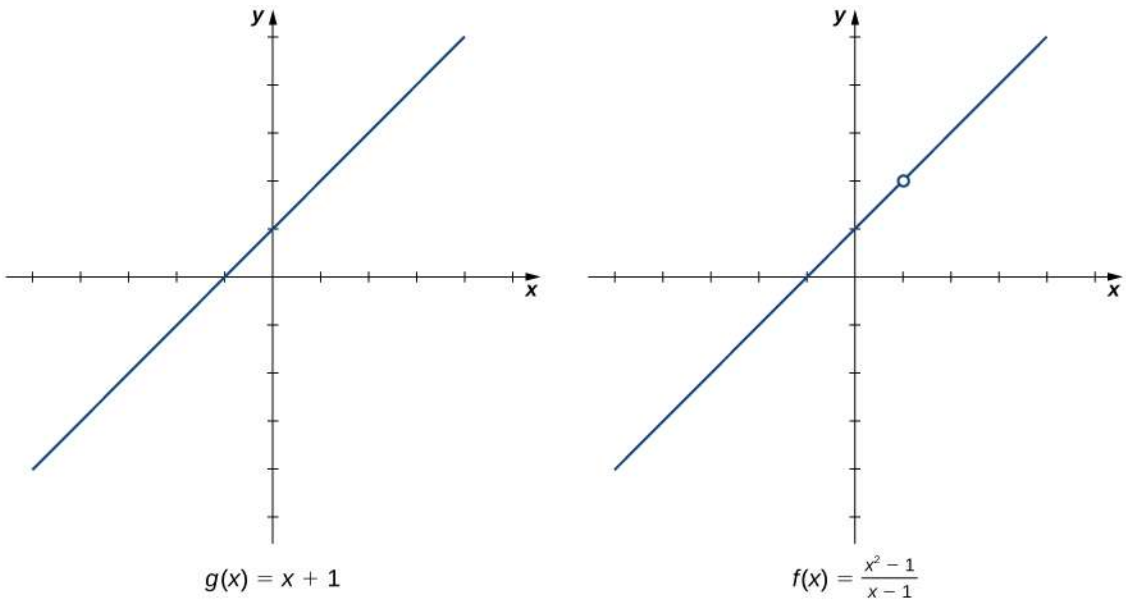


Figure 2.24 The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\begin{aligned}\lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} \\ &= \lim_{x \to 1} (x+1) \\ &= 2.\end{aligned}$$

The limit has the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form 0/0

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x) = f(x)/g(x)$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. **Example 2.17** illustrates the factor-and-cancel technique; **Example 2.18** shows multiplying by a conjugate. In **Example 2.19**, we look at simplifying a complex fraction.

Example 2.17

Evaluating a Limit by Factoring and Canceling

Evaluate $\lim\limits_{x \to 3} \frac{x^2 - 3x}{2x^2 - 5x - 3}$.

Solution

Step 1. The function $f(x) = \frac{x^2 - 3x}{2x^2 - 5x - 3}$ is undefined for $x = 3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x \to 3} \frac{x^2 - 3x}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{x(x - 3)}{(x - 3)(2x + 1)}$$

Step 2. For all $x \neq 3$, $\frac{x^2 - 3x}{2x^2 - 5x - 3} = \frac{x}{2x + 1}$. Therefore,

$$\lim_{x \to 3} \frac{x(x - 3)}{(x - 3)(2x + 1)} = \lim_{x \to 3} \frac{x}{2x + 1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x \to 3} \frac{x}{2x + 1} = \frac{3}{7}.$$

2.13 Evaluate $\lim\limits_{x \to -3} \frac{x^2 + 4x + 3}{x^2 - 9}$.

Example 2.18

Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim\limits_{x \to -1} \frac{\sqrt{x + 2} - 1}{x + 1}$.

Solution

Step 1. $\frac{\sqrt{x + 2} - 1}{x + 1}$ has the form $0/0$ at -1. Let's begin by multiplying by $\sqrt{x + 2} + 1$, the conjugate of $\sqrt{x + 2} - 1$, on the numerator and denominator:

$$\lim_{x \to -1} \frac{\sqrt{x + 2} - 1}{x + 1} = \lim_{x \to -1} \frac{\sqrt{x + 2} - 1}{x + 1} \cdot \frac{\sqrt{x + 2} + 1}{\sqrt{x + 2} + 1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x + 1)$ in the denominator cancels out in the end:

$$= \lim_{x \to -1} \frac{x + 1}{(x + 1)\left(\sqrt{x + 2} + 1\right)}.$$

Step 3. Then we cancel:

$$= \lim_{x \to -1} \frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x \to -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{2}.$$

2.14 Evaluate $\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x-5}$.

Example 2.19

Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$.

Solution

Step 1. $\frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} = \lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} \cdot \frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

$$= \lim_{x \to 1} \frac{2-(x+1)}{2(x-1)(x+1)}.$$

Step 3. Then, we simplify the numerator:

$$= \lim_{x \to 1} \frac{-x+1}{2(x-1)(x+1)}.$$

Step 4. Now we factor out -1 from the numerator:

$$= \lim_{x \to 1} \frac{-(x-1)}{2(x-1)(x+1)}.$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$= \lim_{x \to 1} \frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x \to 1} \frac{-1}{2(x+1)} = -\frac{1}{4}.$$

2.15 Evaluate $\lim_{x \to -3} \frac{\frac{1}{x+2} + 1}{x+3}$.

Example 2.20 does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example 2.20

Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim_{x \to 0} \left(\frac{1}{x} + \frac{5}{x(x-5)}\right)$.

Solution

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\begin{aligned} \frac{1}{x} + \frac{5}{x(x-5)} &= \frac{x-5+5}{x(x-5)} \\ &= \frac{x}{x(x-5)}. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{x \to 0} \left(\frac{1}{x} + \frac{5}{x(x-5)}\right) &= \lim_{x \to 0} \frac{x}{x(x-5)} \\ &= \lim_{x \to 0} \frac{1}{x-5} \\ &= -\frac{1}{5}. \end{aligned}$$

2.16 Evaluate $\lim_{x \to 3} \left(\frac{1}{x-3} - \frac{4}{x^2 - 2x - 3}\right)$.

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim_{x \to a^-} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b, a); for a limit of the form $\lim_{x \to a^+} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a, c). **Example 2.21** illustrates this point.

Example 2.21

Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim_{x \to 3^-} \sqrt{x-3}$

b. $\lim_{x \to 3^+} \sqrt{x-3}$

Solution

Figure 2.25 illustrates the function $f(x) = \sqrt{x-3}$ and aids in our understanding of these limits.

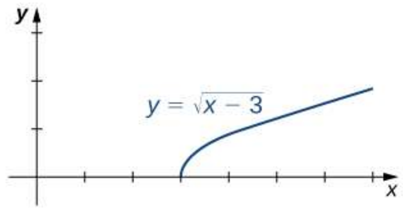


Figure 2.25 The graph shows the function $f(x) = \sqrt{x-3}$.

a. The function $f(x) = \sqrt{x-3}$ is defined over the interval $[3, +\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim_{x \to 3^-} \sqrt{x-3}$. In fact, since $f(x) = \sqrt{x-3}$ is undefined to the left of 3, $\lim_{x \to 3^-} \sqrt{x-3}$ does not exist.

b. Since $f(x) = \sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim_{x \to 3^+} \sqrt{x-3}$. By applying these limit laws we obtain $\lim_{x \to 3^+} \sqrt{x-3} = 0$.

In Example 2.22 we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example 2.22

Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x-3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

c. $\lim_{x \to 2} f(x)$

Solution

Figure 2.26 illustrates the function $f(x)$ and aids in our understanding of these limits.

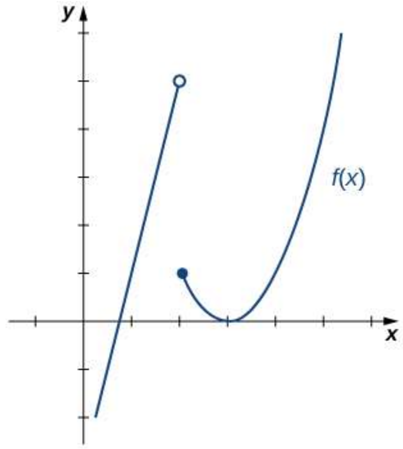


Figure 2.26 This graph shows a function $f(x)$.

a. Since $f(x) = 4x - 3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x - 3$ and apply the limit laws:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x - 3) = 5.$$

b. Since $f(x) = (x - 3)^2$ for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x - 3)^2$ and apply the limit laws:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 3)^2 = 1.$$

c. Since $\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 1$, we conclude that $\lim_{x \to 2} f(x)$ does not exist.

2.17 Graph $f(x) = \begin{cases} -x - 2 \text{ if } x < -1 \\ 2 \text{ if } x = -1 \\ x^3 \text{ if } x > -1 \end{cases}$ and evaluate $\lim_{x \to -1^-} f(x)$.

We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0, K \neq 0$ at a.

Example 2.23

Evaluating a Limit of the Form $K/0, K \neq 0$ Using the Limit Laws

Evaluate $\lim_{x \to 2^-} \frac{x - 3}{x^2 - 2x}$.

Solution

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as x approaches 2 from the left, the numerator approaches −1; and the denominator approaches 0. Consequently, the magnitude of $\frac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = \lim_{x \to 2^-} \frac{x-3}{x(x-2)}.$$

Step 2. Since $x-2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x-2)$ from the rest of the function:

$$= \lim_{x \to 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}.$$

Step 3. $\lim_{x \to 2^-} \frac{x-3}{x} = -\frac{1}{2}$ and $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = +\infty.$$

2.18 Evaluate $\lim_{x \to 1} \frac{x+2}{(x-1)^2}$.

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem**, proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by "squeezing" a function, with a limit at a point *a* that is unknown, between two functions having a common known limit at *a*. **Figure 2.27** illustrates this idea.

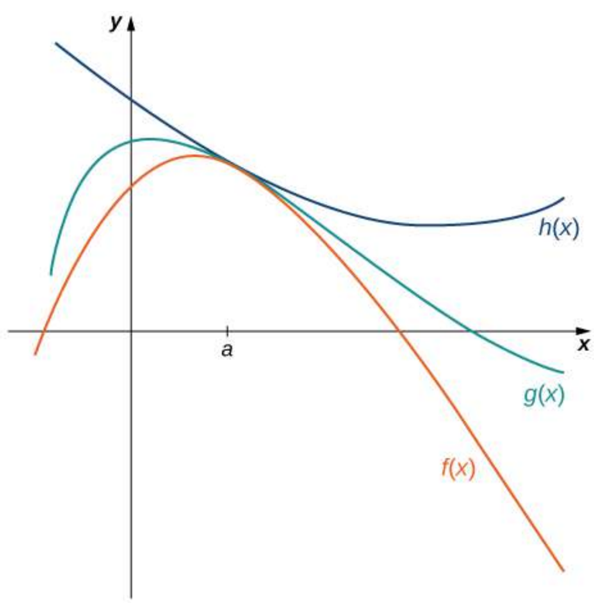


Figure 2.27 The Squeeze Theorem applies when $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Theorem 2.7: The Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.

Example 2.24

Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim_{x \to 0} x \cos x$.

Solution

Because $-1 \leq \cos x \leq 1$ for all x, we have $-|x| \leq x \cos x \leq |x|$. Since $\lim_{x \to 0}(-|x|) = 0 = \lim_{x \to 0}|x|$, from the squeeze theorem, we obtain $\lim_{x \to 0} x \cos x = 0$. The graphs of $f(x) = -|x|$, $g(x) = x \cos x$, and $h(x) = |x|$ are shown in **Figure 2.28**.

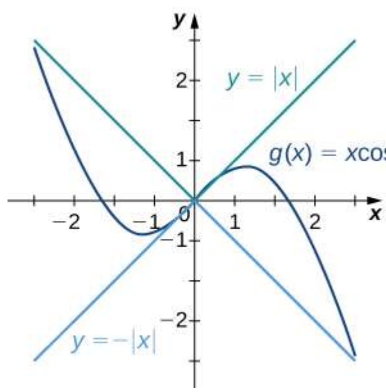


Figure 2.28 The graphs of $f(x)$, $g(x)$, and $h(x)$ are shown around the point $x = 0$.

2.19 Use the squeeze theorem to evaluate $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim_{\theta \to 0} \sin \theta$. Consider the unit circle shown in **Figure 2.29**. In the figure, we see that $\sin \theta$ is the y-coordinate on the unit circle and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \frac{\pi}{2}$, $0 < \sin \theta < \theta$.

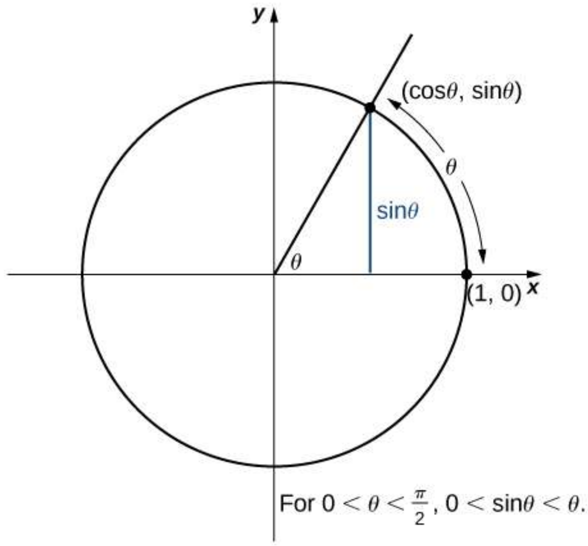


Figure 2.29 The sine function is shown as a line on the unit circle.

Because $\lim_{\theta \to 0^+} 0 = 0$ and $\lim_{\theta \to 0^+} \theta = 0,$ by using the squeeze theorem we conclude that

$$\lim_{\theta \to 0^+} \sin\theta = 0.$$

To see that $\lim_{\theta \to 0^-} \sin\theta = 0$ as well, observe that for $-\frac{\pi}{2} < \theta < 0,\ 0 < -\theta < \frac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta.$ Consequently, $0 < -\sin\theta < -\theta.$ It follows that $0 > \sin\theta > \theta.$ An application of the squeeze theorem produces the desired limit. Thus, since $\lim_{\theta \to 0^+} \sin\theta = 0$ and $\lim_{\theta \to 0^-} \sin\theta = 0,$

$$\lim_{\theta \to 0} \sin\theta = 0. \tag{2.16}$$

Next, using the identity $\cos\theta = \sqrt{1 - \sin^2\theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2},$ we see that

$$\lim_{\theta \to 0} \cos\theta = \lim_{\theta \to 0} \sqrt{1 - \sin^2\theta} = 1. \tag{2.17}$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}.$ To evaluate this limit, we use the unit circle in **Figure 2.30**. Notice that this figure adds one additional triangle to **Figure 2.30**. We see that the length of the side opposite angle θ in this new triangle is $\tan\theta.$ Thus, we see that for $0 < \theta < \frac{\pi}{2},\ \sin\theta < \theta < \tan\theta.$

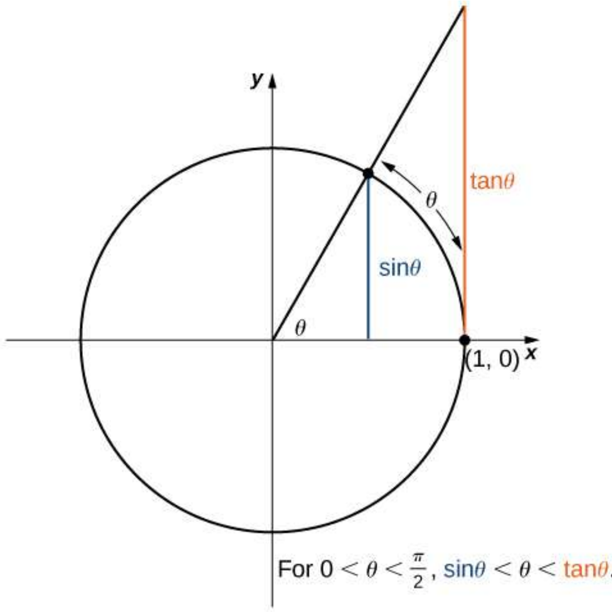


Figure 2.30 The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin\theta$ in all parts of the inequality, we obtain

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Equivalently, we have

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim\limits_{\theta \to 0^+} 1 = 1 = \lim\limits_{\theta \to 0^+} \cos\theta,$ we conclude that $\lim\limits_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1.$ By applying a manipulation similar to that used in demonstrating that $\lim\limits_{\theta \to 0^-} \sin\theta = 0,$ we can show that $\lim\limits_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1.$ Thus,

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1. \tag{2.18}$$

In **Example 2.25** we use this limit to establish $\lim\limits_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0.$ This limit also proves useful in later chapters.

Example 2.25

Evaluating an Important Trigonometric Limit

Evaluate $\lim\limits_{\theta \to 0} \frac{1 - \cos\theta}{\theta}.$

Solution

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

$$\begin{aligned}\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} &= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta} \\ &= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta(1 + \cos\theta)} \\ &= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1 + \cos\theta)} \\ &= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1 + \cos\theta} \\ &= 1 \cdot \frac{0}{2} = 0.\end{aligned}$$

Therefore,

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0. \tag{2.19}$$

2.20 Evaluate $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin\theta}$.

Student PROJECT

Deriving the Formula for the Area of a Circle

Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician Archimedes (ca. 287−212; BCE) was particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit, but we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of n triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1. Express the height h and the base b of the isosceles triangle in **Figure 2.31** in terms of θ and r.

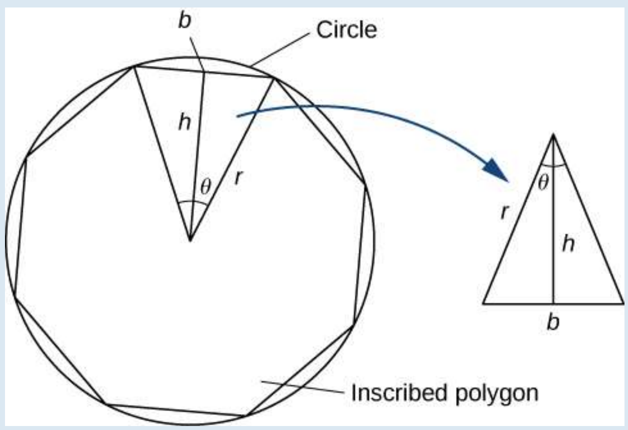


Figure 2.31

2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and r. (Substitute $(1/2)\sin\theta$ for $\sin(\theta/2)\cos(\theta/2)$ in your expression.)
3. If an n-sided regular polygon is inscribed in a circle of radius r, find a relationship between θ and n. Solve this for n. Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)
4. Find an expression for the area of the n-sided polygon in terms of r and θ.
5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero. (*Hint:* $\lim\limits_{\theta \to 0} \frac{(\sin\theta)}{\theta} = 1$).

The technique of estimating areas of regions by using polygons is revisited in **Introduction to Integration**.

2.3 EXERCISES

In the following exercises, use the limit laws to evaluate each limit. Justify each step by indicating the appropriate limit law(s).

83. $\lim_{x \to 0}\left(4x^2 - 2x + 3\right)$

84. $\lim_{x \to 1}\frac{x^3 + 3x^2 + 5}{4 - 7x}$

85. $\lim_{x \to -2}\sqrt{x^2 - 6x + 3}$

86. $\lim_{x \to -1}(9x + 1)^2$

In the following exercises, use direct substitution to evaluate each limit.

87. $\lim_{x \to 7}x^2$

88. $\lim_{x \to -2}\left(4x^2 - 1\right)$

89. $\lim_{x \to 0}\frac{1}{1 + \sin x}$

90. $\lim_{x \to 2}e^{2x - x^2}$

91. $\lim_{x \to 1}\frac{2 - 7x}{x + 6}$

92. $\lim_{x \to 3}\ln e^{3x}$

In the following exercises, use direct substitution to show that each limit leads to the indeterminate form $0/0$. Then, evaluate the limit.

93. $\lim_{x \to 4}\frac{x^2 - 16}{x - 4}$

94. $\lim_{x \to 2}\frac{x - 2}{x^2 - 2x}$

95. $\lim_{x \to 6}\frac{3x - 18}{2x - 12}$

96. $\lim_{h \to 0}\frac{(1 + h)^2 - 1}{h}$

97. $\lim_{t \to 9}\frac{t - 9}{\sqrt{t} - 3}$

98. $\lim_{h \to 0}\frac{\frac{1}{a + h} - \frac{1}{a}}{h}$, where a is a non-zero real-valued constant

99. $\lim_{\theta \to \pi}\frac{\sin\theta}{\tan\theta}$

100. $\lim_{x \to 1}\frac{x^3 - 1}{x^2 - 1}$

101. $\lim_{x \to 1/2}\frac{2x^2 + 3x - 2}{2x - 1}$

102. $\lim_{x \to -3}\frac{\sqrt{x + 4} - 1}{x + 3}$

In the following exercises, use direct substitution to obtain an undefined expression. Then, use the method of **Example 2.23** to simplify the function to help determine the limit.

103. $\lim_{x \to -2^-}\frac{2x^2 + 7x - 4}{x^2 + x - 2}$

104. $\lim_{x \to -2^+}\frac{2x^2 + 7x - 4}{x^2 + x - 2}$

105. $\lim_{x \to 1^-}\frac{2x^2 + 7x - 4}{x^2 + x - 2}$

106. $\lim_{x \to 1^+}\frac{2x^2 + 7x - 4}{x^2 + x - 2}$

In the following exercises, assume that $\lim_{x \to 6}f(x) = 4$, $\lim_{x \to 6}g(x) = 9$, and $\lim_{x \to 6}h(x) = 6$. Use these three facts and the limit laws to evaluate each limit.

107. $\lim_{x \to 6}2f(x)g(x)$

108. $\lim_{x \to 6}\frac{g(x) - 1}{f(x)}$

109. $\lim_{x \to 6}\left(f(x) + \frac{1}{3}g(x)\right)$

110. $\lim_{x \to 6}\frac{(h(x))^3}{2}$

111. $\lim_{x \to 6}\sqrt{g(x) - f(x)}$

112. $\lim_{x \to 6}x \cdot h(x)$

113. $\lim_{x \to 6} [(x + 1) \cdot f(x)]$

114. $\lim_{x \to 6} (f(x) \cdot g(x) - h(x))$

[T] In the following exercises, use a calculator to draw the graph of each piecewise-defined function and study the graph to evaluate the given limits.

115. $f(x) = \begin{cases} x^2, & x \le 3 \\ x + 4, & x > 3 \end{cases}$

a. $\lim_{x \to 3^-} f(x)$

b. $\lim_{x \to 3^+} f(x)$

116. $g(x) = \begin{cases} x^3 - 1, & x \le 0 \\ 1, & x > 0 \end{cases}$

a. $\lim_{x \to 0^-} g(x)$

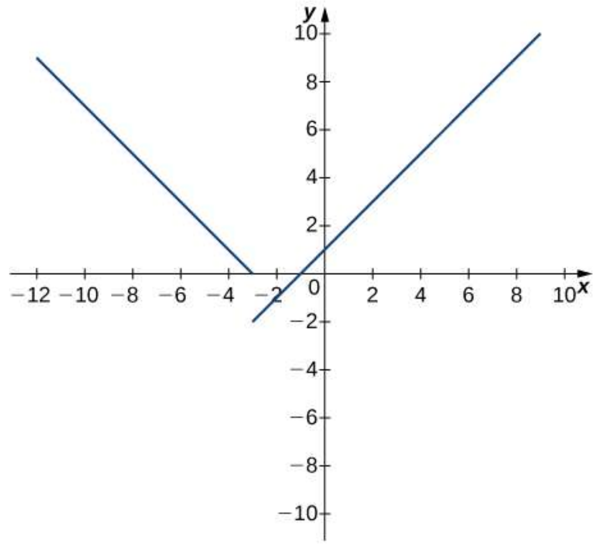
b. $\lim_{x \to 0^+} g(x)$

117. $h(x) = \begin{cases} x^2 - 2x + 1, & x < 2 \\ 3 - x, & x \ge 2 \end{cases}$

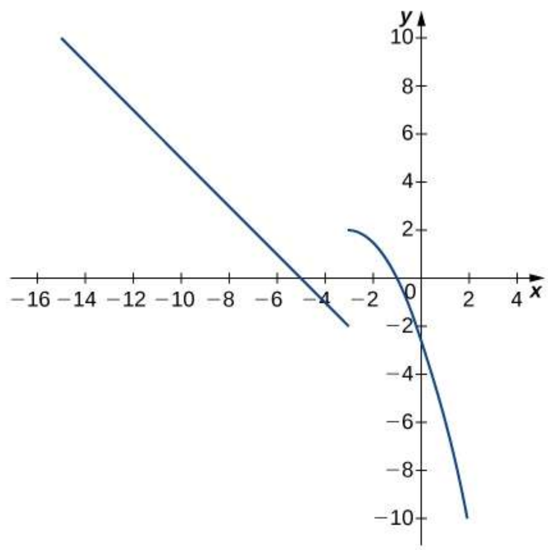
a. $\lim_{x \to 2^-} h(x)$

b. $\lim_{x \to 2^+} h(x)$

In the following exercises, use the following graphs and the limit laws to evaluate each limit.



$y = f(x)$



$y = g(x)$

118. $\lim_{x \to -3^+} (f(x) + g(x))$

119. $\lim_{x \to -3^-} (f(x) - 3g(x))$

120. $\lim_{x \to 0} \frac{f(x)g(x)}{3}$

121. $\lim_{x \to -5} \frac{2 + g(x)}{f(x)}$

122. $\lim_{x \to 1} (f(x))^2$

123. $\lim_{x \to 1} \sqrt[3]{f(x) - g(x)}$

124. $\lim_{x \to -7} (x \cdot g(x))$

125. $\lim_{x \to -9} [x \cdot f(x) + 2 \cdot g(x)]$

126. **[T]** True or False? If $2x - 1 \le g(x) \le x^2 - 2x + 3,$ then $\lim_{x \to 2} g(x) = 0.$

For the following problems, evaluate the limit using the squeeze theorem. Use a calculator to graph the functions $f(x), g(x),$ and $h(x)$ when possible.

127. **[T]** $\lim_{\theta \to 0} \theta^2 \cos\left(\frac{1}{\theta}\right)$

128. $\lim_{x \to 0} f(x),$ where $f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrrational} \end{cases}$

129. **[T]** In physics, the magnitude of an electric field generated by a point charge at a distance r in vacuum is governed by Coulomb's law: $E(r) = \frac{q}{4\pi\varepsilon_0 r^2},$ where E represents the magnitude of the electric field, q is the charge of the particle, r is the distance between the particle and where the strength of the field is measured, and $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's constant: $8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2.$

a. Use a graphing calculator to graph $E(r)$ given that the charge of the particle is $q = 10^{-10}.$
b. Evaluate $\lim_{r \to 0^+} E(r).$ What is the physical meaning of this quantity? Is it physically relevant? Why are you evaluating from the right?

130. **[T]** The density of an object is given by its mass divided by its volume: $\rho = m/V.$

a. Use a calculator to plot the volume as a function of density $(V = m/\rho),$ assuming you are examining something of mass 8 kg ($m = 8$).
b. Evaluate $\lim_{\rho \to 0^+} V(\rho)$ and explain the physical meaning.

2.4 | Continuity

Learning Objectives

- **2.4.1** Explain the three conditions for continuity at a point.
- **2.4.2** Describe three kinds of discontinuities.
- **2.4.3** Define continuity on an interval.
- **2.4.4** State the theorem for limits of composite functions.
- **2.4.5** Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity at a point* where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in **Figure 2.32**. We see that the graph of $f(x)$ has a hole at a. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a, we need the following condition:

$$\text{i. } f(a) \text{ is defined.}$$

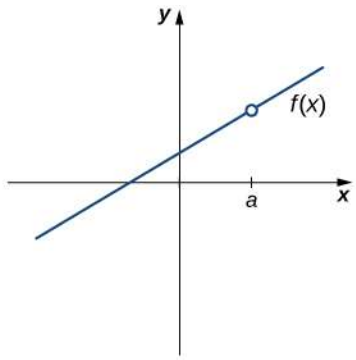


Figure 2.32 The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

However, as we see in **Figure 2.33**, this condition alone is insufficient to guarantee continuity at the point a. Although $f(a)$ is defined, the function has a gap at a. In this example, the gap exists because $\lim_{x \to a} f(x)$ does not exist. We must add another condition for continuity at a—namely,

$$\text{ii. } \lim_{x \to a} f(x) \text{ exists.}$$

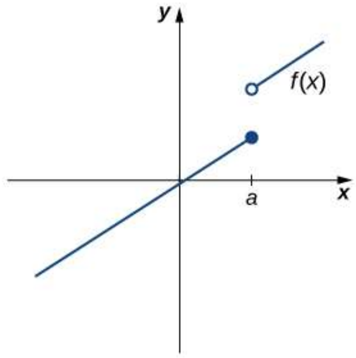


Figure 2.33 The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x)$ does not exist.

However, as we see in **Figure 2.34**, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a. We must add a third condition to our list:

iii. $\lim_{x \to a} f(x) = f(a)$.

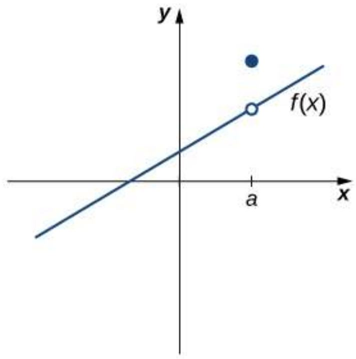


Figure 2.34 The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Definition

A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

i. $f(a)$ is defined

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** a if it fails to be continuous at a.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If $\lim_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x \to a} f(x)$. If $\lim_{x \to a} f(x) \neq f(a)$, then the function is not continuous at a. If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at a.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

Example 2.26

Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = (x^2 - 4)/(x - 2)$ is continuous at $x = 2$. Justify the conclusion.

Solution

Let’s begin by trying to calculate $f(2)$. We can see that $f(2) = 0/0$, which is undefined. Therefore, $f(x) = \frac{x^2 - 4}{x - 2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in **Figure 2.35**.

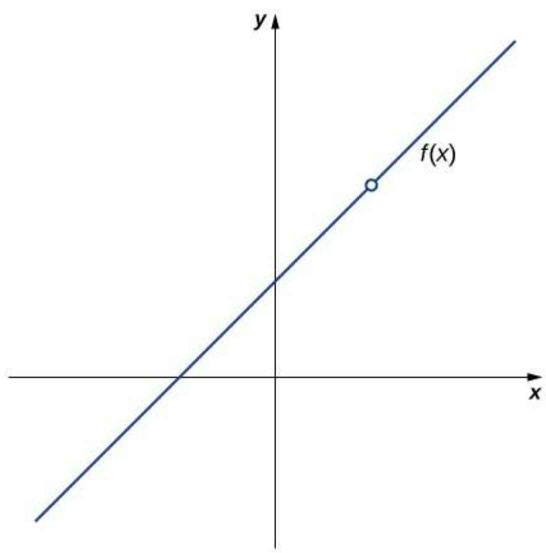


Figure 2.35 The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

Example 2.27

Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \le 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(3)$.

$$f(3) = -(3^2) + 4 = -5.$$

Thus, $f(3)$ is defined. Next, we calculate $\lim_{x \to 3} f(x)$. To do this, we must compute $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$:

$$\lim_{x \to 3^-} f(x) = -(3^2) + 4 = -5$$

and

$$\lim_{x \to 3^+} f(x) = 4(3) - 8 = 4.$$

Therefore, $\lim_{x \to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in **Figure 2.36**.

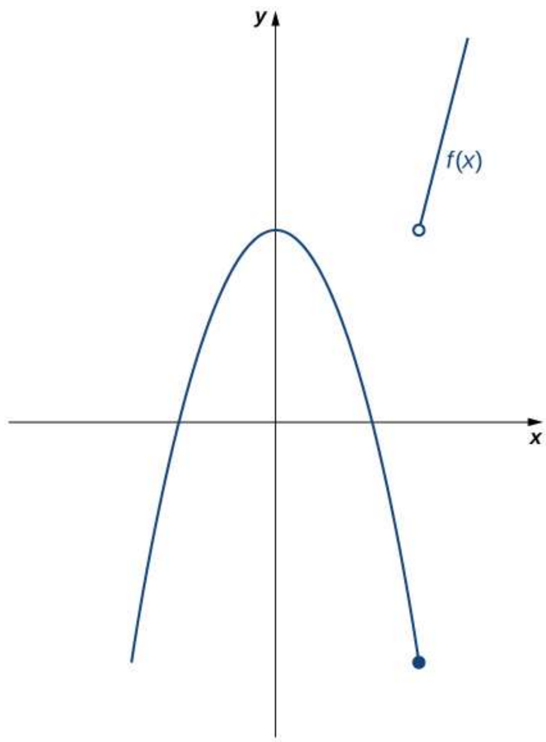


Figure 2.36 The function $f(x)$ is not continuous at 3 because $\lim_{x \to 3} f(x)$ does not exist.

Example 2.28

Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution

First, observe that

$$f(0) = 1.$$

Next,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Last, compare $f(0)$ and $\lim_{x \to 1} f(x)$. We see that

$$f(0) = 1 = \lim_{x \to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

2.21 Using the definition, determine whether the function $f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x + 4 & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$.

If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Theorem 2.8: Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x \to a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply **Continuity of Polynomials and Rational Functions** to determine the points at which a given rational function is continuous.

Example 2.29

Continuity of a Rational Function

For what values of x is $f(x) = \frac{x+1}{x-5}$ continuous?

Solution

The rational function $f(x) = \frac{x+1}{x-5}$ is continuous for every value of x except $x = 5$.

2.22 For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Types of Discontinuities

As we have seen in **Example 2.26** and **Example 2.27**, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. **Figure 2.37** illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

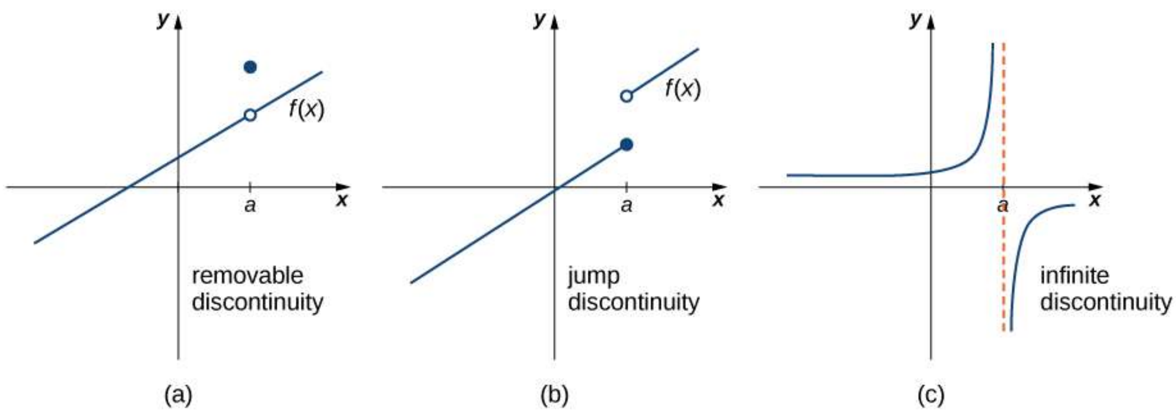


Figure 2.37 Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x \to a} f(x)$ exists. (Note: When we state that $\lim_{x \to a} f(x)$ exists, we mean that $\lim_{x \to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$. (Note: When we state that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

Example 2.30

Classifying a Discontinuity

In **Example 2.26**, we showed that $f(x) = \frac{x^2 - 4}{x - 2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution

To classify the discontinuity at 2 we must evaluate $\lim_{x \to 2} f(x)$:

$$\begin{aligned} \lim_{x \to 2} f(x) &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} \\ &= \lim_{x \to 2} (x + 2) \\ &= 4. \end{aligned}$$

Since f is discontinuous at 2 and $\lim_{x \to 2} f(x)$ exists, f has a removable discontinuity at $x = 2$.

Example 2.31

Classifying a Discontinuity

In **Example 2.27**, we showed that $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \le 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

Solution

Earlier, we showed that f is discontinuous at 3 because $\lim_{x \to 3} f(x)$ does not exist. However, since $\lim_{x \to 3^-} f(x) = -5$ and $\lim_{x \to 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example 2.32

Classifying a Discontinuity

Determine whether $f(x) = \frac{x + 2}{x + 1}$ is continuous at -1. If the function is discontinuous at -1, classify the discontinuity as removable, jump, or infinite.

Solution

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at -1. To determine the type of discontinuity, we must determine the limit at −1. We see that $\lim_{x \to -1^-} \frac{x+2}{x+1} = -\infty$ and $\lim_{x \to -1^+} \frac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at −1.

2.23 For $f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$, decide whether f is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at a if $\lim_{x \to a^-} f(x) = f(a)$.

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b. Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim_{x \to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

Example 2.33

Continuity on an Interval

State the interval(s) over which the function $f(x) = \frac{x-1}{x^2+2x}$ is continuous.

Solution

Since $f(x) = \frac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$. Thus, $f(x)$ is continuous over each of the intervals

$$(-\infty, -2), (-2, 0), \text{ and } (0, +\infty).$$

Example 2.34

Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4 - x^2}$ is continuous.

Solution

From the limit laws, we know that $\lim_{x \to a} \sqrt{4 - x^2} = \sqrt{4 - a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ exists and $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

2.24 State the interval(s) over which the function $f(x) = \sqrt{x + 3}$ is continuous.

The **Composite Function Theorem** allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Theorem 2.9: Composite Function Theorem

If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

Before we move on to **Example 2.35**, recall that earlier, in the section on limit laws, we showed $\lim_{x \to 0} \cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In **Example 2.35** we see how to combine this result with the composite function theorem.

Example 2.35

Limit of a Composite Cosine Function

Evaluate $\lim_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right)$.

Solution

The given function is a composite of $\cos x$ and $x - \frac{\pi}{2}$. Since $\lim_{x \to \pi/2}\left(x - \frac{\pi}{2}\right) = 0$ and $\cos x$ is continuous at 0, we may apply the composite function theorem. Thus,

$$\lim_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\lim_{x \to \pi/2}\left(x - \frac{\pi}{2}\right)\right) = \cos(0) = 1.$$

2.25 Evaluate $\lim_{x \to \pi} \sin(x - \pi)$.

The proof of the next theorem uses the composite function theorem as well as the continuity of $f(x) = \sin x$ and $g(x) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Theorem 2.10: Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim_{x \to a} \cos x = \cos a$ for all values of a.

$$\begin{aligned}
\lim_{x \to a} \cos x &= \lim_{x \to a} \cos((x - a) + a) && \text{rewrite } x = x - a + a \\
&= \lim_{x \to a} (\cos(x - a)\cos a - \sin(x - a)\sin a) && \text{apply the identity for the cosine of the sum of two angles} \\
&= \cos\left(\lim_{x \to a}(x - a)\right)\cos a - \sin\left(\lim_{x \to a}(x - a)\right)\sin a && \lim_{x \to a}(x - a) = 0, \text{ and } \sin x \text{ and } \cos x \text{ are continuous at } 0 \\
&= \cos(0)\cos a - \sin(0)\sin a && \text{evaluate } \cos(0) \text{ and } \sin(0) \text{ and simplify} \\
&= 1 \cdot \cos a - 0 \cdot \sin a = \cos a.
\end{aligned}$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

□

As you can see, the composite function theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem**.

Theorem 2.11: The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$ in **Figure 2.38**.

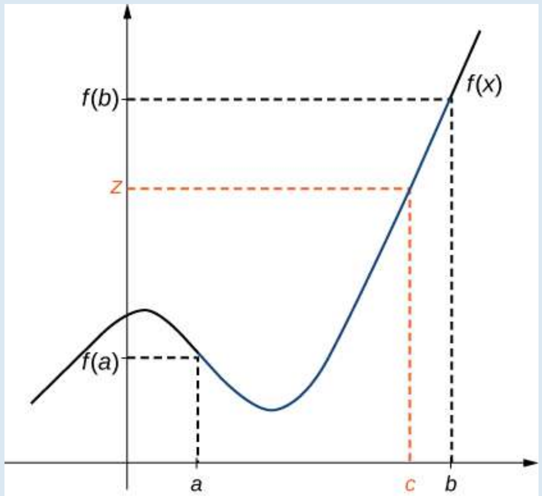


Figure 2.38 There is a number $c \in [a, b]$ that satisfies $f(c) = z.$

Example 2.36

Application of the Intermediate Value Theorem

Show that $f(x) = x - \cos x$ has at least one zero.

Solution

Since $f(x) = x - \cos x$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a, b) that satisfies $f(c) = 0$. Note that

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0.$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0, \pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

Example 2.37

When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$ and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

Solution

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x) = (x - 1)^2$. It satisfies $f(0) = 1 > 0, f(2) = 1 > 0,$ and $f(1) = 0$.

Example 2.38

When Can You Apply the Intermediate Value Theorem?

For $f(x) = 1/x, f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

Solution

No. The function is not continuous over $[-1, 1]$. The Intermediate Value Theorem does not apply here.

2.26 Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

2.4 EXERCISES

For the following exercises, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as jump, removable, infinite, or other.

131. $f(x) = \frac{1}{\sqrt{x}}$

132. $f(x) = \frac{2}{x^2 + 1}$

133. $f(x) = \frac{x}{x^2 - x}$

134. $g(t) = t^{-1} + 1$

135. $f(x) = \frac{5}{e^x - 2}$

136. $f(x) = \frac{|x - 2|}{x - 2}$

137. $H(x) = \tan 2x$

138. $f(t) = \frac{t + 3}{t^2 + 5t + 6}$

For the following exercises, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

139. $f(x)\frac{2x^2 - 5x + 3}{x - 1}$ at $x = 1$

140. $h(\theta) = \frac{\sin\theta - \cos\theta}{\tan\theta}$ at $\theta = \pi$

141. $g(u) = \begin{cases} \frac{6u^2 + u - 2}{2u - 1} & \text{if } u \neq \frac{1}{2} \\ \frac{7}{2} & \text{if } u = \frac{1}{2} \end{cases}$, at $u = \frac{1}{2}$

142. $f(y) = \frac{\sin(\pi y)}{\tan(\pi y)}$, at $y = 1$

143. $f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$, at $x = 0$

144. $f(x) = \begin{cases} x\sin(x) \text{ if } x \leq \pi \\ x\tan(x) \text{ if } x > \pi \end{cases}$, at $x = \pi$

In the following exercises, find the value(s) of k that makes each function continuous over the given interval.

145. $f(x) = \begin{cases} 3x + 2, & x < k \\ 2x - 3, & k \leq x \leq 8 \end{cases}$

146. $f(\theta) = \begin{cases} \sin\theta, & 0 \leq \theta < \frac{\pi}{2} \\ \cos(\theta + k), & \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$

147. $f(x) = \begin{cases} \frac{x^2 + 3x + 2}{x + 2}, & x \neq -2 \\ k, & x = -2 \end{cases}$

148. $f(x) = \begin{cases} e^{kx}, & 0 \leq x < 4 \\ x + 3, & 4 \leq x \leq 8 \end{cases}$

149. $f(x) = \begin{cases} \sqrt{kx}, & 0 \leq x \leq 3 \\ x + 1, & 3 < x \leq 10 \end{cases}$

In the following exercises, use the Intermediate Value Theorem (IVT).

150. Let $h(x) = \begin{cases} 3x^2 - 4, & x \leq 2 \\ 5 + 4x, & x > 2 \end{cases}$ Over the interval $[0, 4]$, there is no value of x such that $h(x) = 10$, although $h(0) < 10$ and $h(4) > 10$. Explain why this does not contradict the IVT.

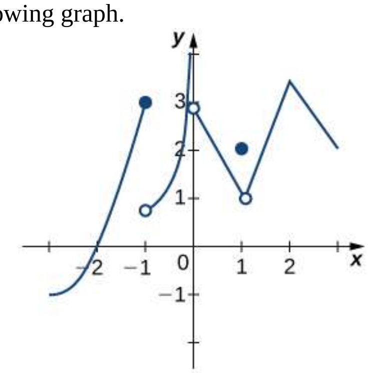
151. A particle moving along a line has at each time t a position function $s(t)$, which is continuous. Assume $s(2) = 5$ and $s(5) = 2$. Another particle moves such that its position is given by $h(t) = s(t) - t$. Explain why there must be a value c for $2 < c < 5$ such that $h(c) = 0$.

152. **[T]** Use the statement "The cosine of t is equal to t cubed."

a. Write a mathematical equation of the statement.
b. Prove that the equation in part a. has at least one real solution.
c. Use a calculator to find an interval of length 0.01 that contains a solution.

153. Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

154. Consider the graph of the function $y = f(x)$ shown in the following graph.



a. Find all values for which the function is discontinuous.
b. For each value in part a., state why the formal definition of continuity does not apply.
c. Classify each discontinuity as either jump, removable, or infinite.

155. Let $f(x) = \begin{cases} 3x, & x > 1 \\ x^3, & x < 1 \end{cases}$.

a. Sketch the graph of *f*.
b. Is it possible to find a value *k* such that $f(1) = k$, which makes $f(x)$ continuous for all real numbers? Briefly explain.

156. Let $f(x) = \dfrac{x^4 - 1}{x^2 - 1}$ for $x \neq -1, 1$.

a. Sketch the graph of *f*.
b. Is it possible to find values k_1 and k_2 such that $f(-1) = k_1$ and $f(1) = k_2$, and that makes $f(x)$ continuous for all real numbers? Briefly explain.

157. Sketch the graph of the function $y = f(x)$ with properties i. through vii.

i. The domain of *f* is $(-\infty, +\infty)$.
ii. *f* has an infinite discontinuity at $x = -6$.
iii. $f(-6) = 3$
iv. $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x) = 2$
v. $f(-3) = 3$
vi. *f* is left continuous but not right continuous at $x = 3$.
vii. $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$

158. Sketch the graph of the function $y = f(x)$ with properties i. through iv.

i. The domain of *f* is $[0, 5]$.
ii. $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ exist and are equal.
iii. $f(x)$ is left continuous but not continuous at $x = 2$, and right continuous but not continuous at $x = 3$.
iv. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and the following limits hold: $\lim_{x \to 3^-} f(x) = -\infty$ and $\lim_{x \to 3^+} f(x) = 2$.

In the following exercises, suppose $y = f(x)$ is defined for all *x*. For each description, sketch a graph with the indicated property.

159. Discontinuous at $x = 1$ with $\lim_{x \to -1} f(x) = -1$ and $\lim_{x \to 2} f(x) = 4$

160. Discontinuous at $x = 2$ but continuous elsewhere with $\lim_{x \to 0} f(x) = \frac{1}{2}$

Determine whether each of the given statements is true. Justify your response with an explanation or counterexample.

161. $f(t) = \dfrac{2}{e^t - e^{-t}}$ is continuous everywhere.

162. If the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal, then *f* cannot be discontinuous at $x = a$.

163. If a function is not continuous at a point, then it is not defined at that point.

164. According to the IVT, $\cos x - \sin x - x = 2$ has a solution over the interval $[-1, 1]$.

165. If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has exactly one solution in $[a, b]$.

166. The function $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 1}$ is continuous over the interval $[0, 3]$.

167. If $f(x)$ is continuous everywhere and $f(a),\ f(b) > 0,$ then there is no root of $f(x)$ in the interval $[a,\ b]$.

[T] The following problems consider the scalar form of Coulomb's law, which describes the electrostatic force between two point charges, such as electrons. It is given by the equation $F(r) = k_e \frac{|q_1 q_2|}{r^2},$ where k_e is Coulomb's constant, q_i are the magnitudes of the charges of the two particles, and *r* is the distance between the two particles.

168. To simplify the calculation of a model with many interacting particles, after some threshold value $r = R,$ we approximate *F* as zero.

a. Explain the physical reasoning behind this assumption.
b. What is the force equation?
c. Evaluate the force *F* using both Coulomb's law and our approximation, assuming two protons with a charge magnitude of 1.6022×10^{-19} coulombs (C), and the Coulomb constant $k_e = 8.988 \times 10^9 \text{Nm}^2/\text{C}^2$ are 1 m apart. Also, assume $R < 1\,\text{m}.$ How much inaccuracy does our approximation generate? Is our approximation reasonable?
d. Is there any finite value of *R* for which this system remains continuous at *R*?

169. Instead of making the force 0 at *R*, instead we let the force be 10^{-20} for $r \geq R.$ Assume two protons, which have a magnitude of charge 1.6022×10^{-19} C, and the Coulomb constant $k_e = 8.988 \times 10^9 \text{Nm}^2/\text{C}^2.$ Is there a value *R* that can make this system continuous? If so, find it.

Recall the discussion on spacecraft from the chapter opener. The following problems consider a rocket launch from Earth's surface. The force of gravity on the rocket is given by $F(d) = -mk/d^2,$ where *m* is the mass of the rocket, *d* is the distance of the rocket from the center of Earth, and *k* is a constant.

170. **[T]** Determine the value and units of *k* given that the mass of the rocket is 3 million kg. (*Hint*: The distance from the center of Earth to its surface is 6378 km.)

171. **[T]** After a certain distance *D* has passed, the gravitational effect of Earth becomes quite negligible, so we can approximate the force function by $F(d) = \begin{cases} -\frac{mk}{d^2} & \text{if } d < D \\ 10{,}000 & \text{if } d \geq D \end{cases}.$ Using the value of k found in the previous exercise, find the necessary condition *D* such that the force function remains continuous.

172. As the rocket travels away from Earth's surface, there is a distance *D* where the rocket sheds some of its mass, since it no longer needs the excess fuel storage. We can write this function as $F(d) = \begin{cases} -\frac{m_1 k}{d^2} & \text{if } d < D \\ -\frac{m_2 k}{d^2} & \text{if } d \geq D \end{cases}.$ Is there a *D* value such that this function is continuous, assuming $m_1 \neq m_2$?

Prove the following functions are continuous everywhere

173. $f(\theta) = \sin\theta$

174. $g(x) = |x|$

175. Where is $f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ 1 \text{ if } x \text{ is rational} \end{cases}$ continuous?

2.5 | The Precise Definition of a Limit

Learning Objectives

- **2.5.1** Describe the epsilon-delta definition of a limit.
- **2.5.2** Apply the epsilon-delta definition to find the limit of a function.
- **2.5.3** Describe the epsilon-delta definitions of one-sided limits and infinite limits.
- **2.5.4** Use the epsilon-delta definition to prove the limit laws.

By now you have progressed from the very informal definition of a limit in the introduction of this chapter to the intuitive understanding of a limit. At this point, you should have a very strong intuitive sense of what the limit of a function means and how you can find it. In this section, we convert this intuitive idea of a limit into a formal definition using precise mathematical language. The formal definition of a limit is quite possibly one of the most challenging definitions you will encounter early in your study of calculus; however, it is well worth any effort you make to reconcile it with your intuitive notion of a limit. Understanding this definition is the key that opens the door to a better understanding of calculus.

Quantifying Closeness

Before stating the formal definition of a limit, we must introduce a few preliminary ideas. Recall that the distance between two points *a* and *b* on a number line is given by $|a - b|$.

- The statement $|f(x) - L| < \varepsilon$ may be interpreted as: *The distance between* $f(x)$ *and L is less than ε.*
- The statement $0 < |x - a| < \delta$ may be interpreted as: $x \neq a$ *and the distance between x and a is less than δ.*

It is also important to look at the following equivalences for absolute value:

- The statement $|f(x) - L| < \varepsilon$ is equivalent to the statement $L - \varepsilon < f(x) < L + \varepsilon$.
- The statement $0 < |x - a| < \delta$ is equivalent to the statement $a - \delta < x < a + \delta$ and $x \neq a$.

With these clarifications, we can state the formal **epsilon-delta definition of the limit**.

Definition

Let $f(x)$ be defined for all $x \neq a$ over an open interval containing *a*. Let *L* be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if, for every $\varepsilon > 0,$ there exists a $\delta > 0,$ such that if $0 < |x - a| < \delta,$ then $|f(x) - L| < \varepsilon$.

This definition may seem rather complex from a mathematical point of view, but it becomes easier to understand if we break it down phrase by phrase. The statement itself involves something called a *universal quantifier* (for every $\varepsilon > 0$), an *existential quantifier* (there exists a $\delta > 0$), and, last, a *conditional statement* (if $0 < |x - a| < \delta,$ then $|f(x) - L| < \varepsilon$). Let’s take a look at Table 2.9, which breaks down the definition and translates each part.

Definition	Translation
1. For every $\varepsilon > 0,$	1. For every positive distance ε from *L*,
2. there exists a $\delta > 0,$	2. There is a positive distance δ from *a*,
3. such that	3. such that
4. if $0 < \lvert x - a \rvert < \delta,$ then $\lvert f(x) - L \rvert < \varepsilon.$	4. if *x* is closer than δ to *a* and $x \neq a,$ then $f(x)$ is closer than ε to *L*.

Table 2.9 Translation of the Epsilon-Delta Definition of the Limit

We can get a better handle on this definition by looking at the definition geometrically. **Figure 2.39** shows possible values of δ for various choices of $\varepsilon > 0$ for a given function $f(x),$ a number *a*, and a limit *L* at *a*. Notice that as we choose smaller values of ε (the distance between the function and the limit), we can always find a δ small enough so that if we have chosen an *x* value within δ of *a*, then the value of $f(x)$ is within ε of the limit *L*.

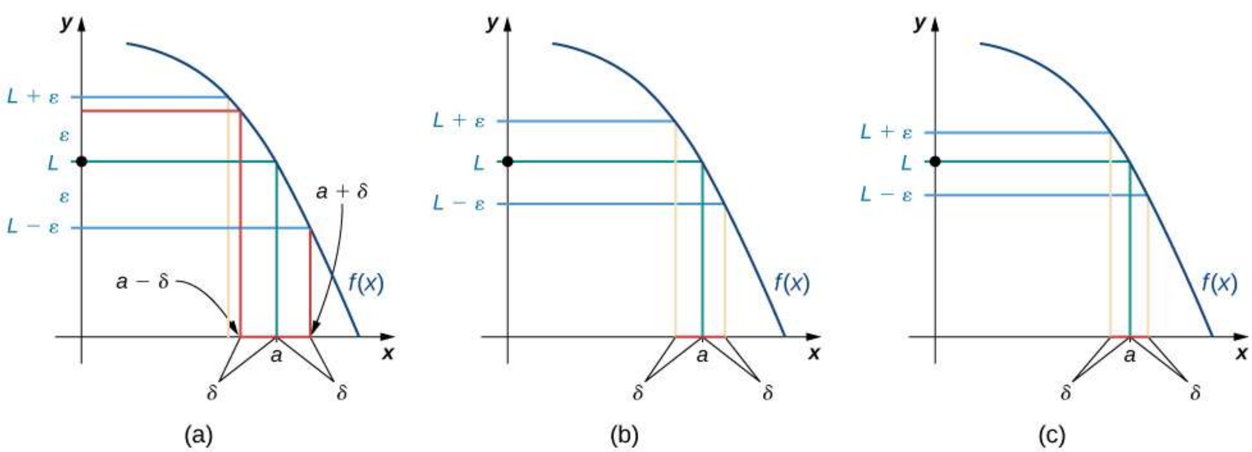


Figure 2.39 These graphs show possible values of δ, given successively smaller choices of ε.

Visit the following applet to experiment with finding values of δ for selected values of ε:

- **The epsilon-delta definition of limit (http://www.openstax.org/l/20_epsilondelt)**

Example 2.39 shows how you can use this definition to prove a statement about the limit of a specific function at a specified value.

Example 2.39

Proving a Statement about the Limit of a Specific Function

Prove that $\lim\limits_{x \to 1} (2x + 1) = 3.$

Solution

Let $\varepsilon > 0$.

The first part of the definition begins "For every $\varepsilon > 0$." This means we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon > 0$," we signal our intent to do so.

Choose $\delta = \frac{\varepsilon}{2}$.

The definition continues with "there exists a $\delta > 0$." The phrase "there exists" in a mathematical statement is always a signal for a scavenger hunt. In other words, we must go and find δ. So, where exactly did $\delta = \varepsilon/2$ come from? There are two basic approaches to tracking down δ. One method is purely algebraic and the other is geometric.

We begin by tackling the problem from an algebraic point of view. Since ultimately we want $|(2x+1)-3| < \varepsilon$, we begin by manipulating this expression: $|(2x+1)-3| < \varepsilon$ is equivalent to $|2x-2| < \varepsilon$, which in turn is equivalent to $|2||x-1| < \varepsilon$. Last, this is equivalent to $|x-1| < \varepsilon/2$. Thus, it would seem that $\delta = \varepsilon/2$ is appropriate.

We may also find δ through geometric methods. **Figure 2.40** demonstrates how this is done.

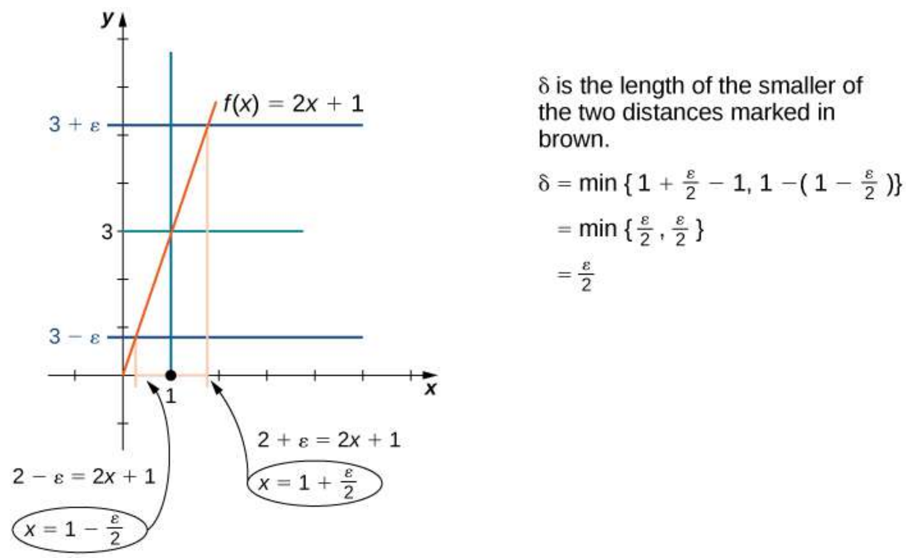


Figure 2.40 This graph shows how we find δ geometrically.

Assume $0 < |x-1| < \delta$. When δ has been chosen, our goal is to show that if $0 < |x-1| < \delta$, then $|(2x+1)-3| < \varepsilon$. To prove any statement of the form "If this, then that," we begin by assuming "this" and trying to get "that."

Thus,

$\lvert(2x+1)-3\rvert$	$= \lvert 2x-2\rvert$	property of absolute value
	$= \lvert 2(x-1)\rvert$	
	$= \lvert 2\rvert\lvert x-1\rvert$	$\lvert 2\rvert = 2$
	$= 2\lvert x-1\rvert$	
	$< 2\cdot\delta$	here's where we use the assumption that $0 < \lvert x-1\rvert < \delta$
	$= 2\cdot\frac{\varepsilon}{2} = \varepsilon$	here's where we use our choice of $\delta = \varepsilon/2$

Analysis

In this part of the proof, we started with $|(2x+1)-3|$ and used our assumption $0 < |x-1| < \delta$ in a key part of the chain of inequalities to get $|(2x+1)-3|$ to be less than ε. We could just as easily have manipulated the assumed inequality $0 < |x-1| < \delta$ to arrive at $|(2x+1)-3| < \varepsilon$ as follows:

$$\begin{aligned}
0 < |x-1| < \delta \quad & \Rightarrow |x-1| < \delta \\
& \Rightarrow -\delta < x-1 < \delta \\
& \Rightarrow -\frac{\varepsilon}{2} < x-1 < \frac{\varepsilon}{2} \\
& \Rightarrow -\varepsilon < 2x-2 < \varepsilon \\
& \Rightarrow -\varepsilon < 2x-2 < \varepsilon \\
& \Rightarrow |2x-2| < \varepsilon \\
& \Rightarrow |(2x+1)-3| < \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x \to 1}(2x+1) = 3.$ (Having completed the proof, we state what we have accomplished.)

After removing all the remarks, here is a final version of the proof:

Let $\varepsilon > 0.$

Choose $\delta = \varepsilon/2.$

Assume $0 < |x-1| < \delta.$

Thus,

$$\begin{aligned}
|(2x+1)-3| \quad & = |2x-2| \\
& = |2(x-1)| \\
& = |2||x-1| \\
& = 2|x-1| \\
& < 2 \cdot \delta \\
& = 2 \cdot \frac{\varepsilon}{2} \\
& = \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x \to 1}(2x+1) = 3.$

The following Problem-Solving Strategy summarizes the type of proof we worked out in **Example 2.39**.

Problem-Solving Strategy: Proving That $\lim_{x \to a} f(x) = L$ for a Specific Function $f(x)$

1. Let's begin the proof with the following statement: Let $\varepsilon > 0.$
2. Next, we need to obtain a value for $\delta.$ After we have obtained this value, we make the following statement, filling in the blank with our choice of δ: Choose $\delta =$ _______.
3. The next statement in the proof should be (at this point, we fill in our given value for a): Assume $0 < |x-a| < \delta.$
4. Next, based on this assumption, we need to show that $|f(x)-L| < \varepsilon,$ where $f(x)$ and L are our function $f(x)$ and our limit L. At some point, we need to use $0 < |x-a| < \delta.$
5. We conclude our proof with the statement: Therefore, $\lim_{x \to a} f(x) = L.$

Example 2.40

Proving a Statement about a Limit

Complete the proof that $\lim_{x \to -1} (4x + 1) = -3$ by filling in the blanks.

Let _____.

Choose $\delta =$ _______.

Assume $0 < |x -$ _______$| < \delta$.

Thus, |________ − ________| = ______________________________________ε.

Solution

We begin by filling in the blanks where the choices are specified by the definition. Thus, we have

Let $\varepsilon > 0$.

Choose $\delta =$ _______.

Assume $0 < |x - (-1)| < \delta$. (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta$_______$\varepsilon$.

Focusing on the final line of the proof, we see that we should choose $\delta = \frac{\varepsilon}{4}$.

We now complete the final write-up of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \frac{\varepsilon}{4}$.

Assume $0 < |x - (-1)| < \delta$ (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta = 4(\varepsilon/4) = \varepsilon$.

2.27 Complete the proof that $\lim_{x \to 2} (3x - 2) = 4$ by filling in the blanks.

Let _______.

Choose $\delta =$ _______.

Assume $0 < |x -$ ____$| <$ ____.

Thus,

|_______ − ____| = ______________________________ε.

Therefore, $\lim_{x \to 2} (3x - 2) = 4$.

In **Example 2.39** and **Example 2.40**, the proofs were fairly straightforward, since the functions with which we were working were linear. In **Example 2.41**, we see how to modify the proof to accommodate a nonlinear function.

Example 2.41

Proving a Statement about the Limit of a Specific Function (Geometric Approach)

Prove that $\lim_{x \to 2} x^2 = 4.$

Solution

1. Let $\varepsilon > 0$. The first part of the definition begins "For every $\varepsilon > 0$," so we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon > 0$," we signal our intent to do so.
2. Without loss of generality, assume $\varepsilon \le 4$. Two questions present themselves: Why do we want $\varepsilon \le 4$ and why is it okay to make this assumption? In answer to the first question: Later on, in the process of solving for δ, we will discover that δ involves the quantity $\sqrt{4 - \varepsilon}$. Consequently, we need $\varepsilon \le 4$. In answer to the second question: If we can find $\delta > 0$ that "works" for $\varepsilon \le 4$, then it will "work" for any $\varepsilon > 4$ as well. Keep in mind that, although it is always okay to put an upper bound on ε, it is never okay to put a lower bound (other than zero) on ε.
3. Choose $\delta = \min\{2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2\}$. **Figure 2.41** shows how we made this choice of δ.

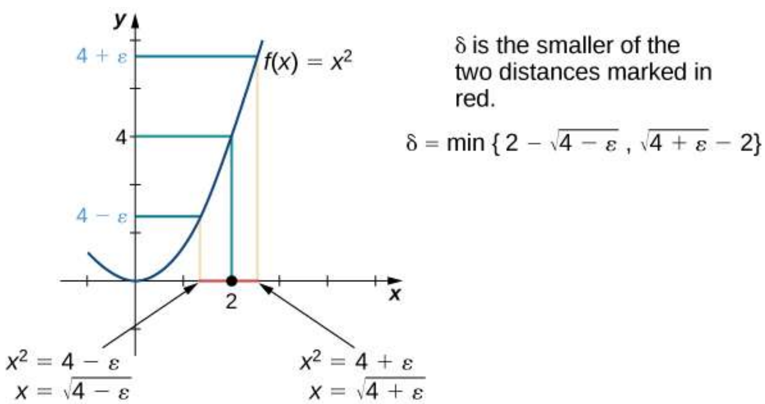


Figure 2.41 This graph shows how we find δ geometrically for a given ε for the proof in **Example 2.41**.

4. We must show: If $0 < |x - 2| < \delta$, then $|x^2 - 4| < \varepsilon$, so we must begin by assuming

$$0 < |x - 2| < \delta.$$

We don't really need $0 < |x - 2|$ (in other words, $x \ne 2$) for this proof. Since $0 < |x - 2| < \delta \Rightarrow |x - 2| < \delta$, it is okay to drop $0 < |x - 2|$.

$$|x - 2| < \delta.$$

Hence,

$$-\delta < x - 2 < \delta.$$

Recall that $\delta = \min\{2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2\}$. Thus, $\delta \le 2 - \sqrt{4 - \varepsilon}$ and consequently $-(2 - \sqrt{4 - \varepsilon}) \le -\delta$. We also use $\delta \le \sqrt{4 + \varepsilon} - 2$ here. We might ask at this point: Why did we substitute $2 - \sqrt{4 - \varepsilon}$ for δ on the left-hand side of the inequality and $\sqrt{4 + \varepsilon} - 2$ on the right-hand side of the inequality? If we look at **Figure 2.41**, we see that $2 - \sqrt{4 - \varepsilon}$ corresponds to the distance on the left of 2 on the x-axis and $\sqrt{4+\varepsilon}-2$ corresponds to the distance on the right. Thus,

$$-\left(2-\sqrt{4-\varepsilon}\right) \le -\delta < x-2 < \delta \le \sqrt{4+\varepsilon}-2.$$

We simplify the expression on the left:

$$-2+\sqrt{4-\varepsilon} < x-2 < \sqrt{4+\varepsilon}-2.$$

Then, we add 2 to all parts of the inequality:

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}.$$

We square all parts of the inequality. It is okay to do so, since all parts of the inequality are positive:

$$4-\varepsilon < x^2 < 4+\varepsilon.$$

We subtract 4 from all parts of the inequality:

$$-\varepsilon < x^2-4 < \varepsilon.$$

Last,

$$|x^2-4| < \varepsilon.$$

5. Therefore,

$$\lim_{x\to 2} x^2 = 4.$$

2.28 Find δ corresponding to $\varepsilon > 0$ for a proof that $\lim_{x\to 9}\sqrt{x} = 3.$

The geometric approach to proving that the limit of a function takes on a specific value works quite well for some functions. Also, the insight into the formal definition of the limit that this method provides is invaluable. However, we may also approach limit proofs from a purely algebraic point of view. In many cases, an algebraic approach may not only provide us with additional insight into the definition, it may prove to be simpler as well. Furthermore, an algebraic approach is the primary tool used in proofs of statements about limits. For **Example 2.42**, we take on a purely algebraic approach.

Example 2.42

Proving a Statement about the Limit of a Specific Function (Algebraic Approach)

Prove that $\lim_{x\to -1}\left(x^2-2x+3\right) = 6.$

Solution

Let's use our outline from the Problem-Solving Strategy:

1. Let $\varepsilon > 0.$
2. Choose $\delta = \min\{1, \varepsilon/5\}.$ This choice of δ may appear odd at first glance, but it was obtained by

taking a look at our ultimate desired inequality: $\left|\left(x^2 - 2x + 3\right) - 6\right| < \varepsilon$. This inequality is equivalent to $|x + 1| \cdot |x - 3| < \varepsilon$. At this point, the temptation simply to choose $\delta = \frac{\varepsilon}{x - 3}$ is very strong. Unfortunately, our choice of δ must depend on ε only and no other variable. If we can replace $|x - 3|$ by a numerical value, our problem can be resolved. This is the place where assuming $\delta \leq 1$ comes into play. The choice of $\delta \leq 1$ here is arbitrary. We could have just as easily used any other positive number. In some proofs, greater care in this choice may be necessary. Now, since $\delta \leq 1$ and $|x + 1| < \delta \leq 1$, we are able to show that $|x - 3| < 5$. Consequently, $|x + 1| \cdot |x - 3| < |x + 1| \cdot 5$. At this point we realize that we also need $\delta \leq \varepsilon/5$. Thus, we choose $\delta = \min\{1, \varepsilon/5\}$.

3. Assume $0 < |x + 1| < \delta$. Thus,

$$|x + 1| < 1 \text{ and } |x + 1| < \frac{\varepsilon}{5}.$$

Since $|x + 1| < 1$, we may conclude that $-1 < x + 1 < 1$. Thus, by subtracting 4 from all parts of the inequality, we obtain $-5 < x - 3 < -1$. Consequently, $|x - 3| < 5$. This gives us

$$\left|\left(x^2 - 2x + 3\right) - 6\right| = |x + 1| \cdot |x - 3| < \frac{\varepsilon}{5} \cdot 5 = \varepsilon.$$

Therefore,

$$\lim_{x \to -1} \left(x^2 - 2x + 3\right) = 6.$$

2.29 Complete the proof that $\lim_{x \to 1} x^2 = 1$.

Let $\varepsilon > 0$; choose $\delta = \min\{1, \varepsilon/3\}$; assume $0 < |x - 1| < \delta$.

Since $|x - 1| < 1$, we may conclude that $-1 < x - 1 < 1$. Thus, $1 < x + 1 < 3$. Hence, $|x + 1| < 3$.

You will find that, in general, the more complex a function, the more likely it is that the algebraic approach is the easiest to apply. The algebraic approach is also more useful in proving statements about limits.

Proving Limit Laws

We now demonstrate how to use the epsilon-delta definition of a limit to construct a rigorous proof of one of the limit laws. The **triangle inequality** is used at a key point of the proof, so we first review this key property of absolute value.

Definition

The **triangle inequality** states that if a and b are any real numbers, then $|a + b| \leq |a| + |b|$.

Proof

We prove the following limit law: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} (f(x) + g(x)) = L + M$.

Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ so that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \varepsilon/2$.

Choose $\delta_2 > 0$ so that if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \varepsilon/2$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.

Assume $0 < |x - a| < \delta$.

Thus,

$$0 < |x - a| < \delta_1 \text{ and } 0 < |x - a| < \delta_2.$$

Hence,

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

We now explore what it means for a limit not to exist. The limit $\lim_{x \to a} f(x)$ does not exist if there is no real number L for which $\lim_{x \to a} f(x) = L$. Thus, for all real numbers L, $\lim_{x \to a} f(x) \ne L$. To understand what this means, we look at each part of the definition of $\lim_{x \to a} f(x) = L$ together with its opposite. A translation of the definition is given in **Table 2.10**.

Definition	**Opposite**
1. For every $\varepsilon > 0,$	1. There exists $\varepsilon > 0$ so that
2. there exists a $\delta > 0,$ so that	2. for every $\delta > 0,$
3. if $0 < |x - a| < \delta,$ then $|f(x) - L| < \varepsilon.$	3. There is an x satisfying $0 < |x - a| < \delta$ so that $|f(x) - L| \ge \varepsilon.$

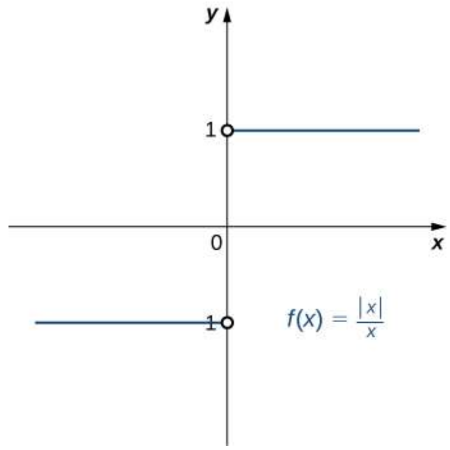
Table 2.10 Translation of the Definition of $\lim_{x \to a} f(x) = L$ and its Opposite

Finally, we may state what it means for a limit not to exist. The limit $\lim_{x \to a} f(x)$ does not exist if for every real number L, there exists a real number $\varepsilon > 0$ so that for all $\delta > 0,$ there is an x satisfying $0 < |x - a| < \delta,$ so that $|f(x) - L| \ge \varepsilon.$ Let's apply this in **Example 2.43** to show that a limit does not exist.

Example 2.43

Showing That a Limit Does Not Exist

Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. The graph of $f(x) = |x|/x$ is shown here:



Solution

Suppose that L is a candidate for a limit. Choose $\varepsilon = 1/2$.

Let $\delta > 0$. Either $L \geq 0$ or $L < 0$. If $L \geq 0$, then let $x = -\delta/2$. Thus,

$$|x - 0| = \left|-\frac{\delta}{2} - 0\right| = \frac{\delta}{2} < \delta$$

and

$$\left|\frac{\left|-\frac{\delta}{2}\right|}{-\frac{\delta}{2}} - L\right| = |-1 - L| = L + 1 \geq 1 > \frac{1}{2} = \varepsilon.$$

On the other hand, if $L < 0$, then let $x = \delta/2$. Thus,

$$|x - 0| = \left|\frac{\delta}{2} - 0\right| = \frac{\delta}{2} < \delta$$

and

$$\left|\frac{\left|\frac{\delta}{2}\right|}{\frac{\delta}{2}} - L\right| = |1 - L| = |L| + 1 \geq 1 > \frac{1}{2} = \varepsilon.$$

Thus, for any value of L, $\lim\limits_{x \to 0} \frac{|x|}{x} \neq L$.

One-Sided and Infinite Limits

Just as we first gained an intuitive understanding of limits and then moved on to a more rigorous definition of a limit, we now revisit one-sided limits. To do this, we modify the epsilon-delta definition of a limit to give formal epsilon-delta definitions for limits from the right and left at a point. These definitions only require slight modifications from the definition of the limit. In the definition of the limit from the right, the inequality $0 < x - a < \delta$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are greater than (to the right of) a. Similarly, in the definition of the limit from the left, the inequality $-\delta < x - a < 0$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are less than (to the left of) a.

Definition

Limit from the Right: Let $f(x)$ be defined over an open interval of the form (a, b) where $a < b$. Then,

$$\lim_{x \to a^+} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$.

Limit from the Left: Let $f(x)$ be defined over an open interval of the form (b, c) where $b < c$. Then,

$$\lim_{x \to a^-} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $-\delta < x - a < 0$, then $|f(x) - L| < \varepsilon$.

Example 2.44

Proving a Statement about a Limit From the Right

Prove that $\lim_{x \to 4^+} \sqrt{x - 4} = 0$.

Solution

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon^2$. Since we ultimately want $\left|\sqrt{x - 4} - 0\right| < \varepsilon$, we manipulate this inequality to get $\sqrt{x - 4} < \varepsilon$ or, equivalently, $0 < x - 4 < \varepsilon^2$, making $\delta = \varepsilon^2$ a clear choice. We may also determine δ geometrically, as shown in Figure 2.42.

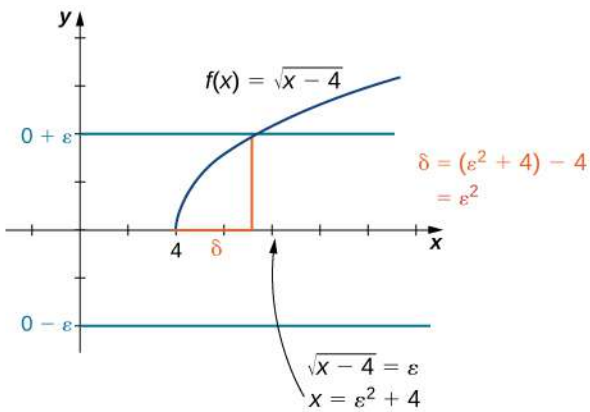


Figure 2.42 This graph shows how we find δ for the proof in **Example 2.44**.

Assume $0 < x - 4 < \delta$. Thus, $0 < x - 4 < \varepsilon^2$. Hence, $0 < \sqrt{x - 4} < \varepsilon$. Finally, $\left|\sqrt{x - 4} - 0\right| < \varepsilon$.

Therefore, $\lim_{x \to 4^+} \sqrt{x - 4} = 0$.

2.30 Find δ corresponding to ε for a proof that $\lim_{x \to 1^-} \sqrt{1 - x} = 0$.

We conclude the process of converting our intuitive ideas of various types of limits to rigorous formal definitions by pursuing a formal definition of infinite limits. To have $\lim_{x \to a} f(x) = +\infty$, we want the values of the function $f(x)$ to get larger and larger as x approaches a. Instead of the requirement that $|f(x) - L| < \varepsilon$ for arbitrarily small ε when $0 < |x - a| < \delta$ for small enough δ, we want $f(x) > M$ for arbitrarily large positive M when $0 < |x - a| < \delta$ for small enough δ. **Figure 2.43** illustrates this idea by showing the value of δ for successively larger values of M.

