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Calc-ulus

Volume 1

5 | INTEGRATION

Figure 5.1 Iceboating is a popular winter sport in parts of the northern United States and Europe. (credit: modification of work by Carter Brown, Flickr)

Chapter Outline

5.1 Approximating Areas
5.2 The Definite Integral
5.3 The Fundamental Theorem of Calculus
5.4 Integration Formulas and the Net Change Theorem
5.5 Substitution
5.6 Integrals Involving Exponential and Logarithmic Functions
5.7 Integrals Resulting in Inverse Trigonometric Functions

Introduction

Iceboats are a common sight on the lakes of Wisconsin and Minnesota on winter weekends. Iceboats are similar to sailboats, but they are fitted with runners, or “skates,” and are designed to run over the ice, rather than on water. Iceboats can move very quickly, and many ice boating enthusiasts are drawn to the sport because of the speed. Top iceboat racers can attain speeds up to five times the wind speed. If we know how fast an iceboat is moving, we can use integration to determine how far it travels. We revisit this question later in the chapter (see **Example 5.27**).

Determining distance from velocity is just one of many applications of integration. In fact, integrals are used in a wide variety of mechanical and physical applications. In this chapter, we first introduce the theory behind integration and use integrals to calculate areas. From there, we develop the Fundamental Theorem of Calculus, which relates differentiation and integration. We then study some basic integration techniques and briefly examine some applications.

5.1 | Approximating Areas

Learning Objectives

5.1.1 Use sigma (summation) notation to calculate sums and powers of integers.
5.1.2 Use the sum of rectangular areas to approximate the area under a curve.
5.1.3 Use Riemann sums to approximate area.

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the *method of exhaustion*, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x),$ and the *x*-axis on a closed interval $[a, b].$ Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter $\Sigma,$ sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

We could probably skip writing a couple of terms and write

$$1 + 2 + 3 + 4 + \cdots + 19 + 20,$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i,$$

which is much more compact.

Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i$$

where a_i describes the terms to be added, and the *i* is called the *index*. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n.$ For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2 + s_3 + s_4 + s_5 + s_6 + s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a *dummy variable*. We can use any letter we like for the index. Typically, mathematicians use *i*, *j*, *k*, *m*, and *n* for indices.

Let's try a couple of examples of using sigma notation.

Example 5.1

Using Sigma Notation

a. Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.

b. Write the sum in sigma notation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

Solution

a. Write

$$\sum_{i=1}^{5} 3^i = 3 + 3^2 + 3^3 + 3^4 + 3^5$$
$$= 363.$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

5.1 Write in sigma notation and evaluate the sum of terms 2^i for $i = 3, 4, 5, 6$.

The properties associated with the summation process are given in the following rule.

Rule: Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \le m \le n$.

1.
$$\sum_{i=1}^{n} c = nc \tag{5.1}$$

2.
$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i \tag{5.2}$$

3.
$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \tag{5.3}$$

4.
$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \tag{5.4}$$

5.

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i \tag{5.5}$$

Proof

We prove properties 2. and 3. here, and leave proof of the other properties to the Exercises.

2. We have

$$\begin{aligned}\sum_{i=1}^{n} ca_i &= ca_1 + ca_2 + ca_3 + \cdots + ca_n \\ &= c(a_1 + a_2 + a_3 + \cdots + a_n) \\ &= c\sum_{i=1}^{n} a_i.\end{aligned}$$

3. We have

$$\begin{aligned}\sum_{i=1}^{n} (a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n) \\ &= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n) \\ &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.\end{aligned}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Example 5.2

Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

a. The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$.

b. The sum of the terms $\left(i^3 - i^2\right)$ for $i = 1, 2, 3, 4, 5, 6.$

Solution

a. Multiplying out $(i-3)^2,$ we can break the expression into three terms.

$$\begin{aligned}\sum_{i=1}^{200}(i-3)^2 &= \sum_{i=1}^{200}\left(i^2 - 6i + 9\right)\\ &= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9\\ &= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9\\ &= \frac{200(200+1)(400+1)}{6} - 6\left[\frac{200(200+1)}{2}\right] + 9(200)\\ &= 2{,}686{,}700 - 120{,}600 + 1800\\ &= 2{,}567{,}900\end{aligned}$$

b. Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

$$\begin{aligned}\sum_{i=1}^{6}\left(i^3 - i^2\right) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2\\ &= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6}\\ &= \frac{1764}{4} - \frac{546}{6}\\ &= 350\end{aligned}$$

5.2 Find the sum of the values of $4 + 3i$ for $i = 1, 2, \ldots, 100.$

Example 5.3

Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10.$

Solution

Using the formula, we have

$$\begin{aligned}\sum_{i=0}^{10} i^3 &= \frac{(10)^2(10+1)^2}{4}\\ &= \frac{100(121)}{4}\\ &= 3025.\end{aligned}$$

5.3 Evaluate the sum indicated by the notation $\sum_{k=1}^{20}(2k+1)$.

Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x-axis below, the line $x = a$ on the left, and the line $x = b$ on the right (**Figure 5.2**).

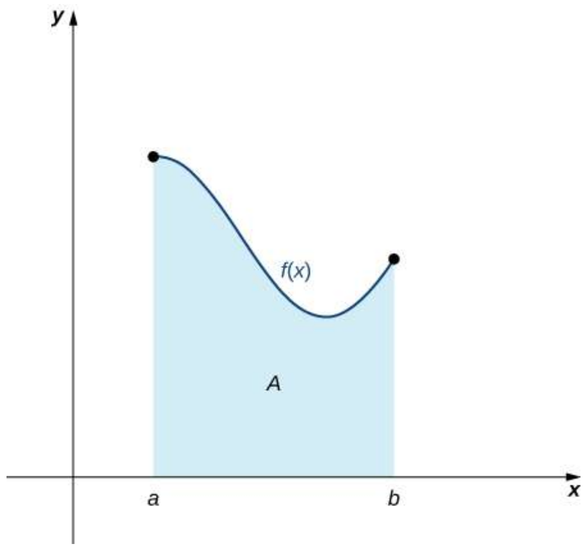


Figure 5.2 An area (shaded region) bounded by the curve $f(x)$ at top, the x-axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b,$ and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

for $i = 1, 2, 3, \ldots, n.$

We denote the width of each subinterval with the notation Δx, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

for $i = 1, 2, 3, \ldots, n.$ This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let’s define some relevant terminology.

Definition

A set of points $P = \{x_i\}$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b,$ which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

Rule: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for *A* (**Figure 5.3**). We use the notation L_n to denote that this is a **left-endpoint approximation** of *A* using *n* subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x \quad (5.6)$$

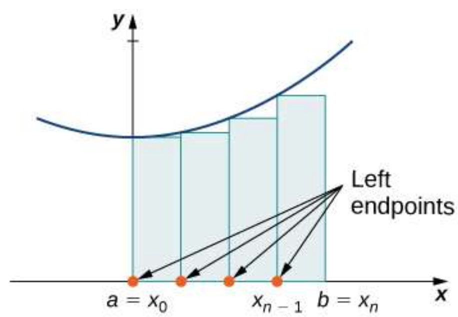


Figure 5.3 In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Rule: Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\Delta x$ and the approximation for *A* is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x. \quad (5.7)$$

The notation R_n indicates this is a **right-endpoint approximation** for *A* (**Figure 5.4**).

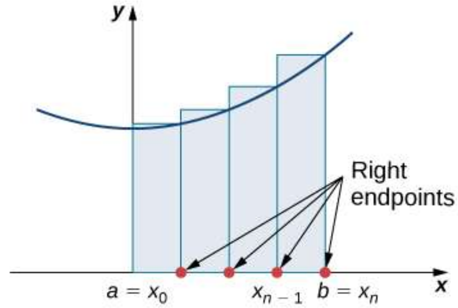


Figure 5.4 In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in **Figure 5.3**.

The graphs in **Figure 5.5** represent the curve $f(x) = \frac{x^2}{2}$. In graph (a) we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width 0.5. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1}, the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\ &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375. \end{aligned}$$

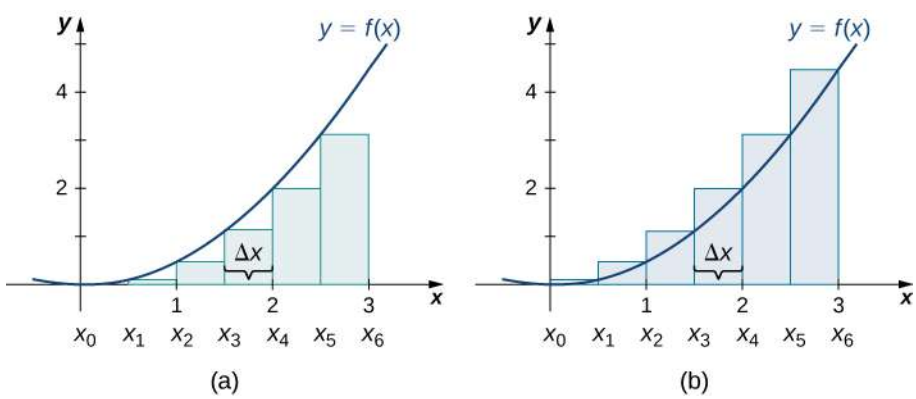


Figure 5.5 Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In **Figure 5.5**(b), we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\ &= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875. \end{aligned}$$

Example 5.4

Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \frac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$ are shown in **Figure 5.6**. Using a left-endpoint approximation, the heights are $f(0) = 0,\ f(0.5) = 0.25,\ f(1) = 1,\ f(1.5) = 2.25$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 1.75. \end{aligned}$$

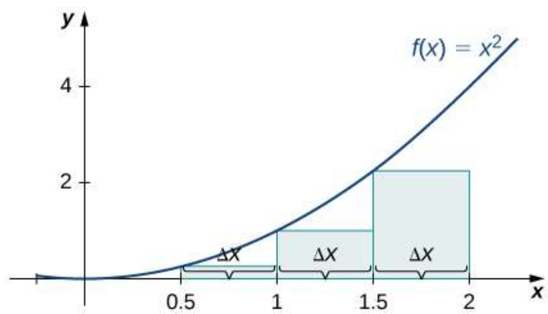


Figure 5.6 The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in **Figure 5.7**. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 3.75. \end{aligned}$$

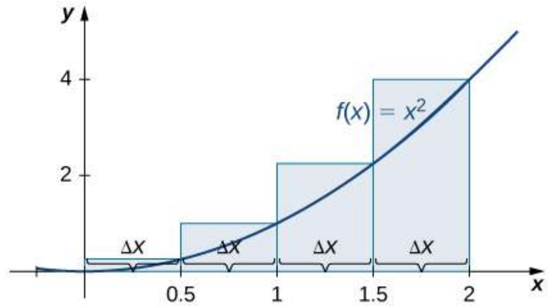


Figure 5.7 The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is 1.75; the right-endpoint approximation is 3.75.

5.4 Sketch left-endpoint and right-endpoint approximations for $f(x) = \frac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Looking at **Figure 5.5** and the graphs in **Example 5.4**, we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n, first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. **Figure 5.8** shows the area of the region under the curve $f(x) = (x - 1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

$$\Delta x = \frac{2 - 0}{4} = \frac{1}{2}.$$

The area is approximated by the summed areas of the rectangles, or

$$\begin{aligned} L_4 &= f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)0.5 \\ &= 7.5. \end{aligned}$$

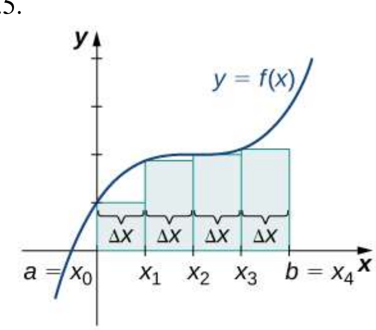


Figure 5.8 With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

Figure 5.9 shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in **Figure 5.8** with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

$$\begin{aligned} L_8 &= f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) \\ &\quad + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) \\ &= 7.75. \end{aligned}$$

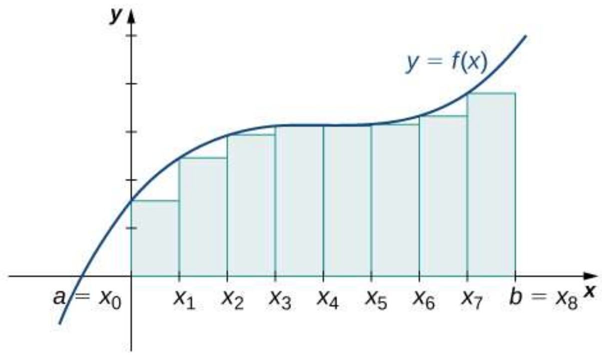


Figure 5.9 The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in **Figure 5.10** shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$\begin{aligned} L_{32} &= f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) \\ &= 7.9375. \end{aligned}$$

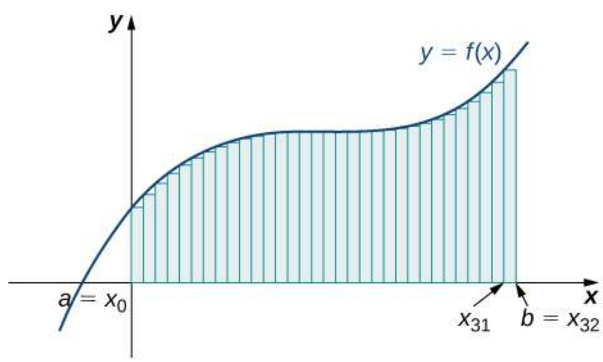


Figure 5.10 Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles (**Figure 5.11**), yields an area

$$\begin{aligned} R_4 &= f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) \\ &= 8.5. \end{aligned}$$

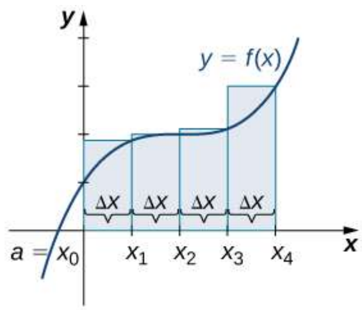


Figure 5.11 Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \frac{2-0}{8} = 0.25$. The graph is shown in Figure 5.12. The area is

$$\begin{aligned} R_8 &= f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) \\ &\quad + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) \\ &= 8.25. \end{aligned}$$

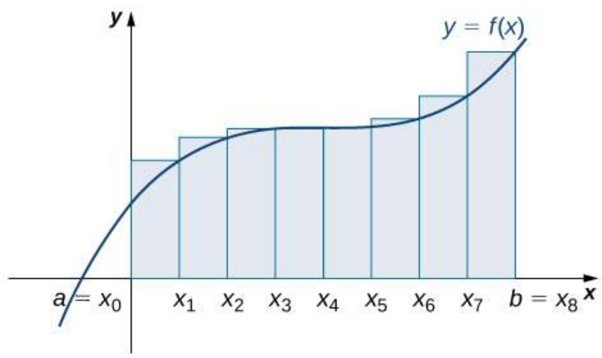


Figure 5.12 Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area (Figure 5.13). The area is approximately

$$\begin{aligned} R_{32} &= f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) \\ &= 8.0625. \end{aligned}$$

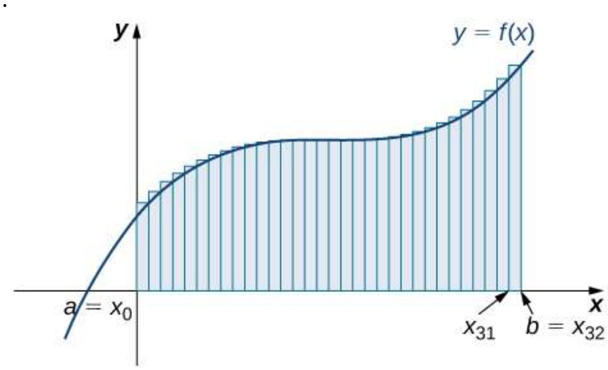


Figure 5.13 The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as n gets larger. Furthermore, as n increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. Table 5.1 shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as *n* gets larger and larger is very important, and we now explore this idea in more detail.

Values of n	Approximate Area L_n	Approximate Area R_n
$n=4$	7.5	8.5
$n=8$	7.75	8.25
$n=32$	7.94	8.06

Table 5.1 Converging Values of Left- and Right-Endpoint Approximations as *n* Increases

Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point x_i in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be a regular partition of $[a, b]$. Let Δx be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A **Riemann sum** is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as *n* get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of *n*. We are now ready to define the area under a curve in terms of Riemann sums.

Definition

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\Delta x$ be a Riemann sum for $f(x)$. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

See a **graphical demonstration (http://www.openstax.org/l/20_riemannsums)** of the construction of a Riemann sum.

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in detail in the chapter on **Sequences and Series (http://cnx.org/content/m53756/latest/)** ; however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of $\{x_i^*\}$. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (in other words, it does not depend on the choice of $\{x_i^*\}$).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for $\{x_i^*\}$. Although any choice for $\{x_i^*\}$ gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high (overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for $\{x_i^*\}$ to guarantee one result or the other.

If we want an overestimate, for example, we can choose $\{x_i^*\}$ such that for $i = 1, 2, 3,\ldots, n,$ $f(x_i^*) \geq f(x)$ for all $x \in [x_{i-1}, x_i]$. In other words, we choose $\{x_i^*\}$ so that for $i = 1, 2, 3,\ldots, n,$ $f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select $\{x_i^*\}$ in this way, then the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose $\{x_i^*\}$ so that for $i = 1, 2, 3,\ldots, n,$ $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Example 5.5

Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

Solution

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \frac{1}{4}$. We can list the intervals as $[1, 1.25], [1.25, 1.5], [1.5, 1.75], [1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, **Figure 5.14** shows that a lower sum is obtained by using the right endpoints.

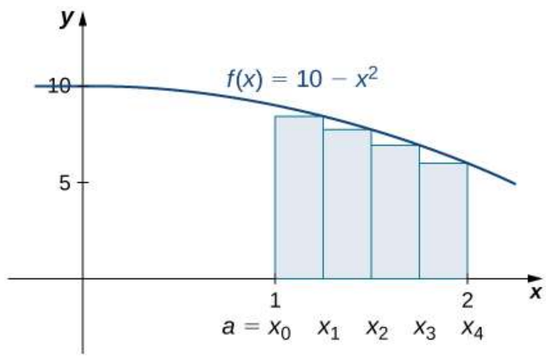


Figure 5.14 The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x-axis on $[1, 2],$ and it shows a lower sum.

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4} \left(10 - x^2\right)(0.25) &= 0.25\left[10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2\right] \\ &= 0.25[8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28.\end{aligned}$$

The area of 7.28 is a lower sum and an underestimate.

5.5 a. Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2];$ let $n = 4.$

b. Sketch the approximation.

Example 5.6

Finding Lower and Upper Sums for $f(x) = \sin x$

Find a lower sum for $f(x) = \sin x$ over the interval $[a, b] = \left[0, \frac{\pi}{2}\right];$ let $n = 6.$

Solution

Let's first look at the graph in **Figure 5.15** to get a better idea of the area of interest.

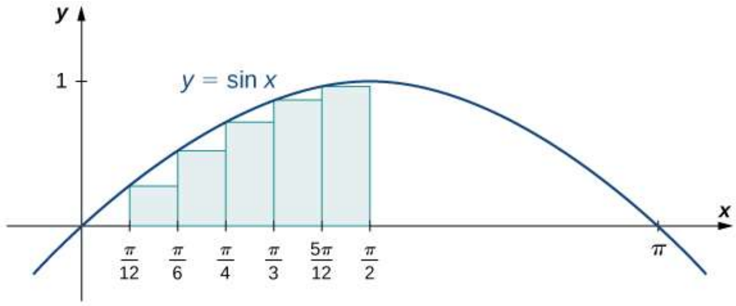


Figure 5.15 The graph of $y = \sin x$ is divided into six regions: $\Delta x = \frac{\pi/2}{6} = \frac{\pi}{12}$.

The intervals are $\left[0, \frac{\pi}{12}\right], \left[\frac{\pi}{12}, \frac{\pi}{6}\right], \left[\frac{\pi}{6}, \frac{\pi}{4}\right], \left[\frac{\pi}{4}, \frac{\pi}{3}\right], \left[\frac{\pi}{3}, \frac{5\pi}{12}\right],$ and $\left[\frac{5\pi}{12}, \frac{\pi}{2}\right]$. Note that $f(x) = \sin x$ is increasing on the interval $\left[0, \frac{\pi}{2}\right],$ so a left-endpoint approximation gives us the lower sum. A left-endpoint approximation is the Riemann sum $\sum_{i=0}^{5} \sin x_i \left(\frac{\pi}{12}\right)$. We have

$$\begin{aligned} A &\approx \sin(0)\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{6}\right)\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right)\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{3}\right)\left(\frac{\pi}{12}\right) + \sin\left(\frac{5\pi}{12}\right)\left(\frac{\pi}{12}\right) \\ &= 0.863. \end{aligned}$$

5.6 Using the function $f(x) = \sin x$ over the interval $\left[0, \frac{\pi}{2}\right],$ find an upper sum; let $n = 6$.

5.1 EXERCISES

1. State whether the given sums are equal or unequal.

   a. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$

   b. $\sum_{i=1}^{10} i$ and $\sum_{i=6}^{15} (i-5)$

   c. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{j=0}^{9} (j+1)j$

   d. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{k=1}^{10} \left(k^2-k\right)$

In the following exercises, use the rules for sums of powers of integers to compute the sums.

2. $\sum_{i=5}^{10} i$

3. $\sum_{i=5}^{10} i^2$

Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$. In the following exercises, compute the sums.

4. $\sum_{i=1}^{100} (a_i + b_i)$

5. $\sum_{i=1}^{100} (a_i - b_i)$

6. $\sum_{i=1}^{100} (3a_i - 4b_i)$

7. $\sum_{i=1}^{100} (5a_i + 4b_i)$

In the following exercises, use summation properties and formulas to rewrite and evaluate the sums.

8. $\sum_{k=1}^{20} 100\left(k^2 - 5k + 1\right)$

9. $\sum_{j=1}^{50} \left(j^2 - 2j\right)$

10. $\sum_{j=11}^{20} \left(j^2 - 10j\right)$

11. $\sum_{k=1}^{25} \left[(2k)^2 - 100k\right]$

Let L_n denote the left-endpoint sum using n subintervals and let R_n denote the corresponding right-endpoint sum. In the following exercises, compute the indicated left and right sums for the given functions on the indicated interval.

12. L_4 for $f(x) = \frac{1}{x-1}$ on $[2, 3]$

13. R_4 for $g(x) = \cos(\pi x)$ on $[0, 1]$

14. L_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2, 5]$

15. R_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2, 5]$

16. R_4 for $\frac{1}{x^2+1}$ on $[-2, 2]$

17. L_4 for $\frac{1}{x^2+1}$ on $[-2, 2]$

18. R_4 for $x^2 - 2x + 1$ on $[0, 2]$

19. L_8 for $x^2 - 2x + 1$ on $[0, 2]$

20. Compute the left and right Riemann sums—L_4 and R_4, respectively—for $f(x) = (2 - |x|)$ on $[-2, 2]$. Compute their average value and compare it with the area under the graph of *f*.

21. Compute the left and right Riemann sums—L_6 and R_6, respectively—for $f(x) = (3 - |3 - x|)$ on $[0, 6]$. Compute their average value and compare it with the area under the graph of *f*.

22. Compute the left and right Riemann sums—L_4 and R_4, respectively—for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ and compare their values.

23. Compute the left and right Riemann sums—L_6 and R_6, respectively—for $f(x) = \sqrt{9 - (x-3)^2}$ on $[0, 6]$ and compare their values.

Express the following endpoint sums in sigma notation but do not evaluate them.

24. L_{30} for $f(x) = x^2$ on $[1, 2]$

25. L_{10} for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$

26. R_{20} for $f(x) = \sin x$ on $[0, \pi]$

27. R_{100} for $\ln x$ on $[1, e]$

In the following exercises, graph the function then use a calculator or a computer program to evaluate the following left and right endpoint sums. Is the area under the curve on the given interval better approximated by the left Riemann sum or right Riemann sum? If the two agree, say "neither."

28. **[T]** L_{100} and R_{100} for $y = x^2 - 3x + 1$ on the interval $[-1, 1]$

29. **[T]** L_{100} and R_{100} for $y = x^2$ on the interval $[0, 1]$

30. **[T]** L_{50} and R_{50} for $y = \dfrac{x+1}{x^2 - 1}$ on the interval $[2, 4]$

31. **[T]** L_{100} and R_{100} for $y = x^3$ on the interval $[-1, 1]$

32. **[T]** L_{50} and R_{50} for $y = \tan(x)$ on the interval $\left[0, \frac{\pi}{4}\right]$

33. **[T]** L_{100} and R_{100} for $y = e^{2x}$ on the interval $[-1, 1]$

34. Let t_j denote the time that it took Tejay van Garteren to ride the *j*th stage of the Tour de France in 2014. If there were a total of 21 stages, interpret $\sum_{j=1}^{21} t_j$.

35. Let r_j denote the total rainfall in Portland on the *j*th day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.

36. Let d_j denote the hours of daylight and δ_j denote the increase in the hours of daylight from day $j - 1$ to day *j* in Fargo, North Dakota, on the *j*th day of the year. Interpret $d_1 + \sum_{j=2}^{365} \delta_j$.

37. To help get in shape, Joe gets a new pair of running shoes. If Joe runs 1 mi each day in week 1 and adds $\frac{1}{10}$ mi to his daily routine each week, what is the total mileage on Joe's shoes after 25 weeks?

38. The following table gives approximate values of the average annual atmospheric rate of increase in carbon dioxide (CO_2) each decade since 1960, in parts per million (ppm). Estimate the total increase in atmospheric CO_2 between 1964 and 2013.

Decade	Ppm/y
1964–1973	1.07
1974–1983	1.34
1984–1993	1.40
1994–2003	1.87
2004–2013	2.07

Table 5.2 Average Annual Atmospheric CO_2 Increase, 1964–2013 ***Source*****: http://www.esrl.noaa.gov/gmd/ccgg/trends/.**

39. The following table gives the approximate increase in sea level in inches over 20 years starting in the given year. Estimate the net change in mean sea level from 1870 to 2010.

Starting Year	20-Year Change
1870	0.3
1890	1.5
1910	0.2
1930	2.8
1950	0.7
1970	1.1
1990	1.5

Table 5.3 Approximate 20-Year Sea Level Increases, 1870–1990 ***Source*: http://link.springer.com/article/10.1007%2Fs10712-011-9119-1**

40. The following table gives the approximate increase in dollars in the average price of a gallon of gas per decade since 1950. If the average price of a gallon of gas in 2010 was $2.60, what was the average price of a gallon of gas in 1950?

Starting Year	10-Year Change
1950	0.03
1960	0.05
1970	0.86
1980	−0.03
1990	0.29
2000	1.12

Table 5.4 Approximate 10-Year Gas Price Increases, 1950–2000 ***Source*: http://epb.lbl.gov/homepages/Rick_Diamond/docs/lbnl55011-trends.pdf.**

41. The following table gives the percent growth of the U.S. population beginning in July of the year indicated. If the U.S. population was 281,421,906 in July 2000, estimate the U.S. population in July 2010.

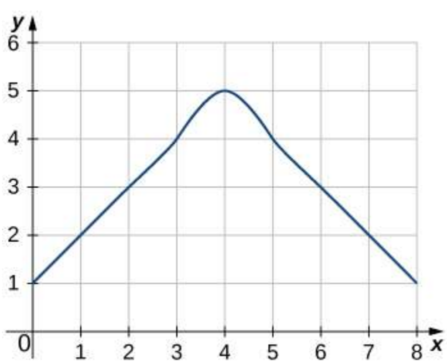
Year	% Change/Year
2000	1.12
2001	0.99
2002	0.93
2003	0.86
2004	0.93
2005	0.93
2006	0.97
2007	0.96
2008	0.95
2009	0.88

Table 5.5 Annual Percentage Growth of U.S. Population, 2000–2009 ***Source*: http://www.census.gov/popest/data.**

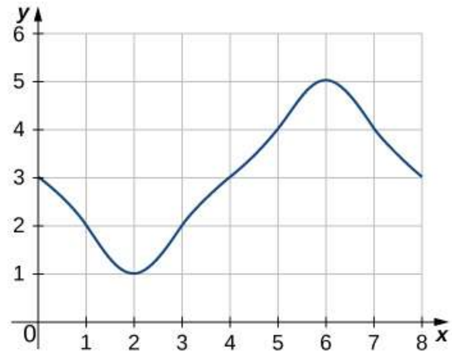
(*Hint:* To obtain the population in July 2001, multiply the population in July 2000 by 1.0112 to get 284,573,831.)

In the following exercises, estimate the areas under the curves by computing the left Riemann sums, L_8.

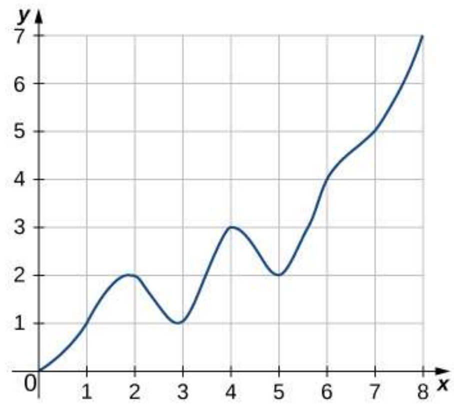
42.



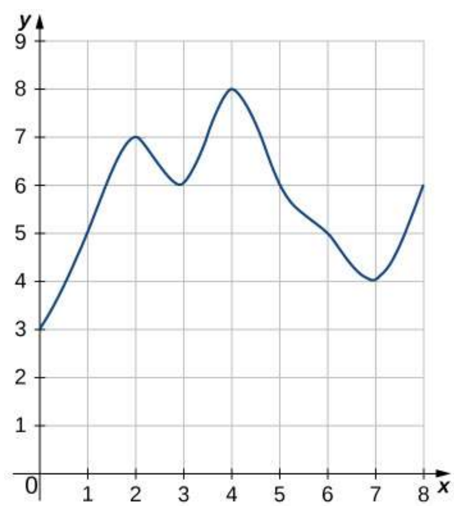
43.



44.



45.



46. **[T]** Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$.

47. **[T]** Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \dfrac{1}{\sqrt{1 + x^2}}$ on $[-1, 1]$.

48. **[T]** Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10,\ 30,\ 50$ for $f(x) = \sin^2 x$ on $[0,\ 2\pi]$. Compare these estimates with π.

In the following exercises, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1{,}10{,}100$. How do these estimates compare with the exact answers, which you can find via geometry?

49. **[T]** $y = \cos(\pi x)$ on the interval $[0,\ 1]$

50. **[T]** $y = 3x + 2$ on the interval $[3,\ 5]$

In the following exercises, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1{,}10{,}100$.

51. **[T]** $y = x^4 - 5x^2 + 4$ on the interval $[-2,\ 2]$, which has an exact area of $\frac{32}{15}$

52. **[T]** $y = \ln x$ on the interval $[1,\ 2]$, which has an exact area of $2\ln(2) - 1$

53. Explain why, if $f(a) \geq 0$ and f is increasing on $[a,\ b]$, that the left endpoint estimate is a lower bound for the area below the graph of f on $[a,\ b]$.

54. Explain why, if $f(b) \geq 0$ and f is decreasing on $[a,\ b]$, that the left endpoint estimate is an upper bound for the area below the graph of f on $[a,\ b]$.

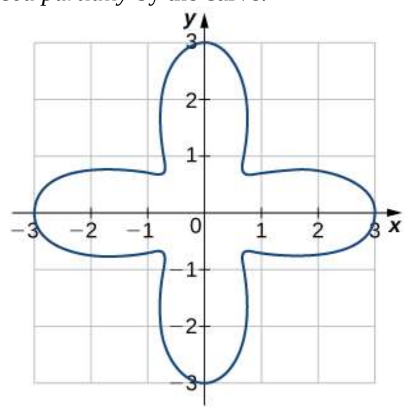
55. Show that, in general, $R_N - L_N = (b - a) \times \frac{f(b) - f(a)}{N}$.

56. Explain why, if f is increasing on $[a,\ b]$, the error between either L_N or R_N and the area A below the graph of f is at most $(b - a)\frac{f(b) - f(a)}{N}$.

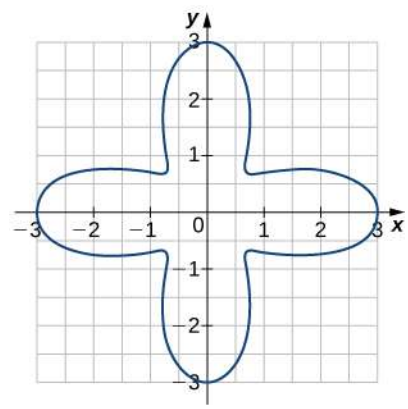
57. For each of the three graphs:

a. Obtain a lower bound $L(A)$ for the area enclosed by the curve by adding the areas of the squares *enclosed completely* by the curve.

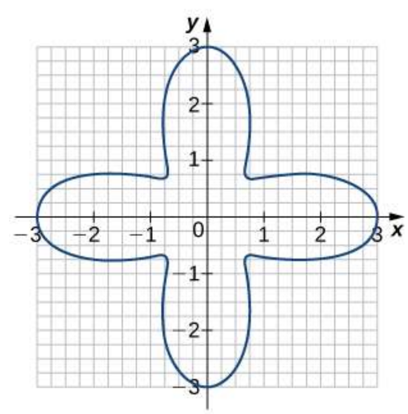
b. Obtain an upper bound $U(A)$ for the area by adding to $L(A)$ the areas $B(A)$ of the squares *enclosed partially* by the curve.



Graph 1



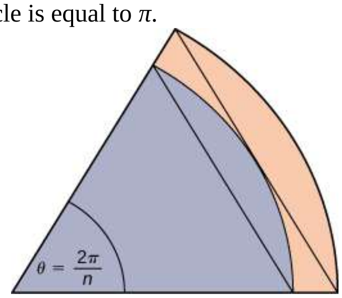
Graph 2



Graph 3

58. In the previous exercise, explain why $L(A)$ gets no smaller while $U(A)$ gets no larger as the squares are subdivided into four boxes of equal area.

59. A unit circle is made up of n wedges equivalent to the inner wedge in the figure. The base of the inner triangle is 1 unit and its height is $\sin\left(\frac{\pi}{n}\right)$. The base of the outer triangle is $B = \cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)$ and the height is $H = B\sin\left(\frac{2\pi}{n}\right)$. Use this information to argue that the area of a unit circle is equal to π.



5.2 | The Definite Integral

Learning Objectives

5.2.1 State the definition of the definite integral.
5.2.2 Explain the terms integrand, limits of integration, and variable of integration.
5.2.3 Explain when a function is integrable.
5.2.4 Describe the relationship between the definite integral and net area.
5.2.5 Use geometry and the properties of definite integrals to evaluate them.
5.2.6 Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Definition

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of *f* from *a* to *b* is given by

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \quad (5.8)$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an **integrable function**.

The integral symbol in the previous definition should look familiar. We have seen similar notation in the chapter on **Applications of Derivatives**, where we used the indefinite integral symbol (without the *a* and *b* above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter we examine how these concepts are related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century and is one of the contributions of Gottfried Wilhelm Leibniz, who is often considered to be the codiscoverer of calculus, along with Isaac Newton. The integration symbol ∫ is an elongated S, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers *a* and *b* are *x*-values and are called the **limits of integration**; specifically, *a* is the lower limit and *b* is the upper limit. To clarify, we are using the word *limit* in two different ways in the context of the definite integral. First, we talk about the limit of a sum as $n \to \infty$. Second, the boundaries of the region are called the *limits of integration*.

We call the function $f(x)$ the **integrand**, and the *dx* indicates that $f(x)$ is a function with respect to *x*, called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a dummy variable, and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a, b]$, then the limit $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

Theorem 5.1: Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Functions that are not continuous on $[a, b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities on a closed interval are integrable.

It is also worth noting here that we have retained the use of a regular partition in the Riemann sums. This restriction is not strictly necessary. Any partition can be used to form a Riemann sum. However, if a nonregular partition is used to define the definite integral, it is not sufficient to take the limit as the number of subintervals goes to infinity. Instead, we must take the limit as the width of the largest subinterval goes to zero. This introduces a little more complex notation in our limits and makes the calculations more difficult without really gaining much additional insight, so we stick with regular partitions for the Riemann sums.

Example 5.7

Evaluating an Integral Using the Definition

Use the definition of the definite integral to evaluate $\int_0^2 x^2\,dx.$ Use a right-endpoint approximation to generate the Riemann sum.

Solution

We first want to set up a Riemann sum. Based on the limits of integration, we have $a = 0$ and $b = 2.$ For $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of $[0, 2].$ Then

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}.$$

Since we are using a right-endpoint approximation to generate Riemann sums, for each i, we need to calculate the function value at the right endpoint of the interval $[x_{i-1}, x_i].$ The right endpoint of the interval is $x_i,$ and since P is a regular partition,

$$x_i = x_0 + i\Delta x = 0 + i\left[\frac{2}{n}\right] = \frac{2i}{n}.$$

Thus, the function value at the right endpoint of the interval is

$$f(x_i) = x_i^2 = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}.$$

Then the Riemann sum takes the form

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right)\frac{2}{n} = \sum_{i=1}^{n}\frac{8i^2}{n^3} = \frac{8}{n^3}\sum_{i=1}^{n} i^2.$$

Using the summation formula for $\sum_{i=1}^{n} i^2,$ we have

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right]$$

$$= \frac{16n^3+24n^2+8n}{6n^3}$$

$$= \frac{8}{3}+\frac{4}{n}+\frac{8}{6n^2}.$$

Now, to calculate the definite integral, we need to take the limit as $n \to \infty$. We get

$$\int_0^2 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{8}{6n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{8}{3}\right)+\lim_{n\to\infty}\left(\frac{4}{n}\right)+\lim_{n\to\infty}\left(\frac{8}{6n^2}\right)$$

$$= \frac{8}{3}+0+0=\frac{8}{3}.$$

5.7 Use the definition of the definite integral to evaluate $\int_0^3 (2x-1)dx$. Use a right-endpoint approximation to generate the Riemann sum.

Evaluating Definite Integrals

Evaluating definite integrals this way can be quite tedious because of the complexity of the calculations. Later in this chapter we develop techniques for evaluating definite integrals *without* taking limits of Riemann sums. However, for now, we can rely on the fact that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area. We do this to confirm that definite integrals do, indeed, represent areas, so we can then discuss what to do in the case of a curve of a function dropping below the x-axis.

Example 5.8

Using Geometric Formulas to Calculate Definite Integrals

Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\,dx.$

Solution

The function describes a semicircle with radius 3. To find

$$\int_3^6 \sqrt{9-(x-3)^2}dx,$$

we want to find the area under the curve over the interval $[3, 6]$. The formula for the area of a circle is $A = \pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A = \left(\frac{1}{2}\right)\pi r^2$. The shaded area in **Figure 5.16** covers one-half of the semicircle, or $A = \left(\frac{1}{4}\right)\pi r^2$. Thus,

$$\begin{aligned}\int_3^6 \sqrt{9-(x-3)^2} &= \frac{1}{4}\pi(3)^2 \\ &= \frac{9}{4}\pi \\ &\approx 7.069.\end{aligned}$$

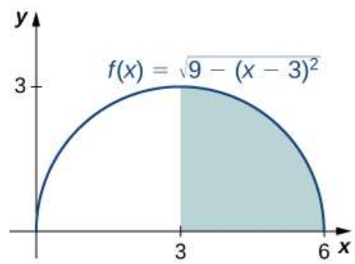


Figure 5.16 The value of the integral of the function $f(x)$ over the interval $[3, 6]$ is the area of the shaded region.

5.8 Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x+3)dx$.

Area and the Definite Integral

When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret "the area under the curve" when $f(x)$ is negative?

Net Signed Area

Let us return to the Riemann sum. Consider, for example, the function $f(x) = 2 - 2x^2$ (shown in **Figure 5.17**) on the interval $[0, 2]$. Use $n = 8$ and choose $\{x_i^*\}$ as the left endpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width Δx. When $f(x_i^*)$ is positive, the product $f(x_i^*)\Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*)\Delta x$ represents the *negative* of the area of the rectangle. The Riemann sum then becomes

$$\sum_{i=1}^{8} f(x_i^*)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

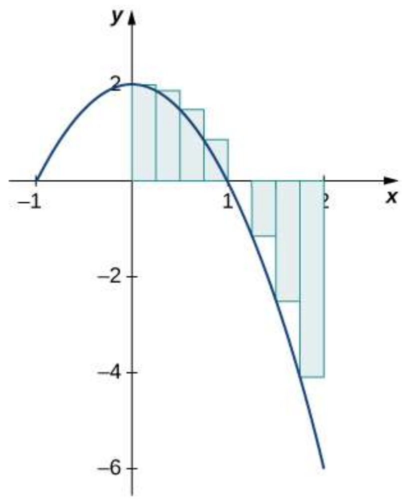


Figure 5.17 For a function that is partly negative, the Riemann sum is the area of the rectangles above the *x*-axis less the area of the rectangles below the *x*-axis.

Taking the limit as $n \to \infty$, the Riemann sum approaches the area between the curve above the *x*-axis and the *x*-axis, less the area between the curve below the *x*-axis and the *x*-axis, as shown in **Figure 5.18**. Then,

$$\begin{aligned}\int_0^2 f(x)dx &= \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x \\ &= A_1 - A_2.\end{aligned}$$

The quantity $A_1 - A_2$ is called the **net signed area**.

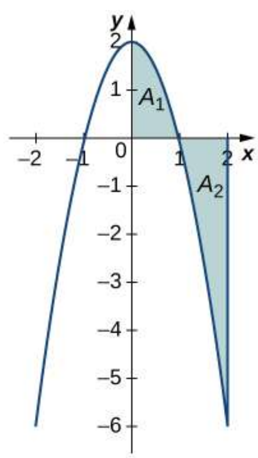


Figure 5.18 In the limit, the definite integral equals area A_1 less area A_2, or the net signed area.

Notice that net signed area can be positive, negative, or zero. If the area above the *x*-axis is larger, the net signed area is positive. If the area below the *x*-axis is larger, the net signed area is negative. If the areas above and below the *x*-axis are equal, the net signed area is zero.

Example 5.9

Finding the Net Signed Area

Find the net signed area between the curve of the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution

The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$ (**Figure 5.19**). Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle A_1, above the axis, is

$$A_1 = \frac{1}{2}3(6) = 9,$$

where 3 is the base and $2(3) = 6$ is the height. The area of triangle A_2, below the axis, is

$$A_2 = \frac{1}{2}(3)(6) = 9,$$

where 3 is the base and 6 is the height. Thus, the net area is

$$\int_{-3}^{3} 2x dx = A_1 - A_2 = 9 - 9 = 0.$$

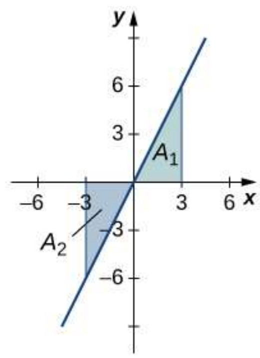
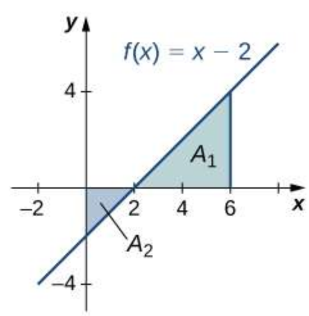


Figure 5.19 The area above the curve and below the x-axis equals the area below the curve and above the x-axis.

Analysis

If A_1 is the area above the x-axis and A_2 is the area below the x-axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

5.9 Find the net signed area of $f(x) = x - 2$ over the interval $[0, 6]$, illustrated in the following image.



Total Area

One application of the definite integral is finding displacement when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. This is a very important application of the definite integral, and we examine it in more detail later in the chapter. For now, we're just going to look at some basics to get a feel for how this works by studying constant velocities.

When velocity is a constant, the area under the curve is just velocity times time. This idea is already very familiar. If a car travels away from its starting position in a straight line at a speed of 75 mph for 2 hours, then it is 150 mi away from its original position (**Figure 5.20**). Using integral notation, we have

$$\int_0^2 75\,dt = 150.$$

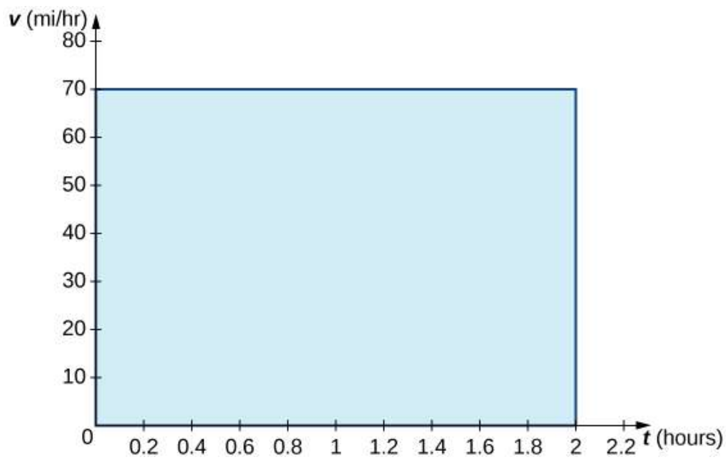


Figure 5.20 The area under the curve $v(t) = 75$ tells us how far the car is from its starting point at a given time.

In the context of displacement, net signed area allows us to take direction into account. If a car travels straight north at a speed of 60 mph for 2 hours, it is 120 mi north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at it starting position (**Figure 5.21**). Again, using integral notation, we have

$$\begin{aligned}\int_0^2 60\,dt + \int_2^5 -40\,dt &= 120 - 120 \\ &= 0.\end{aligned}$$

In this case the displacement is zero.

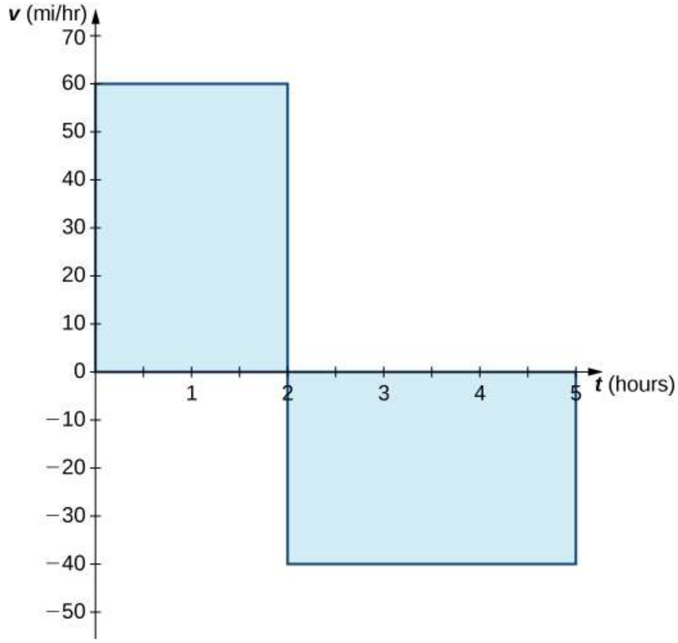


Figure 5.21 The area above the axis and the area below the axis are equal, so the net signed area is zero.

Suppose we want to know how far the car travels overall, regardless of direction. In this case, we want to know the area between the curve and the x-axis, regardless of whether that area is above or below the axis. This is called the **total area**.

Graphically, it is easiest to think of calculating total area by adding the areas above the axis and the areas below the axis (rather than subtracting the areas below the axis, as we did with net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is

$$\begin{aligned}\int_0^2 |60|dt + \int_2^5 |-40|dt &= \int_0^2 60dt + \int_2^5 40dt\\ &= 120 + 120\\ &= 240.\end{aligned}$$

Bringing these ideas together formally, we state the following definitions.

Definition

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x-axis that lies *above* the axis and let A_2 represent the area between $f(x)$ and the x-axis that lies *below* the axis. Then, the **net signed area** between $f(x)$ and the x-axis is given by

$$\int_a^b f(x)dx = A_1 - A_2.$$

The **total area** between $f(x)$ and the x-axis is given by

$$\int_a^b |f(x)|dx = A_1 + A_2.$$

Example 5.10

Finding the Total Area

Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0, 6]$.

Solution

Calculate the x-intercept as $(2, 0)$ (set $y = 0,$ solve for x). To find the total area, take the area below the x-axis over the subinterval $[0, 2]$ and add it to the area above the x-axis on the subinterval $[2, 6]$ (**Figure 5.22**).

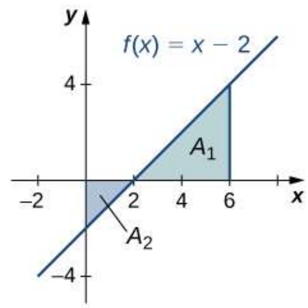


Figure 5.22 The total area between the line and the x-axis over $[0, 6]$ is A_2 plus A_1.

We have

$$\int_0^6 |(x-2)|dx = A_2 + A_1.$$

Then, using the formula for the area of a triangle, we obtain

$$A_2 = \frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 2 = 2$$

$$A_1 = \frac{1}{2}bh = \frac{1}{2} \cdot 4 \cdot 4 = 8.$$

The total area, then, is

$$A_1 + A_2 = 8 + 2 = 10.$$

5.10 Find the total area between the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Properties of the Definite Integral

The properties of indefinite integrals apply to definite integrals as well. Definite integrals also have properties that relate to the limits of integration. These properties, along with the rules of integration that we examine later in this chapter, help us manipulate expressions to evaluate definite integrals.

Rule: Properties of the Definite Integral

1.

$$\int_a^a f(x)dx = 0 \quad (5.9)$$

If the limits of integration are the same, the integral is just a line and contains no area.

2.

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \tag{5.10}$$

If the limits are reversed, then place a negative sign in front of the integral.

3.

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx \tag{5.11}$$

The integral of a sum is the sum of the integrals.

4.

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx \tag{5.12}$$

The integral of a difference is the difference of the integrals.

5.

$$\int_a^b cf(x)dx = c\int_a^b f(x) \tag{5.13}$$

for constant c. The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

6.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{5.14}$$

Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

Example 5.11

Using the Properties of the Definite Integral

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2,\ 1]$ as the sum of three definite integrals.

Solution

Using integral notation, we have $\int_{-2}^{1}\left(-3x^3 + 2x + 2\right)dx.$ We apply properties 3. and 5. to get

$$\begin{aligned}\int_{-2}^{1}\left(-3x^3+2x+2\right)dx &= \int_{-2}^{1}-3x^3\,dx+\int_{-2}^{1}2x\,dx+\int_{-2}^{1}2\,dx\\ &= -3\int_{-2}^{1}x^3\,dx+2\int_{-2}^{1}x\,dx+\int_{-2}^{1}2\,dx.\end{aligned}$$

5.11 Use the properties of the definite integral to express the definite integral of $f(x)=6x^3-4x^2+2x-3$ over the interval $[1,\ 3]$ as the sum of four definite integrals.

Example 5.12

Using the Properties of the Definite Integral

If it is known that $\int_0^8 f(x)dx=10$ and $\int_0^5 f(x)dx=5,$ find the value of $\int_5^8 f(x)dx.$

Solution

By property 6.,

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx.$$

Thus,

$$\begin{aligned}\int_0^8 f(x)dx &= \int_0^5 f(x)dx+\int_5^8 f(x)dx\\ 10 &= 5+\int_5^8 f(x)dx\\ 5 &= \int_5^8 f(x)dx.\end{aligned}$$

5.12 If it is known that $\int_1^5 f(x)dx=-3$ and $\int_2^5 f(x)dx=4,$ find the value of $\int_1^2 f(x)dx.$

Comparison Properties of Integrals

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration. Intuitively, we might say that if a function $f(x)$ is above another function $g(x),$ then the area between $f(x)$ and the x-axis is greater than the area between $g(x)$ and the x-axis. This is true depending on the interval over which the comparison is made. The properties of definite integrals are valid whether $a<b,\ a=b,$ or $a>b.$ The following properties, however, concern only the case $a\le b,$ and are used when we want to compare the sizes of integrals.

Theorem 5.2: Comparison Theorem

i. If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)dx \geq 0.$$

ii. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

iii. If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Example 5.13

Comparing Two Functions over a Given Interval

Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0, 1]$.

Solution

Graphing these functions is necessary to understand how they compare over the interval $[0, 1]$. Initially, when graphed on a graphing calculator, $f(x)$ appears to be above $g(x)$ everywhere. However, on the interval $[0, 1]$, the graphs appear to be on top of each other. We need to zoom in to see that, on the interval $[0, 1]$, $g(x)$ is above $f(x)$. The two functions intersect at $x = 0$ and $x = 1$ (**Figure 5.23**).

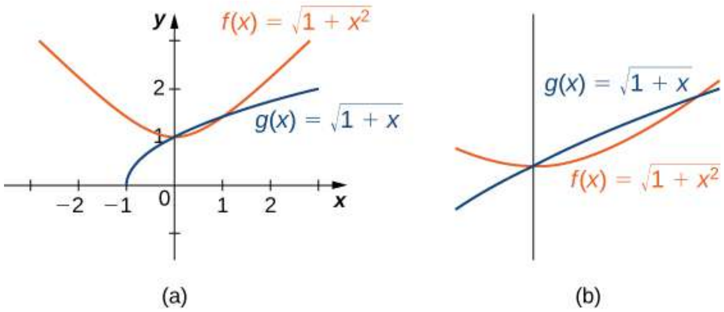


Figure 5.23 (a) The function $f(x)$ appears above the function $g(x)$ except over the interval $[0, 1]$ (b) Viewing the same graph with a greater zoom shows this more clearly.

We can see from the graph that over the interval $[0, 1]$, $g(x) \geq f(x)$. Comparing the integrals over the specified interval $[0, 1]$, we also see that $\int_0^1 g(x)dx \geq \int_0^1 f(x)dx$ (**Figure 5.24**). The thin, red-shaded area shows just how much difference there is between these two integrals over the interval $[0, 1]$.

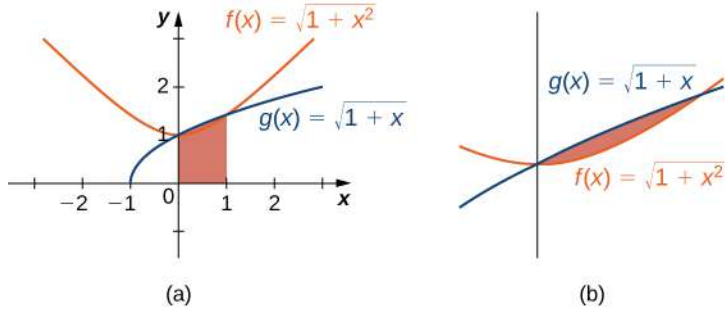


Figure 5.24 (a) The graph shows that over the interval $[0, 1]$, $g(x) \geq f(x)$, where equality holds only at the endpoints of the interval. (b) Viewing the same graph with a greater zoom shows this more clearly.

Average Value of a Function

We often need to find the average of a set of numbers, such as an average test grade. Suppose you received the following test scores in your algebra class: 89, 90, 56, 78, 100, and 69. Your semester grade is your average of test scores and you want to know what grade to expect. We can find the average by adding all the scores and dividing by the number of scores. In this case, there are six test scores. Thus,

$$\frac{89 + 90 + 56 + 78 + 100 + 69}{6} = \frac{482}{6} \approx 80.33.$$

Therefore, your average test grade is approximately 80.33, which translates to a B− at most schools.

Suppose, however, that we have a function $v(t)$ that gives us the speed of an object at any time t, and we want to find the object's average speed. The function $v(t)$ takes on an infinite number of values, so we can't use the process just described. Fortunately, we can use a definite integral to find the average value of a function such as this.

Let $f(x)$ be continuous over the interval $[a, b]$ and let $[a, b]$ be divided into n subintervals of width $\Delta x = (b - a)/n$. Choose a representative x_i^* in each subinterval and calculate $f(x_i^*)$ for $i = 1, 2,\ldots, n$. In other words, consider each $f(x_i^*)$ as a sampling of the function over each subinterval. The average value of the function may then be approximated as

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n},$$

which is basically the same expression used to calculate the average of discrete values.

But we know $\Delta x = \frac{b - a}{n}$, so $n = \frac{b - a}{\Delta x}$, and we get

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\frac{(b - a)}{\Delta x}}.$$

Following through with the algebra, the numerator is a sum that is represented as $\sum_{i=1}^{n} f(x_i^*)$, and we are dividing by a fraction. To divide by a fraction, invert the denominator and multiply. Thus, an approximate value for the average value of the function is given by

$$\frac{\sum_{i=1}^{n} f(x_i^*)}{\frac{(b-a)}{\Delta x}} = \left(\frac{\Delta x}{b-a}\right)\sum_{i=1}^{n} f(x_i^*)$$
$$= \left(\frac{1}{b-a}\right)\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

This is a Riemann sum. Then, to get the *exact* average value, take the limit as n goes to infinity. Thus, the average value of a function is given by

$$\frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx.$$

Definition

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the **average value of the function** $f(x)$ (or $\mathbf{f_{ave}}$) on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example 5.14

Finding the Average Value of a Linear Function

Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.

Solution

First, graph the function on the stated interval, as shown in **Figure 5.25**.

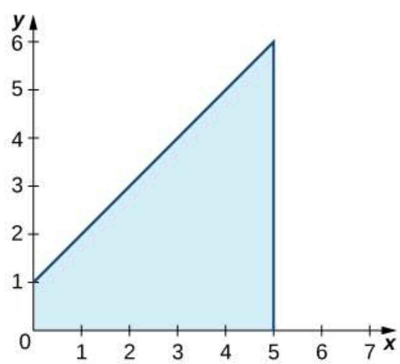


Figure 5.25 The graph shows the area under the function $f(x) = x + 1$ over $[0, 5]$.

The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a + b)$, where h represents height, and a and b represent the two parallel sides. Then,

$$\begin{aligned}\int_0^5 x + 1dx &= \frac{1}{2}h(a+b) \\ &= \frac{1}{2}\cdot 5\cdot(1+6) \\ &= \frac{35}{2}.\end{aligned}$$

Thus the average value of the function is

$$\frac{1}{5-0}\int_0^5 x + 1dx = \frac{1}{5}\cdot\frac{35}{2} = \frac{7}{2}.$$

5.13 Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$.

5.2 EXERCISES

In the following exercises, express the limits as integrals.

60. $\lim_{n \to \infty} \sum_{i=1}^{n} (x_i^*)\Delta x$ over $[1, 3]$

61. $\lim_{n \to \infty} \sum_{i=1}^{n} \left(5(x_i^*)^2 - 3(x_i^*)^3\right)\Delta x$ over $[0, 2]$

62. $\lim_{n \to \infty} \sum_{i=1}^{n} \sin^2(2\pi x_i^*)\Delta x$ over $[0, 1]$

63. $\lim_{n \to \infty} \sum_{i=1}^{n} \cos^2(2\pi x_i^*)\Delta x$ over $[0, 1]$

In the following exercises, given L_n or R_n as indicated, express their limits as $n \to \infty$ as definite integrals, identifying the correct intervals.

64. $L_n = \frac{1}{n}\sum_{i=1}^{n} \frac{i-1}{n}$

65. $R_n = \frac{1}{n}\sum_{i=1}^{n} \frac{i}{n}$

66. $L_n = \frac{2}{n}\sum_{i=1}^{n} \left(1 + 2\frac{i-1}{n}\right)$

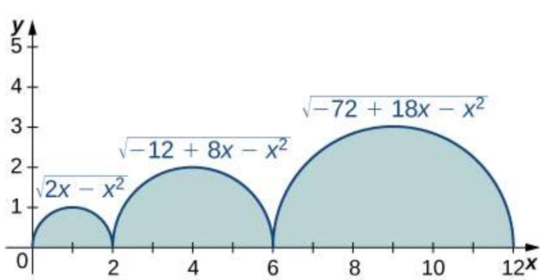
67. $R_n = \frac{3}{n}\sum_{i=1}^{n} \left(3 + 3\frac{i}{n}\right)$

68. $L_n = \frac{2\pi}{n}\sum_{i=1}^{n} 2\pi\frac{i-1}{n}\cos\left(2\pi\frac{i-1}{n}\right)$

69. $R_n = \frac{1}{n}\sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)\log\left(\left(1 + \frac{i}{n}\right)^2\right)$

In the following exercises, evaluate the integrals of the functions graphed using the formulas for areas of triangles and circles, and subtracting the areas below the *x*-axis.

70.



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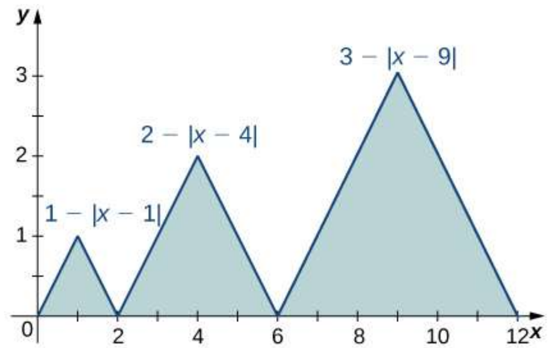

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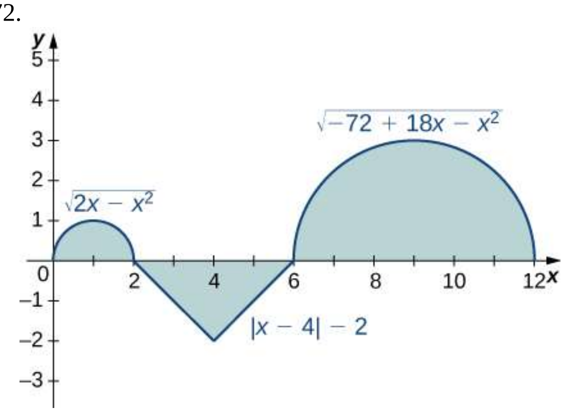

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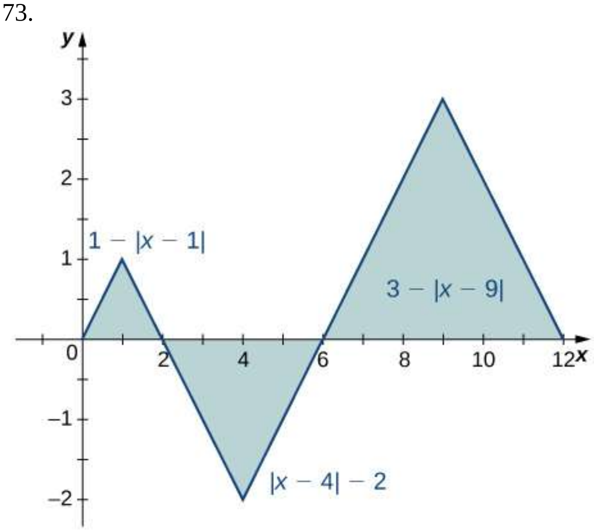

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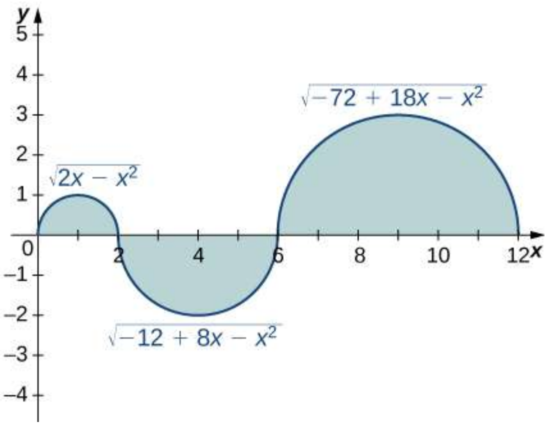

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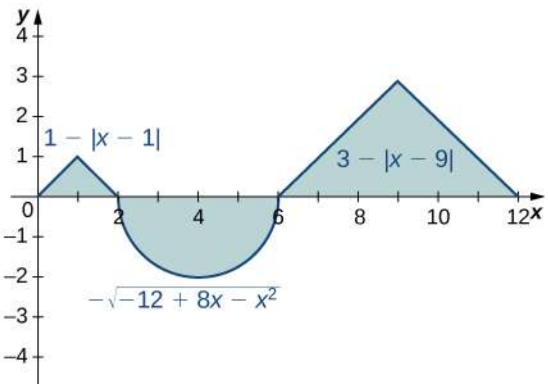


In the following exercises, evaluate the integral using area formulas.

76. $\int_0^3 (3 - x)dx$

77. $\int_2^3 (3 - x)dx$

78. $\int_{-3}^3 (3 - |x|)dx$

79. $\int_0^6 (3 - |x - 3|)dx$

80. $\int_{-2}^2 \sqrt{4 - x^2}dx$

81. $\int_1^5 \sqrt{4 - (x - 3)^2}dx$

82. $\int_0^{12} \sqrt{36 - (x - 6)^2}dx$

83. $\int_{-2}^3 (3 - |x|)dx$

In the following exercises, use averages of values at the left (L) and right (R) endpoints to compute the integrals of the piecewise linear functions with graphs that pass through the given list of points over the indicated intervals.

84. $\{(0, 0), (2, 1), (4, 3), (5, 0), (6, 0), (8, 3)\}$ over $[0, 8]$

85. $\{(0, 2), (1, 0), (3, 5), (5, 5), (6, 2), (8, 0)\}$ over $[0, 8]$

86. $\{(-4, -4), (-2, 0), (0, -2), (3, 3), (4, 3)\}$ over $[-4, 4]$

87. $\{(-4, 0), (-2, 2), (0, 0), (1, 2), (3, 2), (4, 0)\}$ over $[-4, 4]$

Suppose that $\int_0^4 f(x)dx = 5$ and $\int_0^2 f(x)dx = -3$, and $\int_0^4 g(x)dx = -1$ and $\int_0^2 g(x)dx = 2$. In the following exercises, compute the integrals.

88. $\int_0^4 (f(x) + g(x))dx$

89. $\int_2^4 (f(x) + g(x))dx$

90. $\int_0^2 (f(x) - g(x))dx$

91. $\int_2^4 (f(x) - g(x))dx$

92. $\int_0^2 (3f(x) - 4g(x))dx$

93. $\int_2^4 (4f(x) - 3g(x))dx$

In the following exercises, use the identity $\int_{-A}^A f(x)dx = \int_{-A}^0 f(x)dx + \int_0^A f(x)dx$ to compute the integrals.

94. $\int_{-\pi}^{\pi} \frac{\sin t}{1 + t^2}dt$ $(Hint: \sin(-t) = -\sin(t))$

95. $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{t}{1+\cos t} dt$

In the following exercises, find the net signed area between $f(x)$ and the x-axis.

96. $\int_1^3 (2-x)dx$ (*Hint:* Look at the graph of *f*.)

97. $\int_2^4 (x-3)^3 dx$ (*Hint:* Look at the graph of *f*.)

In the following exercises, given that $\int_0^1 x dx = \frac{1}{2}$, $\int_0^1 x^2 dx = \frac{1}{3}$, and $\int_0^1 x^3 dx = \frac{1}{4}$, compute the integrals.

98. $\int_0^1 \left(1+x+x^2+x^3\right)dx$

99. $\int_0^1 \left(1-x+x^2-x^3\right)dx$

100. $\int_0^1 (1-x)^2 dx$

101. $\int_0^1 (1-2x)^3 dx$

102. $\int_0^1 \left(6x - \frac{4}{3}x^2\right)dx$

103. $\int_0^1 \left(7-5x^3\right)dx$

In the following exercises, use the **comparison theorem**.

104. Show that $\int_0^3 \left(x^2-6x+9\right)dx \geq 0$.

105. Show that $\int_{-2}^3 (x-3)(x+2)dx \leq 0$.

106. Show that $\int_0^1 \sqrt{1+x^3}dx \leq \int_0^1 \sqrt{1+x^2}dx$.

107. Show that $\int_1^2 \sqrt{1+x}dx \leq \int_1^2 \sqrt{1+x^2}dx$.

108. Show that $\int_0^{\pi/2} \sin t dt \geq \frac{\pi}{4}$. ($Hint$: $\sin t \geq \frac{2t}{\pi}$ over $\left[0, \frac{\pi}{2}\right]$)

109. Show that $\int_{-\pi/4}^{\pi/4} \cos t dt \geq \pi\sqrt{2}/4$.

In the following exercises, find the average value f_{ave} of f between a and b, and find a point c, where $f(c) = f_{\text{ave}}$.

110. $f(x) = x^2$, $a = -1$, $b = 1$

111. $f(x) = x^5$, $a = -1$, $b = 1$

112. $f(x) = \sqrt{4-x^2}$, $a = 0$, $b = 2$

113. $f(x) = (3-|x|)$, $a = -3$, $b = 3$

114. $f(x) = \sin x$, $a = 0$, $b = 2\pi$

115. $f(x) = \cos x$, $a = 0$, $b = 2\pi$

In the following exercises, approximate the average value using Riemann sums L_{100} and R_{100}. How does your answer compare with the exact given answer?

116. **[T]** $y = \ln(x)$ over the interval $[1, 4]$; the exact solution is $\frac{\ln(256)}{3} - 1$.

117. **[T]** $y = e^{x/2}$ over the interval $[0, 1]$; the exact solution is $2(\sqrt{e}-1)$.

118. **[T]** $y = \tan x$ over the interval $\left[0, \frac{\pi}{4}\right]$; the exact solution is $\frac{2\ln(2)}{\pi}$.

119. **[T]** $y = \frac{x+1}{\sqrt{4-x^2}}$ over the interval $[-1, 1]$; the exact solution is $\frac{\pi}{6}$.

In the following exercises, compute the average value using the left Riemann sums L_N for $N = 1, 10, 100$. How does the accuracy compare with the given exact value?

120. **[T]** $y = x^2 - 4$ over the interval $[0, 2]$; the exact solution is $-\frac{8}{3}$.

121. **[T]** $y = xe^{x^2}$ over the interval $[0, 2]$; the exact solution is $\frac{1}{4}\left(e^4 - 1\right)$.

122. **[T]** $y = \left(\frac{1}{2}\right)^x$ over the interval $[0, 4]$; the exact solution is $\frac{15}{64\ln(2)}$.

123. **[T]** $y = x\sin\left(x^2\right)$ over the interval $[-\pi, 0]$; the exact solution is $\frac{\cos\left(\pi^2\right) - 1}{2\pi}$.

124. Suppose that $A = \int_0^{2\pi} \sin^2 t dt$ and $B = \int_0^{2\pi} \cos^2 t dt$. Show that $A + B = 2\pi$ and $A = B$.

125. Suppose that $A = \int_{-\pi/4}^{\pi/4} \sec^2 t dt = \pi$ and $B = \int_{-\pi/4}^{\pi/4} \tan^2 t dt$. Show that $A - B = \frac{\pi}{2}$.

126. Show that the average value of $\sin^2 t$ over $[0, 2\pi]$ is equal to 1/2 Without further calculation, determine whether the average value of $\sin^2 t$ over $[0, \pi]$ is also equal to 1/2.

127. Show that the average value of $\cos^2 t$ over $[0, 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\cos^2 (t)$ over $[0, \pi]$ is also equal to $1/2$.

128. Explain why the graphs of a quadratic function (parabola) $p(x)$ and a linear function $\ell(x)$ can intersect in at most two points. Suppose that $p(a) = \ell(a)$ and $p(b) = \ell(b)$, and that $\int_a^b p(t)dt > \int_a^b \ell(t)dt$. Explain why $\int_c^d p(t) > \int_c^d \ell(t)dt$ whenever $a \le c < d \le b$.

129. Suppose that parabola $p(x) = ax^2 + bx + c$ opens downward $(a < 0)$ and has a vertex of $y = \frac{-b}{2a} > 0$. For which interval $[A, B]$ is $\int_A^B \left(ax^2 + bx + c\right)dx$ as large as possible?

130. Suppose $[a, b]$ can be subdivided into subintervals $a = a_0 < a_1 < a_2 < \cdots < a_N = b$ such that either $f \ge 0$ over $[a_{i-1}, a_i]$ or $f \le 0$ over $[a_{i-1}, a_i]$. Set $A_i = \int_{a_{i-1}}^{a_i} f(t)dt$.

a. Explain why $\int_a^b f(t)dt = A_1 + A_2 + \cdots + A_N$.

b. Then, explain why $\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt$.

131. Suppose f and g are continuous functions such that $\int_c^d f(t)dt \le \int_c^d g(t)dt$ for every subinterval $[c, d]$ of $[a, b]$. Explain why $f(x) \le g(x)$ for all values of x.

132. Suppose the average value of f over $[a, b]$ is 1 and the average value of f over $[b, c]$ is 1 where $a < c < b$. Show that the average value of f over $[a, c]$ is also 1.

133. Suppose that $[a, b]$ can be partitioned. taking $a = a_0 < a_1 < \cdots < a_N = b$ such that the average value of f over each subinterval $[a_{i-1}, a_i] = 1$ is equal to 1 for each $i = 1,\ldots, N$. Explain why the average value of f over $[a, b]$ is also equal to 1.

134. Suppose that for each i such that $1 \le i \le N$ one has $\int_{i-1}^{i} f(t)dt = i$. Show that $\int_0^N f(t)dt = \frac{N(N+1)}{2}$.

135. Suppose that for each i such that $1 \le i \le N$ one has $\int_{i-1}^{i} f(t)dt = i^2$. Show that $\int_0^N f(t)dt = \frac{N(N+1)(2N+1)}{6}$.

136. **[T]** Compute the left and right Riemann sums L_{10} and R_{10} and their average $\frac{L_{10} + R_{10}}{2}$ for $f(t) = t^2$ over $[0, 1]$. Given that $\int_0^1 t^2 dt = 0.\overline{33}$, to how many decimal places is $\frac{L_{10} + R_{10}}{2}$ accurate?

137. **[T]** Compute the left and right Riemann sums, L_{10} and R_{10}, and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t) = (4 - t^2)$ over $[1, 2]$. Given that $\int_1^2 (4 - t^2)dt = 1.\overline{66}$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

138. If $\int_1^5 \sqrt{1 + t^4}dt = 41.7133...$, what is $\int_1^5 \sqrt{1 + u^4}du$?

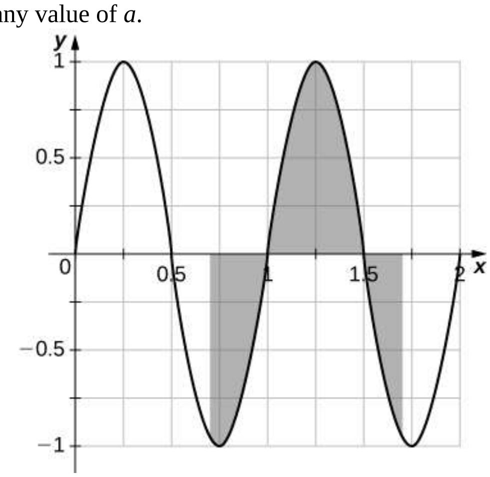
139. Estimate $\int_0^1 tdt$ using the left and right endpoint sums, each with a single rectangle. How does the average of these left and right endpoint sums compare with the actual value $\int_0^1 tdt$?

140. Estimate $\int_0^1 tdt$ by comparison with the area of a single rectangle with height equal to the value of t at the midpoint $t = \frac{1}{2}$. How does this midpoint estimate compare with the actual value $\int_0^1 tdt$?

141. From the graph of $\sin(2\pi x)$ shown:

a. Explain why $\int_0^1 \sin(2\pi t)dt = 0$.

b. Explain why, in general, $\int_a^{a+1} \sin(2\pi t)dt = 0$ for any value of a.



142. If f is 1-periodic $(f(t + 1) = f(t))$, odd, and integrable over $[0, 1]$, is it always true that $\int_0^1 f(t)dt = 0$?

143. If f is 1-periodic and $\int_0^1 f(t)dt = A$, is it necessarily true that $\int_a^{1+a} f(t)dt = A$ for all A?

5.3 | The Fundamental Theorem of Calculus

Learning Objectives

- **5.3.1** Describe the meaning of the Mean Value Theorem for Integrals.
- **5.3.2** State the meaning of the Fundamental Theorem of Calculus, Part 1.
- **5.3.3** Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- **5.3.4** State the meaning of the Fundamental Theorem of Calculus, Part 2.
- **5.3.5** Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- **5.3.6** Explain the relationship between differentiation and integration.

In the previous two sections, we looked at the definite integral and its relationship to the area under the curve of a function. Unfortunately, so far, the only tools we have available to calculate the value of a definite integral are geometric area formulas and limits of Riemann sums, and both approaches are extremely cumbersome. In this section we look at some more powerful and useful techniques for evaluating definite integrals.

These new techniques rely on the relationship between differentiation and integration. This relationship was discovered and explored by both Sir Isaac Newton and Gottfried Wilhelm Leibniz (among others) during the late 1600s and early 1700s, and it is codified in what we now call the **Fundamental Theorem of Calculus**, which has two parts that we examine in this section. Its very name indicates how central this theorem is to the entire development of calculus.

Isaac Newton's contributions to mathematics and physics changed the way we look at the world. The relationships he discovered, codified as Newton's laws and the law of universal gravitation, are still taught as foundational material in physics today, and his calculus has spawned entire fields of mathematics. To learn more, read a **brief biography (http://www.openstax.org/l/20_newtonbio)** of Newton with multimedia clips.

Before we get to this crucial theorem, however, let's examine another important theorem, the Mean Value Theorem for Integrals, which is needed to prove the Fundamental Theorem of Calculus.

The Mean Value Theorem for Integrals

The **Mean Value Theorem for Integrals** states that a continuous function on a closed interval takes on its average value at some point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point *c* exists in an interval $[a, b]$ such that the value of the function at *c* is equal to the average value of $f(x)$ over $[a, b].$ We state this theorem mathematically with the help of the formula for the average value of a function that we presented at the end of the preceding section.

Theorem 5.3: The Mean Value Theorem for Integrals

If $f(x)$ is continuous over an interval $[a, b],$ then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{5.15}$$

This formula can also be stated as

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof

Since $f(x)$ is continuous on $[a, b],$ by the extreme value theorem (see **Maxima and Minima**), it assumes minimum and maximum values—*m* and *M*, respectively—on $[a, b].$ Then, for all *x* in $[a, b],$ we have $m \le f(x) \le M.$ Therefore, by the comparison theorem (see **The Definite Integral**), we have

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Dividing by $b - a$ gives us

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M.$$

Since $\frac{1}{b-a}\int_a^b f(x)dx$ is a number between m and M, and since $f(x)$ is continuous and assumes the values m and M over $[a, b]$, by the Intermediate Value Theorem (see **Continuity**), there is a number c over $[a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx,$$

and the proof is complete.

□

Example 5.15

Finding the Average Value of a Function

Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0, 4]$ and find c such that $f(c)$ equals the average value of the function over $[0, 4]$.

Solution

The formula states the mean value of $f(x)$ is given by

$$\frac{1}{4-0}\int_0^4 (8 - 2x)dx.$$

We can see in **Figure 5.26** that the function represents a straight line and forms a right triangle bounded by the x- and y-axes. The area of the triangle is $A = \frac{1}{2}(\text{base})(\text{height})$. We have

$$A = \frac{1}{2}(4)(8) = 16.$$

The average value is found by multiplying the area by $1/(4 - 0)$. Thus, the average value of the function is

$$\frac{1}{4}(16) = 4.$$

Set the average value equal to $f(c)$ and solve for c.

$$\begin{aligned} 8 - 2c &= 4 \\ c &= 2 \end{aligned}$$

At $c = 2, f(2) = 4$.

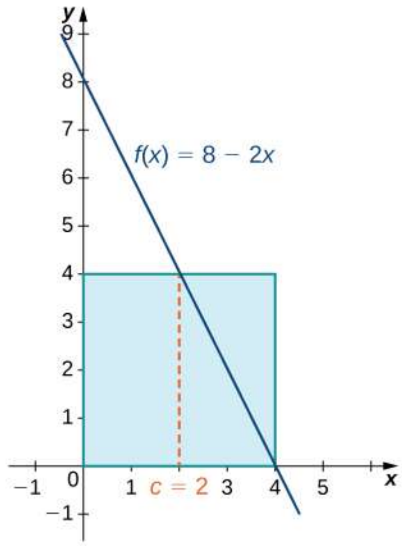


Figure 5.26 By the Mean Value Theorem, the continuous function $f(x)$ takes on its average value at c at least once over a closed interval.

5.14 Find the average value of the function $f(x) = \frac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

Example 5.16

Finding the Point Where a Function Takes on Its Average Value

Given $\int_0^3 x^2\,dx = 9$, find c such that $f(c)$ equals the average value of $f(x) = x^2$ over $[0, 3]$.

Solution

We are looking for the value of c such that

$$f(c) = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}(9) = 3.$$

Replacing $f(c)$ with c^2, we have

$$\begin{aligned} c^2 &= 3 \\ c &= \pm\sqrt{3}. \end{aligned}$$

Since $-\sqrt{3}$ is outside the interval, take only the positive value. Thus, $c = \sqrt{3}$ (**Figure 5.27**).

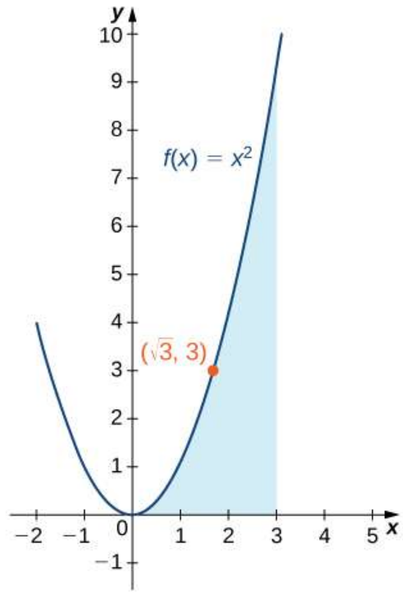


Figure 5.27 Over the interval $[0, 3]$, the function $f(x) = x^2$ takes on its average value at $c = \sqrt{3}$.

5.15 Given $\int_0^3 \left(2x^2 - 1\right)dx = 15$, find c such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[0, 3]$.

Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums or calculating areas. The theorem is comprised of two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1**, is stated here. Part 1 establishes the relationship between differentiation and integration.

Theorem 5.4: Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)dt, \tag{5.16}$$

then $F'(x) = f(x)$ over $[a, b]$.

Before we delve into the proof, a couple of subtleties are worth mentioning here. First, a comment on the notation. Note that we have defined a function, $F(x)$, as the definite integral of another function, $f(t)$, from the point *a* to the point *x*. At first glance, this is confusing, because we have said several times that a definite integral is a number, and here it looks like it's a function. The key here is to notice that for any particular value of *x*, the definite integral is a number. So the function $F(x)$ returns a number (the value of the definite integral) for each value of *x*.

Second, it is worth commenting on some of the key implications of this theorem. There is a reason it is called the *Fundamental* Theorem of Calculus. Not only does it establish a relationship between integration and differentiation, but also it guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative.

Proof

Applying the definition of the derivative, we have

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right] \\ &= \lim_{h \to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt + \int_x^a f(t)dt\right] \\ &= \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)dt. \end{aligned}$$

Looking carefully at this last expression, we see $\frac{1}{h}\int_x^{x+h} f(t)dt$ is just the average value of the function $f(x)$ over the interval $[x, x+h]$. Therefore, by **The Mean Value Theorem for Integrals**, there is some number *c* in $[x, x+h]$ such that

$$\frac{1}{h}\int_x^{x+h} f(x)dx = f(c).$$

In addition, since *c* is between *x* and *x* + *h*, *c* approaches *x* as *h* approaches zero. Also, since $f(x)$ is continuous, we have $\lim_{h \to 0} f(c) = \lim_{c \to x} f(c) = f(x)$. Putting all these pieces together, we have

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(x)dx \\ &= \lim_{h \to 0} f(c) \\ &= f(x), \end{aligned}$$

and the proof is complete.

□

Example 5.17

Finding a Derivative with the Fundamental Theorem of Calculus

Use the **Fundamental Theorem of Calculus, Part 1** to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3 + 1}dt.$$

Solution

According to the Fundamental Theorem of Calculus, the derivative is given by

$$g'(x) = \frac{1}{x^3 + 1}.$$

5.16 Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of $g(r) = \int_0^r \sqrt{x^2 + 4}\,dx.$

Example 5.18

Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives

Let $F(x) = \int_1^{\sqrt{x}} \sin t\,dt.$ Find $F'(x).$

Solution

Letting $u(x) = \sqrt{x},$ we have $F(x) = \int_1^{u(x)} \sin t\,dt.$ Thus, by the Fundamental Theorem of Calculus and the chain rule,

$$\begin{aligned} F'(x) &= \sin(u(x))\frac{du}{dx} \\ &= \sin(u(x)) \cdot \left(\frac{1}{2}x^{-1/2}\right) \\ &= \frac{\sin\sqrt{x}}{2\sqrt{x}}. \end{aligned}$$

5.17 Let $F(x) = \int_1^{x^3} \cos t\,dt.$ Find $F'(x).$

Example 5.19

Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration

Let $F(x) = \int_x^{2x} t^3\,dt.$ Find $F'(x).$

Solution

We have $F(x) = \int_x^{2x} t^3\,dt.$ Both limits of integration are variable, so we need to split this into two integrals. We get

$$\begin{aligned} F(x) &= \int_x^{2x} t^3\,dt \\ &= \int_x^0 t^3\,dt + \int_0^{2x} t^3\,dt \\ &= -\int_0^x t^3\,dt + \int_0^{2x} t^3\,dt. \end{aligned}$$

Differentiating the first term, we obtain

$$\frac{d}{dx}\left[-\int_0^x t^3\,dt\right] = -x^3.$$

Differentiating the second term, we first let $u(x) = 2x$. Then,

$$\begin{aligned}\frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right] &= \frac{d}{dx}\left[\int_0^{u(x)} t^3\,dt\right]\\ &= (u(x))^3\frac{du}{dx}\\ &= (2x)^3\cdot 2\\ &= 16x^3.\end{aligned}$$

Thus,

$$\begin{aligned}F'(x) &= \frac{d}{dx}\left[-\int_0^x t^3\,dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right]\\ &= -x^3 + 16x^3\\ &= 15x^3.\end{aligned}$$

5.18 Let $F(x) = \int_x^{x^2} \cos t\,dt$. Find $F'(x)$.

Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

After finding approximate areas by adding the areas of n rectangles, the application of this theorem is straightforward by comparison. It almost seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

Theorem 5.5: The Fundamental Theorem of Calculus, Part 2

If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a). \tag{5.17}$$

We often see the notation $F(x)|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the upper limit (in this case, b), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, a).

The **Fundamental Theorem of Calculus, Part 2** (also known as the **evaluation theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Proof

Let $P = \{x_i\},\ i = 0,\ 1,\ldots,n$ be a regular partition of $[a,\ b]$. Then, we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)] \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

Now, we know F is an antiderivative of f over $[a,\ b]$, so by the Mean Value Theorem (see **The Mean Value Theorem**) for $i = 0,\ 1,\ldots,n$ we can find c_i in $[x_{i-1},\ x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x.$$

Then, substituting into the previous equation, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x.$$

Taking the limit of both sides as $n \to \infty$, we obtain

$$\begin{aligned} F(b) - F(a) &= \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x \\ &= \int_a^b f(x)dx. \end{aligned}$$

□

Example 5.20

Evaluating an Integral with the Fundamental Theorem of Calculus

Use **The Fundamental Theorem of Calculus, Part 2** to evaluate

$$\int_{-2}^{2} (t^2 - 4)dt.$$

Solution

Recall the power rule for **Antiderivatives**:

$$\text{If } y = x^n,\ \int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Use this rule to find the antiderivative of the function and then apply the theorem. We have

$$\begin{aligned}\int_{-2}^{2}\left(t^2-4\right)dt &= \frac{t^3}{3}-4t\Big|_{-2}^{2} \\ &= \left[\frac{(2)^3}{3}-4(2)\right]-\left[\frac{(-2)^3}{3}-4(-2)\right] \\ &= \left(\frac{8}{3}-8\right)-\left(-\frac{8}{3}+8\right) \\ &= \frac{8}{3}-8+\frac{8}{3}-8 \\ &= \frac{16}{3}-16 \\ &= -\frac{32}{3}.\end{aligned}$$

Analysis

Notice that we did not include the "+ C" term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2, *any* antiderivative works. So, for convenience, we chose the antiderivative with $C = 0$. If we had chosen another antiderivative, the constant term would have canceled out. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in **Figure 5.28**. Note that the region between the curve and the *x*-axis is all below the *x*-axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.

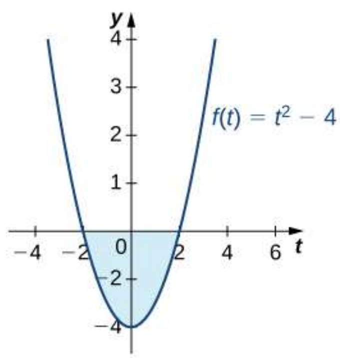


Figure 5.28 The evaluation of a definite integral can produce a negative value, even though area is always positive.

Example 5.21

Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2:

$$\int_{1}^{9}\frac{x-1}{\sqrt{x}}dx.$$

Solution

First, eliminate the radical by rewriting the integral using rational exponents. Then, separate the numerator terms by writing each one over the denominator:

$$\int_1^9 \frac{x-1}{x^{1/2}}dx = \int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right)dx.$$

Use the properties of exponents to simplify:

$$\int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right)dx = \int_1^9 \left(x^{1/2} - x^{-1/2}\right)dx.$$

Now, integrate using the power rule:

$$\begin{aligned}\int_1^9 \left(x^{1/2} - x^{-1/2}\right)dx &= \left(\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}}\right)\Bigg|_1^9 \\ &= \left[\frac{(9)^{3/2}}{\frac{3}{2}} - \frac{(9)^{1/2}}{\frac{1}{2}}\right] - \left[\frac{(1)^{3/2}}{\frac{3}{2}} - \frac{(1)^{1/2}}{\frac{1}{2}}\right] \\ &= \left[\frac{2}{3}(27) - 2(3)\right] - \left[\frac{2}{3}(1) - 2(1)\right] \\ &= 18 - 6 - \frac{2}{3} + 2 \\ &= \frac{40}{3}.\end{aligned}$$

See **Figure 5.29**.