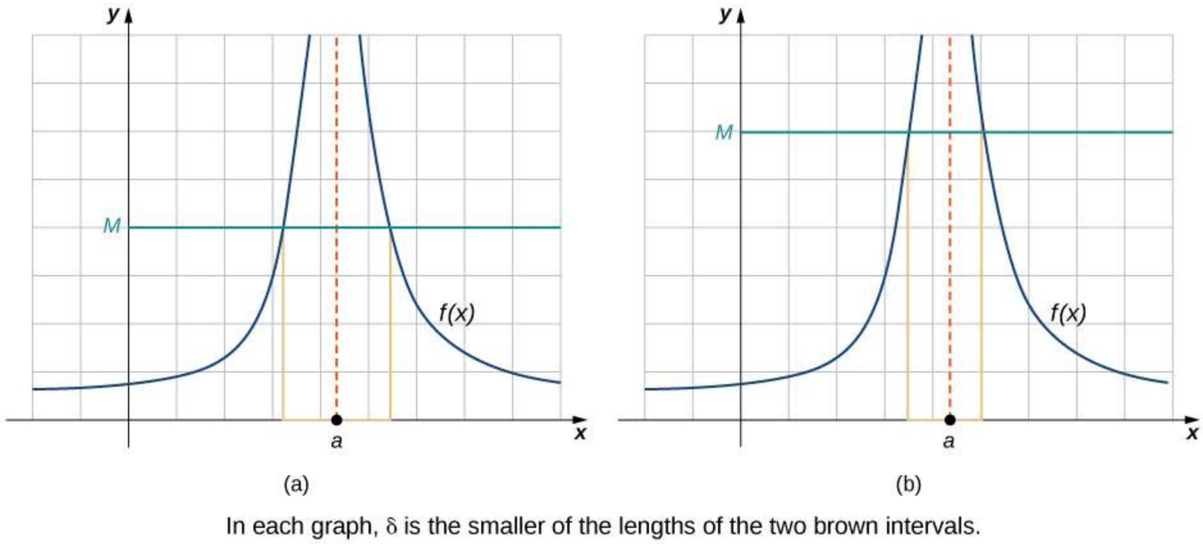


Figure 2.43 These graphs plot values of δ for M to show that $\lim_{x \to a} f(x) = +\infty$.

Definition

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have an infinite limit

$$\lim_{x \to a} f(x) = +\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$.

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have a negative infinite limit

$$\lim_{x \to a} f(x) = -\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) < -M$.

2.5 EXERCISES

In the following exercises, write the appropriate $\varepsilon - \delta$ definition for each of the given statements.

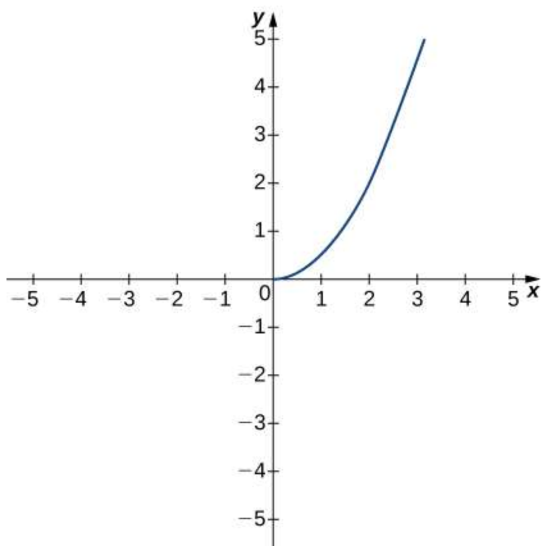
176. $\lim\limits_{x \to a} f(x) = N$

177. $\lim\limits_{t \to b} g(t) = M$

178. $\lim\limits_{x \to c} h(x) = L$

179. $\lim\limits_{x \to a} \varphi(x) = A$

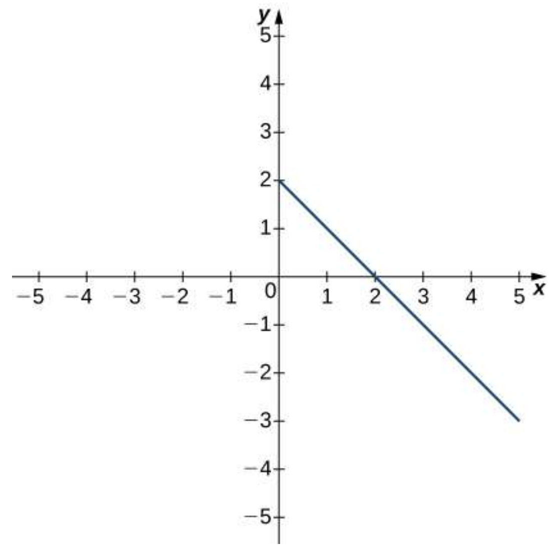
The following graph of the function f satisfies $\lim\limits_{x \to 2} f(x) = 2$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



180. If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 1$.

181. If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 0.5$.

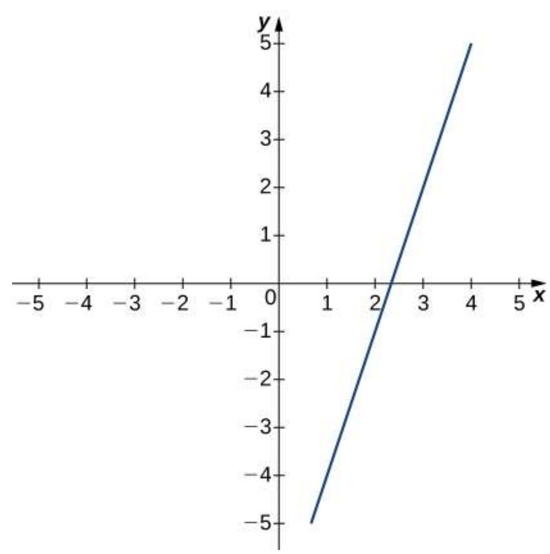
The following graph of the function f satisfies $\lim\limits_{x \to 3} f(x) = -1$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



182. If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 1$.

183. If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 2$.

The following graph of the function f satisfies $\lim\limits_{x \to 3} f(x) = 2$. In the following exercises, for each value of ε, find a value of $\delta > 0$ such that the precise definition of limit holds true.



184. $\varepsilon = 1.5$

185. $\varepsilon = 3$

[T] In the following exercises, use a graphing calculator to find a number δ such that the statements hold true.

186. $\left|\sin(2x) - \frac{1}{2}\right| < 0.1$, whenever $\left|x - \frac{\pi}{12}\right| < \delta$

187. $\left|\sqrt{x-4}-2\right| < 0.1,$ whenever $|x-8| < \delta$

In the following exercises, use the precise definition of limit to prove the given limits.

188. $\lim\limits_{x \to 2}(5x+8) = 18$

189. $\lim\limits_{x \to 3}\frac{x^2-9}{x-3} = 6$

190. $\lim\limits_{x \to 2}\frac{2x^2-3x-2}{x-2} = 5$

191. $\lim\limits_{x \to 0}x^4 = 0$

192. $\lim\limits_{x \to 2}(x^2+2x) = 8$

In the following exercises, use the precise definition of limit to prove the given one-sided limits.

193. $\lim\limits_{x \to 5^-}\sqrt{5-x} = 0$

194. $\lim\limits_{x \to 0^+}f(x) = -2,$ where $f(x) = \begin{cases} 8x-3, & \text{if } x < 0 \\ 4x-2, & \text{if } x \ge 0 \end{cases}.$

195. $\lim\limits_{x \to 1^-}f(x) = 3,$ where $f(x) = \begin{cases} 5x-2, & \text{if } x < 1 \\ 7x-1, & \text{if } x \ge 1 \end{cases}.$

In the following exercises, use the precise definition of limit to prove the given infinite limits.

196. $\lim\limits_{x \to 0}\frac{1}{x^2} = \infty$

197. $\lim\limits_{x \to -1}\frac{3}{(x+1)^2} = \infty$

198. $\lim\limits_{x \to 2} -\frac{1}{(x-2)^2} = -\infty$

199. An engineer is using a machine to cut a flat square of Aerogel of area 144 cm^2. If there is a maximum error tolerance in the area of 8 cm^2, how accurately must the engineer cut on the side, assuming all sides have the same length? How do these numbers relate to $\delta,$ ε, a, and L?

200. Use the precise definition of limit to prove that the following limit does not exist: $\lim\limits_{x \to 1}\frac{|x-1|}{x-1}.$

201. Using precise definitions of limits, prove that $\lim\limits_{x \to 0}f(x)$ does not exist, given that $f(x)$ is the ceiling function. (*Hint*: Try any $\delta < 1.$)

202. Using precise definitions of limits, prove that $\lim\limits_{x \to 0}f(x)$ does not exist: $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}.$ (*Hint*: Think about how you can always choose a rational number $0 < r < d,$ but $|f(r)-0| = 1.$)

203. Using precise definitions of limits, determine $\lim\limits_{x \to 0}f(x)$ for $f(x) = \begin{cases} x \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}.$ (*Hint*: Break into two cases, *x* rational and *x* irrational.)

204. Using the function from the previous exercise, use the precise definition of limits to show that $\lim\limits_{x \to a}f(x)$ does not exist for $a \ne 0.$

For the following exercises, suppose that $\lim\limits_{x \to a}f(x) = L$ and $\lim\limits_{x \to a}g(x) = M$ both exist. Use the precise definition of limits to prove the following limit laws:

205. $\lim\limits_{x \to a}(f(x)+g(x)) = L+M$

206. $\lim\limits_{x \to a}[cf(x)] = cL$ for any real constant c (*Hint*: Consider two cases: $c = 0$ and $c \ne 0.$)

207. $\lim\limits_{x \to a}[f(x)g(x)] = LM.$ (*Hint*: $|f(x)g(x)-LM| = |f(x)g(x)-f(x)M+f(x)M-LM| \le |f(x)||g(x)-M|+|M||f(x)-L|.$)

CHAPTER 2 REVIEW

KEY TERMS

average velocity the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval $[t, a]$ (if $t < a$ or $[a, t]$ if $t > a$), with a position given by $s(t)$, that is $v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$

constant multiple law for limits the limit law $\lim_{x \to a} c f(x) = c \cdot \lim_{x \to a} f(x) = cL$

continuity at a point A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim_{x \to a} f(x)$ exists, and (3) $\lim_{x \to a} f(x) = f(a)$

continuity from the left A function is continuous from the left at *b* if $\lim_{x \to b^-} f(x) = f(b)$

continuity from the right A function is continuous from the right at *a* if $\lim_{x \to a^+} f(x) = f(a)$

continuity over an interval a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b), and it is continuous from the right at *a* and from the left at *b*

difference law for limits the limit law $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

differential calculus the field of calculus concerned with the study of derivatives and their applications

discontinuity at a point A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

epsilon-delta definition of the limit $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

infinite discontinuity An infinite discontinuity occurs at a point *a* if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

infinite limit A function has an infinite limit at a point *a* if it either increases or decreases without bound as it approaches *a*

instantaneous velocity The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t, a]$ and $[a, t]$ approach as the values of *t* move closer to a, provided such a value exists

integral calculus the study of integrals and their applications

Intermediate Value Theorem Let *f* be continuous over a closed bounded interval $[a, b]$; if *z* is any real number between $f(a)$ and $f(b)$, then there is a number *c* in $[a, b]$ satisfying $f(c) = z$

intuitive definition of the limit If all values of the function $f(x)$ approach the real number *L* as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

jump discontinuity A jump discontinuity occurs at a point *a* if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

limit the process of letting *x* or *t* approach *a* in an expression; the limit of a function $f(x)$ as *x* approaches *a* is the value that $f(x)$ approaches as *x* approaches *a*

limit laws the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing *a*; assume that *L* and *M* are real numbers so that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$; let *c* be a constant

multivariable calculus the study of the calculus of functions of two or more variables

one-sided limit A one-sided limit of a function is a limit taken from either the left or the right

power law for limits the limit law $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every positive integer *n*

product law for limits the limit law $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

quotient law for limits the limit law $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

removable discontinuity A removable discontinuity occurs at a point *a* if $f(x)$ is discontinuous at *a*, but $\lim_{x \to a} f(x)$ exists

root law for limits the limit law $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all *L* if *n* is odd and for $L \geq 0$ if *n* is even

secant A secant line to a function $f(x)$ at *a* is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$

squeeze theorem states that if $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ over an open interval containing *a* and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ where *L* is a real number, then $\lim_{x \to a} g(x) = L$

sum law for limits The limit law $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

tangent A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with *x*-values that approach *a*; the slope of the tangent line to a graph at *a* measures the rate of change of the function at *a*

triangle inequality If *a* and *b* are any real numbers, then $|a + b| \leq |a| + |b|$

vertical asymptote A function has a vertical asymptote at $x = a$ if the limit as *x* approaches *a* from the right or left is infinite

KEY EQUATIONS

- **Slope of a Secant Line**
  $m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$
- **Average Velocity over Interval** $[a, t]$
  $v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$
- **Intuitive Definition of the Limit**
  $\lim_{x \to a} f(x) = L$
- **Two Important Limits**
  $\lim_{x \to a} x = a \quad \lim_{x \to a} c = c$
- **One-Sided Limits**

$\lim_{x \to a^-} f(x) = L \quad \lim_{x \to a^+} f(x) = L$

- **Infinite Limits from the Left**
  $\lim_{x \to a^-} f(x) = +\infty \quad \lim_{x \to a^-} f(x) = -\infty$
- **Infinite Limits from the Right**
  $\lim_{x \to a^+} f(x) = +\infty \quad \lim_{x \to a^+} f(x) = -\infty$
- **Two-Sided Infinite Limits**
  $\lim_{x \to a} f(x) = +\infty : \lim_{x \to a^-} f(x) = +\infty$ and $\lim_{x \to a^+} f(x) = +\infty$

  $\lim_{x \to a} f(x) = -\infty : \lim_{x \to a^-} f(x) = -\infty$ and $\lim_{x \to a^+} f(x) = -\infty$
- **Basic Limit Results**
  $\lim_{x \to a} x = a \; \lim_{x \to a} c = c$
- **Important Limits**
  $\lim_{\theta \to 0} \sin\theta = 0$

  $\lim_{\theta \to 0} \cos\theta = 1$

  $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

  $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$

KEY CONCEPTS

2.1 A Preview of Calculus

- Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the rate of change of the function, also called the *derivative*. Calculating a derivative requires finding a limit.
- Integral calculus arose from trying to solve the problem of finding the area of a region between the graph of a function and the x-axis. We can approximate the area by dividing it into thin rectangles and summing the areas of these rectangles. This summation leads to the value of a function called the *integral*. The integral is also calculated by finding a limit and, in fact, is related to the derivative of a function.
- Multivariable calculus enables us to solve problems in three-dimensional space, including determining motion in space and finding volumes of solids.

2.2 The Limit of a Function

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

2.3 The Limit Laws

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions, $\lim_{x \to a} f(x) = f(a)$.

- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.
- The squeeze theorem allows you to find the limit of a function if the function is always greater than one function and less than another function with limits that are known.

2.4 Continuity

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The composite function theorem states: If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L)$.
- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

2.5 The Precise Definition of a Limit

- The intuitive notion of a limit may be converted into a rigorous mathematical definition known as the *epsilon-delta definition of the limit*.
- The epsilon-delta definition may be used to prove statements about limits.
- The epsilon-delta definition of a limit may be modified to define one-sided limits.

CHAPTER 2 REVIEW EXERCISES

True or False. In the following exercises, justify your answer with a proof or a counterexample.

208. A function has to be continuous at $x = a$ if the $\lim_{x \to a} f(x)$ exists.

209. You can use the quotient rule to evaluate $\lim_{x \to 0} \frac{\sin x}{x}$.

210. If there is a vertical asymptote at $x = a$ for the function $f(x)$, then f is undefined at the point $x = a$.

211. If $\lim_{x \to a} f(x)$ does not exist, then f is undefined at the point $x = a$.

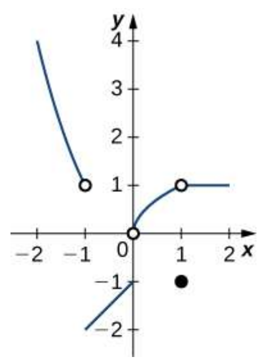
212. Using the graph, find each limit or explain why the limit does not exist.

a. $\lim_{x \to -1} f(x)$

b. $\lim_{x \to 1} f(x)$

c. $\lim_{x \to 0^+} f(x)$

d. $\lim_{x \to 2} f(x)$



In the following exercises, evaluate the limit algebraically or explain why the limit does not exist.

213. $\lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2}$

214. $\lim_{x \to 0} 3x^2 - 2x + 4$

215. $\lim_{x \to 3} \frac{x^3 - 2x^2 - 1}{3x - 2}$

216. $\lim_{x \to \pi/2} \frac{\cot x}{\cos x}$

217. $\lim_{x \to -5} \frac{x^2 + 25}{x + 5}$

218. $\lim_{x \to 2} \frac{3x^2 - 2x - 8}{x^2 - 4}$

219. $\lim_{x \to 1} \frac{x^2 - 1}{x^3 - 1}$

220. $\lim_{x \to 1} \frac{x^2 - 1}{\sqrt{x} - 1}$

221. $\lim_{x \to 4} \frac{4 - x}{\sqrt{x} - 2}$

222. $\lim_{x \to 4} \frac{1}{\sqrt{x} - 2}$

In the following exercises, use the squeeze theorem to prove the limit.

223. $\lim_{x \to 0} x^2 \cos(2\pi x) = 0$

224. $\lim_{x \to 0} x^3 \sin\left(\frac{\pi}{x}\right) = 0$

225. Determine the domain such that the function $f(x) = \sqrt{x - 2} + xe^x$ is continuous over its domain.

In the following exercises, determine the value of c such that the function remains continuous. Draw your resulting function to ensure it is continuous.

226. $f(x) = \begin{cases} x^2 + 1, & x > c \\ 2x, & x \le c \end{cases}$

227. $f(x) = \begin{cases} \sqrt{x + 1}, & x > -1 \\ x^2 + c, & x \le -1 \end{cases}$

In the following exercises, use the precise definition of limit to prove the limit.

228. $\lim_{x \to 1} (8x + 16) = 24$

229. $\lim_{x \to 0} x^3 = 0$

230. A ball is thrown into the air and the vertical position is given by $x(t) = -4.9t^2 + 25t + 5$. Use the Intermediate Value Theorem to show that the ball must land on the ground sometime between 5 sec and 6 sec after the throw.

231. A particle moving along a line has a displacement according to the function $x(t) = t^2 - 2t + 4$, where x is measured in meters and t is measured in seconds. Find the average velocity over the time period $t = [0, 2]$.

232. From the previous exercises, estimate the instantaneous velocity at $t = 2$ by checking the average velocity within $t = 0.01$ sec.

3 | DERIVATIVES

Figure 3.1 The Hennessey Venom GT can go from 0 to 200 mph in 14.51 seconds. (credit: modification of work by Codex41, Flickr)

Chapter Outline

Introduction

The Hennessey Venom GT is one of the fastest cars in the world. In 2014, it reached a record-setting speed of 270.49 mph. It can go from 0 to 200 mph in 14.51 seconds. The techniques in this chapter can be used to calculate the acceleration the Venom achieves in this feat (see **Example 3.8**.)

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

3.1 | Defining the Derivative

Learning Objectives

- **3.1.1** Recognize the meaning of the tangent to a curve at a point.
- **3.1.2** Calculate the slope of a tangent line.
- **3.1.3** Identify the derivative as the limit of a difference quotient.
- **3.1.4** Calculate the derivative of a given function at a point.
- **3.1.5** Describe the velocity as a rate of change.
- **3.1.6** Explain the difference between average velocity and instantaneous velocity.
- **3.1.7** Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac Newton $(1643–1727)$ and the German Gottfried Leibniz $(1646–1716)$, whose images appear in **Figure 3.2**. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.

Figure 3.2 Newton and Leibniz are credited with developing calculus independently.

Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in **Figure 3.3**. The slope of this line is given by an equation in the form of a difference quotient:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with $a + h,$ where h is a value close to 0. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a + h, f(a + h)).$ In this case, we find the secant line has a slope given by the following difference quotient with increment h:

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}.$$

Definition

Let f be a function defined on an interval I containing a. If $x \neq a$ is in $I,$ then

$$Q = \frac{f(x) - f(a)}{x - a} \tag{3.1}$$

is a **difference quotient**.

Also, if $h \neq 0$ is chosen so that $a + h$ is in $I,$ then

$$Q = \frac{f(a + h) - f(a)}{h} \tag{3.2}$$

is a difference quotient with increment h.

View the development of the **derivative (http://www.openstax.org/l/20_calcapplets)** with this applet.

These two expressions for calculating the slope of a secant line are illustrated in **Figure 3.3**. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.

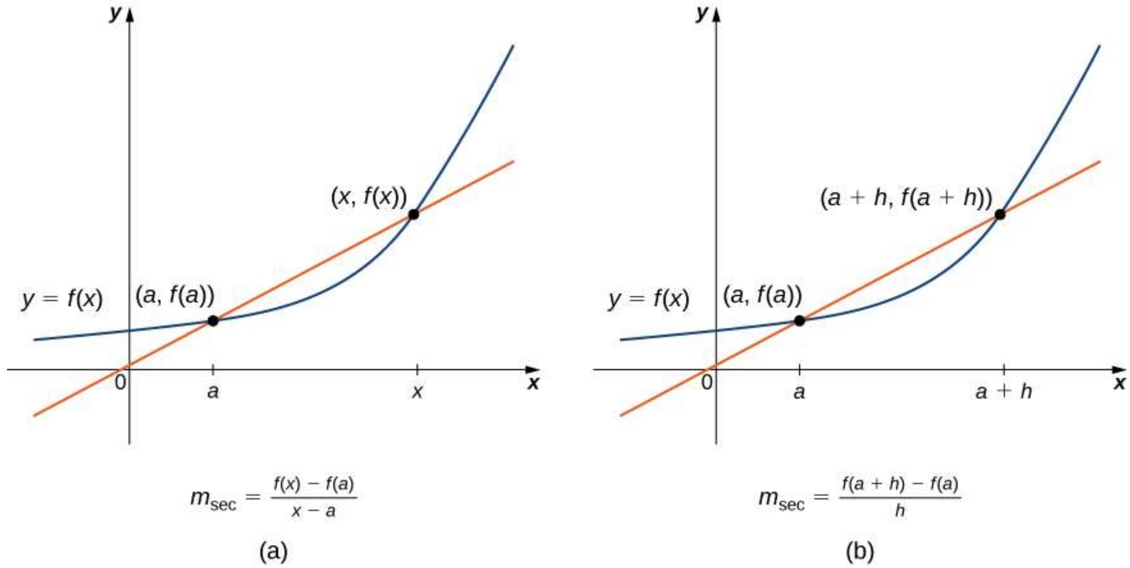


Figure 3.3 We can calculate the slope of a secant line in either of two ways.

In **Figure 3.4**(a) we see that, as the values of x approach a, the slopes of the secant lines provide better estimates of the rate of change of the function at a. Furthermore, the secant lines themselves approach the tangent line to the function at a, which represents the limit of the secant lines. Similarly, **Figure 3.4**(b) shows that as the values of h get closer to 0, the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at a, as shown in **Figure 3.4**(c).

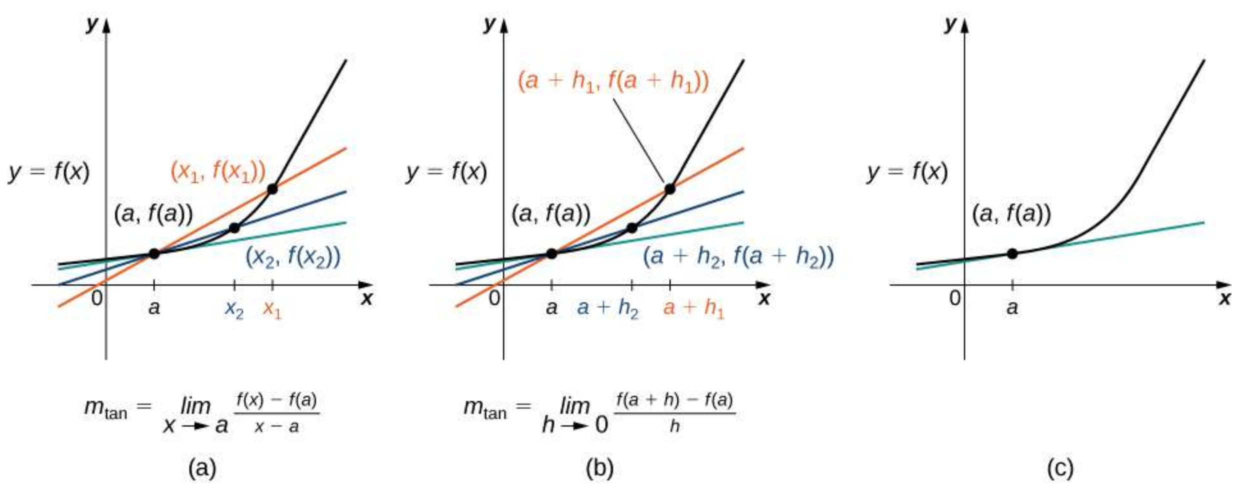


Figure 3.4 The secant lines approach the tangent line (shown in green) as the second point approaches the first.

You can use this **site (http://www.openstax.org/l/20_diffmicros)** to explore graphs to see if they have a tangent line at a point.

In **Figure 3.5** we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to 1. In fact, the graph of $f(x)$ itself appears to be locally linear in the immediate vicinity of $x = 1$.

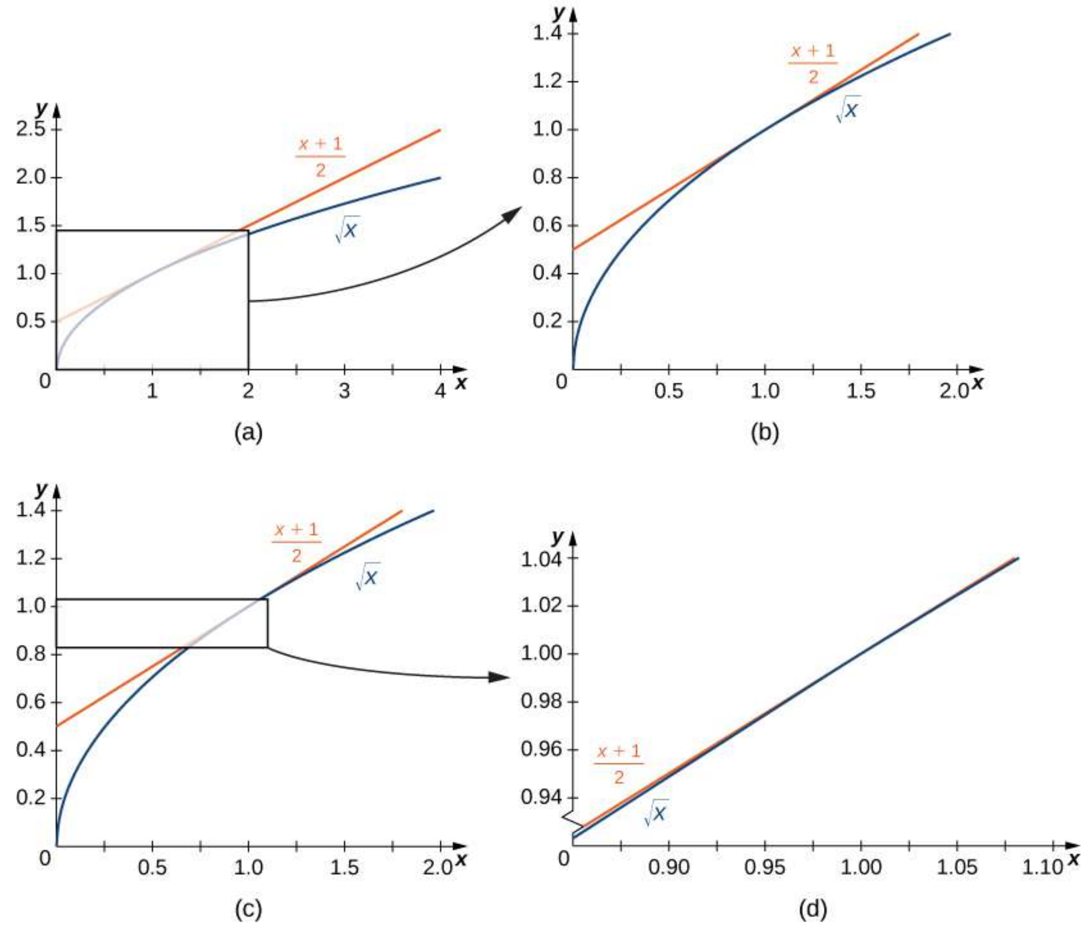


Figure 3.5 For values of x close to $1,$ the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

Definition

Let $f(x)$ be a function defined in an open interval containing a. The *tangent line* to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.3}$$

provided this limit exists.

Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{3.4}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

Example 3.1

Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

First find the slope of the tangent line. In this example, use **Equation 3.3**.

$$\begin{aligned} m_{\tan} &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} && \text{Apply the definition.} \\ &= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} && \text{Substitute } f(x) = x^2 \text{ and } f(3) = 9. \\ &= \lim_{x \to 3} \frac{(x-3)(x+3)}{x - 3} = \lim_{x \to 3}(x + 3) = 6 && \text{Factor the numerator to evaluate the limit.} \end{aligned}$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x = 3$, it passes through the point $(3, f(3))$. We have $f(3) = 9$, so the tangent line passes through the point $(3, 9)$.

Using the point-slope equation of the line with the slope $m = 6$ and the point $(3, 9)$, we obtain the line $y - 9 = 6(x - 3)$. Simplifying, we have $y = 6x - 9$. The graph of $f(x) = x^2$ and its tangent line at 3 are shown in **Figure 3.6**.

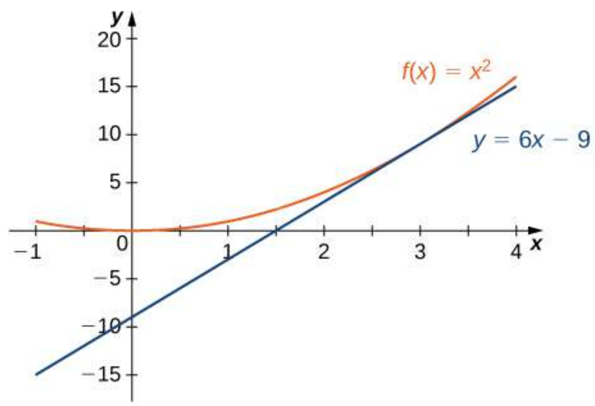


Figure 3.6 The tangent line to $f(x)$ at $x = 3$.

Example 3.2

The Slope of a Tangent Line Revisited

Use **Equation 3.4** to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

The steps are very similar to **Example 3.1**. See **Equation 3.4** for the definition.

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} && \text{Apply the definition.} \\ &= \lim_{h \to 0} \frac{(3+h)^2 - 9}{h} && \text{Substitute } f(3+h) = (3+h)^2 \text{ and } f(3) = 9. \\ &= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} && \text{Expand and simplify to evaluate the limit.} \\ &= \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0} (6+h) = 6 \end{aligned}$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

Example 3.3

Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

Solution

We can use **Equation 3.3**, but as we have seen, the results are the same if we use **Equation 3.4**.

$$\begin{aligned} m_{\tan} &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} && \text{Apply the definition.} \\ &= \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} && \text{Substitute } f(x) = \tfrac{1}{x} \text{ and } f(2) = \tfrac{1}{2}. \\ &= \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \cdot \frac{2x}{2x} && \text{Multiply numerator and denominator by } 2x \text{ to simplify fractions.} \\ &= \lim_{x \to 2} \frac{(2 - x)}{(x - 2)(2x)} && \text{Simplify.} \\ &= \lim_{x \to 2} \frac{-1}{2x} && \text{Simplify using } \tfrac{2 - x}{x - 2} = -1, \text{ for } x \neq 2. \\ &= -\frac{1}{4} && \text{Evaluate the limit.} \end{aligned}$$

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f(2) = \frac{1}{2}$. Since the tangent line passes through the point $(2, \frac{1}{2})$ we can use the point-slope equation of a line to find the equation of the tangent line. Thus the tangent line has the equation $y = -\frac{1}{4}x + 1$. The graphs of $f(x) = \frac{1}{x}$ and $y = -\frac{1}{4}x + 1$ are shown in **Figure 3.7**.

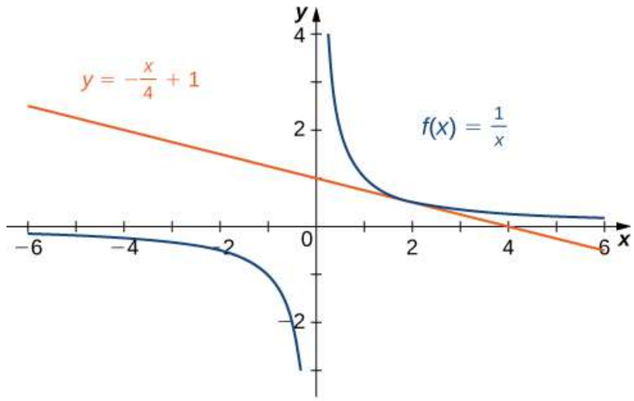


Figure 3.7 The line is tangent to $f(x)$ at $x = 2.$

3.1 Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4.$

The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative**. The process of finding a derivative is called **differentiation**.

Definition

Let $f(x)$ be a function defined in an open interval containing a. The derivative of the function $f(x)$ at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.5}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{3.6}$$

Example 3.4

Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f'(3)$ using **Equation 3.5**.

Solution

Create a table using values of x just below 3 and just above 3.

x	$\frac{x^2-9}{x-3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

After examining the table, we see that a good estimate is $f'(3) = 6$.

3.2 For $f(x) = x^2$, use a table to estimate $f'(3)$ using **Equation 3.6**.

Example 3.5

Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using **Equation 3.5**.

Solution

Substitute the given function and value directly into the equation.

$$\begin{aligned}
f'(x) &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} && \text{Apply the definition.} \\
&= \lim_{x \to 2} \frac{\left(3x^2 - 4x + 1\right) - 5}{x - 2} && \text{Substitute } f(x) = 3x^2 - 4x + 1 \text{ and } f(2) = 5. \\
&= \lim_{x \to 2} \frac{(x-2)(3x+2)}{x-2} && \text{Simplify and factor the numerator.} \\
&= \lim_{x \to 2} (3x + 2) && \text{Cancel the common factor.} \\
&= 8 && \text{Evaluate the limit.}
\end{aligned}$$

Example 3.6

Revisiting the Derivative

For $f(x) = 3x^2 - 4x + 1,$ find $f'(2)$ by using **Equation 3.6**.

Solution

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in **Example 3.5**.

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} && \text{Apply the definition.} \\ &= \lim_{h \to 0} \frac{(3(2+h)^2 - 4(2+h) + 1) - 5}{h} && \text{Substitute } f(2) = 5 \text{ and } f(2+h) = 3(2+h)^2 - 4(2+h) + 1. \\ &= \lim_{h \to 0} \frac{3h^2 + 8h}{h} && \text{Simplify the numerator.} \\ &= \lim_{h \to 0} \frac{h(3h+8)}{h} && \text{Factor the numerator.} \\ &= \lim_{h \to 0} (3h + 8) && \text{Cancel the common factor.} \\ &= 8 && \text{Evaluate the limit.} \end{aligned}$$

The results are the same whether we use **Equation 3.5** or **Equation 3.6**.

3.3 For $f(x) = x^2 + 3x + 2,$ find $f'(1).$

Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the average velocity of the object over a time interval $[a, t]$ if $t > a$ or $[t, a]$ if $t < a$ is given by the difference quotient

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}. \quad \textbf{(3.7)}$$

As the values of t approach $a,$ the values of v_{ave} approach the value we call the instantaneous velocity at $a.$ That is, instantaneous velocity at $a,$ denoted $v(a),$ is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}. \quad \textbf{(3.8)}$$

To better understand the relationship between average velocity and instantaneous velocity, see **Figure 3.8**. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t = a$ whose position at time t is given by the function $s(t).$ The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a, t].$

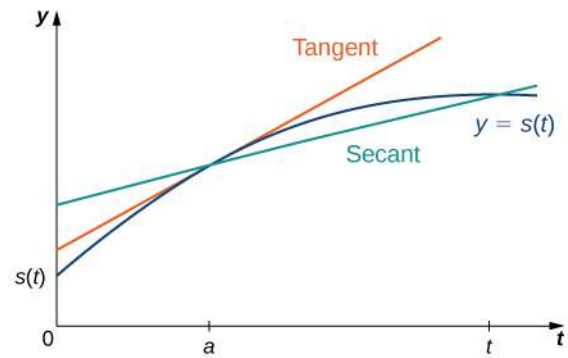


Figure 3.8 The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

We can use **Equation 3.5** to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using **Equation 3.7**.

Example 3.7

Estimating Velocity

A lead weight on a spring is oscillating up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin t$ (**Figure 3.9**). Use a table of values to estimate $v(0)$. Check the estimate by using **Equation 3.5**.

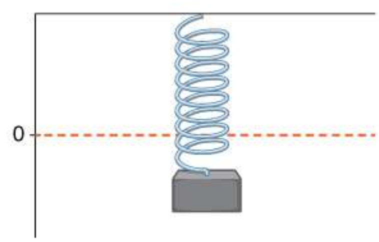


Figure 3.9 A lead weight suspended from a spring in vertical oscillatory motion.

Solution

We can estimate the instantaneous velocity at $t = 0$ by computing a table of average velocities using values of t approaching 0, as shown in **Table 3.1**.

t	$\frac{\sin t - \sin 0}{t - 0} = \frac{\sin t}{t}$
−0.1	0.998334166
−0.01	0.9999833333
−0.001	0.999999833
0.001	0.999999833
0.01	0.9999833333
0.1	0.998334166

Table 3.1
Average velocities using values of t approaching 0

From the table we see that the average velocity over the time interval $[-0.1, 0]$ is $0.998334166,$ the average velocity over the time interval $[-0.01, 0]$ is $0.9999833333,$ and so forth. Using this table of values, it appears that a good estimate is $v(0) = 1.$

By using **Equation 3.5**, we can see that

$$v(0) = s'(0) = \lim_{t \to 0} \frac{\sin t - \sin 0}{t - 0} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$$

Thus, in fact, $v(0) = 1.$

3.4 A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64,\ 0 \le t \le 2.$ Find its instantaneous velocity 1 second after it is dropped, using **Equation 3.5**.

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

Definition

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Example 3.8

Chapter Opener: Estimating Rate of Change of Velocity

Figure 3.10 (credit: modification of work by Codex41, Flickr)

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its acceleration) as it approaches 229.9 mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?

Solution

First observe that 60 mph $= 88$ ft/s, 100 mph ≈ 146.67 ft/s, 200 mph ≈ 293.33 ft/s, and 229.9 mph ≈ 337.19 ft/s. We can summarize the information in a table.

t	$v(t)$
0	0
3.05	88
5.88	147.67
14.51	293.33
19.96	337.19

Table 3.2 $v(t)$ at different values of t

Now compute the average acceleration of the car in feet per second per second on intervals of the form $[t, 19.96]$ as t approaches $19.96,$ as shown in the following table.

t	$\frac{v(t) - v(19.96)}{t - 19.96} = \frac{v(t) - 337.19}{t - 19.96}$
0.0	16.89
3.05	14.74
5.88	13.46
14.51	8.05

Table 3.3
Average acceleration

The rate at which the car is accelerating is decreasing as its velocity approaches 229.9 mph $(337.19$ ft/s).

Example 3.9

Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70°\text{F}$ at 9 p.m., reaches a low of $60°$ during the night, and rises back to $70°$ by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10,$ where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution

Since midnight is 3 hours past 9 p.m., we want to compute $T'(3)$. Refer to **Equation 3.5**.

$$\begin{aligned}
T'(3) &= \lim_{t \to 3} \frac{T(t) - T(3)}{t - 3} && \text{Apply the definition.} \\
&= \lim_{t \to 3} \frac{0.4t^2 - 4t + 70 - 61.6}{t - 3} && \text{Substitute } T(t) = 0.4t^2 - 4t + 70 \text{ and } T(3) = 61.6. \\
&= \lim_{t \to 3} \frac{0.4t^2 - 4t + 8.4}{t - 3} && \text{Simplify.} \\
&= \lim_{t \to 3} \frac{0.4(t-3)(t-7)}{t - 3} && = \lim_{t \to 3} \frac{0.4(t-3)(t-7)}{t - 3} \\
&= \lim_{t \to 3} 0.4(t - 7) && \text{Cancel.} \\
&= -1.6 && \text{Evaluate the limit.}
\end{aligned}$$

The instantaneous rate of change of the temperature at midnight is $-1.6°\text{F}$ per hour.

Example 3.10

Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10{,}000$ dollars. Find the rate of change of profit when $10{,}000$ games are produced. Should the toy company increase or decrease production?

Solution

The profit $P(x)$ earned by producing x gaming systems is $R(x) - C(x),$ where $R(x)$ is the revenue obtained from the sale of x games. Since the company can sell x games at $p = -0.01x + 400$ per game,

$$R(x) = xp = x(-0.01x + 400) = -0.01x^2 + 400x.$$

Consequently,

$$P(x) = -0.01x^2 + 300x - 10{,}000.$$

Therefore, evaluating the rate of change of profit gives

$$\begin{aligned} P'(10000) &= \lim_{x \to 10000} \frac{P(x) - P(10000)}{x - 10000} \\ &= \lim_{x \to 10000} \frac{-0.01x^2 + 300x - 10000 - 1990000}{x - 10000} \\ &= \lim_{x \to 10000} \frac{-0.01x^2 + 300x - 2000000}{x - 10000} \\ &= 100. \end{aligned}$$

Since the rate of change of profit $P'(10{,}000) > 0$ and $P(10{,}000) > 0,$ the company should increase production.

3.5 A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10.$ The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25),$ the rate of change of profit when the price is $\$3.25$ and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

3.1 EXERCISES

For the following exercises, use **Equation 3.1** to find the slope of the secant line between the values x_1 and x_2 for each function $y = f(x)$.

1. $f(x) = 4x + 7;\ x_1 = 2,\ x_2 = 5$

2. $f(x) = 8x - 3;\ x_1 = -1,\ x_2 = 3$

3. $f(x) = x^2 + 2x + 1;\ x_1 = 3,\ x_2 = 3.5$

4. $f(x) = -x^2 + x + 2;\ x_1 = 0.5,\ x_2 = 1.5$

5. $f(x) = \frac{4}{3x - 1};\ x_1 = 1,\ x_2 = 3$

6. $f(x) = \frac{x - 7}{2x + 1};\ x_1 = 0,\ x_2 = 2$

7. $f(x) = \sqrt{x};\ x_1 = 1,\ x_2 = 16$

8. $f(x) = \sqrt{x - 9};\ x_1 = 10,\ x_2 = 13$

9. $f(x) = x^{1/3} + 1;\ x_1 = 0,\ x_2 = 8$

10. $f(x) = 6x^{2/3} + 2x^{1/3};\ x_1 = 1,\ x_2 = 27$

For the following functions,

a. use **Equation 3.4** to find the slope of the tangent line $m_{\tan} = f'(a)$, and

b. find the equation of the tangent line to f at $x = a$.

11. $f(x) = 3 - 4x,\ a = 2$

12. $f(x) = \frac{x}{5} + 6,\ a = -1$

13. $f(x) = x^2 + x,\ a = 1$

14. $f(x) = 1 - x - x^2,\ a = 0$

15. $f(x) = \frac{7}{x},\ a = 3$

16. $f(x) = \sqrt{x + 8},\ a = 1$

17. $f(x) = 2 - 3x^2,\ a = -2$

18. $f(x) = \frac{-3}{x - 1},\ a = 4$

19. $f(x) = \frac{2}{x + 3},\ a = -4$

20. $f(x) = \frac{3}{x^2},\ a = 3$

For the following functions $y = f(x)$, find $f'(a)$ using **Equation 3.1**.

21. $f(x) = 5x + 4,\ a = -1$

22. $f(x) = -7x + 1,\ a = 3$

23. $f(x) = x^2 + 9x,\ a = 2$

24. $f(x) = 3x^2 - x + 2,\ a = 1$

25. $f(x) = \sqrt{x},\ a = 4$

26. $f(x) = \sqrt{x - 2},\ a = 6$

27. $f(x) = \frac{1}{x},\ a = 2$

28. $f(x) = \frac{1}{x - 3},\ a = -1$

29. $f(x) = \frac{1}{x^3},\ a = 1$

30. $f(x) = \frac{1}{\sqrt{x}},\ a = 4$

For the following exercises, given the function $y = f(x)$,

a. find the slope of the secant line PQ for each point $Q(x,\ f(x))$ with x value given in the table.

b. Use the answers from a. to estimate the value of the slope of the tangent line at P.

c. Use the answer from b. to find the equation of the tangent line to f at point P.

31. **[T]** $f(x) = x^2 + 3x + 4$, $P(1, 8)$ (Round to 6 decimal places.)

x	Slope m_{PQ}	x	Slope m_{PQ}
1.1	(i)	0.9	(vii)
1.01	(ii)	0.99	(viii)
1.001	(iii)	0.999	(ix)
1.0001	(iv)	0.9999	(x)
1.00001	(v)	0.99999	(xi)
1.000001	(vi)	0.999999	(xii)

32. **[T]** $f(x) = \frac{x+1}{x^2-1}$, $P(0, -1)$

x	Slope m_{PQ}	x	Slope m_{PQ}
0.1	(i)	−0.1	(vii)
0.01	(ii)	−0.01	(viii)
0.001	(iii)	−0.001	(ix)
0.0001	(iv)	−0.0001	(x)
0.00001	(v)	−0.00001	(xi)
0.000001	(vi)	−0.000001	(xii)

33. **[T]** $f(x) = 10e^{0.5x}$, $P(0, 10)$ (Round to 4 decimal places.)

x	Slope m_{PQ}
−0.1	(i)
−0.01	(ii)
−0.001	(iii)
−0.0001	(iv)
−0.00001	(v)
−0.000001	(vi)

34. **[T]** $f(x) = \tan(x)$, $P(\pi, 0)$

x	Slope m_{PQ}
3.1	(i)
3.14	(ii)
3.141	(iii)
3.1415	(iv)
3.14159	(v)
3.141592	(vi)

[T] For the following position functions $y = s(t)$, an object is moving along a straight line, where t is in seconds and s is in meters. Find

a. the simplified expression for the average velocity from $t = 2$ to $t = 2 + h$;

b. the average velocity between $t = 2$ and $t = 2 + h$, where (i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$, and (iv) $h = 0.0001$; and

c. use the answer from a. to estimate the instantaneous velocity at $t = 2$ second.

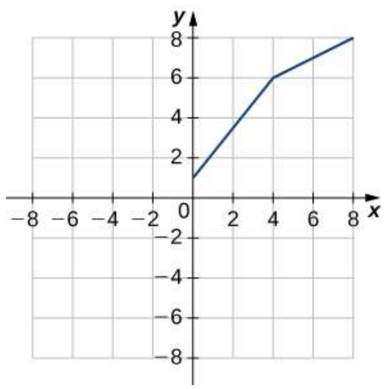
35. $s(t) = \frac{1}{3}t + 5$

36. $s(t) = t^2 - 2t$

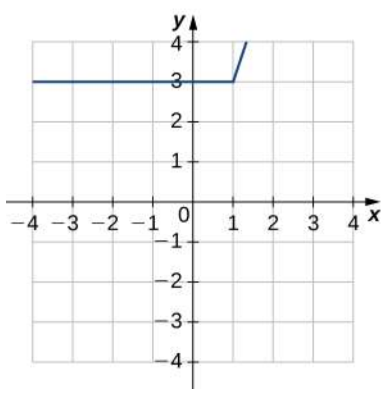
37. $s(t) = 2t^3 + 3$

38. $s(t) = \frac{16}{t^2} - \frac{4}{t}$

39. Use the following graph to evaluate a. $f'(1)$ and b. $f'(6)$.



40. Use the following graph to evaluate a. $f'(-3)$ and b. $f'(1.5)$.



For the following exercises, use the limit definition of derivative to show that the derivative does not exist at $x = a$ for each of the given functions.

41. $f(x) = x^{1/3}, x = 0$

42. $f(x) = x^{2/3}, x = 0$

43. $f(x) = \begin{cases} 1, & x < 1 \\ x, & x \geq 1 \end{cases}, x = 1$

44. $f(x) = \frac{|x|}{x}, x = 0$

45. **[T]** The position in feet of a race car along a straight track after t seconds is modeled by the function $s(t) = 8t^2 - \frac{1}{16}t^3$.

a. Find the average velocity of the vehicle over the following time intervals to four decimal places:
   i. [4, 4.1]
   ii. [4, 4.01]
   iii. [4, 4.001]
   iv. [4, 4.0001]

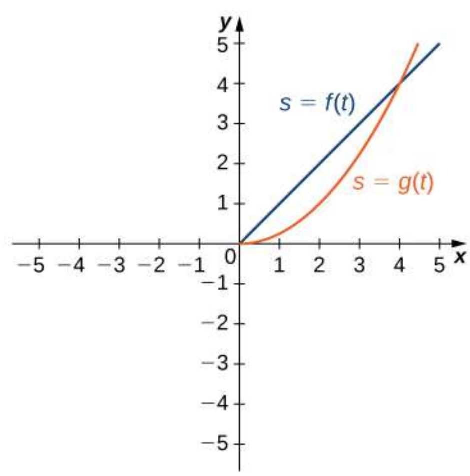
b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at $t = 4$ seconds.

46. **[T]** The distance in feet that a ball rolls down an incline is modeled by the function $s(t) = 14t^2$, where t is seconds after the ball begins rolling.

a. Find the average velocity of the ball over the following time intervals:
   i. [5, 5.1]
   ii. [5, 5.01]
   iii. [5, 5.001]
   iv. [5, 5.0001]

b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at $t = 5$ seconds.

47. Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by $s = f(t)$ and $s = g(t)$, where s is measured in feet and t is measured in seconds.



a. Which vehicle has traveled farther at $t = 2$ seconds?

b. What is the approximate velocity of each vehicle at $t = 3$ seconds?

c. Which vehicle is traveling faster at $t = 4$ seconds?

d. What is true about the positions of the vehicles at $t = 4$ seconds?

48. **[T]** The total cost $C(x)$, in hundreds of dollars, to produce x jars of mayonnaise is given by $C(x) = 0.000003x^3 + 4x + 300.$

a. Calculate the average cost per jar over the following intervals:
   i. [100, 100.1]
   ii. [100, 100.01]
   iii. [100, 100.001]
   iv. [100, 100.0001]
b. Use the answers from a. to estimate the average cost to produce 100 jars of mayonnaise.

49. **[T]** For the function $f(x) = x^3 - 2x^2 - 11x + 12,$ do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the ZOOM feature on the calculator to approximate the two values of $x = a$ for which $m_{\tan} = f'(a) = 0.$

50. **[T]** For the function $f(x) = \frac{x}{1 + x^2},$ do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the ZOOM feature on the calculator to approximate the values of $x = a$ for which $m_{\tan} = f'(a) = 0.$

51. Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle traveling x miles. Suppose the vehicle gets 30 mpg.

a. Find a mathematical expression for $N(x).$
b. What is $N(100)?$ Explain the physical meaning.
c. What is $N'(100)$? Explain the physical meaning.

52. **[T]** For the function $f(x) = x^4 - 5x^2 + 4,$ do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the nDeriv function, which numerically finds the derivative, on a graphing calculator to estimate $f'(-2),\ f'(-0.5),\ f'(1.7),$ and $f'(2.718).$

53. **[T]** For the function $f(x) = \frac{x^2}{x^2 + 1},$ do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the nDeriv function on a graphing calculator to find $f'(-4),\ f'(-2),\ f'(2),$ and $f'(4).$

3.2 | The Derivative as a Function

Learning Objectives

3.2.1 Define the derivative function of a given function.
3.2.2 Graph a derivative function from the graph of a given function.
3.2.3 State the connection between derivatives and continuity.
3.2.4 Describe three conditions for when a function does not have a derivative.
3.2.5 Explain the meaning of a higher-order derivative.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. If we differentiate a position function at a given time, we obtain the velocity at that time. It seems reasonable to conclude that knowing the derivative of the function at every point would produce valuable information about the behavior of the function. However, the process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section we define the derivative function and learn a process for finding it.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Definition

Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.9}$$

A function $f(x)$ is said to be **differentiable at** a if $f'(a)$ exists. More generally, a function is said to be **differentiable on** S if it is differentiable at every point in an open set S, and a **differentiable function** is one in which $f'(x)$ exists on its domain.

In the next few examples we use **Equation 3.9** to find the derivative of a function.

Example 3.11

Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.

Solution

Start directly with the definition of the derivative function. Use **Equation 3.1**.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Substitute $f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

Multiply numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ without distributing in the denominator.

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

Multiply the numerators and simplify.

$$= \lim_{h \to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})}$$

Cancel the h.

$$= \frac{1}{2\sqrt{x}}$$

Evaluate the limit.

Example 3.12

Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

Solution

Follow the same procedure here, but without having to multiply by the conjugate.

$$f'(x) = \lim_{h \to 0} \frac{((x+h)^2 - 2(x+h)) - (x^2 - 2x)}{h}$$

Substitute $f(x+h) = (x+h)^2 - 2(x+h)$ and $f(x) = x^2 - 2x$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$$

Expand $(x+h)^2 - 2(x+h)$.

$$= \lim_{h \to 0} \frac{2xh - 2h + h^2}{h}$$

Simplify.

$$= \lim_{h \to 0} \frac{h(2x - 2 + h)}{h}$$

Factor out h from the numerator.

$$= \lim_{h \to 0} (2x - 2 + h)$$

Cancel the common factor of h.

$$= 2x - 2$$

Evaluate the limit.

3.6 Find the derivative of $f(x) = x^2$.

We use a variety of different notations to express the derivative of a function. In **Example 3.12** we showed that if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$. If we had expressed this function in the form $y = x^2 - 2x$, we could have expressed the derivative as $y' = 2x - 2$ or $\frac{dy}{dx} = 2x - 2$. We could have conveyed the same information by writing $\frac{d}{dx}(x^2 - 2x) = 2x - 2$. Thus, for the function $y = f(x)$, each of the following notations represents the derivative of $f(x)$:

$$f'(x),\ \frac{dy}{dx},\ y',\ \frac{d}{dx}(f(x)).$$

In place of $f'(a)$ we may also use $\left.\frac{dy}{dx}\right|_{x=a}$ Use of the $\frac{dy}{dx}$ notation (called Leibniz notation) is quite common in engineering and physics. To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\frac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx (**Figure 3.11**). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to x, is expressed as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

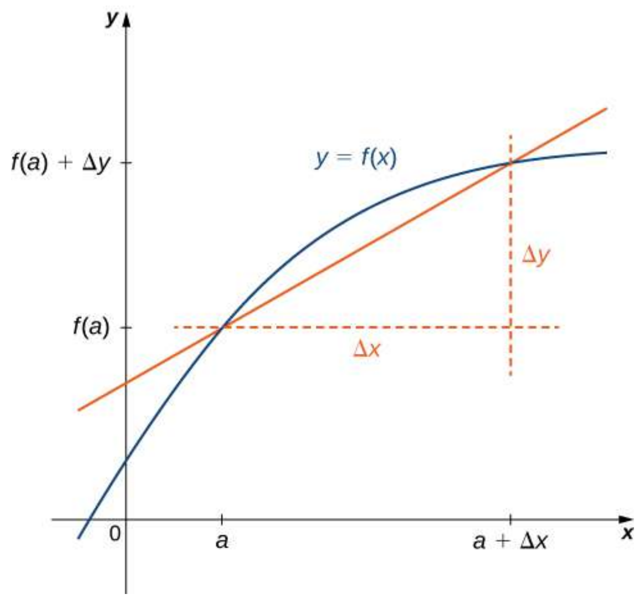


Figure 3.11 The derivative is expressed as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Graphing a Derivative

We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In **Example 3.11** we found that for $f(x) = \sqrt{x}$, $f'(x) = 1/2\sqrt{x}$. If we graph these functions on the same axes, as in **Figure 3.12**, we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \to 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.

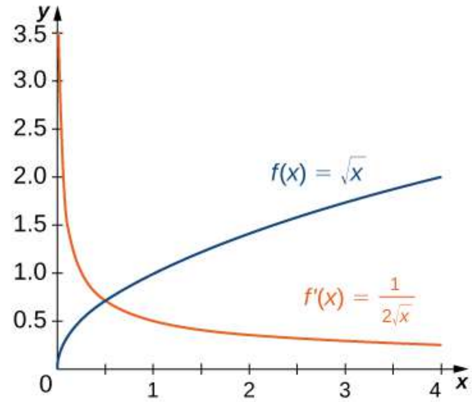


Figure 3.12 The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.

In **Example 3.12** we found that for $f(x) = x^2 - 2x$, $f'(x) = 2x - 2$. The graphs of these functions are shown in **Figure 3.13**. Observe that $f(x)$ is decreasing for $x < 1$. For these same values of x, $f'(x) < 0$. For values of $x > 1$, $f(x)$ is increasing and $f'(x) > 0$. Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.

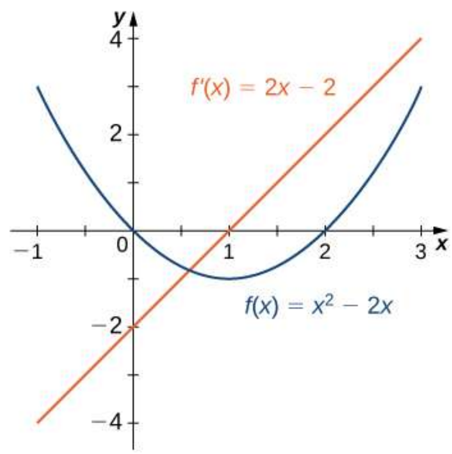
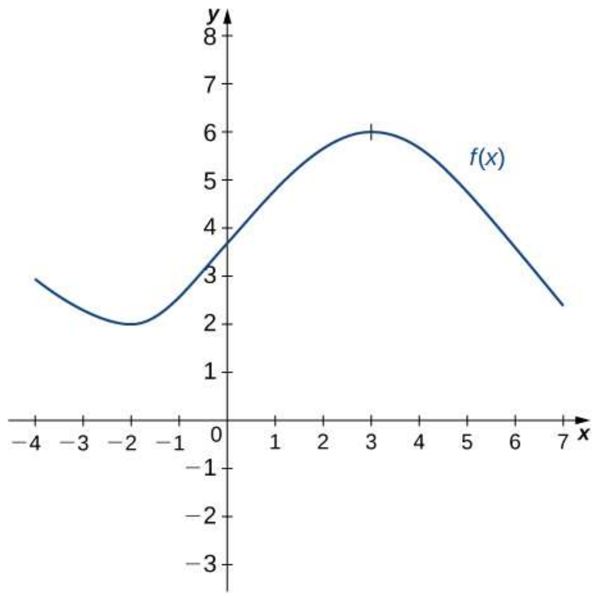


Figure 3.13 The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent.

Example 3.13

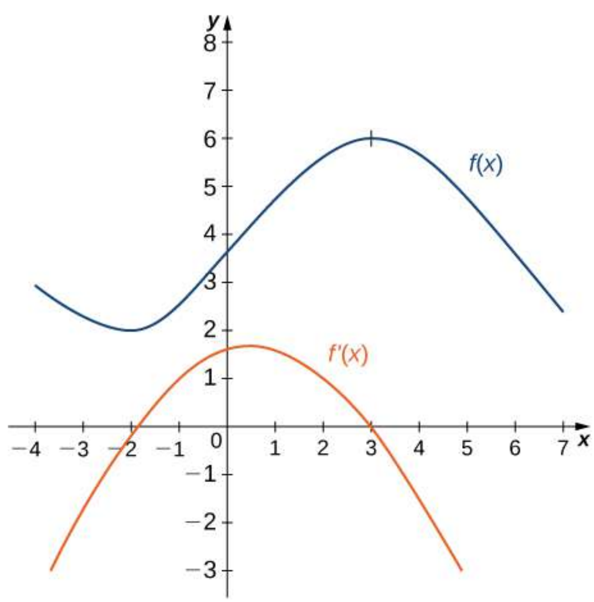
Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



Solution

The solution is shown in the following graph. Observe that $f(x)$ is increasing and $f'(x) > 0$ on $(-2, 3)$. Also, $f(x)$ is decreasing and $f'(x) < 0$ on $(-\infty, -2)$ and on $(3, +\infty)$. Also note that $f(x)$ has horizontal tangents at -2 and $3,$ and $f'(-2) = 0$ and $f'(3) = 0.$



3.7 Sketch the graph of $f(x) = x^2 - 4.$ On what interval is the graph of $f'(x)$ above the x-axis?

Derivatives and Continuity

Now that we can graph a derivative, let's examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there;

however, a function that is continuous at a point need not be differentiable at that point. In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Theorem 3.1: Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

Proof

If $f(x)$ is differentiable at a, then $f'(a)$ exists and

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim_{x \to a} f(x) = f(a)$. Thus,

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left(f(x) - f(a) + f(a)\right) \\
&= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a)\right) && \text{Multiply and divide } f(x) - f(a) \text{ by } x - a. \\
&= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right) \cdot \left(\lim_{x \to a} (x - a)\right) + \lim_{x \to a} f(a) \\
&= f'(a) \cdot 0 + f(a) \\
&= f(a).
\end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a)$, we conclude that f is continuous at a.

□

We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To determine an answer to this question, we examine the function $f(x) = |x|$. This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f(x) = |x|$,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

This limit does not exist because

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \text{ and } \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

See **Figure 3.14**.

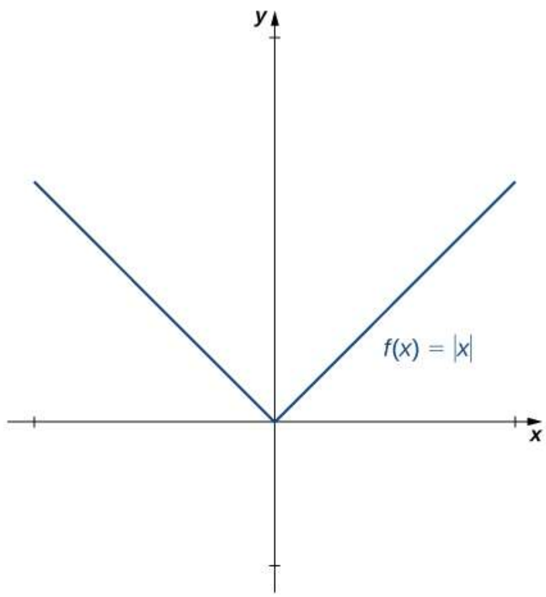


Figure 3.14 The function $f(x) = |x|$ is continuous at 0 but is not differentiable at 0.

Let's consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f(x) = \sqrt[3]{x}$:

$$f'(0) = \lim_{x \to 0} \frac{\sqrt[3]{x} - 0}{x - 0} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Thus $f'(0)$ does not exist. A quick look at the graph of $f(x) = \sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at 0 (**Figure 3.15**).

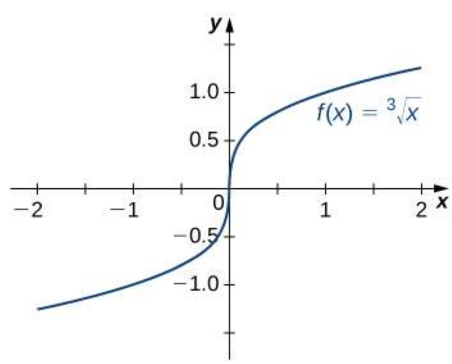


Figure 3.15 The function $f(x) = \sqrt[3]{x}$ has a vertical tangent at $x = 0$. It is continuous at 0 but is not differentiable at 0.

The function $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ also has a derivative that exhibits interesting behavior at 0. We see that

$$f'(0) = \lim_{x \to 0} \frac{x\sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right).$$

This limit does not exist, essentially because the slopes of the secant lines continuously change direction as they approach zero (**Figure 3.16**).

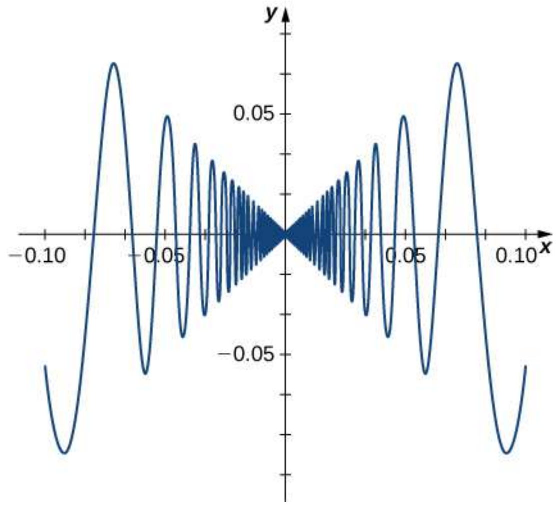


Figure 3.16 The function $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ is not differentiable at 0.

In summary:

1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f(x) = |x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be “smooth” at that point.
3. As we saw in the example of $f(x) = \sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.
4. As we saw with $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

Example 3.14

A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line (**Figure 3.17**). The function that describes the track is to have the form $f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c \text{ if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2} \text{ if } x \geq -10 \end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.

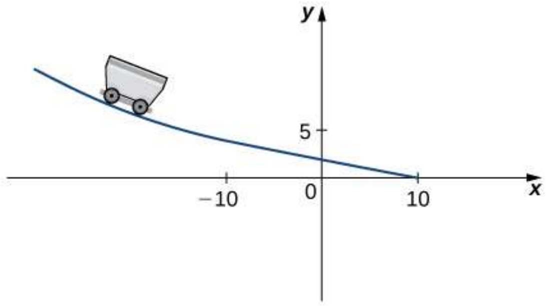


Figure 3.17 For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution

For the function to be continuous at $x = -10$, $\lim_{x \to 10^-} f(x) = f(-10)$. Thus, since

$$\lim_{x \to -10^-} f(x) = \frac{1}{10}(-10)^2 - 10b + c = 10 - 10b + c$$

and $f(-10) = 5$, we must have $10 - 10b + c = 5$. Equivalently, we have $c = 10b - 5$.

For the function to be differentiable at -10,

$$f'(10) = \lim_{x \to -10} \frac{f(x) - f(-10)}{x + 10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

$$\begin{aligned}
\lim_{x \to -10^-} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x \to -10^-} \frac{\frac{1}{10}x^2 + bx + c - 5}{x + 10} \\
&= \lim_{x \to -10^-} \frac{\frac{1}{10}x^2 + bx + (10b - 5) - 5}{x + 10} && \text{Substitute } c = 10b - 5. \\
&= \lim_{x \to -10^-} \frac{x^2 - 100 + 10bx + 100b}{10(x + 10)} \\
&= \lim_{x \to -10^-} \frac{(x + 10)(x - 10 + 10b)}{10(x + 10)} && \text{Factor by grouping.} \\
&= b - 2.
\end{aligned}$$

We also have

$$\begin{aligned}
\lim_{x \to -10^+} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x \to -10^+} \frac{-\frac{1}{4}x + \frac{5}{2} - 5}{x + 10} \\
&= \lim_{x \to -10^+} \frac{-(x + 10)}{4(x + 10)} \\
&= -\frac{1}{4}.
\end{aligned}$$

This gives us $b - 2 = -\frac{1}{4}$. Thus $b = \frac{7}{4}$ and $c = 10\left(\frac{7}{4}\right) - 5 = \frac{25}{2}$.

3.8 Find values of a and b that make $f(x) = \begin{cases} ax + b & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$ both continuous and differentiable at 3.

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y = f(x)$ can be expressed in any of the following forms:

$$f''(x),\ f'''(x),\ f^{(4)}(x),\dots,\ f^{(n)}(x)$$

$$y''(x),\ y'''(x),\ y^{(4)}(x),\dots,\ y^{(n)}(x)$$

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4},\dots, \frac{d^ny}{dx^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly. Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$.

Example 3.15

Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution

First find $f'(x)$.

$f'(x) = \lim\limits_{h \to 0} \frac{\left(2(x+h)^2 - 3(x+h) + 1\right) - (2x^2 - 3x + 1)}{h}$	Substitute $f(x) = 2x^2 - 3x + 1$ and $f(x+h) = 2(x+h)^2 - 3(x+h) + 1$ into $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
$= \lim\limits_{h \to 0} \frac{4xh + 2h^2 - 3h}{h}$	Simplify the numerator.
$= \lim\limits_{h \to 0} (4x + 2h - 3)$	Factor out the h in the numerator and cancel with the h in the denominator.
$= 4x - 3$	Take the limit.

Next, find $f''(x)$ by taking the derivative of $f'(x) = 4x - 3$.

$f''(x) = \lim\limits_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$	Use $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ with $f'(x)$ in place of $f(x)$.
$= \lim\limits_{h \to 0} \frac{(4(x+h) - 3) - (4x - 3)}{h}$	Substitute $f'(x+h) = 4(x+h) - 3$ and $f'(x) = 4x - 3$.
$= \lim\limits_{h \to 0} 4$	Simplify.
$= 4$	Take the limit.

3.9 Find $f''(x)$ for $f(x) = x^2$.

Example 3.16

Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t.

Solution

Since $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$, we begin by finding the derivative of $s(t)$:

$$\begin{aligned} s'(t) &= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h \to 0} \frac{3(t+h)^2 - 4(t+h) + 1 - \left(3t^2 - 4t + 1\right)}{h} \\ &= 6t - 4. \end{aligned}$$

Next,

$$\begin{aligned} s''(t) &= \lim_{h \to 0} \frac{s'(t+h) - s'(t)}{h} \\ &= \lim_{h \to 0} \frac{6(t+h) - 4 - (6t - 4)}{h} \\ &= 6. \end{aligned}$$

Thus, $a = 6 \text{ m/s}^2$.

3.10 For $s(t) = t^3$, find $a(t)$.

3.2 EXERCISES

For the following exercises, use the definition of a derivative to find $f'(x)$.

54. $f(x) = 6$

55. $f(x) = 2 - 3x$

56. $f(x) = \frac{2x}{7} + 1$

57. $f(x) = 4x^2$

58. $f(x) = 5x - x^2$

59. $f(x) = \sqrt{2x}$

60. $f(x) = \sqrt{x - 6}$

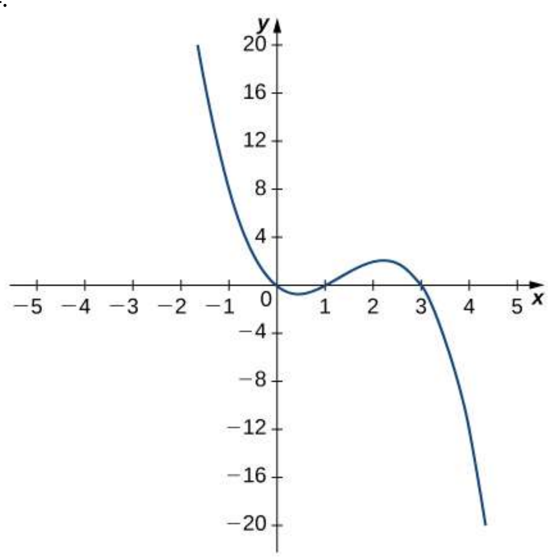
61. $f(x) = \frac{9}{x}$

62. $f(x) = x + \frac{1}{x}$

63. $f(x) = \frac{1}{\sqrt{x}}$

For the following exercises, use the graph of $y = f(x)$ to sketch the graph of its derivative $f'(x)$.

64.



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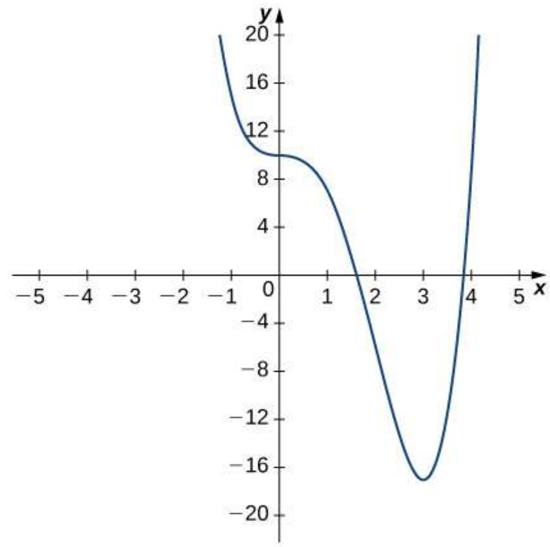

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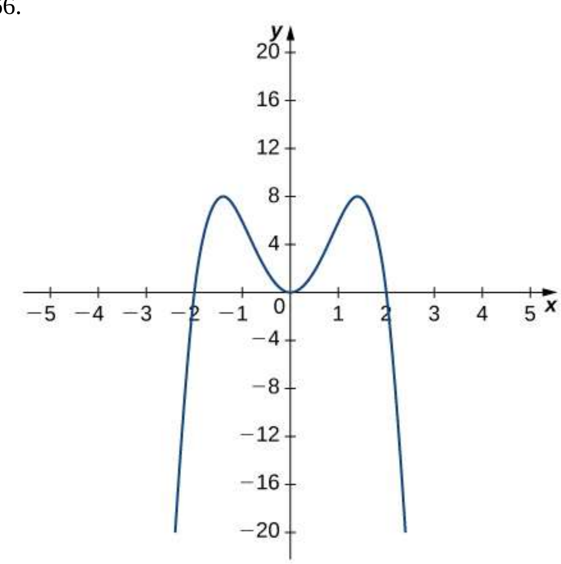

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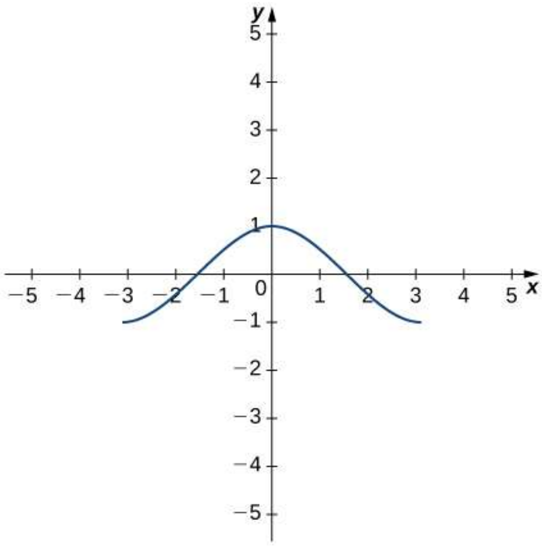


For the following exercises, the given limit represents the derivative of a function $y = f(x)$ at $x = a$. Find $f(x)$ and a.

68. $\lim_{h \to 0} \frac{(1+h)^{2/3} - 1}{h}$

69. $\lim_{h \to 0} \frac{\left[3(2+h)^2 + 2\right] - 14}{h}$

70. $\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

71. $\lim_{h \to 0} \frac{(2+h)^4 - 16}{h}$

72. $\lim_{h \to 0} \frac{[2(3+h)^2 - (3+h)] - 15}{h}$

73. $\lim_{h \to 0} \frac{e^h - 1}{h}$

For the following functions,

a. sketch the graph and
b. use the definition of a derivative to show that the function is not differentiable at $x = 1$.

74. $f(x) = \begin{cases} 2\sqrt{x}, & 0 \le x \le 1 \\ 3x - 1, & x > 1 \end{cases}$

75. $f(x) = \begin{cases} 3, & x < 1 \\ 3x, & x \ge 1 \end{cases}$

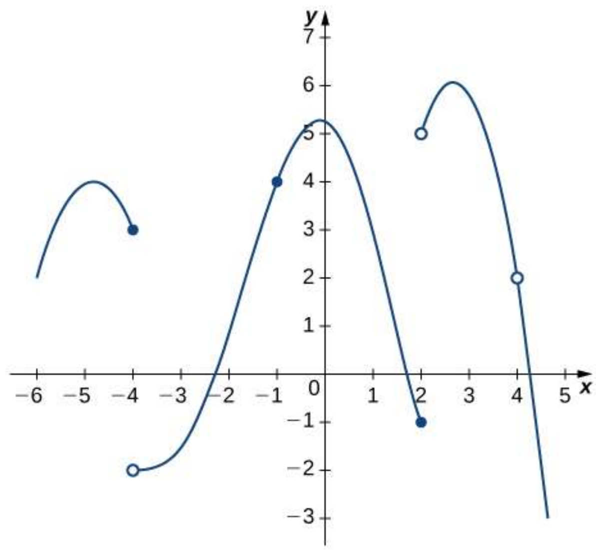
76. $f(x) = \begin{cases} -x^2 + 2, & x \le 1 \\ x, & x > 1 \end{cases}$

77. $f(x) = \begin{cases} 2x, & x \le 1 \\ \frac{2}{x}, & x > 1 \end{cases}$

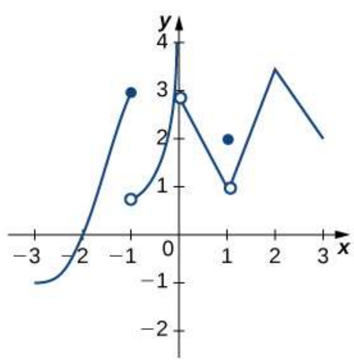
For the following graphs,

a. determine for which values of $x = a$ the $\lim_{x \to a} f(x)$ exists but f is not continuous at $x = a$, and
b. determine for which values of $x = a$ the function is continuous but not differentiable at $x = a$.

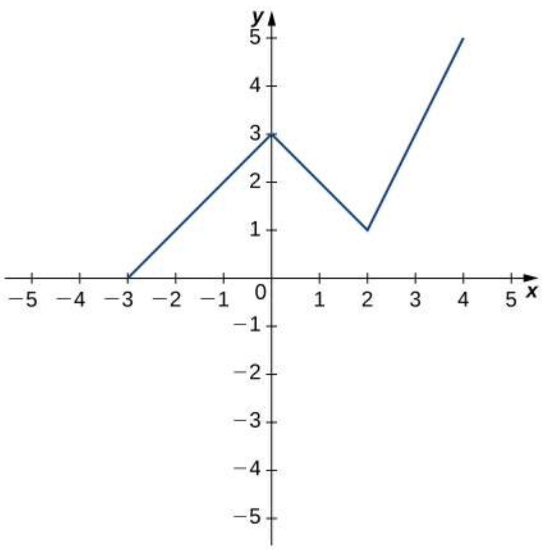
78.



79.



80. Use the graph to evaluate a. $f'(-0.5)$, b. $f'(0)$, c. $f'(1)$, d. $f'(2)$, and e. $f'(3)$, if it exists.



For the following functions, use $f''(x) = \lim\limits_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$ to find $f''(x)$.

81. $f(x) = 2 - 3x$

82. $f(x) = 4x^2$

83. $f(x) = x + \frac{1}{x}$

For the following exercises, use a calculator to graph $f(x)$. Determine the function $f'(x)$, then use a calculator to graph $f'(x)$.