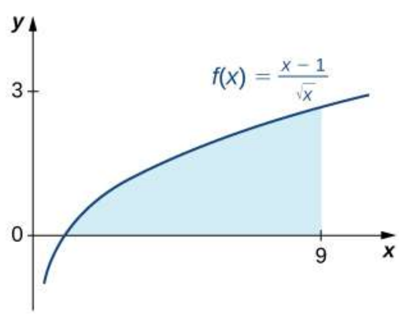


Figure 5.29 The area under the curve from $x = 1$ to $x = 9$ can be calculated by evaluating a definite integral.

5.19 Use **The Fundamental Theorem of Calculus, Part 2** to evaluate $\int_1^2 x^{-4}\,dx$.

Example 5.22

A Roller-Skating Race

James and Kathy are racing on roller skates. They race along a long, straight track, and whoever has gone the farthest after 5 sec wins a prize. If James can skate at a velocity of $f(t) = 5 + 2t$ ft/sec and Kathy can skate at a velocity of $g(t) = 10 + \cos\left(\frac{\pi}{2}t\right)$ ft/sec, who is going to win the race?

Solution

We need to integrate both functions over the interval $[0, 5]$ and see which value is bigger. For James, we want to calculate

$$\int_0^5 (5 + 2t)dt.$$

Using the power rule, we have

$$\begin{aligned}\int_0^5 (5 + 2t)dt &= \left(5t + t^2\right)\Big|_0^5 \\ &= (25 + 25) = 50.\end{aligned}$$

Thus, James has skated 50 ft after 5 sec. Turning now to Kathy, we want to calculate

$$\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)dt.$$

We know $\sin t$ is an antiderivative of $\cos t,$ so it is reasonable to expect that an antiderivative of $\cos\left(\frac{\pi}{2}t\right)$ would involve $\sin\left(\frac{\pi}{2}t\right).$ However, when we differentiate $\sin\left(\frac{\pi}{2}t\right),$ we get $\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)$ as a result of the chain rule, so we have to account for this additional coefficient when we integrate. We obtain

$$\begin{aligned}\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)dt &= \left(10t + \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\right)\Big|_0^5 \\ &= \left(50 + \frac{2}{\pi}\right) - \left(0 - \frac{2}{\pi}\sin 0\right) \\ &\approx 50.6.\end{aligned}$$

Kathy has skated approximately 50.6 ft after 5 sec. Kathy wins, but not by much!

5.20 Suppose James and Kathy have a rematch, but this time the official stops the contest after only 3 sec. Does this change the outcome?

Student PROJECT

A Parachutist in Free Fall

Figure 5.30 Skydivers can adjust the velocity of their dive by changing the position of their body during the free fall. (credit: Jeremy T. Lock)

Julie is an avid skydiver. She has more than 300 jumps under her belt and has mastered the art of making adjustments to her body position in the air to control how fast she falls. If she arches her back and points her belly toward the ground, she reaches a terminal velocity of approximately 120 mph (176 ft/sec). If, instead, she orients her body with her head straight down, she falls faster, reaching a terminal velocity of 150 mph (220 ft/sec).

Since Julie will be moving (falling) in a downward direction, we assume the downward direction is positive to simplify our calculations. Julie executes her jumps from an altitude of 12,500 ft. After she exits the aircraft, she immediately starts falling at a velocity given by $v(t) = 32t.$ She continues to accelerate according to this velocity function until she reaches terminal velocity. After she reaches terminal velocity, her speed remains constant until she pulls her ripcord and slows down to land.

On her first jump of the day, Julie orients herself in the slower “belly down” position (terminal velocity is 176 ft/sec). Using this information, answer the following questions.

1. How long after she exits the aircraft does Julie reach terminal velocity?
2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.
3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?
4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from the time she leaves the airplane until the time her feet touch the ground.

On Julie's second jump of the day, she decides she wants to fall a little faster and orients herself in the "head down" position. Her terminal velocity in this position is 220 ft/sec. Answer these questions based on this velocity:

5. How long does it take Julie to reach terminal velocity in this case?
6. Before pulling her ripcord, Julie reorients her body in the "belly down" position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Some jumpers wear " wingsuits" (see **Figure 5.31**). These suits have fabric panels between the arms and legs and allow the wearer to glide around in a free fall, much like a flying squirrel. (Indeed, the suits are sometimes called "flying squirrel suits.") When wearing these suits, terminal velocity can be reduced to about 30 mph (44 ft/sec), allowing the wearers a much longer time in the air. Wingsuit flyers still use parachutes to land; although the vertical velocities are within the margin of safety, horizontal velocities can exceed 70 mph, much too fast to land safely.

Figure 5.31 The fabric panels on the arms and legs of a wingsuit work to reduce the vertical velocity of a skydiver's fall. (credit: Richard Schneider)

Answer the following question based on the velocity in a wingsuit.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air?

5.3 EXERCISES

144. Consider two athletes running at variable speeds $v_1(t)$ and $v_2(t)$. The runners start and finish a race at exactly the same time. Explain why the two runners must be going the same speed at some point.

145. Two mountain climbers start their climb at base camp, taking two different routes, one steeper than the other, and arrive at the peak at exactly the same time. Is it necessarily true that, at some point, both climbers increased in altitude at the same rate?

146. To get on a certain toll road a driver has to take a card that lists the mile entrance point. The card also has a timestamp. When going to pay the toll at the exit, the driver is surprised to receive a speeding ticket along with the toll. Explain how this can happen.

147. Set $F(x) = \int_1^x (1-t)dt$. Find $F'(2)$ and the average value of F' over $[1, 2]$.

In the following exercises, use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

148. $\frac{d}{dx}\int_1^x e^{-t^2}dt$

149. $\frac{d}{dx}\int_1^x e^{\cos t}dt$

150. $\frac{d}{dx}\int_3^x \sqrt{9-y^2}dy$

151. $\frac{d}{dx}\int_4^x \frac{ds}{\sqrt{16-s^2}}$

152. $\frac{d}{dx}\int_x^{2x} tdt$

153. $\frac{d}{dx}\int_0^{\sqrt{x}} tdt$

154. $\frac{d}{dx}\int_0^{\sin x} \sqrt{1-t^2}dt$

155. $\frac{d}{dx}\int_{\cos x}^1 \sqrt{1-t^2}dt$

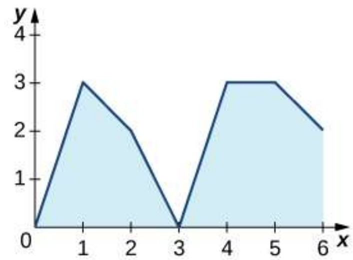
156. $\frac{d}{dx}\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}dt$

157. $\frac{d}{dx}\int_1^{x^2} \frac{\sqrt{t}}{1+t}dt$

158. $\frac{d}{dx}\int_0^{\ln x} e^t dt$

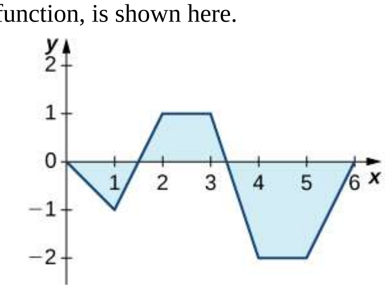
159. $\frac{d}{dx}\int_1^{e^x} \ln u^2 du$

160. The graph of $y = \int_0^x f(t)dt$, where *f* is a piecewise constant function, is shown here.



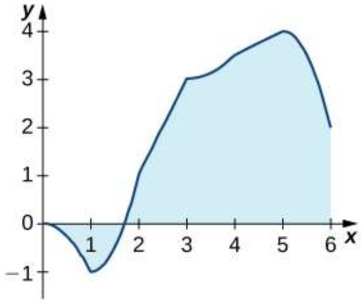
a. Over which intervals is *f* positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?
b. What are the maximum and minimum values of *f*?
c. What is the average value of *f*?

161. The graph of $y = \int_0^x f(t)dt$, where *f* is a piecewise constant function, is shown here.



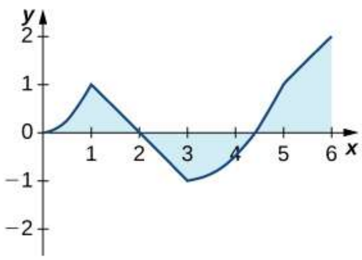
a. Over which intervals is *f* positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?
b. What are the maximum and minimum values of *f*?
c. What is the average value of *f*?

162. The graph of $y = \int_0^x \ell(t)dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?
b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which, if any, is it constant?
c. What is the average value of ℓ?

163. The graph of $y = \int_0^x \ell(t)dt$, where ℓ is a piecewise linear function, is shown here.



a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?
b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which intervals, if any, is it constant?
c. What is the average value of ℓ?

In the following exercises, use a calculator to estimate the area under the curve by computing T_{10}, the average of the left- and right-endpoint Riemann sums using $N = 10$ rectangles. Then, using the Fundamental Theorem of Calculus, Part 2, determine the exact area.

164. **[T]** $y = x^2$ over $[0, 4]$

165. **[T]** $y = x^3 + 6x^2 + x - 5$ over $[-4, 2]$

166. **[T]** $y = \sqrt{x^3}$ over $[0, 6]$

167. **[T]** $y = \sqrt{x} + x^2$ over $[1, 9]$

168. **[T]** $\int (\cos x - \sin x)dx$ over $[0, \pi]$

169. **[T]** $\int \frac{4}{x^2}dx$ over $[1, 4]$

In the following exercises, evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.

170. $\int_{-1}^{2} \left(x^2 - 3x\right)dx$

171. $\int_{-2}^{3} \left(x^2 + 3x - 5\right)dx$

172. $\int_{-2}^{3} (t + 2)(t - 3)dt$

173. $\int_{2}^{3} \left(t^2 - 9\right)\left(4 - t^2\right)dt$

174. $\int_{1}^{2} x^9\, dx$

175. $\int_{0}^{1} x^{99}\, dx$

176. $\int_{4}^{8} \left(4t^{5/2} - 3t^{3/2}\right)dt$

177. $\int_{1/4}^{4} \left(x^2 - \frac{1}{x^2}\right)dx$

178. $\int_{1}^{2} \frac{2}{x^3}dx$

179. $\int_{1}^{4} \frac{1}{2\sqrt{x}}dx$

180. $\int_{1}^{4} \frac{2 - \sqrt{t}}{t^2}dt$

181. $\int_{1}^{16} \frac{dt}{t^{1/4}}$

182. $\int_{0}^{2\pi} \cos\theta d\theta$

183. $\int_{0}^{\pi/2} \sin\theta d\theta$

184. $\int_0^{\pi/4} \sec^2 \theta d\theta$

185. $\int_0^{\pi/4} \sec\theta \tan d\theta$

186. $\int_{\pi/3}^{\pi/4} \csc\theta \cot\theta d\theta$

187. $\int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta$

188. $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

189. $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

In the following exercises, use the evaluation theorem to express the integral as a function $F(x)$.

190. $\int_a^x t^2\, dt$

191. $\int_1^x e^t\, dt$

192. $\int_0^x \cos t dt$

193. $\int_{-x}^x \sin t dt$

In the following exercises, identify the roots of the integrand to remove absolute values, then evaluate using the Fundamental Theorem of Calculus, Part 2.

194. $\int_{-2}^3 |x| dx$

195. $\int_{-2}^4 \left|t^2 - 2t - 3\right| dt$

196. $\int_0^\pi |\cos t| dt$

197. $\int_{-\pi/2}^{\pi/2} |\sin t| dt$

198. Suppose that the number of hours of daylight on a given day in Seattle is modeled by the function $-3.75\cos\left(\frac{\pi t}{6}\right) + 12.25,$ with t given in months and $t = 0$ corresponding to the winter solstice.

a. What is the average number of daylight hours in a year?
b. At which times t_1 and t_2, where $0 \le t_1 < t_2 < 12,$ do the number of daylight hours equal the average number?
c. Write an integral that expresses the total number of daylight hours in Seattle between t_1 and t_2.
d. Compute the mean hours of daylight in Seattle between t_1 and t_2, where $0 \le t_1 < t_2 < 12,$ and then between t_2 and t_1, and show that the average of the two is equal to the average day length.

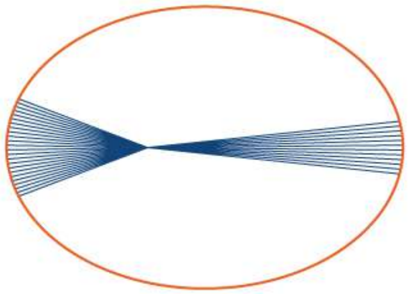
199. Suppose the rate of gasoline consumption over the course of a year in the United States can be modeled by a sinusoidal function of the form $\left(11.21 - \cos\left(\frac{\pi t}{6}\right)\right) \times 10^9$ gal/mo.

a. What is the average monthly consumption, and for which values of t is the rate at time t equal to the average rate?
b. What is the number of gallons of gasoline consumed in the United States in a year?
c. Write an integral that expresses the average monthly U.S. gas consumption during the part of the year between the beginning of April $(t = 3)$ and the end of September $(t = 9)$.

200. Explain why, if f is continuous over $[a, b]$, there is at least one point $c \in [a, b]$ such that $f(c) = \frac{1}{b - a}\int_a^b f(t)dt.$

201. Explain why, if f is continuous over $[a, b]$ and is not equal to a constant, there is at least one point $M \in [a, b]$ such that $f(M) = \frac{1}{b - a}\int_a^b f(t)dt$ and at least one point $m \in [a, b]$ such that $f(m) < \frac{1}{b - a}\int_a^b f(t)dt.$

202. Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus. The closest point of a planetary orbit to the Sun is called the *perihelion* (for Earth, it currently occurs around January 3) and the farthest point is called the *aphelion* (for Earth, it currently occurs around July 4). Kepler's second law states that planets sweep out equal areas of their elliptical orbits in equal times. Thus, the two arcs indicated in the following figure are swept out in equal times. At what time of year is Earth moving fastest in its orbit? When is it moving slowest?



203. A point on an ellipse with major axis length $2a$ and minor axis length $2b$ has the coordinates $(a\cos\theta, b\sin\theta), 0 \le \theta \le 2\pi.$

a. Show that the distance from this point to the focus at $(-c, 0)$ is $d(\theta) = a + c\cos\theta,$ where $c = \sqrt{a^2 - b^2}.$

b. Use these coordinates to show that the average distance $\bar{d}$ from a point on the ellipse to the focus at $(-c, 0),$ with respect to angle θ, is a.

204. As implied earlier, according to Kepler's laws, Earth's orbit is an ellipse with the Sun at one focus. The perihelion for Earth's orbit around the Sun is 147,098,290 km and the aphelion is 152,098,232 km.

a. By placing the major axis along the x-axis, find the average distance from Earth to the Sun.

b. The classic definition of an astronomical unit (AU) is the distance from Earth to the Sun, and its value was computed as the average of the perihelion and aphelion distances. Is this definition justified?

205. The force of gravitational attraction between the Sun and a planet is $F(\theta) = \dfrac{GmM}{r^2(\theta)},$ where m is the mass of the planet, M is the mass of the Sun, G is a universal constant, and $r(\theta)$ is the distance between the Sun and the planet when the planet is at an angle θ with the major axis of its orbit. Assuming that M, m, and the ellipse parameters a and b (half-lengths of the major and minor axes) are given, set up—but do not evaluate—an integral that expresses in terms of G, m, M, a, b the average gravitational force between the Sun and the planet.

206. The displacement from rest of a mass attached to a spring satisfies the simple harmonic motion equation $x(t) = A\cos(\omega t - \phi),$ where ϕ is a phase constant, ω is the angular frequency, and A is the amplitude. Find the average velocity, the average speed (magnitude of velocity), the average displacement, and the average distance from rest (magnitude of displacement) of the mass.

5.4 | Integration Formulas and the Net Change Theorem

Learning Objectives

- **5.4.1** Apply the basic integration formulas.
- **5.4.2** Explain the significance of the net change theorem.
- **5.4.3** Use the net change theorem to solve applied problems.
- **5.4.4** Apply the integrals of odd and even functions.

In this section, we use some basic integration formulas studied previously to solve some key applied problems. It is important to note that these formulas are presented in terms of *indefinite* integrals. Although definite and indefinite integrals are closely related, there are some key differences to keep in mind. A definite integral is either a number (when the limits of integration are constants) or a single function (when one or both of the limits of integration are variables). An indefinite integral represents a family of functions, all of which differ by a constant. As you become more familiar with integration, you will get a feel for when to use definite integrals and when to use indefinite integrals. You will naturally select the correct approach for a given problem without thinking too much about it. However, until these concepts are cemented in your mind, think carefully about whether you need a definite integral or an indefinite integral and make sure you are using the proper notation based on your choice.

Basic Integration Formulas

Recall the integration formulas given in the **table in Antiderivatives** and the rule on properties of definite integrals. Let's look at a few examples of how to apply these rules.

Example 5.23

Integrating a Function Using the Power Rule

Use the power rule to integrate the function $\int_1^4 \sqrt{t}(1+t)dt$.

Solution

The first step is to rewrite the function and simplify it so we can apply the power rule:

$$\begin{aligned}\int_1^4 \sqrt{t}(1+t)dt &= \int_1^4 t^{1/2}(1+t)dt \\ &= \int_1^4 \left(t^{1/2}+t^{3/2}\right)dt.\end{aligned}$$

Now apply the power rule:

$$\begin{aligned}\int_1^4 \left(t^{1/2}+t^{3/2}\right)dt &= \left(\frac{2}{3}t^{3/2}+\frac{2}{5}t^{5/2}\right)\Big|_1^4 \\ &= \left[\frac{2}{3}(4)^{3/2}+\frac{2}{5}(4)^{5/2}\right]-\left[\frac{2}{3}(1)^{3/2}+\frac{2}{5}(1)^{5/2}\right] \\ &= \frac{256}{15}.\end{aligned}$$

5.21 Find the definite integral of $f(x)=x^2-3x$ over the interval $[1, 3]$.

The Net Change Theorem

The **net change theorem** considers the integral of a *rate of change*. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

Theorem 5.6: Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)dx \tag{5.18}$$

or

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Subtracting $F(a)$ from both sides of the first equation yields the second equation. Since they are equivalent formulas, which one we use depends on the application.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let's apply the net change theorem to a velocity function in which the result is displacement.

We looked at a simple example of this in **The Definite Integral**. Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in **Figure 5.32**.

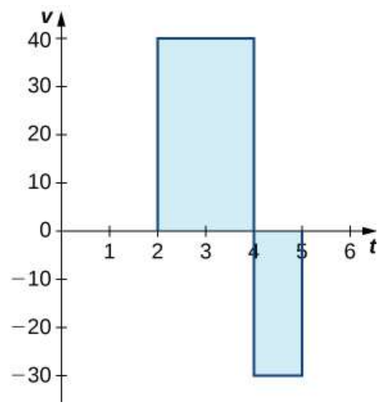


Figure 5.32 The graph shows speed versus time for the given motion of a car.

Just as we did before, we can use definite integrals to calculate the net displacement as well as the total distance traveled. The net displacement is given by

$$\begin{aligned}\int_2^5 v(t)dt &= \int_2^4 40dt + \int_4^5 -30dt \\ &= 80 - 30 \\ &= 50.\end{aligned}$$

Thus, at 5 p.m. the car is 50 mi north of its starting position. The total distance traveled is given by

$$\begin{aligned}\int_2^5 |v(t)|dt &= \int_2^4 40dt + \int_4^5 30dt \\ &= 80 + 30 \\ &= 110.\end{aligned}$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include both positive and negative values. In other words, the velocity function accounts for both forward distance and backward distance. To find net displacement, integrate the velocity function over the interval. Total distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.

Example 5.24

Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3,$ find the net displacement of the particle.

Solution

Applying the net change theorem, we have

$$\begin{aligned}\int_0^3 (3t - 5)dt &= \frac{3t^2}{2} - 5t\Big|_0^3 \\ &= \left[\frac{3(3)^2}{2} - 5(3)\right] - 0 \\ &= \frac{27}{2} - 15 \\ &= \frac{27}{2} - \frac{30}{2} \\ &= -\frac{3}{2}.\end{aligned}$$

The net displacement is $-\frac{3}{2}$ m (**Figure 5.33**).

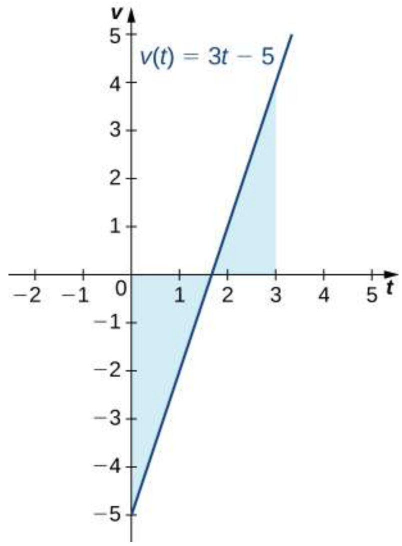


Figure 5.33 The graph shows velocity versus time for a particle moving with a linear velocity function.

Example 5.25

Finding the Total Distance Traveled

Use **Example 5.24** to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

Solution

The total distance traveled includes both the positive and the negative values. Therefore, we must integrate the absolute value of the velocity function to find the total distance traveled.

To continue with the example, use two integrals to find the total distance. First, find the t-intercept of the function, since that is where the division of the interval occurs. Set the equation equal to zero and solve for t. Thus,

$$\begin{aligned} 3t - 5 &= 0 \\ 3t &= 5 \\ t &= \frac{5}{3}. \end{aligned}$$

The two subintervals are $\left[0, \frac{5}{3}\right]$ and $\left[\frac{5}{3}, 3\right]$. To find the total distance traveled, integrate the absolute value of the function. Since the function is negative over the interval $\left[0, \frac{5}{3}\right]$, we have $|v(t)| = -v(t)$ over that interval. Over $\left[\frac{5}{3}, 3\right]$, the function is positive, so $|v(t)| = v(t)$. Thus, we have

$$\begin{aligned}
\int_0^3 |v(t)|dt &= \int_0^{5/3} -v(t)dt + \int_{5/3}^3 v(t)dt \\
&= \int_0^{5/3} 5 - 3tdt + \int_{5/3}^3 3t - 5dt \\
&= \left(5t - \frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2} - 5t\right)\Big|_{5/3}^3 \\
&= \left[5\left(\frac{5}{3}\right) - \frac{3(5/3)^2}{2}\right] - 0 + \left[\frac{27}{2} - 15\right] - \left[\frac{3(5/3)^2}{2} - \frac{25}{3}\right] \\
&= \frac{25}{3} - \frac{25}{6} + \frac{27}{2} - 15 - \frac{25}{6} + \frac{25}{3} \\
&= \frac{41}{6}.
\end{aligned}$$

So, the total distance traveled is $\frac{14}{6}$ m.

5.22 Find the net displacement and total distance traveled in meters given the velocity function $f(t) = \frac{1}{2}e^t - 2$ over the interval $[0, 2]$.

Applying the Net Change Theorem

The net change theorem can be applied to the flow and consumption of fluids, as shown in **Example 5.26**.

Example 5.26

How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t = 0$ and the boat consumes gasoline at $5 - t^3$ gal/hr for the first hour, how much gasoline is used in the first hour?

Solution

Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval $[0, 1]$. We have

$$\begin{aligned}
\int_0^1 \left(5 - t^3\right)dt &= \left(5t - \frac{t^4}{4}\right)\Big|_0^1 \\
&= \left[5(1) - \frac{(1)^4}{4}\right] - 0 \\
&= 5 - \frac{1}{4} \\
&= 4.75.
\end{aligned}$$

Thus, the motorboat uses 4.75 gal of gas in 1 hour.

Example 5.27

Chapter Opener: Iceboats

Figure 5.34 (credit: modification of work by Carter Brown, Flickr)

As we saw at the beginning of the chapter, top iceboat racers (**Figure 5.1**) can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed. Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew gets his iceboat set up, though, the wind begins to pick up. During his first half hour of iceboating, the wind speed increases according to the function $v(t) = 20t + 5.$ For the second half hour of Andrew's outing, the wind remains steady at 15 mph. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5 & \text{for } 0 \le t \le \frac{1}{2} \\ 15 & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

Solution

To figure out how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\text{Distance } = \int_0^1 2v(t)dt.$$

Substituting the expressions we were given for $v(t),$ we get

$$\begin{aligned}
\int_0^1 2v(t)dt &= \int_0^{1/2} 2v(t)dt + \int_{1/2}^1 2v(t)dt \\
&= \int_0^{1/2} 2(20t+5)dt + \int_{1/3}^1 2(15)dt \\
&= \int_0^{1/2} (40t+10)dt + \int_{1/2}^1 30dt \\
&= \left[20t^2+10t\right]\Big|_0^{1/2} + [30t]\Big|_{1/2}^1 \\
&= \left(\frac{20}{4}+5\right) - 0 + (30-15) \\
&= 25.
\end{aligned}$$

Andrew is 25 mi from his starting point after 1 hour.

5.23 Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15.$ In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t+5 & \text{for } 0 \le t \le \frac{1}{2} \\ -10t+15 & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

Integrating Even and Odd Functions

We saw in **Functions and Graphs** that an even function is a function in which $f(-x) = f(x)$ for all x in the domain—that is, the graph of the curve is unchanged when x is replaced with $-x$. The graphs of even functions are symmetric about the y-axis. An odd function is one in which $f(-x) = -f(x)$ for all x in the domain, and the graph of the function is symmetric about the origin.

Integrals of even functions, when the limits of integration are from $-a$ to a, involve two equal areas, because they are symmetric about the y-axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a],$ evaluate to zero because the areas above and below the x-axis are equal.

Rule: Integrals of Even and Odd Functions

For continuous even functions such that $f(-x) = f(x),$

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$$

For continuous odd functions such that $f(-x) = -f(x),$

$$\int_{-a}^{a} f(x)dx = 0.$$

Example 5.28

Integrating an Even Function

Integrate the even function $\int_{-2}^{2} (3x^8 - 2)dx$ and verify that the integration formula for even functions holds.

Solution

The symmetry appears in the graphs in **Figure 5.35**. Graph (a) shows the region below the curve and above the *x*-axis. We have to zoom in to this graph by a huge amount to see the region. Graph (b) shows the region above the curve and below the *x*-axis. The signed area of this region is negative. Both views illustrate the symmetry about the *y*-axis of an even function. We have

$$\begin{aligned} \int_{-2}^{2} (3x^8 - 2)dx &= \left(\frac{x^9}{3} - 2x\right)\Big|_{-2}^{2} \\ &= \left[\frac{(2)^9}{3} - 2(2)\right] - \left[\frac{(-2)^9}{3} - 2(-2)\right] \\ &= \left(\frac{512}{3} - 4\right) - \left(-\frac{512}{3} + 4\right) \\ &= \frac{1000}{3}. \end{aligned}$$

To verify the integration formula for even functions, we can calculate the integral from 0 to 2 and double it, then check to make sure we get the same answer.

$$\begin{aligned} \int_{0}^{2} (3x^8 - 2)dx &= \left(\frac{x^9}{3} - 2x\right)\Big|_{0}^{2} \\ &= \frac{512}{3} - 4 \\ &= \frac{500}{3} \end{aligned}$$

Since $2 \cdot \frac{500}{3} = \frac{1000}{3}$, we have verified the formula for even functions in this particular example.

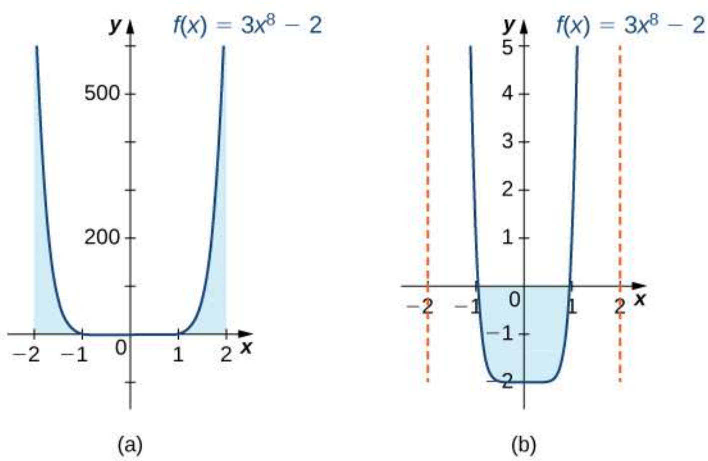


Figure 5.35 Graph (a) shows the positive area between the curve and the *x*-axis, whereas graph (b) shows the negative area between the curve and the *x*-axis. Both views show the symmetry about the *y*-axis.

Example 5.29

Integrating an Odd Function

Evaluate the definite integral of the odd function $-5\sin x$ over the interval $[-\pi, \pi]$.

Solution

The graph is shown in **Figure 5.36**. We can see the symmetry about the origin by the positive area above the x-axis over $[-\pi, 0]$, and the negative area below the x-axis over $[0, \pi]$. We have

$$\begin{aligned}\int_{-\pi}^{\pi} -5\sin x\,dx &= -5(-\cos x)\Big|_{-\pi}^{\pi} \\ &= 5\cos x\Big|_{-\pi}^{\pi} \\ &= [5\cos\pi] - [5\cos(-\pi)] \\ &= -5 - (-5) \\ &= 0.\end{aligned}$$

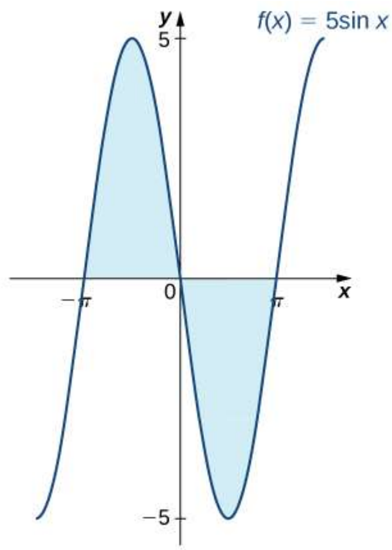


Figure 5.36 The graph shows areas between a curve and the x-axis for an odd function.

5.24 Integrate the function $\int_{-2}^{2} x^4\,dx.$

5.4 EXERCISES

Use basic integration formulas to compute the following antiderivatives or definite integrals.

207. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$

208. $\int \left(e^{2x} - \frac{1}{2}e^{x/2}\right) dx$

209. $\int \frac{dx}{2x}$

210. $\int \frac{x-1}{x^2} dx$

211. $\int_0^{\pi} (\sin x - \cos x) dx$

212. $\int_0^{\pi/2} (x - \sin x) dx$

213. Write an integral that expresses the increase in the perimeter $P(s)$ of a square when its side length s increases from 2 units to 4 units and evaluate the integral.

214. Write an integral that quantifies the change in the area $A(s) = s^2$ of a square when the side length doubles from S units to $2S$ units and evaluate the integral.

215. A regular N-gon (an N-sided polygon with sides that have equal length s, such as a pentagon or hexagon) has perimeter Ns. Write an integral that expresses the increase in perimeter of a regular N-gon when the length of each side increases from 1 unit to 2 units and evaluate the integral.

216. The area of a regular pentagon with side length $a > 0$ is pa^2 with $p = \frac{1}{4}\sqrt{5 + \sqrt{5 + 2\sqrt{5}}}$. The Pentagon in Washington, DC, has inner sides of length 360 ft and outer sides of length 920 ft. Write an integral to express the area of the roof of the Pentagon according to these dimensions and evaluate this area.

217. A dodecahedron is a Platonic solid with a surface that consists of 12 pentagons, each of equal area. By how much does the surface area of a dodecahedron increase as the side length of each pentagon doubles from 1 unit to 2 units?

218. An icosahedron is a Platonic solid with a surface that consists of 20 equilateral triangles. By how much does the surface area of an icosahedron increase as the side length of each triangle doubles from a unit to $2a$ units?

219. Write an integral that quantifies the change in the area of the surface of a cube when its side length doubles from s unit to $2s$ units and evaluate the integral.

220. Write an integral that quantifies the increase in the volume of a cube when the side length doubles from s unit to $2s$ units and evaluate the integral.

221. Write an integral that quantifies the increase in the surface area of a sphere as its radius doubles from R unit to $2R$ units and evaluate the integral.

222. Write an integral that quantifies the increase in the volume of a sphere as its radius doubles from R unit to $2R$ units and evaluate the integral.

223. Suppose that a particle moves along a straight line with velocity $v(t) = 4 - 2t$, where $0 \le t \le 2$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 2$.

224. Suppose that a particle moves along a straight line with velocity defined by $v(t) = t^2 - 3t - 18$, where $0 \le t \le 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

225. Suppose that a particle moves along a straight line with velocity defined by $v(t) = |2t - 6|$, where $0 \le t \le 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

226. Suppose that a particle moves along a straight line with acceleration defined by $a(t) = t - 3$, where $0 \le t \le 6$ (in meters per second). Find the velocity and displacement at time t and the total distance traveled up to $t = 6$ if $v(0) = 3$ and $d(0) = 0$.

227. A ball is thrown upward from a height of 1.5 m at an initial speed of 40 m/sec. Acceleration resulting from gravity is -9.8 m/sec^2. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball t seconds after it is thrown and before it returns to the ground.

228. A ball is thrown upward from a height of 3 m at an initial speed of 60 m/sec. Acceleration resulting from gravity is -9.8 m/sec^2. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball t seconds after it is thrown and before it returns to the ground.

229. The area $A(t)$ of a circular shape is growing at a constant rate. If the area increases from 4π units to 9π units between times $t = 2$ and $t = 3$, find the net change in the radius during that time.

230. A spherical balloon is being inflated at a constant rate. If the volume of the balloon changes from 36π in.^{3} to 288π in.^{3} between time $t = 30$ and $t = 60$ seconds, find the net change in the radius of the balloon during that time.

231. Water flows into a conical tank with cross-sectional area πx^2 at height x and volume $\frac{\pi x^3}{3}$ up to height x. If water flows into the tank at a rate of 1 m^3/min, find the height of water in the tank after 5 min. Find the change in height between 5 min and 10 min.

232. A horizontal cylindrical tank has cross-sectional area $A(x) = 4\left(6x - x^2\right)m^2$ at height x meters above the bottom when $x \le 3$.

a. The volume V between heights a and b is $\int_a^b A(x)dx.$ Find the volume at heights between 2 m and 3 m.
b. Suppose that oil is being pumped into the tank at a rate of 50 L/min. Using the chain rule, $\frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt},$ at how many meters per minute is the height of oil in the tank changing, expressed in terms of x, when the height is at x meters?
c. How long does it take to fill the tank to 3 m starting from a fill level of 2 m?

233. The following table lists the electrical power in gigawatts—the rate at which energy is consumed—used in a certain city for different hours of the day, in a typical 24-hour period, with hour 1 corresponding to midnight to 1 a.m.

Hour	**Power**	**Hour**	**Power**
1	28	13	48
2	25	14	49
3	24	15	49
4	23	16	50
5	24	17	50
6	27	18	50
7	29	19	46
8	32	20	43
9	34	21	42
10	39	22	40
11	42	23	37
12	46	24	34

Find the total amount of energy in gigawatt-hours (gW-h) consumed by the city in a typical 24-hour period.

234. The average residential electrical power use (in hundreds of watts) per hour is given in the following table.

Hour	Power	Hour	Power
1	8	13	12
2	6	14	13
3	5	15	14
4	4	16	15
5	5	17	17
6	6	18	19
7	7	19	18
8	8	20	17
9	9	21	16
10	10	22	16
11	10	23	13
12	11	24	11

a. Compute the average total energy used in a day in kilowatt-hours (kWh).
b. If a ton of coal generates 1842 kWh, how long does it take for an average residence to burn a ton of coal?
c. Explain why the data might fit a plot of the form $p(t) = 11.5 - 7.5\sin\left(\frac{\pi t}{12}\right)$.

235. The data in the following table are used to estimate the average power output produced by Peter Sagan for each of the last 18 sec of Stage 1 of the 2012 Tour de France.

Second	Watts	Second	Watts
1	600	10	1200
2	500	11	1170
3	575	12	1125
4	1050	13	1100
5	925	14	1075
6	950	15	1000
7	1050	16	950
8	950	17	900
9	1100	18	780

Table 5.6 Average Power Output ***Source*:** **sportsexercisengineering.com**

Estimate the net energy used in kilojoules (kJ), noting that 1W = 1 J/s, and the average power output by Sagan during this time interval.

236. The data in the following table are used to estimate the average power output produced by Peter Sagan for each 15-min interval of Stage 1 of the 2012 Tour de France.

Minutes	Watts	Minutes	Watts
15	200	165	170
30	180	180	220
45	190	195	140
60	230	210	225
75	240	225	170
90	210	240	210
105	210	255	200
120	220	270	220
135	210	285	250
150	150	300	400

Table 5.7 Average Power Output ***Source*:** **sportsexercisengineering.com**

Estimate the net energy used in kilojoules, noting that 1W = 1 J/s.

237. The distribution of incomes as of 2012 in the United States in $5000 increments is given in the following table. The kth row denotes the percentage of households with incomes between $\$5000xk$ and $5000xk + 4999.$ The row $k = 40$ contains all households with income between $200,000 and $250,000 and $k = 41$ accounts for all households with income exceeding $250,000.

0	3.5	21	1.5
1	4.1	22	1.4
2	5.9	23	1.3
3	5.7	24	1.3
4	5.9	25	1.1
5	5.4	26	1.0
6	5.5	27	0.75
7	5.1	28	0.8
8	4.8	29	1.0
9	4.1	30	0.6
10	4.3	31	0.6
11	3.5	32	0.5
12	3.7	33	0.5
13	3.2	34	0.4
14	3.0	35	0.3
15	2.8	36	0.3
16	2.5	37	0.3
17	2.2	38	0.2
18	2.2	39	1.8
19	1.8	40	2.3
20	2.1	41	

Table 5.8 Income Distributions ***Source*:** **http://www.census.gov/prod/2013pubs/p60-245.pdf**

Table 5.8 Income Distributions ***Source*:** **http://www.census.gov/prod/2013pubs/p60-245.pdf**

a. Estimate the percentage of U.S. households in 2012 with incomes less than $55,000.
b. What percentage of households had incomes exceeding $85,000?
c. Plot the data and try to fit its shape to that of a graph of the form $a(x+c)e^{-b(x+e)}$ for suitable $a,\ b,\ c$.

238. Newton's law of gravity states that the gravitational force exerted by an object of mass M and one of mass m with centers that are separated by a distance r is $F = G\frac{mM}{r^2}$, with G an empirical constant $G = 6.67x10^{-11}\ m^3/\left(kg \cdot s^2\right)$. The work done by a variable force over an interval $[a,\ b]$ is defined as $W = \int_a^b F(x)dx$. If Earth has mass 5.97219×10^{24} and radius 6371 km, compute the amount of work to elevate a polar weather satellite of mass 1400 kg to its orbiting altitude of 850 km above Earth.

239. For a given motor vehicle, the maximum achievable deceleration from braking is approximately 7 m/sec^2 on dry concrete. On wet asphalt, it is approximately 2.5 m/sec^2. Given that 1 mph corresponds to 0.447 m/sec, find the total distance that a car travels in meters on dry concrete after the brakes are applied until it comes to a complete stop if the initial velocity is 67 mph (30 m/sec) or if the initial braking velocity is 56 mph (25 m/sec). Find the corresponding distances if the surface is slippery wet asphalt.

240. John is a 25-year old man who weighs 160 lb. He burns $500 - 50t$ calories/hr while riding his bike for t hours. If an oatmeal cookie has 55 cal and John eats $4t$ cookies during the tth hour, how many net calories has he lost after 3 hours riding his bike?

241. Sandra is a 25-year old woman who weighs 120 lb. She burns $300 - 50t$ cal/hr while walking on her treadmill. Her caloric intake from drinking Gatorade is $100t$ calories during the tth hour. What is her net decrease in calories after walking for 3 hours?

242. A motor vehicle has a maximum efficiency of 33 mpg at a cruising speed of 40 mph. The efficiency drops at a rate of 0.1 mpg/mph between 40 mph and 50 mph, and at a rate of 0.4 mpg/mph between 50 mph and 80 mph. What is the efficiency in miles per gallon if the car is cruising at 50 mph? What is the efficiency in miles per gallon if the car is cruising at 80 mph? If gasoline costs $3.50/gal, what is the cost of fuel to drive 50 mi at 40 mph, at 50 mph, and at 80 mph?

243. Although some engines are more efficient at given a horsepower than others, on average, fuel efficiency decreases with horsepower at a rate of $1/25$ mpg/horsepower. If a typical 50-horsepower engine has an average fuel efficiency of 32 mpg, what is the average fuel efficiency of an engine with the following horsepower: 150, 300, 450?

244. **[T]** The following table lists the 2013 schedule of federal income tax versus taxable income.

Taxable Income Range	**The Tax Is …**	**… Of the Amount Over**
$0–$8925	10%	$0
$8925–$36,250	$892.50 + 15%	$8925
$36,250–$87,850	$4,991.25 + 25%	$36,250
$87,850–$183,250	$17,891.25 + 28%	$87,850
$183,250–$398,350	$44,603.25 + 33%	$183,250
$398,350–$400,000	$115,586.25 + 35%	$398,350
> $400,000	$116,163.75 + 39.6%	$400,000

Table 5.9 Federal Income Tax Versus Taxable Income ***Source*****: http://www.irs.gov/pub/irs-prior/i1040tt--2013.pdf.**

Suppose that Steve just received a $10,000 raise. How much of this raise is left after federal taxes if Steve's salary before receiving the raise was $40,000? If it was $90,000? If it was $385,000?

245. **[T]** The following table provides hypothetical data regarding the level of service for a certain highway.

Highway Speed Range (mph)	**Vehicles per Hour per Lane**	**Density Range (vehicles/mi)**
> 60	< 600	< 10
60–57	600–1000	10–20
57–54	1000–1500	20–30
54–46	1500–1900	30–45
46–30	1900–2100	45–70
<30	Unstable	70–200

Table 5.10

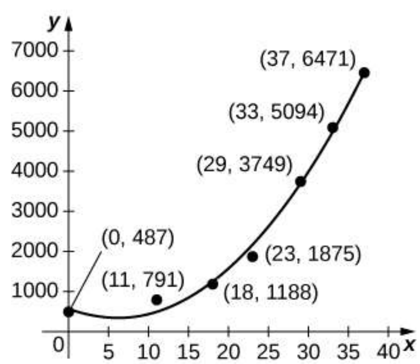
a. Plot vehicles per hour per lane on the *x*-axis and highway speed on the *y*-axis.
b. Compute the average decrease in speed (in miles per hour) per unit increase in congestion (vehicles per hour per lane) as the latter increases from 600 to 1000, from 1000 to 1500, and from 1500 to 2100. Does the decrease in miles per hour depend linearly on the increase in vehicles per hour per lane?
c. Plot minutes per mile (60 times the reciprocal of miles per hour) as a function of vehicles per hour per lane. Is this function linear?

For the next two exercises use the data in the following table, which displays bald eagle populations from 1963 to 2000 in the continental United States.

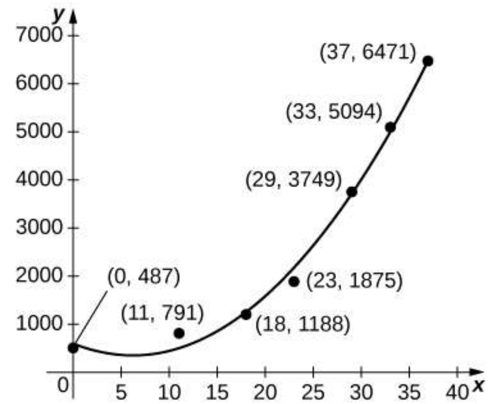
Year	Population of Breeding Pairs of Bald Eagles
1963	487
1974	791
1981	1188
1986	1875
1992	3749
1996	5094
2000	6471

Table 5.11 Population of Breeding Bald Eagle Pairs ***Source*: http://www.fws.gov/Midwest/eagle/population/chtofprs.html.**

246. **[T]** The graph below plots the quadratic $p(t) = 6.48t^2 - 80.3\,1t + 585.69$ against the data in preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of p over $[0, 37]$.

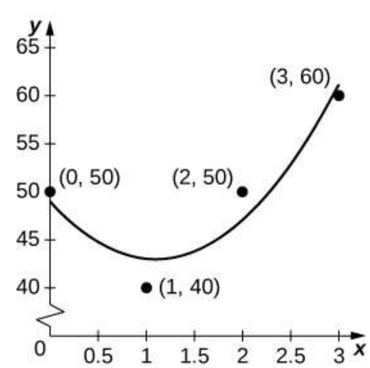


247. **[T]** The graph below plots the cubic $p(t) = 0.07t^3 + 2.42t^2 - 25.63t + 521.23$ against the data in the preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of p over $[0, 37]$.



248. **[T]** Suppose you go on a road trip and record your speed at every half hour, as compiled in the following table. The best quadratic fit to the data is $q(t) = 5x^2 - 11x + 49,$ shown in the accompanying graph. Integrate q to estimate the total distance driven over the 3 hours.

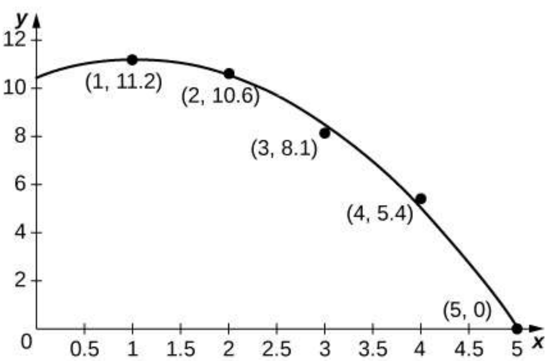
Time (hr)	Speed (mph)
0 (start)	50
1	40
2	50
3	60



As a car accelerates, it does not accelerate at a constant rate; rather, the acceleration is variable. For the following exercises, use the following table, which contains the acceleration measured at every second as a driver merges onto a freeway.

Time (sec)	Acceleration (mph/sec)
1	11.2
2	10.6
3	8.1
4	5.4
5	0

249. **[T]** The accompanying graph plots the best quadratic fit, $a(t) = -0.70t^2 + 1.44t + 10.44,$ to the data from the preceding table. Compute the average value of $a(t)$ to estimate the average acceleration between $t = 0$ and $t = 5.$

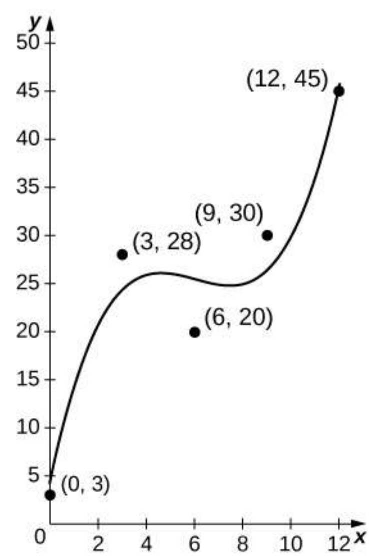


250. **[T]** Using your acceleration equation from the previous exercise, find the corresponding velocity equation. Assuming the final velocity is 0 mph, find the velocity at time $t = 0.$

251. **[T]** Using your velocity equation from the previous exercise, find the corresponding distance equation, assuming your initial distance is 0 mi. How far did you travel while you accelerated your car? (*Hint:* You will need to convert time units.)

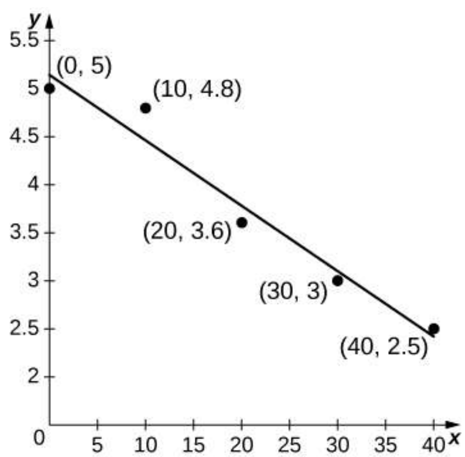
252. **[T]** The number of hamburgers sold at a restaurant throughout the day is given in the following table, with the accompanying graph plotting the best cubic fit to the data, $b(t) = 0.12t^3 - 2.13t^3 + 12.13t + 3.91,$ with $t = 0$ corresponding to 9 a.m. and $t = 12$ corresponding to 9 p.m. Compute the average value of $b(t)$ to estimate the average number of hamburgers sold per hour.

Hours Past Midnight	No. of Burgers Sold
9	3
12	28
15	20
18	30
21	45



253. **[T]** An athlete runs by a motion detector, which records her speed, as displayed in the following table. The best linear fit to this data, $\ell(t) = -0.068t + 5.14$, is shown in the accompanying graph. Use the average value of $\ell(t)$ between $t = 0$ and $t = 40$ to estimate the runner's average speed.

Minutes	Speed (m/sec)
0	5
10	4.8
20	3.6
30	3.0
40	2.5



5.5 | Substitution

Learning Objectives

5.5.1 Use substitution to evaluate indefinite integrals.
5.5.2 Use substitution to evaluate definite integrals.

The Fundamental Theorem of Calculus gave us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called **integration by substitution**, to help us find antiderivatives. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f[g(x)]g'(x)dx.$ For example, in the integral $\int \left(x^2 - 3\right)^3 2x dx,$ we have $f(x) = x^3,\ g(x) = x^2 - 3,$ and $g'(x) = 2x.$ Then,

$$f[g(x)]g'(x) = \left(x^2 - 3\right)^3 (2x),$$

and we see that our integrand is in the correct form.

The method is called *substitution* because we substitute part of the integrand with the variable *u* and part of the integrand with *du*. It is also referred to as **change of variables** because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

Theorem 5.7: Substitution with Indefinite Integrals

Let $u = g(x),$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of *g*, and let $F(x)$ be an antiderivative of $f(x).$ Then,

$$\begin{aligned} \int f[g(x)]g'(x)dx &= \int f(u)du \\ &= F(u) + C \\ &= F(g(x)) + C. \end{aligned} \tag{5.19}$$

Proof

Let *f*, *g*, *u*, and *F* be as specified in the theorem. Then

$$\begin{aligned} \frac{d}{dx}F(g(x)) &= F'(g(x))g'(x) \\ &= f[g(x)]g'(x). \end{aligned}$$

Integrating both sides with respect to *x*, we see that

$$\int f[g(x)]g'(x)dx = F(g(x)) + C.$$

If we now substitute $u = g(x),$ and $du = g'(x)dx,$ we get

$$\begin{aligned} \int f[g(x)]g'(x)dx &= \int f(u)du \\ &= F(u) + C \\ &= F(g(x)) + C. \end{aligned}$$

□

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2xdx$. Rewrite the integral in terms of u:

$$\int \underbrace{\left(x^2 - 3\right)}_{u}^{3} (2\underbrace{xdx}_{du}) = \int u^3 \, du.$$

Using the power rule for integrals, we have

$$\int u^3 \, du = \frac{u^4}{4} + C.$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{\left(x^2 - 3\right)^4}{4} + C.$$

We can generalize the procedure in the following Problem-Solving Strategy.

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x).$ such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx.$ into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Write the result in terms of x and the expression $g(x)$.

Example 5.30

Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative $\int 6x\left(3x^2 + 4\right)^4 dx.$

Solution

The first step is to choose an expression for u. We choose $u = 3x^2 + 4$ because then $du = 6xdx,$ and we already have du in the integrand. Write the integral in terms of u:

$$\int 6x\left(3x^2 + 4\right)^4 dx = \int u^4 \, du.$$

Remember that du is the derivative of the expression chosen for u, regardless of what is inside the integrand. Now we can evaluate the integral with respect to u:

$$\int u^4\,du = \frac{u^5}{5} + C$$
$$= \frac{\left(3x^2+4\right)^5}{5} + C.$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for C of 1, we let $y = \frac{1}{5}\left(3x^2+4\right)^5 + 1.$ We have

$$y = \frac{1}{5}\left(3x^2+4\right)^5 + 1,$$

so

$$y' = \left(\frac{1}{5}\right)5\left(3x^2+4\right)^4 6x$$
$$= 6x\left(3x^2+4\right)^4.$$

This is exactly the expression we started with inside the integrand.

5.25 Use substitution to find the antiderivative $\int 3x^2\left(x^3-3\right)^2 dx.$

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

Example 5.31

Using Substitution with Alteration

Use substitution to find $\int z\sqrt{z^2-5}\,dz.$

Solution

Rewrite the integral as $\int z\left(z^2-5\right)^{1/2} dz.$ Let $u = z^2-5$ and $du = 2z\,dz.$ Now we have a problem because $du = 2z\,dz$ and the original expression has only $z\,dz.$ We have to alter our expression for du or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}.$ we can solve this problem. Thus,

$$u = z^2-5$$
$$du = 2z\,dz$$
$$\frac{1}{2}du = \frac{1}{2}(2z)dz = z\,dz.$$

Write the integral in terms of u, but pull the $\frac{1}{2}$ outside the integration symbol:

$$\int z\left(z^2 - 5\right)^{1/2} dz = \frac{1}{2}\int u^{1/2}\, du.$$

Integrate the expression in u:

$$\begin{aligned}\frac{1}{2}\int u^{1/2}\, du &= \left(\frac{1}{2}\right)\frac{u^{3/2}}{\frac{3}{2}} + C \\ &= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2} + C \\ &= \frac{1}{3}u^{3/2} + C \\ &= \frac{1}{3}\left(z^2 - 5\right)^{3/2} + C.\end{aligned}$$

5.26 Use substitution to find $\int x^2\left(x^3 + 5\right)^9 dx.$

Example 5.32

Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t} dt.$

Solution

We know the derivative of $\cos t$ is $-\sin t,$ so we set $u = \cos t.$ Then $du = -\sin t\, dt.$ Substituting into the integral, we have

$$\int \frac{\sin t}{\cos^3 t} dt = -\int \frac{du}{u^3}.$$

Evaluating the integral, we get

$$\begin{aligned}-\int \frac{du}{u^3} &= -\int u^{-3}\, du \\ &= -\left(-\frac{1}{2}\right)u^{-2} + C.\end{aligned}$$

Putting the answer back in terms of t, we get

$$\begin{aligned}\int \frac{\sin t}{\cos^3 t} dt &= \frac{1}{2u^2} + C \\ &= \frac{1}{2\cos^2 t} + C.\end{aligned}$$

5.27 Use substitution to evaluate the integral $\int \frac{\cos t}{\sin^2 t} dt.$

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When we are done, u should be the only variable in the integrand. In some cases, this means solving for the original variable in terms of u. This technique should become clear in the next example.

Example 5.33

Finding an Antiderivative Using *u*-Substitution

Use substitution to find the antiderivative $\int \frac{x}{\sqrt{x-1}} dx.$

Solution

If we let $u = x - 1,$ then $du = dx.$ But this does not account for the x in the numerator of the integrand. We need to express x in terms of u. If $u = x - 1,$ then $x = u + 1.$ Now we can rewrite the integral in terms of u:

$$\begin{aligned} \int \frac{x}{\sqrt{x-1}} dx &= \int \frac{u+1}{\sqrt{u}} du \\ &= \int \sqrt{u} + \frac{1}{\sqrt{u}} du \\ &= \int \left(u^{1/2} + u^{-1/2}\right) du. \end{aligned}$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned} \int \left(u^{1/2} + u^{-1/2}\right) du &= \frac{2}{3} u^{3/2} + 2u^{1/2} + C \\ &= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C \\ &= (x-1)^{1/2}\left[\frac{2}{3}(x-1) + 2\right] + C \\ &= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + \frac{6}{3}\right) \\ &= (x-1)^{1/2}\left(\frac{2}{3}x + \frac{4}{3}\right) \\ &= \frac{2}{3}(x-1)^{1/2}(x+2) + C. \end{aligned}$$

5.28 Use substitution to evaluate the indefinite integral $\int \cos^3 t \sin t \, dt.$

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Theorem 5.8: Substitution with Definite Integrals

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

Then

$$\begin{aligned} \int_a^b f[g(x)]g'(x)dx &= F(g(x))\big|_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u)\big|_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)du, \end{aligned} \tag{5.20}$$

and we have the desired result.

Example 5.34

Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate $\int_0^1 x^2\left(1 + 2x^3\right)^5 dx.$

Solution

Let $u = 1 + 2x^3$, so $du = 6x^2 dx$. Since the original function includes one factor of x^2 and $du = 6x^2 dx$, multiply both sides of the du equation by $1/6$. Then,

$$\begin{aligned} du &= 6x^2 dx \\ \frac{1}{6}du &= x^2 dx. \end{aligned}$$

To adjust the limits of integration, note that when $x = 0, u = 1 + 2(0) = 1$, and when $x = 1, u = 1 + 2(1) = 3$. Then

$$\int_0^1 x^2\left(1 + 2x^3\right)^5 dx = \frac{1}{6}\int_1^3 u^5 du.$$

Evaluating this expression, we get

$$\begin{aligned}\frac{1}{6}\int_1^3 u^5\,du &= \left(\frac{1}{6}\right)\left(\frac{u^6}{6}\right)\Big|_1^3 \\ &= \frac{1}{36}\left[(3)^6 - (1)^6\right] \\ &= \frac{182}{9}.\end{aligned}$$

5.29 Use substitution to evaluate the definite integral $\int_{-1}^{0} y\left(2y^2 - 3\right)^5 dy.$

Example 5.35

Using Substitution with an Exponential Function

Use substitution to evaluate $\int_0^1 xe^{4x^2 + 3}\,dx.$

Solution

Let $u = 4x^3 + 3.$ Then, $du = 8xdx.$ To adjust the limits of integration, we note that when $x = 0,\ u = 3,$ and when $x = 1,\ u = 7.$ So our substitution gives

$$\begin{aligned}\int_0^1 xe^{4x^2+3}\,dx &= \frac{1}{8}\int_3^7 e^u\,du \\ &= \frac{1}{8}e^u\Big|_3^7 \\ &= \frac{e^7 - e^3}{8} \\ &\approx 134.568.\end{aligned}$$

5.30 Use substitution to evaluate $\int_0^1 x^2 \cos\left(\frac{\pi}{2}x^3\right)dx.$

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution. Also, we have the option of replacing the original expression for *u* after we find the antiderivative, which means that we do not have to change the limits of integration. These two approaches are shown in **Example 5.36**.

Example 5.36

Using Substitution to Evaluate a Trigonometric Integral

Use substitution to evaluate $\int_0^{\pi/2} \cos^2 \theta \, d\theta$.

Solution

Let us first use a trigonometric identity to rewrite the integral. The trig identity $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$ allows us to rewrite the integral as

$$\int_0^{\pi/2} \cos^2 \theta d\theta = \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta.$$

Then,

$$\begin{aligned}\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta &= \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta d\theta.\end{aligned}$$

We can evaluate the first integral as it is, but we need to make a substitution to evaluate the second integral. Let $u = 2\theta$. Then, $du = 2d\theta$, or $\frac{1}{2}du = d\theta$. Also, when $\theta = 0$, $u = 0$, and when $\theta = \pi/2$, $u = \pi$. Expressing the second integral in terms of u, we have

$$\begin{aligned}\frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta d\theta &= \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\left(\frac{1}{2}\right)\int_0^{\pi} \cos u du \\ &= \frac{\theta}{2}\Big|_{\theta = 0}^{\theta = \pi/2} + \frac{1}{4}\sin u\Big|_{u = 0}^{u = \theta} \\ &= \left(\frac{\pi}{4} - 0\right) + (0 - 0) = \frac{\pi}{4}.\end{aligned}$$

5.5 EXERCISES

254. Why is u-substitution referred to as *change of variable*?

255. 2. If $f = g \circ h$, when reversing the chain rule, $\frac{d}{dx}(g \circ h)(x) = g'(h(x))h'(x)$, should you take $u = g(x)$ or $u = h(x)$?

In the following exercises, verify each identity using differentiation. Then, using the indicated u-substitution, identify f such that the integral takes the form $\int f(u)du$.

256. $\int x\sqrt{x+1}dx = \frac{2}{15}(x+1)^{3/2}(3x-2) + C;\ u = x+1$

257. For $x > 1: \int \frac{x^2}{\sqrt{x-1}}dx = \frac{2}{15}\sqrt{x-1}\left(3x^2+4x+8\right) + C;\ u = x-1$

258. $\int x\sqrt{4x^2+9}dx = \frac{1}{12}\left(4x^2+9\right)^{3/2} + C;\ u = 4x^2+9$

259. $\int \frac{x}{\sqrt{4x^2+9}}dx = \frac{1}{4}\sqrt{4x^2+9} + C;\ u = 4x^2+9$

260. $\int \frac{x}{(4x^2+9)^2}dx = -\frac{1}{8(4x^2+9)};\ u = 4x^2+9$

In the following exercises, find the antiderivative using the indicated substitution.

261. $\int (x+1)^4 dx;\ u = x+1$

262. $\int (x-1)^5 dx;\ u = x-1$

263. $\int (2x-3)^{-7} dx;\ u = 2x-3$

264. $\int (3x-2)^{-11} dx;\ u = 3x-2$

265. $\int \frac{x}{\sqrt{x^2+1}}dx;\ u = x^2+1$

266. $\int \frac{x}{\sqrt{1-x^2}}dx;\ u = 1-x^2$

267. $\int (x-1)\left(x^2-2x\right)^3 dx;\ u = x^2-2x$

268. $\int \left(x^2-2x\right)\left(x^3-3x^2\right)^2 dx;\ u = x^3-3x^2$

269. $\int \cos^3\theta d\theta;\ u = \sin\theta\ \ (Hint: \cos^2\theta = 1-\sin^2\theta)$

270. $\int \sin^3\theta d\theta;\ u = \cos\theta\ \ (Hint: \sin^2\theta = 1-\cos^2\theta)$

In the following exercises, use a suitable change of variables to determine the indefinite integral.

271. $\int x(1-x)^{99} dx$

272. $\int t\left(1-t^2\right)^{10} dt$

273. $\int (11x-7)^{-3} dx$

274. $\int (7x-11)^4 dx$

275. $\int \cos^3\theta \sin\theta d\theta$

276. $\int \sin^7\theta \cos\theta d\theta$

277. $\int \cos^2(\pi t)\sin(\pi t)dt$

278. $\int \sin^2 x\cos^3 x dx\ \ (Hint: \sin^2 x + \cos^2 x = 1)$

279. $\int t\sin\left(t^2\right)\cos\left(t^2\right)dt$

280. $\int t^2\cos^2\left(t^3\right)\sin\left(t^3\right)dt$

281. $\int \frac{x^2}{\left(x^3-3\right)^2}dx$

282. $\int \frac{x^3}{\sqrt{1-x^2}}dx$

283. $\displaystyle\int \frac{y^5}{\left(1-y^3\right)^{3/2}}dy$

284. $\displaystyle\int \cos\theta(1-\cos\theta)^{99}\sin\theta d\theta$

285. $\displaystyle\int \left(1-\cos^3\theta\right)^{10}\cos^2\theta\sin\theta d\theta$

286. $\displaystyle\int (\cos\theta-1)\left(\cos^2\theta-2\cos\theta\right)^3\sin\theta d\theta$

287. $\displaystyle\int \left(\sin^2\theta-2\sin\theta\right)\left(\sin^3\theta-3\sin^2\theta\right)^3\cos\theta d\theta$

In the following exercises, use a calculator to estimate the area under the curve using left Riemann sums with 50 terms, then use substitution to solve for the exact answer.

288. **[T]** $y=3(1-x)^2$ over $[0, 2]$

289. **[T]** $y=x\left(1-x^2\right)^3$ over $[-1, 2]$

290. **[T]** $y=\sin x(1-\cos x)^2$ over $[0, \pi]$

291. **[T]** $y=\dfrac{x}{\left(x^2+1\right)^2}$ over $[-1, 1]$

In the following exercises, use a change of variables to evaluate the definite integral.

292. $\displaystyle\int_0^1 x\sqrt{1-x^2}dx$

293. $\displaystyle\int_0^1 \frac{x}{\sqrt{1+x^2}}dx$

294. $\displaystyle\int_0^2 \frac{t^2}{\sqrt{5+t^2}}dt$

295. $\displaystyle\int_0^1 \frac{t^2}{\sqrt{1+t^3}}dt$

296. $\displaystyle\int_0^{\pi/4} \sec^2\theta\tan\theta d\theta$

297. $\displaystyle\int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}d\theta$

In the following exercises, evaluate the indefinite integral $\int f(x)dx$ with constant $C=0$ using u-substitution. Then, graph the function and the antiderivative over the indicated interval. If possible, estimate a value of C that would need to be added to the antiderivative to make it equal to the definite integral $F(x)=\int_a^x f(t)dt,$ with a the left endpoint of the given interval.

298. **[T]** $\displaystyle\int (2x+1)e^{x^2+x-6}dx$ over $[-3, 2]$

299. **[T]** $\displaystyle\int \frac{\cos(\ln(2x))}{x}dx$ on $[0, 2]$

300. **[T]** $\displaystyle\int \frac{3x^2+2x+1}{\sqrt{x^3+x^2+x+4}}dx$ over $[-1, 2]$

301. **[T]** $\displaystyle\int \frac{\sin x}{\cos^3 x}dx$ over $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

302. **[T]** $\displaystyle\int (x+2)e^{-x^2-4x+3}dx$ over $[-5, 1]$

303. **[T]** $\displaystyle\int 3x^2\sqrt{2x^3+1}dx$ over $[0, 1]$

304. If $h(a)=h(b)$ in $\int_a^b g'(h(x))h(x)dx,$ what can you say about the value of the integral?

305. Is the substitution $u=1-x^2$ in the definite integral $\displaystyle\int_0^2 \frac{x}{1-x^2}dx$ okay? If not, why not?

In the following exercises, use a change of variables to show that each definite integral is equal to zero.

306. $\displaystyle\int_0^{\pi} \cos^2(2\theta)\sin(2\theta)d\theta$

307. $\displaystyle\int_0^{\sqrt{\pi}} t\cos\left(t^2\right)\sin\left(t^2\right)dt$

308. $\displaystyle\int_0^1 (1-2t)dt$

309. $\displaystyle\int_0^1 \frac{1-2t}{\left(1+\left(t-\frac{1}{2}\right)^2\right)}dt$

310. $\displaystyle\int_0^{\pi} \sin\left(\left(t-\frac{\pi}{2}\right)^3\right)\cos\left(t-\frac{\pi}{2}\right)dt$

311. $\displaystyle\int_0^2 (1-t)\cos(\pi t)dt$

312. $\displaystyle\int_{\pi/4}^{3\pi/4} \sin^2 t \cos t dt$

313. Show that the average value of $f(x)$ over an interval $[a, b]$ is the same as the average value of $f(cx)$ over the interval $\left[\frac{a}{c}, \frac{b}{c}\right]$ for $c > 0$.

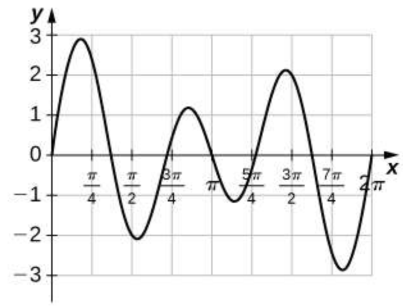
314. Find the area under the graph of $f(t) = \frac{t}{\left(1+t^2\right)^a}$ between $t = 0$ and $t = x$ where $a > 0$ and $a \neq 1$ is fixed, and evaluate the limit as $x \to \infty$.

315. Find the area under the graph of $g(t) = \frac{t}{\left(1-t^2\right)^a}$ between $t = 0$ and $t = x$, where $0 < x < 1$ and $a > 0$ is fixed. Evaluate the limit as $x \to 1$.

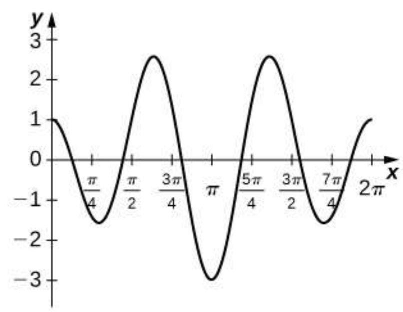
316. The area of a semicircle of radius 1 can be expressed as $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}dx$. Use the substitution $x = \cos t$ to express the area of a semicircle as the integral of a trigonometric function. You do not need to compute the integral.

317. The area of the top half of an ellipse with a major axis that is the x-axis from $x = a$ to a and with a minor axis that is the y-axis from $y = -b$ to b can be written as $\displaystyle\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}dx$. Use the substitution $x = a\cos t$ to express this area in terms of an integral of a trigonometric function. You do not need to compute the integral.

318. **[T]** The following graph is of a function of the form $f(t) = a\sin(nt) + b\sin(mt)$. Estimate the coefficients a and b, and the frequency parameters n and m. Use these estimates to approximate $\displaystyle\int_0^{\pi} f(t)dt$.



319. **[T]** The following graph is of a function of the form $f(x) = a\cos(nt) + b\cos(mt)$. Estimate the coefficients a and b and the frequency parameters n and m. Use these estimates to approximate $\displaystyle\int_0^{\pi} f(t)dt$.



5.6 | Integrals Involving Exponential and Logarithmic Functions

Learning Objectives

5.6.1 Integrate functions involving exponential functions.
5.6.2 Integrate functions involving logarithmic functions.

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

Integrals of Exponential Functions

The exponential function is perhaps the most efficient function in terms of the operations of calculus. The exponential function, $y = e^x$, is its own derivative and its own integral.

Rule: Integrals of Exponential Functions

Exponential functions can be integrated using the following formulas.

$$\begin{aligned} \int e^x\,dx &= e^x + C \\ \int a^x\,dx &= \frac{a^x}{\ln a} + C \end{aligned} \tag{5.21}$$

Example 5.37

Finding an Antiderivative of an Exponential Function

Find the antiderivative of the exponential function e^{-x}.

Solution

Use substitution, setting $u = -x$, and then $du = -1dx$. Multiply the du equation by -1, so you now have $-du = dx$. Then,

$$\begin{aligned} \int e^{-x}dx &= -\int e^u\,du \\ &= -e^u + C \\ &= -e^{-x} + C. \end{aligned}$$

5.31 Find the antiderivative of the function using substitution: $x^2 e^{-2x^3}$.

A common mistake when dealing with exponential expressions is treating the exponent on e the same way we treat exponents in polynomial expressions. We cannot use the power rule for the exponent on e. This can be especially confusing when we have both exponentials and polynomials in the same expression, as in the previous checkpoint. In these cases, we should always double-check to make sure we're using the right rules for the functions we're integrating.

Example 5.38

Square Root of an Exponential Function

Find the antiderivative of the exponential function $e^x\sqrt{1+e^x}$.

Solution

First rewrite the problem using a rational exponent:

$$\int e^x\sqrt{1+e^x}dx = \int e^x(1+e^x)^{1/2}dx.$$

Using substitution, choose $u = 1 + e^x . u = 1 + e^x$. Then, $du = e^x dx$. We have (Figure 5.37)

$$\int e^x(1+e^x)^{1/2}dx = \int u^{1/2}du.$$

Then

$$\int u^{1/2}du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C.$$

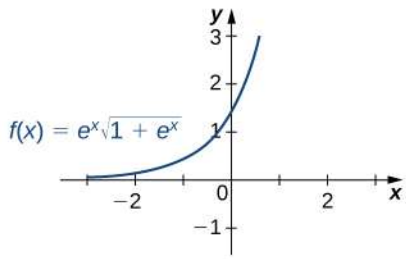


Figure 5.37 The graph shows an exponential function times the square root of an exponential function.

5.32 Find the antiderivative of $e^x(3e^x - 2)^2$.

Example 5.39

Using Substitution with an Exponential Function

Use substitution to evaluate the indefinite integral $\int 3x^2 e^{2x^3}dx$.

Solution

Here we choose to let u equal the expression in the exponent on e. Let $u = 2x^3$ and $du = 6x^2 dx.$. Again, du is off by a constant multiplier; the original function contains a factor of $3x^2$, not $6x^2$. Multiply both sides of the equation by $\frac{1}{2}$ so that the integrand in u equals the integrand in x. Thus,

$$\int 3x^2 e^{2x^3}dx = \frac{1}{2}\int e^u du.$$

Integrate the expression in u and then substitute the original expression in x back into the u integral:

$$\frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x^3} + C.$$

5.33 Evaluate the indefinite integral $\int 2x^3 e^{x^4} dx.$

As mentioned at the beginning of this section, exponential functions are used in many real-life applications. The number e is often associated with compounded or accelerating growth, as we have seen in earlier sections about the derivative. Although the derivative represents a rate of change or a growth rate, the integral represents the total change or the total growth. Let's look at an example in which integration of an exponential function solves a common business application.

A price–demand function tells us the relationship between the quantity of a product demanded and the price of the product. In general, price decreases as quantity demanded increases. The marginal price–demand function is the derivative of the price–demand function and it tells us how fast the price changes at a given level of production. These functions are used in business to determine the price–elasticity of demand, and to help companies determine whether changing production levels would be profitable.

Example 5.40

Finding a Price–Demand Equation

Find the price–demand equation for a particular brand of toothpaste at a supermarket chain when the demand is 50 tubes per week at $2.35 per tube, given that the marginal price—demand function, $p'(x)$, for x number of tubes per week, is given as

$$p'(x) = -0.015e^{-0.01x}.$$

If the supermarket chain sells 100 tubes per week, what price should it set?

Solution

To find the price–demand equation, integrate the marginal price–demand function. First find the antiderivative, then look at the particulars. Thus,

$$\begin{aligned} p(x) &= \int -0.015e^{-0.01x} dx \\ &= -0.015\int e^{-0.01x} dx. \end{aligned}$$

Using substitution, let $u = -0.01x$ and $du = -0.01dx$. Then, divide both sides of the du equation by −0.01. This gives

$$\begin{aligned} \frac{-0.015}{-0.01}\int e^u du &= 1.5\int e^u du \\ &= 1.5e^u + C \\ &= 1.5e^{-0.01x} + C. \end{aligned}$$

The next step is to solve for C. We know that when the price is $2.35 per tube, the demand is 50 tubes per week. This means

$$\begin{aligned} p(50) &= 1.5e^{-0.01(50)} + C \\ &= 2.35. \end{aligned}$$

Now, just solve for C:

$$\begin{aligned} C &= 2.35 - 1.5e^{-0.5} \\ &= 2.35 - 0.91 \\ &= 1.44. \end{aligned}$$

Thus,

$$p(x) = 1.5e^{-0.01x} + 1.44.$$

If the supermarket sells 100 tubes of toothpaste per week, the price would be

$$p(100) = 1.5e^{-0.01(100)} + 1.44 = 1.5e^{-1} + 1.44 \approx 1.99.$$

The supermarket should charge $1.99 per tube if it is selling 100 tubes per week.

Example 5.41

Evaluating a Definite Integral Involving an Exponential Function

Evaluate the definite integral $\int_1^2 e^{1-x}\,dx$.

Solution

Again, substitution is the method to use. Let $u = 1 - x$, so $du = -1dx$ or $-du = dx$. Then $\int e^{1-x}\,dx = -\int e^u\,du$. Next, change the limits of integration. Using the equation $u = 1 - x$, we have

$$\begin{aligned} u &= 1 - (1) = 0 \\ u &= 1 - (2) = -1. \end{aligned}$$

The integral then becomes

$$\begin{aligned} \int_1^2 e^{1-x}\,dx &= -\int_0^{-1} e^u\,du \\ &= \int_{-1}^0 e^u\,du \\ &= e^u\Big|_{-1}^0 \\ &= e^0 - \left(e^{-1}\right) \\ &= -e^{-1} + 1. \end{aligned}$$

See **Figure 5.38**.

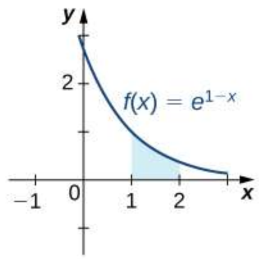


Figure 5.38 The indicated area can be calculated by evaluating a definite integral using substitution.

5.34 Evaluate $\int_0^2 e^{2x}\,dx.$

Example 5.42

Growth of Bacteria in a Culture

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t,$ where t is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with 10,000 bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time t. How many bacteria are in the dish after 2 hours?

Solution

We have

$$Q(t) = \int 3^t\,dt = \frac{3^t}{\ln 3} + C.$$

Then, at $t = 0$ we have $Q(0) = 10 = \frac{1}{\ln 3} + C,$ so $C \approx 9.090$ and we get

$$Q(t) = \frac{3^t}{\ln 3} + 9.090.$$

At time $t = 2,$ we have

$$\begin{aligned} Q(2) &= \frac{3^2}{\ln 3} + 9.090 \\ &= 17.282. \end{aligned}$$

After 2 hours, there are 17,282 bacteria in the dish.

5.35 From **Example 5.42**, suppose the bacteria grow at a rate of $q(t) = 2^t.$ Assume the culture still starts with 10,000 bacteria. Find $Q(t).$ How many bacteria are in the dish after 3 hours?

Example 5.43

Fruit Fly Population Growth

Suppose a population of fruit flies increases at a rate of $g(t) = 2e^{0.02t}$, in flies per day. If the initial population of fruit flies is 100 flies, how many flies are in the population after 10 days?

Solution

Let $G(t)$ represent the number of flies in the population at time t. Applying the net change theorem, we have

$$\begin{aligned} G(10) &= G(0) + \int_0^{10} 2e^{0.02t}\,dt \\ &= 100 + \left[\frac{2}{0.02}e^{0.02t}\right]\Bigg|_0^{10} \\ &= 100 + \left[100e^{0.02t}\right]\Big|_0^{10} \\ &= 100 + 100e^{0.2} - 100 \\ &\approx 122. \end{aligned}$$

There are 122 flies in the population after 10 days.

5.36 Suppose the rate of growth of the fly population is given by $g(t) = e^{0.01t}$, and the initial fly population is 100 flies. How many flies are in the population after 15 days?

Example 5.44

Evaluating a Definite Integral Using Substitution

Evaluate the definite integral using substitution: $\int_1^2 \frac{e^{1/x}}{x^2}dx.$

Solution

This problem requires some rewriting to simplify applying the properties. First, rewrite the exponent on e as a power of x, then bring the x^2 in the denominator up to the numerator using a negative exponent. We have

$$\int_1^2 \frac{e^{1/x}}{x^2}dx = \int_1^2 e^{x^{-1}}\,x^{-2}\,dx.$$

Let $u = x^{-1}$, the exponent on e. Then

$$\begin{aligned} du &= -x^{-2}\,dx \\ -du &= x^{-2}\,dx. \end{aligned}$$

Bringing the negative sign outside the integral sign, the problem now reads

$$-\int e^u\,du.$$

Next, change the limits of integration:

$$u = (1)^{-1} = 1$$
$$u = (2)^{-1} = \frac{1}{2}.$$

Notice that now the limits begin with the larger number, meaning we must multiply by −1 and interchange the limits. Thus,

$$\begin{aligned} -\int_1^{1/2} e^u\,du &= \int_{1/2}^1 e^u\,du \\ &= e^u\big|_{1/2}^1 \\ &= e - e^{1/2} \\ &= e - \sqrt{e}. \end{aligned}$$

5.37 Evaluate the definite integral using substitution: $\int_1^2 \frac{1}{x^3} e^{4x^{-2}}\,dx.$

Integrals Involving Logarithmic Functions

Integrating functions of the form $f(x) = x^{-1}$ result in the absolute value of the natural log function, as shown in the following rule. Integral formulas for other logarithmic functions, such as $f(x) = \ln x$ and $f(x) = \log_a x,$ are also included in the rule.

Rule: Integration Formulas Involving Logarithmic Functions

The following formulas can be used to evaluate integrals involving logarithmic functions.

$$\begin{aligned} \int x^{-1}\,dx &= \ln|x| + C \\ \int \ln x\,dx &= x\ln x - x + C = x(\ln x - 1) + C \\ \int \log_a x\,dx &= \frac{x}{\ln a}(\ln x - 1) + C \end{aligned} \quad (5.22)$$

Example 5.45

Finding an Antiderivative Involving $\ln x$

Find the antiderivative of the function $\frac{3}{x - 10}$.

Solution

First factor the 3 outside the integral symbol. Then use the u^{-1} rule. Thus,

$$\begin{aligned}\int \frac{3}{x-10}dx &= 3\int \frac{1}{x-10}dx \\ &= 3\int \frac{du}{u} \\ &= 3\ln|u| + C \\ &= 3\ln|x-10| + C,\ x \neq 10.\end{aligned}$$

See **Figure 5.39**.

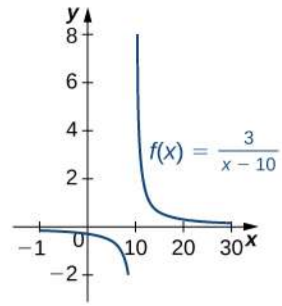


Figure 5.39 The domain of this function is $x \neq 10$.

5.38 Find the antiderivative of $\frac{1}{x+2}$.

Example 5.46

Finding an Antiderivative of a Rational Function

Find the antiderivative of $\frac{2x^3+3x}{x^4+3x^2}$.

Solution

This can be rewritten as $\int (2x^3+3x)(x^4+3x^2)^{-1}dx$. Use substitution. Let $u = x^4 + 3x^2$, then $du = 4x^3 + 6x$. Alter du by factoring out the 2. Thus,

$$\begin{aligned} du &= (4x^3+6x)dx \\ &= 2(2x^3+3x)dx \\ \frac{1}{2}du &= (2x^3+3x)dx.\end{aligned}$$

Rewrite the integrand in u:

$$\int (2x^3 + 3x)(x^4 + 3x^2)^{-1} dx = \frac{1}{2} \int u^{-1} du.$$

Then we have

$$\begin{aligned} \frac{1}{2} \int u^{-1} du &= \frac{1}{2} \ln|u| + C \\ &= \frac{1}{2} \ln\left|x^4 + 3x^2\right| + C. \end{aligned}$$

Example 5.47

Finding an Antiderivative of a Logarithmic Function

Find the antiderivative of the log function $\log_2 x$.

Solution

Follow the format in the formula listed in the rule on integration formulas involving logarithmic functions. Based on this format, we have

$$\int \log_2 x dx = \frac{x}{\ln 2}(\ln x - 1) + C.$$

5.39 Find the antiderivative of $\log_3 x$.

Example 5.48 is a definite integral of a trigonometric function. With trigonometric functions, we often have to apply a trigonometric property or an identity before we can move forward. Finding the right form of the integrand is usually the key to a smooth integration.

Example 5.48

Evaluating a Definite Integral

Find the definite integral of $\int_0^{\pi/2} \frac{\sin x}{1 + \cos x} dx.$

Solution

We need substitution to evaluate this problem. Let $u = 1 + \cos x,$, so $du = -\sin x\, dx.$ Rewrite the integral in terms of u, changing the limits of integration as well. Thus,

$$\begin{aligned} u &= 1 + \cos(0) = 2 \\ u &= 1 + \cos\left(\frac{\pi}{2}\right) = 1. \end{aligned}$$

Then

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin x}{1+\cos x} &= -\int_2^1 u^{-1}\,du \\ &= \int_1^2 u^{-1}\,du \\ &= \ln|u|\Big|_1^2 \\ &= [\ln 2 - \ln 1] \\ &= \ln 2.\end{aligned}$$

5.6 EXERCISES

In the following exercises, compute each indefinite integral.

320. $\int e^{2x} dx$

321. $\int e^{-3x} dx$

322. $\int 2^x dx$

323. $\int 3^{-x} dx$

324. $\int \frac{1}{2x} dx$

325. $\int \frac{2}{x} dx$

326. $\int \frac{1}{x^2} dx$

327. $\int \frac{1}{\sqrt{x}} dx$

In the following exercises, find each indefinite integral by using appropriate substitutions.

328. $\int \frac{\ln x}{x} dx$

329. $\int \frac{dx}{x(\ln x)^2}$

330. $\int \frac{dx}{x \ln x} \ (x > 1)$

331. $\int \frac{dx}{x \ln x \ln(\ln x)}$

332. $\int \tan\theta \, d\theta$

333. $\int \frac{\cos x - x \sin x}{x \cos x} dx$

334. $\int \frac{\ln(\sin x)}{\tan x} dx$

335. $\int \ln(\cos x) \tan x dx$

336. $\int x e^{-x^2} dx$

337. $\int x^2 e^{-x^3} dx$

338. $\int e^{\sin x} \cos x dx$

339. $\int e^{\tan x} \sec^2 x dx$

340. $\int e^{\ln x} \frac{dx}{x}$

341. $\int \frac{e^{\ln(1-t)}}{1-t} dt$

In the following exercises, verify by differentiation that $\int \ln x \, dx = x(\ln x - 1) + C$, then use appropriate changes of variables to compute the integral.

342. $\int \ln x dx$ $\left(\textit{Hint}: \int \ln x dx = \frac{1}{2} \int x \ln\left(x^2\right) dx\right)$

343. $\int x^2 \ln^2 x \, dx$

344. $\int \frac{\ln x}{x^2} dx$ $\left(\textit{Hint}: \text{Set } u = \frac{1}{x}.\right)$

345. $\int \frac{\ln x}{\sqrt{x}} dx$ $\left(\textit{Hint}: \text{Set } u = \sqrt{x}.\right)$

346. Write an integral to express the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to e^x and evaluate the integral.

347. Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x$, and evaluate the integral.

In the following exercises, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.

348. $\int \tan(2x) dx$

349. $\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)} dx$

350. $\int \frac{x \sin\left(x^2\right)}{\cos\left(x^2\right)} dx$

351. $\int x \csc\left(x^2\right) dx$

352. $\int \ln(\cos x)\tan x\, dx$

353. $\int \ln(\csc x)\cot x dx$

354. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

In the following exercises, evaluate the definite integral.

355. $\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3} dx$

356. $\int_0^{\pi/4} \tan x\, dx$

357. $\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x} dx$

358. $\int_{\pi/6}^{\pi/2} \csc x dx$

359. $\int_{\pi/4}^{\pi/3} \cot x dx$

In the following exercises, integrate using the indicated substitution.

360. $\int \frac{x}{x - 100} dx;\ u = x - 100$

361. $\int \frac{y - 1}{y + 1} dy;\ u = y + 1$

362. $\int \frac{1 - x^2}{3x - x^3} dx;\ u = 3x - x^3$

363. $\int \frac{\sin x + \cos x}{\sin x - \cos x} dx;\ u = \sin x - \cos x$

364. $\int e^{2x}\sqrt{1 - e^{2x}} dx;\ u = e^{2x}$

365. $\int \ln(x)\frac{\sqrt{1 - (\ln x)^2}}{x} dx;\ u = \ln x$

In the following exercises, does the right-endpoint approximation overestimate or underestimate the exact area? Calculate the right endpoint estimate R_{50} and solve for the exact area.

366. **[T]** $y = e^x$ over $[0, 1]$

367. **[T]** $y = e^{-x}$ over $[0, 1]$

368. **[T]** $y = \ln(x)$ over $[1, 2]$

369. **[T]** $y = \frac{x + 1}{x^2 + 2x + 6}$ over $[0, 1]$

370. **[T]** $y = 2^x$ over $[-1, 0]$

371. **[T]** $y = -2^{-x}$ over $[0, 1]$

In the following exercises, $f(x) \geq 0$ for $a \leq x \leq b$. Find the area under the graph of $f(x)$ between the given values a and b by integrating.

372. $f(x) = \frac{\log_{10}(x)}{x};\ a = 10,\ b = 100$

373. $f(x) = \frac{\log_2(x)}{x};\ a = 32,\ b = 64$

374. $f(x) = 2^{-x};\ a = 1,\ b = 2$

375. $f(x) = 2^{-x};\ a = 3,\ b = 4$

376. Find the area under the graph of the function $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

377. Compute the integral of $f(x) = xe^{-x^2}$ and find the smallest value of N such that the area under the graph $f(x) = xe^{-x^2}$ between $x = N$ and $x = N + 1$ is, at most, 0.01.

378. Find the limit, as N tends to infinity, of the area under the graph of $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

379. Show that $\int_a^b \frac{dt}{t} = \int_{1/b}^{1/a} \frac{dt}{t}$ when $0 < a \leq b$.

380. Suppose that $f(x) > 0$ for all x and that f and g are differentiable. Use the identity $f^g = e^{g \ln f}$ and the chain rule to find the derivative of f^g.

381. Use the previous exercise to find the antiderivative of $h(x) = x^x(1 + \ln x)$ and evaluate $\int_2^3 x^x(1 + \ln x)dx$.

382. Show that if $c > 0$, then the integral of $1/x$ from ac to bc $(0 < a < b)$ is the same as the integral of $1/x$ from a to b.

The following exercises are intended to derive the fundamental properties of the natural log starting from the *definition* $\ln(x) = \int_1^x \frac{dt}{t}$, using properties of the definite integral and making no further assumptions.

383. Use the identity $\ln(x) = \int_1^x \frac{dt}{t}$ to derive the identity $\ln\left(\frac{1}{x}\right) = -\ln x$.

384. Use a change of variable in the integral $\int_1^{xy} \frac{1}{t} dt$ to show that $\ln xy = \ln x + \ln y$ for $x, y > 0$.