84. **[T]** $f(x) = -\frac{5}{x}$

85. **[T]** $f(x) = 3x^2 + 2x + 4.$

86. **[T]** $f(x) = \sqrt{x} + 3x$

87. **[T]** $f(x) = \frac{1}{\sqrt{2x}}$

88. **[T]** $f(x) = 1 + x + \frac{1}{x}$

89. **[T]** $f(x) = x^3 + 1$

For the following exercises, describe what the two expressions represent in terms of each of the given situations. Be sure to include units.

a. $\frac{f(x+h) - f(x)}{h}$

b. $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

90. $P(x)$ denotes the population of a city at time x in years.

91. $C(x)$ denotes the total amount of money (in thousands of dollars) spent on concessions by x customers at an amusement park.

92. $R(x)$ denotes the total cost (in thousands of dollars) of manufacturing x clock radios.

93. $g(x)$ denotes the grade (in percentage points) received on a test, given x hours of studying.

94. $B(x)$ denotes the cost (in dollars) of a sociology textbook at university bookstores in the United States in x years since 1990.

95. $p(x)$ denotes atmospheric pressure at an altitude of x feet.

96. Sketch the graph of a function $y = f(x)$ with all of the following properties:

a. $f'(x) > 0$ for $-2 \le x < 1$

b. $f'(2) = 0$

c. $f'(x) > 0$ for $x > 2$

d. $f(2) = 2$ and $f(0) = 1$

e. $\lim\limits_{x \to -\infty} f(x) = 0$ and $\lim\limits_{x \to \infty} f(x) = \infty$

f. $f'(1)$ does not exist.

97. Suppose temperature T in degrees Fahrenheit at a height x in feet above the ground is given by $y = T(x)$.

a. Give a physical interpretation, with units, of $T'(x)$.

b. If we know that $T'(1000) = -0.1$, explain the physical meaning.

98. Suppose the total profit of a company is $y = P(x)$ thousand dollars when x units of an item are sold.

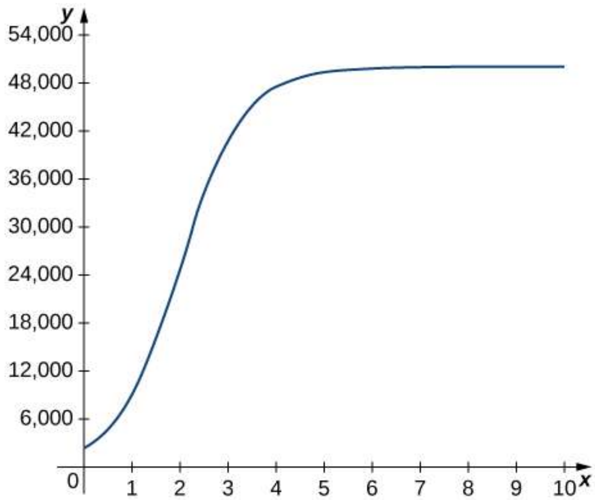
a. What does $\frac{P(b) - P(a)}{b - a}$ for $0 < a < b$ measure, and what are the units?

b. What does $P'(x)$ measure, and what are the units?

c. Suppose that $P'(30) = 5$, what is the approximate change in profit if the number of items sold increases from 30 to 31 ?

99. The graph in the following figure models the number of people $N(t)$ who have come down with the flu t weeks after its initial outbreak in a town with a population of $50,000$ citizens.

a. Describe what $N'(t)$ represents and how it behaves as t increases.
b. What does the derivative tell us about how this town is affected by the flu outbreak?



For the following exercises, use the following table, which shows the height h of the Saturn V rocket for the Apollo 11 mission t seconds after launch.

Time (seconds)	Height (meters)
0	0
1	2
2	4
3	13
4	25
5	32

100. What is the physical meaning of $h'(t)$? What are the units?

101. **[T]** Construct a table of values for $h'(t)$ and graph both $h(t)$ and $h'(t)$ on the same graph. (*Hint:* for **interior points**, estimate both the left limit and right limit and average them. An interior point of an interval I is an element of I which is not an endpoint of I.)

102. **[T]** The best linear fit to the data is given by $H(t) = 7.229t - 4.905,$ where H is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $H'(t)$. Graph $H(t)$ with the given data and, on a separate coordinate plane, graph $H'(t)$.

103. **[T]** The best quadratic fit to the data is given by $G(t) = 1.429t^2 + 0.0857t - 0.1429,$ where G is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $G'(t)$. Graph $G(t)$ with the given data and, on a separate coordinate plane, graph $G'(t)$.

104. **[T]** The best cubic fit to the data is given by $F(t) = 0.2037t^3 + 2.956t^2 - 2.705t + 0.4683,$ where F is the height of the rocket (in m) and t is the time elapsed since take off. From this equation, determine $F'(t)$. Graph $F(t)$ with the given data and, on a separate coordinate plane, graph $F'(t)$. Does the linear, quadratic, or cubic function fit the data best?

105. Using the best linear, quadratic, and cubic fits to the data, determine what $H''(t)$, $G''(t)$ and $F''(t)$ are. What are the physical meanings of $H''(t)$, $G''(t)$ and $F''(t)$, and what are their units?

3.3 | Differentiation Rules

Learning Objectives

- **3.3.1** State the constant, constant multiple, and power rules.
- **3.3.2** Apply the sum and difference rules to combine derivatives.
- **3.3.3** Use the product rule for finding the derivative of a product of functions.
- **3.3.4** Use the quotient rule for finding the derivative of a quotient of functions.
- **3.3.5** Extend the power rule to functions with negative exponents.
- **3.3.6** Combine the differentiation rules to find the derivative of a polynomial or rational function.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit. The process that we could use to evaluate $\frac{d}{dx}\left(\sqrt[3]{x}\right)$ using the definition, while similar, is more complicated. In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c.$ For this function, both $f(x) = c$ and $f(x + h) = c,$ so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= \lim_{h \to 0} 0 = 0. \end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is $0.$ We restate this rule in the following theorem.

Theorem 3.2: The Constant Rule

Let c be a constant.

If $f(x) = c,$ then $f'(c) = 0.$

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example 3.17

Applying the Constant Rule

Find the derivative of $f(x) = 8$.

Solution

This is just a one-step application of the rule:

$$f'(x) = 0.$$

3.11 Find the derivative of $g(x) = -3$.

The Power Rule

We have shown that

$$\frac{d}{dx}\left(x^2\right) = 2x \text{ and } \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}(x^n)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x) = x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$. As we go through this derivation, note that the technique used in this case is essentially the same as the technique used to prove the general case.

Example 3.18

Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

Solution

$$\frac{d}{dx}(x^3) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2 h + 3xh^2 + h^3 - x^3}{h}$$

Notice that the first term in the expansion of $(x+h)^3$ is x^3 and the second term is $3x^2 h$. All other terms contain powers of h that are two or greater.

$$= \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3}{h}$$

In this step the x^3 terms have been cancelled, leaving only terms containing h.

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

Factor out the common factor of h.

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

After cancelling the common factor of h, the only term not containing h is $3x^2$.

$$= 3x^2$$

Let h go to 0.

3.12 Find $\frac{d}{dx}(x^4)$.

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x) = 3^x$.

Theorem 3.3: The Power Rule

Let n be a positive integer. If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof

For $f(x) = x^n$ where n is a positive integer, we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

$$\text{Since } (x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n,$$

we see that

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n.$$

Next, divide both sides by h:

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n}{h}.$$

Thus,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}.$$

Finally,

$$\begin{aligned} f'(x) &= \lim_{h \to 0}\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-1} + h^n\right) \\ &= nx^{n-1}. \end{aligned}$$

□

Example 3.19

Applying the Power Rule

Find the derivative of the function $f(x) = x^{10}$ by applying the power rule.

Solution

Using the power rule with $n = 10,$ we obtain

$$f'(x) = 10x^{10-1} = 10x^9.$$

3.13 Find the derivative of $f(x) = x^7$.

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Theorem 3.4: Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x));$$

that is,

$$\text{for } j(x) = f(x) + g(x),\ j'(x) = f'(x) + g'(x).$$

Difference Rule. The derivative of the difference of a function *f* and a function *g* is the same as the difference of the derivative of f and the derivative of g:

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

that is,

$$\text{for } j(x) = f(x) - g(x),\ j'(x) = f'(x) - g'(x).$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x));$$

that is,

$$\text{for } j(x) = kf(x),\ j'(x) = kf'(x).$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $j(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

$$j'(x) = \lim_{h \to 0} \frac{j(x+h) - j(x)}{h}.$$

By substituting $j(x+h) = f(x+h) + g(x+h)$ and $j(x) = f(x) + g(x)$, we obtain

$$j'(x) = \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}.$$

Rearranging and regrouping the terms, we have

$$j'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right).$$

We now apply the sum law for limits and the definition of the derivative to obtain

$$j'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right) + \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h}\right) = f'(x) + g'(x).$$

□

Example 3.20

Applying the Constant Multiple Rule

Find the derivative of $g(x) = 3x^2$ and compare it to the derivative of $f(x) = x^2$.

Solution

We use the power rule directly:

$$g'(x) = \frac{d}{dx}\left(3x^2\right) = 3\frac{d}{dx}\left(x^2\right) = 3(2x) = 6x.$$

Since $f(x) = x^2$ has derivative $f'(x) = 2x$, we see that the derivative of $g(x)$ is 3 times the derivative of

$f(x)$. This relationship is illustrated in **Figure 3.18**.

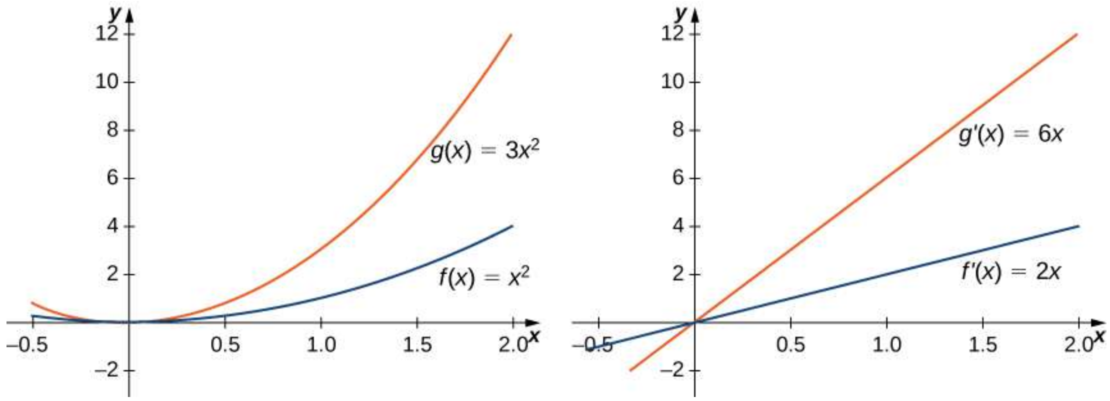


Figure 3.18 The derivative of $g(x)$ is 3 times the derivative of $f(x)$.

Example 3.21

Applying Basic Derivative Rules

Find the derivative of $f(x) = 2x^5 + 7$.

Solution

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^5 + 7\right) \\ &= \frac{d}{dx}\left(2x^5\right) + \frac{d}{dx}(7) && \text{Apply the sum rule.} \\ &= 2\frac{d}{dx}\left(x^5\right) + \frac{d}{dx}(7) && \text{Apply the constant multiple rule.} \\ &= 2\left(5x^4\right) + 0 && \text{Apply the power rule and the constant rule.} \\ &= 10x^4. && \text{Simplify.} \end{aligned}$$

3.14 Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$.

Example 3.22

Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$.

Solution

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

$$f(1) = 1^2 - 4(1) + 6 = 3.$$

This gives us the point $(1, 3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have

$$f'(x) = 2x - 4$$

so the slope of the tangent line is $f'(1) = -2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y - 3 = -2(x - 1).$$

Putting the equation of the line in slope-intercept form, we obtain

$$y = -2x + 5.$$

3.15 Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

Theorem 3.5: Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\big(f(x)g(x)\big) = \frac{d}{dx}\big(f(x)\big) \cdot g(x) + \frac{d}{dx}\big(g(x)\big) \cdot f(x).$$

That is,

$$\text{if } j(x) = f(x)g(x), \text{ then } j'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim_{h \to 0} g(x + h) = g(x)$.

By applying the limit definition of the derivative to $j(x) = f(x)g(x)$, we obtain

$$j'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$j'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

$$j'(x) = \lim_{h \to 0}\left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h}\right) + \lim_{h \to 0}\left(\frac{f(x)g(x+h) - f(x)g(x)}{h}\right).$$

Rearranging, we obtain

$$j'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h)\right) + \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h} \cdot f(x)\right).$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

$$j'(x) = f'(x)g(x) + g'(x)f(x).$$

□

Example 3.23

Applying the Product Rule to Functions at a Point

For $j(x) = f(x)g(x)$, use the product rule to find $j'(2)$ if $f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, and $g'(2) = 6$.

Solution

Since $j(x) = f(x)g(x)$, $j'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

$$j'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14.$$

Example 3.24

Applying the Product Rule to Binomials

For $j(x) = (x^2 + 2)(3x^3 - 5x)$, find $j'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$. Thus,

$$j'(x) = f'(x)g(x) + g'(x)f(x) = (2x)\left(3x^3 - 5x\right) + (9x^2 - 5)(x^2 + 2).$$

Simplifying, we have

$$j'(x) = 15x^4 + 3x^2 - 10.$$

To check, we see that $j(x) = 3x^5 + x^3 - 10x$ and, consequently, $j'(x) = 15x^4 + 3x^2 - 10$.

3.16 Use the product rule to obtain the derivative of $j(x) = 2x^5\left(4x^2 + x\right)$.

The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}\left(x^2\right) = 2x, \text{ not } \frac{\frac{d}{dx}\left(x^3\right)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

Theorem 3.6: The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}.$$

That is,

$$\text{if } j(x) = \frac{f(x)}{g(x)}, \text{ then } j'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}.$$

The proof of the **quotient rule** is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example 3.25

Applying the Quotient Rule

Use the quotient rule to find the derivative of $k(x) = \dfrac{5x^2}{4x + 3}$.

Solution

Let $f(x) = 5x^2$ and $g(x) = 4x + 3$. Thus, $f'(x) = 10x$ and $g'(x) = 4$. Substituting into the quotient rule, we have

$$k'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x + 3) - 4(5x^2)}{(4x + 3)^2}.$$

Simplifying, we obtain

$$k'(x) = \frac{20x^2 + 30x}{(4x+3)^2}.$$

3.17 Find the derivative of $h(x) = \frac{3x+1}{4x-3}$.

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

Theorem 3.7: Extended Power Rule

If k is a negative integer, then

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

Proof

If k is a negative integer, we may set $n = -k$, so that n is a positive integer with $k = -n$. Since for each positive integer n, $x^{-n} = \frac{1}{x^n}$, we may now apply the quotient rule by setting $f(x) = 1$ and $g(x) = x^n$. In this case, $f'(x) = 0$ and $g'(x) = nx^{n-1}$. Thus,

$$\frac{d}{dx}(x^{-n}) = \frac{0(x^n) - 1\left(nx^{n-1}\right)}{(x^n)^2}.$$

Simplifying, we see that

$$\frac{d}{dx}(x^{-n}) = \frac{-nx^{n-1}}{x^{2n}} = -nx^{(n-1)-2n} = -nx^{-n-1}.$$

Finally, observe that since $k = -n$, by substituting we have

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

□

Example 3.26

Using the Extended Power Rule

Find $\frac{d}{dx}\left(x^{-4}\right)$.

Solution

By applying the extended power rule with $k = -4$, we obtain

$$\frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5}.$$

Example 3.27

Using the Extended Power Rule and the Constant Multiple Rule

Use the extended power rule and the constant multiple rule to find the derivative of $f(x) = \frac{6}{x^2}$.

Solution

It may seem tempting to use the quotient rule to find this derivative, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this function by first rewriting it as $f(x) = 6x^{-2}$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{6}{x^2}\right) = \frac{d}{dx}\left(6x^{-2}\right) && \text{Rewrite } \frac{6}{x^2} \text{ as } 6x^{-2}. \\ &= 6\frac{d}{dx}(x^{-2}) && \text{Apply the constant multiple rule.} \\ &= 6(-2x^{-3}) && \text{Use the extended power rule to differentiate } x^{-2}. \\ &= -12x^{-3} && \text{Simplify.} \end{aligned}$$

3.18 Find the derivative of $g(x) = \frac{1}{x^7}$ using the extended power rule.

Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example 3.28

Combining Differentiation Rules

For $k(x) = 3h(x) + x^2 g(x)$, find $k'(x)$.

Solution

Finding this derivative requires the sum rule, the constant multiple rule, and the product rule.

$$\begin{aligned} k'(x) &= \frac{d}{dx}\left(3h(x) + x^2 g(x)\right) = \frac{d}{dx}(3h(x)) + \frac{d}{dx}\left(x^2 g(x)\right) && \text{Apply the sum rule.} \\ &= 3\frac{d}{dx}(h(x)) + \left(\frac{d}{dx}\left(x^2\right)g(x) + \frac{d}{dx}(g(x))x^2\right) && \text{Apply the constant multiple rule to differentiate } 3h(x) \text{ and the product rule to differentiate } x^2 g(x). \\ &= 3h'(x) + 2xg(x) + g'(x)x^2 \end{aligned}$$

Example 3.29

Extending the Product Rule

For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x)$, $g(x)$, $h(x)$, and their derivatives.

Solution

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x) = (f(x)g(x)) \cdot h(x)$. Thus,

$$\begin{aligned} k'(x) &= \frac{d}{dx}(f(x)g(x)) \cdot h(x) + \frac{d}{dx}(h(x)) \cdot (f(x)g(x)) && \text{Apply the product rule to the product of } f(x)g(x) \text{ and } h(x). \\ &= (f'(x)g(x) + g'(x)f(x))h(x) + h'(x)f(x)g(x) && \text{Apply the product rule to } f(x)g(x). \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x). && \text{Simplify.} \end{aligned}$$

Example 3.30

Combining the Quotient Rule and the Product Rule

For $h(x) = \dfrac{2x^3 k(x)}{3x + 2}$, find $h'(x)$.

Solution

This procedure is typical for finding the derivative of a rational function.

$$\begin{aligned} h'(x) &= \frac{\frac{d}{dx}\left(2x^3 k(x)\right) \cdot (3x+2) - \frac{d}{dx}(3x+2) \cdot \left(2x^3 k(x)\right)}{(3x+2)^2} && \text{Apply the quotient rule.} \\ &= \frac{\left(6x^2 k(x) + k'(x) \cdot 2x^3\right)(3x+2) - 3\left(2x^3 k(x)\right)}{(3x+2)^2} && \text{Apply the product rule to find } \frac{d}{dx}\left(2x^3 k(x)\right). \text{ Use } \frac{d}{dx}(3x+2) = 3. \\ &= \frac{-6x^3 k(x) + 18x^3 k(x) + 12x^2 k(x) + 6x^4 k'(x) + 4x^3 k'(x)}{(3x+2)^2} && \text{Simplify.} \end{aligned}$$

3.19 Find $\frac{d}{dx}(3f(x) - 2g(x))$.

Example 3.31

Determining Where a Function Has a Horizontal Tangent

Determine the values of x for which $f(x) = x^3 - 7x^2 + 8x + 1$ has a horizontal tangent line.

Solution

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'(x) = 0$. Since

$$f'(x) = 3x^2 - 14x + 8 = (3x - 2)(x - 4),$$

we must solve $(3x - 2)(x - 4) = 0$. Thus we see that the function has horizontal tangent lines at $x = \frac{2}{3}$ and $x = 4$ as shown in the following graph.

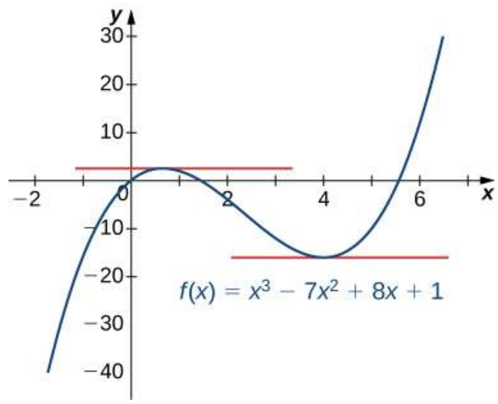


Figure 3.19 This function has horizontal tangent lines at x = 2/3 and x = 4.

Example 3.32

Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s(t) = \frac{t}{t^2 + 1}$. What is the initial velocity of the object?

Solution

Since the initial velocity is $v(0) = s'(0)$, begin by finding $s'(t)$ by applying the quotient rule:

$$s'(t) = \frac{1\left(t^2 + 1\right) - 2t(t)}{\left(t^2 + 1\right)^2} = \frac{1 - t^2}{\left(t^2 + 1\right)^2}.$$

After evaluating, we see that $v(0) = 1$.

3.20 Find the values of x for which the graph of $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to the line $y = 2x + 3$.

Student PROJECT

Formula One Grandstands

Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car (Figure 3.20).

Figure 3.20 The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ (Figure 3.21). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9,\ 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.

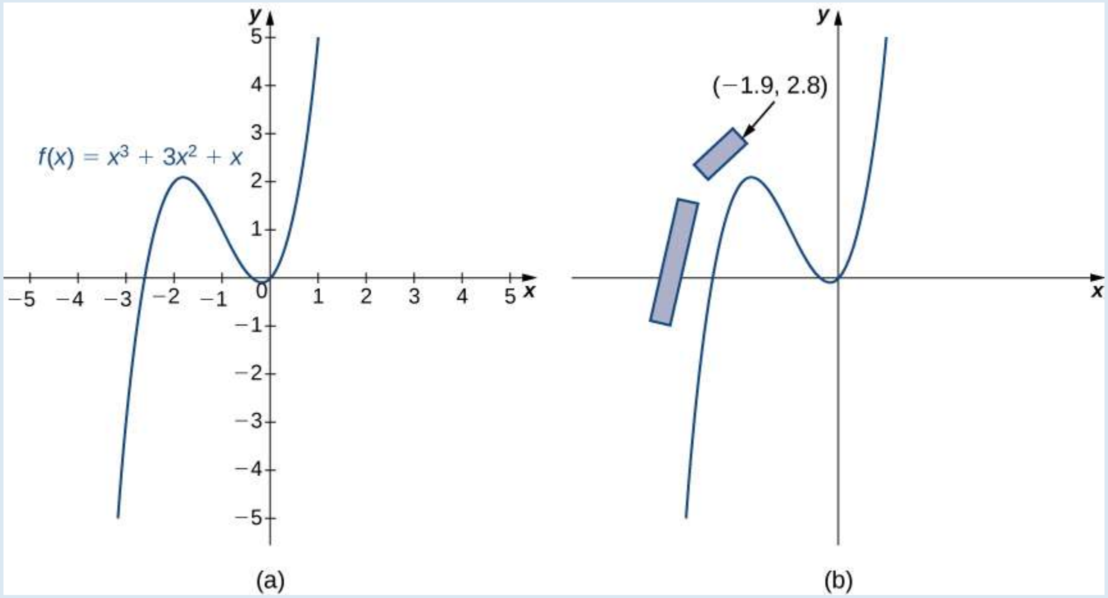


Figure 3.21 (a) One section of the racetrack can be modeled by the function $f(x) = x^3 + 3x^2 + x$. (b) The front corner of the grandstand is located at $(-1.9, 2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x, y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the *x*-coordinate of the point where the tangent line crosses the line $y = 2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?
4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

3.3 EXERCISES

For the following exercises, find $f'(x)$ for each function.

106. $f(x) = x^7 + 10$

107. $f(x) = 5x^3 - x + 1$

108. $f(x) = 4x^2 - 7x$

109. $f(x) = 8x^4 + 9x^2 - 1$

110. $f(x) = x^4 + \frac{2}{x}$

111. $f(x) = 3x\left(18x^4 + \frac{13}{x+1}\right)$

112. $f(x) = (x+2)\left(2x^2 - 3\right)$

113. $f(x) = x^2\left(\frac{2}{x^2} + \frac{5}{x^3}\right)$

114. $f(x) = \frac{x^3 + 2x^2 - 4}{3}$

115. $f(x) = \frac{4x^3 - 2x + 1}{x^2}$

116. $f(x) = \frac{x^2 + 4}{x^2 - 4}$

117. $f(x) = \frac{x + 9}{x^2 - 7x + 1}$

For the following exercises, find the equation of the tangent line $T(x)$ to the graph of the given function at the indicated point. Use a graphing calculator to graph the function and the tangent line.

118. **[T]** $y = 3x^2 + 4x + 1$ at $(0, 1)$

119. **[T]** $y = 2\sqrt{x} + 1$ at $(4, 5)$

120. **[T]** $y = \frac{2x}{x-1}$ at $(-1, 1)$

121. **[T]** $y = \frac{2}{x} - \frac{3}{x^2}$ at $(1, -1)$

For the following exercises, assume that $f(x)$ and $g(x)$ are both differentiable functions for all x. Find the derivative of each of the functions $h(x)$.

122. $h(x) = 4f(x) + \frac{g(x)}{7}$

123. $h(x) = x^3 f(x)$

124. $h(x) = \frac{f(x)g(x)}{2}$

125. $h(x) = \frac{3f(x)}{g(x) + 2}$

For the following exercises, assume that $f(x)$ and $g(x)$ are both differentiable functions with values as given in the following table. Use the following table to calculate the following derivatives.

x	1	2	3	4
$f(x)$	3	5	−2	0
$g(x)$	2	3	−4	6
$f'(x)$	−1	7	8	−3
$g'(x)$	4	1	2	9

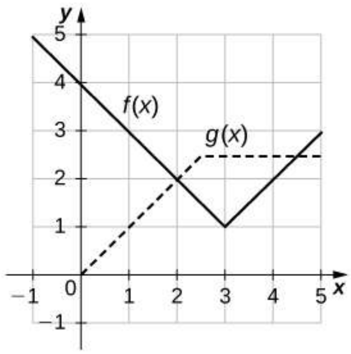
126. Find $h'(1)$ if $h(x) = xf(x) + 4g(x)$.

127. Find $h'(2)$ if $h(x) = \frac{f(x)}{g(x)}$.

128. Find $h'(3)$ if $h(x) = 2x + f(x)g(x)$.

129. Find $h'(4)$ if $h(x) = \frac{1}{x} + \frac{g(x)}{f(x)}$.

For the following exercises, use the following figure to find the indicated derivatives, if they exist.



130. Let $h(x) = f(x) + g(x).$ Find
a. $h'(1),$
b. $h'(3),$ and
c. $h'(4).$

131. Let $h(x) = f(x)g(x).$ Find
a. $h'(1),$
b. $h'(3),$ and
c. $h'(4).$

132. Let $h(x) = \frac{f(x)}{g(x)}.$ Find
a. $h'(1),$
b. $h'(3),$ and
c. $h'(4).$

For the following exercises,
a. evaluate $f'(a),$ and
b. graph the function $f(x)$ and the tangent line at $x = a.$

133. **[T]** $f(x) = 2x^3 + 3x - x^2,\ a = 2$

134. **[T]** $f(x) = \frac{1}{x} - x^2,\ a = 1$

135. **[T]** $f(x) = x^2 - x^{12} + 3x + 2,\ a = 0$

136. **[T]** $f(x) = \frac{1}{x} - x^{2/3},\ a = -1$

137. Find the equation of the tangent line to the graph of $f(x) = 2x^3 + 4x^2 - 5x - 3$ at $x = -1.$

138. Find the equation of the tangent line to the graph of $f(x) = x^2 + \frac{4}{x} - 10$ at $x = 8.$

139. Find the equation of the tangent line to the graph of $f(x) = (3x - x^2)(3 - x - x^2)$ at $x = 1.$

140. Find the point on the graph of $f(x) = x^3$ such that the tangent line at that point has an x intercept of 6.

141. Find the equation of the line passing through the point $P(3,\ 3)$ and tangent to the graph of $f(x) = \frac{6}{x - 1}.$

142. Determine all points on the graph of $f(x) = x^3 + x^2 - x - 1$ for which
a. the tangent line is horizontal
b. the tangent line has a slope of $-1.$

143. Find a quadratic polynomial such that $f(1) = 5,\ f'(1) = 3$ and $f''(1) = -6.$

144. A car driving along a freeway with traffic has traveled $s(t) = t^3 - 6t^2 + 9t$ meters in t seconds.
a. Determine the time in seconds when the velocity of the car is 0.
b. Determine the acceleration of the car when the velocity is 0.

145. **[T]** A herring swimming along a straight line has traveled $s(t) = \frac{t^2}{t^2 + 2}$ feet in t seconds. Determine the velocity of the herring when it has traveled 3 seconds.

146. The population in millions of arctic flounder in the Atlantic Ocean is modeled by the function $P(t) = \frac{8t + 3}{0.2t^2 + 1},$ where t is measured in years.
a. Determine the initial flounder population.
b. Determine $P'(10)$ and briefly interpret the result.

147. **[T]** The concentration of antibiotic in the bloodstream t hours after being injected is given by the function $C(t) = \frac{2t^2 + t}{t^3 + 50},$ where C is measured in milligrams per liter of blood.
a. Find the rate of change of $C(t).$
b. Determine the rate of change for $t = 8,\ 12,\ 24,$ and $36.$
c. Briefly describe what seems to be occurring as the number of hours increases.

148. A book publisher has a cost function given by $C(x) = \frac{x^3 + 2x + 3}{x^2},$ where x is the number of copies of a book in thousands and C is the cost, per book, measured in dollars. Evaluate $C'(2)$ and explain its meaning.

149. **[T]** According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F = \frac{Gm_1 m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.

a. Suppose that $G, m_1,$ and m_2 are constants. Find the rate of change of force F with respect to distance d.

b. Find the rate of change of force F with gravitational constant $G = 6.67 \times 10^{-11}$ $\text{Nm}^2/\text{kg}^2,$ on two bodies 10 meters apart, each with a mass of 1000 kilograms.

3.4 | Derivatives as Rates of Change

Learning Objectives

3.4.1 Determine a new value of a quantity from the old value and the amount of change.
3.4.2 Calculate the average rate of change and explain how it differs from the instantaneous rate of change.
3.4.3 Apply rates of change to displacement, velocity, and acceleration of an object moving along a straight line.
3.4.4 Predict the future population from the present value and the population growth rate.
3.4.5 Use derivatives to calculate marginal cost and revenue in a business situation.

In this section we look at some applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include **acceleration** and velocity in physics, **population growth rates** in biology, and marginal functions in economics.

Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the given point. If $f(x)$ is a function defined on an interval $[a, a+h],$ then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by

$$f(a+h) - f(a).$$

The **average rate of change** of the function f over that same interval is the ratio of the amount of change over that interval to the corresponding change in the x values. It is given by

$$\frac{f(a+h) - f(a)}{h}.$$

As we already know, the instantaneous rate of change of $f(x)$ at a is its derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For small enough values of $h,$ $f'(a) \approx \frac{f(a+h) - f(a)}{h}.$ We can then solve for $f(a+h)$ to get the amount of change formula:

$$f(a+h) \approx f(a) + f'(a)h. \qquad (3.10)$$

We can use this formula if we know only $f(a)$ and $f'(a)$ and wish to estimate the value of $f(a+h).$ For example, we may use the current population of a city and the rate at which it is growing to estimate its population in the near future. As we can see in **Figure 3.22**, we are approximating $f(a+h)$ by the y coordinate at $a+h$ on the line tangent to $f(x)$ at $x = a.$ Observe that the accuracy of this estimate depends on the value of h as well as the value of $f'(a).$

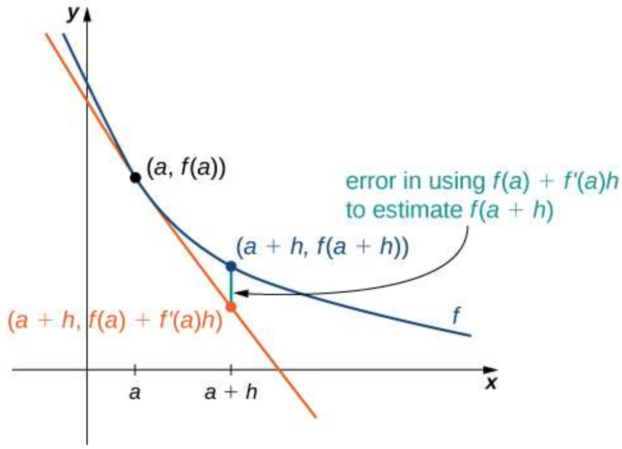


Figure 3.22 The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f(a+h) \approx f(a) + f'(a)h$.

Here is an interesting **demonstration (http://www.openstax.org/l/20_chainrule)** of rate of change.

Example 3.33

Estimating the Value of a Function

If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$.

Solution

Begin by finding h. We have $h = 3.2 - 3 = 0.2$. Thus,

$$f(3.2) = f(3 + 0.2) \approx f(3) + (0.2)f'(3) = 2 + 0.2(5) = 3.$$

3.21 Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

Definition

Let $s(t)$ be a function giving the position of an object at time t.

The velocity of the object at time t is given by $v(t) = s'(t)$.

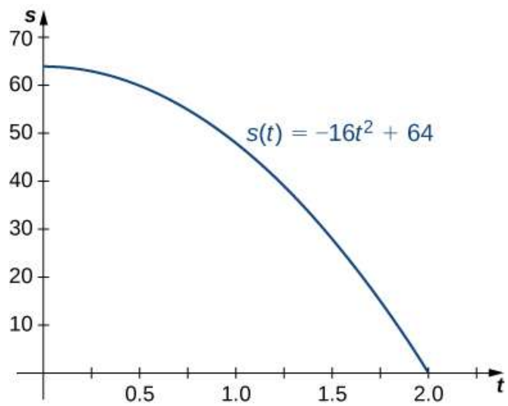
The speed of the object at time t is given by $|v(t)|$.

The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

Example 3.34

Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64.$



a. What is the instantaneous velocity of the ball when it hits the ground?

b. What is the average velocity during its fall?

Solution

The first thing to do is determine how long it takes the ball to reach the ground. To do this, set $s(t) = 0.$ Solving $-16t^2 + 64 = 0,$ we get $t = 2,$ so it take 2 seconds for the ball to reach the ground.

a. The instantaneous velocity of the ball as it strikes the ground is $v(2).$ Since $v(t) = s'(t) = -32t,$ we obtain $v(t) = -64$ ft/s.

b. The average velocity of the ball during its fall is

$$v_{ave} = \frac{s(2) - s(0)}{2 - 0} = \frac{0 - 64}{2} = -32 \text{ ft/s.}$$

Example 3.35

Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2.$ Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

a. Is the particle moving from left to right or from right to left at time $t = 1$?

b. Is the particle speeding up or slowing down at time $t = 1$?

Solution

Begin by finding $v(t)$ and $a(t).$

$$v(t) = s'(t) = 3t^2 - 4 \text{ and } a(t) = v'(t) = s''(t) = 6t.$$

Evaluating these functions at $t = 1,$ we obtain $v(1) = -1$ and $a(1) = 6.$

a. Because $v(1) < 0,$ the particle is moving from right to left.

b. Because $v(1) < 0$ and $a(1) > 0,$ velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing $|v(t)|$ to decrease. The particle is slowing down.

Example 3.36

Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4,\ t \geq 0.$

a. Find $v(t).$

b. At what time(s) is the particle at rest?

c. On what time intervals is the particle moving from left to right? From right to left?

d. Use the information obtained to sketch the path of the particle along a coordinate axis.

Solution

a. The velocity is the derivative of the position function:

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

b. The particle is at rest when $v(t) = 0,$ so set $3t^2 - 18t + 24 = 0.$ Factoring the left-hand side of the equation produces $3(t-2)(t-4) = 0.$ Solving, we find that the particle is at rest at $t = 2$ and $t = 4.$

c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0.$ **Figure 3.23** gives the analysis of the sign of $v(t)$ for $t \geq 0,$ but it does not represent the axis along which the particle is moving.

Figure 3.23 The sign of v(t) determines the direction of the particle.

Since $3t^2 - 18t + 24 > 0$ on $[0, 2) \cup (2, +\infty),$ the particle is moving from left to right on these intervals.
Since $3t^2 - 18t + 24 < 0$ on $(2, 4),$ the particle is moving from right to left on this interval.

d. Before we can sketch the graph of the particle, we need to know its position at the time it starts moving $(t = 0)$ and at the times that it changes direction $(t = 2, 4).$ We have $s(0) = 4,\ s(2) = 24,$ and $s(4) = 20.$ This means that the particle begins on the coordinate axis at 4 and changes direction at 0 and

20 on the coordinate axis. The path of the particle is shown on a coordinate axis in **Figure 3.24**.

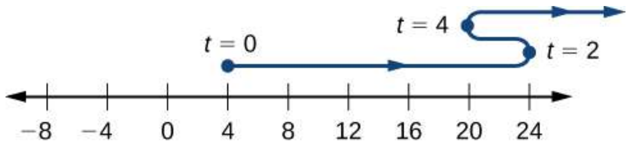


Figure 3.24 The path of the particle can be determined by analyzing v(t).

3.22 A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1$. Is the particle moving from right to left or from left to right at time $t = 3$?

Population Change

In addition to analyzing velocity, speed, acceleration, and position, we can use derivatives to analyze various types of populations, including those as diverse as bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and consequently can be represented by the derivative of the size of the population.

Definition

If $P(t)$ is the number of entities present in a population, then the population growth rate of $P(t)$ is defined to be $P'(t)$.

Example 3.37

Estimating a Population

The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

Solution

Let $P(t)$ be the population (in thousands) t years from now. Thus, we know that $P(0) = 10$ and based on the information, we anticipate $P(5) = 30$. Now estimate $P'(0)$, the current growth rate, using

$$P'(0) \approx \frac{P(5) - P(0)}{5 - 0} = \frac{30 - 10}{5} = 4.$$

By applying **Equation 3.10** to $P(t)$, we can estimate the population 2 years from now by writing

$$P(2) \approx P(0) + (2)P'(0) \approx 10 + 2(4) = 18;$$

thus, in 2 years the population will be 18,000.

3.23 The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3{,}000$. If $P'(0) = 100$, estimate the size of the population in 3 days, where t is measured in days.

Changes in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

Definition

If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.

If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.

If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

We can roughly approximate

$$MC(x) = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

by choosing an appropriate value for h. Since x represents objects, a reasonable and small value for h is 1. Thus, by substituting $h = 1$, we get the approximation $MC(x) = C'(x) \approx C(x+1) - C(x)$. Consequently, $C'(x)$ for a given value of x can be thought of as the change in cost associated with producing one additional item. In a similar way, $MR(x) = R'(x)$ approximates the revenue obtained by selling one additional item, and $MP(x) = P'(x)$ approximates the profit obtained by producing and selling one additional item.

Example 3.38

Applying Marginal Revenue

Assume that the number of barbeque dinners that can be sold, x, can be related to the price charged, p, by the equation $p(x) = 9 - 0.03x, \; 0 \le x \le 300$.

In this case, the revenue in dollars obtained by selling x barbeque dinners is given by

$$R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x \text{ for } 0 \le x \le 300.$$

Use the marginal revenue function to estimate the revenue obtained from selling the 101st barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

Solution

First, find the marginal revenue function: $MR(x) = R'(x) = -0.06x + 9$.

Next, use $R'(100)$ to approximate $R(101) - R(100)$, the revenue obtained from the sale of the 101st dinner. Since $R'(100) = 3$, the revenue obtained from the sale of the 101st dinner is approximately \$3.

The actual revenue obtained from the sale of the 101st dinner is

$$R(101) - R(100) = 602.97 - 600 = 2.97, \text{ or } \$2.97.$$

The marginal revenue is a fairly good estimate in this case and has the advantage of being easy to compute.

3.24 Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50.$ Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

3.4 EXERCISES

For the following exercises, the given functions represent the position of a particle traveling along a horizontal line.

a. Find the velocity and acceleration functions.

b. Determine the time intervals when the object is slowing down or speeding up.

150. $s(t) = 2t^3 - 3t^2 - 12t + 8$

151. $s(t) = 2t^3 - 15t^2 + 36t - 10$

152. $s(t) = \frac{t}{1 + t^2}$

153. A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by $s(t) = -16t^2 + 560t$.

a. Find the velocity of the rocket 3 seconds after being fired.
b. Find the acceleration of the rocket 3 seconds after being fired.

154. A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After t seconds, its height above the ground is given by $s(t) = -16t^2 - 8t + 64$.

a. Determine how long it takes for the ball to hit the ground.
b. Determine the velocity of the ball when it hits the ground.

155. The position function $s(t) = t^2 - 3t - 4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line, where s is in feet and t is in seconds. In this case, $s(t) = 0$ represents the time at which the back of the car is at the garage door, so $s(0) = -4$ is the starting position of the car, 4 feet inside the garage.

a. Determine the velocity of the car when $s(t) = 0$.
b. Determine the velocity of the car when $s(t) = 14$.

156. The position of a hummingbird flying along a straight line in t seconds is given by $s(t) = 3t^3 - 7t$ meters.

a. Determine the velocity of the bird at $t = 1$ sec.
b. Determine the acceleration of the bird at $t = 1$ sec.
c. Determine the acceleration of the bird when the velocity equals 0.

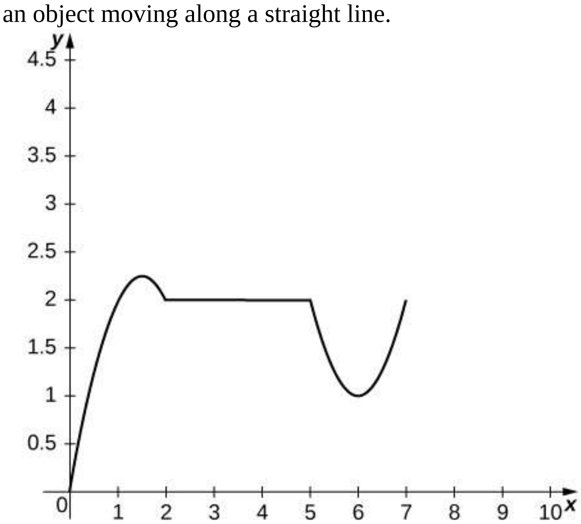
157. A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

a. Find the velocity of the potato after 0.5 s and 5.75 s.
b. Find the speed of the potato at 0.5 s and 5.75 s.
c. Determine when the potato reaches its maximum height.
d. Find the acceleration of the potato at 0.5 s and 1.5 s.
e. Determine how long the potato is in the air.
f. Determine the velocity of the potato upon hitting the ground.

158. The position function $s(t) = t^3 - 8t$ gives the position in miles of a freight train where east is the positive direction and t is measured in hours.

a. Determine the direction the train is traveling when $s(t) = 0$.
b. Determine the direction the train is traveling when $a(t) = 0$.
c. Determine the time intervals when the train is slowing down or speeding up.

159. The following graph shows the position $y = s(t)$ of an object moving along a straight line.



a. Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.
b. Sketch the graph of the velocity function.
c. Use the graph of the velocity function to determine the time intervals when the acceleration is positive, negative, or zero.
d. Determine the time intervals when the object is speeding up or slowing down.

160. The cost function, in dollars, of a company that manufactures food processors is given by $C(x) = 200 + \frac{7}{x} + \frac{x^2}{7}$, where x is the number of food processors manufactured.

a. Find the marginal cost function.
b. Use the marginal cost function to estimate the cost of manufacturing the thirteenth food processor.
c. Find the actual cost of manufacturing the thirteenth food processor.

161. The price p (in dollars) and the demand x for a certain digital clock radio is given by the price–demand function $p = 10 - 0.001x$.

a. Find the revenue function $R(x)$.
b. Find the marginal revenue function.
c. Find the marginal revenue at $x = 2000$ and 5000.

162. **[T]** A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$, where x is the number of skateboards sold.

a. Find the exact profit from the sale of the thirtieth skateboard.
b. Find the marginal profit function and use it to estimate the profit from the sale of the thirtieth skateboard.

163. **[T]** In general, the profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$. Suppose the price-demand and cost functions for the production of cordless drills is given respectively by $p = 143 - 0.03x$ and $C(x) = 75{,}000 + 65x$, where x is the number of cordless drills that are sold at a price of p dollars per drill and $C(x)$ is the cost of producing x cordless drills.

a. Find the marginal cost function.
b. Find the revenue and marginal revenue functions.
c. Find $R'(1000)$ and $R'(4000)$. Interpret the results.
d. Find the profit and marginal profit functions.
e. Find $P'(1000)$ and $P'(4000)$. Interpret the results.

164. A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

a. Find the rate of change function $P'(t)$ of the population function.
b. Find $P'(1), P'(2), P'(3),$ and $P'(4)$. Interpret what the results mean for the town.
c. Find $P''(1), P''(2), P''(3),$ and $P''(4)$. Interpret what the results mean for the town's population.

165. **[T]** A culture of bacteria grows in number according to the function $N(t) = 3000\left(1 + \frac{4t}{t^2 + 100}\right)$, where t is measured in hours.

a. Find the rate of change of the number of bacteria.
b. Find $N'(0), N'(10), N'(20),$ and $N'(30)$.
c. Interpret the results in (b).
d. Find $N''(0), N''(10), N''(20),$ and $N''(30)$. Interpret what the answers imply about the bacteria population growth.

166. The centripetal force of an object of mass m is given by $F(r) = \frac{mv^2}{r}$, where v is the speed of rotation and r is the distance from the center of rotation.

a. Find the rate of change of centripetal force with respect to the distance from the center of rotation.
b. Find the rate of change of centripetal force of an object with mass 1000 kilograms, velocity of 13.89 m/s, and a distance from the center of rotation of 200 meters.

The following questions concern the population (in millions) of London by decade in the 19th century, which is listed in the following table.

Years since 1800	Population (millions)
1	0.8795
11	1.040
21	1.264
31	1.516
41	1.661
51	2.000
61	2.634
71	3.272
81	3.911
91	4.422

Table 3.4 Population of London **Source: http://en.wikipedia.org/wiki/Demographics_of_London.**

167. **[T]**
   a. Using a calculator or a computer program, find the best-fit linear function to measure the population.
   b. Find the derivative of the equation in a. and explain its physical meaning.
   c. Find the second derivative of the equation and explain its physical meaning.

168. **[T]**
   a. Using a calculator or a computer program, find the best-fit quadratic curve through the data.
   b. Find the derivative of the equation and explain its physical meaning.
   c. Find the second derivative of the equation and explain its physical meaning.

For the following exercises, consider an astronaut on a large planet in another galaxy. To learn more about the composition of this planet, the astronaut drops an electronic sensor into a deep trench. The sensor transmits its vertical position every second in relation to the astronaut's position. The summary of the falling sensor data is displayed in the following table.

Time after dropping (s)	Position (m)
0	0
1	−1
2	−2
3	−5
4	−7
5	−14

169. **[T]**
   a. Using a calculator or computer program, find the best-fit quadratic curve to the data.
   b. Find the derivative of the position function and explain its physical meaning.
   c. Find the second derivative of the position function and explain its physical meaning.

170. **[T]**
   a. Using a calculator or computer program, find the best-fit cubic curve to the data.
   b. Find the derivative of the position function and explain its physical meaning.
   c. Find the second derivative of the position function and explain its physical meaning.
   d. Using the result from c. explain why a cubic function is not a good choice for this problem.

The following problems deal with the Holling type I, II, and III equations. These equations describe the ecological event of growth of a predator population given the amount of prey available for consumption.

171. **[T]** The Holling type I equation is described by $f(x) = ax$, where x is the amount of prey available and $a > 0$ is the rate at which the predator meets the prey for consumption.
   a. Graph the Holling type I equation, given $a = 0.5$.
   b. Determine the first derivative of the Holling type I equation and explain physically what the derivative implies.
   c. Determine the second derivative of the Holling type I equation and explain physically what the derivative implies.
   d. Using the interpretations from b. and c. explain why the Holling type I equation may not be realistic.

172. **[T]** The Holling type II equation is described by $f(x) = \frac{ax}{n + x}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type II equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type I and II equations?
b. Take the first derivative of the Holling type II equation and interpret the physical meaning of the derivative.
c. Show that $f(n) = \frac{1}{2}a$ and interpret the meaning of the parameter n.
d. Find and interpret the meaning of the second derivative. What makes the Holling type II function more realistic than the Holling type I function?

173. **[T]** The Holling type III equation is described by $f(x) = \frac{ax^2}{n^2 + x^2}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

a. Graph the Holling type III equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type II and III equations?
b. Take the first derivative of the Holling type III equation and interpret the physical meaning of the derivative.
c. Find and interpret the meaning of the second derivative (it may help to graph the second derivative).
d. What additional ecological phenomena does the Holling type III function describe compared with the Holling type II function?

174. **[T]** The populations of the snowshoe hare (in thousands) and the lynx (in hundreds) collected over 7 years from 1937 to 1943 are shown in the following table. The snowshoe hare is the primary prey of the lynx.

Population of snowshoe hare (thousands)	Population of lynx (hundreds)
20	10
55	15
65	55
95	60

Table 3.5 Snowshoe Hare and Lynx Populations **Source: http://www.biotopics.co.uk/newgcse/predatorprey.html.**

a. Graph the data points and determine which Holling-type function fits the data best.
b. Using the meanings of the parameters a and n, determine values for those parameters by examining a graph of the data. Recall that n measures what prey value results in the half-maximum of the predator value.
c. Plot the resulting Holling-type I, II, and III functions on top of the data. Was the result from part a. correct?

3.5 | Derivatives of Trigonometric Functions

Learning Objectives

- **3.5.1** Find the derivatives of the sine and cosine function.
- **3.5.2** Find the derivatives of the standard trigonometric functions.
- **3.5.3** Calculate the higher-order derivatives of the sine and cosine.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Being able to calculate the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, for values of h very close to 0, $f'(x) \approx \frac{f(x+h) - f(x)}{h}$. We see that by using $h = 0.01$,

$$\frac{d}{dx}(\sin x) \approx \frac{\sin(x + 0.01) - \sin x}{0.01}$$

By setting $D(x) = \frac{\sin(x + 0.01) - \sin x}{0.01}$ and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ (**Figure 3.25**).

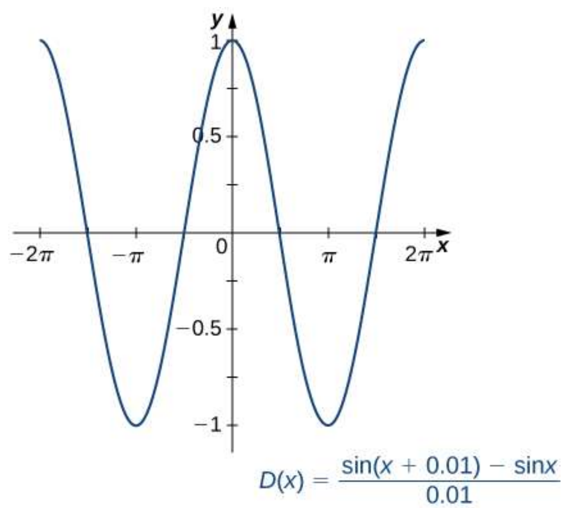


Figure 3.25 The graph of the function $D(x)$ looks a lot like a cosine curve.

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Theorem 3.8: The Derivatives of sin *x* and cos *x*

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

$$\frac{d}{dx}(\sin x) = \cos x \quad (3.11)$$

$$\frac{d}{dx}(\cos x) = -\sin x \quad (3.12)$$

Proof

Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits we learned in **Introduction to Limits**:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

The graphs of $y = \frac{(\sin h)}{h}$ and $y = \frac{(\cos h - 1)}{h}$ are shown in **Figure 3.26**.

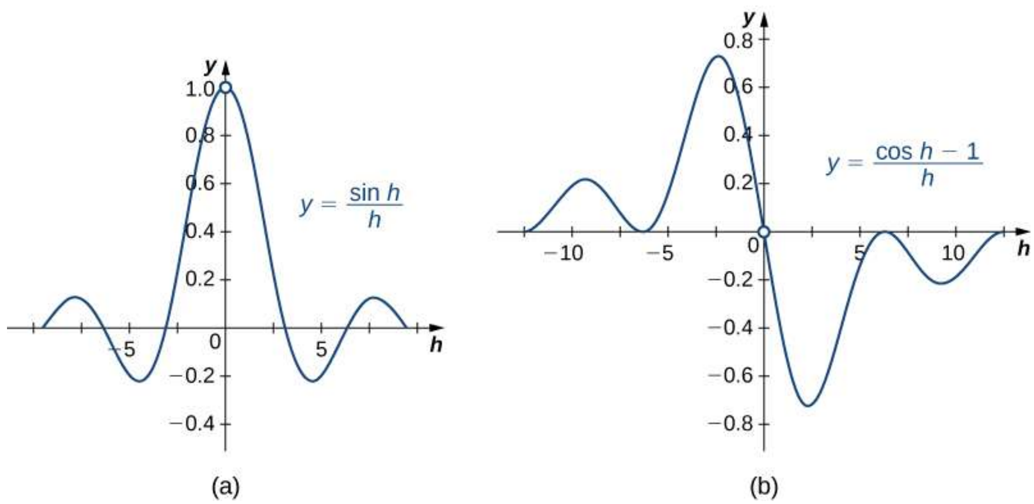


Figure 3.26 These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.

We also recall the following trigonometric identity for the sine of the sum of two angles:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Apply the definition of the derivative.} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{Use trig identity for the sine of the sum of two angles.} \\
&= \lim_{h \to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}\right) && \text{Regroup.} \\
&= \lim_{h \to 0} \left(\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right) && \text{Factor out } \sin x \text{ and } \cos x. \\
&= \sin x \cdot 0 + \cos x \cdot 1 && \text{Apply trig limit formulas.} \\
&= \cos x && \text{Simplify.}
\end{aligned}$$

□

Figure 3.27 shows the relationship between the graph of $f(x) = \sin x$ and its derivative $f'(x) = \cos x$. Notice that at the points where $f(x) = \sin x$ has a horizontal tangent, its derivative $f'(x) = \cos x$ takes on the value zero. We also see that where $f(x) = \sin x$ is increasing, $f'(x) = \cos x > 0$ and where $f(x) = \sin x$ is decreasing, $f'(x) = \cos x < 0$.

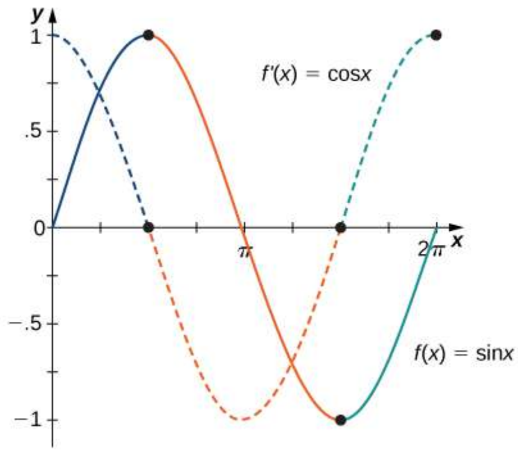


Figure 3.27 Where $f(x)$ has a maximum or a minimum, $f'(x) = 0$ that is, $f'(x) = 0$ where $f(x)$ has a horizontal tangent. These points are noted with dots on the graphs.

Example 3.39

Differentiating a Function Containing sin *x*

Find the derivative of $f(x) = 5x^3 \sin x$.

Solution

Using the product rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(5x^3\right) \cdot \sin x + \frac{d}{dx}(\sin x) \cdot 5x^3 \\ &= 15x^2 \cdot \sin x + \cos x \cdot 5x^3. \end{aligned}$$

After simplifying, we obtain

$$f'(x) = 15x^2 \sin x + 5x^3 \cos x.$$

3.25 Find the derivative of $f(x) = \sin x \cos x$.

Example 3.40

Finding the Derivative of a Function Containing cos *x*

Find the derivative of $g(x) = \frac{\cos x}{4x^2}$.

Solution

By applying the quotient rule, we have

$$g'(x) = \frac{(-\sin x)4x^2 - 8x(\cos x)}{\left(4x^2\right)^2}.$$

Simplifying, we obtain

$$\begin{aligned} g'(x) &= \frac{-4x^2 \sin x - 8x \cos x}{16x^4} \\ &= \frac{-x \sin x - 2 \cos x}{4x^3}. \end{aligned}$$

3.26 Find the derivative of $f(x) = \frac{x}{\cos x}$.

Example 3.41

An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2\sin t - t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Solution

To determine when the particle is at rest, set $s'(t) = v(t) = 0$. Begin by finding $s'(t)$. We obtain

$$s'(t) = 2\cos t - 1,$$

so we must solve

$$2\cos t - 1 = 0 \text{ for } 0 \le t \le 2\pi.$$

The solutions to this equation are $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$. Thus the particle is at rest at times $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

3.27 A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the quotient rule to find formulas for their derivatives.

Example 3.42

The Derivative of the Tangent Function

Find the derivative of $f(x) = \tan x$.

Solution

Start by expressing $\tan x$ as the quotient of $\sin x$ and $\cos x$:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Now apply the quotient rule to obtain

$$f'(x) = \frac{\cos x \cos x - (-\sin x)\sin x}{(\cos x)^2}.$$

Simplifying, we obtain

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Recognizing that $\cos^2 x + \sin^2 x = 1,$ by the Pythagorean theorem, we now have

$$f'(x) = \frac{1}{\cos^2 x}.$$

Finally, use the identity $\sec x = \frac{1}{\cos x}$ to obtain

$$f'(x) = \sec^2 x.$$

3.28 Find the derivative of $f(x) = \cot x$.

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

Theorem 3.9: Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The derivatives of the remaining trigonometric functions are as follows:

$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{3.13}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \tag{3.14}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \tag{3.15}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x. \tag{3.16}$$

Example 3.43

Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution

To find the equation of the tangent line, we need a point and a slope at that point. To find the point, compute

$$f\left(\frac{\pi}{4}\right) = \cot\frac{\pi}{4} = 1.$$

Thus the tangent line passes through the point $\left(\frac{\pi}{4}, 1\right)$. Next, find the slope by finding the derivative of $f(x) = \cot x$ and evaluating it at $\frac{\pi}{4}$:

$$f'(x) = -\csc^2 x \text{ and } f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -2.$$

Using the point-slope equation of the line, we obtain

$$y - 1 = -2\left(x - \frac{\pi}{4}\right)$$

or equivalently,

$$y = -2x + 1 + \frac{\pi}{2}.$$

Example 3.44

Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x\tan x$.

Solution

To find this derivative, we must use both the sum rule and the product rule. Using the sum rule, we find

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x\tan x).$$

In the first term, $\frac{d}{dx}(\csc x) = -\csc x\cot x,$ and by applying the product rule to the second term we obtain

$$\frac{d}{dx}(x\tan x) = (1)(\tan x) + (\sec^2 x)(x).$$

Therefore, we have

$$f'(x) = -\csc x\cot x + \tan x + x\sec^2 x.$$

3.29 Find the derivative of $f(x) = 2\tan x - 3\cot x$.

3.30 Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

Example 3.45

Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

Solution

Each step in the chain is straightforward:

$$\begin{aligned} y &= \sin x \\ \frac{dy}{dx} &= \cos x \\ \frac{d^2y}{dx^2} &= -\sin x \\ \frac{d^3y}{dx^3} &= -\cos x \\ \frac{d^4y}{dx^4} &= \sin x. \end{aligned}$$

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of sin x equals sin x, so

$$\frac{d^4}{dx^4}(\sin x) = \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \ldots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x$$
$$\frac{d^5}{dx^5}(\sin x) = \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \ldots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x.$$

3.31 For $y = \cos x$, find $\frac{d^4y}{dx^4}$.

Example 3.46

Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution

We can see right away that for the 74th derivative of $\sin x$, $74 = 4(18) + 2$, so

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

3.32 For $y = \sin x$, find $\frac{d^{59}}{dx^{59}}(\sin x)$.

Example 3.47

An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

Solution

First find $v(t) = s'(t)$:

$$v(t) = s'(t) = -\cos t.$$

Thus,

$$v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Next, find $a(t) = v'(t)$. Thus, $a(t) = v'(t) = \sin t$ and we have

$$a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Since $v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} < 0$ and $a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} > 0$, we see that velocity and acceleration are acting in opposite directions; that is, the object is being accelerated in the direction opposite to the direction in which it is travelling. Consequently, the particle is slowing down.

3.33 A block attached to a spring is moving vertically. Its position at time t is given by $s(t) = 2\sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

3.5 EXERCISES

For the following exercises, find $\frac{dy}{dx}$ for the given functions.

175. $y = x^2 - \sec x + 1$

176. $y = 3\csc x + \frac{5}{x}$

177. $y = x^2 \cot x$

178. $y = x - x^3 \sin x$

179. $y = \frac{\sec x}{x}$

180. $y = \sin x \tan x$

181. $y = (x + \cos x)(1 - \sin x)$

182. $y = \frac{\tan x}{1 - \sec x}$

183. $y = \frac{1 - \cot x}{1 + \cot x}$

184. $y = \cos x(1 + \csc x)$

For the following exercises, find the equation of the tangent line to each of the given functions at the indicated values of x. Then use a calculator to graph both the function and the tangent line to ensure the equation for the tangent line is correct.

185. **[T]** $f(x) = -\sin x,\ x = 0$

186. **[T]** $f(x) = \csc x,\ x = \frac{\pi}{2}$

187. **[T]** $f(x) = 1 + \cos x,\ x = \frac{3\pi}{2}$

188. **[T]** $f(x) = \sec x,\ x = \frac{\pi}{4}$

189. **[T]** $f(x) = x^2 - \tan x,\ \ x = 0$

190. **[T]** $f(x) = 5\cot x,\ \ x = \frac{\pi}{4}$

For the following exercises, find $\frac{d^2y}{dx^2}$ for the given functions.

191. $y = x\sin x - \cos x$

192. $y = \sin x \cos x$

193. $y = x - \frac{1}{2}\sin x$

194. $y = \frac{1}{x} + \tan x$

195. $y = 2\csc x$

196. $y = \sec^2 x$

197. Find all x values on the graph of $f(x) = -3\sin x \cos x$ where the tangent line is horizontal.

198. Find all x values on the graph of $f(x) = x - 2\cos x$ for $0 < x < 2\pi$ where the tangent line has slope 2.

199. Let $f(x) = \cot x$. Determine the points on the graph of f for $0 < x < 2\pi$ where the tangent line(s) is (are) parallel to the line $y = -2x$.

200. **[T]** A mass on a spring bounces up and down in simple harmonic motion, modeled by the function $s(t) = -6\cos t$ where s is measured in inches and t is measured in seconds. Find the rate at which the spring is oscillating at $t = 5$ s.

201. Let the position of a swinging pendulum in simple harmonic motion be given by $s(t) = a\cos t + b\sin t$ where a and b are constants, t measures time in seconds, and s measures position in centimeters. If the position is 0 cm and the velocity is 3 cm/s when $t = 0$, find the values of a and b.

202. After a diver jumps off a diving board, the edge of the board oscillates with position given by $s(t) = -5\cos t$ cm at t seconds after the jump.

a. Sketch one period of the position function for $t \geq 0$.
b. Find the velocity function.
c. Sketch one period of the velocity function for $t \geq 0$.
d. Determine the times when the velocity is 0 over one period.
e. Find the acceleration function.
f. Sketch one period of the acceleration function for $t \geq 0$.

203. The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is given by $y = 10 + 5\sin x$ where y is the number of hamburgers sold and x represents the number of hours after the restaurant opened at 11 a.m. until 11 p.m., when the store closes. Find y' and determine the intervals where the number of burgers being sold is increasing.

204. **[T]** The amount of rainfall per month in Phoenix, Arizona, can be approximated by $y(t) = 0.5 + 0.3\cos t$, where t is months since January. Find y' and use a calculator to determine the intervals where the amount of rain falling is decreasing.

For the following exercises, use the quotient rule to derive the given equations.

205. $\frac{d}{dx}(\cot x) = -\csc^2 x$

206. $\frac{d}{dx}(\sec x) = \sec x \tan x$

207. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

208. Use the definition of derivative and the identity $\cos(x + h) = \cos x \cos h - \sin x \sin h$ to prove that $\frac{d(\cos x)}{dx} = -\sin x.$

For the following exercises, find the requested higher-order derivative for the given functions.

209. $\frac{d^3 y}{dx^3}$ of $y = 3\cos x$

210. $\frac{d^2 y}{dx^2}$ of $y = 3\sin x + x^2 \cos x$

211. $\frac{d^4 y}{dx^4}$ of $y = 5\cos x$

212. $\frac{d^2 y}{dx^2}$ of $y = \sec x + \cot x$

213. $\frac{d^3 y}{dx^3}$ of $y = x^{10} - \sec x$

3.6 | The Chain Rule

Learning Objectives

3.6.1 State the chain rule for the composition of two functions.
3.6.2 Apply the chain rule together with the power rule.
3.6.3 Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
3.6.4 Recognize the chain rule for a composition of three or more functions.
3.6.5 Describe the proof of the chain rule.

We have seen the techniques for differentiating basic functions $(x^n, \sin x, \cos x, \text{etc.})$ as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x) = \sin\left(x^3\right)$ or $k(x) = \sqrt{3x^2 + 1}$. In this section, we study the rule for finding the derivative of the composition of two or more functions.

Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the **chain rule**, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x) = \sin\left(x^3\right)$. We can think of the derivative of this function with respect to x as the rate of change of $\sin\left(x^3\right)$ relative to the change in x. Consequently, we want to know how $\sin\left(x^3\right)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin\left(x^3\right)$. This chain reaction gives us hints as to what is involved in computing the derivative of $\sin\left(x^3\right)$. First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin\left(x^3\right)$ suggests that the derivative of $\sin(u)$ with respect to u, where $u = x^3$, is also part of the final derivative.

We can take a more formal look at the derivative of $h(x) = \sin\left(x^3\right)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x) = \sin\left(x^3\right)$.

$$h'(a) = \lim_{x \to a} \frac{\sin\left(x^3\right) - \sin\left(a^3\right)}{x - a}.$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression $x^3 - a^3$ to obtain

$$h'(a) = \lim_{x \to a} \frac{\sin\left(x^3\right) - \sin\left(a^3\right)}{x^3 - a^3} \cdot \frac{x^3 - a^3}{x - a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x = a$. That is,

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \frac{d}{dx}\left(x^3\right)_{x = a} = 3a^2.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u = x^3$ and observing that as $x \to a$, $u \to a^3$:

$$\begin{aligned}\lim_{x \to a} \frac{\sin\left(x^3\right) - \sin\left(a^3\right)}{x^3 - a^3} &= \lim_{u \to a^3} \frac{\sin u - \sin\left(a^3\right)}{u - a^3} \\ &= \frac{d}{du}(\sin u)_{u = a^3} \\ &= \cos\left(a^3\right).\end{aligned}$$

Thus, $h'(a) = \cos\left(a^3\right) \cdot 3a^2$.

In other words, if $h(x) = \sin\left(x^3\right)$, then $h'(x) = \cos\left(x^3\right) \cdot 3x^2$. Thus, if we think of $h(x) = \sin\left(x^3\right)$ as the composition $(f \circ g)(x) = f(g(x))$ where $f(x) = \sin x$ and $g(x) = x^3$, then the derivative of $h(x) = \sin\left(x^3\right)$ is the product of the derivative of $g(x) = x^3$ and the derivative of the function $f(x) = \sin x$ evaluated at the function $g(x) = x^3$. At this point, we anticipate that for $h(x) = \sin(g(x))$, it is quite likely that $h'(x) = \cos(g(x))g'(x)$. As we determined above, this is the case for $h(x) = \sin\left(x^3\right)$.

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f(g(x))$$

is given by

$$h'(x) = f'(g(x))g'(x). \tag{3.17}$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Watch an **animation (http://www.openstax.org/l/20_chainrule2)** of the chain rule.

Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h(x) = f(g(x))$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.
3. Find $g'(x)$.
4. Write $h'(x) = f'(g(x)) \cdot g'(x)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = (g(x))^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x) = (g(x))^n$ as $f(g(x))$ where $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Thus, $f'(g(x)) = n(g(x))^{n-1}$. This leads us to the derivative of a power function using the chain rule,

$$h'(x) = n(g(x))^{n-1} g'(x)$$

Rule: Power Rule for Composition of Functions

For all values of x for which the derivative is defined, if

$$h(x) = (g(x))^n.$$

Then

$$h'(x) = n(g(x))^{n-1} g'(x). \quad (3.18)$$

Example 3.48

Using the Chain and Power Rules

Find the derivative of $h(x) = \frac{1}{\left(3x^2 + 1\right)^2}$.

Solution

First, rewrite $h(x) = \frac{1}{\left(3x^2 + 1\right)^2} = \left(3x^2 + 1\right)^{-2}$.

Applying the power rule with $g(x) = 3x^2 + 1$, we have

$$h'(x) = -2\left(3x^2 + 1\right)^{-3}(6x).$$

Rewriting back to the original form gives us

$$h'(x) = \frac{-12x}{(3x^2 + 1)^3}.$$

3.34 Find the derivative of $h(x) = \left(2x^3 + 2x - 1\right)^4$.

Example 3.49

Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

Solution

First recall that $\sin^3 x = (\sin x)^3$, so we can rewrite $h(x) = \sin^3 x$ as $h(x) = (\sin x)^3$.

Applying the power rule with $g(x) = \sin x$, we obtain

$$h'(x) = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x.$$

Example 3.50

Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \frac{1}{(3x-5)^2}$ at $x = 2$.

Solution

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2) = \frac{1}{(3(2)-5)^2} = 1$, the point is $(2, 1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x) = (3x-5)^{-2}$ and apply the power rule to obtain

$$h'(x) = -2(3x-5)^{-3}(3) = -6(3x-5)^{-3}.$$

By substituting, we have $h'(2) = -6(3(2)-5)^{-3} = -6$. Therefore, the line has equation $y - 1 = -6(x-2)$. Rewriting, the equation of the line is $y = -6x + 13$.

3.35 Find the equation of the line tangent to the graph of $f(x) = \left(x^2 - 2\right)^3$ at $x = -2$.

Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the product rule.

Example 3.51

Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos(g(x))$.

Solution

Think of $h(x) = \cos(g(x))$ as $f(g(x))$ where $f(x) = \cos x$. Since $f'(x) = -\sin x$. we have $f'(g(x)) = -\sin(g(x))$. Then we do the following calculation.

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) && \text{Apply the chain rule.} \\ &= -\sin(g(x))g'(x) && \text{Substitute } f'(g(x)) = -\sin(g(x)). \end{aligned}$$

Thus, the derivative of $h(x) = \cos(g(x))$ is given by $h'(x) = -\sin(g(x))g'(x)$.

In the following example we apply the rule that we have just derived.

Example 3.52

Using the Chain Rule on a Cosine Function

Find the derivative of $h(x) = \cos\left(5x^2\right)$.

Solution

Let $g(x) = 5x^2$. Then $g'(x) = 10x$. Using the result from the previous example,

$$\begin{aligned} h'(x) &= -\sin\left(5x^2\right) \cdot 10x \\ &= -10x\sin\left(5x^2\right). \end{aligned}$$

Example 3.53

Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x) = \sec\left(4x^5 + 2x\right)$.

Solution

Apply the chain rule to $h(x) = \sec(g(x))$ to obtain

$$h'(x) = \sec(g(x)\tan(g(x))g'(x).$$

In this problem, $g(x) = 4x^5 + 2x$, so we have $g'(x) = 20x^4 + 2$. Therefore, we obtain

$$\begin{aligned} h'(x) &= \sec\left(4x^5 + 2x\right)\tan\left(4x^5 + 2x\right)\left(20x^4 + 2\right) \\ &= (20x^4 + 2)\sec\left(4x^5 + 2x\right)\tan\left(4x^5 + 2x\right). \end{aligned}$$

3.36 Find the derivative of $h(x) = \sin(7x + 2)$.

At this point we provide a list of derivative formulas that may be obtained by applying the chain rule in conjunction with the formulas for derivatives of trigonometric functions. Their derivations are similar to those used in **Example 3.51** and **Example 3.53**. For convenience, formulas are also given in Leibniz's notation, which some students find easier to remember. (We discuss the chain rule using Leibniz's notation at the end of this section.) It is not absolutely necessary to memorize these as separate formulas as they are all applications of the chain rule to previously learned formulas.

Theorem 3.10: Using the Chain Rule with Trigonometric Functions

For all values of x for which the derivative is defined,

$$\begin{aligned}
\frac{d}{dx}(\sin(g(x)) &= \cos(g(x))g'(x) & \frac{d}{dx}\sin u &= \cos u \frac{du}{dx} \\
\frac{d}{dx}(\cos(g(x)) &= -\sin(g(x))g'(x) & \frac{d}{dx}\cos u &= -\sin u \frac{du}{dx} \\
\frac{d}{dx}(\tan(g(x)) &= \sec^2(g(x))g'(x) & \frac{d}{dx}\tan u &= \sec^2 u \frac{du}{dx} \\
\frac{d}{dx}(\cot(g(x)) &= -\csc^2(g(x))g'(x) & \frac{d}{dx}\cot u &= -\csc^2 u \frac{du}{dx} \\
\frac{d}{dx}(\sec(g(x)) &= \sec(g(x)\tan(g(x))g'(x) & \frac{d}{dx}\sec u &= \sec u \tan u \frac{du}{dx} \\
\frac{d}{dx}(\csc(g(x)) &= -\csc(g(x))\cot(g(x))g'(x) & \frac{d}{dx}\csc u &= -\csc u \cot u \frac{du}{dx}.
\end{aligned}$$

Example 3.54

Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

Solution

First apply the product rule, then apply the chain rule to each term of the product.

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left((2x+1)^5\right)\cdot(3x-2)^7 + \frac{d}{dx}\left((3x-2)^7\right)\cdot(2x+1)^5 && \text{Apply the product rule.} \\
&= 5(2x+1)^4\cdot 2\cdot(3x-2)^7 + 7(3x-2)^6\cdot 3\cdot(2x+1)^5 && \text{Apply the chain rule.} \\
&= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{Simplify.} \\
&= (2x+1)^4(3x-2)^6(10(3x-2)+21(2x+1)) && \text{Factor out } (2x+1)^4(3x-2)^6. \\
&= (2x+1)^4(3x-2)^6(72x+1) && \text{Simplify.}
\end{aligned}$$

3.37 Find the derivative of $h(x) = \dfrac{x}{(2x+3)^3}$.

Composites of Three or More Functions

We can now combine the chain rule with other rules for differentiating functions, but when we are differentiating the composition of three or more functions, we need to apply the chain rule more than once. If we look at this situation in general terms, we can generate a formula, but we do not need to remember it, as we can simply apply the chain rule multiple times.

In general terms, first we let

$$k(x) = h(f(g(x))).$$

Then, applying the chain rule once we obtain

$$k'(x) = \frac{d}{dx}(h(f(g(x))) = h'(f(g(x))) \cdot \frac{d}{dx} f((g(x))).$$

Applying the chain rule again, we obtain

$$k'(x) = h'(f(g(x))f'(g(x))g'(x)).$$

Rule: Chain Rule for a Composition of Three Functions

For all values of x for which the function is differentiable, if

$$k(x) = h(f(g(x))),$$

then

$$k'(x) = h'(f(g(x)))f'(g(x))g'(x).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

Example 3.55

Differentiating a Composite of Three Functions

Find the derivative of $k(x) = \cos^4\left(7x^2 + 1\right)$.

Solution

First, rewrite $k(x)$ as

$$k(x) = \left(\cos\left(7x^2 + 1\right)\right)^4.$$

Then apply the chain rule several times.

$$\begin{aligned} k'(x) &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(\frac{d}{dx}\cos\left(7x^2+1\right)\right) && \text{Apply the chain rule.} \\ &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(-\sin\left(7x^2+1\right)\right)\left(\frac{d}{dx}\left(7x^2+1\right)\right) && \text{Apply the chain rule.} \\ &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(-\sin\left(7x^2+1\right)\right)(14x) && \text{Apply the chain rule.} \\ &= -56x\sin(7x^2+1)\cos^3(7x^2+1) && \text{Simplify.} \end{aligned}$$

3.38 Find the derivative of $h(x) = \sin^6\left(x^3\right)$.

Example 3.56

Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(2t) + \cos(3t)$. What is the velocity of the particle at time $t = \frac{\pi}{6}$?

Solution

To find $v(t)$, the velocity of the particle at time t, we must differentiate $s(t)$. Thus,

$$v(t) = s'(t) = 2\cos(2t) - 3\sin(3t).$$

Substituting $t = \frac{\pi}{6}$ into $v(t)$, we obtain $v\left(\frac{\pi}{6}\right) = -2$.

3.39 A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(4t)$. Find its acceleration at time t.

Proof

At this point, we present a very informal proof of the chain rule. For simplicity's sake we ignore certain issues: For example, we assume that $g(x) \neq g(a)$ for $x \neq a$ in some open interval containing a. We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

Rewriting, we obtain

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}.$$

Although it is clear that

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a),$$

it is not obvious that

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = f'(g(a)).$$

To see that this is true, first recall that since g is differentiable at a, g is also continuous at a. Thus,

$$\lim_{x \to a} g(x) = g(a).$$

Next, make the substitution $y = g(x)$ and $b = g(a)$ and use change of variables in the limit to obtain

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'(g(a)).$$

Finally,

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} = f'(g(a))g'(a).$$

□

Example 3.57

Using the Chain Rule with Functional Values

Let $h(x) = f(g(x))$. If $g(1) = 4$, $g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

Solution

Use the chain rule, then substitute.

$$\begin{aligned} h'(1) &= f'(g(1))g'(1) && \text{Apply the chain rule.} \\ &= f'(4) \cdot 3 && \text{Substitute } g(1) = 4 \text{ and } g'(1) = 3. \\ &= 7 \cdot 3 && \text{Substitute } f'(4) = 7. \\ &= 21 && \text{Simplify.} \end{aligned}$$

3.40 Given $h(x) = f(g(x))$. If $g(2) = -3$, $g'(2) = 4$, and $f'(-3) = 7$, find $h'(2)$.

The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This notation for the chain rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Thus,

$$h'(x) = \frac{dy}{dx},\ f'(g(x)) = f'(u) = \frac{dy}{du} \text{ and } g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'(g(x))g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 3.58

Taking a Derivative Using Leibniz's Notation, Example 1

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the quotient rule,

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and

$$\frac{dy}{du} = 5u^4.$$

Finally, we put it all together.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= 5u^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } \frac{dy}{du} = 5u^4 \text{ and } \frac{du}{dx} = \frac{2}{(3x+2)^2}. \\ &= 5\left(\frac{x}{3x+2}\right)^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } u = \frac{x}{3x+2}. \\ &= \frac{10x^4}{(3x+2)^6} && \text{Simplify.} \end{aligned}$$

It is important to remember that, when using the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Example 3.59

Taking a Derivative Using Leibniz's Notation, Example 2

Find the derivative of $y = \tan\left(4x^2 - 3x + 1\right)$.