385. Use the identity $\ln x = \int_1^x \frac{dt}{x}$ to show that $\ln(x)$ is an increasing function of x on $[0, \infty)$, and use the previous exercises to show that the range of $\ln(x)$ is $(-\infty, \infty)$. Without any further assumptions, conclude that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$.

386. Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln(x)$, but keep in mind that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$. Call it E. Use the identity $\ln xy = \ln x + \ln y$ to deduce that $E(a+b) = E(a)E(b)$ for any real numbers a, b.

387. Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln x$, but keep in mind that $\ln x$ has an inverse function defined on $(-\infty, \infty)$. Call it E. Show that $E'(t) = E(t)$.

388. The sine integral, defined as $S(x) = \int_0^x \frac{\sin t}{t} dt$ is an important quantity in engineering. Although it does not have a simple closed formula, it is possible to estimate its behavior for large x. Show that for $k \geq 1, |S(2\pi k) - S(2\pi(k+1))| \leq \frac{1}{k(2k+1)\pi}$.

$(Hint: \sin(t + \pi) = -\sin t)$

389. **[T]** The normal distribution in probability is given by $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where σ is the standard deviation and μ is the average. The *standard normal distribution* in probability, p_s, corresponds to $\mu = 0$ and $\sigma = 1$. Compute the right endpoint estimates R_{10} and R_{100} of $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^{2/2}} dx$.

390. **[T]** Compute the right endpoint estimates R_{50} and R_{100} of $\int_{-3}^{5} \frac{1}{2\sqrt{2\pi}} e^{-(x-1)^2/8}$.

5.7 | Integrals Resulting in Inverse Trigonometric Functions

Learning Objectives

5.7.1 Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall from **Functions and Graphs** that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also in **Derivatives**, we developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Sine Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

1.
$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{|a|} + C \tag{5.23}$$

2.
$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \tag{5.24}$$

3.
$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{|a|}\sec^{-1}\frac{u}{|a|} + C \tag{5.25}$$

Proof

Let $y = \sin^{-1}\frac{x}{a}$. Then $a\sin y = x$. Now let's use implicit differentiation. We obtain

$$\begin{aligned} \frac{d}{dx}(a\sin y) &= \frac{d}{dx}(x) \\ a\cos y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{a\cos y}. \end{aligned}$$

For $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y \ge 0$. Thus, applying the Pythagorean identity $\sin^2 y + \cos^2 y = 1$, we have $\cos y = \sqrt{1 - \sin^2 y}$. This gives

$$\begin{aligned}\frac{1}{a\cos y} &= \frac{1}{a\sqrt{1-\sin^2 y}} \\ &= \frac{1}{\sqrt{a^2 - a^2\sin^2 y}} \\ &= \frac{1}{\sqrt{a^2 - x^2}}.\end{aligned}$$

Then for $-a \le x \le a,$ and generalizing to u, we have

$$\int \frac{1}{\sqrt{a^2 - u^2}}du = \sin^{-1}\left(\frac{u}{a}\right) + C.$$

□

Example 5.49

Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}.$

Solution

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x\Big|_0^{\frac{1}{2}} \\ &= \sin^{-1}\frac{1}{2} - \sin^{-1} 0 \\ &= \frac{\pi}{6} - 0 \\ &= \frac{\pi}{6}.\end{aligned}$$

5.40 Find the antiderivative of $\int \frac{dx}{\sqrt{1-16x^2}}.$

Example 5.50

Finding an Antiderivative Involving an Inverse Trigonometric Function

Evaluate the integral $\int \frac{dx}{\sqrt{4-9x^2}}.$

Solution

Substitute $u = 3x$. Then $du = 3dx$ and we have

$$\int \frac{dx}{\sqrt{4 - 9x^2}} = \frac{1}{3} \int \frac{du}{\sqrt{4 - u^2}}.$$

Applying the formula with $a = 2$, we obtain

$$\begin{aligned} \int \frac{dx}{\sqrt{4 - 9x^2}} &= \frac{1}{3} \int \frac{du}{\sqrt{4 - u^2}} \\ &= \frac{1}{3} \sin^{-1}\left(\frac{u}{2}\right) + C \\ &= \frac{1}{3} \sin^{-1}\left(\frac{3x}{2}\right) + C. \end{aligned}$$

5.41 Find the indefinite integral using an inverse trigonometric function and substitution for $\int \frac{dx}{\sqrt{9 - x^2}}$.

Example 5.51

Evaluating a Definite Integral

Evaluate the definite integral $\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1 - u^2}}$.

Solution

The format of the problem matches the inverse sine formula. Thus,

$$\begin{aligned} \int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1 - u^2}} &= \sin^{-1} u \Big|_0^{\sqrt{3}/2} \\ &= \left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] - \left[\sin^{-1}(0)\right] \\ &= \frac{\pi}{3}. \end{aligned}$$

Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out -1 and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example 5.52

Finding an Antiderivative Involving the Inverse Tangent Function

Find an antiderivative of $\int \frac{1}{1+4x^2}dx.$

Solution

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1} u + C.$ So we use substitution, letting $u = 2x,$ then $du = 2dx$ and $1/2du = dx.$ Then, we have

$$\frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(2x) + C.$$

5.42 Use substitution to find the antiderivative $\int \frac{dx}{25+4x^2}.$

Example 5.53

Applying the Integration Formulas

Find the antiderivative of $\int \frac{1}{9+x^2}dx.$

Solution

Apply the formula with $a = 3.$ Then,

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

5.43 Find the antiderivative of $\int \frac{dx}{16+x^2}.$

Example 5.54

Evaluating a Definite Integral

Evaluate the definite integral $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2}.$

Solution

Use the formula for the inverse tangent. We have

$$\begin{aligned}\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} &= \tan^{-1} x \Big|_{\sqrt{3}/3}^{\sqrt{3}} \\ &= \left[\tan^{-1}\left(\sqrt{3}\right)\right] - \left[\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right] \\ &= \frac{\pi}{6}.\end{aligned}$$

5.44 Evaluate the definite integral $\int_0^2 \frac{dx}{4+x^2}$.

5.7 EXERCISES

In the following exercises, evaluate each integral in terms of an inverse trigonometric function.

391. $\displaystyle\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

392. $\displaystyle\int_{-1/2}^{1/2} \frac{dx}{1-x^2}$

393. $\displaystyle\int_{\sqrt{3}}^{1} \frac{dx}{1-x^2}$

394. $\displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$

395. $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

396. $\displaystyle\int_1^{2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}}$

In the following exercises, find each indefinite integral, using appropriate substitutions.

397. $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$

398. $\displaystyle\int \frac{dx}{\sqrt{1-16x^2}}$

399. $\displaystyle\int \frac{dx}{9+x^2}$

400. $\displaystyle\int \frac{dx}{25+16x^2}$

401. $\displaystyle\int \frac{dx}{|x|\sqrt{x^2-9}}$

402. $\displaystyle\int \frac{dx}{|x|\sqrt{4x^2-16}}$

403. Explain the relationship $-\cos^{-1}t + C = \displaystyle\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}t + C$. Is it true, in general, that $\cos^{-1}t = -\sin^{-1}t$?

404. Explain the relationship $\sec^{-1}t + C = \displaystyle\int \frac{dt}{|t|\sqrt{t^2-1}} = -\csc^{-1}t + C$. Is it true, in general, that $\sec^{-1}t = -\csc^{-1}t$?

405. Explain what is wrong with the following integral: $\displaystyle\int_1^2 \frac{dt}{\sqrt{1-t^2}}$.

406. Explain what is wrong with the following integral: $\displaystyle\int_{-1}^{1} \frac{dt}{|t|\sqrt{t^2-1}}$.

In the following exercises, solve for the antiderivative $\int f$ of f with $C = 0$, then use a calculator to graph f and the antiderivative over the given interval $[a, b]$. Identify a value of C such that adding C to the antiderivative recovers the definite integral $F(x) = \displaystyle\int_a^x f(t)dt$.

407. **[T]** $\displaystyle\int \frac{1}{\sqrt{9-x^2}}dx$ over $[-3, 3]$

408. **[T]** $\displaystyle\int \frac{9}{9+x^2}dx$ over $[-6, 6]$

409. **[T]** $\displaystyle\int \frac{\cos x}{4+\sin^2 x}dx$ over $[-6, 6]$

410. **[T]** $\displaystyle\int \frac{e^x}{1+e^{2x}}dx$ over $[-6, 6]$

In the following exercises, compute the antiderivative using appropriate substitutions.

411. $\displaystyle\int \frac{\sin^{-1}t\,dt}{\sqrt{1-t^2}}$

412. $\displaystyle\int \frac{dt}{\sin^{-1}t\sqrt{1-t^2}}$

413. $\displaystyle\int \frac{\tan^{-1}(2t)}{1+4t^2}dt$

414. $\displaystyle\int \frac{t\tan^{-1}\left(t^2\right)}{1+t^4}dt$

415. $\displaystyle\int \frac{\sec^{-1}\left(\frac{t}{2}\right)}{|t|\sqrt{t^2-4}}dt$

416. $\displaystyle\int \frac{t\sec^{-1}\left(t^2\right)}{t^2\sqrt{t^4-1}}dt$

In the following exercises, use a calculator to graph the antiderivative $\int f$ with $C=0$ over the given interval $[a, b]$. Approximate a value of C, if possible, such that adding C to the antiderivative gives the same value as the definite integral $F(x)=\int_a^x f(t)dt$.

417. **[T]** $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}dx$ over $[2, 6]$

418. **[T]** $\displaystyle\int \frac{1}{(2x+2)\sqrt{x}}dx$ over $[0, 6]$

419. **[T]** $\displaystyle\int \frac{(\sin x + x\cos x)}{1+x^2\sin^2 x}dx$ over $[-6, 6]$

420. **[T]** $\displaystyle\int \frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}dx$ over $[0, 2]$

421. **[T]** $\displaystyle\int \frac{1}{x+x\ln^2 x}$ over $[0, 2]$

422. **[T]** $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ over $[-1, 1]$

In the following exercises, compute each integral using appropriate substitutions.

423. $\displaystyle\int \frac{e^t}{\sqrt{1-e^{2t}}}dt$

424. $\displaystyle\int \frac{e^t}{1+e^{2t}}dt$

425. $\displaystyle\int \frac{dt}{t\sqrt{1-\ln^2 t}}$

426. $\displaystyle\int \frac{dt}{t\left(1+\ln^2 t\right)}$

427. $\displaystyle\int \frac{\cos^{-1}(2t)}{\sqrt{1-4t^2}}dt$

428. $\displaystyle\int \frac{e^t\cos^{-1}\left(e^t\right)}{\sqrt{1-e^{2t}}}dt$

In the following exercises, compute each definite integral.

429. $\displaystyle\int_0^{1/2} \frac{\tan\left(\sin^{-1} t\right)}{\sqrt{1-t^2}}dt$

430. $\displaystyle\int_{1/4}^{1/2} \frac{\tan\left(\cos^{-1} t\right)}{\sqrt{1-t^2}}dt$

431. $\displaystyle\int_0^{1/2} \frac{\sin\left(\tan^{-1} t\right)}{1+t^2}dt$

432. $\displaystyle\int_0^{1/2} \frac{\cos\left(\tan^{-1} t\right)}{1+t^2}dt$

433. For $A>0$, compute $I(A)=\displaystyle\int_{-A}^{A}\frac{dt}{1+t^2}$ and evaluate $\lim\limits_{a\to\infty} I(A)$, the area under the graph of $\dfrac{1}{1+t^2}$ on $[-\infty, \infty]$.

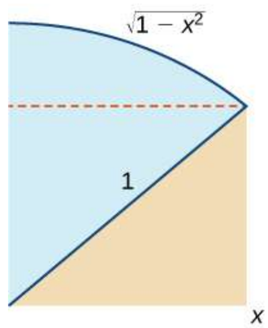
434. For $1<B<\infty$, compute $I(B)=\displaystyle\int_1^B \frac{dt}{t\sqrt{t^2-1}}$ and evaluate $\lim\limits_{B\to\infty} I(B)$, the area under the graph of $\dfrac{1}{t\sqrt{t^2-1}}$ over $[1, \infty)$.

435. Use the substitution $u=\sqrt{2}\cot x$ and the identity $1+\cot^2 x=\csc^2 x$ to evaluate $\displaystyle\int \frac{dx}{1+\cos^2 x}$. (*Hint:* Multiply the top and bottom of the integrand by $\csc^2 x$.)

436. **[T]** Approximate the points at which the graphs of $f(x) = 2x^2 - 1$ and $g(x) = \left(1 + 4x^2\right)^{-3/2}$ intersect to four decimal places, and approximate the area between their graphs to three decimal places.

437. 47. **[T]** Approximate the points at which the graphs of $f(x) = x^2 - 1$ and $g(x) = \left(x^2 + 1\right)^{\frac{1}{2}}$ intersect to four decimal places, and approximate the area between their graphs to three decimal places.

438. Use the following graph to prove that $\int_0^x \sqrt{1 - t^2} dt = \frac{1}{2} x \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x.$



CHAPTER 5 REVIEW

KEY TERMS

average value of a function (or f_{ave}) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

change of variables the substitution of a variable, such as *u*, for an expression in the integrand

definite integral a primary operation of calculus; the area between the curve and the *x*-axis over a given interval is a definite integral

fundamental theorem of calculus the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

fundamental theorem of calculus, part 1 uses a definite integral to define an antiderivative of a function

fundamental theorem of calculus, part 2 (also, **evaluation theorem**) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

integrable function a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as *n* goes to infinity exists

integrand the function to the right of the integration symbol; the integrand includes the function being integrated

integration by substitution a technique for integration that allows integration of functions that are the result of a chain-rule derivative

left-endpoint approximation an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

limits of integration these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

lower sum a sum obtained by using the minimum value of $f(x)$ on each subinterval

mean value theorem for integrals guarantees that a point *c* exists such that $f(c)$ is equal to the average value of the function

net change theorem if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

net signed area the area between a function and the *x*-axis such that the area below the *x*-axis is subtracted from the area above the *x*-axis; the result is the same as the definite integral of the function

partition a set of points that divides an interval into subintervals

regular partition a partition in which the subintervals all have the same width

riemann sum an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

right-endpoint approximation the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

sigma notation (also, **summation notation**) the Greek letter sigma (Σ) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

total area total area between a function and the *x*-axis is calculated by adding the area above the *x*-axis and the area below the *x*-axis; the result is the same as the definite integral of the absolute value of the function

upper sum a sum obtained by using the maximum value of $f(x)$ on each subinterval

variable of integration indicates which variable you are integrating with respect to; if it is *x*, then the function in the integrand is followed by *dx*

KEY EQUATIONS

- **Properties of Sigma Notation**

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$$

- **Sums and Powers of Integers**

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Left-Endpoint Approximation**

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

- **Right-Endpoint Approximation**

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

- **Definite Integral**

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- **Properties of the Definite Integral**

$$\int_a^a f(x)dx = 0$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x) \text{ for constant } c$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

- **Mean Value Theorem for Integrals**

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$

- **Fundamental Theorem of Calculus Part 1**

  If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x).$

- **Fundamental Theorem of Calculus Part 2**

  If *f* is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a).$

- **Net Change Theorem**

  $F(b) = F(a) + \int_a^b F'(x)dx$ or $\int_a^b F'(x)dx = F(b) - F(a)$

- **Substitution with Indefinite Integrals**

  $\int f[g(x)]g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C$

- **Substitution with Definite Integrals**

  $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

- **Integrals of Exponential Functions**

  $\int e^x dx = e^x + C$

  $\int a^x dx = \frac{a^x}{\ln a} + C$

- **Integration Formulas Involving Logarithmic Functions**

  $\int x^{-1} dx = \ln|x| + C$

  $\int \ln x\, dx = x\ln x - x + C = x(\ln x - 1) + C$

  $\int \log_a x\, dx = \frac{x}{\ln a}(\ln x - 1) + C$

- **Integrals That Produce Inverse Trigonometric Functions**

  $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$

  $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

  $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$

KEY CONCEPTS

5.1 Approximating Areas

- The use of sigma (summation) notation of the form $\sum_{i=1}^{n} a_i$ is useful for expressing long sums of values in compact form.

- For a continuous function defined over an interval $[a, b]$, the process of dividing the interval into n equal parts, extending a rectangle to the graph of the function, calculating the areas of the series of rectangles, and then summing the areas yields an approximation of the area of that region.
- The width of each rectangle is $\Delta x = \frac{b-a}{n}$.
- Riemann sums are expressions of the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$, and can be used to estimate the area under the curve $y = f(x)$. Left- and right-endpoint approximations are special kinds of Riemann sums where the values of $\{x_i^*\}$ are chosen to be the left or right endpoints of the subintervals, respectively.
- Riemann sums allow for much flexibility in choosing the set of points $\{x_i^*\}$ at which the function is evaluated, often with an eye to obtaining a lower sum or an upper sum.

5.2 The Definite Integral

- The definite integral can be used to calculate net signed area, which is the area above the *x*-axis less the area below the *x*-axis. Net signed area can be positive, negative, or zero.
- The component parts of the definite integral are the integrand, the variable of integration, and the limits of integration.
- Continuous functions on a closed interval are integrable. Functions that are not continuous may still be integrable, depending on the nature of the discontinuities.
- The properties of definite integrals can be used to evaluate integrals.
- The area under the curve of many functions can be calculated using geometric formulas.
- The average value of a function can be calculated using definite integrals.

5.3 The Fundamental Theorem of Calculus

- The Mean Value Theorem for Integrals states that for a continuous function over a closed interval, there is a value c such that $f(c)$ equals the average value of the function. See **The Mean Value Theorem for Integrals**.
- The Fundamental Theorem of Calculus, Part 1 shows the relationship between the derivative and the integral. See **Fundamental Theorem of Calculus, Part 1**.
- The Fundamental Theorem of Calculus, Part 2 is a formula for evaluating a definite integral in terms of an antiderivative of its integrand. The total area under a curve can be found using this formula. See **The Fundamental Theorem of Calculus, Part 2**.

5.4 Integration Formulas and the Net Change Theorem

- The net change theorem states that when a quantity changes, the final value equals the initial value plus the integral of the rate of change. Net change can be a positive number, a negative number, or zero.
- The area under an even function over a symmetric interval can be calculated by doubling the area over the positive *x*-axis. For an odd function, the integral over a symmetric interval equals zero, because half the area is negative.

5.5 Substitution

- Substitution is a technique that simplifies the integration of functions that are the result of a chain-rule derivative. The term 'substitution' refers to changing variables or substituting the variable *u* and *du* for appropriate expressions in the integrand.
- When using substitution for a definite integral, we also have to change the limits of integration.

5.6 Integrals Involving Exponential and Logarithmic Functions

- Exponential and logarithmic functions arise in many real-world applications, especially those involving growth and decay.
- Substitution is often used to evaluate integrals involving exponential functions or logarithms.

5.7 Integrals Resulting in Inverse Trigonometric Functions

- Formulas for derivatives of inverse trigonometric functions developed in **Derivatives of Exponential and Logarithmic Functions** lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.

CHAPTER 5 REVIEW EXERCISES

True or False. Justify your answer with a proof or a counterexample. Assume all functions f and g are continuous over their domains.

439. If $f(x) > 0,\ f'(x) > 0$ for all x, then the right-hand rule underestimates the integral $\int_a^b f(x)$. Use a graph to justify your answer.

440. $\int_a^b f(x)^2\,dx = \int_a^b f(x)dx \int_a^b f(x)dx$

441. If $f(x) \le g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) \le \int_a^b g(x)$.

442. All continuous functions have an antiderivative.

Evaluate the Riemann sums L_4 and R_4 for the following functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.

443. $y = 3x^2 - 2x + 1$ over $[-1, 1]$

444. $y = \ln\left(x^2 + 1\right)$ over $[0, e]$

445. $y = x^2 \sin x$ over $[0, \pi]$

446. $y = \sqrt{x} + \frac{1}{x}$ over $[1, 4]$

Evaluate the following integrals.

447. $\int_{-1}^{1}\left(x^3 - 2x^2 + 4x\right)dx$

448. $\int_0^4 \frac{3t}{\sqrt{1 + 6t^2}}dt$

449. $\int_{\pi/3}^{\pi/2} 2\sec(2\theta)\tan(2\theta)d\theta$

450. $\int_0^{\pi/4} e^{\cos^2 x}\sin x \cos x\,dx$

Find the antiderivative.

451. $\int \frac{dx}{(x + 4)^3}$

452. $\int x \ln\left(x^2\right)dx$

453. $\int \frac{4x^2}{\sqrt{1 - x^6}}dx$

454. $\int \frac{e^{2x}}{1 + e^{4x}}dx$

Find the derivative.

455. $\frac{d}{dt}\int_0^t \frac{\sin x}{\sqrt{1 + x^2}}dx$

456. $\frac{d}{dx}\int_1^{x^3} \sqrt{4 - t^2}dt$

457. $\frac{d}{dx}\int_1^{\ln(x)} (4t + e^t)dt$

458. $\frac{d}{dx}\int_0^{\cos x} e^{t^2}dt$

The following problems consider the historic average cost per gigabyte of RAM on a computer.

Year	5-Year Change ($)
1980	0
1985	−5,468,750
1990	−755,495
1995	−73,005
2000	−29,768
2005	−918
2010	−177

459. If the average cost per gigabyte of RAM in 2010 is $12, find the average cost per gigabyte of RAM in 1980.

460. The average cost per gigabyte of RAM can be approximated by the function $C(t) = 8,500,000(0.65)^t$, where t is measured in years since 1980, and C is cost in US$. Find the average cost per gigabyte of RAM for 1980 to 2010.

461. Find the average cost of 1GB RAM for 2005 to 2010.

462. The velocity of a bullet from a rifle can be approximated by $v(t) = 6400t^2 - 6505t + 2686,$ where t is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot: $0 \le t \le 0.5.$ What is the total distance the bullet travels in 0.5 sec?

463. What is the average velocity of the bullet for the first half-second?

6 | APPLICATIONS OF INTEGRATION

Figure 6.1 Hoover Dam is one of the United States' iconic landmarks, and provides irrigation and hydroelectric power for millions of people in the southwest United States. (credit: modification of work by Lynn Betts, Wikimedia)

Chapter Outline

6.1 Areas between Curves
6.2 Determining Volumes by Slicing
6.3 Volumes of Revolution: Cylindrical Shells
6.4 Arc Length of a Curve and Surface Area
6.5 Physical Applications
6.6 Moments and Centers of Mass
6.7 Integrals, Exponential Functions, and Logarithms
6.8 Exponential Growth and Decay
6.9 Calculus of the Hyperbolic Functions

Introduction

The Hoover Dam is an engineering marvel. When Lake Mead, the reservoir behind the dam, is full, the dam withstands a great deal of force. However, water levels in the lake vary considerably as a result of droughts and varying water demands. Later in this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force (see **Example 6.28**).

Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

6.1 | Areas between Curves

Learning Objectives

6.1.1 Determine the area of a region between two curves by integrating with respect to the independent variable.

6.1.2 Find the area of a compound region.

6.1.3 Determine the area of a region between two curves by integrating with respect to the dependent variable.

In **Introduction to Integration**, we developed the concept of the definite integral to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of x, beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of y.

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in the following figure.

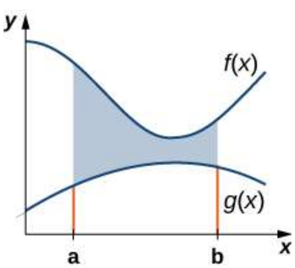


Figure 6.2 The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$.

As we did before, we are going to partition the interval on the x-axis and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. **Figure 6.3**(a) shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$. **Figure 6.3**(b) shows a representative rectangle in detail.

Use this **calculator (http://www.openstax.org/l/20_CurveCalc)** to learn more about the areas between two curves.

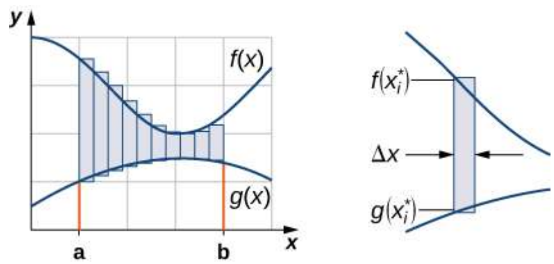


Figure 6.3 (a)We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*)\right]\Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*)\right]\Delta x = \int_a^b [f(x) - g(x)]dx.$$

These findings are summarized in the following theorem.

Theorem 6.1: Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b [f(x) - g(x)]dx. \quad (6.1)$$

We apply this theorem in the following example.

Example 6.1

Finding the Area of a Region between Two Curves 1

If R is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region R.

Solution

The region is depicted in the following figure.

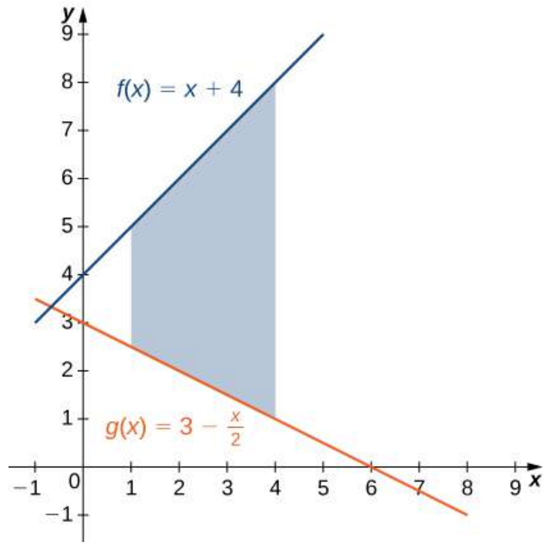


Figure 6.4 A region between two curves is shown where one curve is always greater than the other.

We have

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx \\ &= \int_1^4 \left[(x+4) - \left(3 - \frac{x}{2}\right)\right]dx = \int_1^4 \left[\frac{3x}{2} + 1\right]dx \\ &= \left[\frac{3x^2}{4} + x\right]\Big|_1^4 = \left(16 - \frac{7}{4}\right) = \frac{57}{4}. \end{aligned}$$

The area of the region is $\frac{57}{4}$ units^2.

6.1 If R is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region R.

In **Example 6.1**, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

Example 6.2

Finding the Area of a Region between Two Curves 2

If R is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R.

Solution

The region is depicted in the following figure.

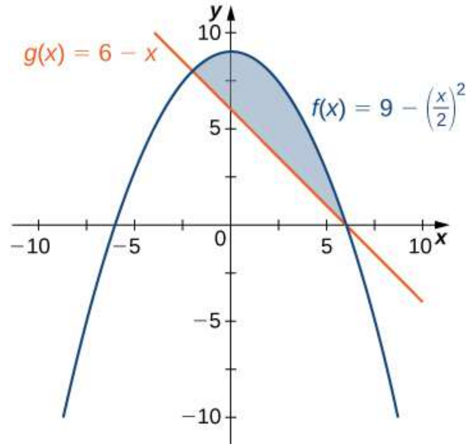


Figure 6.5 This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$\begin{aligned} f(x) &= g(x) \\ 9 - \left(\frac{x}{2}\right)^2 &= 6 - x \\ 9 - \frac{x^2}{4} &= 6 - x \\ 36 - x^2 &= 24 - 4x \\ x^2 - 4x - 12 &= 0 \\ (x-6)(x+2) &= 0. \end{aligned}$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^{6} \left[9 - \left(\frac{x}{2}\right)^2 - (6 - x)\right]dx = \int_{-2}^{6}\left[3 - \frac{x^2}{4} + x\right]dx \\ &= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Bigg|_{-2}^{6} = \frac{64}{3}. \end{aligned}$$

The area of the region is $64/3$ units^2.

6.2 If R is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region R.

Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

Theorem 6.2: Finding the Area of a Region between Curves That Cross

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let R denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b |f(x) - g(x)| dx.$$

In practice, applying this theorem requires us to break up the interval $[a, b]$ and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

Example 6.3

Finding the Area of a Region Bounded by Functions That Cross

If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region R.

Solution

The region is depicted in the following figure.

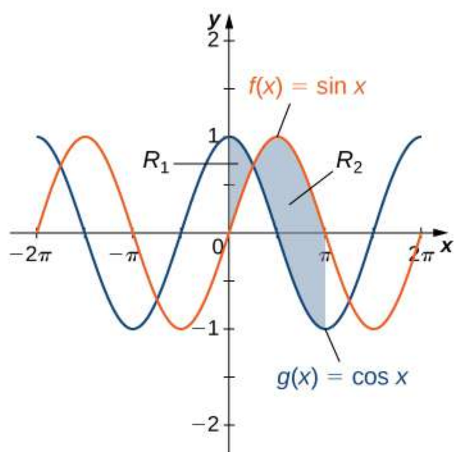


Figure 6.6 The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4]$, $\cos x \geq \sin x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x.$$

On the other hand, for $x \in [\pi/4, \pi]$, $\sin x \geq \cos x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then

$$\begin{aligned} A &= \int_a^b |f(x) - g(x)| dx \\ &= \int_0^{\pi} |\sin x - \cos x| dx = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\ &= [\sin x + \cos x] \Big|_0^{\pi/4} + [-\cos x - \sin x] \Big|_{\pi/4}^{\pi} \\ &= (\sqrt{2} - 1) + \left(1 + \sqrt{2}\right) = 2\sqrt{2}. \end{aligned}$$

The area of the region is $2\sqrt{2}$ units^2.

6.3 If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[\pi/2, 2\pi]$, find the area of region R.

Example 6.4

Finding the Area of a Complex Region

Consider the region depicted in **Figure 6.7**. Find the area of R.

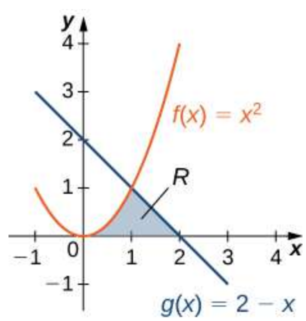


Figure 6.7 Two integrals are required to calculate the area of this region.

Solution

As with **Example 6.3**, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for *x*), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the *x*-axis, so we have

$$A_1 = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the x-axis, so we have

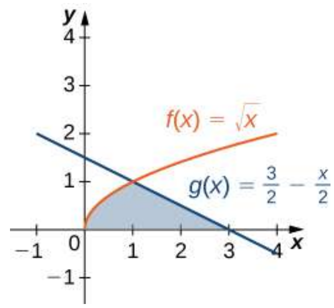
$$A_2 = \int_1^2 (2 - x)dx = \left[2x - \frac{x^2}{2}\right]\bigg|_1^2 = \frac{1}{2}.$$

Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $5/6$ units^2.

6.4 Consider the region depicted in the following figure. Find the area of R.



Regions Defined with Respect to *y*

In **Example 6.4**, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y, instead of as functions of x? Review **Figure 6.7**. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of y. However, based on the graph, it is clear we are interested in the positive square root.) Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of y, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y, we need to evaluate one integral only. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in the following figure.

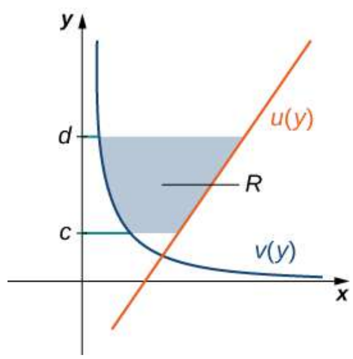


Figure 6.8 We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the y-axis and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2,\ldots, n,$ let $Q = \{y_i\}$ be a regular partition of $[c, d]$. Then, for $i = 1, 2,\ldots, n,$ choose a point $y_i^* \in [y_{i-1}, y_i],$ then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y_i^*)$ to $u(y_i^*)$. **Figure 6.9**(a) shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10.$ **Figure 6.9**(b) shows a representative rectangle in detail.

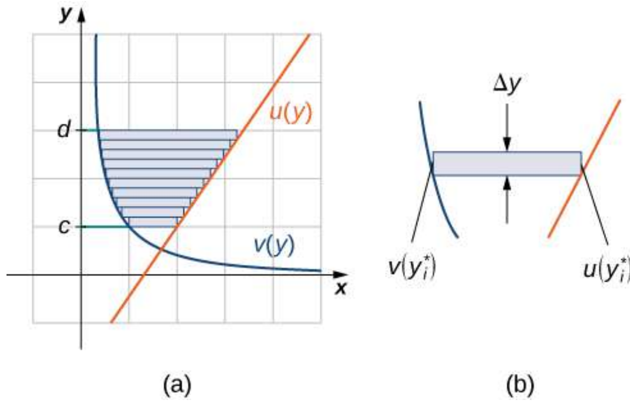


Figure 6.9 (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y),$ with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is Δy and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y.$$

This is a Riemann sum, so we take the limit as $n \to \infty,$ obtaining

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y = \int_c^d [u(y) - v(y)]dy.$$

These findings are summarized in the following theorem.

Theorem 6.3: Finding the Area between Two Curves, Integrating along the *y*-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let R denote the region bounded on the right by the graph of $u(y),$ on the left by the graph of $v(y),$ and above and below by the lines $y = d$ and $y = c,$ respectively. Then, the area of R is given by

$$A = \int_c^d [u(y) - v(y)]dy. \tag{6.2}$$

Example 6.5

Integrating with Respect to *y*

Let's revisit **Example 6.4**, only this time let's integrate with respect to y. Let R be the region depicted in **Figure 6.10**. Find the area of R by integrating with respect to y.

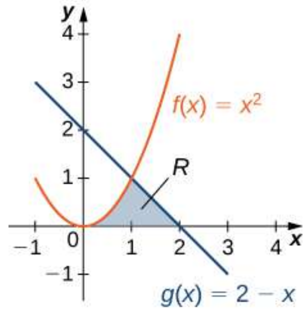


Figure 6.10 The area of region R can be calculated using one integral only when the curves are treated as functions of y.

Solution

We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

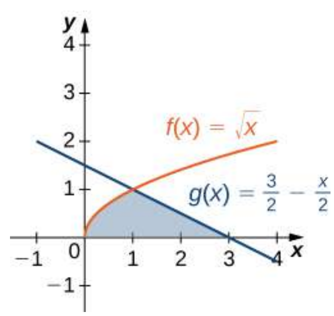
Now we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.

Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y)]dy \\ &= \int_0^1 [(2 - y) - \sqrt{y}]dy = \left[2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\right]\Big|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

The area of the region is $5/6$ units^2.

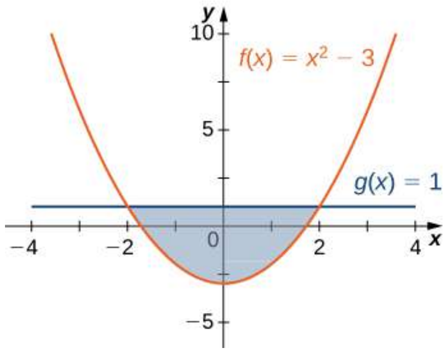
6.5 Let's revisit the checkpoint associated with **Example 6.4**, only this time, let's integrate with respect to y. Let R be the region depicted in the following figure. Find the area of R by integrating with respect to y.



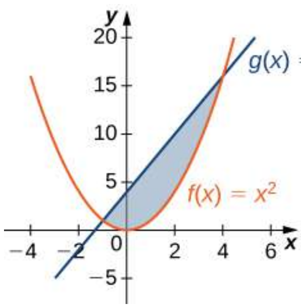
6.1 EXERCISES

For the following exercises, determine the area of the region between the two curves in the given figure by integrating over the x-axis.

1. $y = x^2 - 3$ and $y = 1$

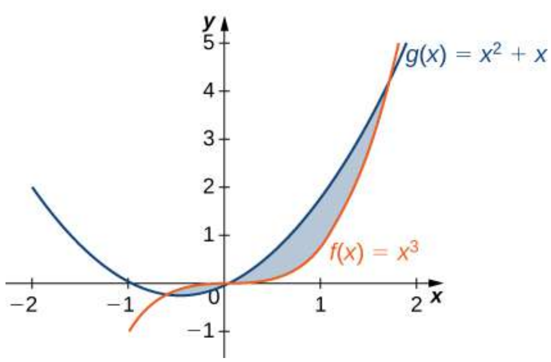


2. $y = x^2$ and $y = 3x + 4$

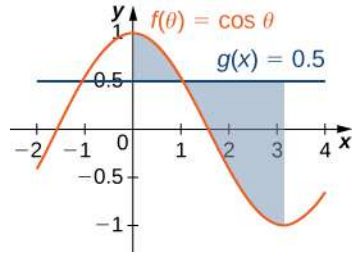


For the following exercises, split the region between the two curves into two smaller regions, then determine the area by integrating over the x-axis. Note that you will have two integrals to solve.

3. $y = x^3$ and $y = x^2 + x$

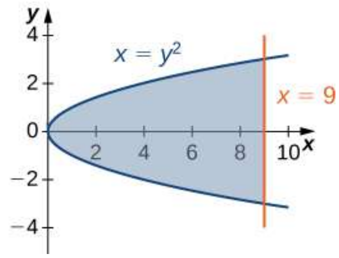


4. $y = \cos\theta$ and $y = 0.5$, for $0 \le \theta \le \pi$

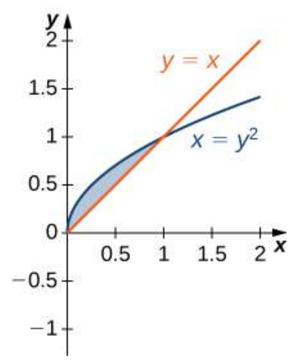


For the following exercises, determine the area of the region between the two curves by integrating over the y-axis.

5. $x = y^2$ and $x = 9$



6. $y = x$ and $x = y^2$



For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis.

7. $y = x^2$ and $y = -x^2 + 18x$

8. $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and $x = 3$

9. $y = \cos x$ and $y = \cos^2 x$ on $x = [-\pi, \pi]$

10. $y = e^x$, $y = e^{2x-1}$, and $x = 0$

11. $y = e^x$, $y = e^{-x}$, $x = -1$ and $x = 1$

12. $y = e,\ y = e^x,$ and $y = e^{-x}$

13. $y = |x|$ and $y = x^2$

For the following exercises, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

14. $y = \sin(\pi x),\ y = 2x,$ and $x > 0$

15. $y = 12 - x,\ y = \sqrt{x},$ and $y = 1$

16. $y = \sin x$ and $y = \cos x$ over $x = [-\pi, \pi]$

17. $y = x^3$ and $y = x^2 - 2x$ over $x = [-1, 1]$

18. $y = x^2 + 9$ and $y = 10 + 2x$ over $x = [-1, 3]$

19. $y = x^3 + 3x$ and $y = 4x$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the y-axis.

20. $x = y^3$ and $x = 3y - 2$

21. $x = 2y$ and $x = y^3 - y$

22. $x = -3 + y^2$ and $x = y - y^2$

23. $y^2 = x$ and $x = y + 2$

24. $x = |y|$ and $2x = -y^2 + 2$

25. $x = \sin y,\ x = \cos(2y),\ y = \pi/2,$ and $y = -\pi/2$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis or y-axis, whichever seems more convenient.

26. $x = y^4$ and $x = y^5$

27. $y = xe^x,\ y = e^x,\ x = 0,$ and $x = 1$

28. $y = x^6$ and $y = x^4$

29. $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$

30. $y = |x|$ and $y = x^2 - 1$

31. $y = 4 - 3x$ and $y = \frac{1}{x}$

32. $y = \sin x,\ x = -\pi/6,\ x = \pi/6,$ and $y = \cos^3 x$

33. $y = x^2 - 3x + 2$ and $y = x^3 - 2x^2 - x + 2$

34. $y = 2\cos^3(3x),\ y = -1,\ x = \frac{\pi}{4},$ and $x = -\frac{\pi}{4}$

35. $y + y^3 = x$ and $2y = x$

36. $y = \sqrt{1 - x^2}$ and $y = x^2 - 1$

37. $y = \cos^{-1} x,\ y = \sin^{-1} x,\ x = -1,$ and $x = 1$

For the following exercises, find the exact area of the region bounded by the given equations if possible. If you are unable to determine the intersection points analytically, use a calculator to approximate the intersection points with three decimal places and determine the approximate area of the region.

38. **[T]** $x = e^y$ and $y = x - 2$

39. **[T]** $y = x^2$ and $y = \sqrt{1 - x^2}$

40. **[T]** $y = 3x^2 + 8x + 9$ and $3y = x + 24$

41. **[T]** $x = \sqrt{4 - y^2}$ and $y^2 = 1 + x^2$

42. **[T]** $x^2 = y^3$ and $x = 3y$

43. **[T]** $y = \sin^3 x + 2,\ y = \tan x,\ x = -1.5,$ and $x = 1.5$

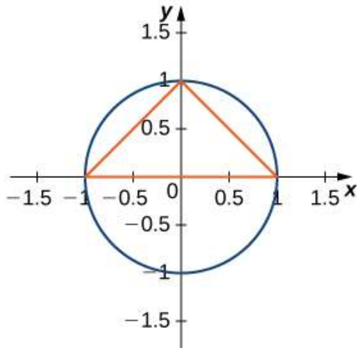
44. **[T]** $y = \sqrt{1 - x^2}$ and $y^2 = x^2$

45. **[T]** $y = \sqrt{1 - x^2}$ and $y = x^2 + 2x + 1$

46. **[T]** $x = 4 - y^2$ and $x = 1 + 3y + y^2$

47. **[T]** $y = \cos x,\ y = e^x,\ x = -\pi,$ and $x = 0$

48. The largest triangle with a base on the x-axis that fits inside the upper half of the unit circle $y^2 + x^2 = 1$ is given by $y = 1 + x$ and $y = 1 - x$. See the following figure. What is the area inside the semicircle but outside the triangle?



49. A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

50. An amusement park has a marginal cost function $C(x) = 1000e^{-x} + 5$, where x represents the number of tickets sold, and a marginal revenue function given by $R(x) = 60 - 0.1x$. Find the total profit generated when selling 550 tickets. Use a calculator to determine intersection points, if necessary, to two decimal places.

51. The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = 1 - \cos((\pi t)/2)$ whereas the speed of the tortoise is $T(t) = (1/2)\tan^{-1}(t/4)$, where t is time measured in hours and the speed is measured in miles per hour. Find the area between the curves from time $t = 0$ to the first time after one hour when the tortoise and hare are traveling at the same speed. What does it represent? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

52. The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = (1/2) - (1/2)\cos(2\pi t)$ whereas the speed of the tortoise is $T(t) = \sqrt{t}$, where t is time measured in hours and speed is measured in kilometers per hour. If the race is over in 1 hour, who won the race and by how much? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

For the following exercises, find the area between the curves by integrating with respect to x and then with respect to y. Is one method easier than the other? Do you obtain the same answer?

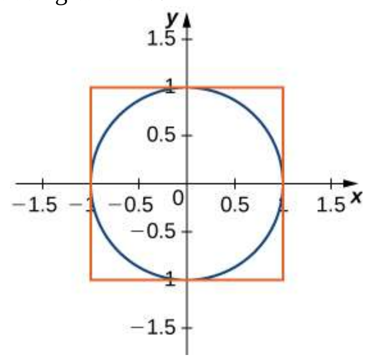
53. $y = x^2 + 2x + 1$ and $y = -x^2 - 3x + 4$

54. $y = x^4$ and $x = y^5$

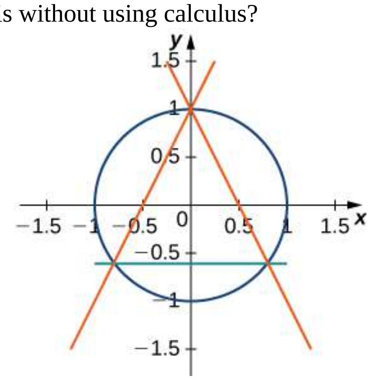
55. $x = y^2 - 2$ and $x = 2y$

For the following exercises, solve using calculus, then check your answer with geometry.

56. Determine the equations for the sides of the square that touches the unit circle on all four sides, as seen in the following figure. Find the area between the perimeter of this square and the unit circle. Is there another way to solve this without using calculus?



57. Find the area between the perimeter of the unit circle and the triangle created from $y = 2x + 1$, $y = 1 - 2x$ and $y = -\frac{3}{5}$, as seen in the following figure. Is there a way to solve this without using calculus?



6.2 | Determining Volumes by Slicing

Learning Objectives

6.2.1 Determine the volume of a solid by integrating a cross-section (the slicing method).
6.2.2 Find the volume of a solid of revolution using the disk method.
6.2.3 Find the volume of a solid of revolution with a cavity using the washer method.

In the preceding section, we used definite integrals to find the area between two curves. In this section, we use definite integrals to find volumes of three-dimensional solids. We consider three approaches—slicing, disks, and washers—for finding these volumes, depending on the characteristics of the solid.

Volume and the Slicing Method

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas. The volume of a rectangular solid, for example, can be computed by multiplying length, width, and height: $V = lwh$. The formulas for the volume of a sphere $\left(V = \frac{4}{3}\pi r^3\right)$, a cone $\left(V = \frac{1}{3}\pi r^2 h\right)$, and a pyramid $\left(V = \frac{1}{3}Ah\right)$ have also been introduced. Although some of these formulas were derived using geometry alone, all these formulas can be obtained by using integration.