Solution

First, let $u = 4x^2 - 3x + 1$. Then $y = \tan u$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$:

$$\frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u.$$

Finally, we put it all together.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= \sec^2 u \cdot (8x-3) && \text{Use } \frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u. \\ &= \sec^2(4x^2 - 3x + 1) \cdot (8x-3) && \text{Substitute } u = 4x^2 - 3x + 1. \end{aligned}$$

3.41 Use Leibniz's notation to find the derivative of $y = \cos\left(x^3\right)$. Make sure that the final answer is expressed entirely in terms of the variable x.

3.6 EXERCISES

For the following exercises, given $y = f(u)$ and $u = g(x)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

214. $y = 3u - 6,\ u = 2x^2$

215. $y = 6u^3,\ u = 7x - 4$

216. $y = \sin u,\ u = 5x - 1$

217. $y = \cos u,\ u = \frac{-x}{8}$

218. $y = \tan u,\ u = 9x + 2$

219. $y = \sqrt{4u + 3},\ u = x^2 - 6x$

For each of the following exercises,

a. decompose each function in the form $y = f(u)$ and $u = g(x)$, and

b. find $\frac{dy}{dx}$ as a function of x.

220. $y = (3x - 2)^6$

221. $y = \left(3x^2 + 1\right)^3$

222. $y = \sin^5(x)$

223. $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

224. $y = \tan(\sec x)$

225. $y = \csc(\pi x + 1)$

226. $y = \cot^2 x$

227. $y = -6\sin^{-3} x$

For the following exercises, find $\frac{dy}{dx}$ for each function.

228. $y = \left(3x^2 + 3x - 1\right)^4$

229. $y = (5 - 2x)^{-2}$

230. $y = \cos^3(\pi x)$

231. $y = \left(2x^3 - x^2 + 6x + 1\right)^3$

232. $y = \frac{1}{\sin^2(x)}$

233. $y = (\tan x + \sin x)^{-3}$

234. $y = x^2\cos^4 x$

235. $y = \sin(\cos 7x)$

236. $y = \sqrt{6 + \sec \pi x^2}$

237. $y = \cot^3(4x + 1)$

238. Let $y = [f(x)]^3$ and suppose that $f'(1) = 4$ and $\frac{dy}{dx} = 10$ for $x = 1$. Find $f(1)$.

239. Let $y = \left(f(x) + 5x^2\right)^4$ and suppose that $f(-1) = -4$ and $\frac{dy}{dx} = 3$ when $x = -1$. Find $f'(-1)$

240. Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x$. If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2$, find $f'(4)$.

241. **[T]** Find the equation of the tangent line to $y = -\sin\left(\frac{x}{2}\right)$ at the origin. Use a calculator to graph the function and the tangent line together.

242. **[T]** Find the equation of the tangent line to $y = \left(3x + \frac{1}{x}\right)^2$ at the point $(1, 16)$. Use a calculator to graph the function and the tangent line together.

243. Find the x-coordinates at which the tangent line to $y = \left(x - \frac{6}{x}\right)^8$ is horizontal.

244. **[T]** Find an equation of the line that is normal to $g(\theta) = \sin^2(\pi\theta)$ at the point $\left(\frac{1}{4}, \frac{1}{2}\right)$. Use a calculator to graph the function and the normal line together.

For the following exercises, use the information in the following table to find $h'(a)$ at the given value for a.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2
1	1	−2	3	0
2	4	4	1	−1
3	3	−3	2	3

245. $h(x) = f(g(x)); a = 0$

246. $h(x) = g(f(x)); a = 0$

247. $h(x) = \left(x^4 + g(x)\right)^{-2}; a = 1$

248. $h(x) = \left(\frac{f(x)}{g(x)}\right)^2; a = 3$

249. $h(x) = f(x + f(x)); a = 1$

250. $h(x) = (1 + g(x))^3; a = 2$

251. $h(x) = g\left(2 + f\left(x^2\right)\right); a = 1$

252. $h(x) = f(g(\sin x)); a = 0$

253. **[T]** The position function of a freight train is given by $s(t) = 100(t + 1)^{-2}$, with s in meters and t in seconds. At time $t = 6$ s, find the train's

a. velocity and
b. acceleration.
c. Using a. and b. is the train speeding up or slowing down?

254. **[T]** A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches: $s(t) = -3\cos\left(\pi t + \frac{\pi}{4}\right)$.

a. Determine the position of the spring at $t = 1.5$ s.
b. Find the velocity of the spring at $t = 1.5$ s.

255. **[T]** The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C = 0.0001x^3 - 0.02x^2 + 3x + 300.$ In t weeks production is estimated to be $x = 1600 + 100t$ boxes.

a. Find the marginal cost $C'(x)$.
b. Use Leibniz's notation for the chain rule, $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.
c. Use b. to determine how fast costs are increasing when $t = 2$ weeks. Include units with the answer.

256. **[T]** The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

a. Suppose $r = 2 - \frac{100}{(t + 7)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the area is expanding.
b. Use a. to find the rate at which the area is expanding at $t = 4$ s.

257. **[T]** The formula for the volume of a sphere is $S = \frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

a. Suppose $r = \frac{1}{(t + 1)^2} - \frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the snowball is melting.
b. Use a. to find the rate at which the volume is changing at $t = 1$ min.

258. **[T]** The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function $T(x) = 94 - 10\cos\left[\frac{\pi}{12}(x - 2)\right]$, where x is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

259. **[T]** The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

3.7 | Derivatives of Inverse Functions

Learning Objectives

3.7.1 Calculate the derivative of an inverse function.

3.7.2 Recognize the derivatives of the standard inverse trigonometric functions.

In this section we explore the relationship between the derivative of a function and the derivative of its inverse. For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without having to use the limit definition of the derivative. In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions. This formula may also be used to extend the power rule to rational exponents.

The Derivative of an Inverse Function

We begin by considering a function and its inverse. If $f(x)$ is both invertible and differentiable, it seems reasonable that the inverse of $f(x)$ is also differentiable. **Figure 3.28** shows the relationship between a function $f(x)$ and its inverse $f^{-1}(x)$. Look at the point $\left(a,\ f^{-1}(a)\right)$ on the graph of $f^{-1}(x)$ having a tangent line with a slope of $\left(f^{-1}\right)'(a) = \frac{p}{q}$. This point corresponds to a point $\left(f^{-1}(a),\ a\right)$ on the graph of $f(x)$ having a tangent line with a slope of $f'\left(f^{-1}(a)\right) = \frac{q}{p}$. Thus, if $f^{-1}(x)$ is differentiable at a, then it must be the case that

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

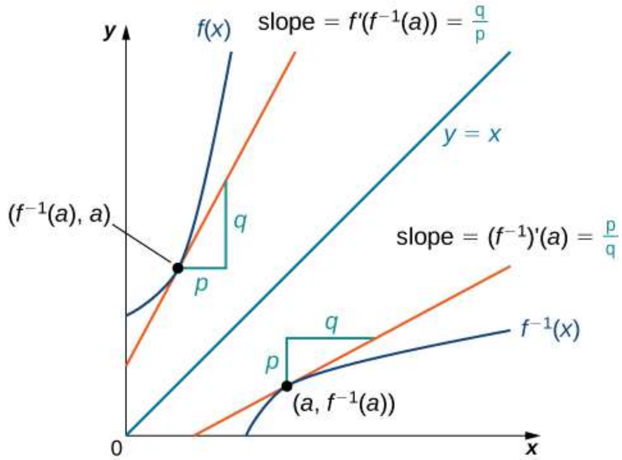


Figure 3.28 The tangent lines of a function and its inverse are related; so, too, are the derivatives of these functions.

We may also derive the formula for the derivative of the inverse by first recalling that $x = f\left(f^{-1}(x)\right)$. Then by differentiating both sides of this equation (using the chain rule on the right), we obtain

$$1 = f'\left(f^{-1}(x)\right)\left(\left(f^{-1}\right)'(x)\right).$$

Solving for $(f^{-1})'(x)$, we obtain

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}. \qquad \textbf{(3.19)}$$

We summarize this result in the following theorem.

Theorem 3.11: Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\left(f^{-1}(x)\right) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Example 3.60

Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \frac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Solution

The inverse of $g(x) = \frac{x+2}{x}$ is $f(x) = \frac{2}{x-1}$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

$$f'(x) = \frac{-2}{(x-1)^2} \text{ and } f'(g(x)) = \frac{-2}{(g(x)-1)^2} = \frac{-2}{\left(\frac{x+2}{x}-1\right)^2} = -\frac{x^2}{2}.$$

Finally,

$$g'(x) = \frac{1}{f'(g(x))} = -\frac{2}{x^2}.$$

We can verify that this is the correct derivative by applying the quotient rule to $g(x)$ to obtain

$$g'(x) = -\frac{2}{x^2}.$$

3.42 Use the inverse function theorem to find the derivative of $g(x) = \frac{1}{x+2}$. Compare the result obtained by differentiating $g(x)$ directly.

Example 3.61

Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \sqrt[3]{x}$.

Solution

The function $g(x) = \sqrt[3]{x}$ is the inverse of the function $f(x) = x^3$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

$$f'(x) = 3x^2 \text{ and } f'(g(x)) = 3\left(\sqrt[3]{x}\right)^2 = 3x^{2/3}.$$

Finally,

$$g'(x) = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}.$$

3.43 Find the derivative of $g(x) = \sqrt[5]{x}$ by applying the inverse function theorem.

From the previous example, we see that we can use the inverse function theorem to extend the power rule to exponents of the form $\frac{1}{n}$, where n is a positive integer. This extension will ultimately allow us to differentiate x^q, where q is any rational number.

Theorem 3.12: Extending the Power Rule to Rational Exponents

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n) - 1}. \tag{3.20}$$

Also, if n is a positive integer and m is an arbitrary integer, then

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n) - 1}. \tag{3.21}$$

Proof

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

$$f'(x) = nx^{n-1} \text{ and } f'(g(x)) = n(x^{1/n})^{n-1} = nx^{(n-1)/n}.$$

Finally,

$$g'(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{(1-n)/n} = \frac{1}{n}x^{(1/n)-1}.$$

To differentiate $x^{m/n}$ we must rewrite it as $\left(x^{1/n}\right)^m$ and apply the chain rule. Thus,

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{d}{dx}\left(\left(x^{1/n}\right)^m\right) = m\left(x^{1/n}\right)^{m-1} \cdot \frac{1}{n}x^{(1/n)-1} = \frac{m}{n}x^{(m/n)-1}.$$

□

Example 3.62

Applying the Power Rule to a Rational Power

Find the equation of the line tangent to the graph of $y = x^{2/3}$ at $x = 8$.

Solution

First find $\frac{dy}{dx}$ and evaluate it at $x = 8$. Since

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3} \text{ and } \frac{dy}{dx}\bigg|_{x=8} = \frac{1}{3}$$

the slope of the tangent line to the graph at $x = 8$ is $\frac{1}{3}$.

Substituting $x = 8$ into the original function, we obtain $y = 4$. Thus, the tangent line passes through the point $(8, 4)$. Substituting into the point-slope formula for a line, we obtain the tangent line

$$y = \frac{1}{3}x + \frac{4}{3}.$$

3.44 Find the derivative of $s(t) = \sqrt{2t + 1}$.

Derivatives of Inverse Trigonometric Functions

We now turn our attention to finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text. The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually algebraic functions. Previously, derivatives of algebraic functions have proven to be algebraic functions and derivatives of trigonometric functions have been shown to be trigonometric functions. Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

Example 3.63

Derivative of the Inverse Sine Function

Use the inverse function theorem to find the derivative of $g(x) = \sin^{-1} x$.

Solution

Since for x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin x$ is the inverse of $g(x) = \sin^{-1} x$, begin by finding $f'(x)$. Since

$$f'(x) = \cos x \text{ and } f'(g(x)) = \cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2},$$

we see that

$$g'(x) = \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{f'(g(x))} = \frac{1}{\sqrt{1 - x^2}}.$$

Analysis

To see that $\cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2}$, consider the following argument. Set $\sin^{-1} x = \theta$. In this case, $\sin\theta = x$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. We begin by considering the case where $0 < \theta < \frac{\pi}{2}$. Since θ is an acute angle, we may construct a right triangle having acute angle θ, a hypotenuse of length 1 and the side opposite angle θ having length x. From the Pythagorean theorem, the side adjacent to angle θ has length $\sqrt{1 - x^2}$. This triangle is shown in **Figure 3.29**. Using the triangle, we see that $\cos\left(\sin^{-1} x\right) = \cos\theta = \sqrt{1 - x^2}$.

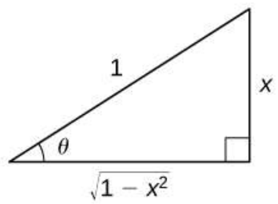


Figure 3.29 Using a right triangle having acute angle θ, a hypotenuse of length 1, and the side opposite angle θ having length x, we can see that $\cos\left(\sin^{-1} x\right) = \cos\theta = \sqrt{1 - x^2}$.

In the case where $-\frac{\pi}{2} < \theta < 0$, we make the observation that $0 < -\theta < \frac{\pi}{2}$ and hence

$$\cos\left(\sin^{-1} x\right) = \cos\theta = \cos(-\theta) = \sqrt{1 - x^2}.$$

Now if $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$, $x = 1$ or $x = -1$, and since in either case $\cos\theta = 0$ and $\sqrt{1 - x^2} = 0$, we have

$$\cos\left(\sin^{-1} x\right) = \cos\theta = \sqrt{1 - x^2}.$$

Finally, if $\theta = -$, $x = 0$ and $\cos\theta = \sqrt{1} = 1$.

Consequently, in all cases, $\cos\left(\sin^{-1} x\right) = \sqrt{1 - x^2}$.

Example 3.64

Applying the Chain Rule to the Inverse Sine Function

Apply the chain rule to the formula derived in **Example 3.61** to find the derivative of $h(x) = \sin^{-1}(g(x))$ and use this result to find the derivative of $h(x) = \sin^{-1}\left(2x^3\right)$.

Solution

Applying the chain rule to $h(x) = \sin^{-1}(g(x))$, we have

$$h'(x) = \frac{1}{\sqrt{1 - (g(x))^2}} g'(x).$$

Now let $g(x) = 2x^3$, so $g'(x) = 6x^2$. Substituting into the previous result, we obtain

$$\begin{aligned} h'(x) &= \frac{1}{\sqrt{1-4x^6}} \cdot 6x^2 \\ &= \frac{6x^2}{\sqrt{1-4x^6}}. \end{aligned}$$

3.45 Use the inverse function theorem to find the derivative of $g(x) = \tan^{-1} x$.

The derivatives of the remaining inverse trigonometric functions may also be found by using the inverse function theorem. These formulas are provided in the following theorem.

Theorem 3.13: Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-(x)^2}} \quad (3.22)$$

$$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-(x)^2}} \quad (3.23)$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+(x)^2} \quad (3.24)$$

$$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+(x)^2} \quad (3.25)$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{(x)^2-1}} \quad (3.26)$$

$$\frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{(x)^2-1}} \quad (3.27)$$

Example 3.65

Applying Differentiation Formulas to an Inverse Tangent Function

Find the derivative of $f(x) = \tan^{-1}\left(x^2\right)$.

Solution

Let $g(x) = x^2$, so $g'(x) = 2x$. Substituting into **Equation 3.24**, we obtain

$$f'(x) = \frac{1}{1+\left(x^2\right)^2} \cdot (2x).$$

Simplifying, we have

$$f'(x) = \frac{2x}{1 + x^4}.$$

Example 3.66

Applying Differentiation Formulas to an Inverse Sine Function

Find the derivative of $h(x) = x^2 \sin^{-1} x$.

Solution

By applying the product rule, we have

$$h'(x) = 2x \sin^{-1} x + \frac{1}{\sqrt{1 - x^2}} \cdot x^2.$$

3.46 Find the derivative of $h(x) = \cos^{-1}(3x - 1)$.

Example 3.67

Applying the Inverse Tangent Function

The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

Solution

Begin by differentiating $s(t)$ in order to find $v(t)$. Thus,

$$v(t) = s'(t) = \frac{1}{1 + \left(\frac{1}{t}\right)^2} \cdot \frac{-1}{t^2}.$$

Simplifying, we have

$$v(t) = -\frac{1}{t^2 + 1}.$$

Thus, $v(1) = -\frac{1}{2}$.

3.47 Find the equation of the line tangent to the graph of $f(x) = \sin^{-1} x$ at $x = 0$.

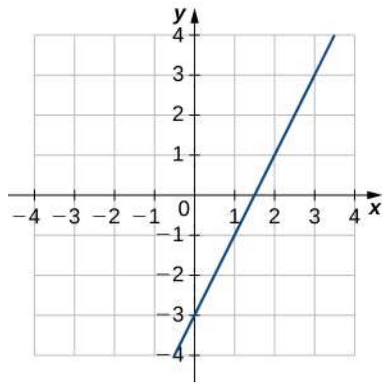
3.7 EXERCISES

For the following exercises, use the graph of $y = f(x)$ to

a. sketch the graph of $y = f^{-1}(x)$, and

b. use part a. to estimate $\left(f^{-1}\right)'(1)$.

260.



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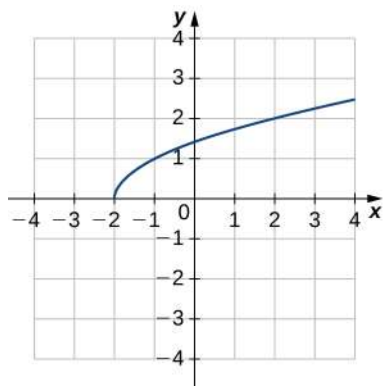

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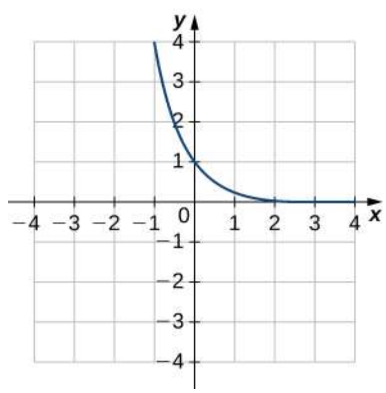

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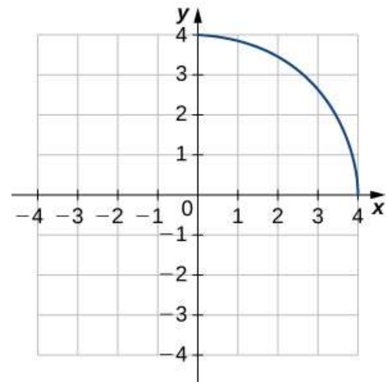


For the following exercises, use the functions $y = f(x)$ to find

a. $\frac{df}{dx}$ at $x = a$ and

b. $x = f^{-1}(y)$.

c. Then use part b. to find $\frac{df^{-1}}{dy}$ at $y = f(a)$.

264. $f(x) = 6x - 1,\ x = -2$

265. $f(x) = 2x^3 - 3,\ x = 1$

266. $f(x) = 9 - x^2,\ 0 \le x \le 3,\ x = 2$

267. $f(x) = \sin x,\ x = 0$

For each of the following functions, find $\left(f^{-1}\right)'(a)$.

268. $f(x) = x^2 + 3x + 2,\ x \ge -\frac{3}{2},\ a = 2$

269. $f(x) = x^3 + 2x + 3,\ a = 0$

270. $f(x) = x + \sqrt{x},\ a = 2$

271. $f(x) = x - \frac{2}{x},\ x < 0,\ a = 1$

272. $f(x) = x + \sin x,\ a = 0$

273. $f(x) = \tan x + 3x^2,\ a = 0$

For each of the given functions $y = f(x)$,

a. find the slope of the tangent line to its inverse function f^{-1} at the indicated point P, and

b. find the equation of the tangent line to the graph of f^{-1} at the indicated point.

274. $f(x) = \frac{4}{1 + x^2},\ P(2,\ 1)$

275. $f(x) = \sqrt{x - 4},\ P(2,\ 8)$

276. $f(x) = \left(x^3 + 1\right)^4,\ P(16,\ 1)$

277. $f(x) = -x^3 - x + 2,\ P(-8,\ 2)$

278. $f(x) = x^5 + 3x^3 - 4x - 8,\ P(-8,\ 1)$

For the following exercises, find $\frac{dy}{dx}$ for the given function.

279. $y = \sin^{-1}\left(x^2\right)$

280. $y = \cos^{-1}(\sqrt{x})$

281. $y = \sec^{-1}\left(\frac{1}{x}\right)$

282. $y = \sqrt{\csc^{-1} x}$

283. $y = \left(1 + \tan^{-1} x\right)^3$

284. $y = \cos^{-1}(2x) \cdot \sin^{-1}(2x)$

285. $y = \frac{1}{\tan^{-1}(x)}$

286. $y = \sec^{-1}(-x)$

287. $y = \cot^{-1}\sqrt{4 - x^2}$

288. $y = x \cdot \csc^{-1} x$

For the following exercises, use the given values to find $\left(f^{-1}\right)'(a)$.

289. $f(\pi) = 0,\ f'(\pi) = -1,\ a = 0$

290. $f(6) = 2,\ f'(6) = \frac{1}{3},\ a = 2$

291. $f\left(\frac{1}{3}\right) = -8,\ f'\left(\frac{1}{3}\right) = 2,\ a = -8$

292. $f(\sqrt{3}) = \frac{1}{2},\ f'(\sqrt{3}) = \frac{2}{3},\ a = \frac{1}{2}$

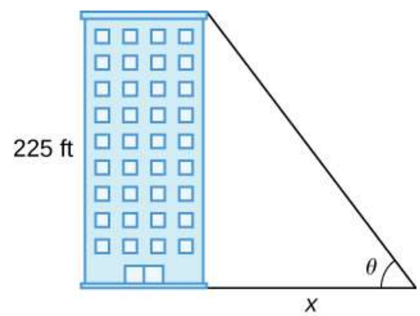
293. $f(1) = -3,\ f'(1) = 10,\ a = -3$

294. $f(1) = 0,\ f'(1) = -2,\ a = 0$

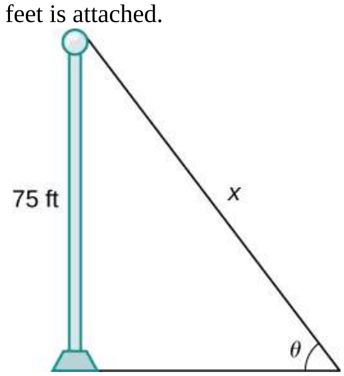
295. **[T]** The position of a moving hockey puck after t seconds is $s(t) = \tan^{-1} t$ where s is in meters.

a. Find the velocity of the hockey puck at any time t.
b. Find the acceleration of the puck at any time t.
c. Evaluate a. and b. for $t = 2,\ 4,$ and 6 seconds.
d. What conclusion can be drawn from the results in c.?

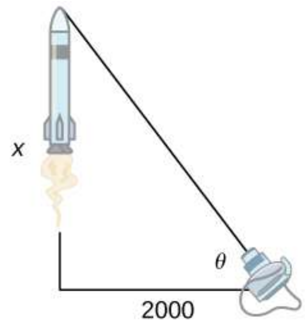
296. **[T]** A building that is 225 feet tall casts a shadow of various lengths x as the day goes by. An angle of elevation θ is formed by lines from the top and bottom of the building to the tip of the shadow, as seen in the following figure. Find the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when $x = 272$ feet.



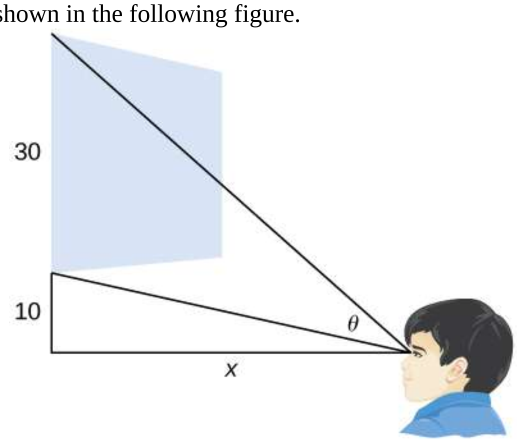
297. **[T]** A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached.



298. **[T]** A television camera at ground level is 2000 feet away from the launching pad of a space rocket that is set to take off vertically, as seen in the following figure. The angle of elevation of the camera can be found by $\theta = \tan^{-1}\left(\frac{x}{2000}\right)$, where x is the height of the rocket. Find the rate of change of the angle of elevation after launch when the camera and the rocket are 5000 feet apart.



299. **[T]** A local movie theater with a 30-foot-high screen that is 10 feet above a person's eye level when seated has a viewing angle θ (in radians) given by $\theta = \cot^{-1}\frac{x}{40} - \cot^{-1}\frac{x}{10}$, where x is the distance in feet away from the movie screen that the person is sitting, as shown in the following figure.



a. Find $\frac{d\theta}{dx}$.
b. Evaluate $\frac{d\theta}{dx}$ for $x = 5, 10, 15,$ and 20.
c. Interpret the results in b..
d. Evaluate $\frac{d\theta}{dx}$ for $x = 25, 30, 35,$ and 40
e. Interpret the results in d. At what distance x should the person stand to maximize his or her viewing angle?

3.8 | Implicit Differentiation

Learning Objectives

3.8.1 Find the derivative of a complicated function by using implicit differentiation.
3.8.2 Use implicit differentiation to determine the equation of a tangent line.

We have already studied how to find equations of tangent lines to functions and the rate of change of a function at a specific point. In all these cases we had the explicit equation for the function and differentiated these functions explicitly. Suppose instead that we want to determine the equation of a tangent line to an arbitrary curve or the rate of change of an arbitrary curve at a point. In this section, we solve these problems by finding the derivatives of functions that define y implicitly in terms of x.

Implicit Differentiation

In most discussions of math, if the dependent variable y is a function of the independent variable x, we express y in terms of x. If this is the case, we say that y is an explicit function of x. For example, when we write the equation $y = x^2 + 1$, we are defining y explicitly in terms of x. On the other hand, if the relationship between the function y and the variable x is expressed by an equation where y is not expressed entirely in terms of x, we say that the equation defines y implicitly in terms of x. For example, the equation $y - x^2 = 1$ defines the function $y = x^2 + 1$ implicitly.

Implicit differentiation allows us to find slopes of tangents to curves that are clearly not functions (they fail the vertical line test). We are using the idea that portions of y are functions that satisfy the given equation, but that y is not actually a function of x.

In general, an equation defines a function implicitly if the function satisfies that equation. An equation may define many different functions implicitly. For example, the functions

$y = \sqrt{25 - x^2}$ and $y = \begin{cases} \sqrt{25 - x^2} & \text{if } -5 < x < 0 \\ -\sqrt{25 - x^2} & \text{if } 0 < x < 25 \end{cases}$, which are illustrated in **Figure 3.30**, are just three of the many functions defined implicitly by the equation $x^2 + y^2 = 25$.

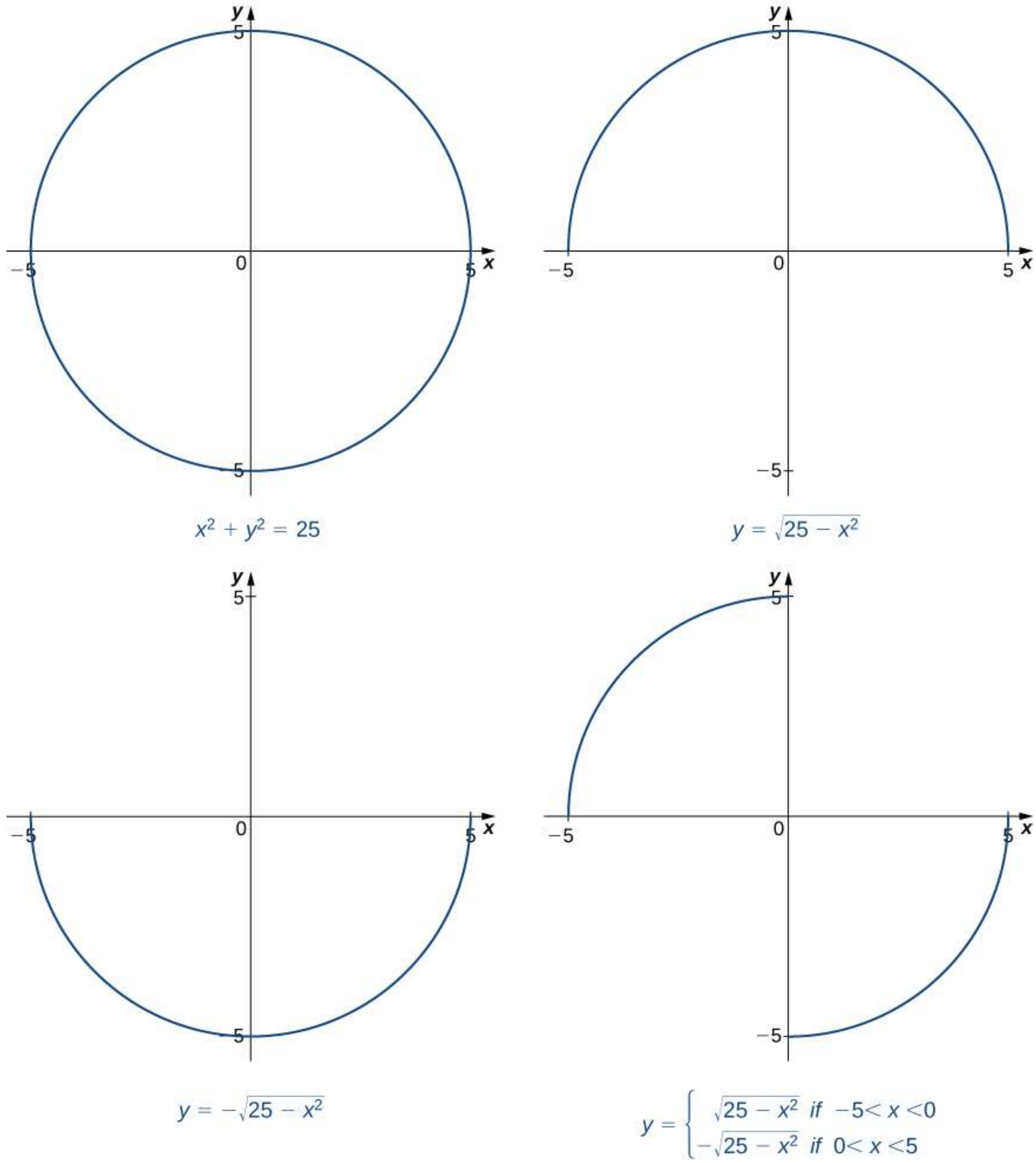


Figure 3.30 The equation $x^2 + y^2 = 25$ defines many functions implicitly.

If we want to find the slope of the line tangent to the graph of $x^2 + y^2 = 25$ at the point $(3, 4),$ we could evaluate the derivative of the function $y = \sqrt{25 - x^2}$ at $x = 3.$ On the other hand, if we want the slope of the tangent line at the point $(3, -4),$ we could use the derivative of $y = -\sqrt{25 - x^2}.$ However, it is not always easy to solve for a function defined implicitly by an equation. Fortunately, the technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly. The process of finding $\frac{dy}{dx}$ using implicit differentiation is described in the following problem-solving strategy.

Problem-Solving Strategy: Implicit Differentiation

To perform implicit differentiation on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently, whereas $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$ because we must use the chain rule to differentiate $\sin y$ with respect to x.
2. Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.
3. Factor out $\frac{dy}{dx}$ on the left.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

Example 3.68

Using Implicit Differentiation

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Solution

Follow the steps in the problem-solving strategy.

$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25)$	Step 1. Differentiate both sides of the equation.
$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = 0$	Step 1.1. Use the sum rule on the left. On the right $\frac{d}{dx}(25) = 0$.
$2x + 2y\frac{dy}{dx} = 0$	Step 1.2. Take the derivatives, so $\frac{d}{dx}\left(x^2\right) = 2x$ and $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$.
$2y\frac{dy}{dx} = -2x$	Step 2. Keep the terms with $\frac{dy}{dx}$ on the left. Move the remaining terms to the right.
$\frac{dy}{dx} = -\frac{x}{y}$	Step 4. Divide both sides of the equation by $2y$. (Step 3 does not apply in this case.)

Analysis

Note that the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y. Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.

Example 3.69

Using Implicit Differentiation and the Product Rule

Assuming that y is defined implicitly by the equation $x^3 \sin y + y = 4x + 3$, find $\frac{dy}{dx}$.

Solution

$\frac{d}{dx}\left(x^3 \sin y + y\right) = \frac{d}{dx}(4x+3)$ Step 1: Differentiate both sides of the equation.

$\frac{d}{dx}\left(x^3 \sin y\right) + \frac{d}{dx}(y) = 4$ Step 1.1: Apply the sum rule on the left. On the right, $\frac{d}{dx}(4x+3) = 4$.

$\left(\frac{d}{dx}\left(x^3\right) \cdot \sin y + \frac{d}{dx}(\sin y) \cdot x^3\right) + \frac{dy}{dx} = 4$ Step 1.2: Use the product rule to find $\frac{d}{dx}\left(x^3 \sin y\right)$. Observe that $\frac{d}{dx}(y) = \frac{dy}{dx}$.

$3x^2 \sin y + \left(\cos y \frac{dy}{dx}\right) \cdot x^3 + \frac{dy}{dx} = 4$ Step 1.3: We know $\frac{d}{dx}\left(x^3\right) = 3x^2$. Use the chain rule to obtain $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$.

$x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y$ Step 2: Keep all terms containing $\frac{dy}{dx}$ on the left. Move all other terms to the right.

$\frac{dy}{dx}\left(x^3 \cos y + 1\right) = 4 - 3x^2 \sin y$ Step 3: Factor out $\frac{dy}{dx}$ on the left.

$\frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$ Step 4: Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by $x^3 \cos y + 1$.

Example 3.70

Using Implicit Differentiation to Find a Second Derivative

Find $\frac{d^2 y}{dx^2}$ if $x^2 + y^2 = 25$.

Solution

In **Example 3.68**, we showed that $\frac{dy}{dx} = -\frac{x}{y}$. We can take the derivative of both sides of this equation to find $\frac{d^2 y}{dx^2}$.

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dy}\left(-\frac{x}{y}\right) && \text{Differentiate both sides of } \frac{dy}{dx} = -\frac{x}{y}. \\
&= -\frac{\left(1 \cdot y - x\frac{dy}{dx}\right)}{y^2} && \text{Use the quotient rule to find } \frac{d}{dy}\left(-\frac{x}{y}\right). \\
&= \frac{-y + x\frac{dy}{dx}}{y^2} && \text{Simplify.} \\
&= \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} && \text{Substitute } \frac{dy}{dx} = -\frac{x}{y}. \\
&= \frac{-y^2 - x^2}{y^3} && \text{Simplify.}
\end{aligned}$$

At this point we have found an expression for $\frac{d^2y}{dx^2}$. If we choose, we can simplify the expression further by recalling that $x^2 + y^2 = 25$ and making this substitution in the numerator to obtain $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$.

3.48 Find $\frac{dy}{dx}$ for y defined implicitly by the equation $4x^5 + \tan y = y^2 + 5x$.

Finding Tangent Lines Implicitly

Now that we have seen the technique of implicit differentiation, we can apply it to the problem of finding equations of tangent lines to curves described by equations.

Example 3.71

Finding a Tangent Line to a Circle

Find the equation of the line tangent to the curve $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

Although we could find this equation without using implicit differentiation, using that method makes it much easier. In **Example 3.68**, we found $\frac{dy}{dx} = -\frac{x}{y}$.

The slope of the tangent line is found by substituting $(3, -4)$ into this expression. Consequently, the slope of the tangent line is $\left.\frac{dy}{dx}\right|_{(3,\,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Using the point $(3, -4)$ and the slope $\frac{3}{4}$ in the point-slope equation of the line, we obtain the equation $y = \frac{3}{4}x - \frac{25}{4}$ (**Figure 3.31**).

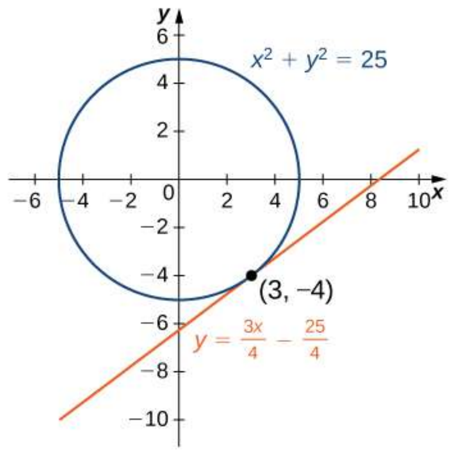


Figure 3.31 The line $y = \frac{3}{4}x - \frac{25}{4}$ is tangent to $x^2 + y^2 = 25$ at the point (3, −4).

Example 3.72

Finding the Equation of the Tangent Line to a Curve

Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ (**Figure 3.32**). This curve is known as the folium (or leaf) of Descartes.

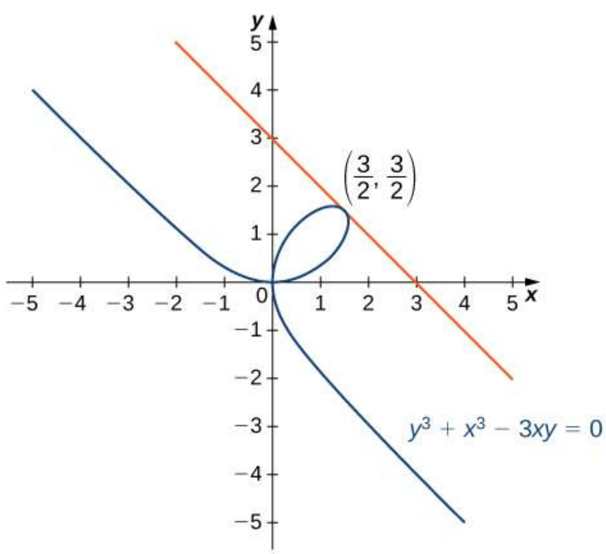


Figure 3.32 Finding the tangent line to the folium of Descartes at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Solution

Begin by finding $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx}\left(y^3 + x^3 - 3xy\right) &= \frac{d}{dx}(0) \\ 3y^2\frac{dy}{dx} + 3x^2 - \left(3y + \frac{dy}{dx}3x\right) &= 0 \\ \frac{dy}{dx} &= \frac{3y - 3x^2}{3y^2 - 3x}. \end{aligned}$$

Next, substitute $\left(\frac{3}{2}, \frac{3}{2}\right)$ into $\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$ to find the slope of the tangent line:

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2}, \frac{3}{2}\right)} = -1.$$

Finally, substitute into the point-slope equation of the line to obtain

$$y = -x + 3.$$

Example 3.73

Applying Implicit Differentiation

In a simple video game, a rocket travels in an elliptical orbit whose path is described by the equation $4x^2 + 25y^2 = 100$. The rocket can fire missiles along lines tangent to its path. The object of the game is to destroy an incoming asteroid traveling along the positive x-axis toward $(0, 0)$. If the rocket fires a missile when it is located at $\left(3, \frac{8}{5}\right)$, where will it intersect the x-axis?

Solution

To solve this problem, we must determine where the line tangent to the graph of $4x^2 + 25y^2 = 100$ at $\left(3, \frac{8}{5}\right)$ intersects the x-axis. Begin by finding $\frac{dy}{dx}$ implicitly.

Differentiating, we have

$$8x + 50y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = -\frac{4x}{25y}.$$

The slope of the tangent line is $\left.\frac{dy}{dx}\right|_{\left(3, \frac{8}{5}\right)} = -\frac{3}{10}$. The equation of the tangent line is $y = -\frac{3}{10}x + \frac{5}{2}$. To determine where the line intersects the x-axis, solve $0 = -\frac{3}{10}x + \frac{5}{2}$. The solution is $x = \frac{25}{3}$. The missile intersects the x-axis at the point $\left(\frac{25}{3}, 0\right)$.

3.49 Find the equation of the line tangent to the hyperbola $x^2 - y^2 = 16$ at the point $(5, 3)$.

3.8 EXERCISES

For the following exercises, use implicit differentiation to find $\frac{dy}{dx}$.

300. $x^2 - y^2 = 4$

301. $6x^2 + 3y^2 = 12$

302. $x^2 y = y - 7$

303. $3x^3 + 9xy^2 = 5x^3$

304. $xy - \cos(xy) = 1$

305. $y\sqrt{x+4} = xy + 8$

306. $-xy - 2 = \frac{x}{7}$

307. $y\sin(xy) = y^2 + 2$

308. $(xy)^2 + 3x = y^2$

309. $x^3 y + xy^3 = -8$

For the following exercises, find the equation of the tangent line to the graph of the given equation at the indicated point. Use a calculator or computer software to graph the function and the tangent line.

310. **[T]** $x^4 y - xy^3 = -2,\ (-1, -1)$

311. **[T]** $x^2 y^2 + 5xy = 14,\ (2, 1)$

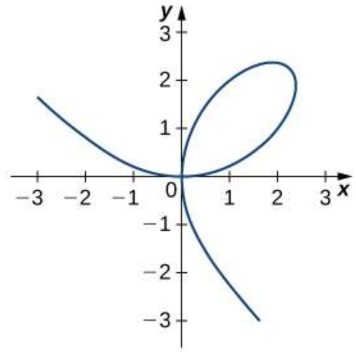
312. **[T]** $\tan(xy) = y,\ \left(\frac{\pi}{4}, 1\right)$

313. **[T]** $xy^2 + \sin(\pi y) - 2x^2 = 10,\ (2, -3)$

314. **[T]** $\frac{x}{y} + 5x - 7 = -\frac{3}{4}y,\ (1, 2)$

315. **[T]** $xy + \sin(x) = 1,\ \left(\frac{\pi}{2}, 0\right)$

316. **[T]** The graph of a folium of Descartes with equation $2x^3 + 2y^3 - 9xy = 0$ is given in the following graph.



a. Find the equation of the tangent line at the point $(2, 1)$. Graph the tangent line along with the folium.
b. Find the equation of the normal line to the tangent line in a. at the point $(2, 1)$.

317. For the equation $x^2 + 2xy - 3y^2 = 0$,
a. Find the equation of the normal to the tangent line at the point $(1, 1)$.
b. At what other point does the normal line in a. intersect the graph of the equation?

318. Find all points on the graph of $y^3 - 27y = x^2 - 90$ at which the tangent line is vertical.

319. For the equation $x^2 + xy + y^2 = 7$,
a. Find the x-intercept(s).
b. Find the slope of the tangent line(s) at the x-intercept(s).
c. What does the value(s) in b. indicate about the tangent line(s)?

320. Find the equation of the tangent line to the graph of the equation $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{6}$ at the point $\left(0, \frac{1}{2}\right)$.

321. Find the equation of the tangent line to the graph of the equation $\tan^{-1}(x + y) = x^2 + \frac{\pi}{4}$ at the point $(0, 1)$.

322. Find y' and y'' for $x^2 + 6xy - 2y^2 = 3$.

323. **[T]** The number of cell phones produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $60x^{3/4}y^{1/4} = 3240.$

a. Find $\frac{dy}{dx}$ and evaluate at the point $(81, 16).$

b. Interpret the result of a.

324. **[T]** The number of cars produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $30x^{1/3}y^{2/3} = 360.$ (Both x and y are measured in thousands of dollars.)

a. Find $\frac{dy}{dx}$ and evaluate at the point $(27, 8).$

b. Interpret the result of a.

325. The volume of a right circular cone of radius x and height y is given by $V = \frac{1}{3}\pi x^2 y.$ Suppose that the volume of the cone is $85\pi\,\text{cm}^3.$ Find $\frac{dy}{dx}$ when $x = 4$ and $y = 16.$

For the following exercises, consider a closed rectangular box with a square base with side x and height $y.$

326. Find an equation for the surface area of the rectangular box, $S(x, y).$

327. If the surface area of the rectangular box is 78 square feet, find $\frac{dy}{dx}$ when $x = 3$ feet and $y = 5$ feet.

For the following exercises, use implicit differentiation to determine $y'.$ Does the answer agree with the formulas we have previously determined?

328. $x = \sin y$

329. $x = \cos y$

330. $x = \tan y$

3.9 | Derivatives of Exponential and Logarithmic Functions

Learning Objectives

3.9.1 Find the derivative of exponential functions.
3.9.2 Find the derivative of logarithmic functions.
3.9.3 Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in **Introduction to Functions and Graphs**, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x,\ b > 0,$ is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of $b^n,$ where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1,\ b^{-n} = \frac{1}{b^n},$ for a positive integer $n,$ and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and $t.$ These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the *continuity* of $B(x) = b^x,\ b > 0,$ we may interpret b^r as $\lim\limits_{x \to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

$$4^3 < 4^\pi < 4^4,\ 4^{3.1} < 4^\pi < 4^{3.2},\ 4^{3.14} < 4^\pi < 4^{3.15},$$
$$4^{3.141} < 4^\pi < 4^{3.142},\ 4^{3.1415} < 4^\pi < 4^{3.1416},\ldots.$$

As we see in the following table, $4^\pi \approx 77.88.$

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Table 3.6 Approximating a Value of 4^{π}

We also assume that for $B(x) = b^x, b > 0,$ the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1.$ We define e to be this unique value, as we did in **Introduction to Functions and Graphs**. **Figure 3.33** provides graphs of the functions $y = 2^x, y = 3^x, y = 2.7^x,$ and $y = 2.8^x.$ A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function**.

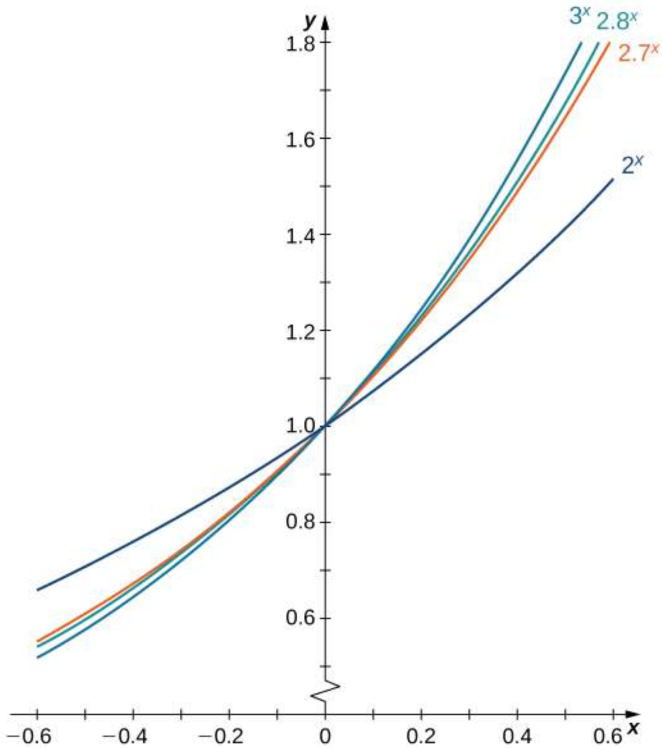


Figure 3.33 The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e, we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$ to obtain the estimate

$$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}.$$

See the following table.

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

Table 3.7 Estimating a Value of e

The evidence from the table suggests that $2.7182 < e < 2.7183$.

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in **Figure 3.34**. This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.

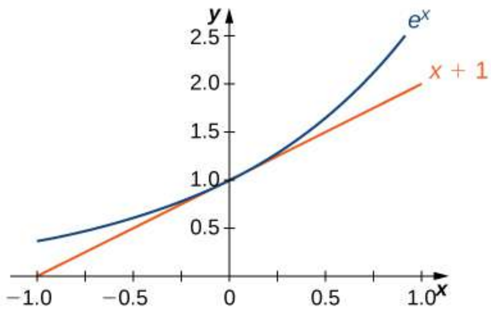


Figure 3.34 The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x$, $b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

$$B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}. \tag{3.28}$$

Turning to $B'(x)$, we obtain the following.

$$\begin{aligned} B'(x) &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} && \text{Apply the limit definition of the derivative.} \\ &= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h. \\ &= \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} && \text{Factor out } b^x. \\ &= b^x \lim_{h \to 0} \frac{b^h - 1}{h} && \text{Apply a property of limits.} \\ &= b^x B'(0) && \text{Use } B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}. \end{aligned}$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is

$$B'(x) = b^x B'(0). \tag{3.29}$$

For $E(x) = e^x$, $E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x$, $b > 0$, will be derived later.)

Theorem 3.14: Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

$$E'(x) = e^x.$$

In general,

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x).$$

Example 3.74

Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution

Using the derivative formula and the chain rule,

$$\begin{aligned} f'(x) &= e^{\tan(2x)} \frac{d}{dx}(\tan(2x)) \\ &= e^{\tan(2x)} \sec^2(2x) \cdot 2. \end{aligned}$$

Example 3.75

Combining Differentiation Rules

Find the derivative of $y = \dfrac{e^{x^2}}{x}$.

Solution

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.

$$\begin{aligned} y' &= \frac{\left(e^{x^2} \cdot 2\right)x \cdot x - 1 \cdot e^{x^2}}{x^2} && \text{Apply the quotient rule.} \\ &= \frac{e^{x^2}\left(2x^2 - 1\right)}{x^2} && \text{Simplify.} \end{aligned}$$

3.50 Find the derivative of $h(x) = xe^{2x}$.

Example 3.76

Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

Solution

First find $A'(t)$. By using the chain rule, we have $A'(t) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by

$$A'(t) = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

3.51 If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Theorem 3.15: The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

$$\frac{dy}{dx} = \frac{1}{x}. \tag{3.30}$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of

$h(x) = \ln(g(x))$ is given by

$$h'(x) = \frac{1}{g(x)} g'(x). \tag{3.31}$$

Proof

If $x > 0$ and $y = \ln x,$ then $e^y = x.$ Differentiating both sides of this equation results in the equation

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

$$\frac{dy}{dx} = \frac{1}{x}.$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$ is the inverse of $f(x) = e^x,$ by applying the inverse function theorem we have

$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields

$$h'(x) = \frac{1}{g(x)} g'(x).$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in **Figure 3.35**.

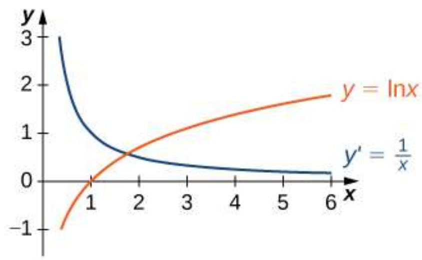


Figure 3.35 The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$.

Example 3.77

Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln\left(x^3 + 3x - 4\right)$.

Solution

Use **Equation 3.31** directly.

$$\begin{aligned} f'(x) &= \frac{1}{x^3 + 3x - 4} \cdot \left(3x^2 + 3\right) && \text{Use } g(x) = x^3 + 3x - 4 \text{ in } h'(x) = \frac{1}{g(x)} g'(x). \\ &= \frac{3x^2 + 3}{x^3 + 3x - 4} && \text{Rewrite.} \end{aligned}$$

Example 3.78

Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x + 1}\right)$.

Solution

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

$$\begin{aligned} f(x) &= \ln\left(\frac{x^2 \sin x}{2x + 1}\right) = 2\ln x + \ln(\sin x) - \ln(2x + 1) && \text{Apply properties of logarithms.} \\ f'(x) &= \frac{2}{x} + \cot x - \frac{2}{2x + 1} && \text{Apply sum rule and } h'(x) = \frac{1}{g(x)} g'(x). \end{aligned}$$

3.52 Differentiate: $f(x) = \ln(3x + 2)^5$.

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = log_b x$ and $y = b^x$ for $b > 0,\ b \neq 1$.

Theorem 3.16: Derivatives of General Exponential and Logarithmic Functions

Let $b > 0,\ b \neq 1,$ and let $g(x)$ be a differentiable function.

i. If, $y = \log_b x,$ then

$$\frac{dy}{dx} = \frac{1}{x \ln b}. \tag{3.32}$$

More generally, if $h(x) = \log_b(g(x)),$ then for all values of x for which $g(x) > 0,$

$$h'(x) = \frac{g'(x)}{g(x) \ln b}. \tag{3.33}$$

ii. If $y = b^x,$ then

$$\frac{dy}{dx} = b^x \ln b. \quad (3.34)$$

More generally, if $h(x) = b^{g(x)}$, then

$$h'(x) = b^{g(x)} g'(x) \ln b. \quad (3.35)$$

Proof

If $y = \log_b x$, then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y, we have $y = \frac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

The derivative in **Equation 3.33** now follows from the chain rule.

If $y = b^x$, then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\frac{1}{y}\frac{dy}{dx} = \ln b$. Solving for $\frac{dy}{dx}$ and substituting $y = b^x$, we see that

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

The more general derivative (**Equation 3.35**) follows from the chain rule.

□

Example 3.79

Applying Derivative Formulas

Find the derivative of $h(x) = \frac{3^x}{3^x + 2}$.

Solution

Use the quotient rule and **Derivatives of General Exponential and Logarithmic Functions**.

$$\begin{aligned} h'(x) &= \frac{3^x \ln 3(3^x + 2) - 3^x \ln 3(3^x)}{(3^x + 2)^2} && \text{Apply the quotient rule.} \\ &= \frac{2 \cdot 3^x \ln 3}{(3^x + 2)^2} && \text{Simplify.} \end{aligned}$$

Example 3.80

Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Solution

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x = 1.$ Using **Equation 3.33**, we see that

$$\frac{dy}{dx} = \frac{3}{(3x+1)\ln 2}.$$

By evaluating the derivative at $x = 1,$ we see that the tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{4\ln 2} = \frac{3}{\ln 16}.$$

3.53 Find the slope for the line tangent to $y = 3^x$ at $x = 2.$

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of $n,$ as well as functions of the form $y = b^{g(x)},$ where $b > 0$ and $b \neq 1.$ Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^{\pi}.$ These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}.$ It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}.$ We outline this technique in the following problem-solving strategy.

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x)).$
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}.$
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}.$
5. Replace y by $h(x).$

Example 3.81

Using Logarithmic Differentiation

Find the derivative of $y = \left(2x^4 + 1\right)^{\tan x}.$

Solution

Use logarithmic differentiation to find this derivative.

$$\ln y = \ln\left(2x^4+1\right)^{\tan x}$$ Step 1. Take the natural logarithm of both sides.

$$\ln y = \tan x \ln\left(2x^4+1\right)$$ Step 2. Expand using properties of logarithms.

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x$$ Step 3. Differentiate both sides. Use the product rule on the right.

$$\frac{dy}{dx} = y\cdot\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x\right)$$ Step 4. Multiply by y on both sides.

$$\frac{dy}{dx} = \left(2x^4+1\right)^{\tan x}\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x\right)$$ Step 5. Substitute $y=\left(2x^4+1\right)^{\tan x}$.

Example 3.82

Using Logarithmic Differentiation

Find the derivative of $y=\frac{x\sqrt{2x+1}}{e^x\sin^3 x}$.

Solution

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

$$\ln y = \ln\frac{x\sqrt{2x+1}}{e^x\sin^3 x}$$ Step 1. Take the natural logarithm of both sides.

$$\ln y = \ln x + \frac{1}{2}\ln(2x+1) - x\ln e - 3\ln\sin x$$ Step 2. Expand using properties of logarithms.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2x+1} - 1 - 3\frac{\cos x}{\sin x}$$ Step 3. Differentiate both sides.

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)$$ Step 4. Multiply by y on both sides.

$$\frac{dy}{dx} = \frac{x\sqrt{2x+1}}{e^x\sin^3 x}\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)$$ Step 5. Substitute $y=\frac{x\sqrt{2x+1}}{e^x\sin^3 x}$.

Example 3.83

Extending the Power Rule

Find the derivative of $y=x^r$ where r is an arbitrary real number.

Solution

The process is the same as in **Example 3.82**, though with fewer complications.

$$\begin{aligned}
\ln y &= \ln x^r && \text{Step 1. Take the natural logarithm of both sides.}\\
\ln y &= r \ln x && \text{Step 2. Expand using properties of logarithms.}\\
\frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x} && \text{Step 3. Differentiate both sides.}\\
\frac{dy}{dx} &= y\frac{r}{x} && \text{Step 4. Multiply by } y \text{ on both sides.}\\
\frac{dy}{dx} &= x^r \frac{r}{x} && \text{Step 5. Substitute } y = x^r.\\
\frac{dy}{dx} &= rx^{r-1} && \text{Simplify.}
\end{aligned}$$

3.54 Use logarithmic differentiation to find the derivative of $y = x^x$.

3.55 Find the derivative of $y = (\tan x)^{\pi}$.

3.9 EXERCISES

For the following exercises, find $f'(x)$ for each function.

331. $f(x) = x^2 e^x$

332. $f(x) = \frac{e^{-x}}{x}$

333. $f(x) = e^{x^3 \ln x}$

334. $f(x) = \sqrt{e^{2x} + 2x}$

335. $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

336. $f(x) = \frac{10^x}{\ln 10}$

337. $f(x) = 2^{4x} + 4x^2$

338. $f(x) = 3^{\sin 3x}$

339. $f(x) = x^{\pi} \cdot \pi^x$

340. $f(x) = \ln\left(4x^3 + x\right)$

341. $f(x) = \ln\sqrt{5x - 7}$

342. $f(x) = x^2 \ln 9x$

343. $f(x) = \log(\sec x)$

344. $f(x) = \log_7\left(6x^4 + 3\right)^5$

345. $f(x) = 2^x \cdot \log_3 7^{x^2 - 4}$

For the following exercises, use logarithmic differentiation to find $\frac{dy}{dx}$.

346. $y = x^{\sqrt{x}}$

347. $y = (\sin 2x)^{4x}$

348. $y = (\ln x)^{\ln x}$

349. $y = x^{\log_2 x}$

350. $y = \left(x^2 - 1\right)^{\ln x}$

351. $y = x^{\cot x}$

352. $y = \frac{x + 11}{\sqrt[3]{x^2 - 4}}$

353. $y = x^{-1/2}\left(x^2 + 3\right)^{2/3}(3x - 4)^4$

354. **[T]** Find an equation of the tangent line to the graph of $f(x) = 4xe^{\left(x^2 - 1\right)}$ at the point where $x = -1$. Graph both the function and the tangent line.

355. **[T]** Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

356. **[T]** Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (*Hint*: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

357. Consider the function $y = x^{1/x}$ for $x > 0$.

   a. Determine the points on the graph where the tangent line is horizontal.
   b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

358. The formula $I(t) = \frac{\sin t}{e^t}$ is the formula for a decaying alternating current.

a. Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\frac{\pi}{2}$	(ii)
π	(iii)
$\frac{3\pi}{2}$	(iv)
2π	(v)
$\frac{5\pi}{2}$	(vi)
3π	(vii)
$\frac{7\pi}{2}$	(viii)
4π	(ix)

b. Using only the values in the table, determine where the tangent line to the graph of $I(t)$ is horizontal.

359. **[T]** The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.

b. Use a. to determine the rate at which the population is increasing in t years.

c. Use b. to determine the rate at which the population is increasing in 10 years.

360. **[T]** An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.

b. Use a. to determine the rate at which the substance is decaying in t hours.

c. Use b. to determine the rate of decay at $t = 4$ hours.

361. **[T]** The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function $N(t) = 5.3e^{0.093t^2 - 0.87t}$, $(0 \le t \le 4)$, where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.

b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in New York City.

362. **[T]** The *relative rate of change* of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}\%$. One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where $a, b,$ and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.

b. Use a. to find the relative rate of change of a population in $x = 20$ months when $a = 204,\ b = 0.0198,$ and $c = 0.15.$

c. Briefly interpret what the result of b. means.

For the following exercises, use the population of New York City from 1790 to 1860, given in the following table.

Years since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

Table 3.8 New York City Population Over Time **Source: http://en.wikipedia.org/wiki/Largest_cities_in_the_United_States_by_population_by_decade.**

363. **[T]** Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

364. **[T]** Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

365. **[T]** Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

366. **[T]** Using the tables of first and second derivatives and the best fit, answer the following questions:
   a. Will the model be accurate in predicting the future population of New York City? Why or why not?
   b. Estimate the population in 2010. Was the prediction correct from a.?

CHAPTER 3 REVIEW

KEY TERMS

acceleration is the rate of change of the velocity, that is, the derivative of velocity

amount of change the amount of a function $f(x)$ over an interval $[x, x+h]$ is $f(x+h) - f(x)$

average rate of change is a function $f(x)$ over an interval $[x, x+h]$ is $\frac{f(x+h) - f(a)}{b - a}$

chain rule the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

constant multiple rule the derivative of a constant *c* multiplied by a function *f* is the same as the constant multiplied by the derivative: $\frac{d}{dx}(cf(x)) = cf'(x)$

constant rule the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0,$ where *c* is a constant

derivative the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

difference quotient of a function $f(x)$ at a is given by

$$\frac{f(a+h) - f(a)}{h} \text{ or } \frac{f(x) - f(a)}{x - a}$$

difference rule the derivative of the difference of a function *f* and a function *g* is the same as the difference of the derivative of *f* and the derivative of *g*: $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

differentiable at *a* a function for which $f'(a)$ exists is differentiable at a

differentiable function a function for which $f'(x)$ exists is a differentiable function

differentiable on *S* a function for which $f'(x)$ exists for each x in the open set S is differentiable on S

differentiation the process of taking a derivative

higher-order derivative a derivative of a derivative, from the second derivative to the *n*th derivative, is called a higher-order derivative

implicit differentiation is a technique for computing $\frac{dy}{dx}$ for a function defined by an equation, accomplished by differentiating both sides of the equation (remembering to treat the variable y as a function) and solving for $\frac{dy}{dx}$

instantaneous rate of change the rate of change of a function at any point along the function $a,$ also called $f'(a),$ or the derivative of the function at a

logarithmic differentiation is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

marginal cost is the derivative of the cost function, or the approximate cost of producing one more item

marginal profit is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

marginal revenue is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

population growth rate is the derivative of the population with respect to time

power rule the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}x^n = nx^{n-1}$

product rule the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x)$

quotient rule the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

speed is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time t whose velocity is given by $v(t)$

sum rule the derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g: $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

KEY EQUATIONS

- **Difference quotient**
$$Q = \frac{f(x) - f(a)}{x - a}$$
- **Difference quotient with increment** h
$$Q = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$
- **Slope of tangent line**
$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
- **Derivative of** $f(x)$ **at** a
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
- **Average velocity**
$$v_{ave} = \frac{s(t) - s(a)}{t - a}$$
- **Instantaneous velocity**
$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$
- **The derivative function**
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
- **Derivative of sine function**
$$\frac{d}{dx}(\sin x) = \cos x$$
- **Derivative of cosine function**
$$\frac{d}{dx}(\cos x) = -\sin x$$
- **Derivative of tangent function**

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

- **Derivative of cotangent function**
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

- **Derivative of secant function**
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

- **Derivative of cosecant function**
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

- **The chain rule**
$$h'(x) = f'(g(x))g'(x)$$

- **The power rule for functions**
$$h'(x) = n(g(x))^{n-1} g'(x)$$

- **Inverse function theorem**
$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} \text{ whenever } f'\left(f^{-1}(x)\right) \neq 0 \text{ and } f(x) \text{ is differentiable.}$$

- **Power rule with rational exponents**
$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n} x^{(m/n)-1}.$$

- **Derivative of inverse sine function**
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-(x)^2}}$$

- **Derivative of inverse cosine function**
$$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-(x)^2}}$$

- **Derivative of inverse tangent function**
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+(x)^2}$$

- **Derivative of inverse cotangent function**
$$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+(x)^2}$$

- **Derivative of inverse secant function**
$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{(x)^2-1}}$$

- **Derivative of inverse cosecant function**
$$\frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{(x)^2-1}}$$

- **Derivative of the natural exponential function**
$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x)$$

- **Derivative of the natural logarithmic function**
$$\frac{d}{dx}(\ln g(x)) = \frac{1}{g(x)} g'(x)$$

- **Derivative of the general exponential function**

$$\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)} g'(x) \ln b$$

- **Derivative of the general logarithmic function**
$$\frac{d}{dx}(\log_b g(x)) = \frac{g'(x)}{g(x) \ln b}$$

KEY CONCEPTS

3.1 Defining the Derivative

- The slope of the tangent line to a curve measures the instantaneous rate of change of a curve. We can calculate it by finding the limit of the difference quotient or the difference quotient with increment h.
- The derivative of a function $f(x)$ at a value a is found using either of the definitions for the slope of the tangent line.
- Velocity is the rate of change of position. As such, the velocity $v(t)$ at time t is the derivative of the position $s(t)$ at time t. Average velocity is given by

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}.$$

Instantaneous velocity is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}.$$

- We may estimate a derivative by using a table of values.

3.2 The Derivative as a Function

- The derivative of a function $f(x)$ is the function whose value at x is $f'(x)$.
- The graph of a derivative of a function $f(x)$ is related to the graph of $f(x)$. Where $f(x)$ has a tangent line with positive slope, $f'(x) > 0$. Where $f(x)$ has a tangent line with negative slope, $f'(x) < 0$. Where $f(x)$ has a horizontal tangent line, $f'(x) = 0$.
- If a function is differentiable at a point, then it is continuous at that point. A function is not differentiable at a point if it is not continuous at the point, if it has a vertical tangent line at the point, or if the graph has a sharp corner or cusp.
- Higher-order derivatives are derivatives of derivatives, from the second derivative to the nth derivative.

3.3 Differentiation Rules

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant *c* multiplied by a function *f* is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function *f* and a function *g* is the same as the sum of the derivative of *f* and the derivative of *g*.
- The derivative of the difference of a function *f* and a function *g* is the same as the difference of the derivative of *f* and the derivative of *g*.
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus

the derivative of the second function times the first function, all divided by the square of the second function.

- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

3.4 Derivatives as Rates of Change

- Using $f(a+h) \approx f(a) + f'(a)h,$ it is possible to estimate $f(a+h)$ given $f'(a)$ and $f(a)$.
- The rate of change of position is velocity, and the rate of change of velocity is acceleration. Speed is the absolute value, or magnitude, of velocity.
- The population growth rate and the present population can be used to predict the size of a future population.
- Marginal cost, marginal revenue, and marginal profit functions can be used to predict, respectively, the cost of producing one more item, the revenue obtained by selling one more item, and the profit obtained by producing and selling one more item.

3.5 Derivatives of Trigonometric Functions

- We can find the derivatives of sin x and cos x by using the definition of derivative and the limit formulas found earlier. The results are

$$\frac{d}{dx}\sin x = \cos x \frac{d}{dx}\cos x = -\sin x.$$

- With these two formulas, we can determine the derivatives of all six basic trigonometric functions.

3.6 The Chain Rule

- The chain rule allows us to differentiate compositions of two or more functions. It states that for $h(x) = f(g(x)),$

$$h'(x) = f'(g(x))g'(x).$$

In Leibniz's notation this rule takes the form

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- We can use the chain rule with other rules that we have learned, and we can derive formulas for some of them.
- The chain rule combines with the power rule to form a new rule:

$$\text{If } h(x) = (g(x))^n, \text{ then } h'(x) = n(g(x))^{n-1} g'(x).$$

- When applied to the composition of three functions, the chain rule can be expressed as follows: If $h(x) = f(g(k(x))),$ then $h'(x) = f'(g(k(x))g'(k(x))k'(x).$

3.7 Derivatives of Inverse Functions

- The inverse function theorem allows us to compute derivatives of inverse functions without using the limit definition of the derivative.
- We can use the inverse function theorem to develop differentiation formulas for the inverse trigonometric functions.

3.8 Implicit Differentiation

- We use implicit differentiation to find derivatives of implicitly defined functions (functions defined by equations).
- By using implicit differentiation, we can find the equation of a tangent line to the graph of a curve.

3.9 Derivatives of Exponential and Logarithmic Functions

- On the basis of the assumption that the exponential function $y = b^x, b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.

- We can use a formula to find the derivative of $y = \ln x,$ and the relationship $\log_b x = \frac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.
- Logarithmic differentiation allows us to differentiate functions of the form $y = g(x)^{f(x)}$ or very complex functions by taking the natural logarithm of both sides and exploiting the properties of logarithms before differentiating.

CHAPTER 3 REVIEW EXERCISES

True or False? Justify the answer with a proof or a counterexample.

367. Every function has a derivative.

368. A continuous function has a continuous derivative.

369. A continuous function has a derivative.

370. If a function is differentiable, it is continuous.

Use the limit definition of the derivative to exactly evaluate the derivative.

371. $f(x) = \sqrt{x+4}$

372. $f(x) = \frac{3}{x}$

Find the derivatives of the following functions.

373. $f(x) = 3x^3 - \frac{4}{x^2}$

374. $f(x) = \left(4 - x^2\right)^3$

375. $f(x) = e^{\sin x}$

376. $f(x) = \ln(x+2)$

377. $f(x) = x^2 \cos x + x \tan(x)$

378. $f(x) = \sqrt{3x^2 + 2}$

379. $f(x) = \frac{x}{4}\sin^{-1}(x)$

380. $x^2 y = (y+2) + xy\sin(x)$

Find the following derivatives of various orders.

381. First derivative of $y = x\ln(x)\cos x$

382. Third derivative of $y = (3x+2)^2$

383. Second derivative of $y = 4^x + x^2\sin(x)$

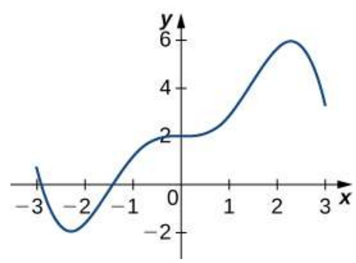
Find the equation of the tangent line to the following equations at the specified point.

384. $y = \cos^{-1}(x) + x$ at $x = 0$

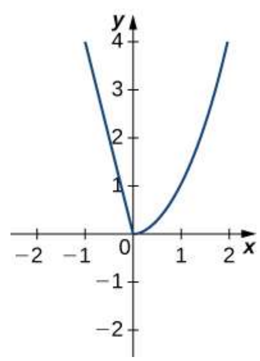
385. $y = x + e^x - \frac{1}{x}$ at $x = 1$

Draw the derivative for the following graphs.

386.



387.



The following questions concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t) = 1.9 + 2.9\cos\left(\frac{\pi}{6}t\right),$ where t is measured in hours after midnight, and the height is measured in feet.

388. Find and graph the derivative. What is the physical meaning?

389. Find $w'(3)$. What is the physical meaning of this value?

The following questions consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

Table 3.9 Wind Speeds of Hurricane Katrina **Source: http://news.nationalgeographic.com/news/2005/09/0914_050914_katrina_timeline.html.**

390. Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

391. Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

4 | APPLICATIONS OF DERIVATIVES

Figure 4.1 As a rocket is being launched, at what rate should the angle of a video camera change to continue viewing the rocket? (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

Chapter Outline

Introduction

A rocket is being launched from the ground and cameras are recording the event. A video camera is located on the ground a certain distance from the launch pad. At what rate should the angle of inclination (the angle the camera makes with the ground) change to allow the camera to record the flight of the rocket as it heads upward? (See **Example 4.3**.)

A rocket launch involves two related quantities that change over time. Being able to solve this type of problem is just one application of derivatives introduced in this chapter. We also look at how derivatives are used to find maximum and minimum values of functions. As a result, we will be able to solve applied optimization problems, such as maximizing revenue and minimizing surface area. In addition, we examine how derivatives are used to evaluate complicated limits, to approximate roots of functions, and to provide accurate graphs of functions.

4.1 | Related Rates

Learning Objectives

4.1.1 Express changing quantities in terms of derivatives.
4.1.2 Find relationships among the derivatives in a given problem.
4.1.3 Use the chain rule to find the rate of change of one quantity that depends on the rate of change of other quantities.

We have seen that for quantities that are changing over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related. For example, if a balloon is being filled with air, both the radius of the balloon and the volume of the balloon are increasing. In this section, we consider several problems in which two or more related quantities are changing and we study how to determine the relationship between the rates of change of these quantities.

Setting up Related-Rates Problems

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, V, is related to the rate of change in the radius, r. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r. Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

Example 4.1

Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\text{ cm}^3\text{/sec}$ (**Figure 4.2**). How fast is the radius increasing when the radius is 3 cm?

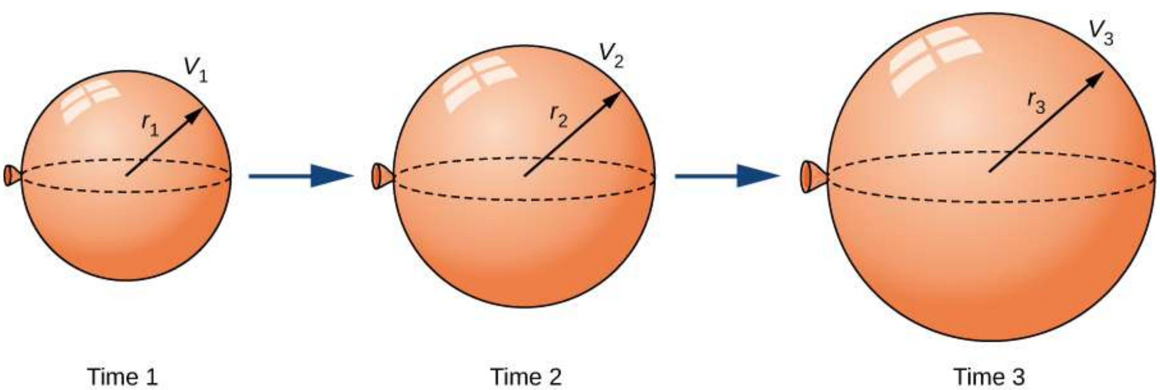


Figure 4.2 As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Solution

The volume of a sphere of radius r centimeters is

$$V = \frac{4}{3}\pi r^3 \text{ cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

$$V(t) = \frac{4}{3}\pi[r(t)]^3 \text{ cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

$$V'(t) = 4\pi[r(t)]^2 r'(t).$$

The balloon is being filled with air at the constant rate of 2 cm^3/sec, so $V'(t) = 2 \text{ cm}^3\text{/sec}$. Therefore,

$$2\text{cm}^3\text{/sec} = \left(4\pi[r(t)]^2 \text{ cm}^2\right) \cdot (r'(t)\text{cm/s}),$$

which implies

$$r'(t) = \frac{1}{2\pi[r(t)]^2} \text{cm/sec}.$$

When the radius $r = 3$ cm,

$$r'(t) = \frac{1}{18\pi} \text{cm/sec}.$$

4.1 What is the instantaneous rate of change of the radius when $r = 6$ cm?

Before looking at other examples, let's outline the problem-solving strategy we will be using to solve related-rates problems.

Problem-Solving Strategy: Solving a Related-Rates Problem

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1.
4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

Note that when solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4. We examine this potential error in the following example.

Examples of the Process

Let's now implement the strategy just described to solve several related-rates problems. The first example involves a plane flying overhead. The relationship we are studying is between the speed of the plane and the rate at which the distance between the plane and a person on the ground is changing.

Example 4.2

An Airplane Flying at a Constant Elevation

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

Solution

Step 1. Draw a picture, introducing variables to represent the different quantities involved.

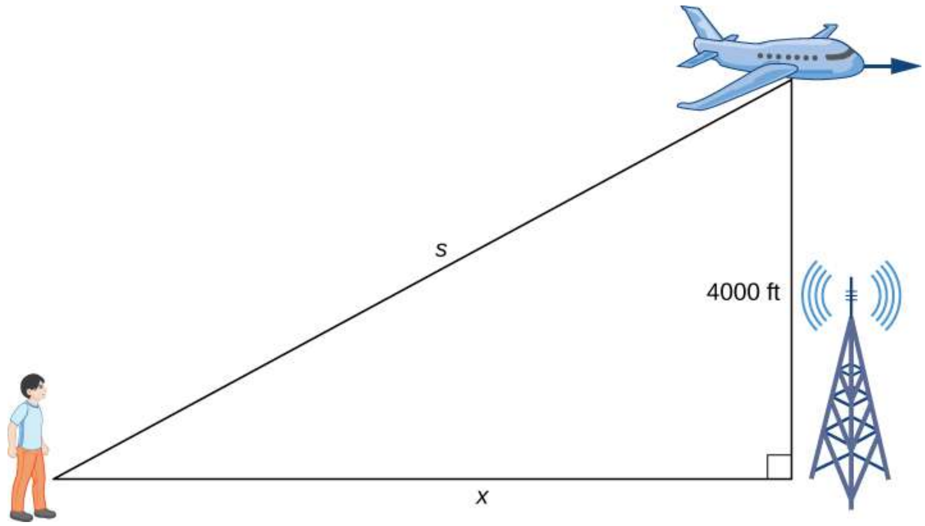


Figure 4.3 An airplane is flying at a constant height of 4000 ft. The distance between the person and the airplane and the person and the place on the ground directly below the airplane are changing. We denote those quantities with the variables s and x, respectively.

As shown, x denotes the distance between the man and the position on the ground directly below the airplane. The variable s denotes the distance between the man and the plane. Note that both x and s are functions of time. We do not introduce a variable for the height of the plane because it remains at a constant elevation of 4000 ft. Since an object's height above the ground is measured as the shortest distance between the object and the ground, the line segment of length 4000 ft is perpendicular to the line segment of length x feet, creating a right triangle.

Step 2. Since x denotes the horizontal distance between the man and the point on the ground below the plane, dx/dt represents the speed of the plane. We are told the speed of the plane is 600 ft/sec. Therefore, $\frac{dx}{dt} = 600$ ft/sec. Since we are asked to find the rate of change in the distance between the man and the plane when the plane is directly above the radio tower, we need to find ds/dt when $x = 3000$ ft.