We can also calculate the volume of a cylinder. Although most of us think of a cylinder as having a circular base, such as a soup can or a metal rod, in mathematics the word *cylinder* has a more general meaning. To discuss cylinders in this more general context, we first need to define some vocabulary.

We define the **cross-section** of a solid to be the intersection of a plane with the solid. A *cylinder* is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the *axis* of the cylinder. Thus, all cross-sections perpendicular to the axis of a cylinder are identical. The solid shown in **Figure 6.11** is an example of a cylinder with a noncircular base. To calculate the volume of a cylinder, then, we simply multiply the area of the cross-section by the height of the cylinder: $V = A \cdot h$. In the case of a right circular cylinder (soup can), this becomes $V = \pi r^2 h$.

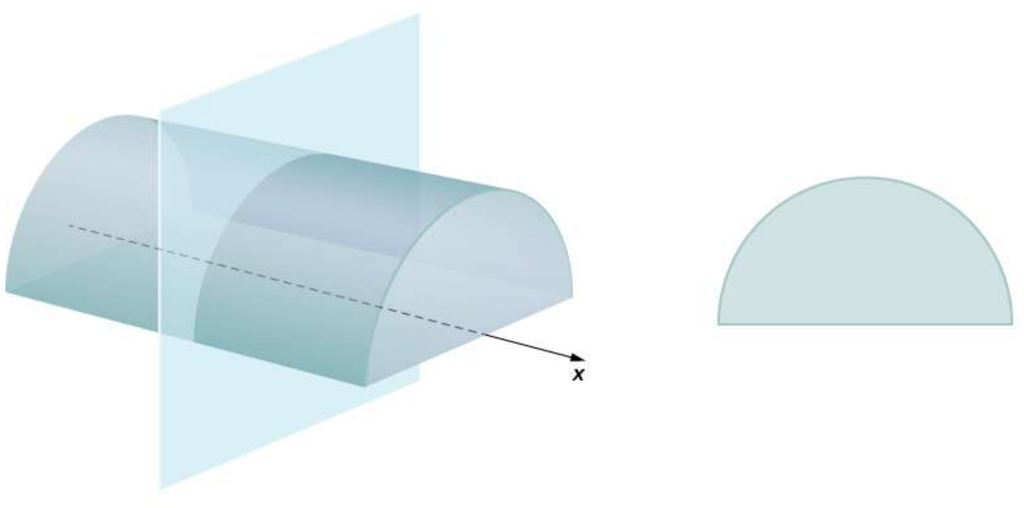


Figure 6.11 Each cross-section of a particular cylinder is identical to the others.

If a solid does not have a constant cross-section (and it is not one of the other basic solids), we may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid. We do this by slicing the solid into pieces, estimating the volume of each slice, and then adding those estimated volumes together. The slices should all be parallel to one another, and when we put all the slices together, we should get the whole solid. Consider, for example, the solid S shown in **Figure 6.12**, extending along the x-axis.

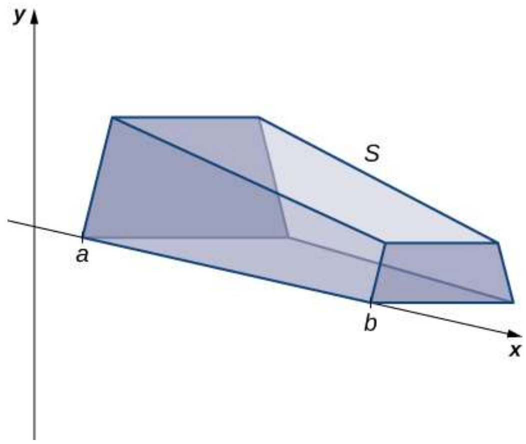


Figure 6.12 A solid with a varying cross-section.

We want to divide S into slices perpendicular to the x-axis. As we see later in the chapter, there may be times when we want to slice the solid in some other direction—say, with slices perpendicular to the y-axis. The decision of which way to slice the solid is very important. If we make the wrong choice, the computations can get quite messy. Later in the chapter, we examine some of these situations in detail and look at how to decide which way to slice the solid. For the purposes of this section, however, we use slices perpendicular to the x-axis.

Because the cross-sectional area is not constant, we let $A(x)$ represent the area of the cross-section at point x. Now let $P = \{x_0, x_1 \ldots, X_n\}$ be a regular partition of $[a, b]$, and for $i = 1, 2,\ldots n,$ let S_i represent the slice of S stretching from x_{i-1} to x_i. The following figure shows the sliced solid with $n = 3$.

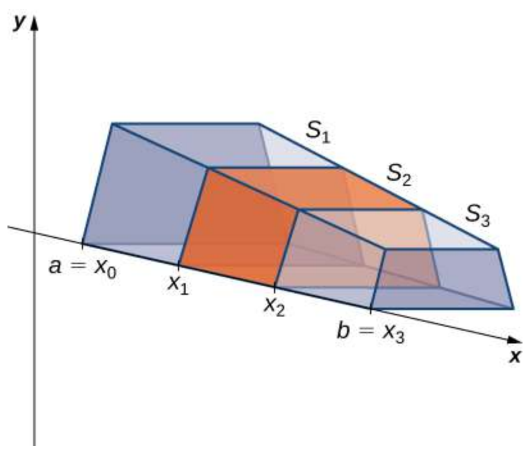


Figure 6.13 The solid S has been divided into three slices perpendicular to the x-axis.

Finally, for $i = 1, 2,\ldots n,$ let x_i^* be an arbitrary point in $[x_{i-1}, x_i]$. Then the volume of slice S_i can be estimated by $V(S_i) \approx A(x_i^*)\Delta x$. Adding these approximations together, we see the volume of the entire solid S can be approximated by

$$V(S) \approx \sum_{i=1}^{n} A(x_i^*)\Delta x.$$

By now, we can recognize this as a Riemann sum, and our next step is to take the limit as $n \to \infty$. Then we have

$$V(S) = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx.$$

The technique we have just described is called the **slicing method**. To apply it, we use the following strategy.

Problem-Solving Strategy: Finding Volumes by the Slicing Method

1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
2. Determine a formula for the area of the cross-section.
3. Integrate the area formula over the appropriate interval to get the volume.

Recall that in this section, we assume the slices are perpendicular to the x-axis. Therefore, the area formula is in terms of x and the limits of integration lie on the x-axis. However, the problem-solving strategy shown here is valid regardless of how we choose to slice the solid.

Example 6.6

Deriving the Formula for the Volume of a Pyramid

We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah.$ If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2 h,$ where a denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

Solution

We want to apply the slicing method to a pyramid with a square base. To set up the integral, consider the pyramid shown in Figure 6.14, oriented along the x-axis.

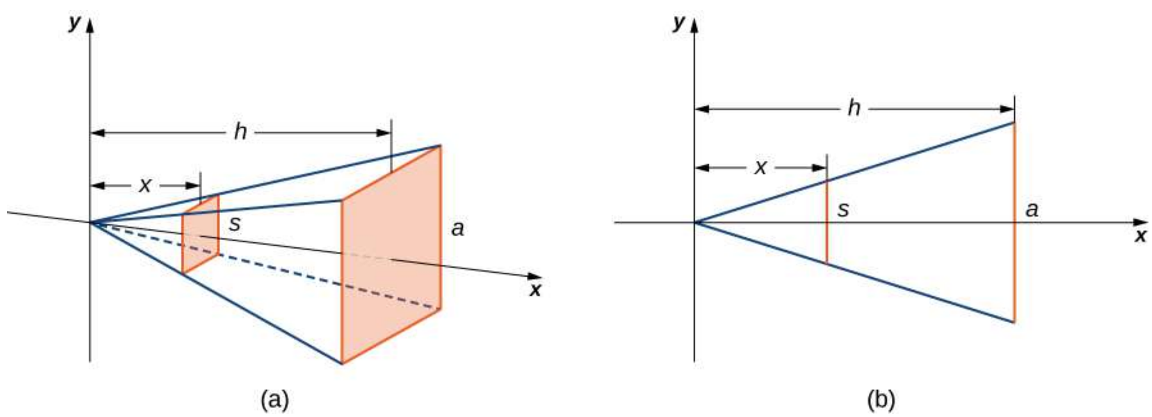


Figure 6.14 (a) A pyramid with a square base is oriented along the x-axis. (b) A two-dimensional view of the pyramid is seen from the side.

We first want to determine the shape of a cross-section of the pyramid. We know the base is a square, so the cross-sections are squares as well (step 1). Now we want to determine a formula for the area of one of these cross-sectional squares. Looking at Figure 6.14(b), and using a proportion, since these are similar triangles, we have

$$\frac{s}{a} = \frac{x}{h} \text{ or } s = \frac{ax}{h}.$$

Therefore, the area of one of the cross-sectional squares is

$$A(x) = s^2 = \left(\frac{ax}{h}\right)^2 \text{(step 2)}.$$

Then we find the volume of the pyramid by integrating from 0 to h (step 3):

$$\begin{aligned} V &= \int_0^h A(x)dx \\ &= \int_0^h \left(\frac{ax}{h}\right)^2 dx = \frac{a^2}{h^2}\int_0^h x^2 dx \\ &= \left[\frac{a^2}{h^2}\left(\frac{1}{3}x^3\right)\right]\Bigg|_0^h = \frac{1}{3}a^2 h. \end{aligned}$$

This is the formula we were looking for.

6.6 Use the slicing method to derive the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a circular cone.

Solids of Revolution

If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**, as shown in the following figure.

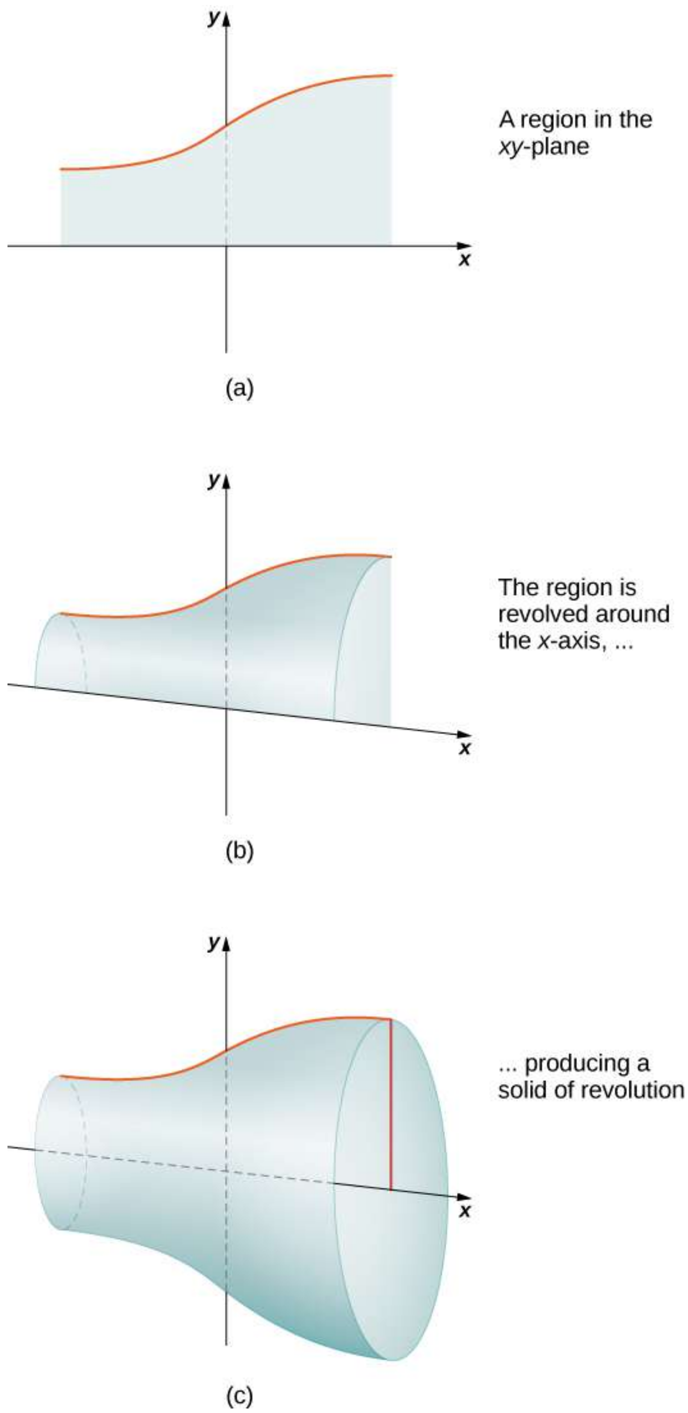


Figure 6.15 (a) This is the region that is revolved around the *x*-axis. (b) As the region begins to revolve around the axis, it sweeps out a solid of revolution. (c) This is the solid that results when the revolution is complete.

Solids of revolution are common in mechanical applications, such as machine parts produced by a lathe. We spend the rest of this section looking at solids of this type. The next example uses the slicing method to calculate the volume of a solid of revolution.

Use an online **integral calculator (http://www.openstax.org/l/20_IntCalc2)** to learn more.

Example 6.7

Using the Slicing Method to find the Volume of a Solid of Revolution

Use the slicing method to find the volume of the solid of revolution bounded by the graphs of $f(x) = x^2 - 4x + 5,\ x = 1,$ and $x = 4,$ and rotated about the x-axis.

Solution

Using the problem-solving strategy, we first sketch the graph of the quadratic function over the interval $[1, 4]$ as shown in the following figure.

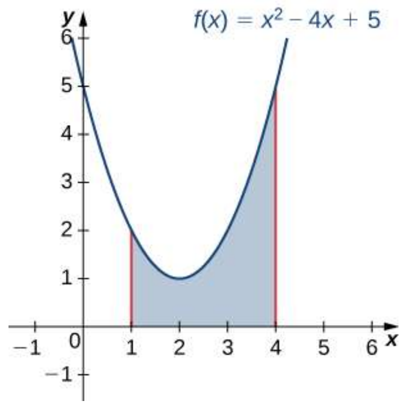


Figure 6.16 A region used to produce a solid of revolution.

Next, revolve the region around the x-axis, as shown in the following figure.

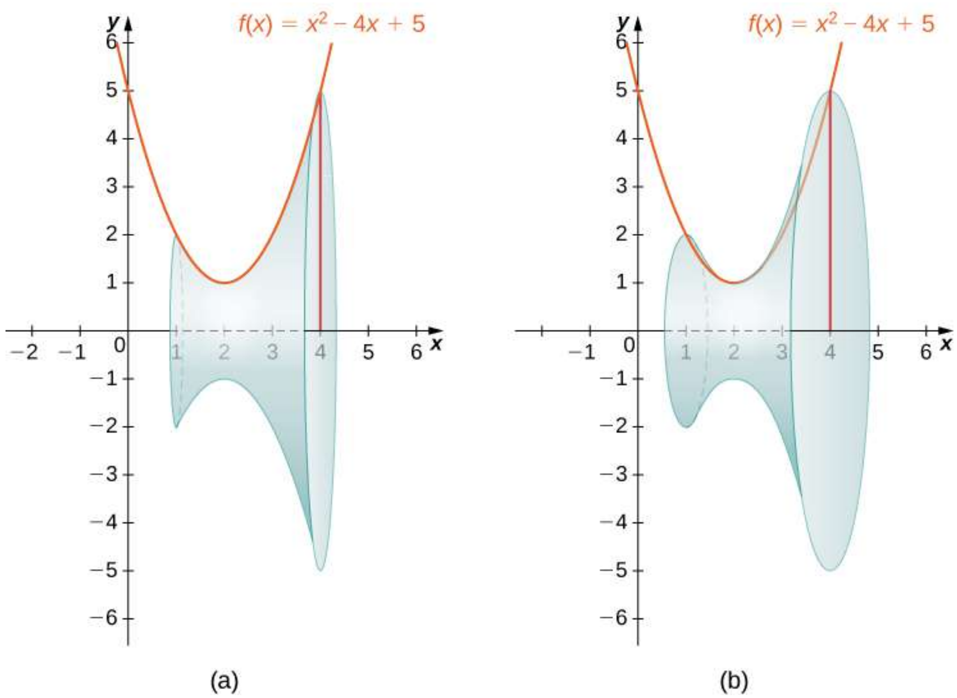


Figure 6.17 Two views, (a) and (b), of the solid of revolution produced by revolving the region in **Figure 6.16** about the x-axis.

Since the solid was formed by revolving the region around the x-axis, the cross-sections are circles (step 1). The area of the cross-section, then, is the area of a circle, and the radius of the circle is given by $f(x)$. Use the formula for the area of the circle:

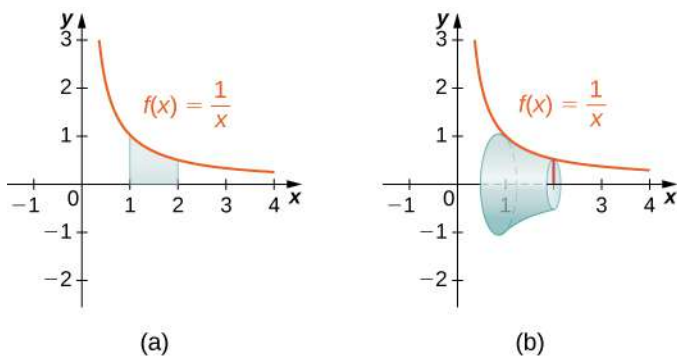
$$A(x) = \pi r^2 = \pi[f(x)]^2 = \pi\left(x^2 - 4x + 5\right)^2 \text{ (step 2).}$$

The volume, then, is (step 3)

$$\begin{aligned} V &= \int_a^b A(x)dx \\ &= \int_1^4 \pi\left(x^2 - 4x + 5\right)^2 dx = \pi\int_1^4 \left(x^4 - 8x^3 + 26x^2 - 40x + 25\right)dx \\ &= \pi\left(\frac{x^5}{5} - 2x^4 + \frac{26x^3}{3} - 20x^2 + 25x\right)\Big|_1^4 = \frac{78}{5}\pi. \end{aligned}$$

The volume is $78\pi/5$.

6.7 Use the method of slicing to find the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = 1/x$ and the x-axis over the interval $[1, 2]$ around the x-axis. See the following figure.



The Disk Method

When we use the slicing method with solids of revolution, it is often called the **disk method** because, for solids of revolution, the slices used to over approximate the volume of the solid are disks. To see this, consider the solid of revolution generated by revolving the region between the graph of the function $f(x) = (x - 1)^2 + 1$ and the x-axis over the interval $[-1, 3]$ around the x-axis. The graph of the function and a representative disk are shown in Figure 6.18(a) and (b). The region of revolution and the resulting solid are shown in Figure 6.18(c) and (d).

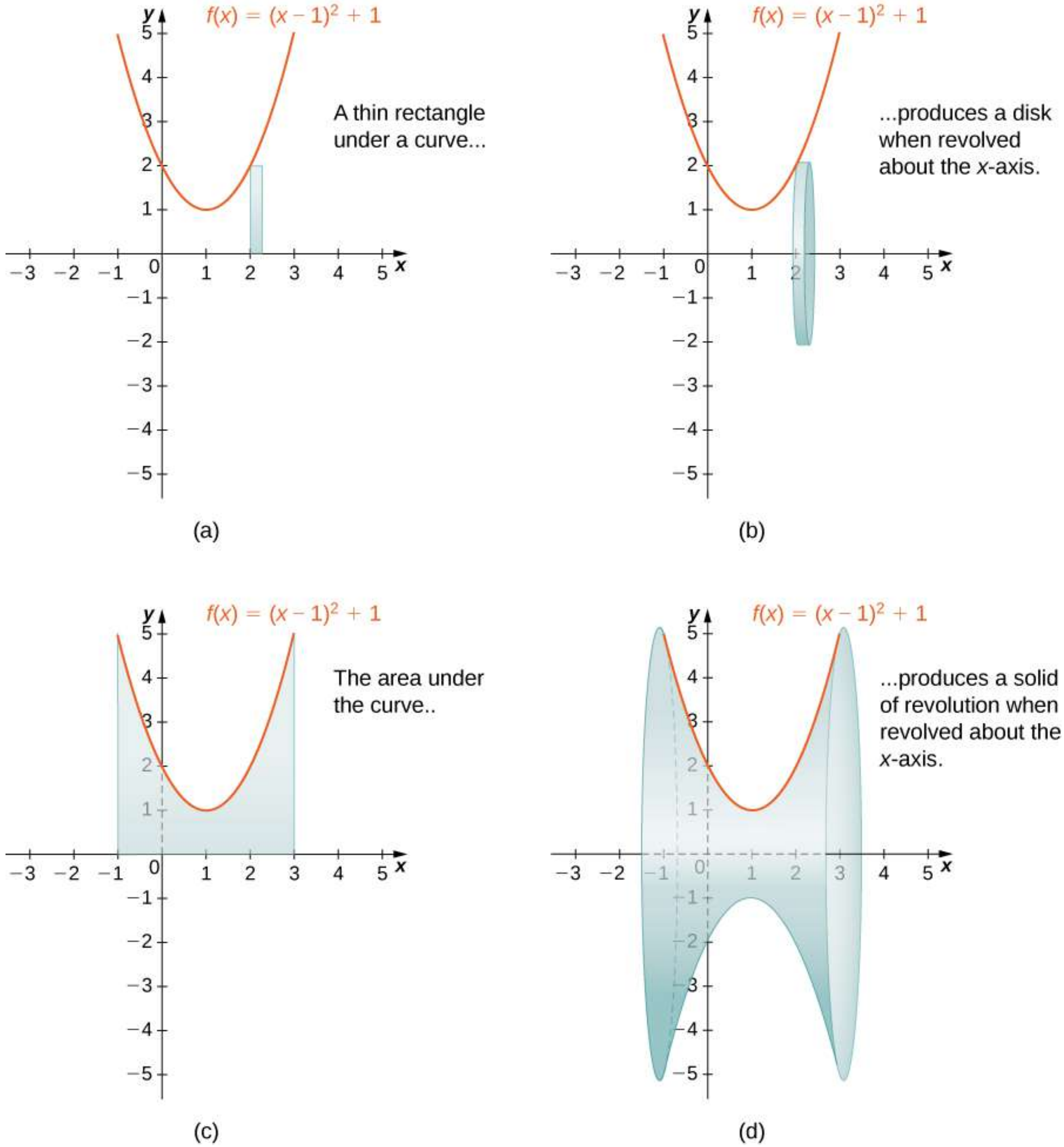


Figure 6.18 (a) A thin rectangle for approximating the area under a curve. (b) A representative disk formed by revolving the rectangle about the x-axis. (c) The region under the curve is revolved about the x-axis, resulting in (d) the solid of revolution.

We already used the formal Riemann sum development of the volume formula when we developed the slicing method. We know that

$$V = \int_a^b A(x)dx.$$

The only difference with the disk method is that we know the formula for the cross-sectional area ahead of time; it is the area of a circle. This gives the following rule.

Rule: The Disk Method

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the

x-axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi[f(x)]^2\,dx. \tag{6.3}$$

The volume of the solid we have been studying (**Figure 6.18**) is given by

$$\begin{aligned} V &= \int_a^b \pi[f(x)]^2\,dx \\ &= \int_{-1}^{3} \pi\left[(x-1)^2+1\right]^2 dx = \pi\int_{-1}^{3}\left[(x-1)^4 + 2(x-1)^2 + 1\right]dx \\ &= \pi\left[\frac{1}{5}(x-1)^5 + \frac{2}{3}(x-1)^3 + x\right]\Big|_{-1}^{3} = \pi\left[\left(\frac{32}{5} + \frac{16}{3} + 3\right) - \left(-\frac{32}{5} - \frac{16}{3} - 1\right)\right] = \frac{412\pi}{15}\text{ units}^3. \end{aligned}$$

Let's look at some examples.

Example 6.8

Using the Disk Method to Find the Volume of a Solid of Revolution 1

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{x}$ and the x-axis over the interval $[1, 4]$ around the x-axis.

Solution

The graphs of the function and the solid of revolution are shown in the following figure.

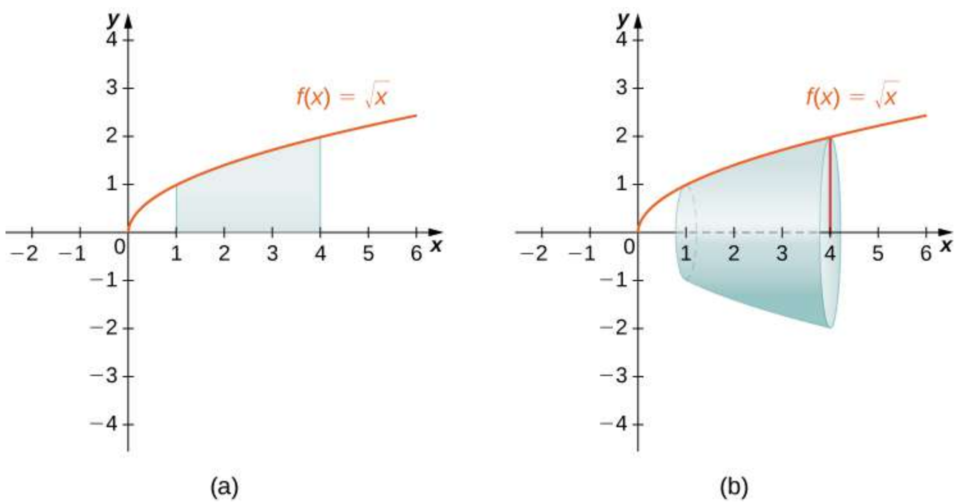


Figure 6.19 (a) The function $f(x) = \sqrt{x}$ over the interval $[1, 4]$. (b) The solid of revolution obtained by revolving the region under the graph of $f(x)$ about the x-axis.

We have

$$\begin{aligned} V &= \int_a^b \pi[f(x)]^2\,dx \\ &= \int_1^4 \pi[\sqrt{x}]^2 dx = \pi\int_1^4 x\,dx \\ &= \frac{\pi}{2}x^2\Big|_1^4 = \frac{15\pi}{2}. \end{aligned}$$

The volume is $(15\pi)/2$ units^3.

6.8 Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{4-x}$ and the x-axis over the interval $[0,\ 4]$ around the x-axis.

So far, our examples have all concerned regions revolved around the x-axis, but we can generate a solid of revolution by revolving a plane region around any horizontal or vertical line. In the next example, we look at a solid of revolution that has been generated by revolving a region around the y-axis. The mechanics of the disk method are nearly the same as when the x-axis is the axis of revolution, but we express the function in terms of y and we integrate with respect to y as well. This is summarized in the following rule.

Rule: The Disk Method for Solids of Revolution around the *y*-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y-axis is given by

$$V = \int_c^d \pi[g(y)]^2\,dy. \tag{6.4}$$

The next example shows how this rule works in practice.

Example 6.9

Using the Disk Method to Find the Volume of a Solid of Revolution 2

Let R be the region bounded by the graph of $g(y) = \sqrt{4-y}$ and the y-axis over the y-axis interval $[0,\ 4]$. Use the disk method to find the volume of the solid of revolution generated by rotating R around the y-axis.

Solution

Figure 6.20 shows the function and a representative disk that can be used to estimate the volume. Notice that since we are revolving the function around the y-axis, the disks are horizontal, rather than vertical.

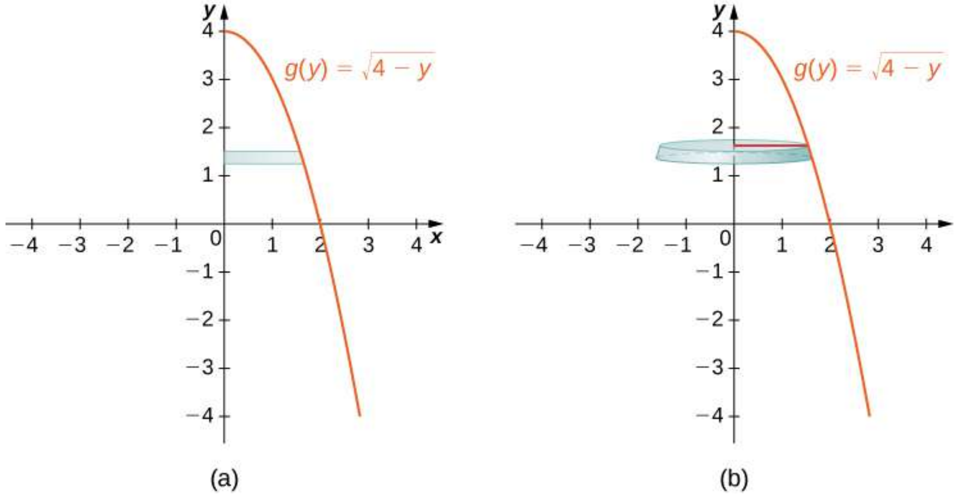


Figure 6.20 (a) Shown is a thin rectangle between the curve of the function $g(y) = \sqrt{4 - y}$ and the $y\text{-axis}$. (b) The rectangle forms a representative disk after revolution around the $y\text{-axis}$.

The region to be revolved and the full solid of revolution are depicted in the following figure.

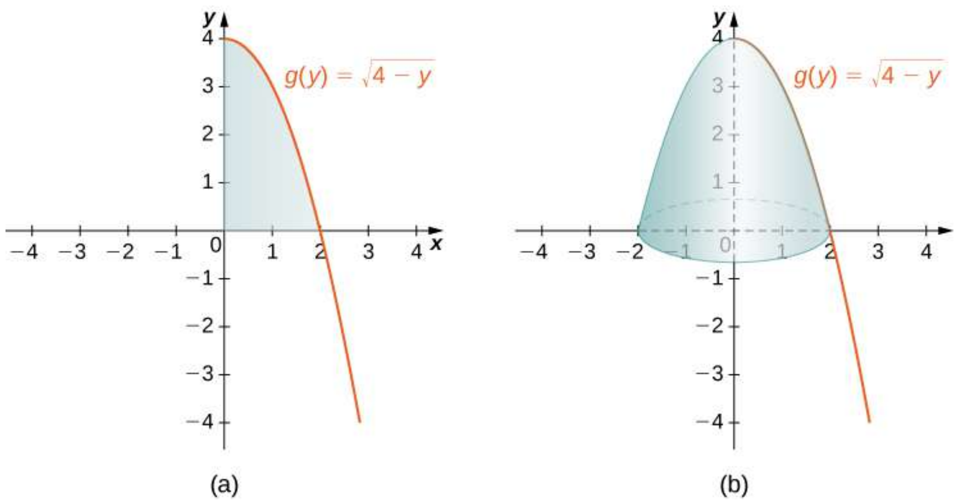


Figure 6.21 (a) The region to the left of the function $g(y) = \sqrt{4 - y}$ over the $y\text{-axis}$ interval $[0,\ 4]$. (b) The solid of revolution formed by revolving the region about the $y\text{-axis}$.

To find the volume, we integrate with respect to y. We obtain

$$\begin{aligned} V &= \int_c^d \pi[g(y)]^2\,dy \\ &= \int_0^4 \pi\left[\sqrt{4 - y}\right]^2 dy = \pi\int_0^4 (4 - y)dy \\ &= \pi\left[4y - \frac{y^2}{2}\right]\Bigg|_0^4 = 8\pi. \end{aligned}$$

The volume is 8π units^3.

6.9 Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $g(y) = y$ and the y-axis over the interval $[1, 4]$ around the y-axis.

The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happen is when an axis of revolution other than the x-axis or y-axis is selected.

When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but washers (disks with holes in the center). For example, consider the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1$ over the interval $[1, 4]$. When this region is revolved around the x-axis, the result is a solid with a cavity in the middle, and the slices are washers. The graph of the function and a representative washer are shown in **Figure 6.22**(a) and (b). The region of revolution and the resulting solid are shown in **Figure 6.22**(c) and (d).

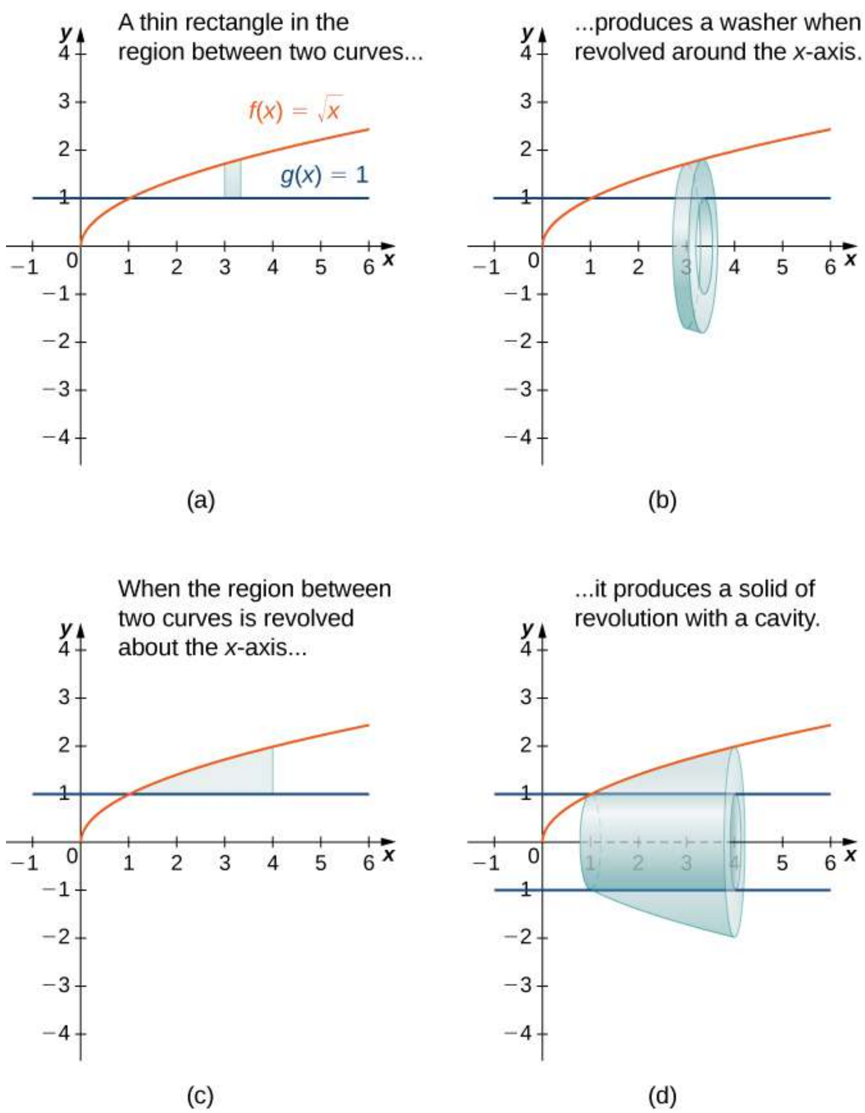


Figure 6.22 (a) A thin rectangle in the region between two curves. (b) A representative disk formed by revolving the rectangle about the x-axis. (c) The region between the curves over the given interval. (d) The resulting solid of revolution.

The cross-sectional area, then, is the area of the outer circle less the area of the inner circle. In this case,

$$A(x) = \pi(\sqrt{x})^2 - \pi(1)^2 = \pi(x - 1).$$

Then the volume of the solid is

$$\begin{aligned} V &= \int_a^b A(x)dx \\ &= \int_1^4 \pi(x - 1)dx = \pi\left[\frac{x^2}{2} - x\right]\Big|_1^4 = \frac{9}{2}\pi \text{ units}^3. \end{aligned}$$

Generalizing this process gives the **washer method**.

Rule: The Washer Method

Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx. \tag{6.5}$$

Example 6.10

Using the Washer Method

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = 1/x$ over the interval $[1, 4]$ around the x-axis.

Solution

The graphs of the functions and the solid of revolution are shown in the following figure.

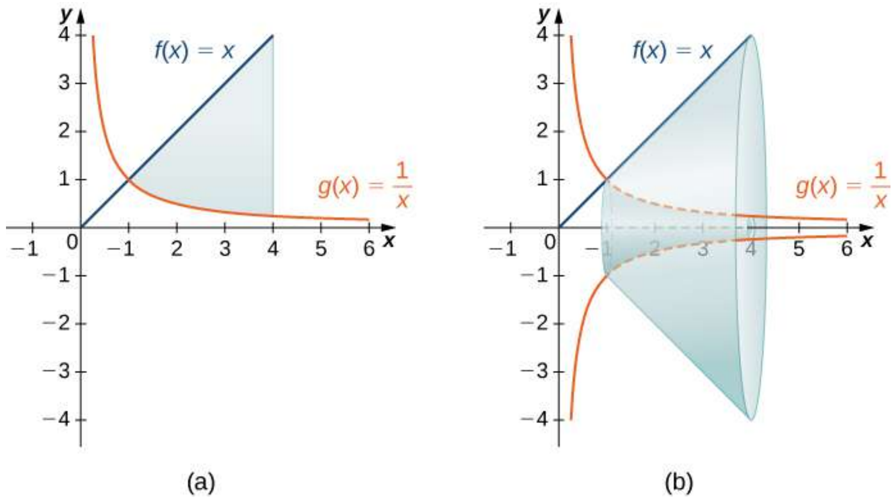


Figure 6.23 (a) The region between the graphs of the functions $f(x) = x$ and $g(x) = 1/x$ over the interval $[1, 4]$. (b) Revolving the region about the x-axis generates a solid of revolution with a cavity in the middle.

We have

$$\begin{aligned} V &= \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx \\ &= \pi\int_1^4 \left[x^2 - \left(\frac{1}{x}\right)^2\right]dx = \pi\left[\frac{x^3}{3} + \frac{1}{x}\right]\Bigg|_1^4 = \frac{81\pi}{4}\text{ units}^3. \end{aligned}$$

6.10 Find the volume of a solid of revolution formed by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = 1/x$ over the interval $[1, 3]$ around the x-axis.

As with the disk method, we can also apply the washer method to solids of revolution that result from revolving a region around the y-axis. In this case, the following rule applies.

Rule: The Washer Method for Solids of Revolution around the y-axis

Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \le u(y)$ for $y \in [c, d]$. Let Q denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y-axis is given by

$$V = \int_c^d \pi\left[(u(y))^2 - (v(y))^2\right]dy.$$

Rather than looking at an example of the washer method with the y-axis as the axis of revolution, we now consider an example in which the axis of revolution is a line other than one of the two coordinate axes. The same general method applies, but you may have to visualize just how to describe the cross-sectional area of the volume.

Example 6.11

The Washer Method with a Different Axis of Revolution

Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the x-axis over the interval $[0, 4]$ around the line $y = -2$.

Solution

The graph of the region and the solid of revolution are shown in the following figure.

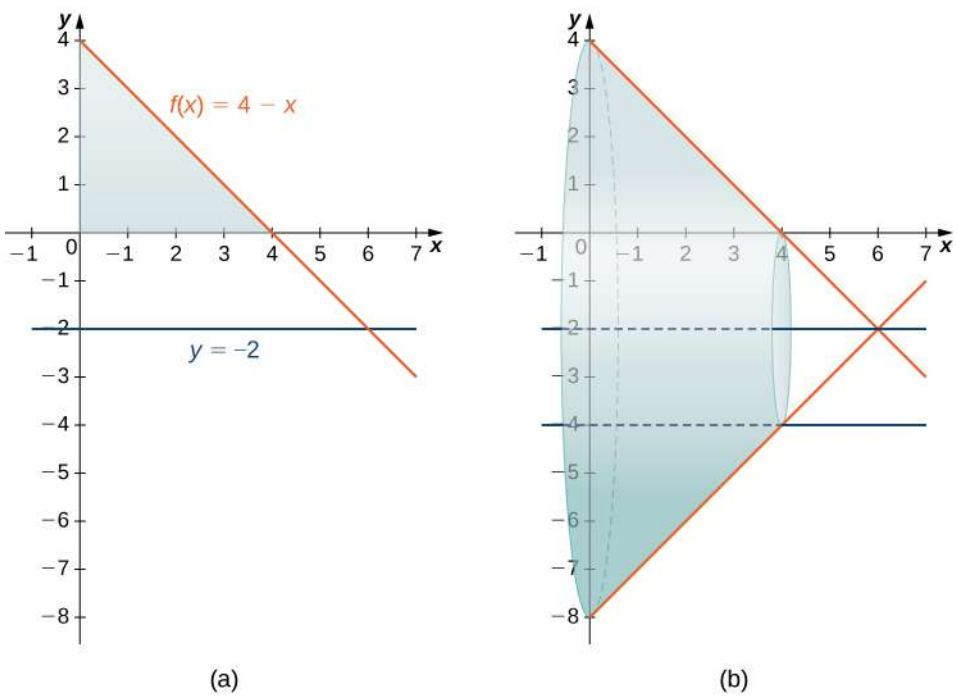


Figure 6.24 (a) The region between the graph of the function $f(x) = 4 - x$ and the x-axis over the interval $[0, 4]$. (b) Revolving the region about the line $y = -2$ generates a solid of revolution with a cylindrical hole through its middle.

We can't apply the volume formula to this problem directly because the axis of revolution is not one of the coordinate axes. However, we still know that the area of the cross-section is the area of the outer circle less the area of the inner circle. Looking at the graph of the function, we see the radius of the outer circle is given by $f(x) + 2,$ which simplifies to

$$f(x) + 2 = (4 - x) + 2 = 6 - x.$$

The radius of the inner circle is $g(x) = 2.$ Therefore, we have

$$\begin{aligned} V &= \int_0^4 \pi\left[(6 - x)^2 - (2)^2\right]dx \\ &= \pi\int_0^4 \left(x^2 - 12x + 32\right)dx = \pi\left[\frac{x^3}{3} - 6x^2 + 32x\right]\Big|_0^4 = \frac{160\pi}{3} \text{ units}^3. \end{aligned}$$

6.11 Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x + 2$ and below by the x-axis over the interval $[0, 3]$ around the line $y = -1.$

6.2 EXERCISES

58. Derive the formula for the volume of a sphere using the slicing method.

59. Use the slicing method to derive the formula for the volume of a cone.

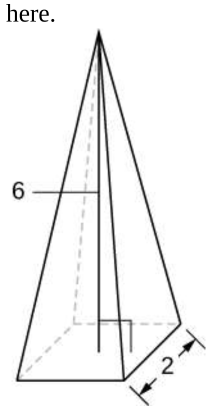
60. Use the slicing method to derive the formula for the volume of a tetrahedron with side length a.

61. Use the disk method to derive the formula for the volume of a trapezoidal cylinder.

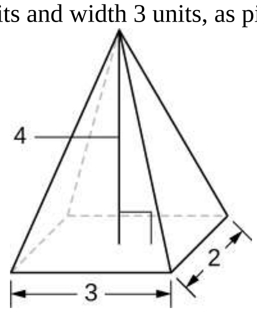
62. Explain when you would use the disk method versus the washer method. When are they interchangeable?

For the following exercises, draw a typical slice and find the volume using the slicing method for the given volume.

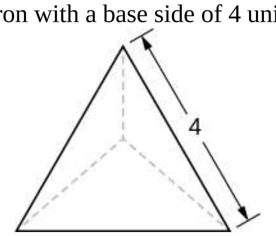
63. A pyramid with height 6 units and square base of side 2 units, as pictured here.



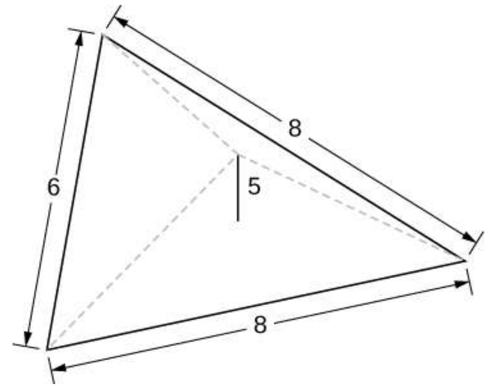
64. A pyramid with height 4 units and a rectangular base with length 2 units and width 3 units, as pictured here.



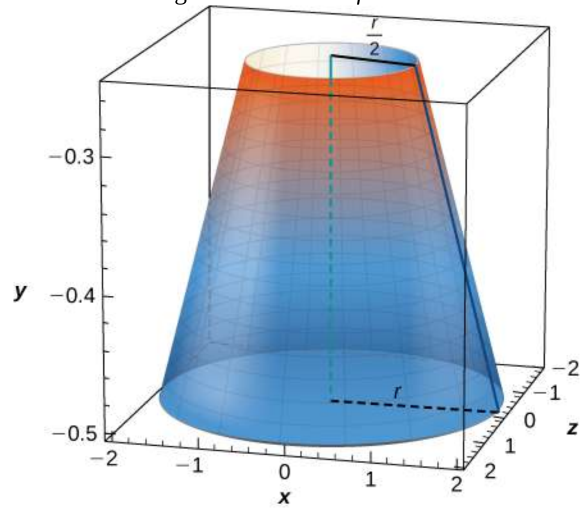
65. A tetrahedron with a base side of 4 units, as seen here.