Step 3. From the figure, we can use the Pythagorean theorem to write an equation relating x and s:

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Step 4. Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation

$$x\frac{dx}{dt} = s\frac{ds}{dt}.$$

Step 5. Find the rate at which the distance between the man and the plane is increasing when the plane is directly over the radio tower. That is, find $\frac{ds}{dt}$ when $x = 3000$ ft. Since the speed of the plane is 600 ft/sec, we know that $\frac{dx}{dt} = 600$ ft/sec. We are not given an explicit value for s; however, since we are trying to find $\frac{ds}{dt}$ when $x = 3000$ ft, we can use the Pythagorean theorem to determine the distance s when $x = 3000$ and the height is 4000 ft. Solving the equation

$$3000^2 + 4000^2 = s^2$$

for s, we have $s = 5000$ ft at the time of interest. Using these values, we conclude that ds/dt is a solution of the equation

$$(3000)(600) = (5000) \cdot \frac{ds}{dt}.$$

Therefore,

$$\frac{ds}{dt} = \frac{3000 \cdot 600}{5000} = 360 \text{ ft/sec.}$$

Note: When solving related-rates problems, it is important not to substitute values for the variables too soon. For example, in step 3, we related the variable quantities $x(t)$ and $s(t)$ by the equation

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Since the plane remains at a constant height, it is not necessary to introduce a variable for the height, and we are allowed to use the constant 4000 to denote that quantity. However, the other two quantities are changing. If we mistakenly substituted $x(t) = 3000$ into the equation before differentiating, our equation would have been

$$3000^2 + 4000^2 = [s(t)]^2.$$

After differentiating, our equation would become

$$0 = s(t)\frac{ds}{dt}.$$

As a result, we would incorrectly conclude that $\frac{ds}{dt} = 0.$

4.2 What is the speed of the plane if the distance between the person and the plane is increasing at the rate of 300 ft/sec?

We now return to the problem involving the rocket launch from the beginning of the chapter.

Example 4.3

Chapter Opener: A Rocket Launch

Figure 4.4 (credit: modification of work by Steve Jurvetson, Wikimedia Commons)

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is $600\text{ ft/sec}.$ Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

Solution

Step 1. Draw a picture introducing the variables.

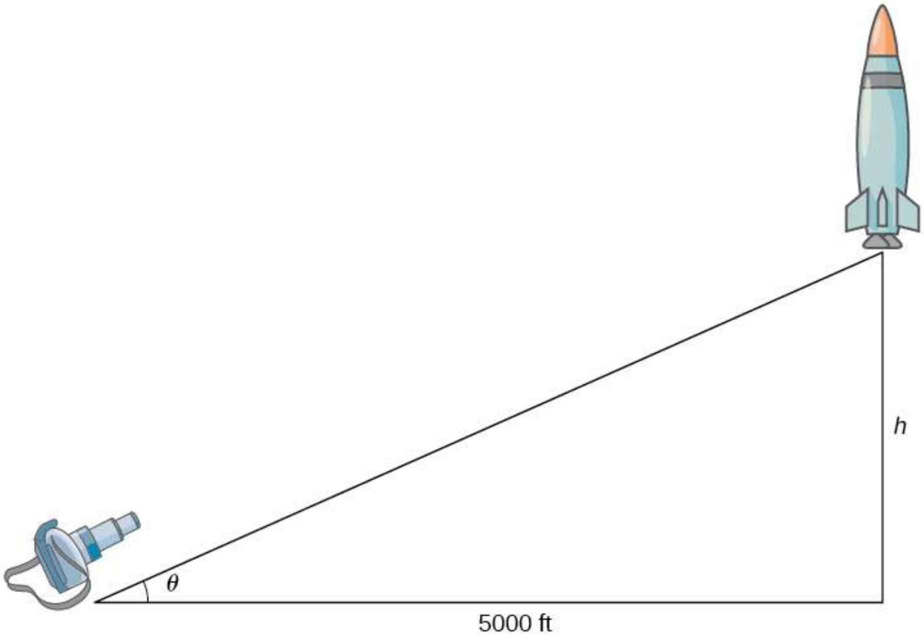


Figure 4.5 A camera is positioned 5000 ft from the launch pad of the rocket. The height of the rocket and the angle of the camera are changing with respect to time. We denote those quantities with the variables h and $\theta,$ respectively.

Let h denote the height of the rocket above the launch pad and θ be the angle between the camera lens and the ground.

Step 2. We are trying to find the rate of change in the angle of the camera with respect to time when the rocket is 1000 ft off the ground. That is, we need to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At that time, we know the velocity of the rocket is $\frac{dh}{dt} = 600$ ft/sec.

Step 3. Now we need to find an equation relating the two quantities that are changing with respect to time: h and θ. How can we create such an equation? Using the fact that we have drawn a right triangle, it is natural to think about trigonometric functions. Recall that $\tan\theta$ is the ratio of the length of the opposite side of the triangle to the length of the adjacent side. Thus, we have

$$\tan\theta = \frac{h}{5000}.$$

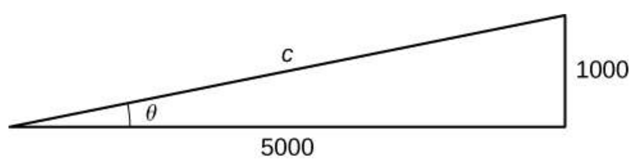
This gives us the equation

$$h = 5000\tan\theta.$$

Step 4. Differentiating this equation with respect to time t, we obtain

$$\frac{dh}{dt} = 5000\sec^2\theta\frac{d\theta}{dt}.$$

Step 5. We want to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At this time, we know that $\frac{dh}{dt} = 600$ ft/sec. We need to determine $\sec^2\theta$. Recall that $\sec\theta$ is the ratio of the length of the hypotenuse to the length of the adjacent side. We know the length of the adjacent side is 5000 ft. To determine the length of the hypotenuse, we use the Pythagorean theorem, where the length of one leg is 5000 ft, the length of the other leg is $h = 1000$ ft, and the length of the hypotenuse is c feet as shown in the following figure.



We see that

$$1000^2 + 5000^2 = c^2$$

and we conclude that the hypotenuse is

$$c = 1000\sqrt{26}\text{ ft}.$$

Therefore, when $h = 1000$, we have

$$\sec^2\theta = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

Recall from step 4 that the equation relating $\frac{d\theta}{dt}$ to our known values is

$$\frac{dh}{dt} = 5000\sec^2\theta\frac{d\theta}{dt}.$$

When $h = 1000$ ft, we know that $\frac{dh}{dt} = 600$ ft/sec and $\sec^2\theta = \frac{26}{25}$. Substituting these values into the previous equation, we arrive at the equation

$$600 = 5000\left(\frac{26}{25}\right)\frac{d\theta}{dt}.$$

Therefore, $\frac{d\theta}{dt} = \frac{3}{26}$ rad/sec.

4.3 What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

In the next example, we consider water draining from a cone-shaped funnel. We compare the rate at which the level of water in the cone is decreasing with the rate at which the volume of water is decreasing.

Example 4.4

Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\text{ ft}^3/\text{sec}$. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}\text{ ft}$?

Solution

Step 1: Draw a picture introducing the variables.

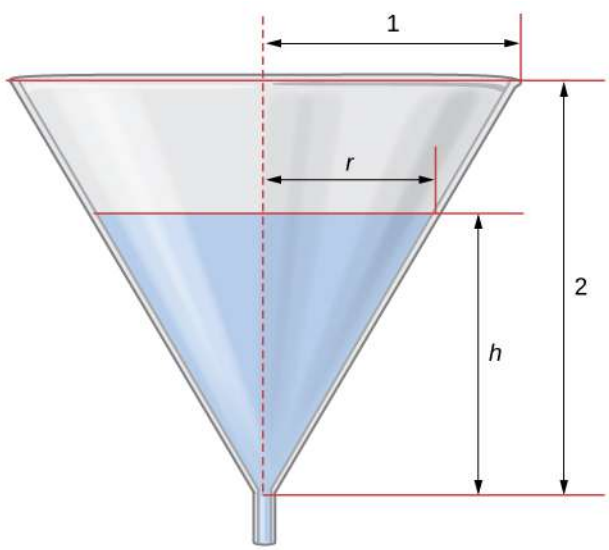


Figure 4.6 Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of water are changing over time. We denote these quantities with the variables h and r, respectively.

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that $\frac{dV}{dt} = -0.03$ ft/sec.

Step 3: The volume of water in the cone is

$$V = \frac{1}{3}\pi r^2 h.$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}.$ Using this fact, the equation for volume can be simplified to

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Step 4: Applying the chain rule while differentiating both sides of this equation with respect to time $t,$ we obtain

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03\text{ ft}^3/\text{sec},$ we know that $\frac{dV}{dt} = -0.03\text{ ft}^3/\text{sec}.$ Therefore,

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2\frac{dh}{dt},$$

which implies

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

It follows that

$$\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153\text{ ft/sec}.$$

4.4 At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

4.1 EXERCISES

For the following exercises, find the quantities for the given equation.

1. Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4.$

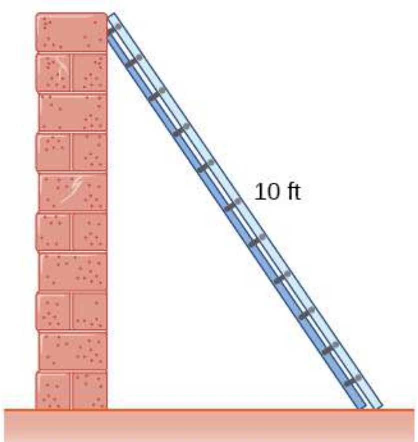
2. Find $\frac{dx}{dt}$ at $x = -2$ and $y = 2x^2 + 1$ if $\frac{dy}{dt} = -1.$

3. Find $\frac{dz}{dt}$ at $(x, y) = (1, 3)$ and $z^2 = x^2 + y^2$ if $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 3.$

For the following exercises, sketch the situation if necessary and used related rates to solve for the quantities.

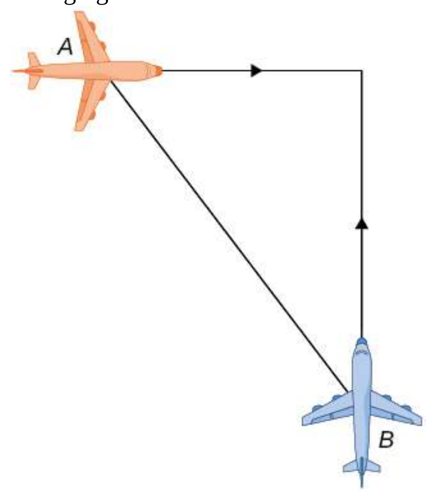
4. **[T]** If two electrical resistors are connected in parallel, the total resistance (measured in ohms, denoted by the Greek capital letter omega, Ω) is given by the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$ If R_1 is increasing at a rate of $0.5\ \Omega/\text{min}$ and R_2 decreases at a rate of $1.1\Omega/\text{min},$ at what rate does the total resistance change when $R_1 = 20\Omega$ and $R_2 = 50\Omega$?

5. A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?



6. A 25-ft ladder is leaning against a wall. If we push the ladder toward the wall at a rate of 1 ft/sec, and the bottom of the ladder is initially 20 ft away from the wall, how fast does the ladder move up the wall 5 sec after we start pushing?

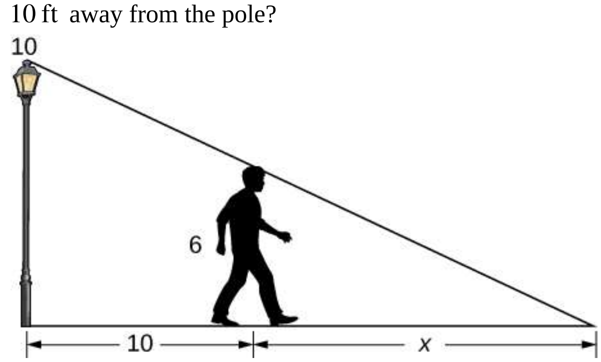
7. Two airplanes are flying in the air at the same height: airplane *A* is flying east at 250 mi/h and airplane *B* is flying north at $300\text{ mi/h}.$ If they are both heading to the same airport, located 30 miles east of airplane *A* and 40 miles north of airplane *B*, at what rate is the distance between the airplanes changing?



8. You and a friend are riding your bikes to a restaurant that you think is east; your friend thinks the restaurant is north. You both leave from the same point, with you riding at 16 mph east and your friend riding 12 mph north. After you traveled $4\text{ mi},$ at what rate is the distance between you changing?

9. Two buses are driving along parallel freeways that are 5 mi apart, one heading east and the other heading west. Assuming that each bus drives a constant $55\text{ mph},$ find the rate at which the distance between the buses is changing when they are 13 mi apart, heading toward each other.

10. A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of $3\text{ ft/sec}.$ What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?



11. Using the previous problem, what is the rate at which the tip of the shadow moves away from the person when the person is 10 ft from the pole?

12. A 5-ft-tall person walks toward a wall at a rate of 2 ft/sec. A spotlight is located on the ground 40 ft from the wall. How fast does the height of the person's shadow on the wall change when the person is 10 ft from the wall?

13. Using the previous problem, what is the rate at which the shadow changes when the person is 10 ft from the wall, if the person is walking away from the wall at a rate of 2 ft/sec?

14. A helicopter starting on the ground is rising directly into the air at a rate of 25 ft/sec. You are running on the ground starting directly under the helicopter at a rate of 10 ft/sec. Find the rate of change of the distance between the helicopter and yourself after 5 sec.

15. Using the previous problem, what is the rate at which the distance between you and the helicopter is changing when the helicopter has risen to a height of 60 ft in the air, assuming that, initially, it was 30 ft above you?

For the following exercises, draw and label diagrams to help solve the related-rates problems.

16. The side of a cube increases at a rate of $\frac{1}{2}$ m/sec. Find the rate at which the volume of the cube increases when the side of the cube is 4 m.

17. The volume of a cube decreases at a rate of 10 m^3/s. Find the rate at which the side of the cube changes when the side of the cube is 2 m.

18. The radius of a circle increases at a rate of 2 m/sec. Find the rate at which the area of the circle increases when the radius is 5 m.

19. The radius of a sphere decreases at a rate of 3 m/sec. Find the rate at which the surface area decreases when the radius is 10 m.

20. The radius of a sphere increases at a rate of 1 m/sec. Find the rate at which the volume increases when the radius is 20 m.

21. The radius of a sphere is increasing at a rate of 9 cm/sec. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate.

22. The base of a triangle is shrinking at a rate of 1 cm/min and the height of the triangle is increasing at a rate of 5 cm/min. Find the rate at which the area of the triangle changes when the height is 22 cm and the base is 10 cm.

23. A triangle has two constant sides of length 3 ft and 5 ft. The angle between these two sides is increasing at a rate of 0.1 rad/sec. Find the rate at which the area of the triangle is changing when the angle between the two sides is $\pi/6$.

24. A triangle has a height that is increasing at a rate of 2 cm/sec and its area is increasing at a rate of 4 cm^2/sec. Find the rate at which the base of the triangle is changing when the height of the triangle is 4 cm and the area is 20 cm^2.

For the following exercises, consider a right cone that is leaking water. The dimensions of the conical tank are a height of 16 ft and a radius of 5 ft.

25. How fast does the depth of the water change when the water is 10 ft high if the cone leaks water at a rate of 10 ft^3/min?

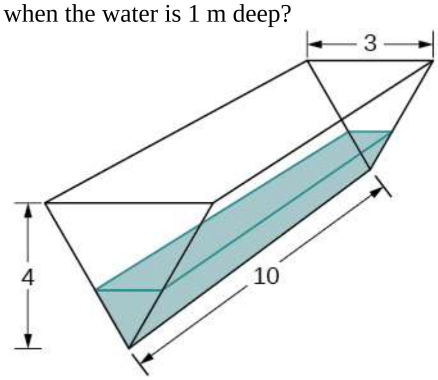
26. Find the rate at which the surface area of the water changes when the water is 10 ft high if the cone leaks water at a rate of 10 ft^3/min.

27. If the water level is decreasing at a rate of 3 in/min when the depth of the water is 8 ft, determine the rate at which water is leaking out of the cone.

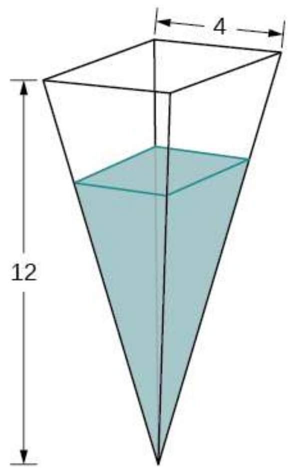
28. A vertical cylinder is leaking water at a rate of 1 ft^3/sec. If the cylinder has a height of 10 ft and a radius of 1 ft, at what rate is the height of the water changing when the height is 6 ft?

29. A cylinder is leaking water but you are unable to determine at what rate. The cylinder has a height of 2 m and a radius of 2 m. Find the rate at which the water is leaking out of the cylinder if the rate at which the height is decreasing is 10 cm/min when the height is 1 m.

30. A trough has ends shaped like isosceles triangles, with width 3 m and height 4 m, and the trough is 10 m long. Water is being pumped into the trough at a rate of $5\,m^3$/min. At what rate does the height of the water change when the water is 1 m deep?



31. A tank is shaped like an upside-down square pyramid, with base of 4 m by 4 m and a height of 12 m (see the following figure). How fast does the height increase when the water is 2 m deep if water is being pumped in at a rate of $\frac{2}{3}$ m/sec?



For the following problems, consider a pool shaped like the bottom half of a sphere, that is being filled at a rate of 25 ft^3/min. The radius of the pool is 10 ft.

32. Find the rate at which the depth of the water is changing when the water has a depth of 5 ft.

33. Find the rate at which the depth of the water is changing when the water has a depth of 1 ft.

34. If the height is increasing at a rate of 1 in./sec when the depth of the water is 2 ft, find the rate at which water is being pumped in.

35. Gravel is being unloaded from a truck and falls into a pile shaped like a cone at a rate of 10 ft^3/min. The radius of the cone base is three times the height of the cone. Find the rate at which the height of the gravel changes when the pile has a height of 5 ft.

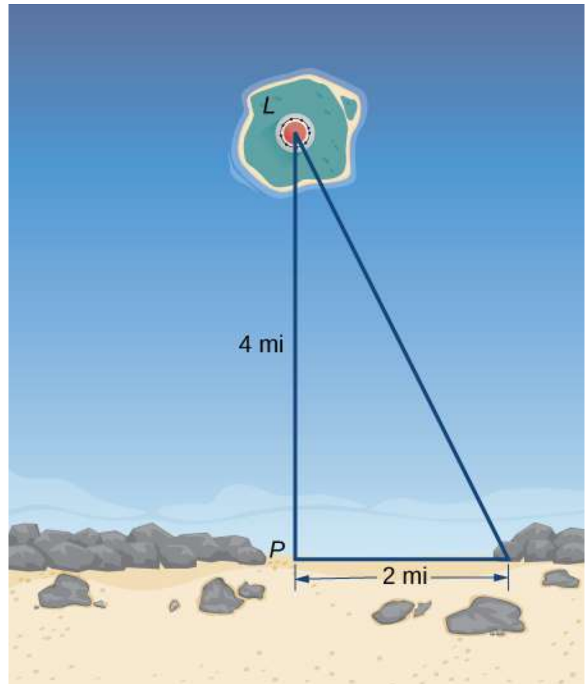
36. Using a similar setup from the preceding problem, find the rate at which the gravel is being unloaded if the pile is 5 ft high and the height is increasing at a rate of 4 in./min.

For the following exercises, draw the situations and solve the related-rate problems.

37. You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec. The bird is located 40 m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?

38. You stand 40 ft from a bottle rocket on the ground and watch as it takes off vertically into the air at a rate of 20 ft/sec. Find the rate at which the angle of elevation changes when the rocket is 30 ft in the air.

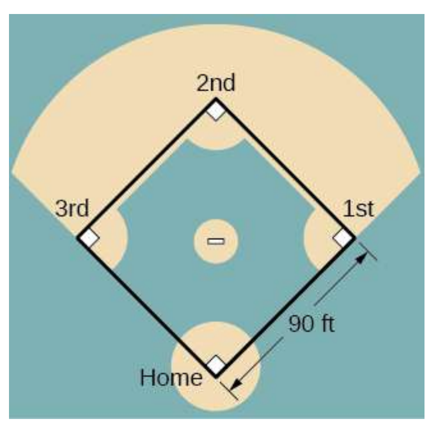
39. A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach (see the following image). If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?



40. Using the same setup as the previous problem, determine at what rate the beam of light moves across the beach 1 mi away from the closest point on the beach.

41. You are walking to a bus stop at a right-angle corner. You move north at a rate of 2 m/sec and are 20 m south of the intersection. The bus travels west at a rate of 10 m/sec away from the intersection – you have missed the bus! What is the rate at which the angle between you and the bus is changing when you are 20 m south of the intersection and the bus is 10 m west of the intersection?

For the following exercises, refer to the figure of baseball diamond, which has sides of 90 ft.



42. **[T]** A batter hits a ball toward third base at 75 ft/sec and runs toward first base at a rate of 24 ft/sec. At what rate does the distance between the ball and the batter change when 2 sec have passed?

43. **[T]** A batter hits a ball toward second base at 80 ft/sec and runs toward first base at a rate of 30 ft/sec. At what rate does the distance between the ball and the batter change when the runner has covered one-third of the distance to first base? (*Hint*: Recall the law of cosines.)

44. **[T]** A batter hits the ball and runs toward first base at a speed of 22 ft/sec. At what rate does the distance between the runner and second base change when the runner has run 30 ft?

45. **[T]** Runners start at first and second base. When the baseball is hit, the runner at first base runs at a speed of 18 ft/sec toward second base and the runner at second base runs at a speed of 20 ft/sec toward third base. How fast is the distance between runners changing 1 sec after the ball is hit?

4.2 | Linear Approximations and Differentials

Learning Objectives

4.2.1 Describe the linear approximation to a function at a point.
4.2.2 Write the linearization of a given function.
4.2.3 Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
4.2.4 Calculate the relative error and percentage error in using a differential approximation.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials **Introduction to Power Series and Functions (http://cnx.org/content/m53760/latest/)** .

Linear Approximation of a Function at a Point

Consider a function f that is differentiable at a point $x = a$. Recall that the tangent line to the graph of f at a is given by the equation

$$y = f(a) + f'(a)(x - a).$$

For example, consider the function $f(x) = \frac{1}{x}$ at $a = 2$. Since f is differentiable at $x = 2$ and $f'(x) = -\frac{1}{x^2}$, we see that $f'(2) = -\frac{1}{4}$. Therefore, the tangent line to the graph of f at $a = 2$ is given by the equation

$$y = \frac{1}{2} - \frac{1}{4}(x - 2).$$

Figure 4.7(a) shows a graph of $f(x) = \frac{1}{x}$ along with the tangent line to f at $x = 2$. Note that for x near 2, the graph of the tangent line is close to the graph of f. As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near 2. For example, if $x = 2.1$, the y value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.475.$$

The actual value of $f(2.1)$ is given by

$$f(2.1) = \frac{1}{2.1} \approx 0.47619.$$

Therefore, the tangent line gives us a fairly good approximation of $f(2.1)$ (**Figure 4.7**(b)). However, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if $x = 10$, the y -value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(10 - 2) = \frac{1}{2} - 2 = -1.5,$$

whereas the value of the function at $x = 10$ is $f(10) = 0.1$.

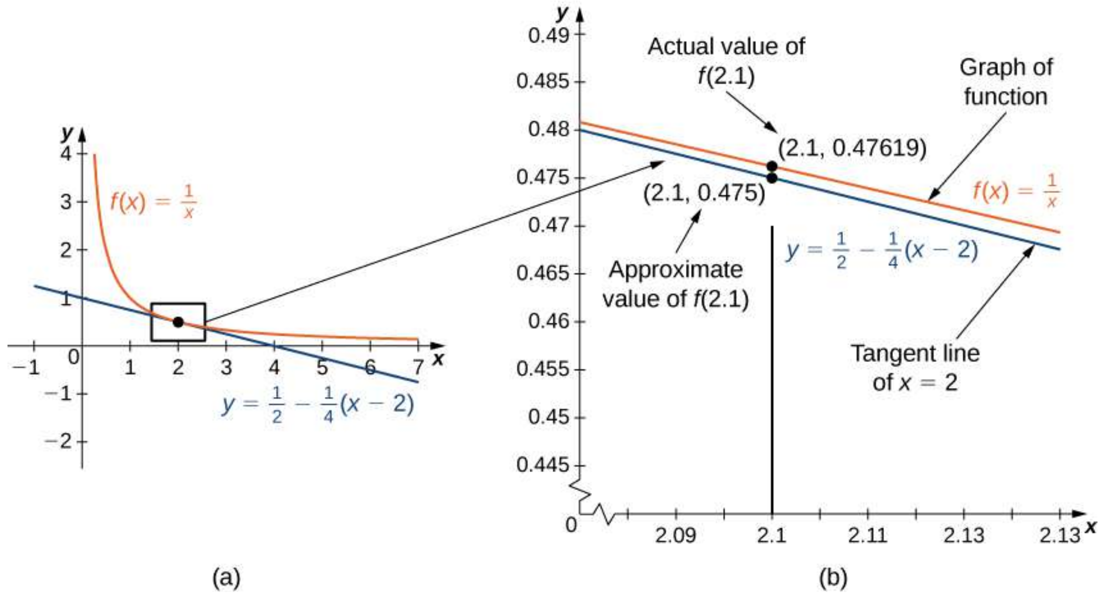


Figure 4.7 (a) The tangent line to $f(x) = 1/x$ at $x = 2$ provides a good approximation to f for x near 2. (b) At $x = 2.1,$ the value of y on the tangent line to $f(x) = 1/x$ is 0.475. The actual value of $f(2.1)$ is $1/2.1,$ which is approximately 0.47619.

In general, for a differentiable function $f,$ the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near $a.$ Therefore, we can write

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

We call the linear function

$$L(x) = f(a) + f'(a)(x - a) \tag{4.1}$$

the **linear approximation**, or **tangent line approximation**, of f at $x = a.$ This function L is also known as the **linearization** of f at $x = a.$

To show how useful the linear approximation can be, we look at how to find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9.$

Example 4.5

Linear Approximation of $\sqrt{x}$

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}.$

Solution

Since we are looking for the linear approximation at $x = 9,$ using **Equation 4.1** we know the linear approximation is given by

$$L(x) = f(9) + f'(9)(x - 9).$$

We need to find $f(9)$ and $f'(9).$

$$f(x) = \sqrt{x} \quad \Rightarrow \quad f(9) = \sqrt{9} = 3$$
$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Therefore, the linear approximation is given by **Figure 4.8**.

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$

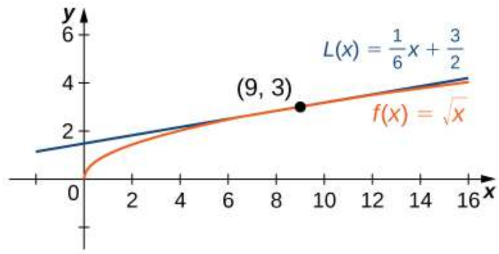


Figure 4.8 The local linear approximation to $f(x) = \sqrt{x}$ at $x = 9$ provides an approximation to f for x near 9.

Analysis

Using a calculator, the value of $\sqrt{9.1}$ to four decimal places is 3.0166. The value given by the linear approximation, 3.0167, is very close to the value obtained with a calculator, so it appears that using this linear approximation is a good way to estimate $\sqrt{x}$, at least for x near 9. At the same time, it may seem odd to use a linear approximation when we can just push a few buttons on a calculator to evaluate $\sqrt{9.1}$. However, how does the calculator evaluate $\sqrt{9.1}$? The calculator uses an approximation! In fact, calculators and computers use approximations all the time to evaluate mathematical expressions; they just use higher-degree approximations.

4.5 Find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$. Use it to approximate $\sqrt[3]{8.1}$ to five decimal places.

Example 4.6

Linear Approximation of $\sin x$

Find the linear approximation of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ and use it to approximate $\sin(62°)$.

Solution

First we note that since $\frac{\pi}{3}$ rad is equivalent to $60°$, using the linear approximation at $x = \pi/3$ seems reasonable. The linear approximation is given by

$$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right).$$

We see that

$$f(x) = \sin x \quad \Rightarrow \quad f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$f'(x) = \cos x \quad \Rightarrow \quad f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Therefore, the linear approximation of f at $x = \pi/3$ is given by **Figure 4.9**.

$$L(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

To estimate $\sin(62°)$ using $L,$ we must first convert $62°$ to radians. We have $62° = \frac{62\pi}{180}$ radians, so the estimate for $\sin(62°)$ is given by

$$\sin(62°) = f\left(\frac{62\pi}{180}\right) \approx L\left(\frac{62\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{62\pi}{180} - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{2\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{180} \approx 0.88348.$$

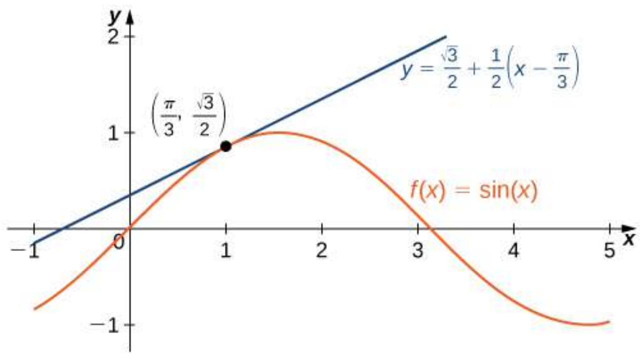


Figure 4.9 The linear approximation to $f(x) = \sin x$ at $x = \pi/3$ provides an approximation to $\sin x$ for x near $\pi/3$.

4.6 Find the linear approximation for $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

Linear approximations may be used in estimating roots and powers. In the next example, we find the linear approximation for $f(x) = (1 + x)^n$ at $x = 0,$ which can be used to estimate roots and powers for real numbers near 1. The same idea can be extended to a function of the form $f(x) = (m + x)^n$ to estimate roots and powers near a different number m.

Example 4.7

Approximating Roots and Powers

Find the linear approximation of $f(x) = (1 + x)^n$ at $x = 0.$ Use this approximation to estimate $(1.01)^3$.

Solution

The linear approximation at $x = 0$ is given by

$$L(x) = f(0) + f'(0)(x - 0).$$

Because

$$f(x) = (1+x)^n \quad \Rightarrow \quad f(0) = 1$$
$$f'(x) = n(1+x)^{n-1} \quad \Rightarrow \quad f'(0) = n,$$

the linear approximation is given by **Figure 4.10**(a).

$$L(x) = 1 + n(x - 0) = 1 + nx$$

We can approximate $(1.01)^3$ by evaluating $L(0.01)$ when $n = 3.$ We conclude that

$$(1.01)^3 = f(1.01) \approx L(1.01) = 1 + 3(0.01) = 1.03.$$

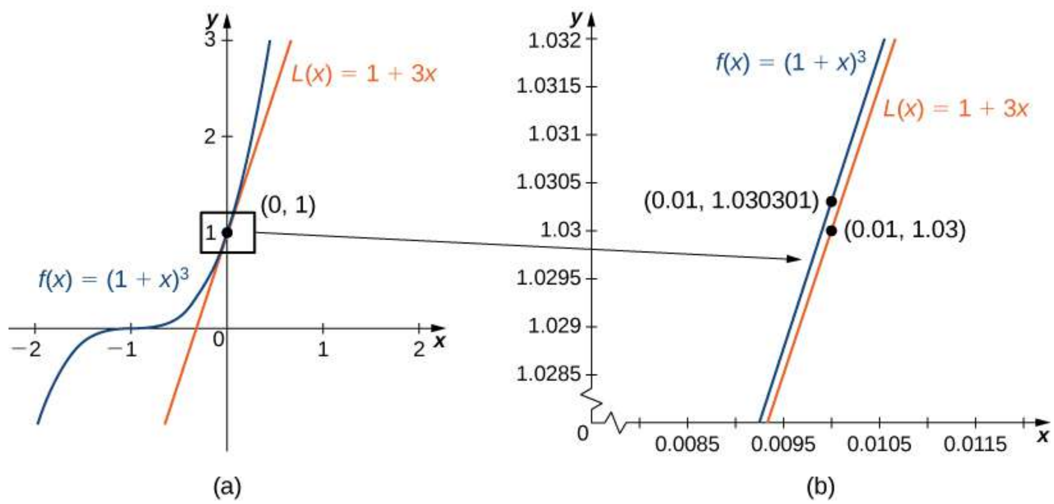


Figure 4.10 (a) The linear approximation of $f(x)$ at $x = 0$ is $L(x)$. (b) The actual value of 1.01^3 is 1.030301. The linear approximation of $f(x)$ at $x = 0$ estimates 1.01^3 to be 1.03.

4.7 Find the linear approximation of $f(x) = (1+x)^4$ at $x = 0$ without using the result from the preceding example.

Differentials

We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input. To discuss this more formally, we define a related concept: **differentials**. Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.

When we first looked at derivatives, we used the Leibniz notation dy/dx to represent the derivative of y with respect to x. Although we used the expressions *dy* and *dx* in this notation, they did not have meaning on their own. Here we see a meaning to the expressions *dy* and *dx*. Suppose $y = f(x)$ is a differentiable function. Let *dx* be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$dy = f'(x)dx. \tag{4.2}$$

It is important to notice that dy is a function of both x and dx. The expressions *dy* and *dx* are called *differentials*. We can divide both sides of **Equation 4.2** by $dx,$ which yields

$$\frac{dy}{dx} = f'(x). \tag{4.3}$$

This is the familiar expression we have used to denote a derivative. **Equation 4.2** is known as the **differential form** of **Equation 4.3**.

Example 4.8

Computing differentials

For each of the following functions, find *dy* and evaluate when $x = 3$ and $dx = 0.1.$

a. $y = x^2 + 2x$

b. $y = \cos x$

Solution

The key step is calculating the derivative. When we have that, we can obtain *dy* directly.

a. Since $f(x) = x^2 + 2x,$ we know $f'(x) = 2x + 2,$ and therefore

$$dy = (2x + 2)dx.$$

When $x = 3$ and $dx = 0.1,$

$$dy = (2 \cdot 3 + 2)(0.1) = 0.8.$$

b. Since $f(x) = \cos x,$ $f'(x) = -\sin(x).$ This gives us

$$dy = -\sin x\, dx.$$

When $x = 3$ and $dx = 0.1,$

$$dy = -\sin(3)(0.1) = -0.1 \sin(3).$$

4.8 For $y = e^{x^2},$ find $dy.$

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point $a.$ Suppose the input x changes by a small amount. We are interested in how much the output y changes. If x changes from a to $a + dx,$ then the change in x is dx (also denoted Δx), and the change in y is given by

$$\Delta y = f(a + dx) - f(a).$$

Instead of calculating the exact change in $y,$ however, it is often easier to approximate the change in y by using a linear approximation. For x near $a,$ $f(x)$ can be approximated by the linear approximation

$$L(x) = f(a) + f'(a)(x - a).$$

Therefore, if dx is small,

$$f(a + dx) \approx L(a + dx) = f(a) + f'(a)(a + dx - a).$$

That is,

$$f(a+dx) - f(a) \approx L(a+dx) - f(a) = f'(a)dx.$$

In other words, the actual change in the function f if x increases from a to $a+dx$ is approximately the difference between $L(a+dx)$ and $f(a)$, where $L(x)$ is the linear approximation of f at a. By definition of $L(x)$, this difference is equal to $f'(a)dx$. In summary,

$$\Delta y = f(a+dx) - f(a) \approx L(a+dx) - f(a) = f'(a)dx = dy.$$

Therefore, we can use the differential $dy = f'(a)dx$ to approximate the change in y if x increases from $x=a$ to $x=a+dx$. We can see this in the following graph.

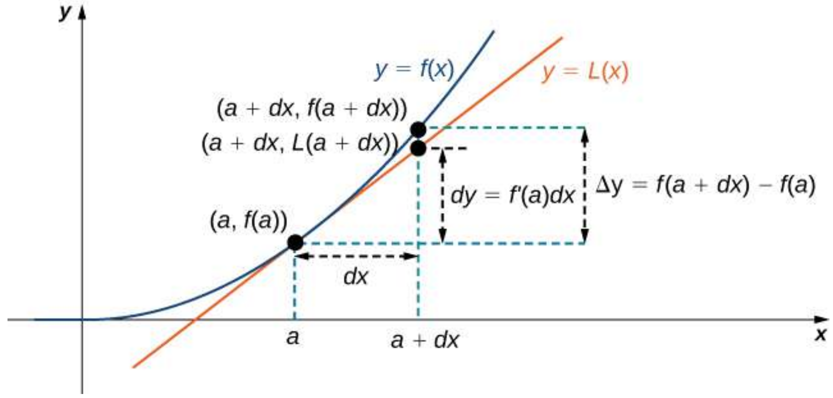


Figure 4.11 The differential $dy = f'(a)dx$ is used to approximate the actual change in y if x increases from a to $a+dx$.

We now take a look at how to use differentials to approximate the change in the value of the function that results from a small change in the value of the input. Note the calculation with differentials is much simpler than calculating actual values of functions and the result is very close to what we would obtain with the more exact calculation.

Example 4.9

Approximating Change with Differentials

Let $y = x^2 + 2x$. Compute Δy and dy at $x = 3$ if $dx = 0.1$.

Solution

The actual change in y if x changes from $x=3$ to $x=3.1$ is given by

$$\Delta y = f(3.1) - f(3) = [(3.1)^2 + 2(3.1)] - [3^2 + 2(3)] = 0.81.$$

The approximate change in y is given by $dy = f'(3)dx$. Since $f'(x) = 2x+2$, we have

$$dy = f'(3)dx = (2(3)+2)(0.1) = 0.8.$$

4.9 For $y = x^2 + 2x$, find Δy and dy at $x=3$ if $dx = 0.2$.

Calculating the Amount of Error

Any type of measurement is prone to a certain amount of error. In many applications, certain quantities are calculated based on measurements. For example, the area of a circle is calculated by measuring the radius of the circle. An error in the measurement of the radius leads to an error in the computed value of the area. Here we examine this type of error and study how differentials can be used to estimate the error.

Consider a function f with an input that is a measured quantity. Suppose the exact value of the measured quantity is a, but the measured value is $a + dx$. We say the measurement error is dx (or Δx). As a result, an error occurs in the calculated quantity $f(x)$. This type of error is known as a **propagated error** and is given by

$$\Delta y = f(a + dx) - f(a).$$

Since all measurements are prone to some degree of error, we do not know the exact value of a measured quantity, so we cannot calculate the propagated error exactly. However, given an estimate of the accuracy of a measurement, we can use differentials to approximate the propagated error Δy. Specifically, if f is a differentiable function at a, the propagated error is

$$\Delta y \approx dy = f'(a)dx.$$

Unfortunately, we do not know the exact value a. However, we can use the measured value $a + dx$, and estimate

$$\Delta y \approx dy \approx f'(a + dx)dx.$$

In the next example, we look at how differentials can be used to estimate the error in calculating the volume of a box if we assume the measurement of the side length is made with a certain amount of accuracy.

Example 4.10

Volume of a Cube

Suppose the side length of a cube is measured to be 5 cm with an accuracy of 0.1 cm.

a. Use differentials to estimate the error in the computed volume of the cube.
b. Compute the volume of the cube if the side length is (i) 4.9 cm and (ii) 5.1 cm to compare the estimated error with the actual potential error.

Solution

a. The measurement of the side length is accurate to within ± 0.1 cm. Therefore,

$$-0.1 \le dx \le 0.1.$$

The volume of a cube is given by $V = x^3$, which leads to

$$dV = 3x^2\,dx.$$

Using the measured side length of 5 cm, we can estimate that

$$-3(5)^2(0.1) \le dV \le 3(5)^2(0.1).$$

Therefore,

$$-7.5 \le dV \le 7.5.$$

b. If the side length is actually 4.9 cm, then the volume of the cube is

$$V(4.9) = (4.9)^3 = 117.649\ \text{cm}^3.$$

If the side length is actually 5.1 cm, then the volume of the cube is

$$V(5.1) = (5.1)^3 = 132.651 \text{ cm}^3.$$

Therefore, the actual volume of the cube is between 117.649 and 132.651. Since the side length is measured to be 5 cm, the computed volume is $V(5) = 5^3 = 125.$ Therefore, the error in the computed volume is

$$117.649 - 125 \le \Delta V \le 132.651 - 125.$$

That is,

$$-7.351 \le \Delta V \le 7.651.$$

We see the estimated error dV is relatively close to the actual potential error in the computed volume.

4.10 Estimate the error in the computed volume of a cube if the side length is measured to be 6 cm with an accuracy of 0.2 cm.

The measurement error dx $(=\Delta x)$ and the propagated error Δy are absolute errors. We are typically interested in the size of an error relative to the size of the quantity being measured or calculated. Given an absolute error Δq for a particular quantity, we define the **relative error** as $\frac{\Delta q}{q},$ where q is the actual value of the quantity. The **percentage error** is the relative error expressed as a percentage. For example, if we measure the height of a ladder to be 63 in. when the actual height is 62 in., the absolute error is 1 in. but the relative error is $\frac{1}{62} = 0.016,$ or $1.6\%.$ By comparison, if we measure the width of a piece of cardboard to be 8.25 in. when the actual width is 8 in., our absolute error is $\frac{1}{4}$ in., whereas the relative error is $\frac{0.25}{8} = \frac{1}{32},$ or $3.1\%.$ Therefore, the percentage error in the measurement of the cardboard is larger, even though 0.25 in. is less than 1 in.

Example 4.11

Relative and Percentage Error

An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ± 80 mi. Let's use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

Solution

If the measurement of the radius is accurate to within $\pm 80,$ we have

$$-80 \le dr \le 80.$$

Since the volume of a sphere is given by $V = \left(\frac{4}{3}\right)\pi r^3,$ we have

$$dV = 4\pi r^2 dr.$$

Using the measured radius of 4000 mi, we can estimate

$$-4\pi(4000)^2(80) \le dV \le 4\pi(4000)^2(80).$$

To estimate the relative error, consider $\frac{dV}{V}$. Since we do not know the exact value of the volume V, use the measured radius $r = 4000$ mi to estimate V. We obtain $V \approx \left(\frac{4}{3}\right)\pi(4000)^3$. Therefore the relative error satisfies

$$\frac{-4\pi(4000)^2(80)}{4\pi(4000)^3/3} \le \frac{dV}{V} \le \frac{4\pi(4000)^2(80)}{4\pi(4000)^3/3},$$

which simplifies to

$$-0.06 \le \frac{dV}{V} \le 0.06.$$

The relative error is 0.06 and the percentage error is 6%.

4.11 Determine the percentage error if the radius of Earth is measured to be 3950 mi with an error of ± 100 mi.

4.2 EXERCISES

46. What is the linear approximation for any generic linear function $y = mx + b$?

47. Determine the necessary conditions such that the linear approximation function is constant. Use a graph to prove your result.

48. Explain why the linear approximation becomes less accurate as you increase the distance between x and a. Use a graph to prove your argument.

49. When is the linear approximation exact?

For the following exercises, find the linear approximation $L(x)$ to $y = f(x)$ near $x = a$ for the function.

50. $f(x) = x + x^4,\ a = 0$

51. $f(x) = \frac{1}{x},\ a = 2$

52. $f(x) = \tan x,\ a = \frac{\pi}{4}$

53. $f(x) = \sin x,\ a = \frac{\pi}{2}$

54. $f(x) = x \sin x,\ a = 2\pi$

55. $f(x) = \sin^2 x,\ a = 0$

For the following exercises, compute the values given within 0.01 by deciding on the appropriate $f(x)$ and a, and evaluating $L(x) = f(a) + f'(a)(x - a)$. Check your answer using a calculator.

56. **[T]** $(2.001)^6$

57. **[T]** $\sin(0.02)$

58. **[T]** $\cos(0.03)$

59. **[T]** $(15.99)^{1/4}$

60. **[T]** $\frac{1}{0.98}$

61. **[T]** $\sin(3.14)$

For the following exercises, determine the appropriate $f(x)$ and a, and evaluate $L(x) = f(a) + f'(a)(x - a)$. Calculate the numerical error in the linear approximations that follow.

62. **[T]** $(1.01)^3$

63. **[T]** $\cos(0.01)$

64. **[T]** $(\sin(0.01))^2$

65. **[T]** $(1.01)^{-3}$

66. **[T]** $\left(1 + \frac{1}{10}\right)^{10}$

67. **[T]** $\sqrt{8.99}$

For the following exercises, find the differential of the function.

68. $y = 3x^4 + x^2 - 2x + 1$

69. $y = x \cos x$

70. $y = \sqrt{1 + x}$

71. $y = \frac{x^2 + 2}{x - 1}$

For the following exercises, find the differential and evaluate for the given x and dx.

72. $y = 3x^2 - x + 6, \quad x = 2, \quad dx = 0.1$

73. $y = \frac{1}{x + 1}, \quad x = 1, \quad dx = 0.25$

74. $y = \tan x, \quad x = 0, \quad dx = \frac{\pi}{10}$

75. $y = \frac{3x^2 + 2}{\sqrt{x + 1}}, \quad x = 0, \quad dx = 0.1$

76. $y = \frac{\sin(2x)}{x}, \quad x = \pi, \quad dx = 0.25$

77. $y = x^3 + 2x + \frac{1}{x}, \quad x = 1, \quad dx = 0.05$

For the following exercises, find the change in volume dV or in surface area dA.

78. dV if the sides of a cube change from 10 to 10.1.

79. dA if the sides of a cube change from x to $x + dx$.

80. dA if the radius of a sphere changes from r by dr.

81. dV if the radius of a sphere changes from r by dr.

82. dV if a circular cylinder with $r = 2$ changes height from 3 cm to $3.05\,\text{cm}$.

83. dV if a circular cylinder of height 3 changes from $r = 2$ to $r = 1.9\,\text{cm}$.

For the following exercises, use differentials to estimate the maximum and relative error when computing the surface area or volume.

84. A spherical golf ball is measured to have a radius of $5\,\text{mm},$ with a possible measurement error of $0.1\,\text{mm}.$ What is the possible change in volume?

85. A pool has a rectangular base of 10 ft by 20 ft and a depth of 6 ft. What is the change in volume if you only fill it up to 5.5 ft?

86. An ice cream cone has height 4 in. and radius 1 in. If the cone is 0.1 in. thick, what is the difference between the volume of the cone, including the shell, and the volume of the ice cream you can fit inside the shell?

For the following exercises, confirm the approximations by using the linear approximation at $x = 0.$

87. $\sqrt{1 - x} \approx 1 - \frac{1}{2}x$

88. $\frac{1}{\sqrt{1 - x^2}} \approx 1$

89. $\sqrt{c^2 + x^2} \approx c$

4.3 | Maxima and Minima

Learning Objectives

4.3.1 Define absolute extrema.
4.3.2 Define local extrema.
4.3.3 Explain how to find the critical points of a function over a closed interval.
4.3.4 Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty$, $f(x) \to \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (see the following figure).

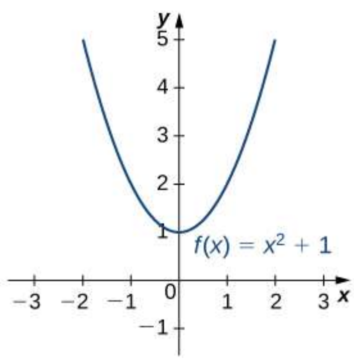


Figure 4.12 The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Definition

Let f be a function defined over an interval I and let $c \in I$. We say f has an **absolute maximum** on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an **absolute minimum** on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an **absolute extremum** on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term *absolute* here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is

$f(0) = 1.$ It occurs at $x = 0,$ as shown in **Figure 4.13**(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. **Figure 4.13** shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.

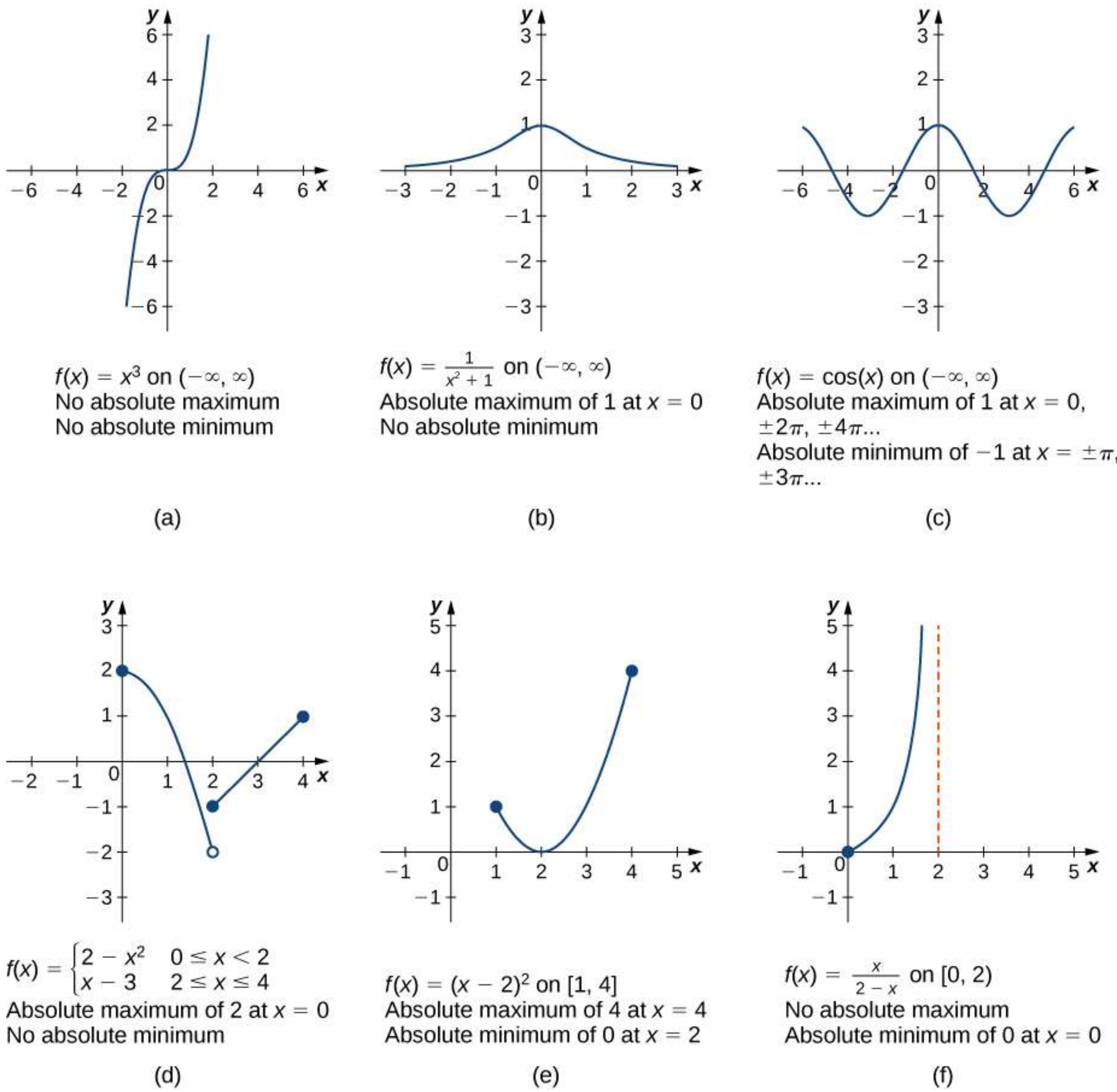


Figure 4.13 Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty).$ Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Theorem 4.1: Extreme Value Theorem

If f is a continuous function over the closed, bounded interval $[a, b],$ then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b].$

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I. For example, consider the functions shown in **Figure 4.13**(d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let's examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

Local Extrema and Critical Points

Consider the function f shown in **Figure 4.14**. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.

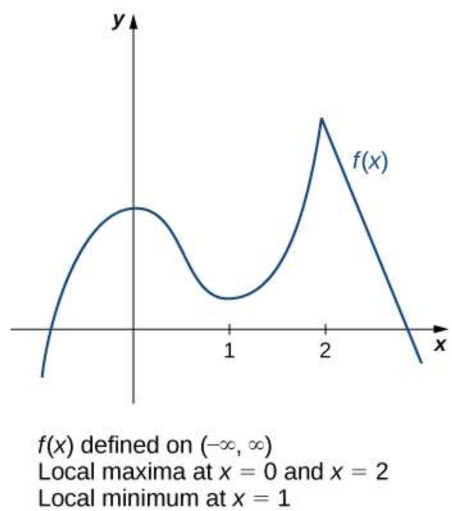


Figure 4.14 This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Definition

A function f has a **local maximum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a **local minimum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a **local extremum** at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at **Figure 4.14** again. The local extrema occur at $x = 0,$ $x = 1,$ and $x = 2.$ Notice that at $x = 0$ and $x = 1,$ the derivative $f'(x) = 0.$ At $x = 2,$ the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c,$ the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Definition

Let c be an interior point in the domain of f. We say that c is a **critical point** of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c,$ then c must be a critical point of f. This fact is known as **Fermat's theorem**.

Theorem 4.2: Fermat's Theorem

If f has a local extremum at c and f is differentiable at $c,$ then $f'(c) = 0.$

Proof

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f'(c) = 0.$ To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0,$ and therefore $f'(c) = 0.$ Since f has a local extremum at $c,$ f has a local maximum or local minimum at c. Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I.$ Since f is differentiable at $c,$ from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c).$ Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}, \tag{4.4}$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}. \tag{4.5}$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near $c,$ but $x > c,$ we have $\frac{f(x) - f(c)}{x - c} \leq 0.$ From **Equation 4.4** we conclude that $f'(c) \leq 0.$ Similarly, it can be shown that $f'(c) \geq 0.$ Therefore, $f'(c) = 0.$

□

From Fermat's theorem, we conclude that if f has a local extremum at $c,$ then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$. In **Figure 4.15**, we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

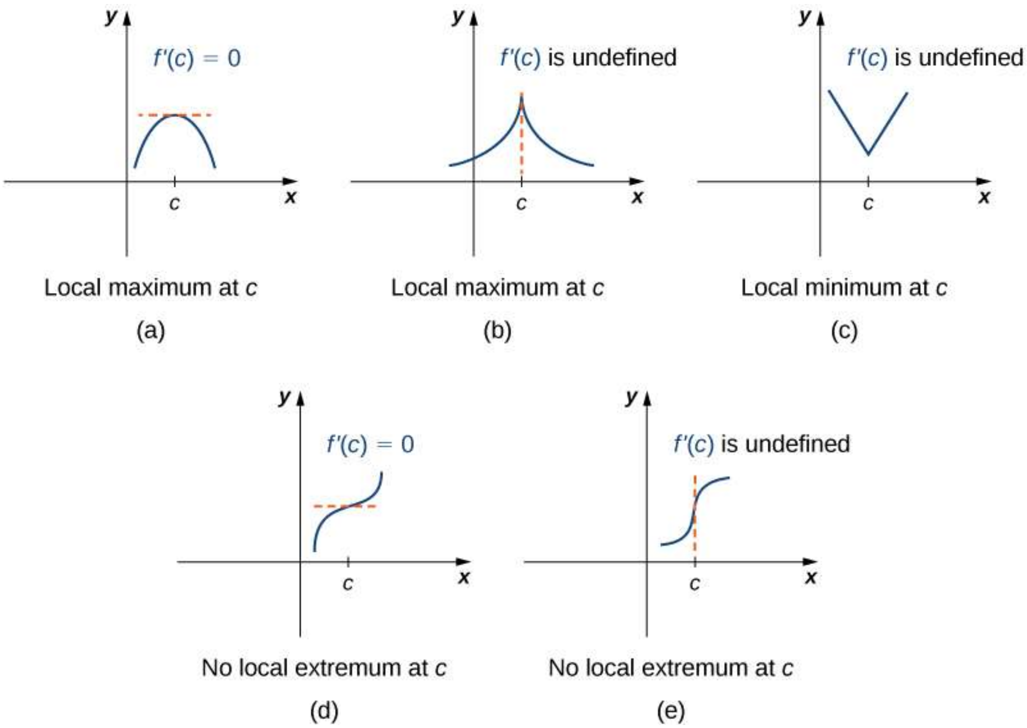


Figure 4.15 (a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 4.12

Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$

b. $f(x) = \left(x^2 - 1\right)^3$

c. $f(x) = \frac{4x}{1 + x^2}$

Solution

a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x - 4)(x - 1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in **Figure 4.16**, we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

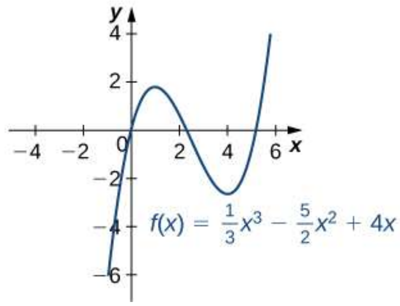


Figure 4.16 This function has a local maximum and a local minimum.

b. Using the chain rule, we see the derivative is

$$f'(x) = 3\left(x^2 - 1\right)^2 (2x) = 6x\left(x^2 - 1\right)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in **Figure 4.17**, we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum at $x = 1$ or $x = -1$.

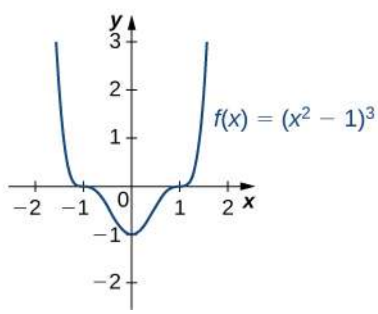


Figure 4.17 This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

c. By the chain rule, we see that the derivative is

$$f'(x) = \frac{\left(1 + x^2 4\right) - 4x(2x)}{\left(1 + x^2\right)^2} = \frac{4 - 4x^2}{\left(1 + x^2\right)^2}.$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in **Figure 4.18**, we see that f has an absolute maximum at $x = 1$

and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I, then f has a local extremum at c.)

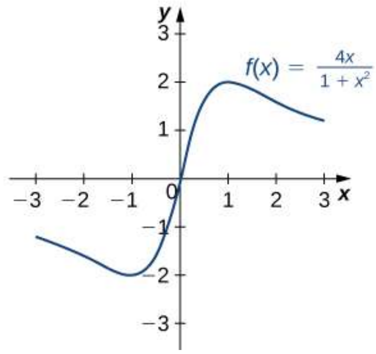


Figure 4.18 This function has an absolute maximum and an absolute minimum.

4.12 Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in **Figure 4.13**, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by **Fermat's Theorem**, the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem 4.3: Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let's examine a procedure for locating absolute extrema.

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). From **Location of Absolute Extrema**, the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example 4.13

Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.

b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3,$ f' is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0.$ Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3],$ $f\left(\frac{3}{2}\right)$ is a candidate for an absolute extremum of f over $[1, 3].$ We evaluate $f\left(\frac{3}{2}\right)$ and find

$$f\left(\frac{3}{2}\right) = \frac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	Conclusion
0	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4},$ and it occurs at $x = \frac{3}{2}.$ The absolute minimum of f over the interval [1, 3] is $-2,$ and it occurs at $x = 3$ as shown in the following graph.

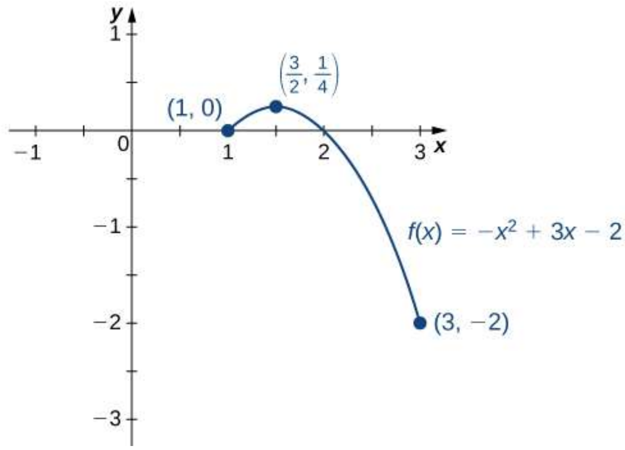


Figure 4.19 This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

$$f(0) = 0 \text{ and } f(2) = 4 - 3\sqrt[3]{4} \approx -0.762$$

Step 2. The derivative of f is given by

$$f'(x) = 2x - \frac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	−2	Absolute minimum
2	−0.762	

We conclude that the absolute maximum of f over the interval [0, 2] is zero, and it occurs at $x = 0$. The absolute minimum is −2, and it occurs at $x = 1$ as shown in the following graph.

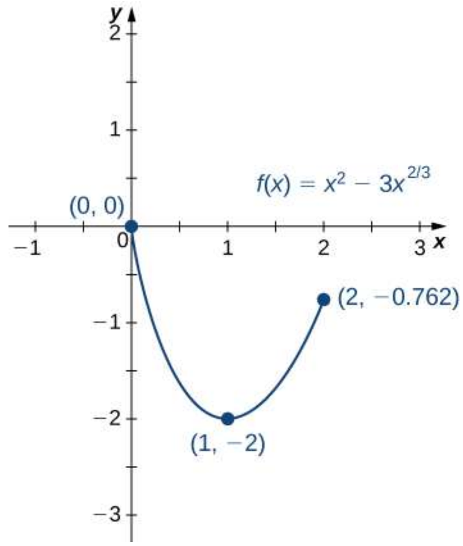


Figure 4.20 This function has an absolute maximum at an endpoint of the interval.

4.13 Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point $c,$ then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

4.3 EXERCISES

90. In precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was $h = -\frac{b}{(2a)}$. Prove this formula using calculus.

91. If you are finding an absolute minimum over an interval $[a, b]$, why do you need to check the endpoints? Draw a graph that supports your hypothesis.

92. If you are examining a function over an interval (a, b), for a and b finite, is it possible not to have an absolute maximum or absolute minimum?

93. When you are checking for critical points, explain why you also need to determine points where $f'(x)$ is undefined. Draw a graph to support your explanation.

94. Can you have a finite absolute maximum for $y = ax^2 + bx + c$ over $(-\infty, \infty)$? Explain why or why not using graphical arguments.

95. Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming a is non-zero? Explain why or why not using graphical arguments.

96. Let m be the number of local minima and M be the number of local maxima. Can you create a function where $M > m + 2$? Draw a graph to support your explanation.

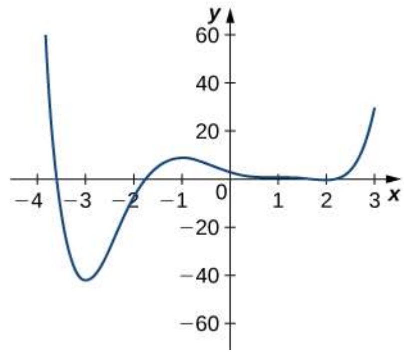
97. Is it possible to have more than one absolute maximum? Use a graphical argument to prove your hypothesis.

98. Is it possible to have no absolute minimum or maximum for a function? If so, construct such a function. If not, explain why this is not possible.

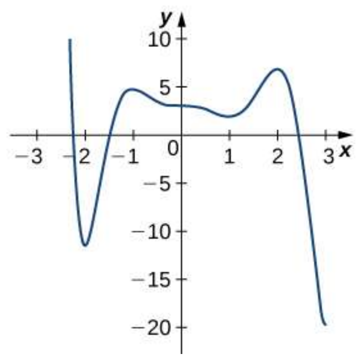
99. **[T]** Graph the function $y = e^{ax}$. For which values of a, on any infinite domain, will you have an absolute minimum and absolute maximum?

For the following exercises, determine where the local and absolute maxima and minima occur on the graph given. Assume the graph represents the entirety of each function.

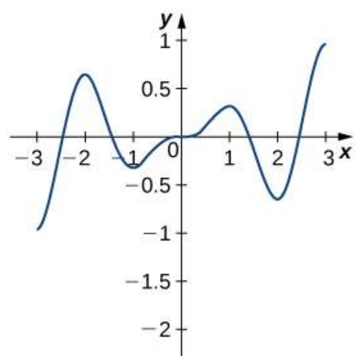
100.



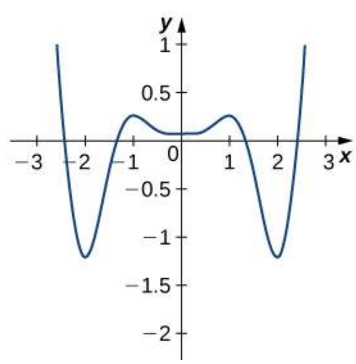
101.



102.



103.



For the following problems, draw graphs of $f(x)$, which is continuous, over the interval $[-4, 4]$ with the following properties:

104. Absolute maximum at $x = 2$ and absolute minima at $x = \pm 3$

105. Absolute minimum at $x = 1$ and absolute maximum at $x = 2$

106. Absolute maximum at $x = 4,$ absolute minimum at $x = -1,$ local maximum at $x = -2,$ and a critical point that is not a maximum or minimum at $x = 2$

107. Absolute maxima at $x = 2$ and $x = -3,$ local minimum at $x = 1,$ and absolute minimum at $x = 4$

For the following exercises, find the critical points in the domains of the following functions.

108. $y = 4x^3 - 3x$

109. $y = 4\sqrt{x} - x^2$

110. $y = \frac{1}{x - 1}$

111. $y = \ln(x - 2)$

112. $y = \tan(x)$

113. $y = \sqrt{4 - x^2}$

114. $y = x^{3/2} - 3x^{5/2}$

115. $y = \frac{x^2 - 1}{x^2 + 2x - 3}$

116. $y = \sin^2(x)$

117. $y = x + \frac{1}{x}$

For the following exercises, find the local and/or absolute maxima for the functions over the specified domain.

118. $f(x) = x^2 + 3$ over $[-1, 4]$

119. $y = x^2 + \frac{2}{x}$ over $[1, 4]$

120. $y = \left(x - x^2\right)^2$ over $[-1, 1]$

121. $y = \frac{1}{\left(x - x^2\right)}$ over $(0, 1)$

122. $y = \sqrt{9 - x}$ over $[1, 9]$

123. $y = x + \sin(x)$ over $[0, 2\pi]$

124. $y = \frac{x}{1 + x}$ over $[0, 100]$

125. $y = |x + 1| + |x - 1|$ over $[-3, 2]$

126. $y = \sqrt{x} - \sqrt{x^3}$ over $[0, 4]$

127. $y = \sin x + \cos x$ over $[0, 2\pi]$

128. $y = 4\sin\theta - 3\cos\theta$ over $[0, 2\pi]$

For the following exercises, find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$.

129. $y = x^2 + 4x + 5$

130. $y = x^3 - 12x$

131. $y = 3x^4 + 8x^3 - 18x^2$

132. $y = x^3 (1 - x)^6$

133. $y = \frac{x^2 + x + 6}{x - 1}$

134. $y = \frac{x^2 - 1}{x - 1}$

For the following functions, use a calculator to graph the function and to estimate the absolute and local maxima and minima. Then, solve for them explicitly.

135. **[T]** $y = 3x\sqrt{1 - x^2}$

136. **[T]** $y = x + \sin(x)$

137. **[T]** $y = 12x^5 + 45x^4 + 20x^3 - 90x^2 - 120x + 3$

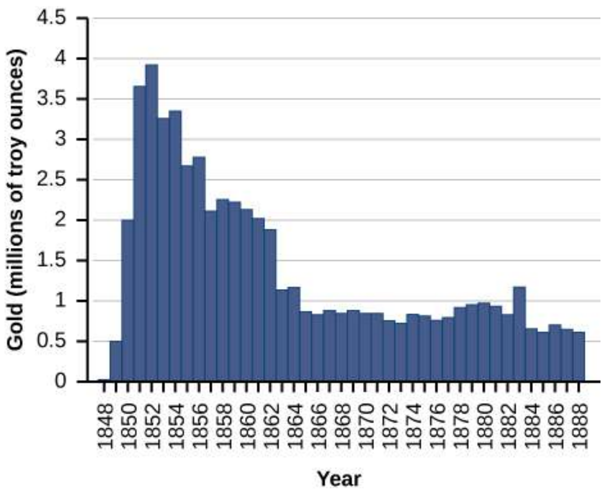
138. **[T]** $y = \frac{x^3 + 6x^2 - x - 30}{x - 2}$

139. **[T]** $y = \frac{\sqrt{4 - x^2}}{\sqrt{4 + x^2}}$

140. A company that produces cell phones has a cost function of $C = x^2 - 1200x + 36{,}400,$ where C is cost in dollars and x is number of cell phones produced (in thousands). How many units of cell phone (in thousands) minimizes this cost function?

141. A ball is thrown into the air and its position is given by $h(t) = -4.9t^2 + 60t + 5$ m. Find the height at which the ball stops ascending. How long after it is thrown does this happen?

For the following exercises, consider the production of gold during the California gold rush (1848–1888). The production of gold can be modeled by $G(t) = \frac{(25t)}{\left(t^2 + 16\right)}$, where t is the number of years since the rush began $(0 \leq t \leq 40)$ and G is ounces of gold produced (in millions). A summary of the data is shown in the following figure.



142. Find when the maximum (local and global) gold production occurred, and the amount of gold produced during that maximum.

143. Find when the minimum (local and global) gold production occurred. What was the amount of gold produced during this minimum?

Find the critical points, maxima, and minima for the following piecewise functions.

144. $y = \begin{cases} x^2 - 4x & 0 \leq x \leq 1 \\ x^2 - 4 & 1 < x \leq 2 \end{cases}$

145. $y = \begin{cases} x^2 + 1 & x \leq 1 \\ x^2 - 4x + 5 & x > 1 \end{cases}$

For the following exercises, find the critical points of the following generic functions. Are they maxima, minima, or neither? State the necessary conditions.

146. $y = ax^2 + bx + c$, given that $a > 0$

147. $y = (x - 1)^a$, given that $a > 1$ and a is an integer.

4.4 | The Mean Value Theorem

Learning Objectives

4.4.1 Explain the meaning of Rolle's theorem.
4.4.2 Describe the significance of the Mean Value Theorem.
4.4.3 State three important consequences of the Mean Value Theorem.

The **Mean Value Theorem** is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0.$ Figure 4.21 illustrates this theorem.

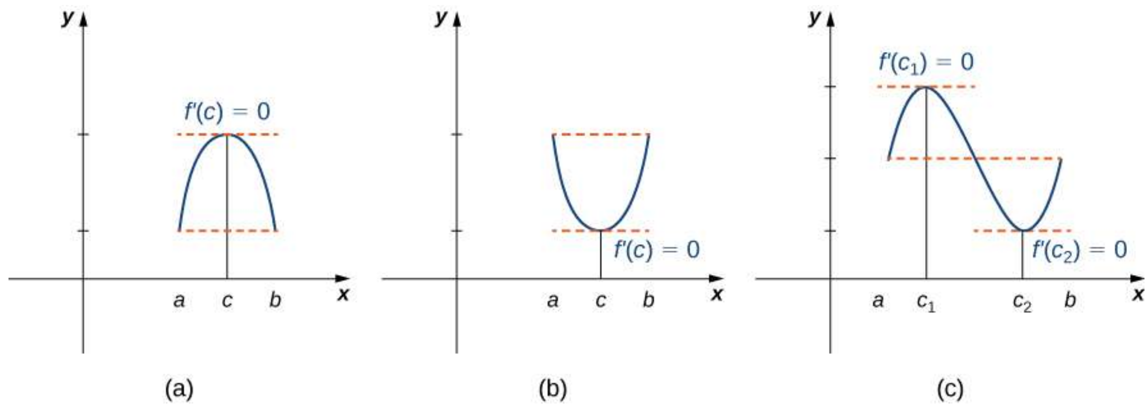


Figure 4.21 If a differentiable function *f* satisfies $f(a) = f(b),$ then its derivative must be zero at some point(s) between a and $b.$

Theorem 4.4: Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b).$ There then exists at least one $c \in (a, b)$ such that $f'(c) = 0.$

Proof

Let $k = f(a) = f(b).$ We consider three cases:

1. $f(x) = k$ for all $x \in (a, b).$
2. There exists $x \in (a, b)$ such that $f(x) > k.$
3. There exists $x \in (a, b)$ such that $f(x) < k.$

Case 1: If $f(x) = k$ for all $x \in (a, b),$ then $f'(x) = 0$ for all $x \in (a, b).$

Case 2: Since f is a continuous function over the closed, bounded interval $[a, b],$ by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k,$ the absolute maximum is greater than $k.$ Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b).$ Because f has a maximum at an interior point $c,$ and f is differentiable at $c,$ by Fermat's theorem, $f'(c) = 0.$

Case 3: The case when there exists a point $x \in (a, b)$ such that $f(x) < k$ is analogous to case 2, with maximum replaced by minimum.

□

An important point about Rolle's theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over $[-1, 1]$ and $f(-1) = 0 = f(1)$, but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown in the following figure.

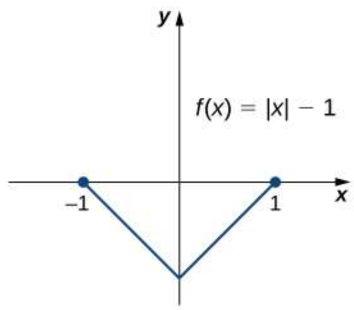


Figure 4.22 Since $f(x) = |x| - 1$ is not differentiable at $x = 0$, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points c where $f'(c) = 0$.

Example 4.14

Using Rolle's Theorem

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

a. $f(x) = x^2 + 2x$ over $[-2, 0]$

b. $f(x) = x^3 - 4x$ over $[-2, 2]$

Solution

a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0)$. Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0$. Since $f'(x) = 2x + 2 = 2(x + 1)$, we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.

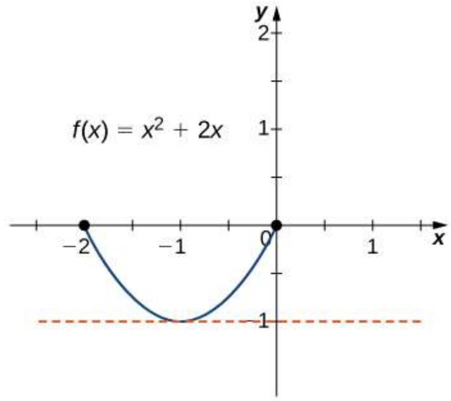


Figure 4.23 This function is continuous and differentiable over $[-2, 0]$, $f'(c) = 0$ when $c = -1$.

b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.

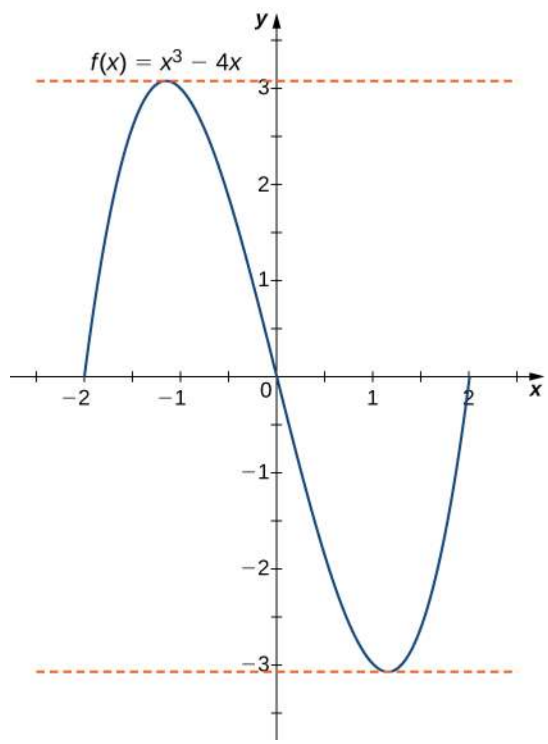


Figure 4.24 For this polynomial over $[-2, 2]$, $f'(c) = 0$ at $x = \pm 2/\sqrt{3}$.

4.14 Verify that the function $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1, 3]$ satisfies the conditions of Rolle's theorem. Find all points c guaranteed by Rolle's theorem.

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f defined on a closed interval $[a, b]$ with $f(a) = f(b)$. The Mean Value Theorem generalizes Rolle's theorem by considering functions that do not necessarily have equal value at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem (**Figure 4.25**). The Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

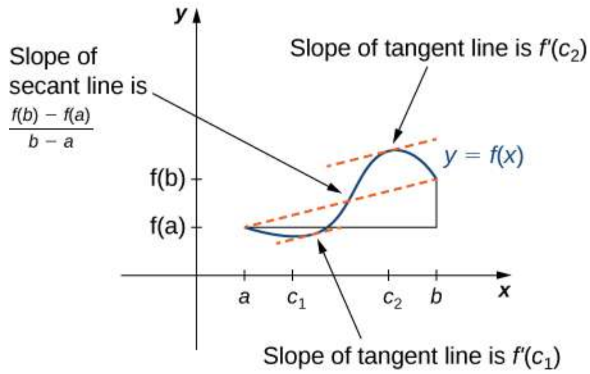


Figure 4.25 The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

Theorem 4.5: Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point (x, y) on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

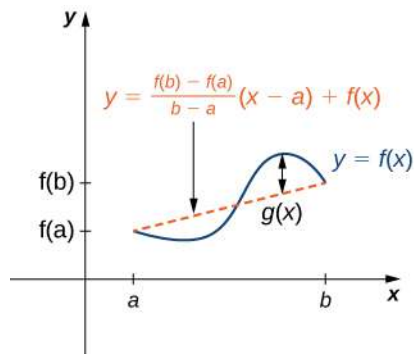


Figure 4.26 The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point (x, y) on the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over $[a, b]$, g is also continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

In the next example, we show how the Mean Value Theorem can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

Example 4.15

Verifying that the Mean Value Theorem Applies

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

Solution

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$ (**Figure 4.27**). To determine which value(s) of c are guaranteed, first calculate the derivative of f. The derivative $f'(x) = \frac{1}{(2\sqrt{x})}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}.$$

We want to find c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Solving this equation for c, we obtain $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

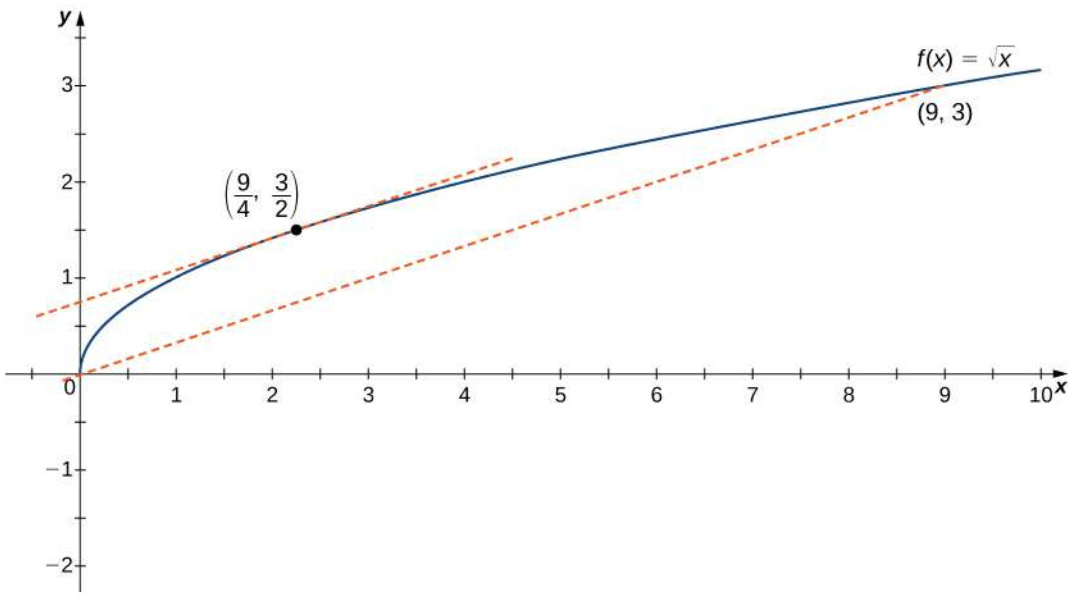


Figure 4.27 The slope of the tangent line at $c = 9/4$ is the same as the slope of the line segment connecting $(0, 0)$ and $(9, 3)$.