66. A pyramid with height 5 units, and an isosceles triangular base with lengths of 6 units and 8 units, as seen here.



67. A cone of radius r and height h has a smaller cone of radius $r/2$ and height $h/2$ removed from the top, as seen here. The resulting solid is called a *frustum*.



For the following exercises, draw an outline of the solid and find the volume using the slicing method.

68. The base is a circle of radius a. The slices perpendicular to the base are squares.

69. The base is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Slices perpendicular to the x-axis are semicircles.

70. The base is the region under the parabola $y = 1 - x^2$ in the first quadrant. Slices perpendicular to the xy-plane are squares.

71. The base is the region under the parabola $y = 1 - x^2$ and above the x-axis. Slices perpendicular to the y-axis are squares.

72. The base is the region enclosed by $y = x^2$ and $y = 9$. Slices perpendicular to the x-axis are right isosceles triangles. The intersection of one of these slices and the base is the leg of the triangle.

73. The base is the area between $y = x$ and $y = x^2$. Slices perpendicular to the x-axis are semicircles.

For the following exercises, draw the region bounded by the curves. Then, use the disk method to find the volume when the region is rotated around the x-axis.

74. $x + y = 8,\ x = 0,$ and $y = 0$

75. $y = 2x^2,\ x = 0,\ x = 4,$ and $y = 0$

76. $y = e^x + 1,\ x = 0,\ x = 1,$ and $y = 0$

77. $y = x^4,\ x = 0,$ and $y = 1$

78. $y = \sqrt{x},\ x = 0,\ x = 4,$ and $y = 0$

79. $y = \sin x,\ y = \cos x,$ and $x = 0$

80. $y = \frac{1}{x},\ x = 2,$ and $y = 3$

81. $x^2 - y^2 = 9$ and $x + y = 9,\ y = 0$ and $x = 0$

For the following exercises, draw the region bounded by the curves. Then, find the volume when the region is rotated around the y-axis.

82. $y = 4 - \frac{1}{2}x,\ x = 0,$ and $y = 0$

83. $y = 2x^3,\ x = 0,\ x = 1,$ and $y = 0$

84. $y = 3x^2,\ x = 0,$ and $y = 3$

85. $y = \sqrt{4 - x^2},\ y = 0,$ and $x = 0$

86. $y = \frac{1}{\sqrt{x + 1}},\ x = 0,$ and $x = 3$

87. $x = \sec(y)$ and $y = \frac{\pi}{4},\ y = 0$ and $x = 0$

88. $y = \frac{1}{x + 1},\ x = 0,$ and $x = 2$

89. $y = 4 - x,\ y = x,$ and $x = 0$

For the following exercises, draw the region bounded by the curves. Then, find the volume when the region is rotated around the x-axis.

90. $y = x + 2,\ y = x + 6,\ x = 0,$ and $x = 5$

91. $y = x^2$ and $y = x + 2$

92. $x^2 = y^3$ and $x^3 = y^2$

93. $y = 4 - x^2$ and $y = 2 - x$

94. **[T]** $y = \cos x,\ y = e^{-x},\ x = 0,$ and $x = 1.2927$

95. $y = \sqrt{x}$ and $y = x^2$

96. $y = \sin x,\ y = 5\sin x,\ x = 0$ and $x = \pi$

97. $y = \sqrt{1 + x^2}$ and $y = \sqrt{4 - x^2}$

For the following exercises, draw the region bounded by the curves. Then, use the washer method to find the volume when the region is revolved around the y-axis.

98. $y = \sqrt{x},\ x = 4,$ and $y = 0$

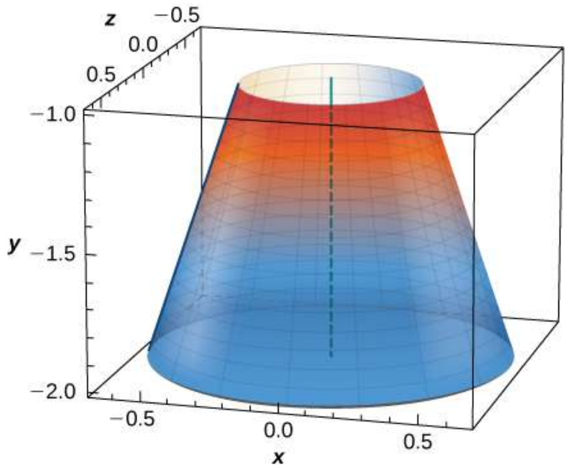
99. $y = x + 2,\ y = 2x - 1,$ and $x = 0$

100. $y = \sqrt[3]{x}$ and $y = x^3$

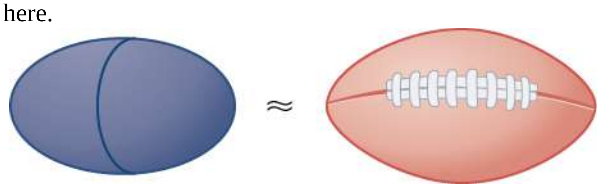
101. $x = e^{2y},\ x = y^2,\ y = 0,$ and $y = \ln(2)$

102. $x = \sqrt{9 - y^2},\ x = e^{-y},\ y = 0,$ and $y = 3$

103. Yogurt containers can be shaped like frustums. Rotate the line $y=\frac{1}{m}x$ around the y-axis to find the volume between $y=a$ and $y=b$.

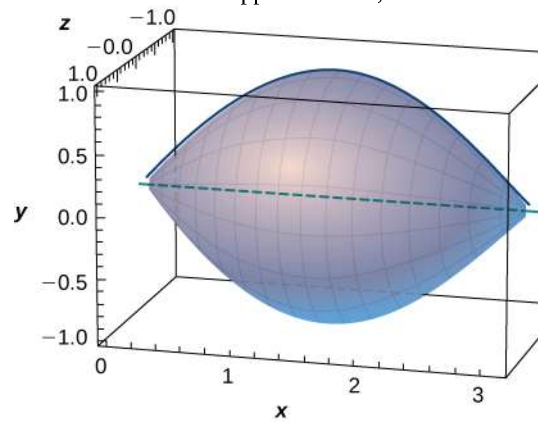


104. Rotate the ellipse $\left(x^2/a^2\right)+\left(y^2/b^2\right)=1$ around the x-axis to approximate the volume of a football, as seen here.

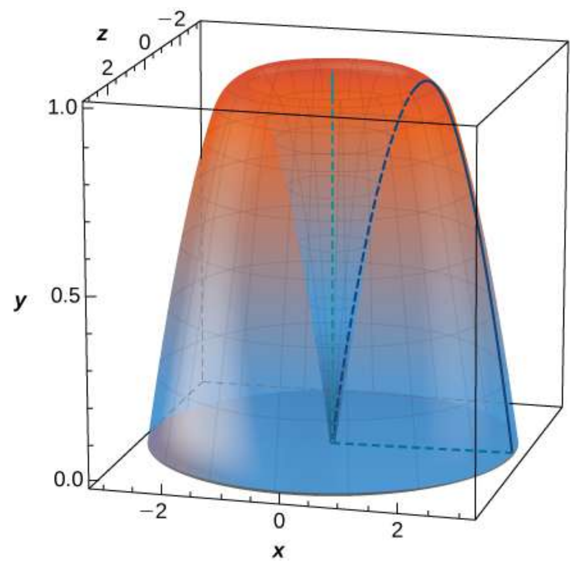


105. Rotate the ellipse $\left(x^2/a^2\right)+\left(y^2/b^2\right)=1$ around the y-axis to approximate the volume of a football.

106. A better approximation of the volume of a football is given by the solid that comes from rotating $y=\sin x$ around the x-axis from $x=0$ to $x=\pi$. What is the volume of this football approximation, as seen here?



107. What is the volume of the Bundt cake that comes from rotating $y=\sin x$ around the y-axis from $x=0$ to $x=\pi$?

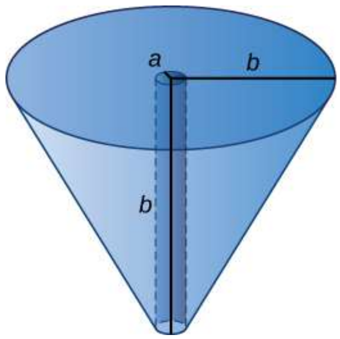


For the following exercises, find the volume of the solid described.

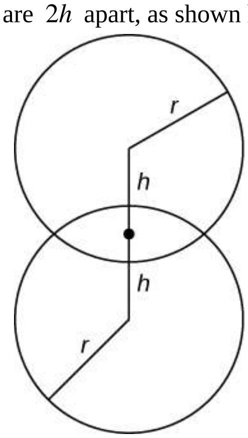
108. The base is the region between $y = x$ and $y = x^2$. Slices perpendicular to the *x*-axis are semicircles.

109. The base is the region enclosed by the generic ellipse $\left(x^2/a^2\right) + \left(y^2/b^2\right) = 1.$ Slices perpendicular to the *x*-axis are semicircles.

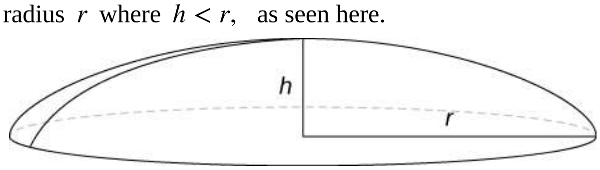
110. Bore a hole of radius a down the axis of a right cone and through the base of radius $b,$ as seen here.



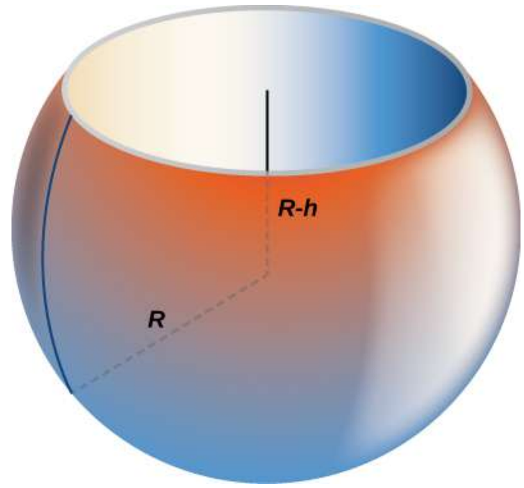
111. Find the volume common to two spheres of radius r with centers that are $2h$ apart, as shown here.



112. Find the volume of a spherical cap of height h and radius r where $h < r,$ as seen here.



113. Find the volume of a sphere of radius R with a cap of height h removed from the top, as seen here.



6.3 | Volumes of Revolution: Cylindrical Shells

Learning Objectives

6.3.1 Calculate the volume of a solid of revolution by using the method of cylindrical shells.
6.3.2 Compare the different methods for calculating a volume of revolution.

In this section, we examine the method of cylindrical shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method; however, with the disk and washer methods, we integrate along the coordinate axis parallel to the axis of revolution. With the method of cylindrical shells, we integrate along the coordinate axis *perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the washer method. In the last part of this section, we review all the methods for finding volume that we have studied and lay out some guidelines to help you determine which method to use in a given situation.

The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region R, bounded above by the graph of a function $y = f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in **Figure 6.25**(a). We then revolve this region around the y-axis, as shown in **Figure 6.25**(b). Note that this is different from what we have done before. Previously, regions defined in terms of functions of x were revolved around the x-axis or a line parallel to it.

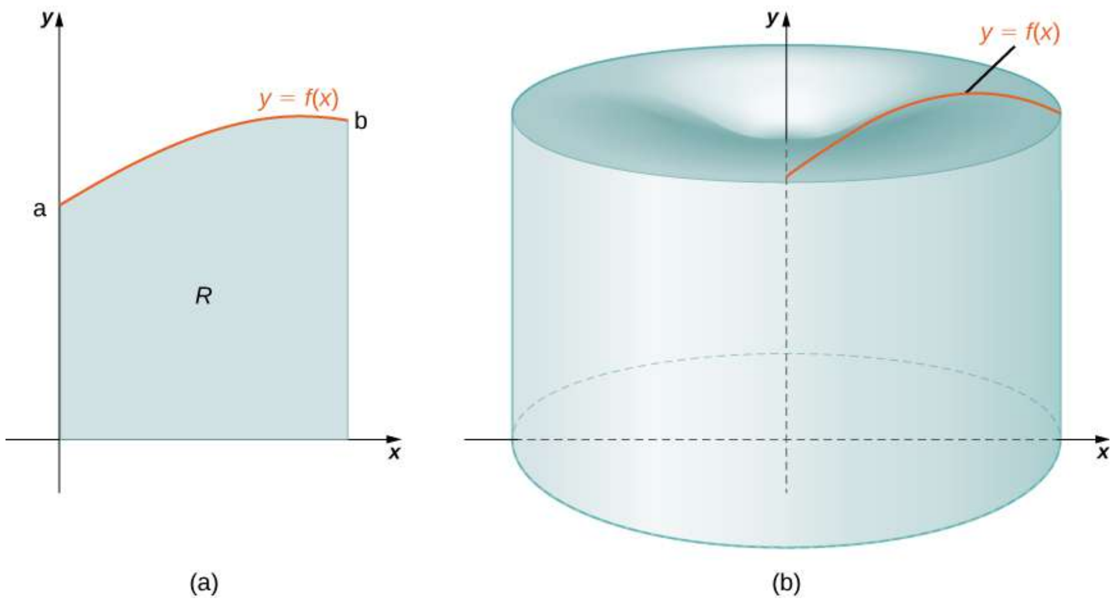


Figure 6.25 (a) A region bounded by the graph of a function of x. (b) The solid of revolution formed when the region is revolved around the y-axis.

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = \{x_0, x_1, \ldots, x_n\}$ and, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width Δx. A representative rectangle is shown in **Figure 6.26**(a). When that rectangle is revolved around the y-axis, instead of a disk or a washer, we get a cylindrical shell, as shown in the following figure.

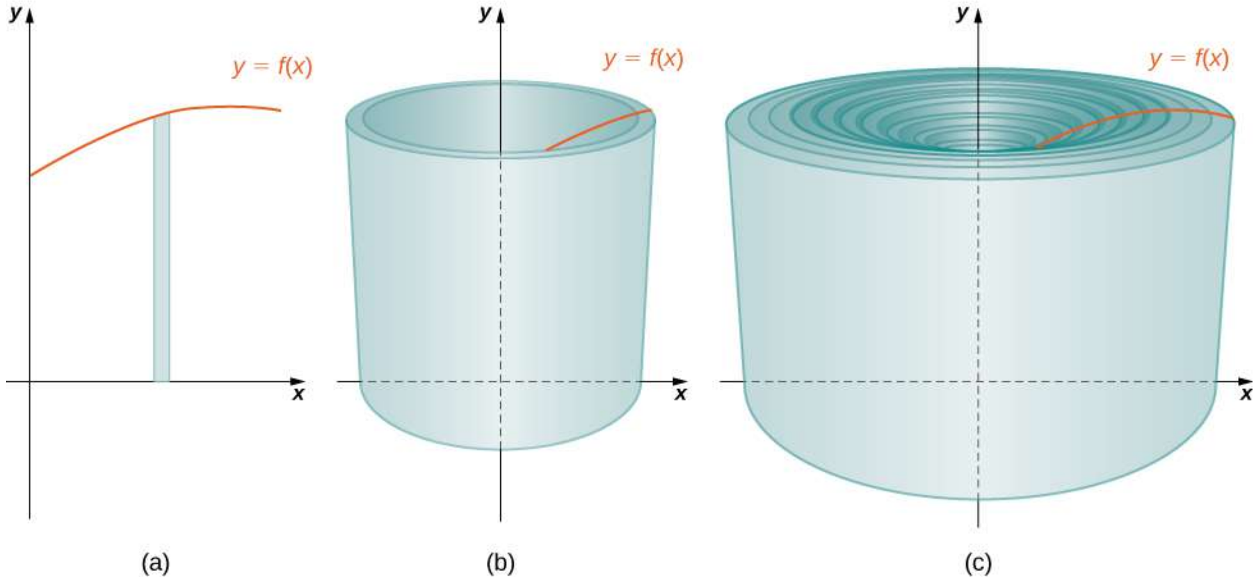


Figure 6.26 (a) A representative rectangle. (b) When this rectangle is revolved around the y-axis, the result is a cylindrical shell. (c) When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider **Figure 6.27**.

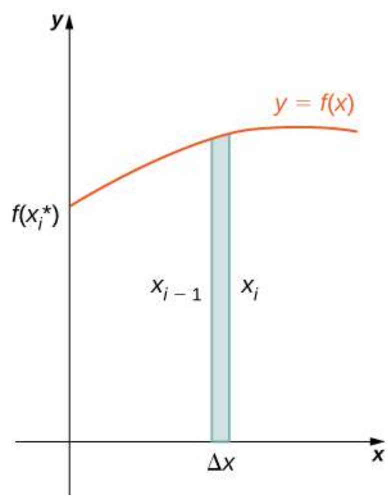


Figure 6.27 Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are annuli (ring-shaped regions—essentially, circles with a hole in the center), with outer radius x_i and inner radius x_{i-1}. Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)\left(x_i^2 - x_{i-1}^2\right) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x$, so we have

$$V_{\text{shell}} = 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x.$$

Furthermore, $\frac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by x_i^*. We then have

$$V_{\text{shell}} \approx 2\pi f(x_i^*) x_i^* \Delta x.$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate (**Figure 6.28**).

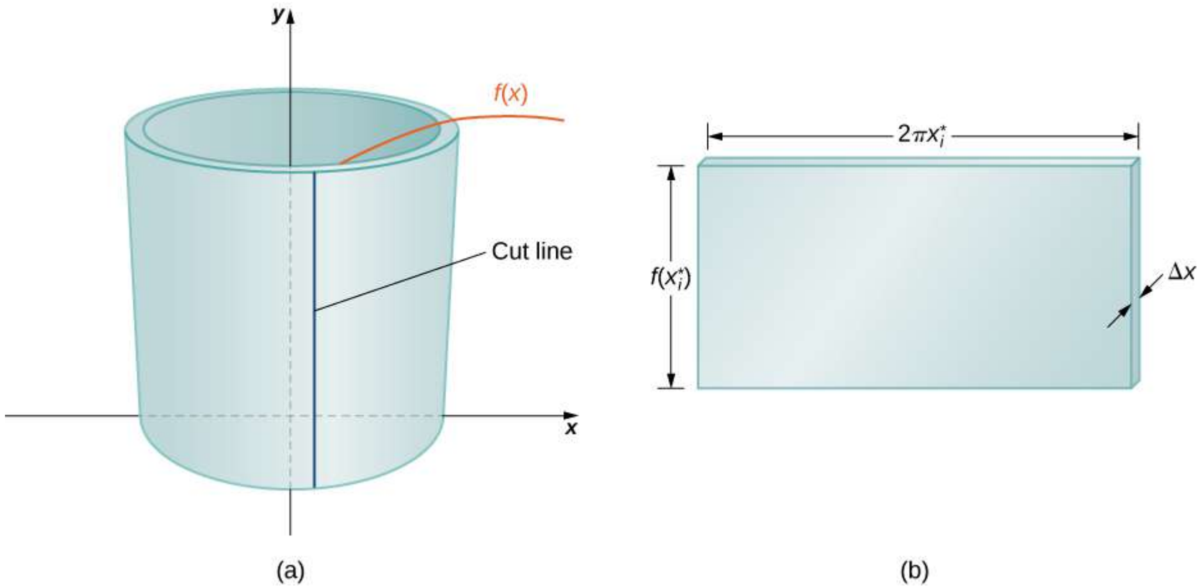


Figure 6.28 (a) Make a vertical cut in a representative shell. (b) Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius, and hence the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness Δx (**Figure 6.28**). The volume of the shell, then, is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

$$V_{\text{shell}} \approx f(x_i^*)(2\pi x_i^*)\Delta x,$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n} \left(2\pi x_i^* f(x_i^*)\Delta x\right).$$

Here we have another Riemann sum, this time for the function $2\pi x f(x)$. Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \left(2\pi x_i^* f(x_i^*)\Delta x\right) = \int_a^b \left(2\pi x f(x)\right) dx.$$

This leads to the following rule for the **method of cylindrical shells.**

Rule: The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x-axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving R around the y-axis is given by

$$V = \int_a^b (2\pi x f(x))dx. \quad (6.6)$$

Now let's consider an example.

Example 6.12

The Method of Cylindrical Shells 1

Define R as the region bounded above by the graph of $f(x) = 1/x$ and below by the x-axis over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

First we must graph the region R and the associated solid of revolution, as shown in the following figure.

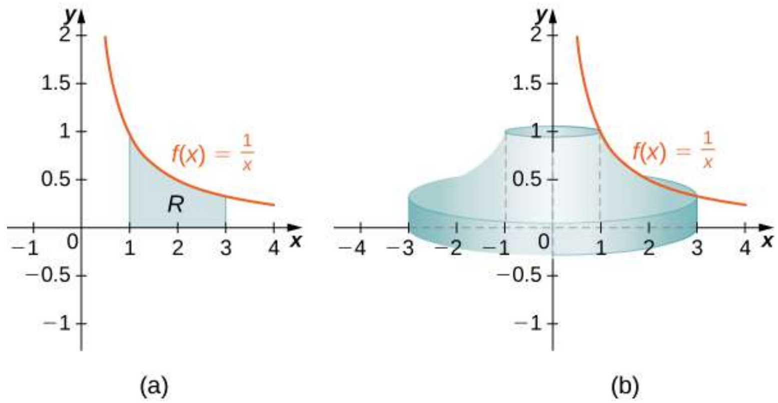


Figure 6.29 (a) The region R under the graph of $f(x) = 1/x$ over the interval $[1, 3]$. (b) The solid of revolution generated by revolving R about the y-axis.

Then the volume of the solid is given by

$$\begin{aligned} V &= \int_a^b (2\pi x f(x))dx \\ &= \int_1^3 \left(2\pi x\left(\frac{1}{x}\right)\right)dx \\ &= \int_1^3 2\pi\, dx = 2\pi x\Big|_1^3 = 4\pi \text{ units}^3. \end{aligned}$$

6.12 Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Example 6.13

The Method of Cylindrical Shells 2

Define R as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

First graph the region R and the associated solid of revolution, as shown in the following figure.

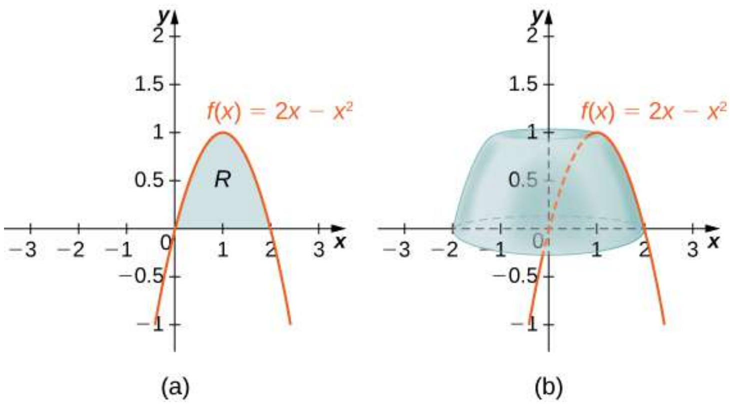


Figure 6.30 (a) The region R under the graph of $f(x) = 2x - x^2$ over the interval $[0, 2]$. (b) The volume of revolution obtained by revolving R about the y-axis.

Then the volume of the solid is given by

$$\begin{aligned} V &= \int_a^b (2\pi x f(x))dx \\ &= \int_0^2 \left(2\pi x\left(2x - x^2\right)\right)dx = 2\pi \int_0^2 \left(2x^2 - x^3\right)dx \\ &= 2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]\Bigg|_0^2 = \frac{8\pi}{3}\,\text{units}^3. \end{aligned}$$

6.13 Define R as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the x-axis, when we want to integrate with respect to y. The analogous rule for this type of solid is given here.

Rule: The Method of Cylindrical Shells for Solids of Revolution around the *x*-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the x-axis is given by

$$V = \int_c^d (2\pi y g(y))dy.$$

Example 6.14

The Method of Cylindrical Shells for a Solid Revolved around the *x*-axis

Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Solution

First, we need to graph the region Q and the associated solid of revolution, as shown in the following figure.

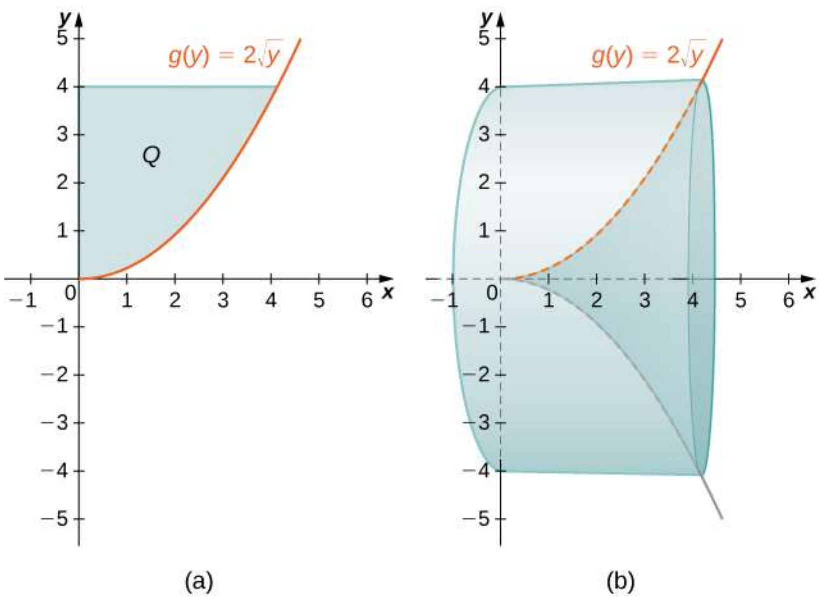


Figure 6.31 (a) The region Q to the left of the function $g(y)$ over the interval $[0, 4]$. (b) The solid of revolution generated by revolving Q around the x-axis.

Label the shaded region Q. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_c^d (2\pi y g(y))dy \\ &= \int_0^4 (2\pi y(2\sqrt{y}))dy = 4\pi \int_0^4 y^{3/2}dy \\ &= 4\pi \left[\frac{2y^{5/2}}{5}\right]\Bigg|_0^4 = \frac{256\pi}{5} \text{ units}^3. \end{aligned}$$

6.14 Define Q as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the y-axis for $y \in [1, 3]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

For the next example, we look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate axes. To set this up, we need to revisit the development of the method of cylindrical shells. Recall that we found the volume of one of the shells to be given by

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)\left(x_i^2 - x_{i-1}^2\right) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

This was based on a shell with an outer radius of x_i and an inner radius of x_{i-1}. If, however, we rotate the region around a line other than the y-axis, we have a different outer and inner radius. Suppose, for example, that we rotate the region around the line $x = -k$, where k is some positive constant. Then, the outer radius of the shell is $x_i + k$ and the inner radius of the shell is $x_{i-1} + k$. Substituting these terms into the expression for volume, we see that when a plane region is rotated around the line $x = -k$, the volume of a shell is given by

$$\begin{aligned} V_{\text{shell}} &= 2\pi f(x_i^*)\left(\frac{(x_i + k) + (x_{i-1} + k)}{2}\right)\left((x_i + k) - (x_{i-1} + k)\right) \\ &= 2\pi f(x_i^*)\left(\left(\frac{x_i + x_{i-2}}{2}\right) + k\right)\Delta x. \end{aligned}$$

As before, we notice that $\frac{x_i + x_{i-1}}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$ and can be approximated by x_i^*. Then, the approximate volume of the shell is

$$V_{\text{shell}} \approx 2\pi\left(x_i^* + k\right)f(x_i^*)\Delta x.$$

The remainder of the development proceeds as before, and we see that

$$V = \int_a^b (2\pi(x + k)f(x))dx.$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the x-term in the integral must be replaced with an expression representing the radius of a shell. To see how this works, consider the following example.

Example 6.15

A Region of Revolution Revolved around a Line

Define R as the region bounded above by the graph of $f(x) = x$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -1$.

Solution

First, graph the region R and the associated solid of revolution, as shown in the following figure.

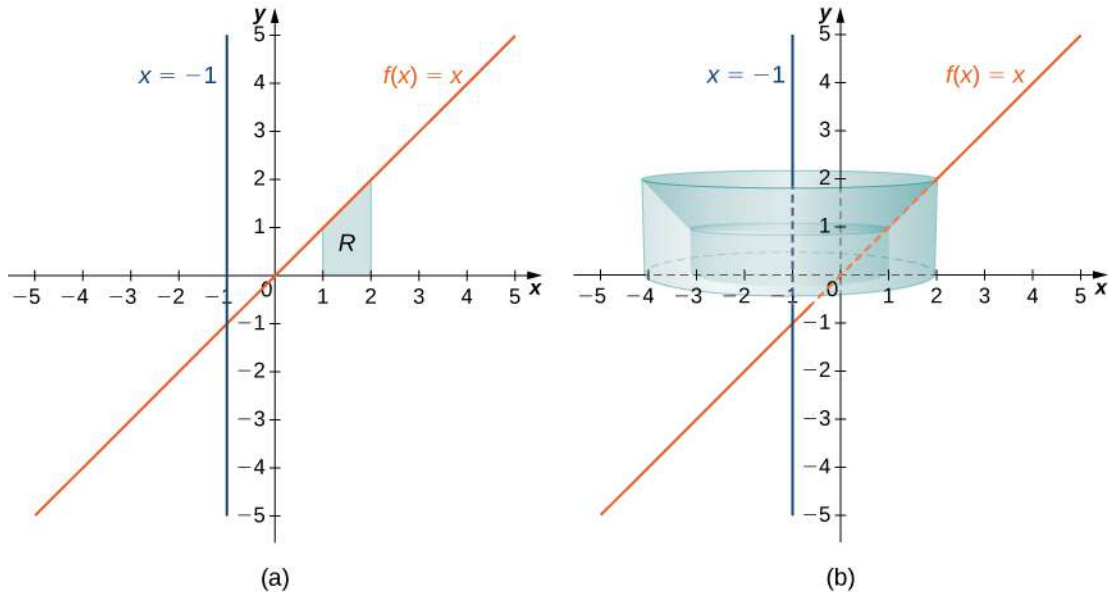


Figure 6.32 (a) The region R between the graph of $f(x)$ and the x-axis over the interval $[1, 2]$. (b) The solid of revolution generated by revolving R around the line $x = -1$.

Note that the radius of a shell is given by $x + 1$. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^2 (2\pi(x+1)f(x))dx \\ &= \int_1^2 (2\pi(x+1)x)dx = 2\pi\int_1^2 \left(x^2 + x\right)dx \\ &= 2\pi\left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Bigg|_1^2 = \frac{23\pi}{3}\text{ units}^3. \end{aligned}$$

6.15 Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -2$.

For our final example in this section, let's look at the volume of a solid of revolution for which the region of revolution is bounded by the graphs of two functions.

Example 6.16

A Region of Revolution Bounded by the Graphs of Two Functions

Define R as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1/x$ over the interval $[1, 4]$. Find the volume of the solid of revolution generated by revolving R around the y-axis.

Solution

First, graph the region R and the associated solid of revolution, as shown in the following figure.

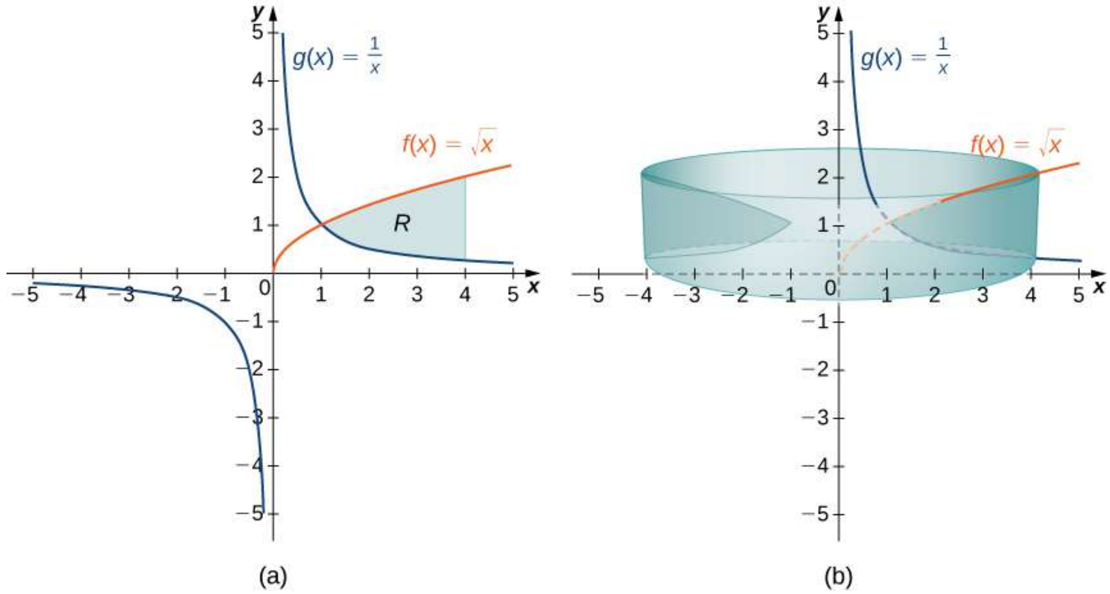


Figure 6.33 (a) The region R between the graph of $f(x)$ and the graph of $g(x)$ over the interval $[1, 4]$. (b) The solid of revolution generated by revolving R around the y-axis.

Note that the axis of revolution is the y-axis, so the radius of a shell is given simply by x. We don't need to make any adjustments to the x-term of our integrand. The height of a shell, though, is given by $f(x) - g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^4 (2\pi x(f(x) - g(x)))dx \\ &= \int_1^4 \left(2\pi x\left(\sqrt{x} - \frac{1}{x}\right)\right)dx = 2\pi \int_1^4 \left(x^{3/2} - 1\right)dx \\ &= 2\pi \left[\frac{2x^{5/2}}{5} - x\right]\Bigg|_1^4 = \frac{94\pi}{5} \text{ units}^3. \end{aligned}$$

6.16 Define R as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Which Method Should We Use?

We have studied several methods for finding the volume of a solid of revolution, but how do we know which method to use? It often comes down to a choice of which integral is easiest to evaluate. **Figure 6.34** describes the different approaches for solids of revolution around the x-axis. It's up to you to develop the analogous table for solids of revolution around the y-axis.

Comparing the Methods for Finding the Volume of a Solid Revolution around the x-axis

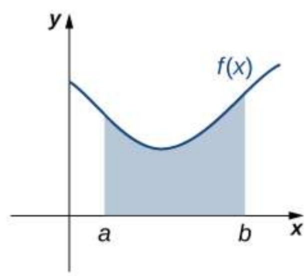
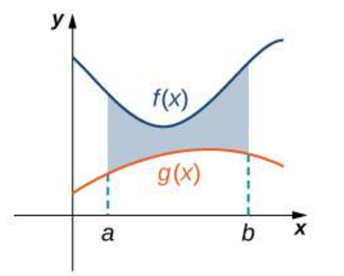
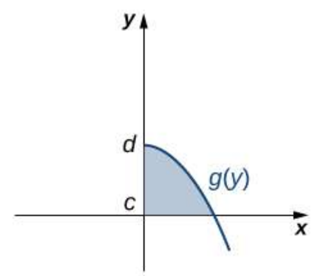
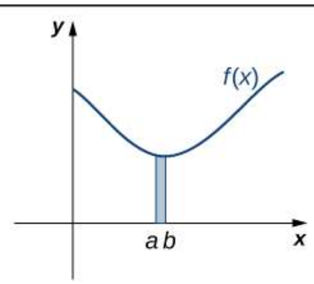
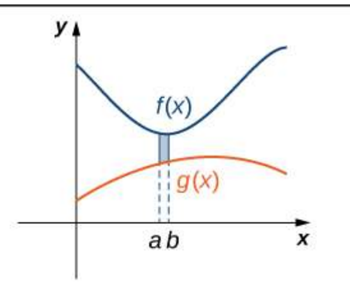
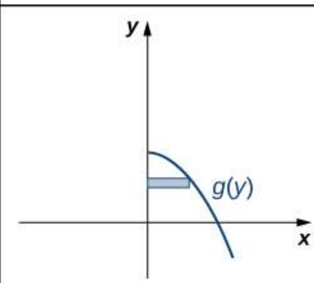
Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V = \int_a^b \pi[f(x)]^2\,dx$	$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$	$V = \int_c^d 2\pi y\, g(y)\, dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on x-axis	$[a, b]$ on x-axis	$[c, d]$ on y-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

Figure 6.34

Let's take a look at a couple of additional problems and decide on the best approach to take for solving them.

Example 6.17

Selecting the Best Method

For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral).

a. The region bounded by the graphs of $y = x,$ $y = 2 - x,$ and the x-axis.

b. The region bounded by the graphs of $y = 4x - x^2$ and the x-axis.

Solution

a. First, sketch the region and the solid of revolution as shown.

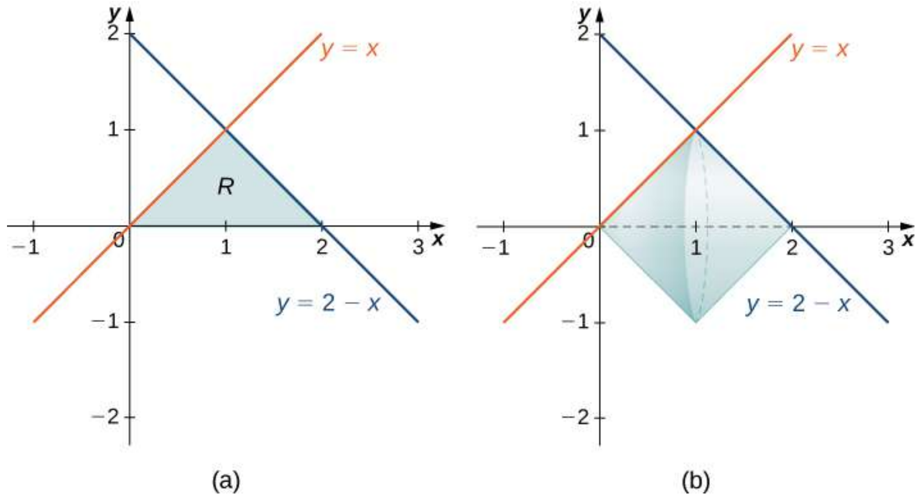


Figure 6.35 (a) The region R bounded by two lines and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, if we want to integrate with respect to x, we would have to break the integral into two pieces, because we have different functions bounding the region over $[0, 1]$ and $[1, 2]$. In this case, using the disk method, we would have

$$V = \int_0^1 \left(\pi x^2\right)dx + \int_1^2 \left(\pi(2-x)^2\right)dx.$$

If we used the shell method instead, we would use functions of y to represent the curves, producing

$$\begin{aligned} V &= \int_0^1 (2\pi y[(2-y)-y])dy \\ &= \int_0^1 (2\pi y[2-2y])dy. \end{aligned}$$

Neither of these integrals is particularly onerous, but since the shell method requires only one integral, and the integrand requires less simplification, we should probably go with the shell method in this case.

b. First, sketch the region and the solid of revolution as shown.

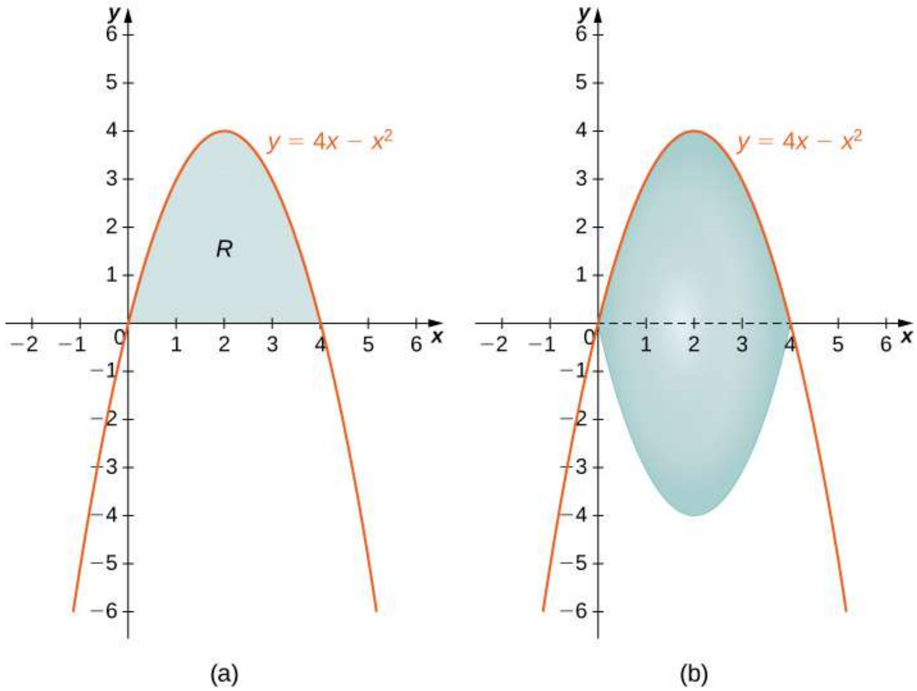


Figure 6.36 (a) The region R between the curve and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, it would be problematic to define a horizontal rectangle; the region is bounded on the left and right by the same function. Therefore, we can dismiss the method of shells. The solid has no cavity in the middle, so we can use the method of disks. Then

$$V = \int_0^4 \pi\left(4x - x^2\right)^2 dx.$$

6.17 Select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral): the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

6.3 EXERCISES

For the following exercise, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the washer method. Use technology to graph the functions and draw a typical slice by hand.

114. **[T]** Over the curve of $y = 3x,\ x = 0,$ and $y = 3$ rotated around the y-axis.

115. **[T]** Under the curve of $y = 3x,\ y = 0,$ and $x = 3$ rotated around the y-axis.

116. **[T]** Over the curve of $y = 3x,\ y = 0,$ and $y = 3$ rotated around the x-axis.

117. **[T]** Under the curve of $y = 3x,\ y = 0,$ and $x = 3$ rotated around the x-axis.

118. **[T]** Under the curve of $y = 2x^3,\ x = 0,$ and $x = 2$ rotated around the y-axis.

119. **[T]** Under the curve of $y = 2x^3,\ x = 0,$ and $x = 2$ rotated around the x-axis.

For the following exercises, use shells to find the volumes of the given solids. Note that the rotated regions lie between the curve and the x-axis and are rotated around the y-axis.

120. $y = 1 - x^2,\ x = 0,$ and $x = 1$

121. $y = 5x^3,\ x = 0,$ and $x = 1$

122. $y = \frac{1}{x},\ x = 1,$ and $x = 100$

123. $y = \sqrt{1 - x^2},\ x = 0,$ and $x = 1$

124. $y = \frac{1}{1 + x^2},\ x = 0,$ and $x = 3$

125. $y = \sin x^2,\ x = 0,$ and $x = \sqrt{\pi}$

126. $y = \frac{1}{\sqrt{1 - x^2}},\ x = 0,$ and $x = \frac{1}{2}$

127. $y = \sqrt{x},\ x = 0,$ and $x = 1$

128. $y = \left(1 + x^2\right)^3,\ x = 0,$ and $x = 1$

129. $y = 5x^3 - 2x^4,\ x = 0,$ and $x = 2$

For the following exercises, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the x-axis.

130. $y = \sqrt{1 - x^2},\ x = 0,\ x = 1$ and the x-axis

131. $y = x^2,\ x = 0,\ x = 2$ and the x-axis

132. $y = \frac{x^3}{2},\ x = 0,\ x = 2,$ and the x-axis

133. $y = \frac{2}{x^2},\ x = 1,\ x = 2,$ and the x-axis

134. $x = \frac{1}{1 + y^2},\ x = \frac{1}{5},$ and $y = 0$

135. $x = \frac{1 + y^2}{y},\ y = 1,\ y = 4,$ and the y-axis

136. $x = \cos y,\ y = 0,$ and $y = \pi$

137. $x = y^3 - 2y^2,\ x = 0,\ x = 9,$ and the y-axis

138. $x = \sqrt{y} + 1,\ x = 1,\ x = 3,$ and the x-axis

139. $x = \sqrt[3]{27y}$ and $x = \frac{3y}{4}$

For the following exercises, find the volume generated when the region between the curves is rotated around the given axis.

140. $y = 3 - x,\ y = 0,\ x = 0,$ and $x = 2$ rotated around the y-axis.

141. $y = x^3,\ x = 0,$ and $y = 8$ rotated around the y-axis.

142. $y = x^2,\ y = x,$ rotated around the y-axis.

143. $y = \sqrt{x},\ y = 0,$ and $x = 1$ rotated around the line $x = 2$.

144. $y = \frac{1}{4 - x},\ x = 1,$ and $x = 2$ rotated around the line $x = 4$.

145. $y = \sqrt{x}$ and $y = x^2$ rotated around the y-axis.

146. $y = \sqrt{x}$ and $y = x^2$ rotated around the line $x = 2$.