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for 1 h down a straight road with an average velocity of 45 mph. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0 \le t \le 1$ h. Assuming that the position function $s(t)$ is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c \in (0, 1)$, the speed of the car was exactly

$$v(c) = s'(c) = \frac{s(1) - s(0)}{1 - 0} = 45 \text{ mph}.$$

Example 4.16

Mean Value Theorem and Velocity

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100$.

a. Determine how long it takes before the rock hits the ground.
b. Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.
c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

Solution

a. When the rock hits the ground, its position is $s(t) = 0$. Solving the equation $-16t^2 + 100 = 0$ for t, we find that $t = \pm\frac{5}{2}$sec. Since we are only considering $t \geq 0$, the ball will hit the ground $\frac{5}{2}$ sec after it is dropped.
b. The average velocity is given by

$$v_{\text{avg}} = \frac{s(5/2) - s(0)}{5/2 - 0} = \frac{0 - 100}{5/2} = -40 \text{ ft/sec.}$$

c. The instantaneous velocity is given by the derivative of the position function. Therefore, we need to find a time t such that $v(t) = s'(t) = v_{\text{avg}} = -40$ ft/sec. Since $s(t)$ is continuous over the interval $[0, 5/2]$ and differentiable over the interval $(0, 5/2)$, by the Mean Value Theorem, there is guaranteed to be a point $c \in (0, 5/2)$ such that

$$s'(c) = \frac{s(5/2) - s(0)}{5/2 - 0} = -40.$$

Taking the derivative of the position function $s(t)$, we find that $s'(t) = -32t$. Therefore, the equation reduces to $s'(c) = -32c = -40$. Solving this equation for c, we have $c = \frac{5}{4}$. Therefore, $\frac{5}{4}$ sec after the rock is dropped, the instantaneous velocity equals the average velocity of the rock during its free fall: -40 ft/sec.

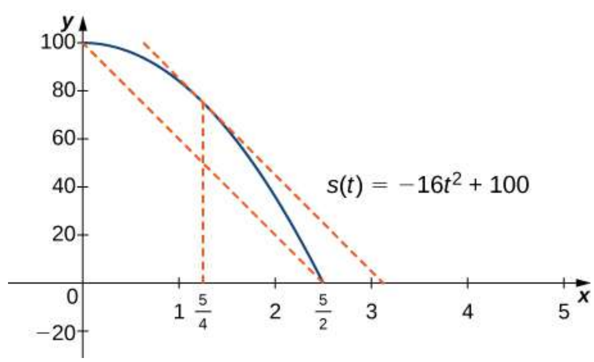


Figure 4.28 At time $t = 5/4$ sec, the velocity of the rock is equal to its average velocity from the time it is dropped until it hits the ground.

4.15 Suppose a ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity of the ball equals its average velocity.

Corollaries of the Mean Value Theorem

Let's now look at three corollaries of the Mean Value Theorem. These results have important consequences, which we use in upcoming sections.

At this point, we know the derivative of any constant function is zero. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f'(x) = 0$ for all x in some interval I, then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

Theorem 4.6: Corollary 1: Functions with a Derivative of Zero

Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x) =$ constant for all $x \in I$.

Proof

Since f is differentiable over I, f must be continuous over I. Suppose $f(x)$ is not constant for all x in I. Then there exist $a, b \in I$, where $a \neq b$ and $f(a) \neq f(b)$. Choose the notation so that $a < b$. Therefore,

$$\frac{f(b) - f(a)}{b - a} \neq 0.$$

Since f is a differentiable function, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in I$ such that $f'(c) \neq 0$, which contradicts the assumption that $f'(x) = 0$ for all $x \in I$.

□

From **Corollary 1: Functions with a Derivative of Zero**, it follows that if two functions have the same derivative, they differ by, at most, a constant.

Theorem 4.7: Corollary 2: Constant Difference Theorem

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C.

Proof

Let $h(x) = f(x) - g(x)$. Then, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By Corollary 1, there is a constant C such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

□

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Recall that a function f is increasing over I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas f is decreasing over I if $f(x)_1 > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive, then the function is increasing; if the derivative is negative, then the function is decreasing (**Figure 4.29**). We make use of this fact in the next section, where we show how to use the derivative of a function to locate local maximum and minimum values of the function, and how to determine the shape of the graph.

This fact is important because it means that for a given function f, if there exists a function F such that $F'(x) = f(x)$; then, the only other functions that have a derivative equal to f are $F(x) + C$ for some constant C. We discuss this result in more detail later in the chapter.

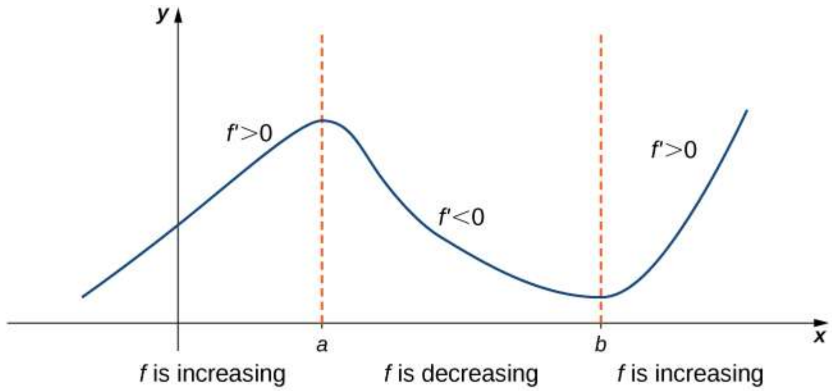


Figure 4.29 If a function has a positive derivative over some interval I, then the function increases over that interval I; if the derivative is negative over some interval I, then the function decreases over that interval I.

Theorem 4.8: Corollary 3: Increasing and Decreasing Functions

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.

ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

Proof

We will prove i.; the proof of ii. is similar. Suppose f is not an increasing function on I. Then there exist a and b in I such that $a < b$, but $f(a) \geq f(b)$. Since f is a differentiable function over I, by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, $a < b$ tells us that $b - a > 0$. We conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction, and therefore f must be an increasing function over I.

□

4.4 EXERCISES

148. Why do you need continuity to apply the Mean Value Theorem? Construct a counterexample.

149. Why do you need differentiability to apply the Mean Value Theorem? Find a counterexample.

150. When are Rolle's theorem and the Mean Value Theorem equivalent?

151. If you have a function with a discontinuity, is it still possible to have $f'(c)(b-a) = f(b) - f(a)$? Draw such an example or prove why not.

For the following exercises, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.

152. $y = \sin(\pi x)$

153. $y = \frac{1}{x^3}$

154. $y = \sqrt{4 - x^2}$

155. $y = \sqrt{x^2 - 4}$

156. $y = \ln(3x - 5)$

For the following exercises, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points c such that $f'(c)(b-a) = f(b) - f(a)$.

157. **[T]** $y = 3x^3 + 2x + 1$ over $[-1, 1]$

158. **[T]** $y = \tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2}, \frac{3}{2}\right]$

159. **[T]** $y = x^2\cos(\pi x)$ over $[-2, 2]$

160. **[T]** $y = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32}$ over $[-1, 1]$

For the following exercises, use the Mean Value Theorem and find all points $0 < c < 2$ such that $f(2) - f(0) = f'(c)(2 - 0)$.

161. $f(x) = x^3$

162. $f(x) = \sin(\pi x)$

163. $f(x) = \cos(2\pi x)$

164. $f(x) = 1 + x + x^2$

165. $f(x) = (x - 1)^{10}$

166. $f(x) = (x - 1)^9$

For the following exercises, show there is no c such that $f(1) - f(-1) = f'(c)(2)$. Explain why the Mean Value Theorem does not apply over the interval $[-1, 1]$.

167. $f(x) = \left|x - \frac{1}{2}\right|$

168. $f(x) = \frac{1}{x^2}$

169. $f(x) = \sqrt{|x|}$

170. $f(x) = \lfloor x \rfloor$ (*Hint*: This is called the *floor function* and it is defined so that $f(x)$ is the largest integer less than or equal to x.)

For the following exercises, determine whether the Mean Value Theorem applies for the functions over the given interval $[a, b]$. Justify your answer.

171. $y = e^x$ over $[0, 1]$

172. $y = \ln(2x + 3)$ over $\left[-\frac{3}{2}, 0\right]$

173. $f(x) = \tan(2\pi x)$ over $[0, 2]$

174. $y = \sqrt{9 - x^2}$ over $[-3, 3]$

175. $y = \frac{1}{|x + 1|}$ over $[0, 3]$

176. $y = x^3 + 2x + 1$ over $[0, 6]$

177. $y = \frac{x^2 + 3x + 2}{x}$ over $[-1, 1]$

178. $y = \frac{x}{\sin(\pi x) + 1}$ over $[0, 1]$

179. $y = \ln(x + 1)$ over $[0, e - 1]$

180. $y = x\sin(\pi x)$ over $[0, 2]$

181. $y = 5 + |x|$ over $[-1, 1]$

For the following exercises, consider the roots of the equation.

182. Show that the equation $y = x^3 + 3x^2 + 16$ has exactly one real root. What is it?

183. Find the conditions for exactly one root (double root) for the equation $y = x^2 + bx + c$

184. Find the conditions for $y = e^x - b$ to have one root. Is it possible to have more than one root?

For the following exercises, use a calculator to graph the function over the interval $[a, b]$ and graph the secant line from a to $b.$ Use the calculator to estimate all values of c as guaranteed by the Mean Value Theorem. Then, find the exact value of $c,$ if possible, or write the final equation and use a calculator to estimate to four digits.

185. **[T]** $y = \tan(\pi x)$ over $\left[-\frac{1}{4}, \frac{1}{4}\right]$

186. **[T]** $y = \frac{1}{\sqrt{x+1}}$ over $[0, 3]$

187. **[T]** $y = \left|x^2 + 2x - 4\right|$ over $[-4, 0]$

188. **[T]** $y = x + \frac{1}{x}$ over $\left[\frac{1}{2}, 4\right]$

189. **[T]** $y = \sqrt{x+1} + \frac{1}{x^2}$ over $[3, 8]$

190. At 10:17 a.m., you pass a police car at 55 mph that is stopped on the freeway. You pass a second police car at 55 mph at 10:53 a.m., which is located 39 mi from the first police car. If the speed limit is 60 mph, can the police cite you for speeding?

191. Two cars drive from one spotlight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

192. Show that $y = \sec^2 x$ and $y = \tan^2 x$ have the same derivative. What can you say about $y = \sec^2 x - \tan^2 x$?

193. Show that $y = \csc^2 x$ and $y = \cot^2 x$ have the same derivative. What can you say about $y = \csc^2 x - \cot^2 x$?

4.5 | Derivatives and the Shape of a Graph

Learning Objectives

4.5.1 Explain how the sign of the first derivative affects the shape of a function's graph.
4.5.2 State the first derivative test for critical points.
4.5.3 Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
4.5.4 Explain the concavity test for a function over an open interval.
4.5.5 Explain the relationship between a function and its first and second derivatives.
4.5.6 State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function f has a local extremum at a point $c,$ then c must be a critical point of f. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0,$ but f does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over I. On the other hand, if the derivative of the function is negative over an interval $I,$ then the function is decreasing over I as shown in the following figure.

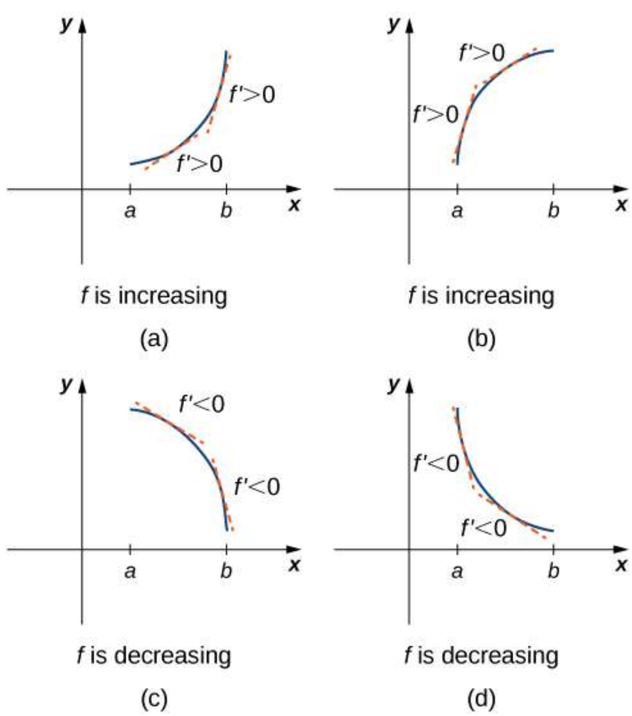


Figure 4.30 Both functions are increasing over the interval (a, b). At each point $x,$ the derivative $f'(x) > 0$. Both functions are decreasing over the interval (a, b). At each point $x,$ the derivative $f'(x) < 0$.

A continuous function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c. Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c. If f is a continuous function over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c. If f is differentiable at c, the only way that f'. can change sign as x increases through c is if $f'(c) = 0$. Therefore, for a function f that is continuous over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is undefined. Consequently, to locate local extrema for a function f, we look for points c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such points are called critical points of f.

Note that f need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In **Figure 4.31**, we show that if a continuous function f has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if f has a local extremum at a critical point, then the sign of f' switches as x increases through that point.

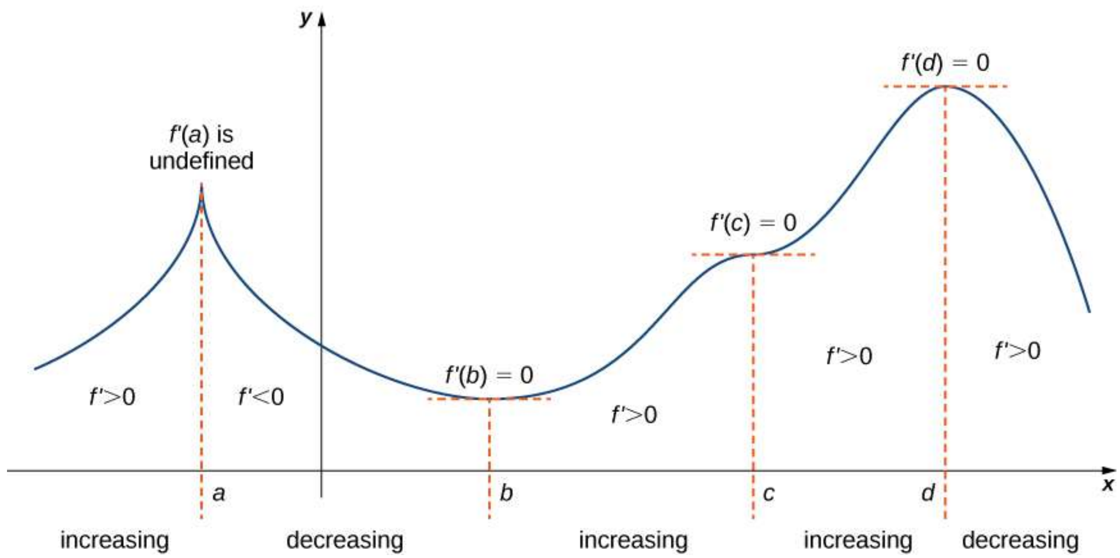


Figure 4.31 The function f has four critical points: $a, b, c,$ and d. The function f has local maxima at a and d, and a local minimum at b. The function f does not have a local extremum at c. The sign of f' changes at all local extrema.

Using **Figure 4.31**, we summarize the main results regarding local extrema.

- If a continuous function f has a local extremum, it must occur at a critical point c.
- The function has a local extremum at the critical point c if and only if the derivative f' switches sign as x increases through c.
- Therefore, to test whether a function has a local extremum at a critical point c, we must determine the sign of $f'(x)$ to the left and right of c.

This result is known as the **first derivative test**.

Theorem 4.9: First Derivative Test

Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possibly at point c, then $f(c)$ satisfies one of the following descriptions:

i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.
ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.
iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f.

We can summarize the first derivative test as a strategy for locating local extrema.

Problem-Solving Strategy: Using the First Derivative Test

Consider a function f that is continuous over an interval I.

1. Find all critical points of f and divide the interval I into smaller intervals using the critical points as endpoints.
2. Analyze the sign of f' in each of the subintervals. If f' is continuous over a given subinterval (which is typically the case), then the sign of f' in that subinterval does not change and, therefore, can be determined by choosing an arbitrary test point x in that subinterval and by evaluating the sign of f' at that test point. Use the sign analysis to determine whether f is increasing or decreasing over that interval.
3. Use **First Derivative Test** and the results of step 2 to determine whether f has a local maximum, a local minimum, or neither at each of the critical points.

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example 4.17

Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is $f'(x) = 3x^2 - 6x - 9$. To find the critical points, we need to find where $f'(x) = 0$. Factoring the polynomial, we conclude that the critical points must satisfy

$$3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0.$$

Therefore, the critical points are $x = 3, -1$. Now divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$.

Step 2. Since f' is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$ and determine the sign of f' at each of these points. For example, let's choose $x = -2$, $x = 0$, and $x = 4$ as test points.

Interval	**Test Point**	**Sign of $f'(x) = 3(x-3)(x+1)$ at Test Point**	**Conclusion**
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing.
$(-1, 3)$	$x = 0$	$(+)(-)(+) = -$	f is decreasing.
$(3, \infty)$	$x = 4$	$(+)(+)(+) = +$	f is increasing.

Step 3. Since f' switches sign from positive to negative as x increases through 1, f has a local maximum at $x = -1$. Since f' switches sign from negative to positive as x increases through 3, f has a local minimum at $x = 3$. These analytical results agree with the following graph.

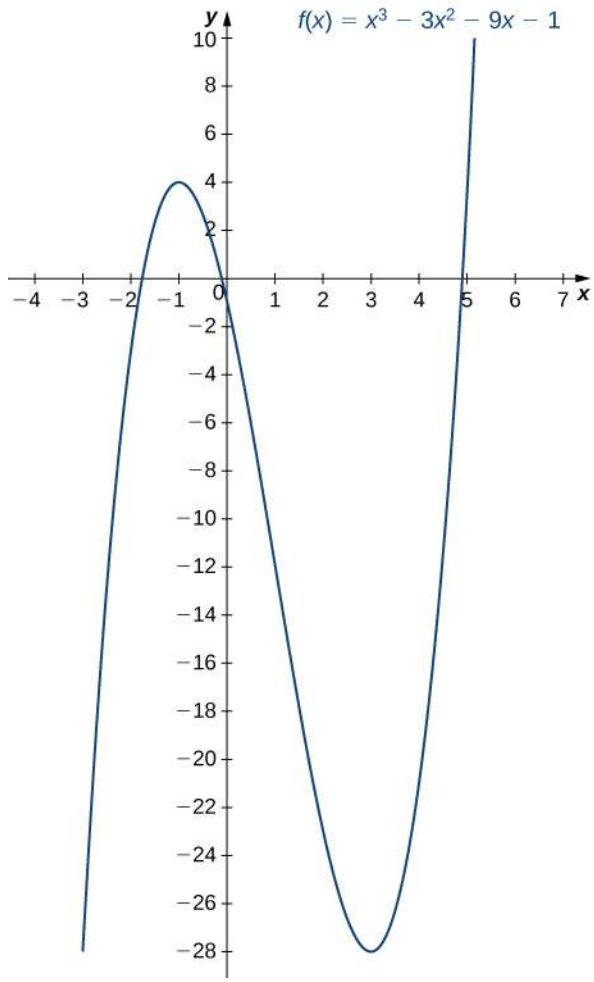


Figure 4.32 The function f has a maximum at $x = -1$ and a minimum at $x = 3$

4.16 Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x.$

Example 4.18

Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}.$ Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3x^{2/3}} - \frac{5x^{2/3}}{3} = \frac{5 - 5x^{4/3}}{3x^{2/3}} = \frac{5\left(1 - x^{4/3}\right)}{3x^{2/3}}.$$

The derivative $f'(x) = 0$ when $1 - x^{4/3} = 0.$ Therefore, $f'(x) = 0$ at $x = \pm 1.$ The derivative $f'(x)$ is undefined at $x = 0.$ Therefore, we have three critical points: $x = 0,$ $x = 1,$ and $x = -1.$ Consequently, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1), (-1, 0), (0, 1),$ and $(1, \infty).$

Step 2: Since f' is continuous over each subinterval, it suffices to choose a test point x in each of the intervals from step 1 and determine the sign of f' at each of these points. The points $x = -2,\ x = -\frac{1}{2},\ x = \frac{1}{2},$ and $x = 2$ are test points for these intervals.

Interval	Test Point	Sign of $f'(x) = \frac{5\left(1 - x^{4/3}\right)}{3x^{2/3}}$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	$\frac{(+)(-)}{+} = -$	f is decreasing.
$(-1, 0)$	$x = -\frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(1, \infty)$	$x = 2$	$\frac{(+)(-)}{+} = -$	f is decreasing.

Step 3: Since f is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0),$ f has a local minimum at $x = -1.$ Since f is increasing over the interval $(-1, 0)$ and the interval $(0, 1),$ f does not have a local extremum at $x = 0.$ Since f is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty),$ f has a local maximum at $x = 1.$ The analytical results agree with the following graph.

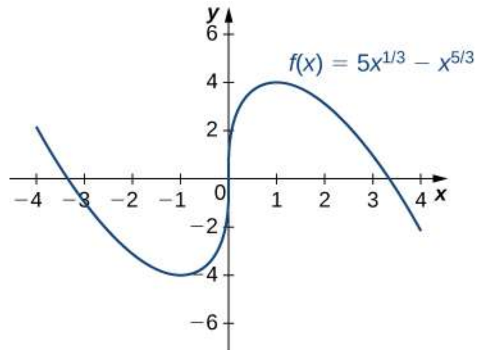


Figure 4.33 The function *f* has a local minimum at $x = -1$ and a local maximum at $x = 1$.

4.17 Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x-1}$.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure 4.34(a) shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. **Figure 4.34**(b) shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Definition

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

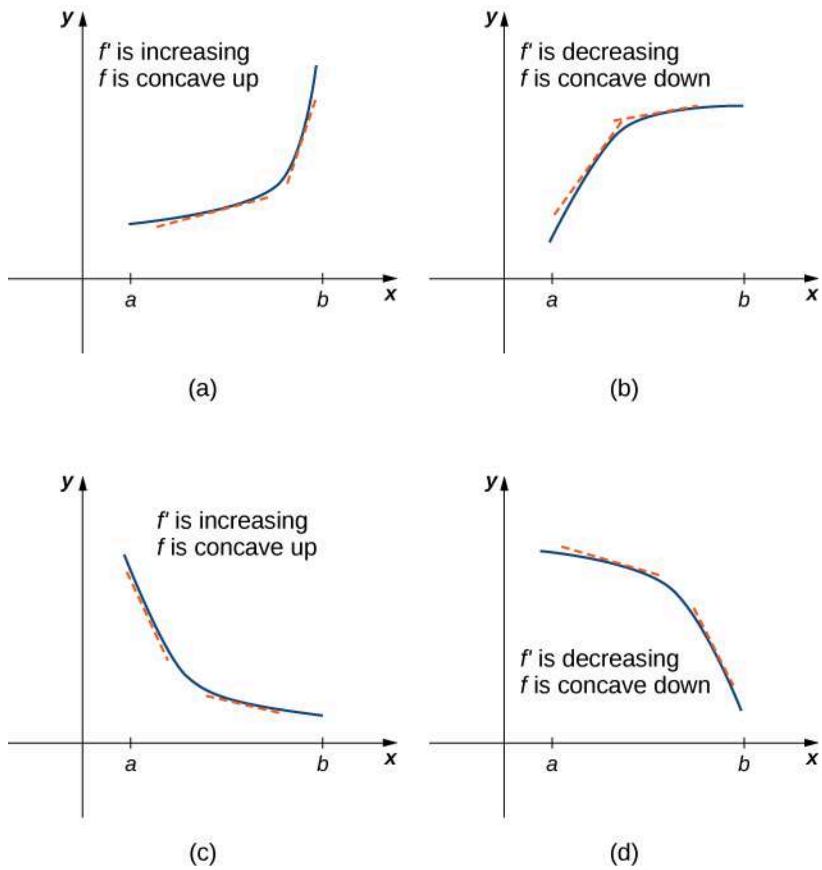


Figure 4.34 (a), (c) Since f' is increasing over the interval (a, b), we say f is concave up over (a, b). (b), (d) Since f' is decreasing over the interval (a, b), we say f is concave down over (a, b).

In general, without having the graph of a function f, how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary 3, we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0$. Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the **concavity test**.

Theorem 4.10: Test for Concavity

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I.

ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (**Figure 4.35**). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point $x,$ then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at $a,$ we say the point $(a, f(a))$ is an inflection point of f.

Definition

If f is continuous at a and f changes concavity at $a,$ the point $(a, f(a))$ is an **inflection point** of f.

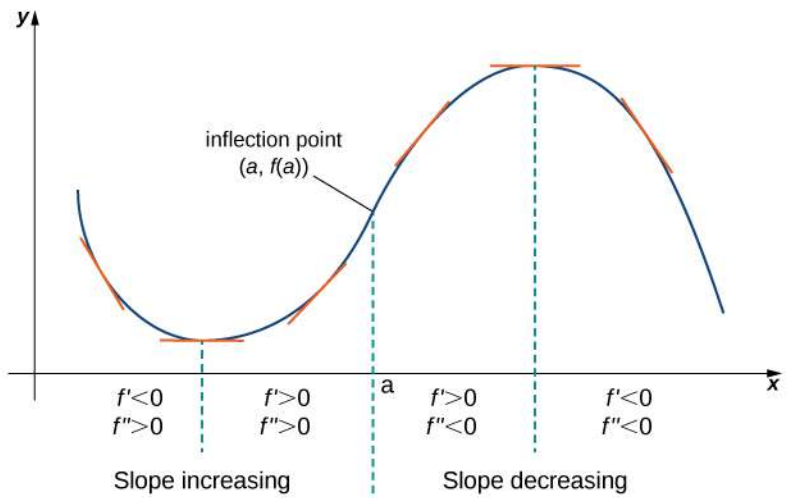


Figure 4.35 Since $f''(x) > 0$ for $x < a,$ the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a,$ the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example 4.19

Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30,$ determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9,$ so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers $x,$ we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0,$ we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	$-$	f is concave down
$(2, \infty)$	$x = 3$	$+$	f is concave up.

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. **Figure 4.36** confirms the analytical results.

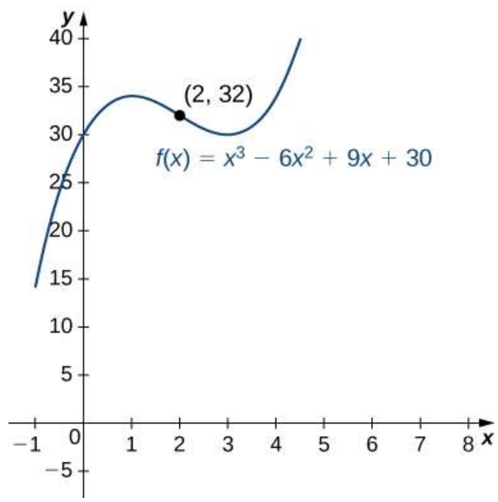


Figure 4.36 The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

4.18 For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where f is concave up and all intervals where f is concave down.

We now summarize, in **Table 4.1**, the information that the first and second derivatives of a function f provide about the graph of f, and illustrate this information in **Figure 4.37**.

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

Table 4.1 What Derivatives Tell Us about Graphs

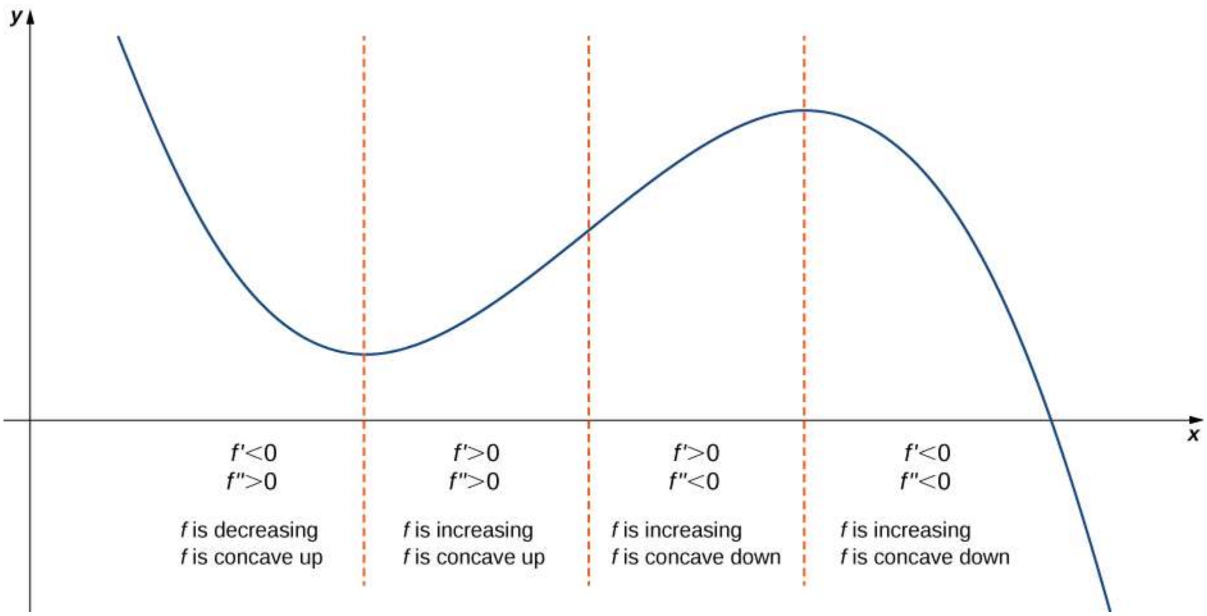


Figure 4.37 Consider a twice-differentiable function f over an open interval I. If $f'(x) > 0$ for all $x \in I$, the function is increasing over I. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I.

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extrema, it must occur at a critical point. However, a function need not have a local extrema at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a. Suppose $f''(a) < 0$. Since f'' is continuous over I, $f''(x) < 0$ for all $x \in I$ (Figure 4.38). Then, by Corollary 3, f' is a decreasing function over I. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$. On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, then $f''(x) > 0$ for all $x \in I$ (Figure 4.38). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.

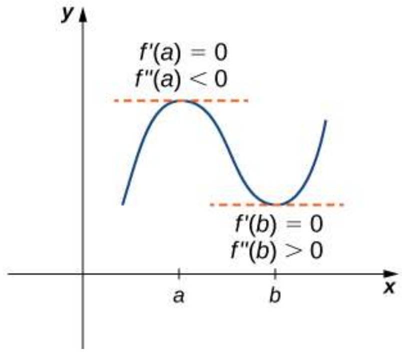


Figure 4.38 Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0,$ there is an interval I containing a such that for all x in $I,$ f is increasing if $x < a$ and f is decreasing if $x > a.$ As a result, f has a local maximum at $x = a.$ Since $f'(b) = 0$ and $f''(b) > 0,$ there is an interval I containing b such that for all x in $I,$ f is decreasing if $x < b$ and f is increasing if $x > b.$ As a result, f has a local minimum at $x = b.$

Theorem 4.11: Second Derivative Test

Suppose $f'(c) = 0,$ f'' is continuous over an interval containing $c.$

i. If $f''(c) > 0,$ then f has a local minimum at $c.$

ii. If $f''(c) < 0,$ then f has a local maximum at $c.$

iii. If $f''(c) = 0,$ then the test is inconclusive.

Note that for case iii. when $f''(c) = 0,$ then f may have a local maximum, local minimum, or neither at $c.$ For example, the functions $f(x) = x^3,$ $f(x) = x^4,$ and $f(x) = -x^4$ all have critical points at $x = 0.$ In each case, the second derivative is zero at $x = 0.$ However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at $x,$ and the function $f(x) = x^3$ does not have a local extremum at $x = 0.$

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0.$

Example 4.20

Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3.$

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0.$ The derivative is

$f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2\left(x^2 - 3\right) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extrema at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x\left(2x^2 - 3\right).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

x	$f''(x)$	**Conclusion**
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2\left(x^2 - 3\right)$ for $x \in \left(-\sqrt{3}, 0\right)$ and $x \in \left(0, \sqrt{3}\right)$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.

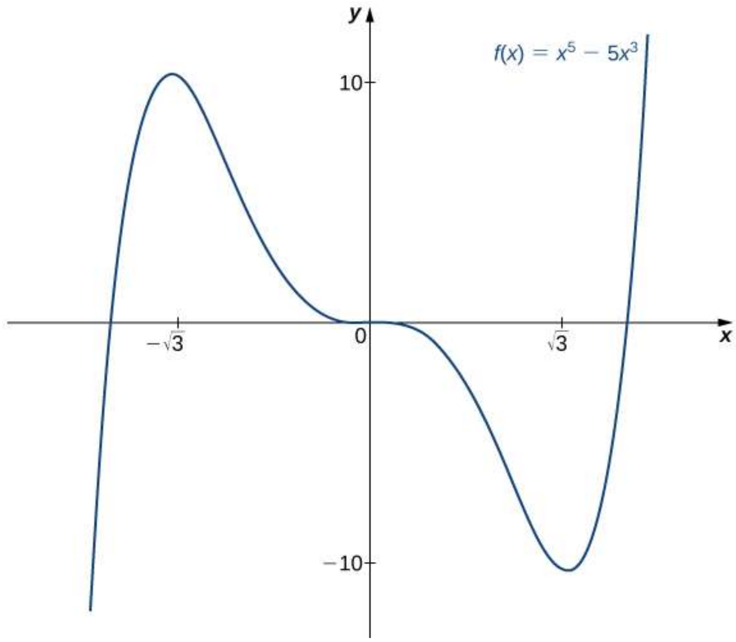


Figure 4.39 The function f has a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$

4.19 Consider the function $f(x) = x^3 - \left(\frac{3}{2}\right)x^2 - 18x.$ The points $c = 3, -2$ satisfy $f'(c) = 0.$ Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x \to \pm\infty.$ At that point, we have enough tools to provide accurate graphs of a large variety of functions.

4.5 EXERCISES

194. If c is a critical point of $f(x)$, when is there no local maximum or minimum at c? Explain.

195. For the function $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum?

196. For the function $y = x^3$, is $x = 0$ an inflection point?

197. Is it possible for a point c to be both an inflection point and a local extrema of a twice differentiable function?

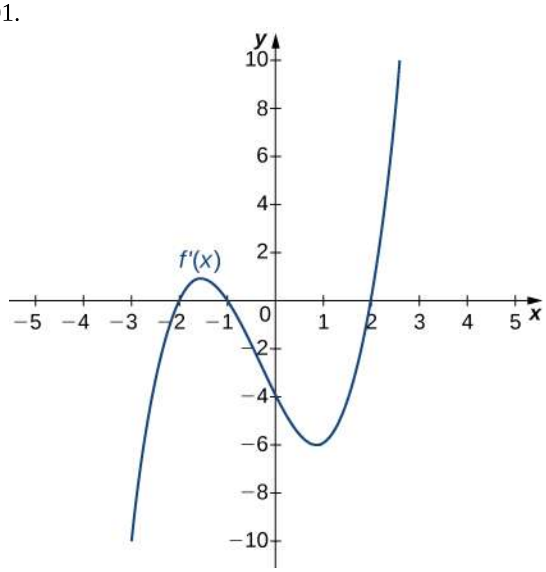
198. Why do you need continuity for the first derivative test? Come up with an example.

199. Explain whether a concave-down function has to cross $y = 0$ for some value of x.

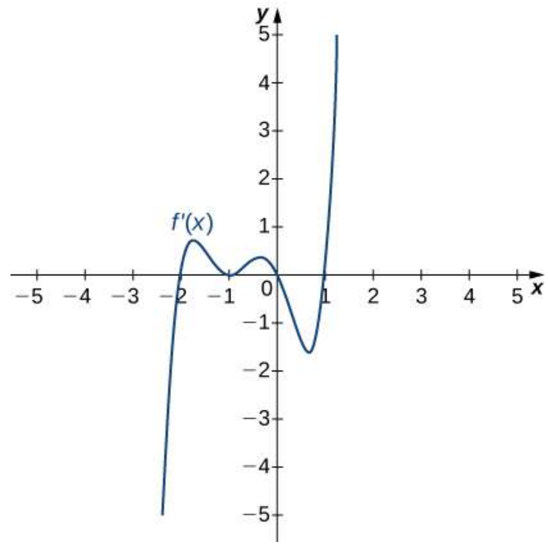
200. Explain whether a polynomial of degree 2 can have an inflection point.

For the following exercises, analyze the graphs of f', then list all intervals where f is increasing or decreasing.

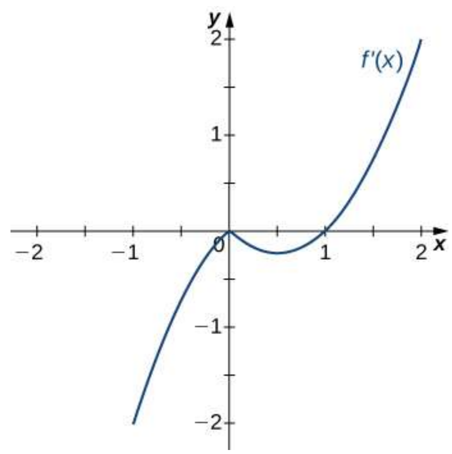
201.



202.



203.



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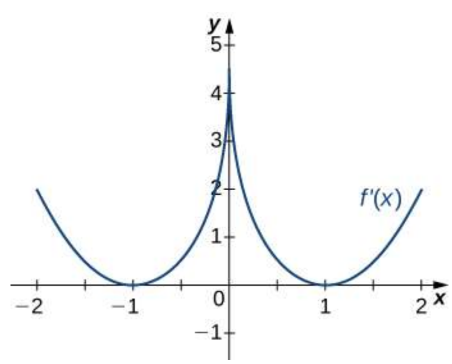

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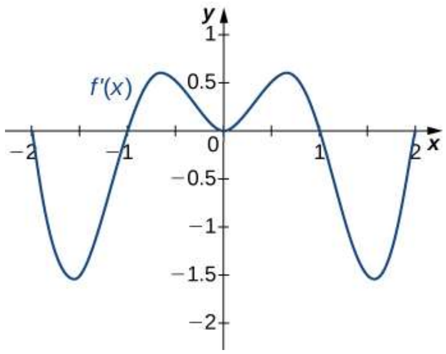
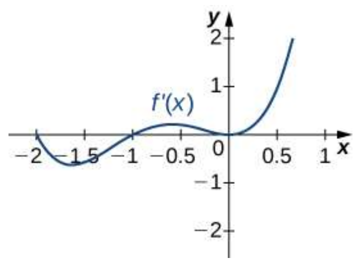


For the following exercises, analyze the graphs of f', then list all intervals where

a. f is increasing and decreasing and

b. the minima and maxima are located.

206.



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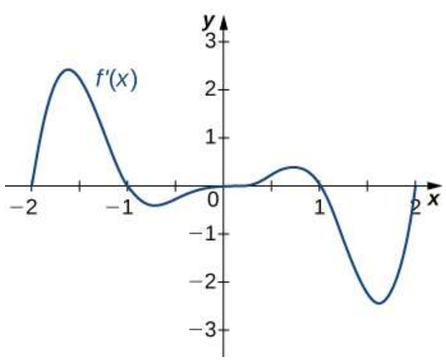

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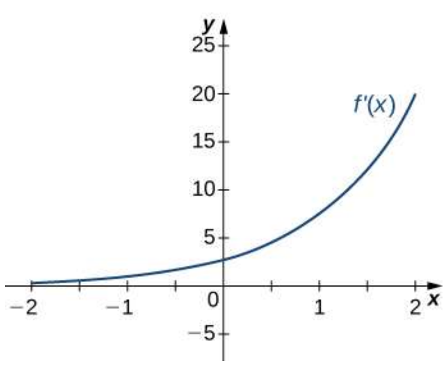

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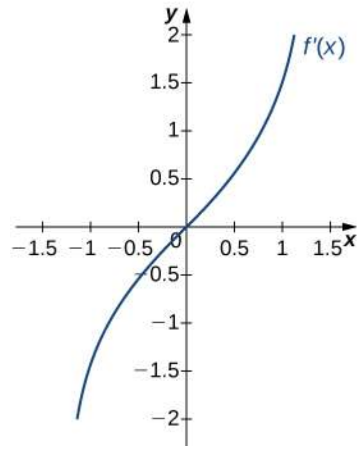

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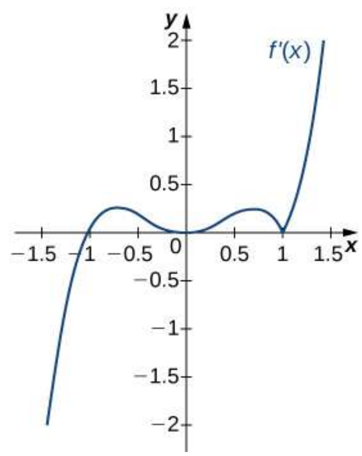
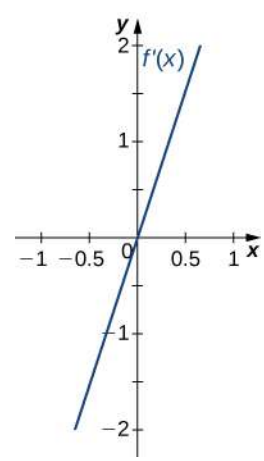
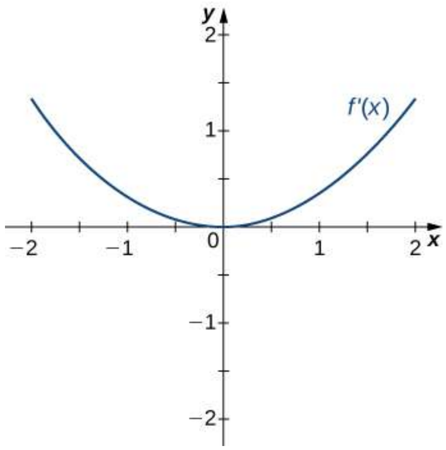


For the following exercises, analyze the graphs of f', then list all inflection points and intervals f that are concave up and concave down.

211.



212.



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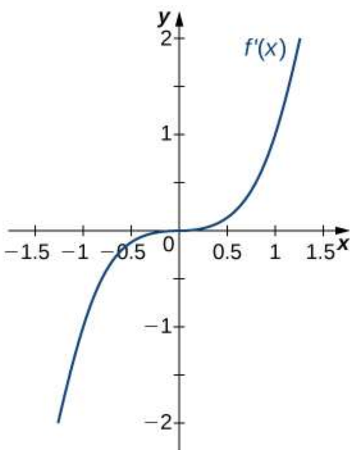

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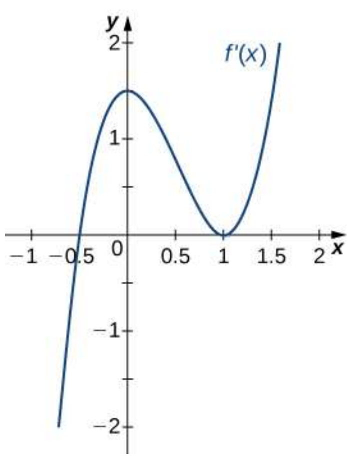
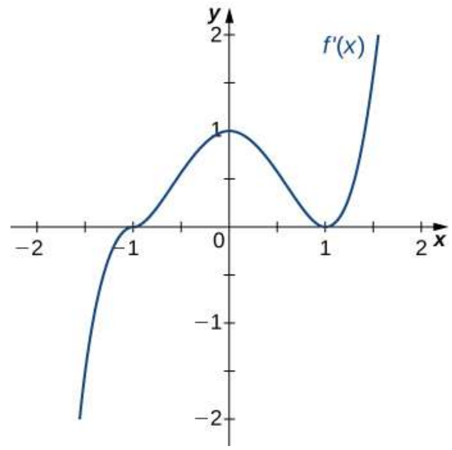


215.



For the following exercises, draw a graph that satisfies the given specifications for the domain $x = [-3, 3]$. The function does not have to be continuous or differentiable.

216. $f(x) > 0,\ f'(x) > 0$ over $x > 1,\ -3 < x < 0,\ f'(x) = 0$ over $0 < x < 1$

217. $f'(x) > 0$ over $x > 2,\ -3 < x < -1,\ f'(x) < 0$ over $-1 < x < 2,\ f''(x) < 0$ for all x

218. $f''(x) < 0$ over $-1 < x < 1,\ f''(x) > 0,\ -3 < x < -1,\ 1 < x < 3,$ local maximum at $x = 0,$ local minima at $x = \pm 2$

219. There is a local maximum at $x = 2,$ local minimum at $x = 1,$ and the graph is neither concave up nor concave down.

220. There are local maxima at $x = \pm 1,$ the function is concave up for all $x,$ and the function remains positive for all $x.$

For the following exercises, determine

a. intervals where f is increasing or decreasing and

b. local minima and maxima of f.

221. $f(x) = \sin x + \sin^3 x$ over $-\pi < x < \pi$

222. $f(x) = x^2 + \cos x$

For the following exercises, determine a. intervals where f is concave up or concave down, and b. the inflection points of f.

223. $f(x) = x^3 - 4x^2 + x + 2$

For the following exercises, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f.

224. $f(x) = x^2 - 6x$

225. $f(x) = x^3 - 6x^2$

226. $f(x) = x^4 - 6x^3$

227. $f(x) = x^{11} - 6x^{10}$

228. $f(x) = x + x^2 - x^3$

229. $f(x) = x^2 + x + 1$

230. $f(x) = x^3 + x^4$

For the following exercises, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f. Sketch the curve, then use a calculator to compare your answer. If you cannot determine the exact answer analytically, use a calculator.

231. **[T]** $f(x) = \sin(\pi x) - \cos(\pi x)$ over $x = [-1, 1]$

232. **[T]** $f(x) = x + \sin(2x)$ over $x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

233. **[T]** $f(x) = \sin x + \tan x$ over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

234. **[T]** $f(x) = (x-2)^2(x-4)^2$

235. **[T]** $f(x) = \frac{1}{1-x}, x \neq 1$

236. **[T]** $f(x) = \frac{\sin x}{x}$ over $x = [2\pi, 0) \cup (0, 2\pi]$

237. $f(x) = \sin(x)e^x$ over $x = [-\pi, \pi]$

238. $f(x) = \ln x\sqrt{x}, x > 0$

239. $f(x) = \frac{1}{4}\sqrt{x} + \frac{1}{x}, x > 0$

240. $f(x) = \frac{e^x}{x}, x \neq 0$

For the following exercises, interpret the sentences in terms of f, f', and f''.

241. The population is growing more slowly. Here f is the population.

242. A bike accelerates faster, but a car goes faster. Here $f =$ Bike's position minus Car's position.

243. The airplane lands smoothly. Here f is the plane's altitude.

244. Stock prices are at their peak. Here f is the stock price.

245. The economy is picking up speed. Here f is a measure of the economy, such as GDP.

For the following exercises, consider a third-degree polynomial $f(x)$, which has the properties $f'(1) = 0$, $f'(3) = 0$. Determine whether the following statements are *true or false*. Justify your answer.

246. $f(x) = 0$ for some $1 \leq x \leq 3$

247. $f''(x) = 0$ for some $1 \leq x \leq 3$

248. There is no absolute maximum at $x = 3$

249. If $f(x)$ has three roots, then it has 1 inflection point.

250. If $f(x)$ has one inflection point, then it has three real roots.

4.6 | Limits at Infinity and Asymptotes

Learning Objectives

- **4.6.1** Calculate the limit of a function as x increases or decreases without bound.
- **4.6.2** Recognize a horizontal asymptote on the graph of a function.
- **4.6.3** Estimate the end behavior of a function as x increases or decreases without bound.
- **4.6.4** Recognize an oblique asymptote on the graph of a function.
- **4.6.5** Analyze a function and its derivatives to draw its graph.

We have shown how to use the first and second derivatives of a function to describe the shape of a graph. To graph a function f defined on an unbounded domain, we also need to know the behavior of f as $x \to \pm\infty$. In this section, we define limits at infinity and show how these limits affect the graph of a function. At the end of this section, we outline a strategy for graphing an arbitrary function f.

Limits at Infinity

We begin by examining what it means for a function to have a finite limit at infinity. Then we study the idea of a function with an infinite limit at infinity. Back in **Introduction to Functions and Graphs**, we looked at vertical asymptotes; in this section we deal with horizontal and oblique asymptotes.

Limits at Infinity and Horizontal Asymptotes

Recall that $\lim_{x \to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as long as x is sufficiently close to a. We can extend this idea to limits at infinity. For example, consider the function $f(x) = 2 + \frac{1}{x}$. As can be seen graphically in **Figure 4.40** and numerically in **Table 4.2**, as the values of x get larger, the values of $f(x)$ approach 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim_{x \to \infty} f(x) = 2$. Similarly, for $x < 0$, as the values $|x|$ get larger, the values of $f(x)$ approaches 2. We say the limit as x approaches $-\infty$ of $f(x)$ is 2 and write $\lim_{x \to -\infty} f(x) = 2$.

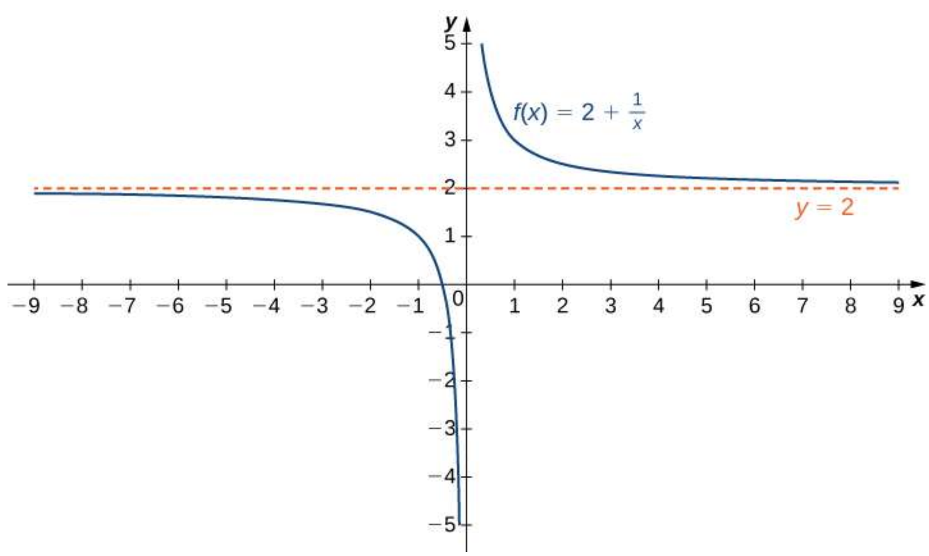


Figure 4.40 The function approaches the asymptote $y = 2$ as x approaches $\pm\infty$.

x	10	100	1,000	10,000
$2+\frac{1}{x}$	2.1	2.01	2.001	2.0001
x	-10	-100	-1000	$-10,000$
$2+\frac{1}{x}$	1.9	1.99	1.999	1.9999

Table 4.2 Values of a function f as $x \to \pm\infty$

More generally, for any function f, we say the limit as $x \to \infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as x is sufficiently large. In that case, we write $\lim_{x \to \infty} f(x) = L.$ Similarly, we say the limit as $x \to -\infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as $x < 0$ and $|x|$ is sufficiently large. In that case, we write $\lim_{x \to -\infty} f(x) = L.$ We now look at the definition of a function having a limit at infinity.

Definition

(Informal) If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, we say the function f has a **limit at infinity** and write

$$\lim_{x \to \infty} f(x) = L.$$

If the values of $f(x)$ becomes arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function f has a limit at negative infinity and write

$$\lim_{x \to -\infty} f(x) = L.$$

If the values $f(x)$ are getting arbitrarily close to some finite value L as $x \to \infty$ or $x \to -\infty$, the graph of f approaches the line $y = L.$ In that case, the line $y = L$ is a horizontal asymptote of f (**Figure 4.41**). For example, for the function $f(x) = \frac{1}{x}$, since $\lim_{x \to \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of $f(x) = \frac{1}{x}$.

Definition

If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of f.

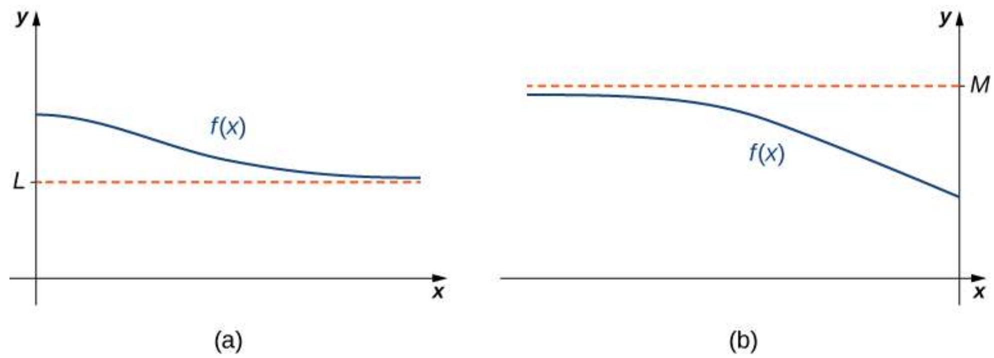


Figure 4.41 (a) As $x \to \infty$, the values of f are getting arbitrarily close to L. The line $y = L$ is a horizontal asymptote of f. (b) As $x \to -\infty$, the values of f are getting arbitrarily close to M. The line $y = M$ is a horizontal asymptote of f.

A function cannot cross a vertical asymptote because the graph must approach infinity (or $-\infty$) from at least one direction as x approaches the vertical asymptote. However, a function may cross a horizontal asymptote. In fact, a function may cross a horizontal asymptote an unlimited number of times. For example, the function $f(x) = \frac{(\cos x)}{x} + 1$ shown in **Figure 4.42** intersects the horizontal asymptote $y = 1$ an infinite number of times as it oscillates around the asymptote with ever-decreasing amplitude.

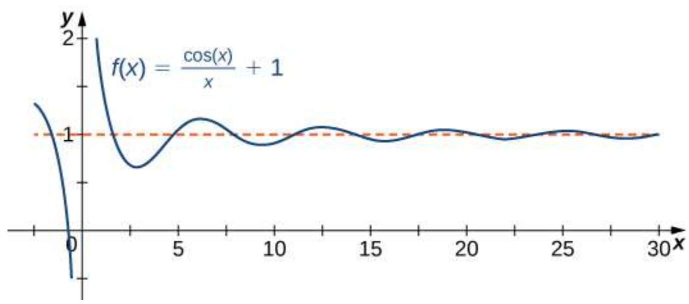


Figure 4.42 The graph of $f(x) = (\cos x)/x + 1$ crosses its horizontal asymptote $y = 1$ an infinite number of times.

The algebraic limit laws and squeeze theorem we introduced in **Introduction to Limits** also apply to limits at infinity. We illustrate how to use these laws to compute several limits at infinity.

Example 4.21

Computing Limits at Infinity

For each of the following functions f, evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. Determine the horizontal asymptote(s) for f.

a. $f(x) = 5 - \frac{2}{x^2}$

b. $f(x) = \frac{\sin x}{x}$

c. $f(x) = \tan^{-1}(x)$

Solution

a. Using the algebraic limit laws, we have $\lim_{x \to \infty}\left(5 - \frac{2}{x^2}\right) = \lim_{x \to \infty} 5 - 2\left(\lim_{x \to \infty} \frac{1}{x}\right) \cdot \left(\lim_{x \to \infty} \frac{1}{x}\right) = 5 - 2 \cdot 0 = 5.$

Similarly, $\lim_{x \to -\infty} f(x) = 5$. Therefore, $f(x) = 5 - \frac{2}{x^2}$ has a horizontal asymptote of $y = 5$ and f approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

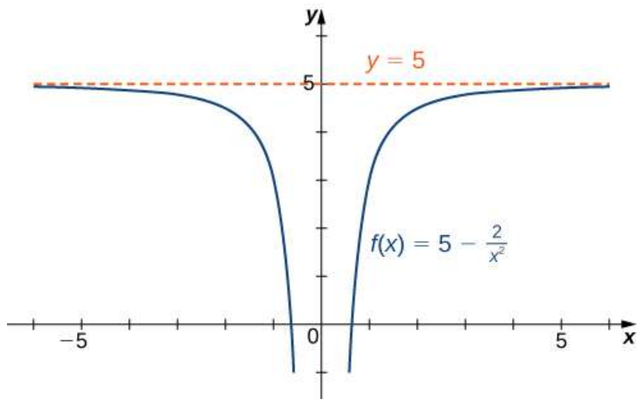


Figure 4.43 This function approaches a horizontal asymptote as $x \to \pm\infty$.

b. Since $-1 \le \sin x \le 1$ for all x, we have

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

for all $x \ne 0$. Also, since

$$\lim_{x \to \infty} \frac{-1}{x} = 0 = \lim_{x \to \infty} \frac{1}{x},$$

we can apply the squeeze theorem to conclude that

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0.$$

Similarly,

$$\lim_{x \to -\infty} \frac{\sin x}{x} = 0.$$

Thus, $f(x) = \frac{\sin x}{x}$ has a horizontal asymptote of $y = 0$ and $f(x)$ approaches this horizontal asymptote as $x \to \pm\infty$ as shown in the following graph.

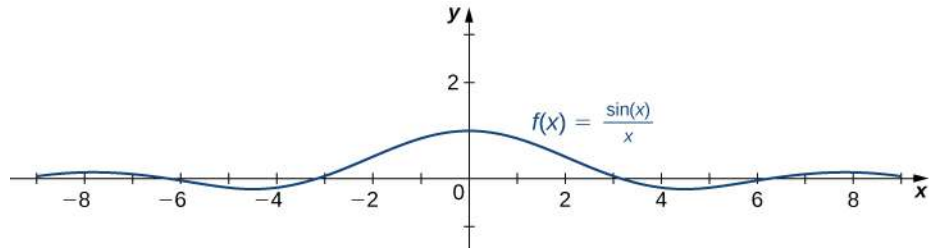


Figure 4.44 This function crosses its horizontal asymptote multiple times.

c. To evaluate $\lim_{x \to \infty} \tan^{-1}(x)$ and $\lim_{x \to -\infty} \tan^{-1}(x)$, we first consider the graph of $y = \tan(x)$ over the interval $(-\pi/2, \pi/2)$ as shown in the following graph.

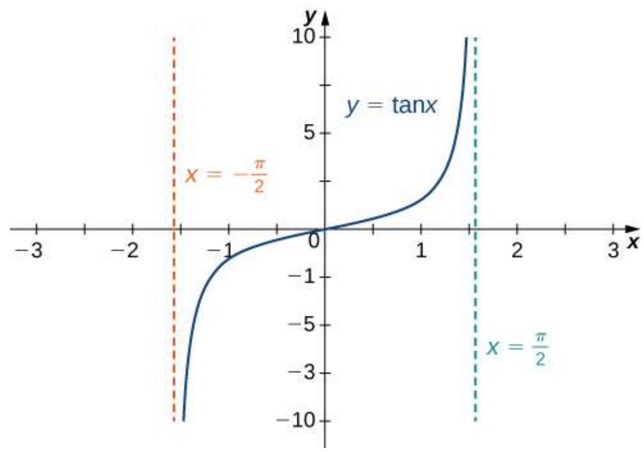


Figure 4.45 The graph of $\tan x$ has vertical asymptotes at $x = \pm\frac{\pi}{2}$

Since

$$\lim_{x \to (\pi/2)^-} \tan x = \infty,$$

it follows that

$$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}.$$

Similarly, since

$$\lim_{x \to (-\pi/2)^+} \tan x = -\infty,$$

it follows that

$$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}.$$

As a result, $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes of $f(x) = \tan^{-1}(x)$ as shown in the following graph.

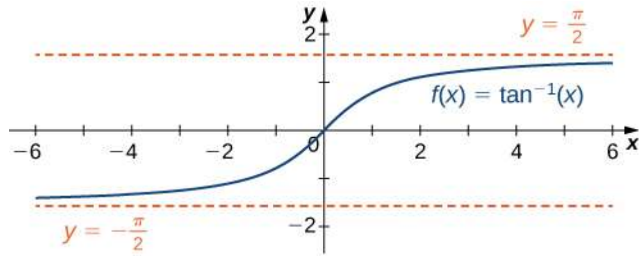


Figure 4.46 This function has two horizontal asymptotes.

4.20 Evaluate $\lim_{x \to -\infty}\left(3 + \frac{4}{x}\right)$ and $\lim_{x \to \infty}\left(3 + \frac{4}{x}\right)$. Determine the horizontal asymptotes of $f(x) = 3 + \frac{4}{x}$, if any.

Infinite Limits at Infinity

Sometimes the values of a function f become arbitrarily large as $x \to \infty$ (or as $x \to -\infty$). In this case, we write $\lim_{x \to \infty} f(x) = \infty$ (or $\lim_{x \to -\infty} f(x) = \infty$). On the other hand, if the values of f are negative but become arbitrarily large in magnitude as $x \to \infty$ (or as $x \to -\infty$), we write $\lim_{x \to \infty} f(x) = -\infty$ (or $\lim_{x \to -\infty} f(x) = -\infty$).

For example, consider the function $f(x) = x^3$. As seen in **Table 4.3** and **Figure 4.47**, as $x \to \infty$ the values $f(x)$ become arbitrarily large. Therefore, $\lim_{x \to \infty} x^3 = \infty$. On the other hand, as $x \to -\infty$, the values of $f(x) = x^3$ are negative but become arbitrarily large in magnitude. Consequently, $\lim_{x \to -\infty} x^3 = -\infty$.

x	10	20	50	100	1000
x^3	1000	8000	125,000	1,000,000	1,000,000,000
x	−10	−20	−50	−100	−1000
x^3	−1000	−8000	−125,000	−1,000,000	−1,000,000,000

Table 4.3 Values of a power function as $x \to \pm\infty$

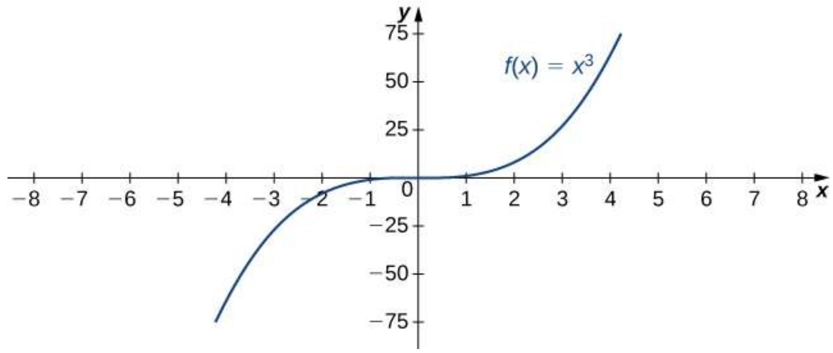


Figure 4.47 For this function, the functional values approach infinity as $x \to \pm\infty$.

Definition

(Informal) We say a function f has an infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = \infty.$$

if $f(x)$ becomes arbitrarily large for x sufficiently large. We say a function has a negative infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = -\infty.$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. Similarly, we can define infinite limits as $x \to -\infty$.

Formal Definitions

Earlier, we used the terms *arbitrarily close*, *arbitrarily large*, and *sufficiently large* to define limits at infinity informally. Although these terms provide accurate descriptions of limits at infinity, they are not precise mathematically. Here are more formal definitions of limits at infinity. We then look at how to use these definitions to prove results involving limits at infinity.

Definition

(Formal) We say a function f has a **limit at infinity**, if there exists a real number L such that for all $\varepsilon > 0,$ there exists $N > 0$ such that

$$|f(x) - L| < \varepsilon$$

for all $x > N.$ In that case, we write

$$\lim_{x \to \infty} f(x) = L$$

(see **Figure 4.48**).

We say a function f has a limit at negative infinity if there exists a real number L such that for all $\varepsilon > 0,$ there exists $N < 0$ such that

$$|f(x) - L| < \varepsilon$$

for all $x < N.$ In that case, we write

$$\lim_{x \to -\infty} f(x) = L.$$

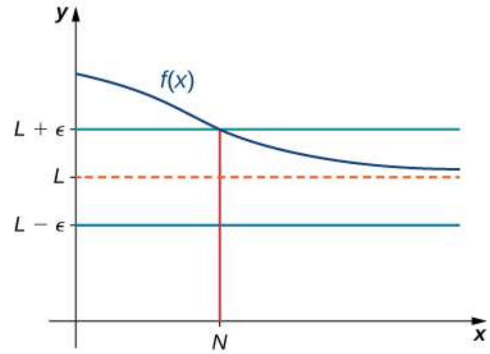


Figure 4.48 For a function with a limit at infinity, for all $x > N, \quad |f(x) - L| < \varepsilon$.

Earlier in this section, we used graphical evidence in **Figure 4.40** and numerical evidence in **Table 4.2** to conclude that $\lim_{x \to \infty} \left(2 + \frac{1}{x}\right) = 2$. Here we use the formal definition of limit at infinity to prove this result rigorously.

Example 4.22 A Finite Limit at Infinity Example

Use the formal definition of limit at infinity to prove that $\lim_{x \to \infty} \left(2 + \frac{1}{x}\right) = 2$.

Solution

Let $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon}$. Therefore, for all $x > N$, we have

$$\left|2 + \frac{1}{x} - 2\right| = \left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{N} = \varepsilon.$$

4.21 Use the formal definition of limit at infinity to prove that $\lim_{x \to \infty} \left(3 - \frac{1}{x^2}\right) = 3$.

We now turn our attention to a more precise definition for an infinite limit at infinity.

Definition

(Formal) We say a function f has an **infinite limit at infinity** and write

$$\lim_{x \to \infty} f(x) = \infty$$

if for all $M > 0$, there exists an $N > 0$ such that

$$f(x) > M$$

for all $x > N$ (see **Figure 4.49**).

We say a function has a negative infinite limit at infinity and write

$$\lim_{x \to \infty} f(x) = -\infty$$

if for all $M < 0$, there exists an $N > 0$ such that

$$f(x) < M$$

for all $x > N$.

Similarly we can define limits as $x \to -\infty$.

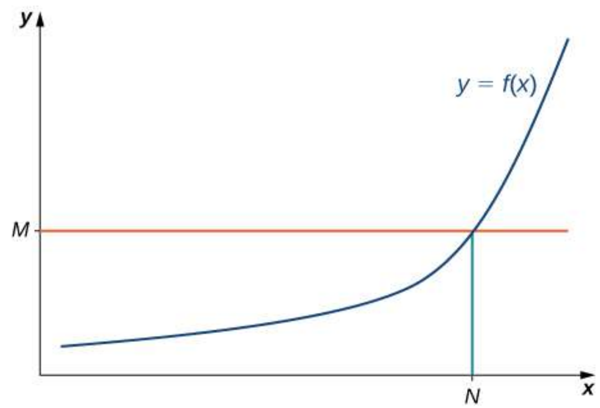


Figure 4.49 For a function with an infinite limit at infinity, for all $x > N$, $f(x) > M$.

Earlier, we used graphical evidence (**Figure 4.47**) and numerical evidence (**Table 4.3**) to conclude that $\lim_{x \to \infty} x^3 = \infty$. Here we use the formal definition of infinite limit at infinity to prove that result.

Example 4.23 An Infinite Limit at Infinity

Use the formal definition of infinite limit at infinity to prove that $\lim_{x \to \infty} x^3 = \infty$.

Solution

Let $M > 0$. Let $N = \sqrt[3]{M}$. Then, for all $x > N$, we have

$$x^3 > N^3 = \left(\sqrt[3]{M}\right)^3 = M.$$

Therefore, $\lim_{x \to \infty} x^3 = \infty$.

4.22 Use the formal definition of infinite limit at infinity to prove that $\lim_{x \to \infty} 3x^2 = \infty$.

End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

1. The function $f(x)$ approaches a horizontal asymptote $y = L$.
2. The function $f(x) \to \infty$ or $f(x) \to -\infty$.
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior.

Let's consider several classes of functions here and look at the different types of end behaviors for these functions.

End Behavior for Polynomial Functions

Consider the power function $f(x) = x^n$ where n is a positive integer. From **Figure 4.50** and **Figure 4.51**, we see that

$$\lim_{x \to \infty} x^n = \infty;\ n = 1, 2, 3, \ldots$$

and

$$\lim_{x \to -\infty} x^n = \begin{cases} \infty;\ n = 2, 4, 6, \ldots \\ -\infty;\ n = 1, 3, 5, \ldots \end{cases}.$$

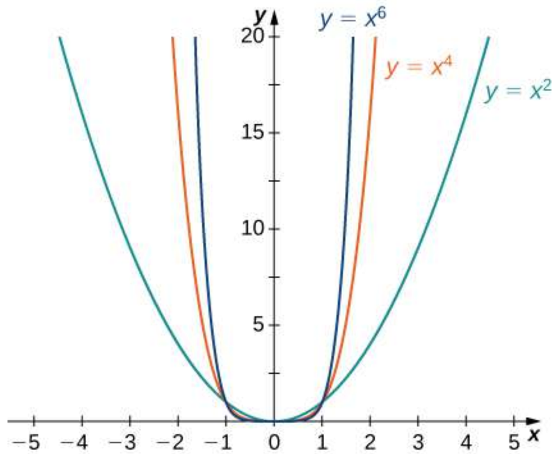


Figure 4.50 For power functions with an even power of n, $\lim_{x \to \infty} x^n = \infty = \lim_{x \to -\infty} x^n$.

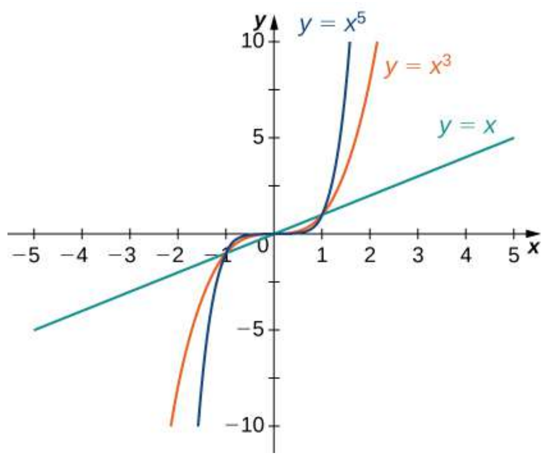


Figure 4.51 For power functions with an odd power of n, $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to -\infty} x^n = -\infty$.

Using these facts, it is not difficult to evaluate $\lim_{x \to \infty} cx^n$ and $\lim_{x \to -\infty} cx^n$, where c is any constant and n is a positive integer. If $c > 0$, the graph of $y = cx^n$ is a vertical stretch or compression of $y = x^n$, and therefore

$$\lim_{x \to \infty} cx^n = \lim_{x \to \infty} x^n \text{ and } \lim_{x \to -\infty} cx^n = \lim_{x \to -\infty} x^n \text{ if } c > 0.$$

If $c < 0$, the graph of $y = cx^n$ is a vertical stretch or compression combined with a reflection about the x-axis, and therefore

$$\lim_{x \to \infty} cx^n = -\lim_{x \to \infty} x^n \text{ and } \lim_{x \to -\infty} cx^n = -\lim_{x \to -\infty} x^n \text{ if } c < 0.$$

If $c = 0,\ y = cx^n = 0,$ in which case $\lim_{x \to \infty} cx^n = 0 = \lim_{x \to -\infty} cx^n.$

Example 4.24

Limits at Infinity for Power Functions

For each function $f,$ evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x).$

a. $f(x) = -5x^3$

b. $f(x) = 2x^4$

Solution

a. Since the coefficient of x^3 is $-5,$ the graph of $f(x) = -5x^3$ involves a vertical stretch and reflection of the graph of $y = x^3$ about the x-axis. Therefore, $\lim_{x \to \infty}\left(-5x^3\right) = -\infty$ and $\lim_{x \to -\infty}\left(-5x^3\right) = \infty.$

b. Since the coefficient of x^4 is $2,$ the graph of $f(x) = 2x^4$ is a vertical stretch of the graph of $y = x^4.$ Therefore, $\lim_{x \to \infty} 2x^4 = \infty$ and $\lim_{x \to -\infty} 2x^4 = \infty.$

4.23 Let $f(x) = -3x^4.$ Find $\lim_{x \to \infty} f(x).$

We now look at how the limits at infinity for power functions can be used to determine $\lim_{x \to \pm\infty} f(x)$ for any polynomial function $f.$ Consider a polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

of degree $n \geq 1$ so that $a_n \neq 0.$ Factoring, we see that

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \ldots + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}\right).$$

As $x \to \pm\infty,$ all the terms inside the parentheses approach zero except the first term. We conclude that

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n.$$

For example, the function $f(x) = 5x^3 - 3x^2 + 4$ behaves like $g(x) = 5x^3$ as $x \to \pm\infty$ as shown in **Figure 4.52** and **Table 4.4**.

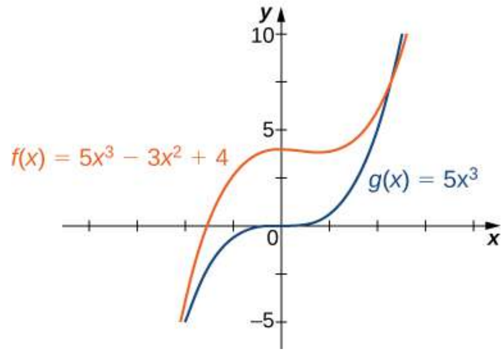


Figure 4.52 The end behavior of a polynomial is determined by the behavior of the term with the largest exponent.

x	10	100	1000
$f(x)=5x^3-3x^2+4$	4704	4,970,004	4,997,000,004
$g(x)=5x^3$	5000	5,000,000	5,000,000,000
x	−10	−100	−1000
$f(x)=5x^3-3x^2+4$	−5296	−5,029,996	−5,002,999,996
$g(x)=5x^3$	−5000	−5,000,000	−5,000,000,000

Table 4.4 A polynomial's end behavior is determined by the term with the largest exponent.

End Behavior for Algebraic Functions

The end behavior for rational functions and functions involving radicals is a little more complicated than for polynomials. In **Example 4.25**, we show that the limits at infinity of a rational function $f(x)=\frac{p(x)}{q(x)}$ depend on the relationship between the degree of the numerator and the degree of the denominator. To evaluate the limits at infinity for a rational function, we divide the numerator and denominator by the highest power of x appearing in the denominator. This determines which term in the overall expression dominates the behavior of the function at large values of x.

Example 4.25

Determining End Behavior for Rational Functions

For each of the following functions, determine the limits as $x \to \infty$ and $x \to -\infty$. Then, use this information to describe the end behavior of the function.

a. $f(x)=\frac{3x-1}{2x+5}$ (*Note:* The degree of the numerator and the denominator are the same.)

b. $f(x) = \dfrac{3x^2 + 2x}{4x^3 - 5x + 7}$ (*Note:* The degree of numerator is less than the degree of the denominator.)

c. $f(x) = \dfrac{3x^2 + 4x}{x + 2}$ (*Note:* The degree of numerator is greater than the degree of the denominator.)

Solution

a. The highest power of x in the denominator is x. Therefore, dividing the numerator and denominator by x and applying the algebraic limit laws, we see that

$$\begin{aligned}\lim_{x \to \pm\infty} \frac{3x - 1}{2x + 5} &= \lim_{x \to \pm\infty} \frac{3 - 1/x}{2 + 5/x} \\ &= \frac{\lim_{x \to \pm\infty}(3 - 1/x)}{\lim_{x \to \pm\infty}(2 + 5/x)} \\ &= \frac{\lim_{x \to \pm\infty} 3 - \lim_{x \to \pm\infty} 1/x}{\lim_{x \to \pm\infty} 2 + \lim_{x \to \pm\infty} 5/x} \\ &= \frac{3 - 0}{2 + 0} = \frac{3}{2}.\end{aligned}$$

Since $\lim_{x \to \pm\infty} f(x) = \frac{3}{2}$, we know that $y = \frac{3}{2}$ is a horizontal asymptote for this function as shown in the following graph.

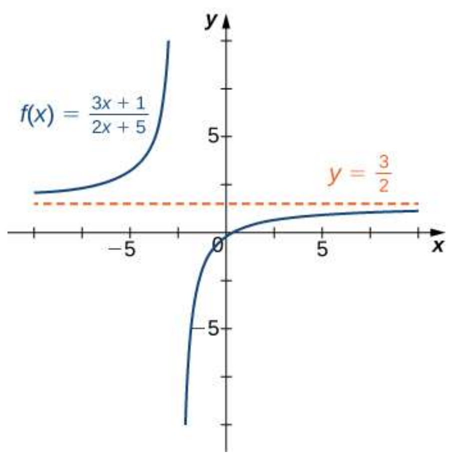


Figure 4.53 The graph of this rational function approaches a horizontal asymptote as $x \to \pm\infty$.

b. Since the largest power of x appearing in the denominator is x^3, divide the numerator and denominator by x^3. After doing so and applying algebraic limit laws, we obtain

$$\lim_{x \to \pm\infty} \frac{3x^2 + 2x}{4x^3 - 5x + 7} = \lim_{x \to \pm\infty} \frac{3/x + 2/x^2}{4 - 5/x^2 + 7/x^3} = \frac{3(0) + 2(0)}{4 - 5(0) + 7(0)} = 0.$$

Therefore f has a horizontal asymptote of $y = 0$ as shown in the following graph.

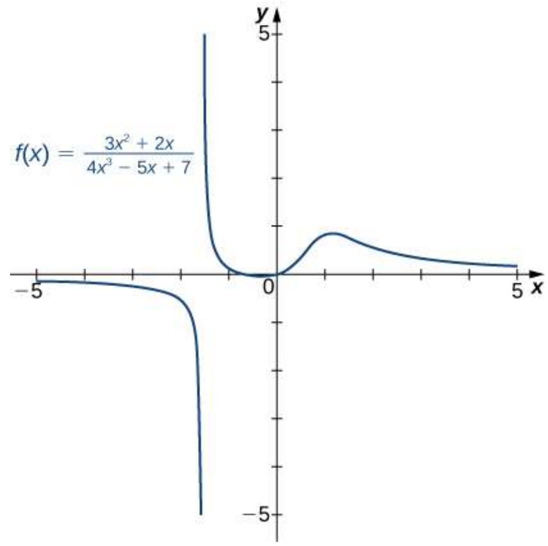


Figure 4.54 The graph of this rational function approaches the horizontal asymptote $y = 0$ as $x \to \pm\infty$.

c. Dividing the numerator and denominator by x, we have

$$\lim_{x \to \pm\infty} \frac{3x^2 + 4x}{x + 2} = \lim_{x \to \pm\infty} \frac{3x + 4}{1 + 2/x}.$$

As $x \to \pm\infty$, the denominator approaches 1. As $x \to \infty$, the numerator approaches $+\infty$. As $x \to -\infty$, the numerator approaches $-\infty$. Therefore $\lim_{x \to \infty} f(x) = \infty$, whereas $\lim_{x \to -\infty} f(x) = -\infty$ as shown in the following figure.

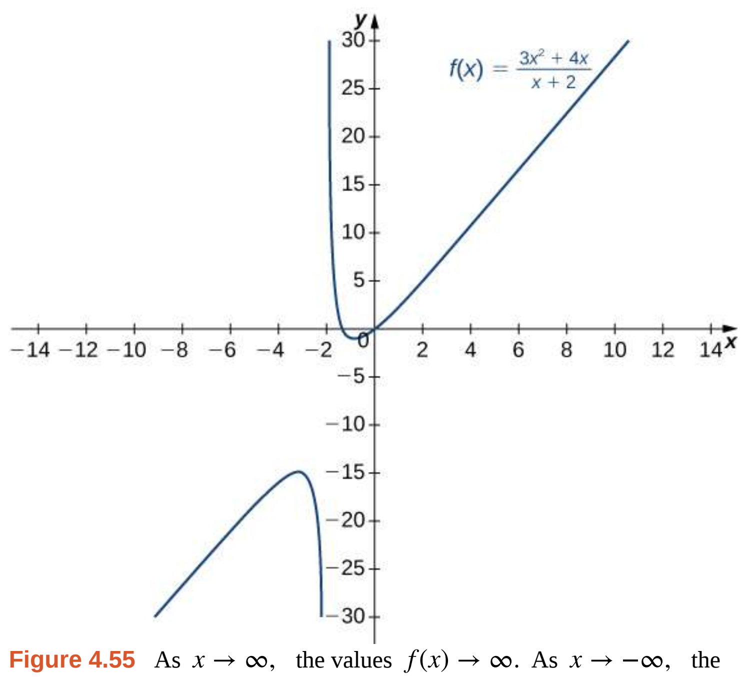


Figure 4.55 As $x \to \infty$, the values $f(x) \to \infty$. As $x \to -\infty$, the values $f(x) \to -\infty$.

4.24 Evaluate $\lim_{x \to \pm\infty} \frac{3x^2 + 2x - 1}{5x^2 - 4x + 7}$ and use these limits to determine the end behavior of $f(x) = \frac{3x^2 + 2x - 1}{5x^2 - 4x + 7}$.

Before proceeding, consider the graph of $f(x) = \frac{(3x^2 + 4x)}{(x + 2)}$ shown in **Figure 4.56**. As $x \to \infty$ and $x \to -\infty$, the graph of f appears almost linear. Although f is certainly not a linear function, we now investigate why the graph of f seems to be approaching a linear function. First, using long division of polynomials, we can write

$$f(x) = \frac{3x^2 + 4x}{x + 2} = 3x - 2 + \frac{4}{x + 2}.$$

Since $\frac{4}{(x + 2)} \to 0$ as $x \to \pm\infty$, we conclude that

$$\lim_{x \to \pm\infty} (f(x) - (3x - 2)) = \lim_{x \to \pm\infty} \frac{4}{x + 2} = 0.$$

Therefore, the graph of f approaches the line $y = 3x - 2$ as $x \to \pm\infty$. This line is known as an **oblique asymptote** for f (**Figure 4.56**).

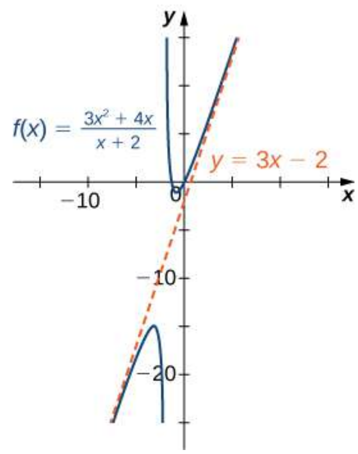


Figure 4.56 The graph of the rational function $f(x) = (3x^2 + 4x)/(x + 2)$ approaches the oblique asymptote $y = 3x - 2$ as $x \to \pm\infty$.

We can summarize the results of **Example 4.25** to make the following conclusion regarding end behavior for rational functions. Consider a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0},$$

where $a_n \neq 0$ and $b_m \neq 0$.

1. If the degree of the numerator is the same as the degree of the denominator $(n = m)$, then f has a horizontal asymptote of $y = a_n/b_m$ as $x \to \pm\infty$.
2. If the degree of the numerator is less than the degree of the denominator $(n < m)$, then f has a horizontal asymptote of $y = 0$ as $x \to \pm\infty$.
3. If the degree of the numerator is greater than the degree of the denominator $(n > m)$, then f does not have a

horizontal asymptote. The limits at infinity are either positive or negative infinity, depending on the signs of the leading terms. In addition, using long division, the function can be rewritten as

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

where the degree of $r(x)$ is less than the degree of $q(x)$. As a result, $\lim_{x \to \pm\infty} r(x)/q(x) = 0$. Therefore, the values of $[f(x) - g(x)]$ approach zero as $x \to \pm\infty$. If the degree of $p(x)$ is exactly one more than the degree of $q(x)$ $(n = m + 1)$, the function $g(x)$ is a linear function. In this case, we call $g(x)$ an oblique asymptote.

Now let's consider the end behavior for functions involving a radical.

Example 4.26

Determining End Behavior for a Function Involving a Radical

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \frac{3x - 2}{\sqrt{4x^2 + 5}}$ and describe the end behavior of f.

Solution

Let's use the same strategy as we did for rational functions: divide the numerator and denominator by a power of x. To determine the appropriate power of x, consider the expression $\sqrt{4x^2 + 5}$ in the denominator. Since

$$\sqrt{4x^2 + 5} \approx \sqrt{4x^2} = 2|x|$$

for large values of x in effect x appears just to the first power in the denominator. Therefore, we divide the numerator and denominator by $|x|$. Then, using the fact that $|x| = x$ for $x > 0$, $|x| = -x$ for $x < 0$, and $|x| = \sqrt{x^2}$ for all x, we calculate the limits as follows:

$$\begin{aligned}
\lim_{x \to \infty} \frac{3x - 2}{\sqrt{4x^2 + 5}} &= \lim_{x \to \infty} \frac{(1/|x|)(3x - 2)}{(1/|x|)\sqrt{4x^2 + 5}} \\
&= \lim_{x \to \infty} \frac{(1/x)(3x - 2)}{\sqrt{\left(1/x^2\right)\left(4x^2 + 5\right)}} \\
&= \lim_{x \to \infty} \frac{3 - 2/x}{\sqrt{4 + 5/x^2}} = \frac{3}{\sqrt{4}} = \frac{3}{2} \\
\lim_{x \to -\infty} \frac{3x - 2}{\sqrt{4x^2 + 5}} &= \lim_{x \to -\infty} \frac{(1/|x|)(3x - 2)}{(1/|x|)\sqrt{4x^2 + 5}} \\
&= \lim_{x \to -\infty} \frac{(-1/x)(3x - 2)}{\sqrt{\left(1/x^2\right)\left(4x^2 + 5\right)}} \\
&= \lim_{x \to -\infty} \frac{-3 + 2/x}{\sqrt{4 + 5/x^2}} = \frac{-3}{\sqrt{4}} = \frac{-3}{2}.
\end{aligned}$$

Therefore, $f(x)$ approaches the horizontal asymptote $y = \frac{3}{2}$ as $x \to \infty$ and the horizontal asymptote $y = -\frac{3}{2}$ as $x \to -\infty$ as shown in the following graph.

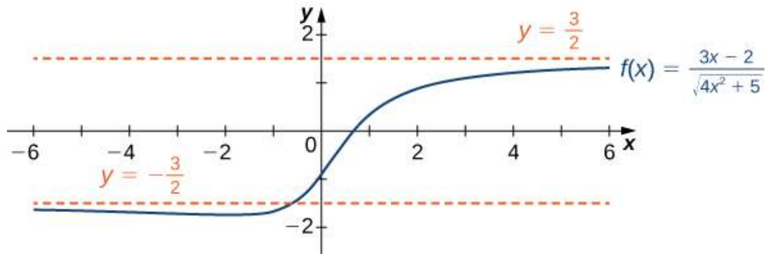


Figure 4.57 This function has two horizontal asymptotes and it crosses one of the asymptotes.

4.25 Evaluate $\lim_{x \to \infty} \frac{\sqrt{3x^2 + 4}}{x + 6}$.

Determining End Behavior for Transcendental Functions

The six basic trigonometric functions are periodic and do not approach a finite limit as $x \to \pm\infty$. For example, $\sin x$ oscillates between $1 \text{ and } -1$ (**Figure 4.58**). The tangent function x has an infinite number of vertical asymptotes as $x \to \pm\infty$; therefore, it does not approach a finite limit nor does it approach $\pm\infty$ as $x \to \pm\infty$ as shown in **Figure 4.59**.

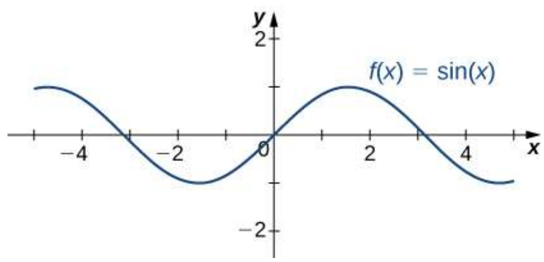


Figure 4.58 The function $f(x) = \sin x$ oscillates between $1 \text{ and } -1$ as $x \to \pm\infty$

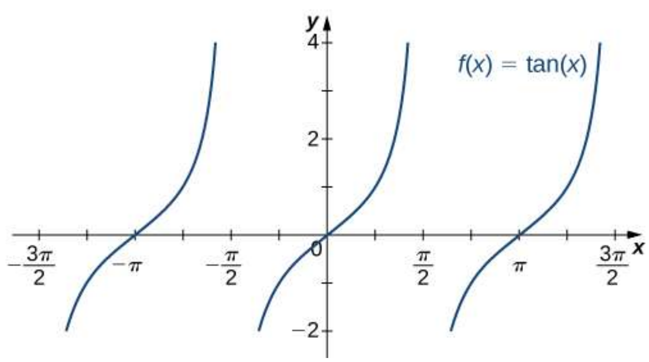


Figure 4.59 The function $f(x) = \tan x$ does not approach a limit and does not approach $\pm\infty$ as $x \to \pm\infty$

Recall that for any base $b > 0, b \neq 1,$ the function $y = b^x$ is an exponential function with domain $(-\infty, \infty)$ and range $(0, \infty)$. If $b > 1, y = b^x$ is increasing over $(-\infty, \infty)$. If $0 < b < 1,$ $y = b^x$ is decreasing over $(-\infty, \infty)$. For the natural exponential function $f(x) = e^x,$ $e \approx 2.718 > 1$. Therefore, $f(x) = e^x$ is increasing on $(-\infty, \infty)$ and the range is $(0, \infty)$. The exponential function $f(x) = e^x$ approaches ∞ as $x \to \infty$ and approaches 0 as $x \to -\infty$ as shown in **Table 4.5** and **Figure 4.60**.

x	-5	-2	0	2	5
e^x	0.00674	0.135	1	7.389	148.413

Table 4.5 End behavior of the natural exponential function

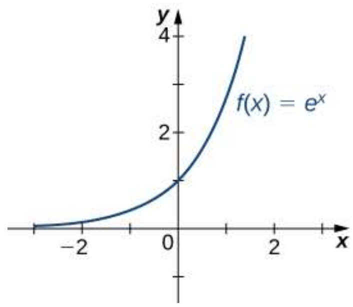


Figure 4.60 The exponential function approaches zero as $x \to -\infty$ and approaches ∞ as $x \to \infty$.

Recall that the natural logarithm function $f(x) = \ln(x)$ is the inverse of the natural exponential function $y = e^x$. Therefore, the domain of $f(x) = \ln(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of $f(x) = \ln(x)$ is the reflection of the graph of $y = e^x$ about the line $y = x$. Therefore, $\ln(x) \to -\infty$ as $x \to 0^+$ and $\ln(x) \to \infty$ as $x \to \infty$ as shown in **Figure 4.61** and **Table 4.6**.

x	0.01	0.1	1	10	100
$\ln(x)$	-4.605	-2.303	0	2.303	4.605

Table 4.6 End behavior of the natural logarithm function

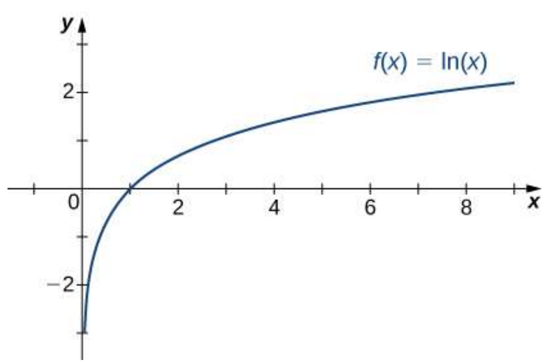


Figure 4.61 The natural logarithm function approaches ∞ as $x \to \infty$.

Example 4.27

Determining End Behavior for a Transcendental Function

Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \frac{(2+3e^x)}{(7-5e^x)}$ and describe the end behavior of f.

Solution

To find the limit as $x \to \infty$, divide the numerator and denominator by e^x:

$$\begin{aligned}\lim_{x \to \infty} f(x) &= \lim_{x \to \infty} \frac{2+3e^x}{7-5e^x} \\ &= \lim_{x \to \infty} \frac{(2/e^x)+3}{(7/e^x)-5}.\end{aligned}$$

As shown in **Figure 4.60**, $e^x \to \infty$ as $x \to \infty$. Therefore,

$$\lim_{x \to \infty} \frac{2}{e^x} = 0 = \lim_{x \to \infty} \frac{7}{e^x}.$$

We conclude that $\lim_{x \to \infty} f(x) = -\frac{3}{5}$, and the graph of f approaches the horizontal asymptote $y = -\frac{3}{5}$ as $x \to \infty$. To find the limit as $x \to -\infty$, use the fact that $e^x \to 0$ as $x \to -\infty$ to conclude that $\lim_{x \to \infty} f(x) = \frac{2}{7}$, and therefore the graph of approaches the horizontal asymptote $y = \frac{2}{7}$ as $x \to -\infty$.

4.26 Find the limits as $x \to \infty$ and $x \to -\infty$ for $f(x) = \frac{(3e^x-4)}{(5e^x+2)}$.

Guidelines for Drawing the Graph of a Function

We now have enough analytical tools to draw graphs of a wide variety of algebraic and transcendental functions. Before showing how to graph specific functions, let's look at a general strategy to use when graphing any function.

Problem-Solving Strategy: Drawing the Graph of a Function

Given a function f, use the following steps to sketch a graph of f:

1. Determine the domain of the function.
2. Locate the x- and y-intercepts.
3. Evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ to determine the end behavior. If either of these limits is a finite number L, then $y = L$ is a horizontal asymptote. If either of these limits is ∞ or $-\infty$, determine whether f has an oblique asymptote. If f is a rational function such that $f(x) = \frac{p(x)}{q(x)}$, where the degree of the numerator is greater than the degree of the denominator, then f can be written as

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

where the degree of $r(x)$ is less than the degree of $q(x)$. The values of $f(x)$ approach the values of $g(x)$ as

$x \to \pm\infty$. If $g(x)$ is a linear function, it is known as an *oblique asymptote*.

4. Determine whether f has any vertical asymptotes.
5. Calculate f'. Find all critical points and determine the intervals where f is increasing and where f is decreasing. Determine whether f has any local extrema.
6. Calculate f''. Determine the intervals where f is concave up and where f is concave down. Use this information to determine whether f has any inflection points. The second derivative can also be used as an alternate means to determine or verify that f has a local extremum at a critical point.

Now let's use this strategy to graph several different functions. We start by graphing a polynomial function.

Example 4.28

Sketching a Graph of a Polynomial

Sketch a graph of $f(x) = (x-1)^2(x+2)$.

Solution

Step 1. Since f is a polynomial, the domain is the set of all real numbers.

Step 2. When $x = 0$, $f(x) = 2$. Therefore, the y-intercept is $(0, 2)$. To find the x-intercepts, we need to solve the equation $(x-1)^2(x+2) = 0$, gives us the x-intercepts $(1, 0)$ and $(-2, 0)$

Step 3. We need to evaluate the end behavior of f. As $x \to \infty$, $(x-1)^2 \to \infty$ and $(x+2) \to \infty$. Therefore, $\lim_{x \to \infty} f(x) = \infty$. As $x \to -\infty$, $(x-1)^2 \to \infty$ and $(x+2) \to -\infty$. Therefore, $\lim_{x \to -\infty} f(x) = -\infty$. To get even more information about the end behavior of f, we can multiply the factors of f. When doing so, we see that

$$f(x) = (x-1)^2(x+2) = x^3 - 3x + 2.$$

Since the leading term of f is x^3, we conclude that f behaves like $y = x^3$ as $x \to \pm\infty$.

Step 4. Since f is a polynomial function, it does not have any vertical asymptotes.

Step 5. The first derivative of f is

$$f'(x) = 3x^2 - 3.$$

Therefore, f has two critical points: $x = 1, -1$. Divide the interval $(-\infty, \infty)$ into the three smaller intervals: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. Then, choose test points $x = -2$, $x = 0$, and $x = 2$ from these intervals and evaluate the sign of $f'(x)$ at each of these test points, as shown in the following table.

Interval	**Test Point**	**Sign of Derivative** $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$	**Conclusion**
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing.
$(-1, 1)$	$x = 0$	$(+)(-)(+) = -$	f is decreasing.
$(1, \infty)$	$x = 2$	$(+)(+)(+) = +$	f is increasing.

From the table, we see that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. Evaluating $f(x)$ at those two points, we find that the local maximum value is $f(-1) = 4$ and the local minimum value is $f(1) = 0$.

Step 6. The second derivative of f is

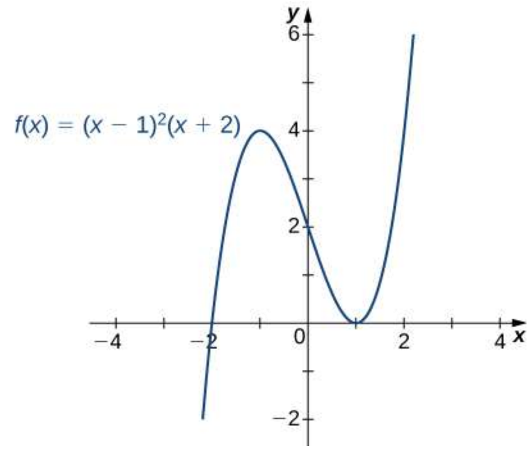
$$f''(x) = 6x.$$

The second derivative is zero at $x = 0$. Therefore, to determine the concavity of f, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$ and $(0, \infty)$, and choose test points $x = -1$ and $x = 1$ to determine the concavity of f on each of these smaller intervals as shown in the following table.

Interval	**Test Point**	**Sign of** $f''(x) = 6x$	**Conclusion**
$(-\infty, 0)$	$x = -1$	$-$	f is concave down.
$(0, \infty)$	$x = 1$	$+$	f is concave up.

We note that the information in the preceding table confirms the fact, found in step 5, that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. In addition, the information found in step 5—namely, f has a local maximum at $x = -1$ and a local minimum at $x = 1$, and $f'(x) = 0$ at those points—combined with the fact that f'' changes sign only at $x = 0$ confirms the results found in step 6 on the concavity of f.

Combining this information, we arrive at the graph of $f(x) = (x-1)^2(x+2)$ shown in the following graph.



4.27 Sketch a graph of $f(x) = (x-1)^3(x+2)$.

Example 4.29

Sketching a Rational Function

Sketch the graph of $f(x) = \frac{x^2}{\left(1-x^2\right)}$.

Solution

Step 1. The function f is defined as long as the denominator is not zero. Therefore, the domain is the set of all real numbers x except $x = \pm 1$.

Step 2. Find the intercepts. If $x = 0$, then $f(x) = 0$, so 0 is an intercept. If $y = 0$, then $\frac{x^2}{\left(1-x^2\right)} = 0$, which implies $x = 0$. Therefore, $(0, 0)$ is the only intercept.

Step 3. Evaluate the limits at infinity. Since f is a rational function, divide the numerator and denominator by the highest power in the denominator: x^2. We obtain

$$\lim_{x \to \pm\infty} \frac{x^2}{1-x^2} = \lim_{x \to \pm\infty} \frac{1}{\frac{1}{x^2} - 1} = -1.$$

Therefore, f has a horizontal asymptote of $y = -1$ as $x \to \infty$ and $x \to -\infty$.

Step 4. To determine whether f has any vertical asymptotes, first check to see whether the denominator has any zeroes. We find the denominator is zero when $x = \pm 1$. To determine whether the lines $x = 1$ or $x = -1$ are vertical asymptotes of f, evaluate $\lim_{x \to 1} f(x)$ and $\lim_{x \to -1} f(x)$. By looking at each one-sided limit as $x \to 1$, we see that

$$\lim_{x \to 1^+} \frac{x^2}{1 - x^2} = -\infty \text{ and } \lim_{x \to 1^-} \frac{x^2}{1 - x^2} = \infty.$$

In addition, by looking at each one-sided limit as $x \to -1$, we find that

$$\lim_{x \to -1^+} \frac{x^2}{1 - x^2} = \infty \text{ and } \lim_{x \to -1^-} \frac{x^2}{1 - x^2} = -\infty.$$

Step 5. Calculate the first derivative:

$$f'(x) = \frac{\left(1 - x^2\right)(2x) - x^2(-2x)}{\left(1 - x^2\right)^2} = \frac{2x}{\left(1 - x^2\right)^2}.$$

Critical points occur at points x where $f'(x) = 0$ or $f'(x)$ is undefined. We see that $f'(x) = 0$ when $x = 0$. The derivative f' is not undefined at any point in the domain of f. However, $x = \pm 1$ are not in the domain of f. Therefore, to determine where f is increasing and where f is decreasing, divide the interval $(-\infty, \infty)$ into four smaller intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$, and choose a test point in each interval to determine the sign of $f'(x)$ in each of these intervals. The values $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are good choices for test points as shown in the following table.

Interval	**Test Point**	**Sign of $f'(x) = \frac{2x}{\left(1 - x^2\right)^2}$**	**Conclusion**
$(-\infty, -1)$	$x = -2$	$-/+ = -$	f is decreasing.
$(-1, 0)$	$x = -1/2$	$-/+ = -$	f is decreasing.
$(0, 1)$	$x = 1/2$	$+/+ = +$	f is increasing.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing.

From this analysis, we conclude that f has a local minimum at $x = 0$ but no local maximum.

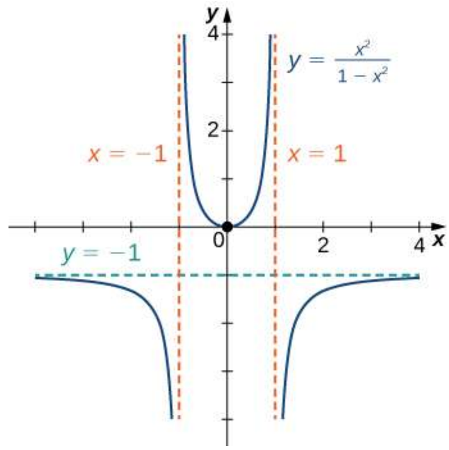
Step 6. Calculate the second derivative:

$$\begin{aligned} f''(x) &= \frac{\left(1-x^2\right)^2(2) - 2x\left(2\left(1-x^2\right)(-2x)\right)}{\left(1-x^2\right)^4} \\ &= \frac{\left(1-x^2\right)\left[2\left(1-x^2\right)+8x^2\right]}{\left(1-x^2\right)^4} \\ &= \frac{2\left(1-x^2\right)+8x^2}{\left(1-x^2\right)^3} \\ &= \frac{6x^2+2}{\left(1-x^2\right)^3}. \end{aligned}$$

To determine the intervals where f is concave up and where f is concave down, we first need to find all points x where $f''(x)=0$ or $f''(x)$ is undefined. Since the numerator $6x^2+2\neq 0$ for any x, $f''(x)$ is never zero. Furthermore, f'' is not undefined for any x in the domain of f. However, as discussed earlier, $x=\pm 1$ are not in the domain of f. Therefore, to determine the concavity of f, we divide the interval $(-\infty, \infty)$ into the three smaller intervals $(-\infty, -1)$, $(-1, -1)$, and $(1, \infty)$, and choose a test point in each of these intervals to evaluate the sign of $f''(x)$. in each of these intervals. The values $x=-2$, $x=0$, and $x=2$ are possible test points as shown in the following table.

Interval	**Test Point**	**Sign of $f''(x)=\frac{6x^2+2}{\left(1-x^2\right)^3}$**	**Conclusion**
$(-\infty, -1)$	$x=-2$	$+/- = -$	f is concave down.
$(-1, -1)$	$x=0$	$+/+ = +$	f is concave up.
$(1, \infty)$	$x=2$	$+/- = -$	f is concave down.

Combining all this information, we arrive at the graph of f shown below. Note that, although f changes concavity at $x=-1$ and $x=1$, there are no inflection points at either of these places because f is not continuous at $x=-1$ or $x=1$.



4.28 Sketch a graph of $f(x) = \frac{(3x + 5)}{(8 + 4x)}$.

Example 4.30

Sketching a Rational Function with an Oblique Asymptote

Sketch the graph of $f(x) = \frac{x^2}{(x - 1)}$

Solution

Step 1. The domain of f is the set of all real numbers x except $x = 1$.

Step 2. Find the intercepts. We can see that when $x = 0$, $f(x) = 0$, so $(0, 0)$ is the only intercept.

Step 3. Evaluate the limits at infinity. Since the degree of the numerator is one more than the degree of the denominator, f must have an oblique asymptote. To find the oblique asymptote, use long division of polynomials to write

$$f(x) = \frac{x^2}{x - 1} = x + 1 + \frac{1}{x - 1}.$$

Since $1/(x - 1) \to 0$ as $x \to \pm\infty$, $f(x)$ approaches the line $y = x + 1$ as $x \to \pm\infty$. The line $y = x + 1$ is an oblique asymptote for f.

Step 4. To check for vertical asymptotes, look at where the denominator is zero. Here the denominator is zero at $x = 1$. Looking at both one-sided limits as $x \to 1$, we find

$$\lim_{x \to 1^+} \frac{x^2}{x - 1} = \infty \text{ and } \lim_{x \to 1^-} \frac{x^2}{x - 1} = -\infty.$$

Therefore, $x = 1$ is a vertical asymptote, and we have determined the behavior of f as x approaches 1 from the right and the left.

Step 5. Calculate the first derivative:

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

We have $f'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$. Therefore, $x = 0$ and $x = 2$ are critical points. Since f is undefined at $x = 1$, we need to divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$, and choose a test point from each interval to evaluate the sign of $f'(x)$ in each of these smaller intervals. For example, let $x = -1$, $x = \frac{1}{2}$, $x = \frac{3}{2}$, and $x = 3$ be the test points as shown in the following table.

Interval	**Test Point**	**Sign of** $f'(x) = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$	**Conclusion**
$(-\infty, 0)$	$x = -1$	$(-)(-)/+ = +$	f is increasing.
$(0, 1)$	$x = 1/2$	$(+)(-)/+ = -$	f is decreasing.
$(1, 2)$	$x = 3/2$	$(+)(-)/+ = -$	f is decreasing.
$(2, \infty)$	$x = 3$	$(+)(+)/+ = +$	f is increasing.

From this table, we see that f has a local maximum at $x = 0$ and a local minimum at $x = 2$. The value of f at the local maximum is $f(0) = 0$ and the value of f at the local minimum is $f(2) = 4$. Therefore, $(0, 0)$ and $(2, 4)$ are important points on the graph.

Step 6. Calculate the second derivative:

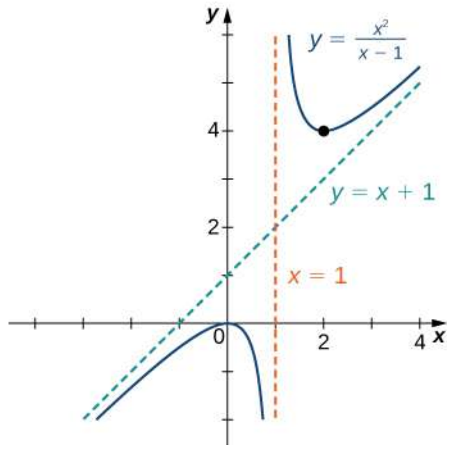
$$\begin{aligned} f''(x) &= \frac{(x-1)^2(2x-2) - (x^2 - 2x)(2(x-1))}{(x-1)^4} \\ &= \frac{(x-1)\left[(x-1)(2x-2) - 2(x^2 - 2x)\right]}{(x-1)^4} \\ &= \frac{(x-1)(2x-2) - 2(x^2 - 2x)}{(x-1)^3} \\ &= \frac{2x^2 - 4x + 2 - (2x^2 - 4x)}{(x-1)^3} \\ &= \frac{2}{(x-1)^3}. \end{aligned}$$

We see that $f''(x)$ is never zero or undefined for x in the domain of f. Since f is undefined at $x = 1$, to check concavity we just divide the interval $(-\infty, \infty)$ into the two smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose a test point from each interval to evaluate the sign of $f''(x)$ in each of these intervals. The values $x = 0$

and $x = 2$ are possible test points as shown in the following table.

Interval	**Test Point**	**Sign of $f''(x) = \frac{2}{(x-1)^3}$**	**Conclusion**
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is concave down.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is concave up.

From the information gathered, we arrive at the following graph for f.



4.29 Find the oblique asymptote for $f(x) = \frac{(3x^3 - 2x + 1)}{(2x^2 - 4)}$.

Example 4.31

Sketching the Graph of a Function with a Cusp

Sketch a graph of $f(x) = (x-1)^{2/3}$.

Solution

Step 1. Since the cube-root function is defined for all real numbers x and $(x-1)^{2/3} = \left(\sqrt[3]{x-1}\right)^2$, the domain of f is all real numbers.

Step 2: To find the y-intercept, evaluate $f(0)$. Since $f(0) = 1$, the y-intercept is $(0, 1)$. To find the x-intercept, solve $(x-1)^{2/3} = 0$. The solution of this equation is $x = 1$, so the x-intercept is $(1, 0)$.

Step 3: Since $\lim_{x \to \pm\infty} (x-1)^{2/3} = \infty$, the function continues to grow without bound as $x \to \infty$ and $x \to -\infty$.

Step 4: The function has no vertical asymptotes.

Step 5: To determine where f is increasing or decreasing, calculate f'. We find

$$f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}.$$

This function is not zero anywhere, but it is undefined when $x = 1$. Therefore, the only critical point is $x = 1$. Divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points in each of these intervals to determine the sign of $f'(x)$ in each of these smaller intervals. Let $x = 0$ and $x = 2$ be the test points as shown in the following table.

Interval	**Test Point**	**Sign of $f'(x) = \frac{2}{3(x-1)^{1/3}}$**	**Conclusion**
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is decreasing.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing.

We conclude that f has a local minimum at $x = 1$. Evaluating f at $x = 1$, we find that the value of f at the local minimum is zero. Note that $f'(1)$ is undefined, so to determine the behavior of the function at this critical point, we need to examine $\lim_{x \to 1} f'(x)$. Looking at the one-sided limits, we have

$$\lim_{x \to 1^+} \frac{2}{3(x-1)^{1/3}} = \infty \text{ and } \lim_{x \to 1^-} \frac{2}{3(x-1)^{1/3}} = -\infty.$$

Therefore, f has a cusp at $x = 1$.

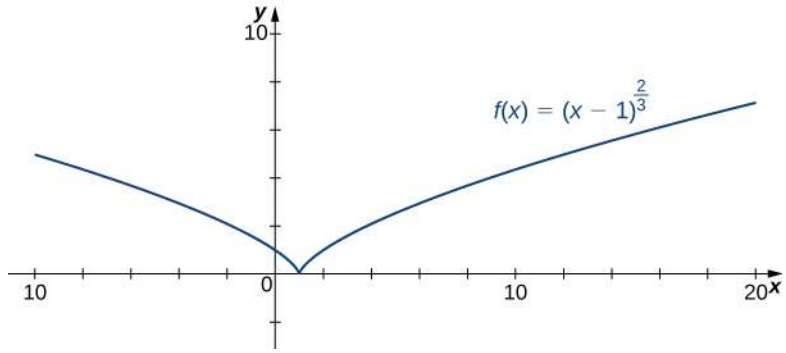
Step 6: To determine concavity, we calculate the second derivative of f:

$$f''(x) = -\frac{2}{9}(x-1)^{-4/3} = \frac{-2}{9(x-1)^{4/3}}.$$

We find that $f''(x)$ is defined for all x, but is undefined when $x = 1$. Therefore, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points to evaluate the sign of $f''(x)$ in each of these intervals. As we did earlier, let $x = 0$ and $x = 2$ be test points as shown in the following table.

Interval	**Test Point**	**Sign of $f''(x) = \frac{-2}{9(x-1)^{4/3}}$**	**Conclusion**
$(-\infty, 1)$	$x = 0$	$-/+ = -$	f is concave down.
$(1, \infty)$	$x = 2$	$-/+ = -$	f is concave down.

From this table, we conclude that f is concave down everywhere. Combining all of this information, we arrive at the following graph for f.

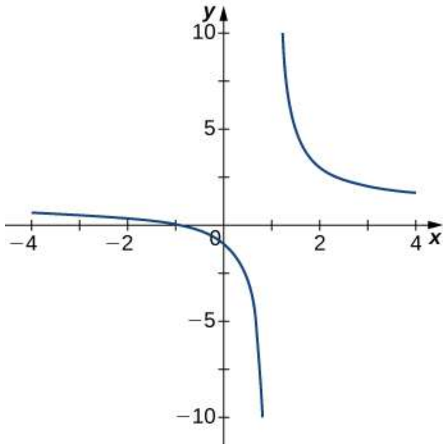


4.30 Consider the function $f(x) = 5 - x^{2/3}$. Determine the point on the graph where a cusp is located. Determine the end behavior of f.

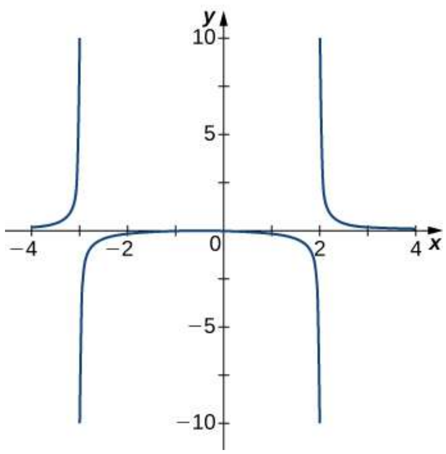
4.6 EXERCISES

For the following exercises, examine the graphs. Identify where the vertical asymptotes are located.

251.



252.



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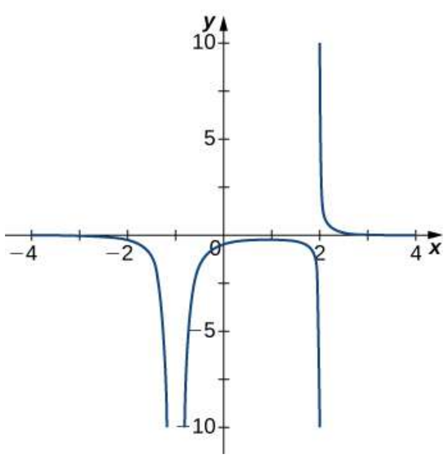
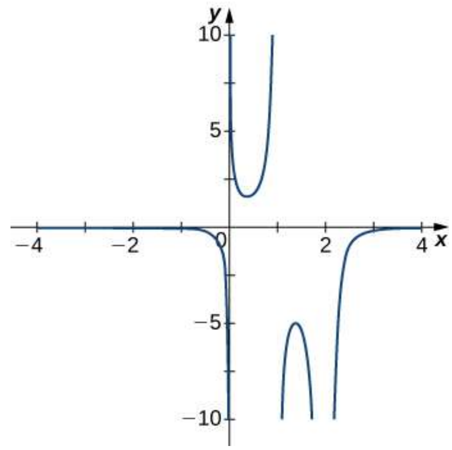
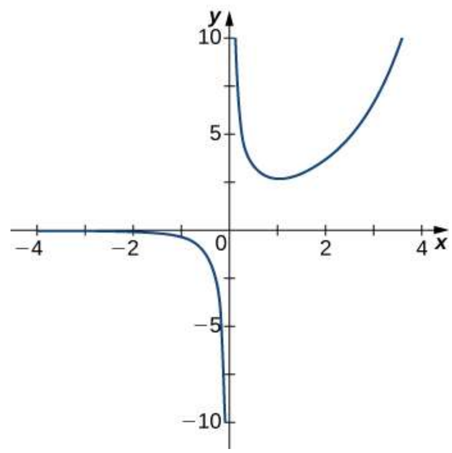


254.



255.



For the following functions $f(x)$, determine whether there is an asymptote at $x = a$. Justify your answer without graphing on a calculator.

256. $f(x) = \dfrac{x+1}{x^2+5x+4}$, $a = -1$

257. $f(x) = \dfrac{x}{x-2}$, $a = 2$

258. $f(x) = (x+2)^{3/2}$, $a = -2$

259. $f(x) = (x-1)^{-1/3}$, $a = 1$

260. $f(x) = 1 + x^{-2/5}$, $a = 1$

For the following exercises, evaluate the limit.

261. $\lim\limits_{x \to \infty} \dfrac{1}{3x+6}$

262. $\lim_{x \to \infty} \frac{2x - 5}{4x}$

263. $\lim_{x \to \infty} \frac{x^2 - 2x + 5}{x + 2}$

264. $\lim_{x \to -\infty} \frac{3x^3 - 2x}{x^2 + 2x + 8}$

265. $\lim_{x \to -\infty} \frac{x^4 - 4x^3 + 1}{2 - 2x^2 - 7x^4}$

266. $\lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 1}}$

267. $\lim_{x \to -\infty} \frac{\sqrt{4x^2 - 1}}{x + 2}$

268. $\lim_{x \to \infty} \frac{4x}{\sqrt{x^2 - 1}}$

269. $\lim_{x \to -\infty} \frac{4x}{\sqrt{x^2 - 1}}$

270. $\lim_{x \to \infty} \frac{2\sqrt{x}}{x - \sqrt{x} + 1}$

For the following exercises, find the horizontal and vertical asymptotes.

271. $f(x) = x - \frac{9}{x}$

272. $f(x) = \frac{1}{1 - x^2}$

273. $f(x) = \frac{x^3}{4 - x^2}$

274. $f(x) = \frac{x^2 + 3}{x^2 + 1}$

275. $f(x) = \sin(x)\sin(2x)$

276. $f(x) = \cos x + \cos(3x) + \cos(5x)$

277. $f(x) = \frac{x\sin(x)}{x^2 - 1}$

278. $f(x) = \frac{x}{\sin(x)}$

279. $f(x) = \frac{1}{x^3 + x^2}$

280. $f(x) = \frac{1}{x - 1} - 2x$

281. $f(x) = \frac{x^3 + 1}{x^3 - 1}$

282. $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$

283. $f(x) = x - \sin x$

284. $f(x) = \frac{1}{x} - \sqrt{x}$

For the following exercises, construct a function $f(x)$ that has the given asymptotes.

285. $x = 1$ and $y = 2$

286. $x = 1$ and $y = 0$

287. $y = 4, \quad x = -1$

288. $x = 0$

For the following exercises, graph the function on a graphing calculator on the window $x = [-5, 5]$ and estimate the horizontal asymptote or limit. Then, calculate the actual horizontal asymptote or limit.

289. **[T]** $f(x) = \frac{1}{x + 10}$

290. **[T]** $f(x) = \frac{x + 1}{x^2 + 7x + 6}$

291. **[T]** $\lim_{x \to -\infty} x^2 + 10x + 25$

292. **[T]** $\lim_{x \to -\infty} \frac{x + 2}{x^2 + 7x + 6}$

293. **[T]** $\lim_{x \to \infty} \frac{3x + 2}{x + 5}$

For the following exercises, draw a graph of the functions without using a calculator. Be sure to notice all important features of the graph: local maxima and minima, inflection points, and asymptotic behavior.

294. $y = 3x^2 + 2x + 4$

295. $y = x^3 - 3x^2 + 4$

296. $y = \frac{2x + 1}{x^2 + 6x + 5}$

297. $y = \frac{x^3 + 4x^2 + 3x}{3x + 9}$

298. $y = \frac{x^2 + x - 2}{x^2 - 3x - 4}$

299. $y = \sqrt{x^2 - 5x + 4}$

300. $y = 2x\sqrt{16 - x^2}$

301. $y = \frac{\cos x}{x}$, on $x = [-2\pi, 2\pi]$

302. $y = e^x - x^3$

303. $y = x\tan x,\ x = [-\pi, \pi]$

304. $y = x\ln(x),\ x > 0$

305. $y = x^2 \sin(x),\ x = [-2\pi, 2\pi]$

306. For $f(x) = \frac{P(x)}{Q(x)}$ to have an asymptote at $y = 2$ then the polynomials $P(x)$ and $Q(x)$ must have what relation?

307. For $f(x) = \frac{P(x)}{Q(x)}$ to have an asymptote at $x = 0,$ then the polynomials $P(x)$ and $Q(x).$ must have what relation?

308. If $f'(x)$ has asymptotes at $y = 3$ and $x = 1,$ then $f(x)$ has what asymptotes?

309. Both $f(x) = \frac{1}{(x - 1)}$ and $g(x) = \frac{1}{(x - 1)^2}$ have asymptotes at $x = 1$ and $y = 0.$ What is the most obvious difference between these two functions?

310. True or false: Every ratio of polynomials has vertical asymptotes.

4.7 | Applied Optimization Problems

Learning Objectives

4.7.1 Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the **optimization problems** that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in Example 4.32, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example 4.32

Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (Figure 4.62). Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

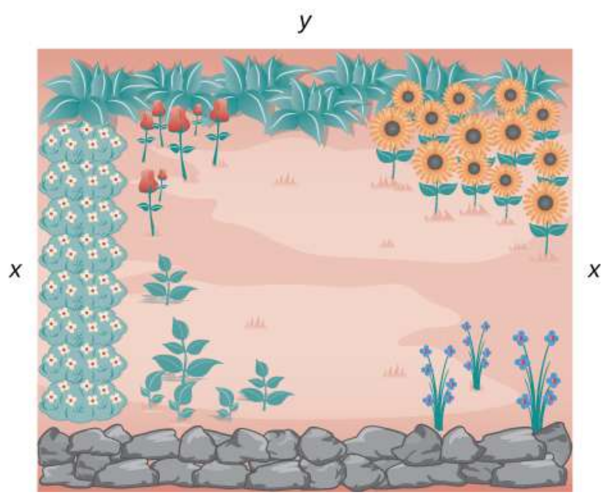


Figure 4.62 We want to determine the measurements x and y that will create a garden with a maximum area using 100 ft of fencing.

Solution

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is 100 ft. From **Figure 4.62**, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden. Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ (**Figure 4.63**). We conclude that the maximum area must occur when $x = 25$. Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50\text{ ft}$. The area of this garden is 1250 ft^2.

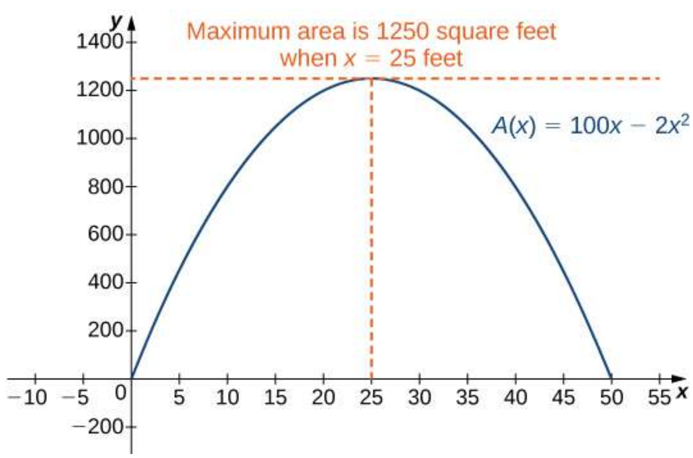


Figure 4.63 To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$.

4.31 Determine the maximum area if we want to make the same rectangular garden as in **Figure 4.63**, but we have 200 ft of fencing.

Now let's look at a general strategy for solving optimization problems similar to **Example 4.32**.

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step $3.$ Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step $4.$ This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example 4.33

Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution

Step 1: Let x be the side length of the square to be removed from each corner (**Figure 4.64**). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.

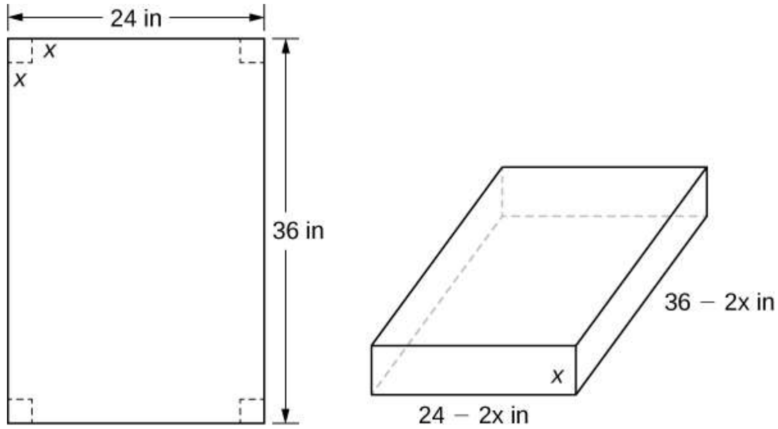


Figure 4.64 A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step $2,$ are trying to maximize the volume of a box. The volume of a box is $V = L \cdot W \cdot H,$ where $L,$ $W,$ and H are the length, width, and height, respectively.

Step 4: From **Figure 4.64**, we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$V(x) = (36 - 2x)(24 - 2x)x = 4x^3 - 120x^2 + 864x.$$

Step 5: To determine the domain of consideration, let's examine **Figure 4.64**. Certainly, we need $x > 0.$ Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12).$ Since V is a continuous function over the closed interval $[0, 12],$ we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12],$ V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12,$ the maximum must occur at a critical point. The derivative is

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by $12,$ the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} = \frac{20 \pm \sqrt{112}}{2} = \frac{20 \pm 4\sqrt{7}}{2} = 10 \pm 2\sqrt{7}.$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}$ in. The maximum volume is $V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825$ in.^{3} as shown in the following graph.

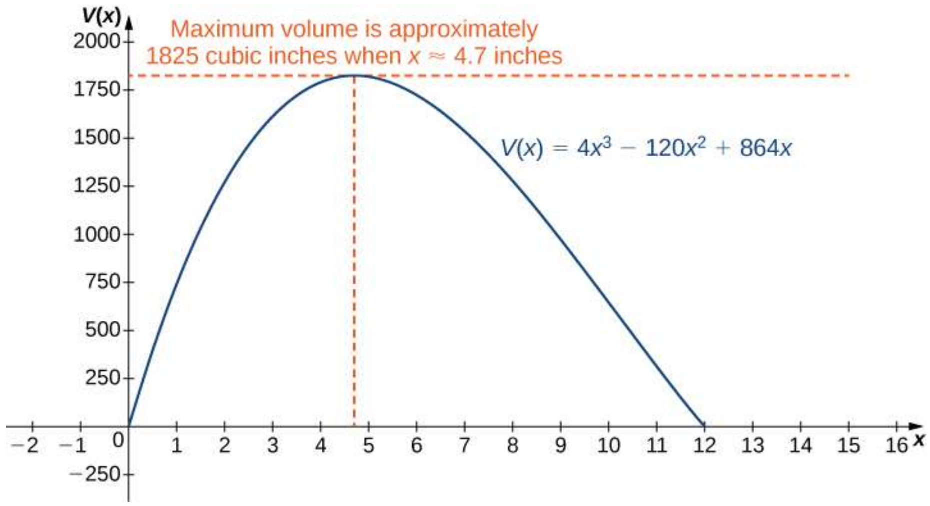


Figure 4.65 Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

Watch a **video (http://www.openstax.org/l/20_boxvolume)** about optimizing the volume of a box.

4.32 Suppose the dimensions of the cardboard in **Example 4.33** are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Example 4.34

Minimizing Travel Time

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution

Step 1: Let x be the distance running and let y be the distance swimming (**Figure 4.66**). Let T be the time it takes to get from the cabin to the island.

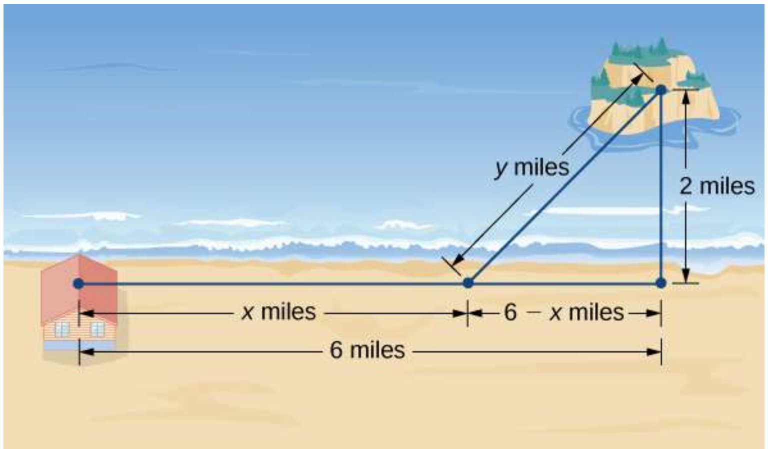


Figure 4.66 How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate × Time $(D = R \times T)$, the time spent running is

$$T_{\text{running}} = \frac{D_{\text{running}}}{R_{\text{running}}} = \frac{x}{8},$$

and the time spent swimming is

$$T_{\text{swimming}} = \frac{D_{\text{swimming}}}{R_{\text{swimming}}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From **Figure 4.66**, the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x\text{ mi}$. Therefore, by the Pythagorean theorem, $2^2 + (6 - x)^2 = y^2$, and we obtain $y = \sqrt{(6 - x)^2 + 4}$. Thus, the total time spent traveling is given by the function

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6 - x)^2 + 4}}{3}.$$

Step 5: From **Figure 4.66**, we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

$$T'(x) = \frac{1}{8} - \frac{1}{2}\frac{\left[(6 - x)^2 + 4\right]^{-1/2}}{3} \cdot 2(6 - x) = \frac{1}{8} - \frac{(6 - x)}{3\sqrt{(6 - x)^2 + 4}}.$$

If $T'(x) = 0$, then

$$\frac{1}{8} = \frac{6 - x}{3\sqrt{(6 - x)^2 + 4}}.$$

Therefore,

$$3\sqrt{(6 - x)^2 + 4} = 8(6 - x). \quad \textbf{(4.6)}$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

$$9\left[(6 - x)^2 + 4\right] = 64(6 - x)^2,$$

which implies

$$55(6 - x)^2 = 36.$$

We conclude that if x is a critical point, then x satisfies

$$(x - 6)^2 = \frac{36}{55}.$$

Therefore, the possibilities for critical points are

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + 6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - 6/\sqrt{55}$ is in the domain. Since we squared both sides of **Equation 4.6** to arrive at the possible critical points, it remains to verify that $x = 6 - 6/\sqrt{55}$ satisfies **Equation 4.6**. Since $x = 6 - 6/\sqrt{55}$ does satisfy that equation, we conclude that $x = 6 - 6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of $x,$ we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6,$ and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}.$ We find that $T(0) \approx 2.108$ h and $T(6) \approx 1.417$ h, whereas $T\left(6 - 6/\sqrt{55}\right) \approx 1.368$ h. Therefore, we conclude that T has a local minimum at $x \approx 5.19$ mi.

4.33 Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the rate of 2.5 mph and runs at a rate of 6 mph. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

In business, companies are interested in maximizing revenue. In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

Example 4.35

Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \le p \le 200,$ the number of cars n they rent per day can be modeled by the linear function

$n(p) = 1000 - 5p.$ If they charge $\$50$ per day or less, they will rent all their cars. If they charge $\$200$ per day or more, they will not rent any cars. Assuming the owners plan to charge customers between \$50 per day and $\$200$ per day to rent a car, how much should they charge to maximize their revenue?

Solution

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize $R.$

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \times p.$

Step 4: Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p,$ the revenue R can be represented by the function

$$R(p) = n \times p = (1000 - 5p)p = -5p^2 + 1000p.$$

Step 5: Since the owners plan to charge between $\$50$ per car per day and $\$200$ per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200].$

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200],$ it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000.$ Therefore, the critical point is $p = 100$ When $p = 100,$ $R(100) = \$50{,}000.$ When $p = 50,$ $R(p) = \$37{,}500.$ When $p = 200,$ $R(p) = \$0.$ Therefore, the absolute maximum occurs at $p = \$100.$ The car rental company should charge $\$100$ per day per car to maximize revenue as shown in the following figure.

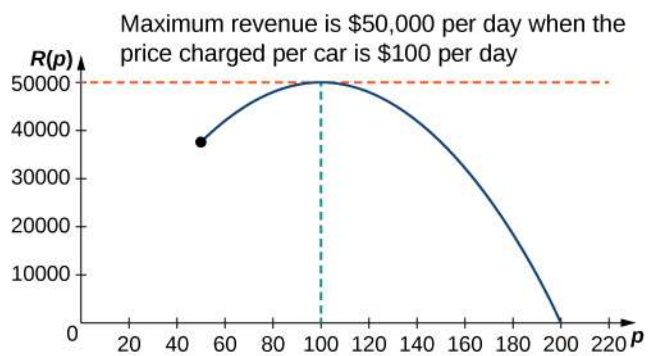


Figure 4.67 To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

4.34 A car rental company charges its customers p dollars per day, where $60 \le p \le 150.$ It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p.$ How much should the company charge each customer to maximize revenue?

Example 4.36

Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

Solution

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.

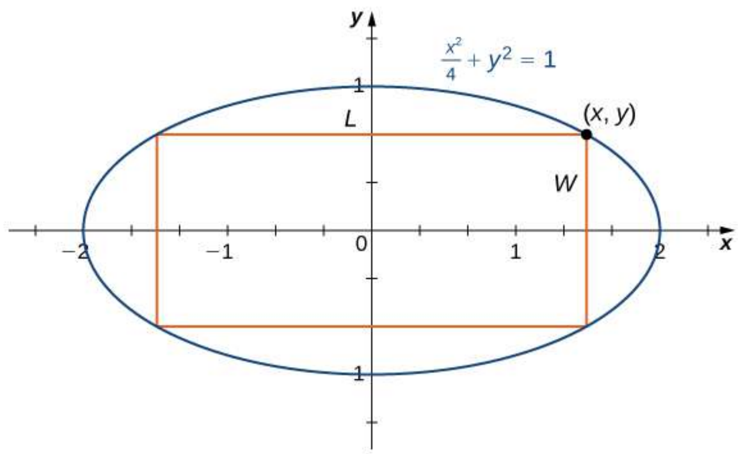


Figure 4.68 We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A.

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in **Figure 4.68**. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4} + y^2 = 1$ and $y > 0$, we have $y = \sqrt{1 - \frac{x^2}{4}}$. Therefore, the area is

$$A = LW = (2x)(2y) = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}.$$

Step 5: From **Figure 4.68**, we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy $0 < x < 2$. Therefore, the problem reduces to looking for the maximum value of $A(x)$ over the open interval $(0, 2)$. Since $A(x)$ will have an absolute maximum (and absolute minimum) over the closed interval $[0, 2]$, we consider $A(x) = 2x\sqrt{4 - x^2}$ over the interval $[0, 2]$. If the absolute maximum occurs at an interior point, then we have found an absolute maximum in the open interval.

Step 6: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A(x) = 0$. For $0 < x < 2$, $A(x) > 0$. Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

$$\begin{aligned} A'(x) &= 2\sqrt{4-x^2} + 2x \cdot \frac{1}{2\sqrt{4-x^2}}(-2x) \\ &= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}} \\ &= \frac{8-4x^2}{\sqrt{4-x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0$. We can see that if x is a solution of

$$\frac{8-4x^2}{\sqrt{4-x^2}} = 0, \tag{4.7}$$

then x must satisfy

$$8 - 4x^2 = 0.$$

Therefore, $x^2 = 2$. Thus, $x = \pm\sqrt{2}$ are the possible solutions of **Equation 4.7**. Since we are considering x over the interval $[0, 2]$, $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of **Equation 4.7**. Since $x = \sqrt{2}$ is a solution of **Equation 4.7**, we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0, 2]$. Therefore, $A(x)$ must have an absolute maximum at the critical point $x = \sqrt{2}$. To determine the dimensions of the rectangle, we need to find the length L and the width W. If $x = \sqrt{2}$ then

$$y = \sqrt{1 - \frac{(\sqrt{2})^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \frac{2}{\sqrt{2}} = \sqrt{2}$. The area of this rectangle is $A = LW = (2\sqrt{2})(\sqrt{2}) = 4$.

4.35 Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

Example 4.37

Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of 216 in.^3 is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.

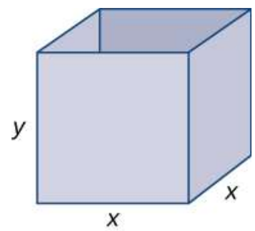


Figure 4.69 We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

$$S = 4xy + x^2.$$

Step 4: Since the volume of this box is $x^2 y$ and the volume is given as 216 in.^3, the constraint equation is

$$x^2 y = 216.$$

Solving the constraint equation for y, we have $y = \frac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2.$$

Therefore, $S(x) = \frac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2 y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2 y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \to 0^+$, $S(x) \to \infty$. Also, as $x \to \infty$, $S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S'(x) = 0$ when $2x = \frac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see **Figure 4.70**). When $x = 6\sqrt[3]{2}$, $y = \frac{216}{\left(6\sqrt[3]{2}\right)^2} = 3\sqrt[3]{2}$ in. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}$ in. and $y = 3\sqrt[3]{2}$ in. With these dimensions, the surface area is

$$S\left(6\sqrt[3]{2}\right) = \frac{864}{6\sqrt[3]{2}} + \left(6\sqrt[3]{2}\right)^2 = 108\sqrt[3]{4}\text{ in.}^2$$

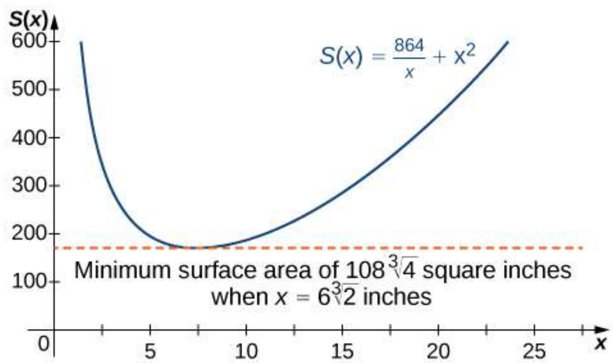


Figure 4.70 We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

4.36 Consider the same open-top box, which is to have volume 216 in.^3. Suppose the cost of the material for the base is $20\,¢/\text{in.}^2$ and the cost of the material for the sides is $30\,¢/\text{in.}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

4.7 EXERCISES

For the following exercises, answer by proof, counterexample, or explanation.

311. When you find the maximum for an optimization problem, why do you need to check the sign of the derivative around the critical points?

312. Why do you need to check the endpoints for optimization problems?

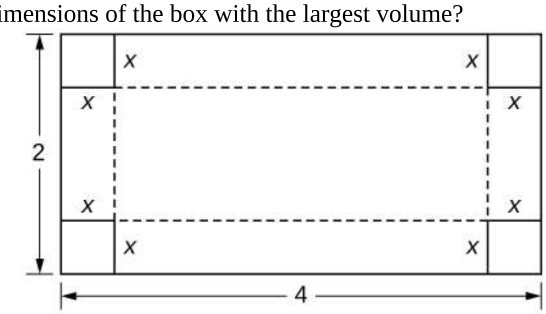
313. *True or False*. For every continuous nonlinear function, you can find the value x that maximizes the function.

314. *True or False*. For every continuous nonconstant function on a closed, finite domain, there exists at least one x that minimizes or maximizes the function.

For the following exercises, set up and evaluate each optimization problem.

315. To carry a suitcase on an airplane, the length $+\text{width}+$ height of the box must be less than or equal to $62\,\text{in}.$ Assuming the height is fixed, show that the maximum volume is $V = h\left(31 - \left(\frac{1}{2}\right)h\right)^2.$ What height allows you to have the largest volume?

316. You are constructing a cardboard box with the dimensions $2\text{ m by }4\text{ m}.$ You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



317. Find the positive integer that minimizes the sum of the number and its reciprocal.

318. Find two positive integers such that their sum is $10,$ and minimize and maximize the sum of their squares.

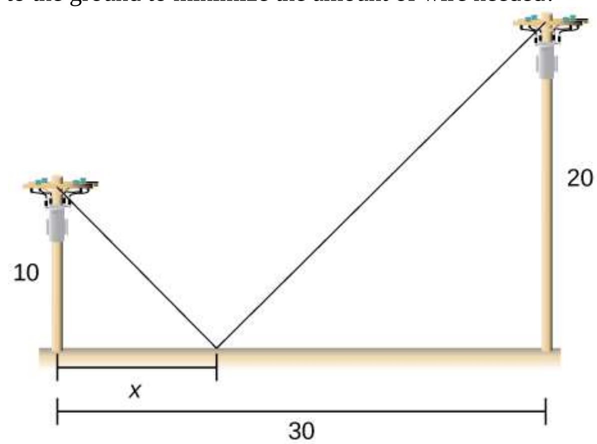
For the following exercises, consider the construction of a pen to enclose an area.

319. You have $400\,\text{ft}$ of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the area?

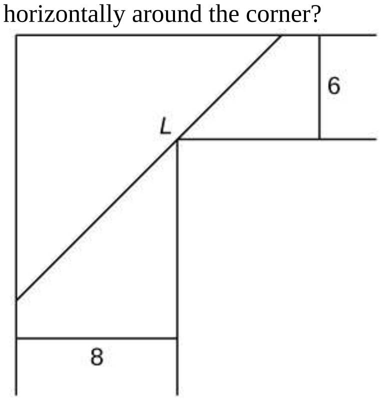
320. You have $800\,\text{ft}$ of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?

321. You need to construct a fence around an area of $1600\,\text{ft}.$ What are the dimensions of the rectangular pen to minimize the amount of material needed?

322. Two poles are connected by a wire that is also connected to the ground. The first pole is $20\,\text{ft}$ tall and the second pole is $10\,\text{ft}$ tall. There is a distance of $30\,\text{ft}$ between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



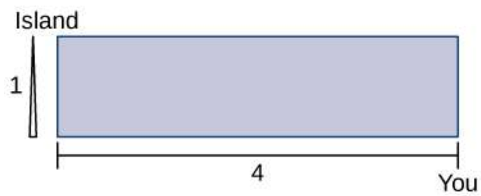
323. **[T]** You are moving into a new apartment and notice there is a corner where the hallway narrows from $8\text{ ft to }6\text{ ft}.$ What is the length of the longest item that can be carried horizontally around the corner?



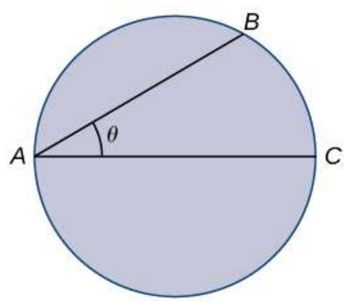
324. A patient's pulse measures $70\text{ bpm}, 80\text{ bpm}, \text{then }120\text{ bpm}.$ To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression $(x-70)^2 + (x-80)^2 + (x-120)^2?$ What value minimizes it?

325. In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes $(x-70)^2+(x-80)^2+\frac{1}{2}(x-120)^2$?

326. You can run at a speed of 6 mph and swim at a speed of 3 mph and are located on the shore, 4 miles east of an island that is 1 mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?



For the following problems, consider a lifeguard at a circular pool with diameter 40 m. He must reach someone who is drowning on the exact opposite side of the pool, at position C. The lifeguard swims with a speed v and runs around the pool at speed $w=3v$.



327. Find a function that measures the total amount of time it takes to reach the drowning person as a function of the swim angle, θ.

328. Find at what angle θ the lifeguard should swim to reach the drowning person in the least amount of time.

329. A truck uses gas as $g(v)=av+\frac{b}{v}$, where v represents the speed of the truck and g represents the gallons of fuel per mile. At what speed is fuel consumption minimized?

For the following exercises, consider a limousine that gets $m(v)=\frac{(120-2v)}{5}$ mi/gal at speed v, the chauffeur costs $\$15$/h, and gas is $\$3.5$/gal.

330. Find the cost per mile at speed v.

331. Find the cheapest driving speed.

For the following exercises, consider a pizzeria that sell pizzas for a revenue of $R(x)=ax$ and costs $C(x)=b+cx+dx^2$, where x represents the number of pizzas.

332. Find the profit function for the number of pizzas. How many pizzas gives the largest profit per pizza?

333. Assume that $R(x)=10x$ and $C(x)=2x+x^2$. How many pizzas sold maximizes the profit?

334. Assume that $R(x)=15x$, and $C(x)=60+3x+\frac{1}{2}x^2$. How many pizzas sold maximizes the profit?

For the following exercises, consider a wire 4 ft long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x.

335. Choose x to maximize the sum of their areas.

336. Choose x to minimize the sum of their areas.

For the following exercises, consider two nonnegative numbers x and y such that $x+y=10$. Maximize and minimize the quantities.

337. xy

338. x^2y^2

339. $y-\frac{1}{x}$

340. x^2-y

For the following exercises, draw the given optimization problem and solve.

341. Find the volume of the largest right circular cylinder that fits in a sphere of radius 1.

342. Find the volume of the largest right cone that fits in a sphere of radius 1.

343. Find the area of the largest rectangle that fits into the triangle with sides $x=0$, $y=0$ and $\frac{x}{4}+\frac{y}{6}=1$.

344. Find the largest volume of a cylinder that fits into a cone that has base radius R and height h.

345. Find the dimensions of the closed cylinder volume $V=16\pi$ that has the least amount of surface area.

346. Find the dimensions of a right cone with surface area $S=4\pi$ that has the largest volume.

For the following exercises, consider the points on the given graphs. Use a calculator to graph the functions.

347. **[T]** Where is the line $y = 5 - 2x$ closest to the origin?

348. **[T]** Where is the line $y = 5 - 2x$ closest to point $(1, 1)$?

349. **[T]** Where is the parabola $y = x^2$ closest to point $(2, 0)$?

350. **[T]** Where is the parabola $y = x^2$ closest to point $(0, 3)$?

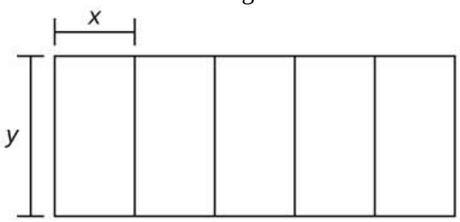
For the following exercises, set up, but do not evaluate, each optimization problem.

351. A window is composed of a semicircle placed on top of a rectangle. If you have 20 ft of window-framing materials for the outer frame, what is the maximum size of the window you can create? Use r to represent the radius of the semicircle.

352. You have a garden row of 20 watermelon plants that produce an average of 30 watermelons apiece. For any additional watermelon plants planted, the output per watermelon plant drops by one watermelon. How many extra watermelon plants should you plant?

353. You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/\text{ft}^2$ and the material for the sides costs $\$2/\text{ft}^2$. You need a box with volume 4 ft^2. Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

354. You are building five identical pens adjacent to each other with a total area of 1000 m^2, as shown in the following figure. What dimensions should you use to minimize the amount of fencing?



355. You are the manager of an apartment complex with 50 units. When you set rent at $\$800/\text{month},$ all apartments are rented. As you increase rent by $\$25/\text{month},$ one fewer apartment is rented. Maintenance costs run $\$50/\text{month}$ for each occupied unit. What is the rent that maximizes the total amount of profit?

4.8 | L'Hôpital's Rule

Learning Objectives

4.8.1 Recognize when to apply L'Hôpital's rule.
4.8.2 Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
4.8.3 Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L'Hôpital's rule**, uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L'Hôpital's Rule

L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

If $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2 \neq 0,$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

However, what happens if $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$? We call this one of the **indeterminate forms**, of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \text{ and } \lim_{x \to 0} \frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4.$$

For $\lim_{x \to 0} \frac{\sin x}{x}$ we were able to show, using a geometric argument, that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more important, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and such that $g'(a) \neq 0$ For x near $a,$ we can write

$$f(x) \approx f(a) + f'(a)(x - a)$$

and

$$g(x) \approx g(a) + g'(a)(x - a).$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)}.$$

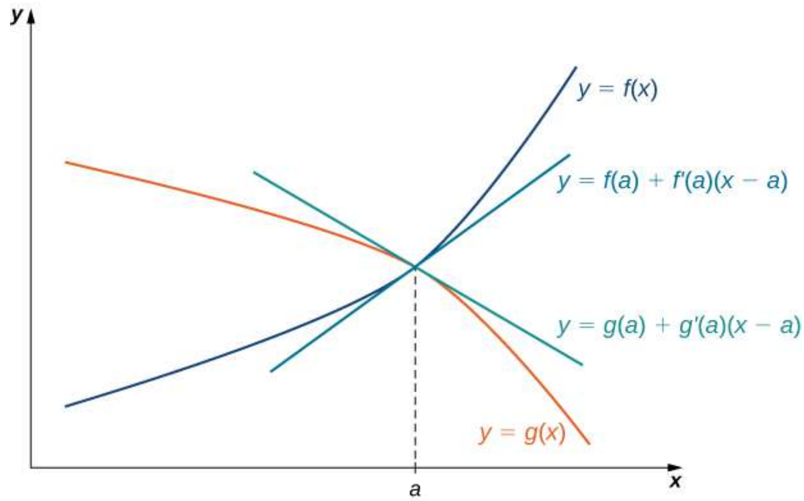


Figure 4.71 If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x),$ then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at $a,$ then f is continuous at $a,$ and therefore $f(a) = \lim_{x \to a} f(x) = 0.$ Similarly, $g(a) = \lim_{x \to a} g(x) = 0.$ If we also assume that f' and g' are continuous at $x = a,$ then $f'(a) = \lim_{x \to a} f'(x)$ and $g'(a) = \lim_{x \to a} g'(x).$ Using these ideas, we conclude that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}.$ Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

Theorem 4.12: L'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing $a,$ except possibly at $a.$ If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0,$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty.$ This result also holds if we are considering one-sided limits, or if $a = \infty$ and $-\infty.$

Proof

We provide a proof of this theorem in the special case when $f, g, f',$ and g' are all continuous over an open interval containing $a.$ In that case, since $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and f and g are continuous at $a,$ it follows that $f(a) = 0 = g(a).$ Therefore,

$$\begin{aligned}
\lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} && \text{since } f(a) = 0 = g(a) \\
&= \lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} && \text{algebra} \\
&= \frac{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} && \text{limit of a quotient} \\
&= \frac{f'(a)}{g'(a)} && \text{definition of the derivative} \\
&= \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)} && \text{continuity of } f' \text{ and } g' \\
&= \lim_{x \to a} \frac{f'(x)}{g'(x)}. && \text{limit of a quotient}
\end{aligned}$$

Note that L’Hôpital’s rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example 4.38

Applying L’Hôpital’s Rule (0/0 Case)

Evaluate each of the following limits by applying L’Hôpital’s rule.

a. $\lim_{x \to 0} \frac{1 - \cos x}{x}$

b. $\lim_{x \to 1} \frac{\sin(\pi x)}{\ln x}$

c. $\lim_{x \to \infty} \frac{e^{1/x} - 1}{1/x}$

d. $\lim_{x \to 0} \frac{\sin x - x}{x^2}$

Solution

a. Since the numerator $1 - \cos x \to 0$ and the denominator $x \to 0$, we can apply L’Hôpital’s rule to evaluate this limit. We have

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(x)} \\
&= \lim_{x \to 0} \frac{\sin x}{1} \\
&= \frac{\lim_{x \to 0} (\sin x)}{\lim_{x \to 0} (1)} \\
&= \frac{0}{1} = 0.
\end{aligned}$$

b. As $x \to 1$, the numerator $\sin(\pi x) \to 0$ and the denominator $\ln(x) \to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\begin{aligned}\lim_{x \to 1} \frac{\sin(\pi x)}{\ln x} &= \lim_{x \to 1} \frac{\pi \cos(\pi x)}{1/x} \\ &= \lim_{x \to 1} (\pi x)\cos(\pi x) \\ &= (\pi \cdot 1)(-1) = -\pi.\end{aligned}$$

c. As $x \to \infty$, the numerator $e^{1/x} - 1 \to 0$ and the denominator $\left(\frac{1}{x}\right) \to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x \to \infty} \frac{e^{1/x} - 1}{\frac{1}{x}} = \lim_{x \to \infty} \frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)} = \lim_{x \to \infty} e^{1/x} = e^0 = 1.$$

d. As $x \to 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} = \lim_{x \to 0} \frac{\cos x - 1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x \to 0$, we apply L'Hôpital's rule again. In doing so, we see that

$$\lim_{x \to 0} \frac{\cos x - 1}{2x} = \lim_{x \to 0} \frac{-\sin x}{2} = 0.$$

Therefore, we conclude that

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} = 0.$$

4.37 Evaluate $\lim_{x \to 0} \frac{x}{\tan x}$.

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$. Limits of this form are classified as *indeterminate forms of type* ∞/∞. Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, ∞/∞. is used to represent a quotient of limits, each of which is ∞ or $-\infty$.

Theorem 4.13: L'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x \to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$). Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or

$-\infty,$ or the limit is one-sided.

Example 4.39

Applying L'Hôpital's Rule $(\infty/\infty$ Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x\to\infty}\frac{3x+5}{2x+1}$

b. $\lim_{x\to 0^+}\frac{\ln x}{\cot x}$

Solution

a. Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim_{x\to\infty}(3x+5)=\infty$ and $\lim_{x\to\infty}(2x+1)=\infty.$ Therefore, we apply L'Hôpital's rule and obtain

$$\lim_{x\to\infty}\frac{3x+5}{2x+1/\ x}=\lim_{x\to\infty}\frac{3}{2}=\frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3+5/x}{2x+1/x}=\frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim_{x\to 0^+}\ln x=-\infty$ and $\lim_{x\to 0^+}\cot x=\infty.$ Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to 0^+}\frac{\ln x}{\cot x}=\lim_{x\to 0^+}\frac{1/x}{-\csc^2 x}=\lim_{x\to 0^+}\frac{1}{-x\csc^2 x}.$$

Now as $x\to 0^+,$ $\csc^2 x\to\infty.$ Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

$$\lim_{x\to 0^+}\frac{1}{-x\csc^2 x}=\lim_{x\to 0^+}\frac{\sin^2 x}{-x}.$$

Now $\lim_{x\to 0^+}\sin^2 x=0$ and $\lim_{x\to 0^+}x=0,$ so we apply L'Hôpital's rule again. We find

$$\lim_{x\to 0^+}\frac{\sin^2 x}{-x}=\lim_{x\to 0^+}\frac{2\sin x\cos x}{-1}=\frac{0}{-1}=0.$$

We conclude that

$$\lim_{x\to 0^+}\frac{\ln x}{\cot x}=0.$$

4.38 Evaluate $\lim_{x \to \infty} \frac{\ln x}{5x}$.

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or ∞/∞. Consider the following example.

Example 4.40

When L'Hôpital's Rule Does Not Apply

Consider $\lim_{x \to 1} \frac{x^2 + 5}{3x + 4}$. Show that the limit cannot be evaluated by applying L'Hôpital's rule.

Solution

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

$$\frac{d}{dx}\left(x^2 + 5\right) = 2x$$

and

$$\frac{d}{dx}(3x + 4) = 3.$$

At which point we would conclude erroneously that

$$\lim_{x \to 1} \frac{x^2 + 5}{3x + 4} = \lim_{x \to 1} \frac{2x}{3} = \frac{2}{3}.$$

However, since $\lim_{x \to 1}\left(x^2 + 5\right) = 6$ and $\lim_{x \to 1}(3x + 4) = 7,$ we actually have

$$\lim_{x \to 1} \frac{x^2 + 5}{3x + 4} = \frac{6}{7}.$$

We can conclude that

$$\lim_{x \to 1} \frac{x^2 + 5}{3x + 4} \neq \lim_{x \to 1} \frac{\frac{d}{dx}\left(x^2 + 5\right)}{\frac{d}{dx}(3x + 4)}.$$

4.39 Explain why we cannot apply L'Hôpital's rule to evaluate $\lim_{x \to 0^+} \frac{\cos x}{x}$. Evaluate $\lim_{x \to 0^+} \frac{\cos x}{x}$ by other means.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and ∞/∞. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0 \cdot \infty$, $\infty - \infty$, 1^∞, ∞^0, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\frac{0}{0}$ or ∞/∞.

Indeterminate Form of Type $0 \cdot \infty$

Suppose we want to evaluate $\lim\limits_{x \to a}(f(x) \cdot g(x))$, where $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0 \cdot \infty$ to denote the form that arises in this situation. The expression $0 \cdot \infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x \to a$. For example, let n be a positive integer and consider

$$f(x) = \frac{1}{(x^n + 1)} \text{ and } g(x) = 3x^2.$$

As $x \to \infty$, $f(x) \to 0$ and $g(x) \to \infty$. However, the limit as $x \to \infty$ of $f(x)g(x) = \frac{3x^2}{(x^n + 1)}$ varies, depending on n. If $n = 2$, then $\lim\limits_{x \to \infty} f(x)g(x) = 3$. If $n = 1$, then $\lim\limits_{x \to \infty} f(x)g(x) = \infty$. If $n = 3$, then $\lim\limits_{x \to \infty} f(x)g(x) = 0$. Here we consider another limit involving the indeterminate form $0 \cdot \infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example 4.41

Indeterminate Form of Type $0 \cdot \infty$

Evaluate $\lim\limits_{x \to 0^+} x \ln x$.

Solution

First, rewrite the function $x \ln x$ as a quotient to apply L'Hôpital's rule. If we write

$$x \ln x = \frac{\ln x}{1/x},$$

we see that $\ln x \to -\infty$ as $x \to 0^+$ and $\frac{1}{x} \to \infty$ as $x \to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

We conclude that

$$\lim_{x \to 0^+} x \ln x = 0.$$

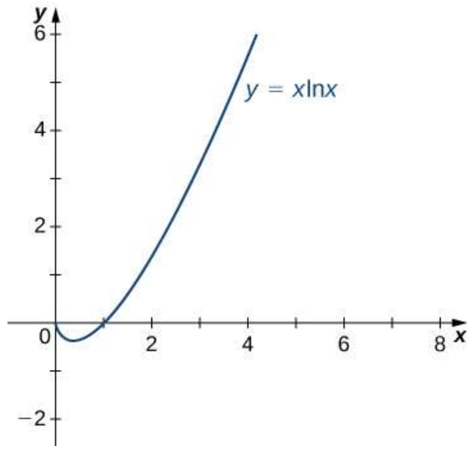


Figure 4.72 Finding the limit at $x = 0$ of the function $f(x) = x\ln x$.

4.40 Evaluate $\lim\limits_{x \to 0} x\cot x$.

Indeterminate Form of Type $\infty - \infty$

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim\limits_{x \to \infty}(f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x \to \infty}(f(x) - g(x)) = \lim_{x \to \infty}\left(3x^3 - 3x^2 - 5\right) = \infty.$$

On the other hand, if $n = 2$, then

$$\lim_{x \to \infty}(f(x) - g(x)) = \lim_{x \to \infty}\left(3x^2 - 3x^2 - 5\right) = -5.$$

However, if $n = 1$, then

$$\lim_{x \to \infty}(f(x) - g(x)) = \lim_{x \to \infty}\left(3x - 3x^2 - 5\right) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example 4.42

Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x}\right).$

Solution

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x,$ we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+,$ the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0.$ Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} = \lim_{x \to 0^+} \frac{\left(\sec^2 x\right) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \to 0^+,$ $\left(\sec^2 x\right) - 2x \to 1$ and $x^2 \sec^2 x + 2x \tan x \to 0.$ Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{\left(\sec^2 x\right) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x}\right) = \infty.$$

4.41 Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right).$

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions $0^0,$ $\infty^0,$ and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x \to a} f(x)^{g(x)}$ and we arrive at the indeterminate form $\infty^0.$ (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

$$y = f(x)^{g(x)}.$$

Then,

$$\ln y = \ln\left(f(x)^{g(x)}\right) = g(x)\ln(f(x)).$$

Therefore,

$$\lim_{x \to a}[\ln(y)] = \lim_{x \to a}[g(x)\ln(f(x))].$$

Since $\lim_{x \to a} f(x) = \infty,$ we know that $\lim_{x \to a} \ln(f(x)) = \infty.$ Therefore, $\lim_{x \to a} g(x)\ln(f(x))$ is of the indeterminate form

$0 \cdot \infty,$ and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x \to a} g(x)\ln(f(x)) = L,$ where L may be ∞ or $-\infty$. Then

$$\lim_{x \to a} [\ln(y)] = L.$$

Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x \to a} y\right) = L,$$

which gives us

$$\lim_{x \to a} y = \lim_{x \to a} f(x)^{g(x)} = e^L.$$

Example 4.43

Indeterminate Form of Type ∞^0

Evaluate $\lim_{x \to \infty} x^{1/x}$.

Solution

Let $y = x^{1/x}$. Then,

$$\ln\left(x^{1/x}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x \to \infty} \frac{\ln x}{x}$. Applying L'Hôpital's rule, we obtain

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0.$$

Therefore, $\lim_{x \to \infty} \ln y = 0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x \to \infty} y\right) = 0,$$

which leads to

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{\ln x}{x} = e^0 = 1.$$

Hence,

$$\lim_{x \to \infty} x^{1/x} = 1.$$

4.42 Evaluate $\lim_{x \to \infty} x^{1/\ln(x)}$.

Example 4.44

Indeterminate Form of Type 0^0

Evaluate $\lim_{x \to 0^+} x^{\sin x}$.

Solution

Let

$$y = x^{\sin x}.$$

Therefore,

$$\ln y = \ln\left(x^{\sin x}\right) = \sin x \ln x.$$

We now evaluate $\lim_{x \to 0^+} \sin x \ln x$. Since $\lim_{x \to 0^+} \sin x = 0$ and $\lim_{x \to 0^+} \ln x = -\infty$, we have the indeterminate form $0 \cdot \infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x \ln x$ as a fraction. We could write

$$\sin x \ln x = \frac{\sin x}{1/\ln x}$$

or

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x}.$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain

$$\lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\sin x}{1/\ln x} = \lim_{x \to 0^+} \frac{\cos x}{-1/\left(x(\ln x)^2\right)} = \lim_{x \to 0^+} \left(-x(\ln x)^2 \cos x\right).$$

Unfortunately, we not only have another expression involving the indeterminate form $0 \cdot \infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x},$$

and applying L'Hôpital's rule, we obtain

$$\lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x \to 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $\csc x = \frac{1}{\sin x}$ and $\cot x = \frac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as

$$\lim_{x \to 0^+} \frac{-\sin^2 x}{x \cos x} = \lim_{x \to 0^+} \left[\frac{\sin x}{x} \cdot (-\tan x)\right] = \left(\lim_{x \to 0^+} \frac{\sin x}{x}\right) \cdot \left(\lim_{x \to 0^+} (-\tan x)\right) = 1 \cdot 0 = 0.$$

We conclude that $\lim_{x \to 0^+} \ln y = 0$. Therefore, $\ln\left(\lim_{x \to 0^+} y\right) = 0$ and we have

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} x^{\sin x} = e^0 = 1.$$

Hence,

$$\lim_{x \to 0^+} x^{\sin x} = 1.$$

4.43 Evaluate $\lim_{x \to 0^+} x^x$.

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as shown in the following table, the values of x^3 are growing much faster than the values of x^2.

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

Table 4.7 Comparing the Growth Rates of x^2 and x^3

In fact,

$$\lim_{x \to \infty} \frac{x^3}{x^2} = \lim_{x \to \infty} x = \infty. \text{ or, equivalently, } \lim_{x \to \infty} \frac{x^2}{x^3} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in the following table. In fact,

$$\lim_{x \to \infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

Table 4.8 Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = \infty; \text{ or, equivalently, } \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 4.45

Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate $\lim_{x \to \infty}\left(\frac{f(x)}{g(x)}\right)$.

a. $f(x) = x^2$ and $g(x) = e^x$

b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solution

a. Since $\lim_{x \to \infty} x^2 = \infty$ and $\lim_{x \to \infty} e^x = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x \to \infty}\left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x \to \infty}\frac{x^2}{e^x} = \lim_{x \to \infty}\frac{2x}{e^x}.$$

Since $\lim_{x \to \infty} 2x = \infty$ and $\lim_{x \to \infty} e^x = \infty$, we can apply L'Hôpital's rule again. Since

$$\lim_{x \to \infty}\frac{2x}{e^x} = \lim_{x \to \infty}\frac{2}{e^x} = 0,$$

we conclude that

$$\lim_{x \to \infty}\frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \to \infty$ (See **Figure 4.73** and **Table 4.9**).

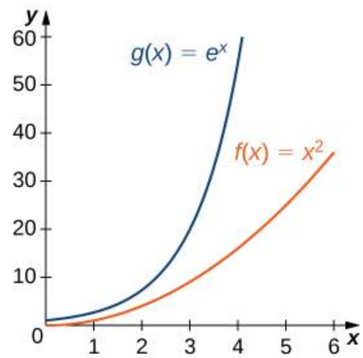


Figure 4.73 An exponential function grows at a faster rate than a power function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22,026	3,269,017	485,165,195

Table 4.9
Growth rates of a power function and an exponential function.

b. Since $\lim_{x \to \infty} \ln x = \infty$ and $\lim_{x \to \infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x \to \infty} \frac{\ln x}{x^2}$. We obtain

$$\lim_{x \to \infty} \frac{\ln x}{x^2} = \lim_{x \to \infty} \frac{1/x}{2x} = \lim_{x \to \infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see **Figure 4.74** and **Table 4.10**).

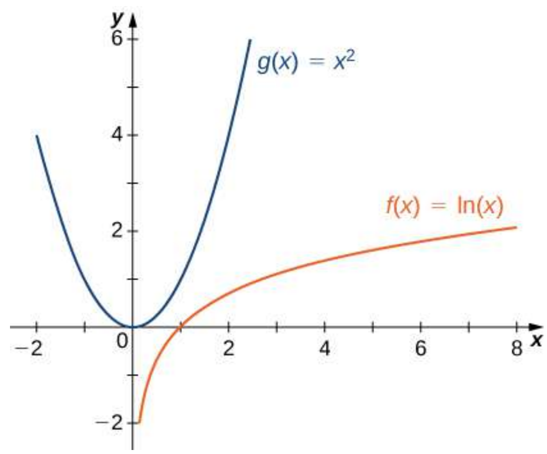


Figure 4.74 A power function grows at a faster rate than a logarithmic function.

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

Table 4.10
Growth rates of a power function and a logarithmic function

4.44 Compare the growth rates of x^{100} and 2^x.

Using the same ideas as in **Example 4.45**a. it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In **Figure 4.75** and **Table 4.11**, we compare e^x with x^3 and x^4 as $x \to \infty$.

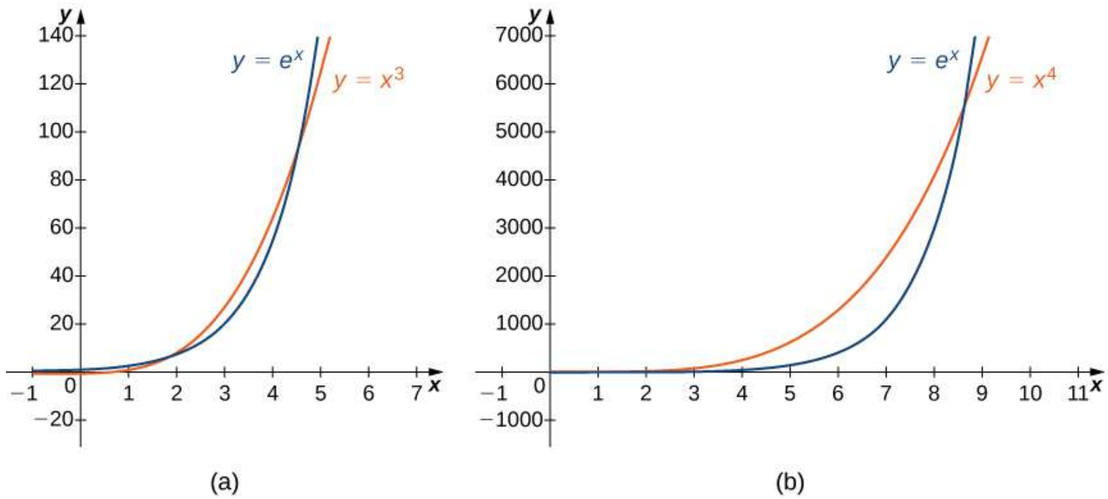


Figure 4.75 The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

Table 4.11 An exponential function grows at a faster rate than any power function

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In **Figure 4.76** and **Table 4.12**, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

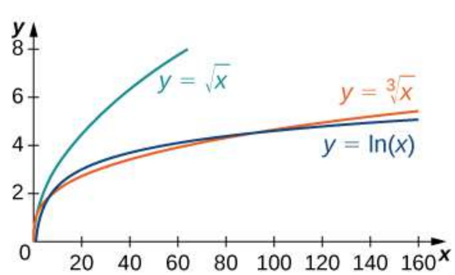


Figure 4.76 The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Table 4.12 A logarithmic function grows at a slower rate than any root function

4.8 EXERCISES

For the following exercises, evaluate the limit.

356. Evaluate the limit $\lim_{x \to \infty} \frac{e^x}{x}$.

357. Evaluate the limit $\lim_{x \to \infty} \frac{e^x}{x^k}$.

358. Evaluate the limit $\lim_{x \to \infty} \frac{\ln x}{x^k}$.

359. Evaluate the limit $\lim_{x \to a} \frac{x-a}{x^2-a^2}, \quad a \neq 0$.

360. Evaluate the limit $\lim_{x \to a} \frac{x-a}{x^3-a^3}, \quad a \neq 0$.

361. Evaluate the limit $\lim_{x \to a} \frac{x-a}{x^n-a^n}, \quad a \neq 0$.

For the following exercises, determine whether you can apply L'Hôpital's rule directly. Explain why or why not. Then, indicate if there is some way you can alter the limit so you can apply L'Hôpital's rule.

362. $\lim_{x \to 0^+} x^2 \ln x$

363. $\lim_{x \to \infty} x^{1/x}$

364. $\lim_{x \to 0} x^{2/x}$

365. $\lim_{x \to 0} \frac{x^2}{1/x}$

366. $\lim_{x \to \infty} \frac{e^x}{x}$

For the following exercises, evaluate the limits with either L'Hôpital's rule or previously learned methods.

367. $\lim_{x \to 3} \frac{x^2-9}{x-3}$

368. $\lim_{x \to 3} \frac{x^2-9}{x+3}$

369. $\lim_{x \to 0} \frac{(1+x)^{-2}-1}{x}$

370. $\lim_{x \to \pi/2} \frac{\cos x}{\frac{\pi}{2}-x}$

371. $\lim_{x \to \pi} \frac{x-\pi}{\sin x}$

372. $\lim_{x \to 1} \frac{x-1}{\sin x}$

373. $\lim_{x \to 0} \frac{(1+x)^n-1}{x}$

374. $\lim_{x \to 0} \frac{(1+x)^n-1-nx}{x^2}$

375. $\lim_{x \to 0} \frac{\sin x - \tan x}{x^3}$

376. $\lim_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

377. $\lim_{x \to 0} \frac{e^x-x-1}{x^2}$

378. $\lim_{x \to 0} \frac{\tan x}{\sqrt{x}}$

379. $\lim_{x \to 1} \frac{x-1}{\ln x}$

380. $\lim_{x \to 0} (x+1)^{1/x}$

381. $\lim_{x \to 1} \frac{\sqrt{x}-\sqrt[3]{x}}{x-1}$

382. $\lim_{x \to 0^+} x^{2x}$

383. $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

384. $\lim_{x \to 0} \frac{\sin x - x}{x^2}$

385. $\lim_{x \to 0^+} x \ln\left(x^4\right)$

386. $\lim_{x \to \infty} (x - e^x)$

387. $\lim_{x \to \infty} x^2 e^{-x}$

388. $\lim_{x \to 0} \frac{3^x-2^x}{x}$

389. $\lim_{x \to 0} \frac{1+1/x}{1-1/x}$

390. $\lim_{x \to \pi/4} (1-\tan x)\cot x$

391. $\lim_{x \to \infty} xe^{1/x}$

392. $\lim_{x \to 0^+} x^{1/\cos x}$

393. $\lim_{x \to 0^+} x^{1/x}$

394. $\lim_{x \to 0^-} \left(1 - \frac{1}{x}\right)^x$

395. $\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x$

For the following exercises, use a calculator to graph the function and estimate the value of the limit, then use L'Hôpital's rule to find the limit directly.

396. **[T]** $\lim_{x \to 0} \frac{e^x - 1}{x}$

397. **[T]** $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

398. **[T]** $\lim_{x \to 1} \frac{x - 1}{1 - \cos(\pi x)}$

399. **[T]** $\lim_{x \to 1} \frac{e^{(x-1)} - 1}{x - 1}$

400. **[T]** $\lim_{x \to 1} \frac{(x-1)^2}{\ln x}$

401. **[T]** $\lim_{x \to \pi} \frac{1 + \cos x}{\sin x}$

402. **[T]** $\lim_{x \to 0} \left(\csc x - \frac{1}{x}\right)$

403. **[T]** $\lim_{x \to 0^+} \tan(x^x)$

404. **[T]** $\lim_{x \to 0^+} \frac{\ln x}{\sin x}$

405. **[T]** $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$

4.9 | Newton's Method

Learning Objectives

4.9.1 Describe the steps of Newton's method.
4.9.2 Explain what an iterative process means.
4.9.3 Recognize when Newton's method does not work.
4.9.4 Apply iterative processes to various situations.

In many areas of pure and applied mathematics, we are interested in finding solutions to an equation of the form $f(x) = 0.$ For most functions, however, it is difficult—if not impossible—to calculate their zeroes explicitly. In this section, we take a look at a technique that provides a very efficient way of approximating the zeroes of functions. This technique makes use of tangent line approximations and is behind the method used often by calculators and computers to find zeroes.

Describing Newton's Method

Consider the task of finding the solutions of $f(x) = 0.$ If f is the first-degree polynomial $f(x) = ax + b,$ then the solution of $f(x) = 0$ is given by the formula $x = -\frac{b}{a}.$ If f is the second-degree polynomial $f(x) = ax^2 + bx + c,$ the solutions of $f(x) = 0$ can be found by using the quadratic formula. However, for polynomials of degree 3 or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree 5 or greater, it is known that no such formulas exist. For example, consider the function

$$f(x) = x^5 + 8x^4 + 4x^3 - 2x - 7.$$

No formula exists that allows us to find the solutions of $f(x) = 0.$ Similar difficulties exist for nonpolynomial functions. For example, consider the task of finding solutions of $\tan(x) - x = 0.$ No simple formula exists for the solutions of this equation. In cases such as these, we can use Newton's method to approximate the roots.

Newton's method makes use of the following idea to approximate the solutions of $f(x) = 0.$ By sketching a graph of $f,$ we can estimate a root of $f(x) = 0.$ Let's call this estimate $x_0.$ We then draw the tangent line to f at $x_0.$ If $f'(x_0) \neq 0,$ this tangent line intersects the x-axis at some point $(x_1, 0).$ Now let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next we draw the tangent line to f at $x_1.$ If $f'(x_1) \neq 0,$ this tangent line also intersects the x-axis, producing another approximation, $x_2.$ We continue in this way, deriving a list of approximations: $x_0, x_1, x_2, \ldots.$ Typically, the numbers $x_0, x_1, x_2, \ldots$ quickly approach an actual root $x^*,$ as shown in the following figure.

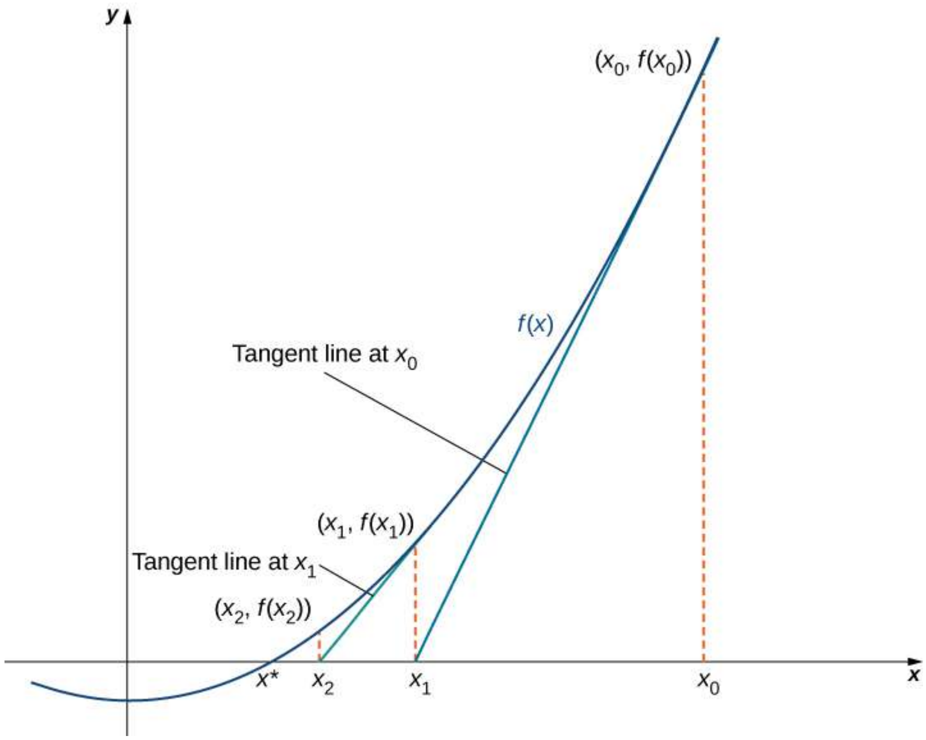


Figure 4.77 The approximations $x_0, x_1, x_2, \ldots$ approach the actual root x^*. The approximations are derived by looking at tangent lines to the graph of f.

Now let's look at how to calculate the approximations $x_0, x_1, x_2, \ldots$. If x_0 is our first approximation, the approximation x_1 is defined by letting $(x_1, 0)$ be the x-intercept of the tangent line to f at x_0. The equation of this tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

Solving this equation for x_1, we conclude that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point $(x_2, 0)$ is the x-intercept of the tangent line to f at x_1. Therefore, x_2 satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for $n > 0$, x_n satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \qquad (4.8)$$

Next we see how to make use of this technique to approximate the root of the polynomial $f(x) = x^3 - 3x + 1$.

Example 4.46

Finding a Root of a Polynomial

Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find $x_1, x_2, x_3, x_4,$ and x_5.

Solution

From **Figure 4.78**, we see that f has one root over the interval $(1, 2)$. Therefore $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use **Equation 4.8**. Since $f(x) = x^3 - 3x + 1,$ the derivative is $f'(x) = 3x^2 - 3$. Using **Equation 4.8** with $n = 1$ (and a calculator that displays 10 digits), we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, $x_2,$ we use **Equation 4.8** with $n = 2$ and the value of x_1 stored on the calculator. We find that

$$x_2 = x_1 = \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Continuing in this way, we obtain the following results:

$$\begin{aligned} x_1 &\approx 1.666666667 \\ x_2 &\approx 1.548611111 \\ x_3 &\approx 1.532390162 \\ x_4 &\approx 1.532088989 \\ x_5 &\approx 1.532088886 \\ x_6 &\approx 1.532088886. \end{aligned}$$

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.

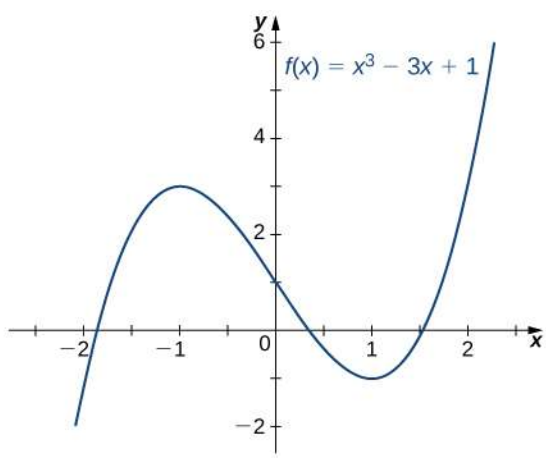


Figure 4.78 The function $f(x) = x^3 - 3x + 1$ has one root over the interval $[1, 2]$.

4.45 Letting $x_0 = 0,$ let's use Newton's method to approximate the root of $f(x) = x^3 - 3x + 1$ over the interval $[0, 1]$ by calculating x_1 and x_2.

Newton's method can also be used to approximate square roots. Here we show how to approximate $\sqrt{2}$. This method can be modified to approximate the square root of any positive number.

Example 4.47

Finding a Square Root

Use Newton's method to approximate $\sqrt{2}$ (**Figure 4.79**). Let $f(x) = x^2 - 2,$ let $x_0 = 2,$ and calculate x_1, x_2, x_3, x_4, x_5. (We note that since $f(x) = x^2 - 2$ has a zero at $\sqrt{2},$ the initial value $x_0 = 2$ is a reasonable choice to approximate $\sqrt{2}$.)

Solution

For $f(x) = x^2 - 2,\ f'(x) = 2x.$ From **Equation 4.8**, we know that

$$\begin{aligned} x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{x^2_{n-1} - 2}{2x_{n-1}} \\ &= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\ &= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} x_1 &= \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = 1.5 \\ x_2 &= \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.416666667. \end{aligned}$$

Continuing in this way, we find that

$$\begin{aligned} x_1 &= 1.5 \\ x_2 &\approx 1.416666667 \\ x_3 &\approx 1.414215686 \\ x_4 &\approx 1.414213562 \\ x_5 &\approx 1.414213562. \end{aligned}$$

Since we obtained the same value for x_4 and $x_5,$ it is unlikely that the value x_n will change on any subsequent application of Newton's method. We conclude that $\sqrt{2} \approx 1.414213562.$

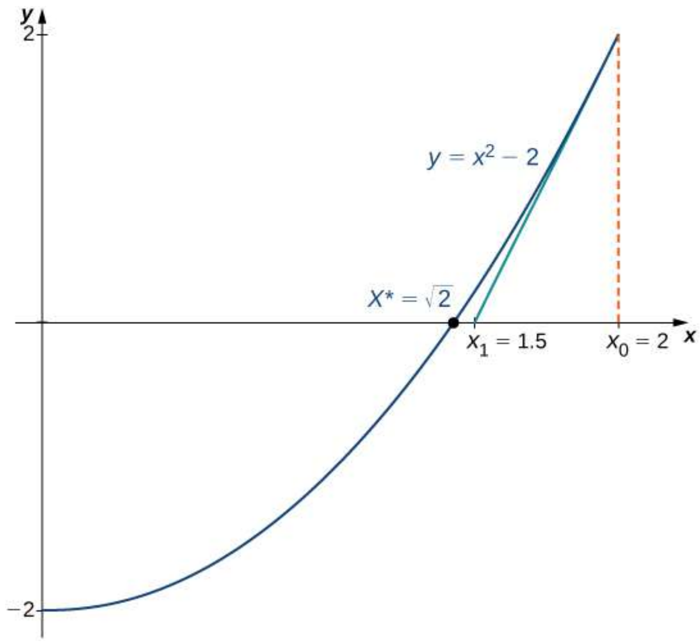


Figure 4.79 We can use Newton's method to find $\sqrt{2}$.

4.46 Use Newton's method to approximate $\sqrt{3}$ by letting $f(x) = x^2 - 3$ and $x_0 = 3$. Find x_1 and x_2.

When using Newton's method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$, we can rewrite **Equation 4.8** as $x_n = F(x_{n-1})$. This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an **iterative process**. Shortly, we examine other iterative processes. First, let's look at the reasons why Newton's method could fail to find a root.

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.
2. The approximations $x_0, x_1, x_2, \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see **Figure 4.80**). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In **Example 4.48**, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

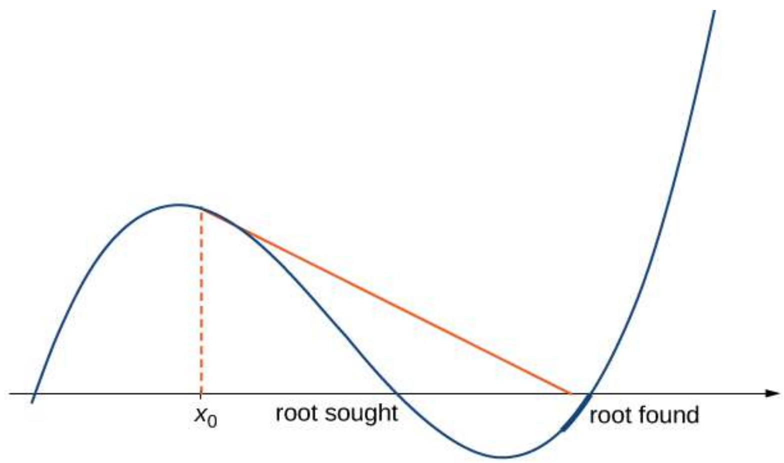


Figure 4.80 If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

Example 4.48

When Newton's Method Fails

Consider the function $f(x) = x^3 - 2x + 2$. Let $x_0 = 0$. Show that the sequence $x_1, x_2, \ldots$ fails to approach a root of f.

Solution

For $f(x) = x^3 - 2x + 2$, the derivative is $f'(x) = 3x^2 - 2$. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers $x_0, x_1, x_2, \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$ (see **Figure 4.81**). Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.

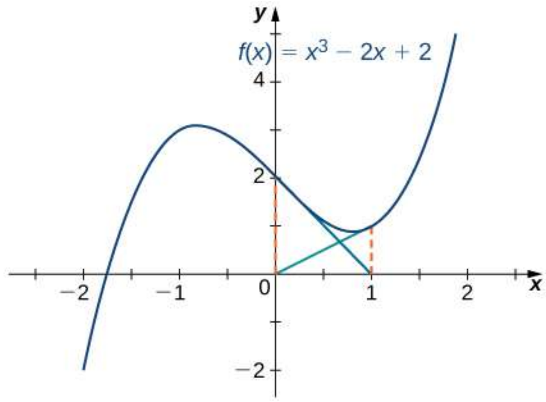


Figure 4.81 The approximations continue to alternate between 0 and 1 and never approach the root of f.

4.47 For $f(x) = x^3 - 2x + 2,$ let $x_0 = -1.5$ and find x_1 and x_2.

From **Example 4.48**, we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

Other Iterative Processes

As mentioned earlier, Newton's method is a type of iterative process. We now look at an example of a different type of iterative process.

Consider a function F and an initial number x_0. Define the subsequent numbers x_n by the formula $x_n = F(x_{n-1})$. This process is an iterative process that creates a list of numbers $x_0, x_1, x_2, \ldots, x_n, \ldots$. This list of numbers may approach a finite number x^* as n gets larger, or it may not. In **Example 4.49**, we see an example of a function F and an initial guess x_0 such that the resulting list of numbers approaches a finite value.

Example 4.49

Finding a Limit for an Iterative Process

Let $F(x) = \frac{1}{2}x + 4$ and let $x_0 = 0$. For all $n \geq 1,$ let $x_n = F(x_{n-1})$. Find the values x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to this list of numbers $x_1, x_2, x_3 \ldots, x_n, \ldots$ as $n \to \infty$. If the list of numbers $x_1, x_2, x_3, \ldots$ approaches a finite number x^*, then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F.

Solution

If $x_0 = 0,$ then

$$x_1 = \frac{1}{2}(0) + 4 = 4$$
$$x_2 = \frac{1}{2}(4) + 4 = 6$$
$$x_3 = \frac{1}{2}(6) + 4 = 7$$
$$x_4 = \frac{1}{2}(7) + 4 = 7.5$$
$$x_5 = \frac{1}{2}(7.5) + 4 = 7.75$$
$$x_6 = \frac{1}{2}(7.75) + 4 = 7.875$$
$$x_7 = \frac{1}{2}(7.875) + 4 = 7.9375$$
$$x_8 = \frac{1}{2}(7.9375) + 4 = 7.96875$$
$$x_9 = \frac{1}{2}(7.96875) + 4 = 7.984375.$$

From this list, we conjecture that the values x_n approach $8.$

Figure 4.82 provides a graphical argument that the values approach 8 as $n \to \infty.$ Starting at the point $(x_0, x_0),$ we draw a vertical line to the point $(x_0, F(x_0)).$ The next number in our list is $x_1 = F(x_0).$ We use x_1 to calculate $x_2.$ Therefore, we draw a horizontal line connecting (x_0, x_1) to the point (x_1, x_1) on the line $y = x,$ and then draw a vertical line connecting (x_1, x_1) to the point $(x_1, F(x_1)).$ The output $F(x_1)$ becomes $x_2.$ Continuing in this way, we could create an infinite number of line segments. These line segments are trapped between the lines $F(x) = \frac{x}{2} + 4$ and $y = x.$ The line segments get closer to the intersection point of these two lines, which occurs when $x = F(x).$ Solving the equation $x = \frac{x}{2} + 4,$ we conclude they intersect at $x = 8.$ Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers $x_0, x_1, x_2, \ldots$ approaches $x^* = 8$ as $n \to \infty.$

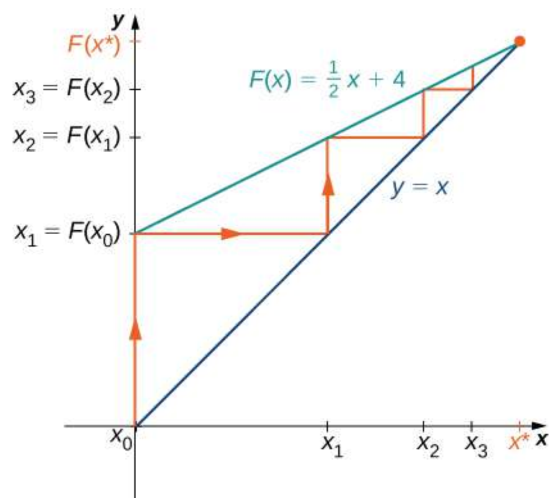


Figure 4.82 This iterative process approaches the value $x^* = 8.$

4.48 Consider the function $F(x) = \frac{1}{3}x + 6$. Let $x_0 = 0$ and let $x_n = F(x_{n-1})$ for $n \geq 2$. Find x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to the list of numbers $x_1, x_2, x_3, \ldots x_n, \ldots$ as $n \to \infty$.

Student PROJECT

Iterative Processes and Chaos

Iterative processes can yield some very interesting behavior. In this section, we have seen several examples of iterative processes that converge to a fixed point. We also saw in **Example 4.48** that the iterative process bounced back and forth between two values. We call this kind of behavior a 2-*cycle*. Iterative processes can converge to cycles with various periodicities, such as $2-\text{cycles},$ $4-\text{cycles}$ (where the iterative process repeats a sequence of four values), 8-cycles, and so on.

Some iterative processes yield what mathematicians call *chaos*. In this case, the iterative process jumps from value to value in a seemingly random fashion and never converges or settles into a cycle. Although a complete exploration of chaos is beyond the scope of this text, in this project we look at one of the key properties of a chaotic iterative process: sensitive dependence on initial conditions. This property refers to the concept that small changes in initial conditions can generate drastically different behavior in the iterative process.

Probably the best-known example of chaos is the Mandelbrot set (see **Figure 4.83**), named after Benoit Mandelbrot (1924–2010), who investigated its properties and helped popularize the field of chaos theory. The Mandelbrot set is usually generated by computer and shows fascinating details on enlargement, including self-replication of the set. Several colorized versions of the set have been shown in museums and can be found online and in popular books on the subject.

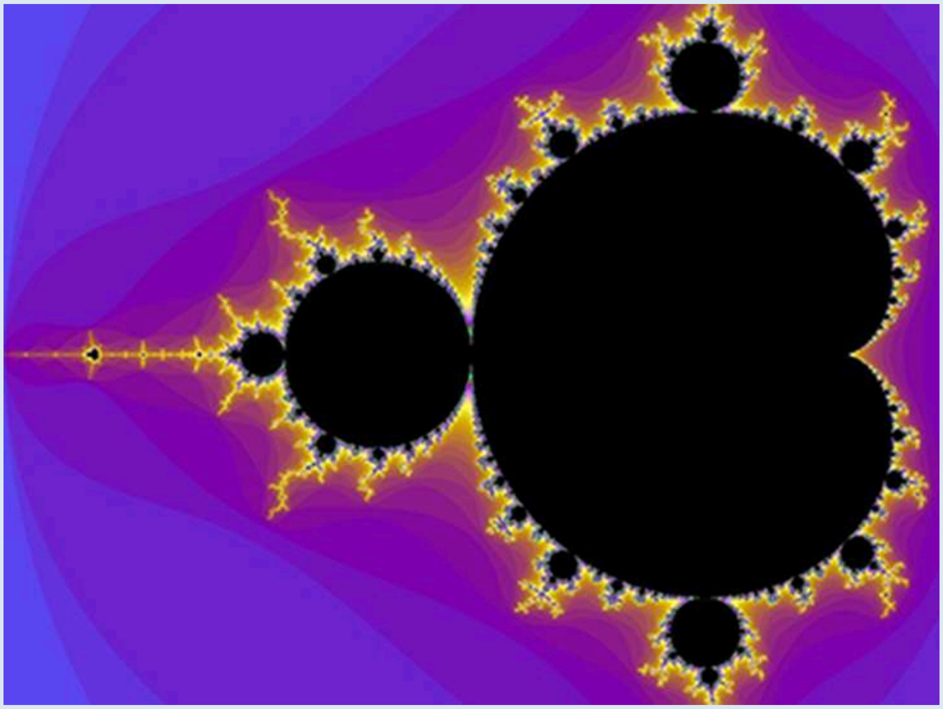


Figure 4.83 The Mandelbrot set is a well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.

In this project we use the logistic map

$$f(x) = rx(1-x), \text{ where } x \in [0,\ 1] \text{ and } r > 0$$

as the function in our iterative process. The logistic map is a deceptively simple function; but, depending on the value of $r,$ the resulting iterative process displays some very interesting behavior. It can lead to fixed points, cycles, and even chaos.

To visualize the long-term behavior of the iterative process associated with the logistic map, we will use a tool called a *cobweb diagram*. As we did with the iterative process we examined earlier in this section, we first draw a vertical line from the point $(x_0, 0)$ to the point $(x_0, f(x_0)) = (x_0, x_1)$. We then draw a horizontal line from that point to the point (x_1, x_1), then draw a vertical line to $(x_1, f(x_1)) = (x_1, x_2)$, and continue the process until the long-term behavior of the system becomes apparent. **Figure 4.84** shows the long-term behavior of the logistic map when $r = 3.55$ and $x_0 = 0.2$. (The first 100 iterations are not plotted.) The long-term behavior of this iterative process is an 8 -cycle.

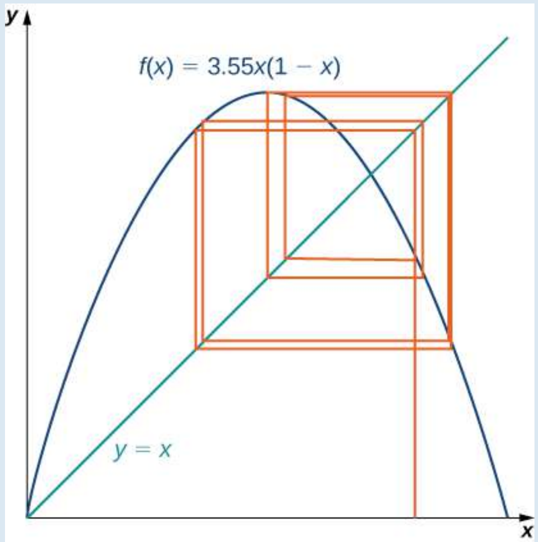


Figure 4.84 A cobweb diagram for $f(x) = 3.55x(1 - x)$ is presented here. The sequence of values results in an 8 -cycle.

1. Let $r = 0.5$ and choose $x_0 = 0.2$. Either by hand or by using a computer, calculate the first 10 values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, $2 - \text{cycle}$, $4 - \text{cycle}.$)?
2. What happens when $r = 2$?
3. For $r = 3.2$ and $r = 3.5$, calculate the first 100 sequence values. Generate a cobweb diagram for each iterative process. (Several free applets are available online that generate cobweb diagrams for the logistic map.) What is the long-term behavior in each of these cases?
4. Now let $r = 4$. Calculate the first 100 sequence values and generate a cobweb diagram. What is the long-term behavior in this case?
5. Repeat the process for $r = 4$, but let $x_0 = 0.201$. How does this behavior compare with the behavior for $x_0 = 0.2$?

4.9 EXERCISES

For the following exercises, write Newton's formula as $x_{n+1} = F(x_n)$ for solving $f(x) = 0$.

406. $f(x) = x^2 + 1$

407. $f(x) = x^3 + 2x + 1$

408. $f(x) = \sin x$

409. $f(x) = e^x$

410. $f(x) = x^3 + 3xe^x$

For the following exercises, solve $f(x) = 0$ using the iteration $x_{n+1} = x_n - cf(x_n)$, which differs slightly from Newton's method. Find a c that works and a c that fails to converge, with the exception of $c = 0$.

411. $f(x) = x^2 - 4$, with $x_0 = 0$

412. $f(x) = x^2 - 4x + 3$, with $x_0 = 2$

413. What is the value of "c" for Newton's method?

For the following exercises, start at

a. $x_0 = 0.6$ and

b. $x_0 = 2$.

Compute x_1 and x_2 using the specified iterative method.

414. $x_{n+1} = {x_n}^2 - \frac{1}{2}$

415. $x_{n+1} = 2x_n(1 - x_n)$

416. $x_{n+1} = \sqrt{x_n}$

417. $x_{n+1} = \frac{1}{\sqrt{x_n}}$

418. $x_{n+1} = 3x_n(1 - x_n)$

419. $x_{n+1} = {x_n}^2 + x_n - 2$

420. $x_{n+1} = \frac{1}{2}x_n - 1$

421. $x_{n+1} = |x_n|$

For the following exercises, solve to four decimal places using Newton's method and a computer or calculator. Choose any initial guess x_0 that is not the exact root.

422. $x^2 - 10 = 0$

423. $x^4 - 100 = 0$

424. $x^2 - x = 0$

425. $x^3 - x = 0$

426. $x + 5\cos(x) = 0$

427. $x + \tan(x) = 0$, choose $x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

428. $\frac{1}{1 - x} = 2$

429. $1 + x + x^2 + x^3 + x^4 = 2$

430. $x^3 + (x + 1)^3 = 10^3$

431. $x = \sin^2(x)$

For the following exercises, use Newton's method to find the fixed points of the function where $f(x) = x$; round to three decimals.

432. $\sin x$

433. $\tan(x)$ on $x = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

434. $e^x - 2$

435. $\ln(x) + 2$

Newton's method can be used to find maxima and minima of functions in addition to the roots. In this case apply Newton's method to the derivative function $f'(x)$ to find its roots, instead of the original function. For the following exercises, consider the formulation of the method.

436. To find candidates for maxima and minima, we need to find the critical points $f'(x) = 0$. Show that to solve for the critical points of a function $f(x)$, Newton's method is given by $x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$.

437. What additional restrictions are necessary on the function f?

For the following exercises, use Newton’s method to find the location of the local minima and/or maxima of the following functions; round to three decimals.

438. Minimum of $f(x) = x^2 + 2x + 4$

439. Minimum of $f(x) = 3x^3 + 2x^2 - 16$

440. Minimum of $f(x) = x^2 e^x$

441. Maximum of $f(x) = x + \frac{1}{x}$

442. Maximum of $f(x) = x^3 + 10x^2 + 15x - 2$

443. Maximum of $f(x) = \frac{\sqrt{x} - \sqrt[3]{x}}{x}$

444. Minimum of $f(x) = x^2 \sin x$, closest non-zero minimum to $x = 0$

445. Minimum of $f(x) = x^4 + x^3 + 3x^2 + 12x + 6$

For the following exercises, use the specified method to solve the equation. If it does not work, explain why it does not work.

446. Newton’s method, $x^2 + 2 = 0$

447. Newton’s method, $0 = e^x$

448. Newton’s method, $0 = 1 + x^2$ starting at $x_0 = 0$

449. Solving $x_{n+1} = -x_n{}^3$ starting at $x_0 = -1$

For the following exercises, use the secant method, an alternative iterative method to Newton’s method. The formula is given by

$$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.$$

450. Find a root to $0 = x^2 - x - 3$ accurate to three decimal places.

451. Find a root to $0 = \sin x + 3x$ accurate to four decimal places.

452. Find a root to $0 = e^x - 2$ accurate to four decimal places.

453. Find a root to $\ln(x + 2) = \frac{1}{2}$ accurate to four decimal places.

454. Why would you use the secant method over Newton’s method? What are the necessary restrictions on f?

For the following exercises, use both Newton’s method and the secant method to calculate a root for the following equations. Use a calculator or computer to calculate how many iterations of each are needed to reach within three decimal places of the exact answer. For the secant method, use the first guess from Newton’s method.

455. $f(x) = x^2 + 2x + 1,\ x_0 = 1$

456. $f(x) = x^2,\ x_0 = 1$

457. $f(x) = \sin x,\ x_0 = 1$

458. $f(x) = e^x - 1,\ x_0 = 2$

459. $f(x) = x^3 + 2x + 4,\ x_0 = 0$

In the following exercises, consider Kepler’s equation regarding planetary orbits, $M = E - \varepsilon \sin(E)$, where M is the mean anomaly, E is eccentric anomaly, and ε measures eccentricity.

460. Use Newton’s method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{\pi}{3}$ and the eccentricity of the orbit $\varepsilon = 0.25$; round to three decimals.

461. Use Newton’s method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{3\pi}{2}$ and the eccentricity of the orbit $\varepsilon = 0.8$; round to three decimals.

The following two exercises consider a bank investment. The initial investment is $10,000. After 25 years, the investment has tripled to $30,000.

462. Use Newton’s method to determine the interest rate if the interest was compounded annually.

463. Use Newton’s method to determine the interest rate if the interest was compounded continuously.

464. The cost for printing a book can be given by the equation $C(x) = 1000 + 12x + \left(\frac{1}{2}\right)x^{2/3}$. Use Newton’s method to find the break-even point if the printer sells each book for $20.

4.10 | Antiderivatives

Learning Objectives

- **4.10.1** Find the general antiderivative of a given function.
- **4.10.2** Explain the terms and notation used for an indefinite integral.
- **4.10.3** State the power rule for integrals.
- **4.10.4** Use antidifferentiation to solve simple initial-value problems.

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function $f,$ how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative $f.$ Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in **Derivatives** of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$—that is, $v(t) = s'(t).$ Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$—that is, $a(t) = v'(t) = s''(t).$ Now suppose we are given an acceleration function $a,$ but not the velocity function v or the position function $s.$ Since $a(t) = v'(t),$ determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s'(t),$ determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text (**Introduction to Techniques of Integration (http://cnx.org/content/m53654/latest/)**).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function $f,$ how can we find a function with derivative $f?$ If we can find a function F derivative $f,$ we call F an antiderivative of $f.$

Definition

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of $f.$

Consider the function $f(x) = 2x.$ Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x.$ Are there any other antiderivatives of $f?$ Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x.$ Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives. Are there any others that are not of the form $x^2 + C$ for some constant $C?$ The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x),$ then $F(x) - G(x) = C$ for some constant $C.$ This fact leads to the following important theorem.

Theorem 4.14: General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

i. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;

ii. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 4.50

Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x) = 3x^2$

b. $f(x) = \frac{1}{x}$

c. $f(x) = \cos x$

d. $f(x) = e^x$

Solution

a. Because

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

then $F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C, and every function of the form $x^3 + C$ is an antiderivative of $3x^2$.

b. Let $f(x) = \ln|x|$. For $x > 0$, $f(x) = \ln(x)$ and

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

For $x < 0$, $f(x) = \ln(-x)$ and

$$\frac{d}{dx}(\ln(-x)) = -\frac{1}{-x} = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\frac{1}{x}$. Therefore, every antiderivative of $\frac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\ln|x| + C$ is an antiderivative of $\frac{1}{x}$.

c. We have

$$\frac{d}{dx}(\sin x) = \cos x,$$

so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x + C$ for some constant C and every function of the form $\sin x + C$ is an antiderivative of $\cos x$.

d. Since

$$\frac{d}{dx}(e^x) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x.

4.49 Find all antiderivatives of $f(x) = \sin x$.

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f, we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here we introduce notation for antiderivatives. If F is an antiderivative of f, we say that $F(x) + C$ is the most general antiderivative of f and write

$$\int f(x)dx = F(x) + C.$$

The symbol $\int$ is called an *integral sign*, and $\int f(x)dx$ is called the indefinite integral of f.

Definition

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)dx,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)dx = F(x) + C.$$

The expression $f(x)$ is called the *integrand* and the variable x is the *variable of integration*.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as *integrating* f.

For a function f and an antiderivative F, the functions $F(x) + C$, where C is any real number, is often referred to as *the family of antiderivatives of* f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form $x^2 + C$, we write

$$\int 2x\,dx = x^2 + C.$$

The collection of all functions of the form $x^2 + C,$ where C is any real number, is known as the *family of antiderivatives of* $2x$. **Figure 4.85** shows a graph of this family of antiderivatives.

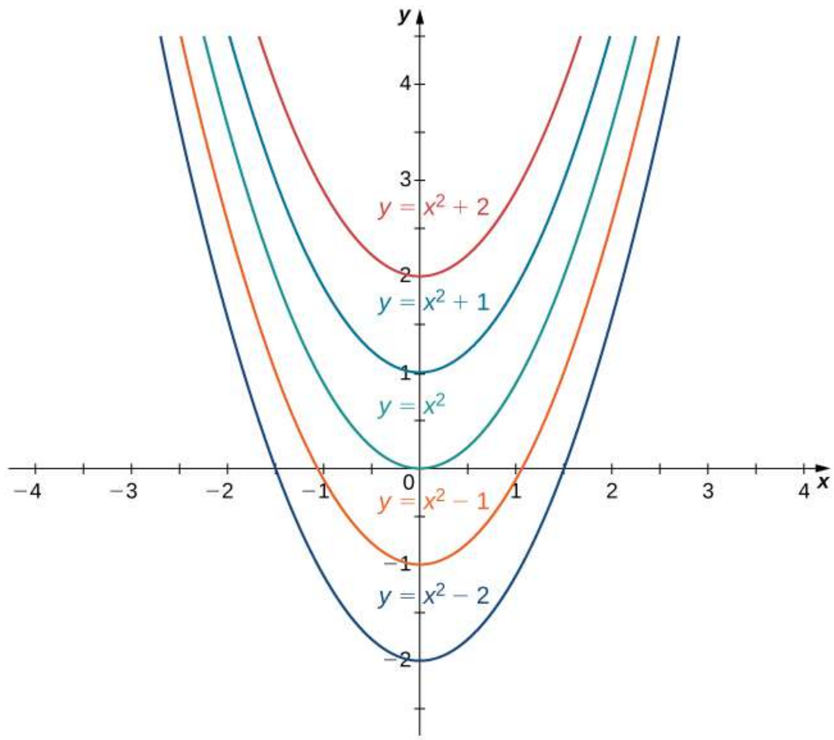


Figure 4.85 The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C,$ where C is any real number.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1,$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n.$$

This fact is known as *the power rule for integrals*.

Theorem 4.15: Power Rule for Integrals

For $n \neq -1,$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in **Appendix B**.

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k dx = \int k x^0 dx = kx + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dn = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}(\ln\|x\|) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\|x\| + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \csc x \cot x dx = -\csc x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x dx = -\cot x + C$
$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$	$\int \frac{1}{\sqrt{1 - x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2}$	$\int \frac{1}{1 + x^2} dx = \tan^{-1} x + C$
$\frac{d}{dx}\left(\sec^{-1} \|x\|\right) = \frac{1}{x\sqrt{x^2 - 1}}$	$\int \frac{1}{x\sqrt{x^2 - 1}} dx = \sec^{-1} \|x\| + C$

Table 4.13 Integration Formulas

From the definition of indefinite integral of f, we know

$$\int f(x)dx = F(x) + C$$

if and only if F is an antiderivative of f. Therefore, when claiming that

$$\int f(x)dx = F(x) + C$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example 4.51

Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$

b. $\int xe^x dx = xe^x - e^x + C$

Solution

a. Since

$$\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x,$$

the statement

$$\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x, respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

$$\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

$$\int xe^x dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2 e^x/2$ is not an antiderivative of xe^x since

$$\frac{d}{dx}\left(\frac{x^2 e^x}{2}\right) = xe^x + \frac{x^2 e^x}{2} \neq xe^x.$$

In general, the product of antiderivatives is not an antiderivative of a product.

4.50 Verify that $\int x \cos x \, dx = x \sin x + \cos x + C.$

In **Table 4.13**, we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f + g.$ In **Example 4.51**a. we showed that an antiderivative of the sum $x + e^x$ is given by the sum $\left(\frac{x^2}{2}\right) + e^x$—that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and $g,$ respectively, then

$$\frac{d}{dx}(F(x) + G(x)) = F'(x) + G'(x) = f(x) + g(x).$$

Therefore, $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and we have

$$\int (f(x) + g(x))dx = F(x) + G(x) + C.$$

Similarly,

$$\int (f(x) - g(x))dx = F(x) - G(x) + C.$$

In addition, consider the task of finding an antiderivative of $kf(x),$ where k is any real number. Since

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}F(x) = kf'(x)$$

for any real number $k,$ we conclude that

$$\int kf(x)dx = kF(x) + C.$$

These properties are summarized next.

Theorem 4.16: Properties of Indefinite Integrals

Let F and G be antiderivatives of f and $g,$ respectively, and let k be any real number.

Sums and Differences

$$\int (f(x) \pm g(x))dx = F(x) \pm G(x) + C$$

Constant Multiples

$$\int kf(x)dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated (see **Example 4.51**b. for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in **Introduction to Integration**. In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example 4.52

Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int (5x^3 - 7x^2 + 3x + 4)dx$

b. $\int \frac{x^2 + 4\sqrt[3]{x}}{x}dx$

c. $\int \frac{4}{1 + x^2}dx$

d. $\int \tan x \cos x\, dx$

Solution

a. Using **Properties of Indefinite Integrals**, we can integrate each of the four terms in the integrand separately. We obtain

$$\int (5x^3 - 7x^2 + 3x + 4)dx = \int 5x^3\, dx - \int 7x^2\, dx + \int 3x\, dx + \int 4dx.$$

From the second part of **Properties of Indefinite Integrals**, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3\, dx - \int 7x^2\, dx + \int 3x\, dx + \int 4dx = 5\int x^3\, dx - 7\int x^2\, dx + 3\int x\, dx + 4\int 1dx.$$

Using the power rule for integrals, we conclude that

$$\int (5x^3 - 7x^2 + 3x + 4)dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x} = 0.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

$$\begin{aligned}\int \left(x + \frac{4}{x^{2/3}}\right)dx &= \int x\, dx + 4\int x^{-2/3}dx \\ &= \frac{1}{2}x^2 + 4\frac{1}{\left(\frac{-2}{3}\right) + 1}x^{(-2/3) + 1} + C \\ &= \frac{1}{2}x^2 + 12x^{1/3} + C.\end{aligned}$$

c. Using **Properties of Indefinite Integrals**, write the integral as

$$4\int \frac{1}{1 + x^2}dx.$$

Then, use the fact that $\tan^{-1}(x)$ is an antiderivative of $\frac{1}{(1 + x^2)}$ to conclude that

$$\int \frac{4}{1 + x^2}dx = 4\tan^{-1}(x) + C.$$

d. Rewrite the integrand as

$$\tan x \cos x = \frac{\sin x}{\cos x}\cos x = \sin x.$$

Therefore,

$$\int \tan x \cos x = \int \sin x = -\cos x + C.$$

4.51 Evaluate $\int \left(4x^3 - 5x^2 + x - 7\right) dx.$

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A *differential equation* is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x) \tag{4.9}$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of **Equation 4.9** are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2\, dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x) \tag{4.10}$$

with the additional condition

$$y(x_0) = y_0 \tag{4.11}$$

is an example of an **initial-value problem**. The condition $y(x_0) = y_0$ is known as an *initial condition*. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C,$ to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5.$ From this equation, we see that $C = 3,$ and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.

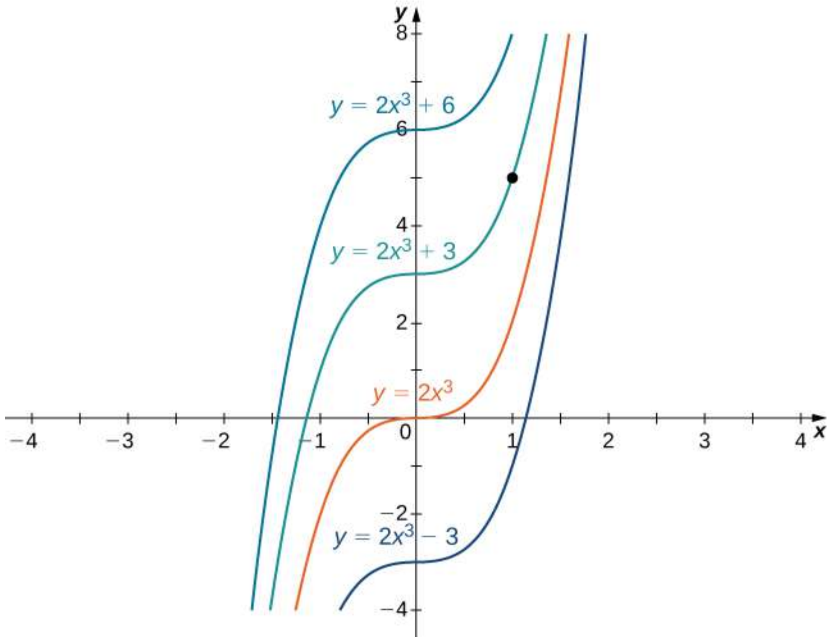


Figure 4.86 Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5.$

Example 4.53

Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin x,\ y(0) = 5.$$

Solution

First we need to solve the differential equation. If $\frac{dy}{dx} = \sin x,$ then

$$y = \int \sin(x)dx = -\cos x + C.$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition $y(0) = 5$ means we need a constant C such that $-\cos x + C = 5.$ Therefore,

$$C = 5 + \cos(0) = 6.$$

The solution of the initial-value problem is $y = -\cos x + 6.$

4.52 Solve the initial value problem $\frac{dy}{dx} = 3x^{-2},\ y(1) = 2.$

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car.

We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 4.54

Decelerating Car

A car is traveling at the rate of 88 ft/sec $(60$ mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec^2.

a. How many seconds elapse before the car stops?

b. How far does the car travel during that time?

Solution

a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.

The car is traveling at a rate of 88 ft/sec. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is

$$a(t) = -15 \text{ ft/s}^2.$$

The acceleration is the derivative of the velocity,

$$v'(t) = -15.$$

Therefore, we have an initial-value problem to solve:

$$v'(t) = -15, \; v(0) = 88.$$

Integrating, we find that

$$v(t) = -15t + C.$$

Since $v(0) = 88,\; C = 88.$ Thus, the velocity function is

$$v(t) = -15t + 88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t + 88 = 0,$ we obtain $t = \frac{88}{15}$ sec.

b. To find how far the car travels during this time, we need to find the position of the car after $\frac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem

$$s'(t) = -15t + 88, \; s(0) = 0.$$

Integrating, we have

$$s(t) = -\frac{15}{2}t^2 + 88t + C.$$

Since $s(0) = 0,$ the constant is $C = 0.$ Therefore, the position function is

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

4.53 Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop? How far will the car travel?

4.10 EXERCISES

For the following exercises, show that $F(x)$ are antiderivatives of $f(x)$.

465. $F(x) = 5x^3 + 2x^2 + 3x + 1,\ f(x) = 15x^2 + 4x + 3$

466. $F(x) = x^2 + 4x + 1,\ f(x) = 2x + 4$

467. $F(x) = x^2 e^x,\ f(x) = e^x\left(x^2 + 2x\right)$

468. $F(x) = \cos x,\ f(x) = -\sin x$

469. $F(x) = e^x,\ f(x) = e^x$

For the following exercises, find the antiderivative of the function.

470. $f(x) = \frac{1}{x^2} + x$

471. $f(x) = e^x - 3x^2 + \sin x$

472. $f(x) = e^x + 3x - x^2$

473. $f(x) = x - 1 + 4\sin(2x)$

For the following exercises, find the antiderivative $F(x)$ of each function $f(x)$.

474. $f(x) = 5x^4 + 4x^5$

475. $f(x) = x + 12x^2$

476. $f(x) = \frac{1}{\sqrt{x}}$

477. $f(x) = (\sqrt{x})^3$

478. $f(x) = x^{1/3} + (2x)^{1/3}$

479. $f(x) = \frac{x^{1/3}}{x^{2/3}}$

480. $f(x) = 2\sin(x) + \sin(2x)$

481. $f(x) = \sec^2(x) + 1$

482. $f(x) = \sin x \cos x$

483. $f(x) = \sin^2(x)\cos(x)$

484. $f(x) = 0$

485. $f(x) = \frac{1}{2}\csc^2(x) + \frac{1}{x^2}$

486. $f(x) = \csc x \cot x + 3x$

487. $f(x) = 4\csc x \cot x - \sec x \tan x$

488. $f(x) = 8\sec x(\sec x - 4\tan x)$

489. $f(x) = \frac{1}{2}e^{-4x} + \sin x$

For the following exercises, evaluate the integral.

490. $\int (-1)dx$

491. $\int \sin x\, dx$

492. $\int (4x + \sqrt{x})dx$

493. $\int \frac{3x^2 + 2}{x^2}dx$

494. $\int (\sec x \tan x + 4x)dx$

495. $\int \left(4\sqrt{x} + \sqrt[4]{x}\right)dx$

496. $\int \left(x^{-1/3} - x^{2/3}\right)dx$

497. $\int \frac{14x^3 + 2x + 1}{x^3}dx$

498. $\int (e^x + e^{-x})dx$

For the following exercises, solve the initial value problem.

499. $f'(x) = x^{-3},\ f(1) = 1$

500. $f'(x) = \sqrt{x} + x^2,\ f(0) = 2$

501. $f'(x) = \cos x + \sec^2(x),\ f\left(\frac{\pi}{4}\right) = 2 + \frac{\sqrt{2}}{2}$

502. $f'(x) = x^3 - 8x^2 + 16x + 1,\ f(0) = 0$

503. $f'(x) = \frac{2}{x^2} - \frac{x^2}{2},\ f(1) = 0$

For the following exercises, find two possible functions f given the second- or third-order derivatives.

504. $f''(x) = x^2 + 2$

505. $f''(x) = e^{-x}$

506. $f''(x) = 1 + x$

507. $f'''(x) = \cos x$

508. $f'''(x) = 8e^{-2x} - \sin x$

509. A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of 10 ft/sec^2. How long before the car stops?

510. In the preceding problem, calculate how far the car travels in the time it takes to stop.

511. You are merging onto the freeway, accelerating at a constant rate of 12 ft/sec^2. How long does it take you to reach merging speed at 60 mph?

512. Based on the previous problem, how far does the car travel to reach merging speed?

513. A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

514. A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

For the following exercises, find the antiderivative of the function, assuming $F(0) = 0$.

515. **[T]** $f(x) = x^2 + 2$

516. **[T]** $f(x) = 4x - \sqrt{x}$

517. **[T]** $f(x) = \sin x + 2x$

518. **[T]** $f(x) = e^x$

519. **[T]** $f(x) = \frac{1}{(x+1)^2}$

520. **[T]** $f(x) = e^{-2x} + 3x^2$

For the following exercises, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

521. If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

522. If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

523. If $f(x)$ is the antiderivative of $v(x)$, then $f(x) + 1$ is the antiderivative of $v(x) + 1$.

524. If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

CHAPTER 4 REVIEW

KEY TERMS

absolute extremum if f has an absolute maximum or absolute minimum at $c,$ we say f has an absolute extremum at c

absolute maximum if $f(c) \geq f(x)$ for all x in the domain of $f,$ we say f has an absolute maximum at c

absolute minimum if $f(c) \leq f(x)$ for all x in the domain of $f,$ we say f has an absolute minimum at c

antiderivative a function F such that $F'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

concave down if f is differentiable over an interval I and f' is decreasing over $I,$ then f is concave down over I

concave up if f is differentiable over an interval I and f' is increasing over $I,$ then f is concave up over I

concavity the upward or downward curve of the graph of a function

concavity test suppose f is twice differentiable over an interval I; if $f'' > 0$ over $I,$ then f is concave up over I; if $f'' < 0$ over $I,$ then f is concave down over I

critical point if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

differential the differential dx is an independent variable that can be assigned any nonzero real number; the differential dy is defined to be $dy = f'(x)dx$

differential form given a differentiable function $y = f'(x),$ the equation $dy = f'(x)dx$ is the differential form of the derivative of y with respect to x

end behavior the behavior of a function as $x \to \infty$ and $x \to -\infty$

extreme value theorem if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem if f has a local extremum at $c,$ then c is a critical point of f

first derivative test let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through $c,$ then f has a local maximum at c; if f' changes sign from negative to positive as x increases through $c,$ then f has a local minimum at c; if f' does not change sign as x increases through $c,$ then f does not have a local extremum at c

horizontal asymptote if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L,$ then $y = L$ is a horizontal asymptote of f

indefinite integral the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)dx$ to denote the indefinite integral of f

indeterminate forms when evaluating a limit, the forms $\frac{0}{0},$ $\infty/\infty,$ $0 \cdot \infty,$ $\infty - \infty,$ $0^0,$ $\infty^0,$ and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is

infinite limit at infinity a function that becomes arbitrarily large as x becomes large

inflection point if f is continuous at c and f changes concavity at $c,$ the point $(c, f(c))$ is an inflection point of f

initial value problem a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

iterative process process in which a list of numbers $x_0, x_1, x_2, x_3 \ldots$ is generated by starting with a number x_0 and defining $x_n = F(x_{n-1})$ for $n \geq 1$

limit at infinity the limiting value, if it exists, of a function as $x \to \infty$ or $x \to -\infty$

linear approximation the linear function $L(x) = f(a) + f'(a)(x - a)$ is the linear approximation of f at $x = a$

local extremum if f has a local maximum or local minimum at $c,$ we say f has a local extremum at c

local maximum if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I,$ we say f has a local maximum at c

local minimum if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I,$ we say f has a local minimum at c

L'Hôpital's rule if f and g are differentiable functions over an interval $a,$ except possibly at $a,$ and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$ assuming the limit on the right exists or is ∞ or $-\infty$

mean value theorem if f is continuous over $[a, b]$ and differentiable over $(a, b),$ then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Newton's method method for approximating roots of $f(x) = 0;$ using an initial guess $x_0;$ each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

oblique asymptote the line $y = mx + b$ if $f(x)$ approaches it as $x \to \infty$ or $x \to -\infty$

optimization problems problems that are solved by finding the maximum or minimum value of a function

percentage error the relative error expressed as a percentage

propagated error the error that results in a calculated quantity $f(x)$ resulting from a measurement error dx

related rates are rates of change associated with two or more related quantities that are changing over time

relative error given an absolute error Δq for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

rolle's theorem if f is continuous over $[a, b]$ and differentiable over $(a, b),$ and if $f(a) = f(b),$ then there exists $c \in (a, b)$ such that $f'(c) = 0$

second derivative test suppose $f'(c) = 0$ and f'' is continuous over an interval containing $c;$ if $f''(c) > 0,$ then f has a local minimum at $c;$ if $f''(c) < 0,$ then f has a local maximum at $c;$ if $f''(c) = 0,$ then the test is inconclusive

tangent line approximation (linearization) since the linear approximation of f at $x = a$ is defined using the equation of the tangent line, the linear approximation of f at $x = a$ is also known as the tangent line approximation to f at $x = a$

KEY EQUATIONS

- **Linear approximation**

$L(x) = f(a) + f'(a)(x - a)$

- **A differential**
  $dy = f'(x)dx.$

KEY CONCEPTS

4.1 Related Rates

- To solve a related rates problem, first draw a picture that illustrates the relationship between the two or more related quantities that are changing with respect to time.
- In terms of the quantities, state the information given and the rate to be found.
- Find an equation relating the quantities.
- Use differentiation, applying the chain rule as necessary, to find an equation that relates the rates.
- Be sure not to substitute a variable quantity for one of the variables until after finding an equation relating the rates.

4.2 Linear Approximations and Differentials

- A differentiable function $y = f(x)$ can be approximated at a by the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

- For a function $y = f(x)$, if x changes from a to $a + dx$, then

$$dy = f'(x)dx$$

  is an approximation for the change in y. The actual change in y is

$$\Delta y = f(a + dx) - f(a).$$

- A measurement error dx can lead to an error in a calculated quantity $f(x)$. The error in the calculated quantity is known as the *propagated error*. The propagated error can be estimated by

$$dy \approx f'(x)dx.$$

- To estimate the relative error of a particular quantity q, we estimate $\frac{\Delta q}{q}$.

4.3 Maxima and Minima

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

4.4 The Mean Value Theorem

- If f is continuous over $[a, b]$ and differentiable over (a, b) and $f(a) = 0 = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$. This is Rolle's theorem.
- If f is continuous over $[a, b]$ and differentiable over (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is the Mean Value Theorem.

- If $f'(x) = 0$ over an interval I, then f is constant over I.
- If two differentiable functions f and g satisfy $f'(x) = g'(x)$ over I, then $f(x) = g(x) + C$ for some constant C.
- If $f'(x) > 0$ over an interval I, then f is increasing over I. If $f'(x) < 0$ over I, then f is decreasing over I.

4.5 Derivatives and the Shape of a Graph

- If c is a critical point of f and $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at c.
- If c is a critical point of f and $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local minimum at c.
- If $f''(x) > 0$ over an interval I, then f is concave up over I.
- If $f''(x) < 0$ over an interval I, then f is concave down over I.
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

4.6 Limits at Infinity and Asymptotes

- The limit of $f(x)$ is L as $x \to \infty$ (or as $x \to -\infty$) if the values $f(x)$ become arbitrarily close to L as x becomes sufficiently large.
- The limit of $f(x)$ is ∞ as $x \to \infty$ if $f(x)$ becomes arbitrarily large as x becomes sufficiently large. The limit of $f(x)$ is $-\infty$ as $x \to \infty$ if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large as x becomes sufficiently large. We can define the limit of $f(x)$ as x approaches $-\infty$ similarly.
- For a polynomial function $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$, the end behavior is determined by the leading term $a_n x^n$. If $n \neq 0$, $p(x)$ approaches ∞ or $-\infty$ at each end.
- For a rational function $f(x) = \frac{p(x)}{q(x)}$, the end behavior is determined by the relationship between the degree of p and the degree of q. If the degree of p is less than the degree of q, the line $y = 0$ is a horizontal asymptote for f. If the degree of p is equal to the degree of q, then the line $y = \frac{a_n}{b_n}$ is a horizontal asymptote, where a_n and b_n are the leading coefficients of p and q, respectively. If the degree of p is greater than the degree of q, then f approaches ∞ or $-\infty$ at each end.

4.7 Applied Optimization Problems

- To solve an optimization problem, begin by drawing a picture and introducing variables.
- Find an equation relating the variables.
- Find a function of one variable to describe the quantity that is to be minimized or maximized.

- Look for critical points to locate local extrema.

4.8 L'Hôpital's Rule

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞.
- The exponential function e^x grows faster than any power function $x^p,\quad p > 0.$
- The logarithmic function $\ln x$ grows more slowly than any power function $x^p,\quad p > 0.$

4.9 Newton's Method

- Newton's method approximates roots of $f(x) = 0$ by starting with an initial approximation $x_0,$ then uses tangent lines to the graph of f to create a sequence of approximations $x_1, x_2, x_3, \ldots.$
- Typically, Newton's method is an efficient method for finding a particular root. In certain cases, Newton's method fails to work because the list of numbers $x_0, x_1, x_2, \ldots$ does not approach a finite value or it approaches a value other than the root sought.
- Any process in which a list of numbers $x_0, x_1, x_2, \ldots$ is generated by defining an initial number x_0 and defining the subsequent numbers by the equation $x_n = F(x_{n-1})$ for some function F is an iterative process. Newton's method is an example of an iterative process, where the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$ for a given function f.

4.10 Antiderivatives

- If F is an antiderivative of $f,$ then every antiderivative of f is of the form $F(x) + C$ for some constant C.
- Solving the initial-value problem

$$\frac{dy}{dx} = f(x),\ y(x_0) = y_0$$

requires us first to find the set of antiderivatives of f and then to look for the particular antiderivative that also satisfies the initial condition.

CHAPTER 4 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample. Assume that $f(x)$ is continuous and differentiable unless stated otherwise.

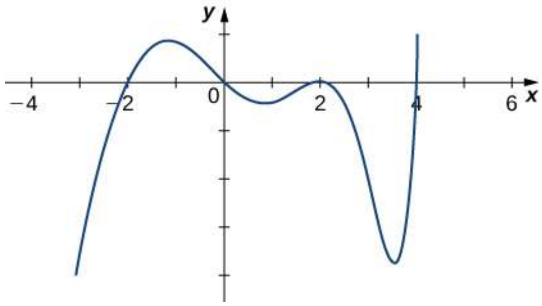
525. If $f(-1) = -6$ and $f(1) = 2,$ then there exists at least one point $x \in [-1, 1]$ such that $f'(x) = 4.$

526. If $f'(c) = 0,$ there is a maximum or minimum at $x = c.$

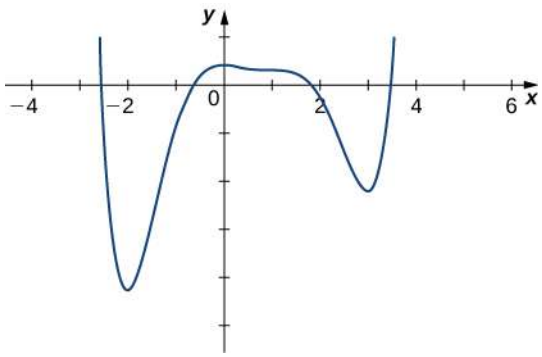
527. There is a function such that $f(x) < 0,\ f'(x) > 0,$ and $f''(x) < 0.$ (A graphical "proof" is acceptable for this answer.)

528. There is a function such that there is both an inflection point and a critical point for some value $x = a.$

529. Given the graph of f', determine where f is increasing or decreasing.



530. The graph of f is given below. Draw f'.



531. Find the linear approximation $L(x)$ to $y = x^2 + \tan(\pi x)$ near $x = \frac{1}{4}$.

532. Find the differential of $y = x^2 - 5x - 6$ and evaluate for $x = 2$ with $dx = 0.1$.

Find the critical points and the local and absolute extrema of the following functions on the given interval.

533. $f(x) = x + \sin^2(x)$ over $[0, \pi]$

534. $f(x) = 3x^4 - 4x^3 - 12x^2 + 6$ over $[-3, 3]$

Determine over which intervals the following functions are increasing, decreasing, concave up, and concave down.

535. $x(t) = 3t^4 - 8t^3 - 18t^2$

536. $y = x + \sin(\pi x)$

537. $g(x) = x - \sqrt{x}$

538. $f(\theta) = \sin(3\theta)$

Evaluate the following limits.

539. $\lim_{x \to \infty} \frac{3x\sqrt{x^2 + 1}}{\sqrt{x^4 - 1}}$

540. $\lim_{x \to \infty} \cos\left(\frac{1}{x}\right)$

541. $\lim_{x \to 1} \frac{x - 1}{\sin(\pi x)}$

542. $\lim_{x \to \infty} (3x)^{1/x}$

Use Newton's method to find the first two iterations, given the starting point.

543. $y = x^3 + 1,\ x_0 = 0.5$

544. $\frac{1}{x + 1} = \frac{1}{2},\ x_0 = 0$

Find the antiderivatives $F(x)$ of the following functions.

545. $g(x) = \sqrt{x} - \frac{1}{x^2}$

546. $f(x) = 2x + 6\cos x,\ F(\pi) = \pi^2 + 2$

Graph the following functions by hand. Make sure to label the inflection points, critical points, zeros, and asymptotes.

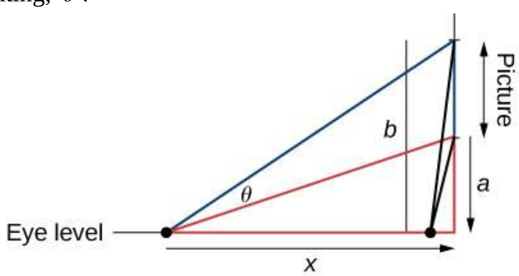
547. $y = \frac{1}{x(x + 1)^2}$

548. $y = x - \sqrt{4 - x^2}$

549. A car is being compacted into a rectangular solid. The volume is decreasing at a rate of 2 m^3/sec. The length and width of the compactor are square, but the height is not the same length as the length and width. If the length and width walls move toward each other at a rate of 0.25 m/sec, find the rate at which the height is changing when the length and width are 2 m and the height is 1.5 m.

550. A rocket is launched into space; its kinetic energy is given by $K(t) = \left(\frac{1}{2}\right)m(t)v(t)^2,$ where K is the kinetic energy in joules, m is the mass of the rocket in kilograms, and v is the velocity of the rocket in meters/second. Assume the velocity is increasing at a rate of 15 m/sec^2 and the mass is decreasing at a rate of 10 kg/sec because the fuel is being burned. At what rate is the rocket's kinetic energy changing when the mass is 2000 kg and the velocity is 5000 m/sec? Give your answer in mega-Joules (MJ), which is equivalent to 10^6 J.

551. The famous Regiomontanus' problem for angle maximization was proposed during the 15 th century. A painting hangs on a wall with the bottom of the painting a distance a feet above eye level, and the top b feet above eye level. What distance x (in feet) from the wall should the viewer stand to maximize the angle subtended by the painting, θ?



552. An airline sells tickets from Tokyo to Detroit for $\$1200.$ There are 500 seats available and a typical flight books 350 seats. For every $\$10$ decrease in price, the airline observes an additional five seats sold. What should the fare be to maximize profit? How many passengers would be onboard?