147. $x = y^3, x = \frac{1}{y}, x = 1,$ and $x = 2$ rotated around the x-axis.

148. $x = y^2$ and $y = x$ rotated around the line $y = 2$.

149. **[T]** Left of $x = \sin(\pi y)$, right of $y = x$, around the y-axis.

For the following exercises, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find the volume.

150. **[T]** $y = x^2$ and $y = 4x$ rotated around the y-axis.

151. **[T]** $y = \cos(\pi x), y = \sin(\pi x), x = \frac{1}{4},$ and $x = \frac{5}{4}$ rotated around the y-axis.

152. **[T]** $y = x^2 - 2x, x = 2,$ and $x = 4$ rotated around the y-axis.

153. **[T]** $y = x^2 - 2x, x = 2,$ and $x = 4$ rotated around the x-axis.

154. **[T]** $y = 3x^3 - 2, y = x,$ and $x = 2$ rotated around the x-axis.

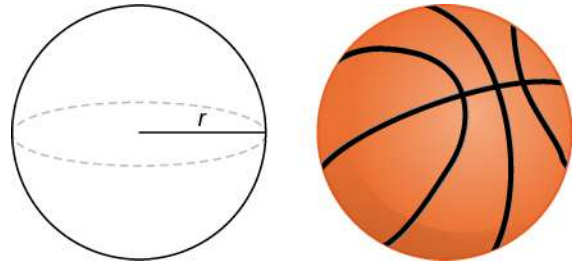
155. **[T]** $y = 3x^3 - 2, y = x,$ and $x = 2$ rotated around the y-axis.

156. **[T]** $x = \sin\left(\pi y^2\right)$ and $x = \sqrt{2}y$ rotated around the x-axis.

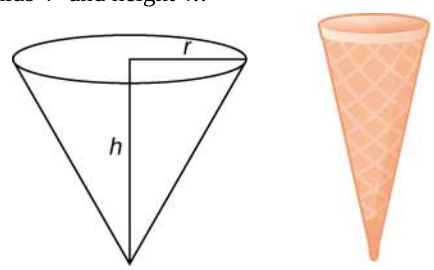
157. **[T]** $x = y^2, x = y^2 - 2y + 1,$ and $x = 2$ rotated around the y-axis.

For the following exercises, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.

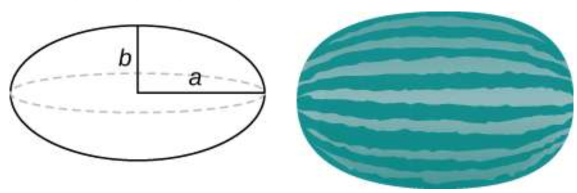
158. Use the method of shells to find the volume of a sphere of radius r.



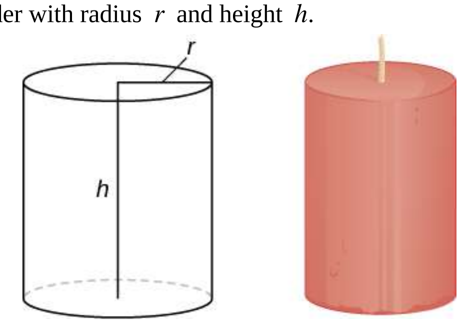
159. Use the method of shells to find the volume of a cone with radius r and height h.



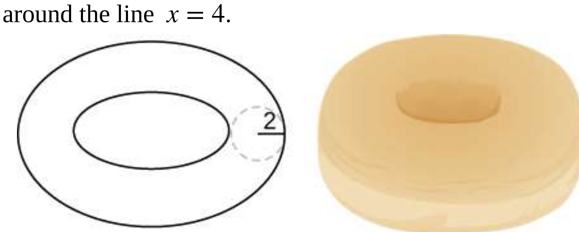
160. Use the method of shells to find the volume of an ellipse $\left(x^2/a^2\right) + \left(y^2/b^2\right) = 1$ rotated around the x-axis.



161. Use the method of shells to find the volume of a cylinder with radius r and height h.



162. Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4$.



163. Consider the region enclosed by the graphs of $y = f(x),\ y = 1 + f(x),\ x = 0,\ y = 0,$ and $x = a > 0.$ What is the volume of the solid generated when this region is rotated around the y-axis? Assume that the function is defined over the interval $[0,\ a].$

164. Consider the function $y = f(x),$ which decreases from $f(0) = b$ to $f(1) = 0.$ Set up the integrals for determining the volume, using both the shell method and the disk method, of the solid generated when this region, with $x = 0$ and $y = 0,$ is rotated around the y-axis. Prove that both methods approximate the same volume. Which method is easier to apply? (*Hint:* Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y).$)

6.4 | Arc Length of a Curve and Surface Area

Learning Objectives

6.4.1 Determine the length of a curve, $y = f(x)$, between two points.

6.4.2 Determine the length of a curve, $x = g(y)$, between two points.

6.4.3 Find the surface area of a solid of revolution.

In this section, we use definite integrals to find the arc length of a curve. We can think of **arc length** as the distance you would travel if you were walking along the path of the curve. Many real-world applications involve arc length. If a rocket is launched along a parabolic path, we might want to know how far the rocket travels. Or, if a curve on a map represents a road, we might want to know how far we have to drive to reach our destination.

We begin by calculating the arc length of curves defined as functions of x, then we examine the same process for curves defined as functions of y. (The process is identical, with the roles of x and y reversed.) The techniques we use to find arc length can be extended to find the surface area of a surface of revolution, and we close the section with an examination of this concept.

Arc Length of the Curve *y* = *f*(*x*)

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and furthermore we require its derivative, $f'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called *smooth*. (This property comes up again in later chapters.)

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0, 1, 2,\ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2,\ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure 6.37 depicts this construct for $n = 5$.

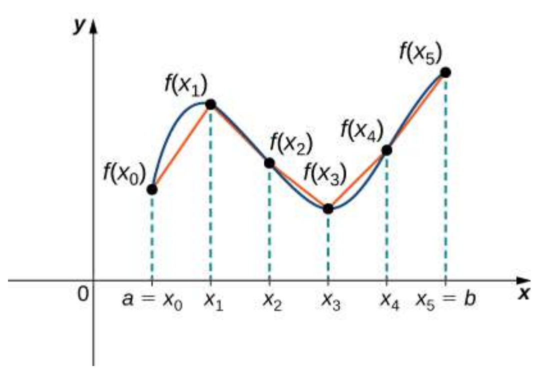


Figure 6.37 We can approximate the length of a curve by adding line segments.

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by Δx. The change in vertical distance varies from interval to interval, though, so we use $\Delta y_i = f(x_i) - f(x_{i-1})$ to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure 6.38. Note that some (or all) Δy_i may be negative.

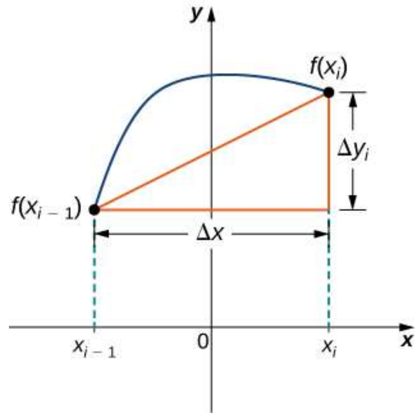


Figure 6.38 A representative line segment approximates the curve over the interval $[x_{i-1}, x_i]$.

By the Pythagorean theorem, the length of the line segment is $\sqrt{(\Delta x)^2 + (\Delta y_i)^2}$. We can also write this as $\Delta x\sqrt{1 + ((\Delta y_i)/(\Delta x))^2}$. Now, by the Mean Value Theorem, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/(\Delta x)$. Then the length of the line segment is given by $\Delta x\sqrt{1 + [f'(x_i^*)]^2}$. Adding up the lengths of all the line segments, we get

$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\text{Arc Length} = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

We summarize these findings in the following theorem.

Theorem 6.4: Arc Length for *y* = *f*(*x*)

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx. \tag{6.7}$$

Note that we are integrating an expression involving $f'(x)$, so we need to be sure $f'(x)$ is integrable. This is why we require $f(x)$ to be smooth. The following example shows how to apply the theorem.

Example 6.18

Calculating the Arc Length of a Function of *x*

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Solution

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \\ &= \int_0^1 \sqrt{1 + 9x}\,dx.\end{aligned}$$

Substitute $u = 1 + 9x$. Then, $du = 9\,dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 10$. Thus,

$$\begin{aligned}\text{Arc Length} &= \int_0^1 \sqrt{1 + 9x}\,dx \\ &= \frac{1}{9}\int_0^1 \sqrt{1 + 9x}9dx = \frac{1}{9}\int_1^{10} \sqrt{u}\,du \\ &= \frac{1}{9} \cdot \frac{2}{3}u^{3/2}\Big|_1^{10} = \frac{2}{27}\left[10\sqrt{10} - 1\right] \approx 2.268 \text{ units.}\end{aligned}$$

6.18 Let $f(x) = (4/3)x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Although it is nice to have a formula for calculating arc length, this particular theorem can generate expressions that are difficult to integrate. We study some techniques for integration in **Introduction to Techniques of Integration (http://cnx.org/content/m53654/latest/)** . In some cases, we may have to use a computer or calculator to approximate the value of the integral.

Example 6.19

Using a Computer or Calculator to Determine the Arc Length of a Function of *x*

Let $f(x) = x^2$. Calculate the arc length of the graph of $f(x)$ over the interval $[1, 3]$.

Solution

We have $f'(x) = 2x$, so $[f'(x)]^2 = 4x^2$. Then the arc length is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx = \int_1^3 \sqrt{1 + 4x^2}\,dx.$$

Using a computer to approximate the value of this integral, we get

$$\int_1^3 \sqrt{1 + 4x^2}\,dx \approx 8.26815.$$

6.19 Let $f(x) = \sin x$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, \pi]$. Use a computer or calculator to approximate the value of the integral.

Arc Length of the Curve *x* = *g*(*y*)

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of y, we can repeat the same process, except we partition the y-axis instead of the x-axis. **Figure 6.39** shows a representative line segment.

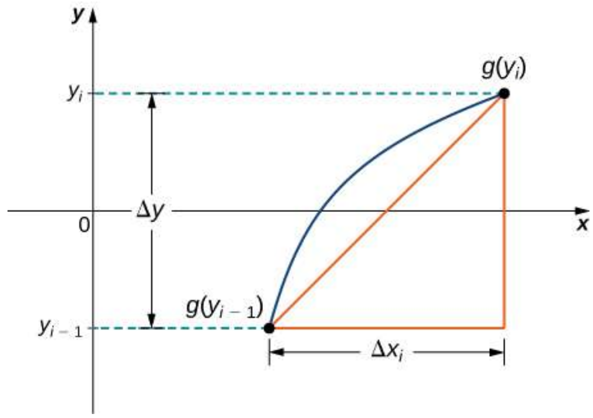


Figure 6.39 A representative line segment over the interval $[y_{i-1}, y_i]$.

Then the length of the line segment is $\sqrt{(\Delta y)^2 + (\Delta x_i)^2}$, which can also be written as $\Delta y\sqrt{1 + ((\Delta x_i)/(\Delta y))^2}$. If we now follow the same development we did earlier, we get a formula for arc length of a function $x = g(y)$.

Theorem 6.5: Arc Length for *x* = *g*(*y*)

Let $g(y)$ be a smooth function over an interval $[c, d]$. Then, the arc length of the graph of $g(y)$ from the point $(c, g(c))$ to the point $(d, g(d))$ is given by

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \tag{6.8}$$

Example 6.20

Calculating the Arc Length of a Function of *y*

Let $g(y) = 3y^3$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 2]$.

Solution

We have $g'(y) = 9y^2$, so $[g'(y)]^2 = 81y^4$. Then the arc length is

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_1^2 \sqrt{1 + 81y^4}\, dy.$$

Using a computer to approximate the value of this integral, we obtain

$$\int_1^2 \sqrt{1 + 81y^4}\, dy \approx 21.0277.$$

6.20 Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 4]$. Use a computer or calculator to approximate the value of the integral.

Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following figure.

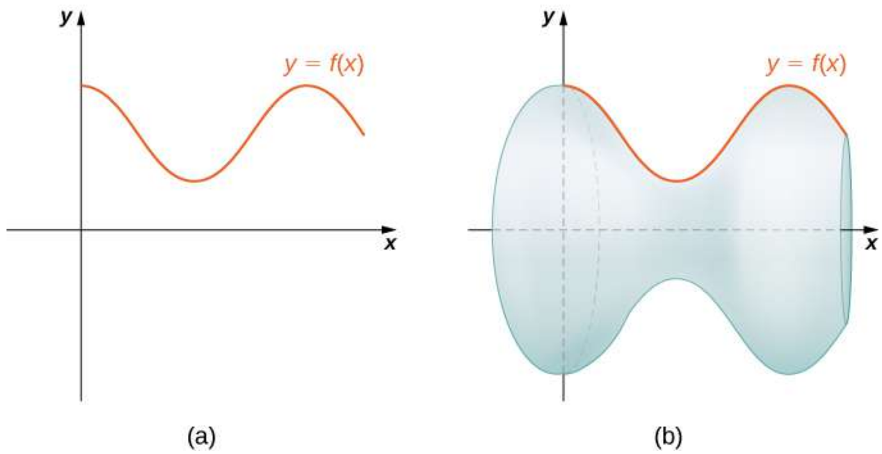


Figure 6.40 (a) A curve representing the function $f(x)$. (b) The surface of revolution formed by revolving the graph of $f(x)$ around the x-axis.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2,\ldots, n,$ construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the x-axis to generate an approximation of the surface of revolution as shown in the following figure.

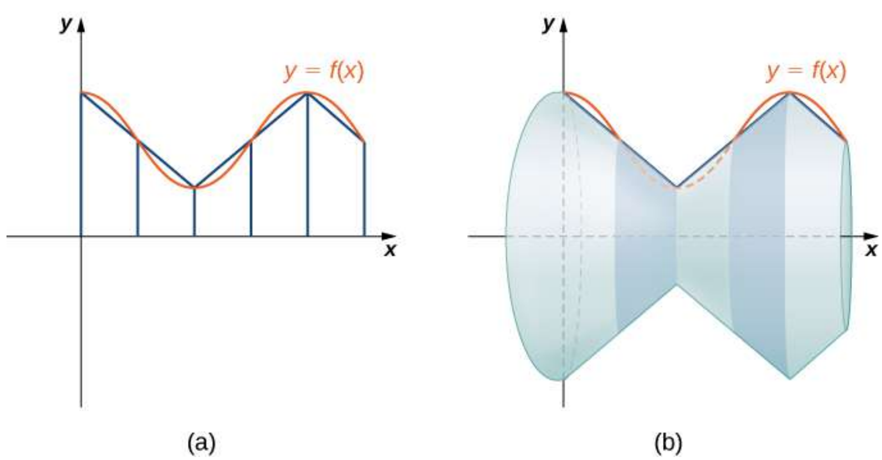


Figure 6.41 (a) Approximating $f(x)$ with line segments. (b) The surface of revolution formed by revolving the line segments around the x-axis.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off). A piece of a cone like this is called a **frustum** of a cone.

To find the surface area of the band, we need to find the lateral surface area, S, of the frustum (the area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces). Let r_1 and r_2 be the radii of the wide end and the narrow end of the frustum, respectively, and let l be the slant height of the frustum as shown in the following figure.

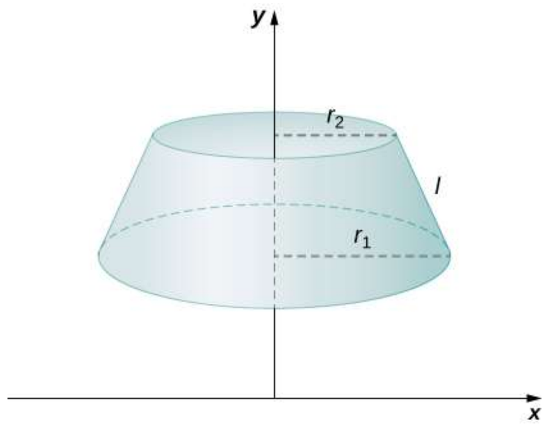


Figure 6.42 A frustum of a cone can approximate a small part of surface area.

We know the lateral surface area of a cone is given by

$$\text{Lateral Surface Area} = \pi rs,$$

where r is the radius of the base of the cone and s is the slant height (see the following figure).

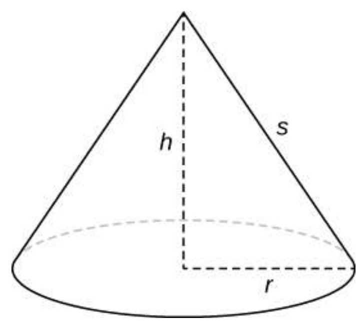


Figure 6.43 The lateral surface area of the cone is given by πrs.

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (see the following figure).

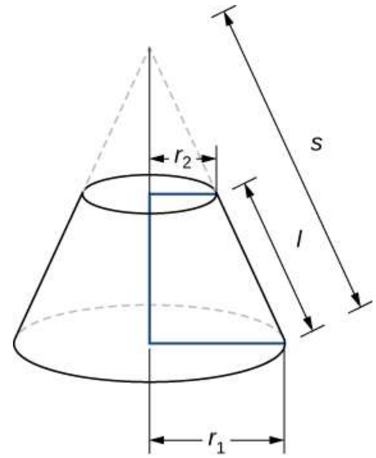


Figure 6.44 Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s}.$$

Solving for $s,$ we get

$$\begin{aligned} \frac{r_2}{r_1} &= \frac{s-l}{s} \\ r_2 s &= r_1(s-l) \\ r_2 s &= r_1 s - r_1 l \\ r_1 l &= r_1 s - r_2 s \\ r_1 l &= (r_1 - r_2)s \\ \frac{r_1 l}{r_1 - r_2} &= s. \end{aligned}$$

Then the lateral surface area (SA) of the frustum is

$$\begin{aligned} S &= \text{(Lateral SA of large cone)} - \text{(Lateral SA of small cone)} \\ &= \pi r_1 s - \pi r_2 (s-l) \\ &= \pi r_1 \left(\frac{r_1 l}{r_1 - r_2}\right) - \pi r_2 \left(\frac{r_1 l}{r_1 - r_2} - l\right) \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \pi r_2 l \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_2 l(r_1 - r_2)}{r_1 - r_2} \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_1 r_2 l}{r_1 - r_2} - \frac{\pi r_2{}^2 l}{r_1 - r_2} \\ &= \frac{\pi\left(r_1^2 - r_2^2\right) l}{r_1 - r_2} = \frac{\pi(r_1 - r_2)(r_1 + r_2) l}{r_1 - r_2} = \pi(r_1 + r_2) l. \end{aligned}$$

Let's now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the x-axis. A representative band is shown in the following figure.

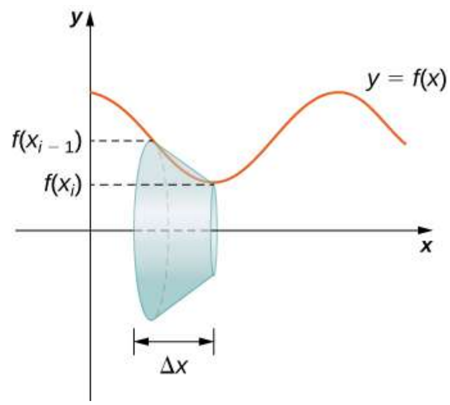


Figure 6.45 A representative band used for determining surface area.

Note that the slant height of this frustum is just the length of the line segment used to generate it. So, applying the surface area formula, we have

$$\begin{aligned} S &= \pi(r_1 + r_2)l \\ &= \pi\left(f(x_{i-1}) + f(x_i)\right)\sqrt{\Delta x^2 + (\Delta y_i)^2} \\ &= \pi\left(f(x_{i-1}) + f(x_i)\right)\Delta x\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}. \end{aligned}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/\Delta x$. This gives us

$$S = \pi\left(f(x_{i-1}) + f(x_i)\right)\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2}.$$

Furthermore, since $f(x)$ is continuous, by the Intermediate Value Theorem, there is a point $x_i^{**} \in [x_{i-1}, x_i]$ such that $f(x_i^{**}) = (1/2)\left[f(x_{i-1}) + f(x_i)\right]$, so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2}.$$

This *almost* looks like a Riemann sum, except we have functions evaluated at two different points, x_i^* and x_i^{**}, over the interval $[x_{i-1}, x_i]$. Although we do not examine the details here, it turns out that because $f(x)$ is smooth, if we let $n \to \infty$, the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both x_i^* and x_i^{**} are in the interval $[x_{i-1}, x_i]$, so it makes sense that as $n \to \infty$, both x_i^* and x_i^{**} approach x. Those of you who are interested in the details should consult an advanced calculus text.

Taking the limit as $n \to \infty$, we get

$$\text{Surface Area} = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2} = \int_a^b \left(2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\right)dx.$$

As with arc length, we can conduct a similar development for functions of y to get a formula for the surface area of surfaces of revolution about the y-axis. These findings are summarized in the following theorem.

Theorem 6.6: Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + (f'(x))^2}\right)dx. \tag{6.9}$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c, d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the y-axis is given by

$$\text{Surface Area} = \int_c^d \left(2\pi g(y)\sqrt{1 + (g'(y))^2}\right)dy.$$

Example 6.21

Calculating the Surface Area of a Surface of Revolution 1

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Solution

The graph of $f(x)$ and the surface of rotation are shown in the following figure.

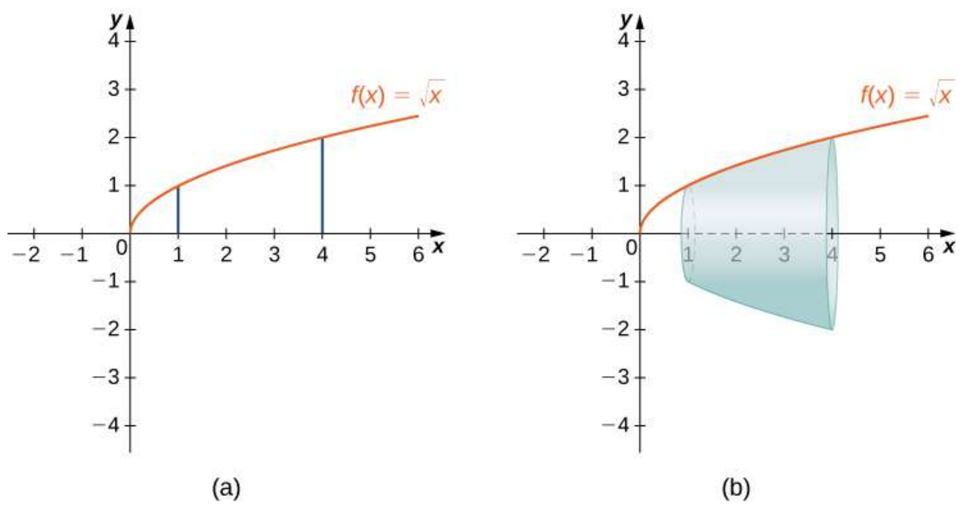


Figure 6.46 (a) The graph of $f(x)$. (b) The surface of revolution.

We have $f(x) = \sqrt{x}$. Then, $f'(x) = 1/(2\sqrt{x})$ and $(f'(x))^2 = 1/(4x)$. Then,

$$\begin{aligned}\text{Surface Area} &= \int_a^b \left(2\pi f(x)\sqrt{1+(f'(x))^2}\right)dx \\ &= \int_1^4 \left(2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\right)dx \\ &= \int_1^4 \left(2\pi\sqrt{x+\frac{1}{4}}\right)dx.\end{aligned}$$

Let $u = x + 1/4$. Then, $du = dx$. When $x = 1$, $u = 5/4$, and when $x = 4$, $u = 17/4$. This gives us

$$\begin{aligned}\int_1^4 \left(2\pi\sqrt{x+\frac{1}{4}}\right)dx &= \int_{5/4}^{17/4} 2\pi\sqrt{u}\,du \\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right]\Big|_{5/4}^{17/4} = \frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right] \approx 30.846.\end{aligned}$$

6.21 Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Example 6.22

Calculating the Surface Area of a Surface of Revolution 2

Let $f(x) = y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \le y \le 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.

Solution

Notice that we are revolving the curve around the y-axis, and the interval is in terms of y, so we want to rewrite the function as a function of y. We get $x = g(y) = (1/3)y^3$. The graph of $g(y)$ and the surface of rotation are shown in the following figure.

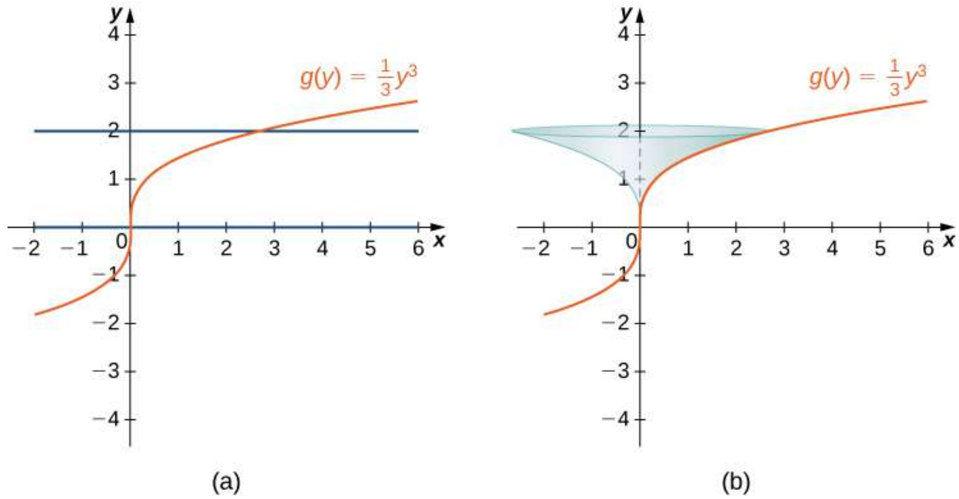


Figure 6.47 (a) The graph of $g(y)$. (b) The surface of revolution.

We have $g(y) = (1/3)y^3$, so $g'(y) = y^2$ and $(g'(y))^2 = y^4$. Then

$$\begin{aligned}\text{Surface Area} &= \int_c^d \left(2\pi g(y)\sqrt{1 + (g'(y))^2}\right)dy \\ &= \int_0^2 \left(2\pi\left(\frac{1}{3}y^3\right)\sqrt{1 + y^4}\right)dy \\ &= \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1 + y^4}\right)dy.\end{aligned}$$

Let $u = y^4 + 1$. Then $du = 4y^3\,dy$. When $y = 0$, $u = 1$, and when $y = 2$, $u = 17$. Then

$$\begin{aligned}\frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1 + y^4}\right)dy &= \frac{2\pi}{3}\int_1^{17} \frac{1}{4}\sqrt{u}\,du \\ &= \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]\Bigg|_1^{17} = \frac{\pi}{9}\left[(17)^{3/2} - 1\right] \approx 24.118.\end{aligned}$$

6.22 Let $g(y) = \sqrt{9 - y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the y-axis.

6.4 EXERCISES

For the following exercises, find the length of the functions over the given interval.

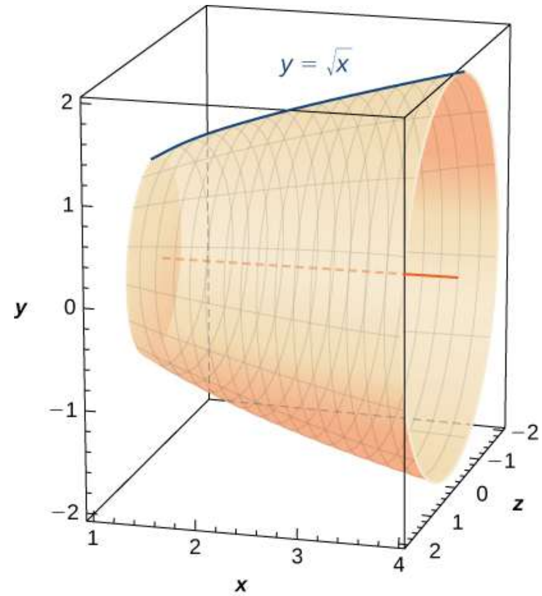
165. $y = 5x$ from $x = 0$ to $x = 2$

166. $y = -\frac{1}{2}x + 25$ from $x = 1$ to $x = 4$

167. $x = 4y$ from $y = -1$ to $y = 1$

168. Pick an arbitrary linear function $x = g(y)$ over any interval of your choice (y_1, y_2). Determine the length of the function and then prove the length is correct by using geometry.

169. Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolves around the x-axis from $(1, 1)$ to $(4, 2)$, as seen here.



170. Find the surface area of the volume generated when the curve $y = x^2$ revolves around the y-axis from $(1, 1)$ to $(3, 9)$.

For the following exercises, find the lengths of the functions of x over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

171. $y = x^{3/2}$ from $(0, 0)$ to $(1, 1)$

172. $y = x^{2/3}$ from $(1, 1)$ to $(8, 4)$

173. $y = \frac{1}{3}\left(x^2 + 2\right)^{3/2}$ from $x = 0$ to $x = 1$

174. $y = \frac{1}{3}\left(x^2 - 2\right)^{3/2}$ from $x = 2$ to $x = 4$

175. **[T]** $y = e^x$ on $x = 0$ to $x = 1$

176. $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 3$

177. $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$

178. $y = \frac{2x^{3/2}}{3} - \frac{x^{1/2}}{2}$ from $x = 1$ to $x = 4$

179. $y = \frac{1}{27}\left(9x^2 + 6\right)^{3/2}$ from $x = 0$ to $x = 2$

180. **[T]** $y = \sin x$ on $x = 0$ to $x = \pi$

For the following exercises, find the lengths of the functions of y over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

181. $y = \frac{5 - 3x}{4}$ from $y = 0$ to $y = 4$

182. $x = \frac{1}{2}\left(e^y + e^{-y}\right)$ from $y = -1$ to $y = 1$

183. $x = 5y^{3/2}$ from $y = 0$ to $y = 1$

184. **[T]** $x = y^2$ from $y = 0$ to $y = 1$

185. $x = \sqrt{y}$ from $y = 0$ to $y = 1$

186. $x = \frac{2}{3}\left(y^2 + 1\right)^{3/2}$ from $y = 1$ to $y = 3$

187. **[T]** $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$

188. **[T]** $x = \cos^2 y$ from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$

189. **[T]** $x = 4^y$ from $y = 0$ to $y = 2$

190. **[T]** $x = \ln(y)$ on $y = \frac{1}{e}$ to $y = e$

For the following exercises, find the surface area of the volume generated when the following curves revolve around the x-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

191. $y = \sqrt{x}$ from $x = 2$ to $x = 6$

192. $y = x^3$ from $x = 0$ to $x = 1$

193. $y = 7x$ from $x = -1$ to $x = 1$

194. **[T]** $y = \frac{1}{x^2}$ from $x = 1$ to $x = 3$

195. $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

196. $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$

197. $y = 5x$ from $x = 1$ to $x = 5$

198. **[T]** $y = \tan x$ from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$

For the following exercises, find the surface area of the volume generated when the following curves revolve around the y-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

199. $y = x^2$ from $x = 0$ to $x = 2$

200. $y = \frac{1}{2}x^2 + \frac{1}{2}$ from $x = 0$ to $x = 1$

201. $y = x + 1$ from $x = 0$ to $x = 3$

202. **[T]** $y = \frac{1}{x}$ from $x = \frac{1}{2}$ to $x = 1$

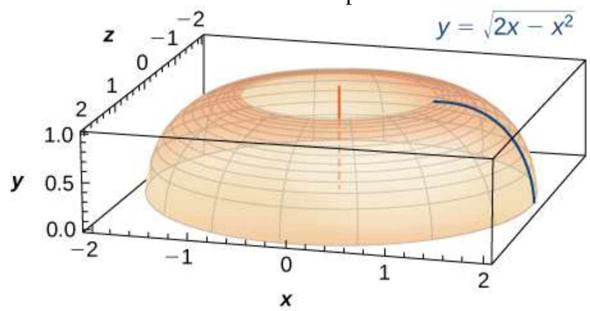
203. $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$

204. **[T]** $y = 3x^4$ from $x = 0$ to $x = 1$

205. **[T]** $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

206. **[T]** $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$

207. The base of a lamp is constructed by revolving a quarter circle $y = \sqrt{2x - x^2}$ around the y-axis from $x = 1$ to $x = 2$, as seen here. Create an integral for the surface area of this curve and compute it.



208. A light bulb is a sphere with radius $1/2$ in. with the bottom sliced off to fit exactly onto a cylinder of radius $1/4$ in. and length $1/3$ in., as seen here. The sphere is cut off at the bottom to fit exactly onto the cylinder, so the radius of the cut is $1/4$ in. Find the surface area (not including the top or bottom of the cylinder).

209. **[T]** A lampshade is constructed by rotating $y = 1/x$ around the x-axis from $y = 1$ to $y = 2$, as seen here. Determine how much material you would need to construct this lampshade—that is, the surface area—accurate to four decimal places.

210. **[T]** An anchor drags behind a boat according to the function $y = 24e^{-x/2} - 24$, where y represents the depth beneath the boat and x is the horizontal distance of the anchor from the back of the boat. If the anchor is 23 ft below the boat, how much rope do you have to pull to reach the anchor? Round your answer to three decimal places.

211. **[T]** You are building a bridge that will span 10 ft. You intend to add decorative rope in the shape of $y = 5|\sin((x\pi)/5)|,$ where x is the distance in feet from one end of the bridge. Find out how much rope you need to buy, rounded to the nearest foot.

For the following exercises, find the exact arc length for the following problems over the given interval.

212. $y = \ln(\sin x)$ from $x = \pi/4$ to $x = (3\pi)/4.$ (*Hint:* Recall trigonometric identities.)

213. Draw graphs of $y = x^2,$ $y = x^6,$ and $y = x^{10}.$ For $y = x^n,$ as n increases, formulate a prediction on the arc length from $(0, 0)$ to $(1, 1).$ Now, compute the lengths of these three functions and determine whether your prediction is correct.

214. Compare the lengths of the parabola $x = y^2$ and the line $x = by$ from $(0, 0)$ to $\left(b^2, b\right)$ as b increases. What do you notice?

215. Solve for the length of $x = y^2$ from $(0, 0)$ to $(1, 1).$ Show that $x = (1/2)y^2$ from $(0, 0)$ to $(2, 2)$ is twice as long. Graph both functions and explain why this is so.

216. **[T]** Which is longer between $(1, 1)$ and $(2, 1/2)$: the hyperbola $y = 1/x$ or the graph of $x + 2y = 3$?

217. Explain why the surface area is infinite when $y = 1/x$ is rotated around the x-axis for $1 \le x < \infty,$ but the volume is finite.

6.5 | Physical Applications

Learning Objectives

- **6.5.1** Determine the mass of a one-dimensional object from its linear density function.
- **6.5.2** Determine the mass of a two-dimensional circular object from its radial density function.
- **6.5.3** Calculate the work done by a variable force acting along a line.
- **6.5.4** Calculate the work done in pumping a liquid from one height to another.
- **6.5.5** Find the hydrostatic force against a submerged vertical plate.

In this section, we examine some physical applications of integration. Let's begin with a look at calculating mass from a density function. We then turn our attention to work, and close the section with a study of hydrostatic force.

Mass and Density

We can use integration to develop a formula for calculating mass based on a density function. First we consider a thin rod or wire. Orient the rod so it aligns with the x-axis, with the left end of the rod at $x = a$ and the right end of the rod at $x = b$ (**Figure 6.48**). Note that although we depict the rod with some thickness in the figures, for mathematical purposes we assume the rod is thin enough to be treated as a one-dimensional object.

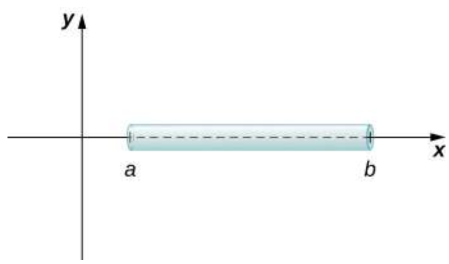


Figure 6.48 We can calculate the mass of a thin rod oriented along the x-axis by integrating its density function.

If the rod has constant density $\rho,$ given in terms of mass per unit length, then the mass of the rod is just the product of the density and the length of the rod: $(b - a)\rho.$ If the density of the rod is not constant, however, the problem becomes a little more challenging. When the density of the rod varies from point to point, we use a linear **density function**, $\rho(x),$ to denote the density of the rod at any point, $x.$ Let $\rho(x)$ be an integrable linear density function. Now, for $i = 0, 1, 2,\ldots, n$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b],$ and for $i = 1, 2,\ldots, n$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i].$ **Figure 6.49** shows a representative segment of the rod.

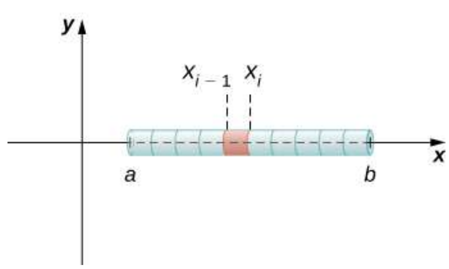


Figure 6.49 A representative segment of the rod.

The mass m_i of the segment of the rod from x_{i-1} to x_i is approximated by

$$m_i \approx \rho(x_i^*)(x_i - x_{i-1}) = \rho(x_i^*)\Delta x.$$

Adding the masses of all the segments gives us an approximation for the mass of the entire rod:

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we get an expression for the exact mass of the rod:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x = \int_a^b \rho(x)dx.$$

We state this result in the following theorem.

Theorem 6.7: Mass–Density Formula of a One-Dimensional Object

Given a thin rod oriented along the x-axis over the interval $[a, b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point x in the interval. Then the mass of the rod is given by

$$m = \int_a^b \rho(x)dx. \tag{6.10}$$

We apply this theorem in the next example.

Example 6.23

Calculating Mass from Linear Density

Consider a thin rod oriented on the x-axis over the interval $[\pi/2, \pi]$. If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

Solution

Applying **Equation 6.10** directly, we have

$$m = \int_a^b \rho(x)dx = \int_{\pi/2}^{\pi} \sin x\, dx = -\cos x|_{\pi/2}^{\pi} = 1.$$

6.23 Consider a thin rod oriented on the x-axis over the interval $[1, 3]$. If the density of the rod is given by $\rho(x) = 2x^2 + 3$, what is the mass of the rod?

We now extend this concept to find the mass of a two-dimensional disk of radius r. As with the rod we looked at in the one-dimensional case, here we assume the disk is thin enough that, for mathematical purposes, we can treat it as a two-dimensional object. We assume the density is given in terms of mass per unit area (called *area density*), and further assume the density varies only along the disk's radius (called *radial density*). We orient the disk in the xy-plane, with the center at the origin. Then, the density of the disk can be treated as a function of x, denoted $\rho(x)$. We assume $\rho(x)$ is integrable. Because density is a function of x, we partition the interval from $[0, r]$ along the x-axis. For $i = 0, 1, 2,\ldots, n$, let $P = \{x_i\}$ be a regular partition of the interval $[0, r]$, and for $i = 1, 2,\ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Now, use the partition to break up the disk into thin (two-dimensional) washers. A disk and a representative washer are depicted in the following figure.

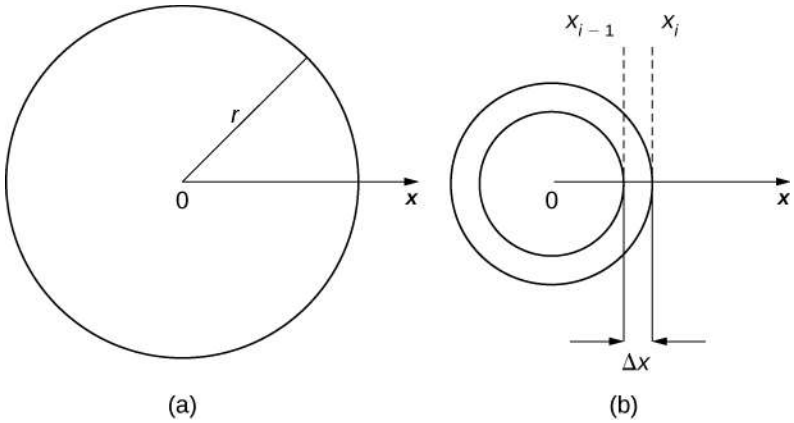


Figure 6.50 (a) A thin disk in the *xy*-plane. (b) A representative washer.

We now approximate the density and area of the washer to calculate an approximate mass, m_i. Note that the area of the washer is given by

$$\begin{aligned} A_i &= \pi(x_i)^2 - \pi(x_{i-1})^2 \\ &= \pi\left[x_i^2 - x_{i-1}^2\right] \\ &= \pi(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \pi(x_i + x_{i-1})\Delta x. \end{aligned}$$

You may recall that we had an expression similar to this when we were computing volumes by shells. As we did there, we use $x_i^* \approx (x_i + x_{i-1})/2$ to approximate the average radius of the washer. We obtain

$$A_i = \pi(x_i + x_{i-1})\Delta x \approx 2\pi x_i^* \Delta x.$$

Using $\rho(x_i^*)$ to approximate the density of the washer, we approximate the mass of the washer by

$$m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x.$$

Adding up the masses of the washers, we see the mass m of the entire disk is approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x.$$

We again recognize this as a Riemann sum, and take the limit as $n \to \infty$. This gives us

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x = \int_0^r 2\pi x \rho(x)dx.$$

We summarize these findings in the following theorem.

Theorem 6.8: Mass–Density Formula of a Circular Object

Let $\rho(x)$ be an integrable function representing the radial density of a disk of radius r. Then the mass of the disk is given by

$$m = \int_0^r 2\pi x \rho(x)dx. \tag{6.11}$$

Example 6.24

Calculating Mass from Radial Density

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

Solution

Applying the formula, we find

$$\begin{aligned} m &= \int_0^r 2\pi x \rho(x) dx \\ &= \int_0^4 2\pi x \sqrt{x} dx = 2\pi \int_0^4 x^{3/2} dx \\ &= 2\pi \frac{2}{5} x^{5/2} \Big|_0^4 = \frac{4\pi}{5}[32] = \frac{128\pi}{5}. \end{aligned}$$

6.24 Let $\rho(x) = 3x + 2$ represent the radial density of a disk. Calculate the mass of a disk of radius 2.

Work Done by a Force

We now consider work. In physics, work is related to force, which is often intuitively defined as a push or pull on an object. When a force moves an object, we say the force does work on the object. In other words, work can be thought of as the amount of energy it takes to move an object. According to physics, when we have a constant force, work can be expressed as the product of force and distance.

In the English system, the unit of force is the pound and the unit of distance is the foot, so work is given in foot-pounds. In the metric system, kilograms and meters are used. One newton is the force needed to accelerate 1 kilogram of mass at the rate of 1 m/sec^2. Thus, the most common unit of work is the newton-meter. This same unit is also called the *joule*. Both are defined as kilograms times meters squared over seconds squared $\left(\text{kg} \cdot \text{m}^2/\text{s}^2\right)$.

When we have a constant force, things are pretty easy. It is rare, however, for a force to be constant. The work done to compress (or elongate) a spring, for example, varies depending on how far the spring has already been compressed (or stretched). We look at springs in more detail later in this section.

Suppose we have a variable force $F(x)$ that moves an object in a positive direction along the *x*-axis from point a to point b. To calculate the work done, we partition the interval $[a, b]$ and estimate the work done over each subinterval. So, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. To calculate the work done to move an object from point x_{i-1} to point $x_i,$ we assume the force is roughly constant over the interval, and use $F(x_i^*)$ to approximate the force. The work done over the interval $[x_{i-1}, x_i],$ then, is given by

$$W_i \approx F(x_i^*)(x_i - x_{i-1}) = F(x_i^*)\Delta x.$$

Therefore, the work done over the interval $[a, b]$ is approximately

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Taking the limit of this expression as $n \to \infty$ gives us the exact value for work:

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i^*)\Delta x = \int_a^b F(x)dx.$$

Thus, we can define work as follows.

Definition

If a variable force $F(x)$ moves an object in a positive direction along the *x*-axis from point *a* to point *b*, then the **work** done on the object is

$$W = \int_a^b F(x)dx. \quad (6.12)$$

Note that if *F* is constant, the integral evaluates to $F \cdot (b - a) = F \cdot d,$ which is the formula we stated at the beginning of this section.

Now let's look at the specific example of the work done to compress or elongate a spring. Consider a block attached to a horizontal spring. The block moves back and forth as the spring stretches and compresses. Although in the real world we would have to account for the force of friction between the block and the surface on which it is resting, we ignore friction here and assume the block is resting on a frictionless surface. When the spring is at its natural length (at rest), the system is said to be at equilibrium. In this state, the spring is neither elongated nor compressed, and in this equilibrium position the block does not move until some force is introduced. We orient the system such that $x = 0$ corresponds to the equilibrium position (see the following figure).

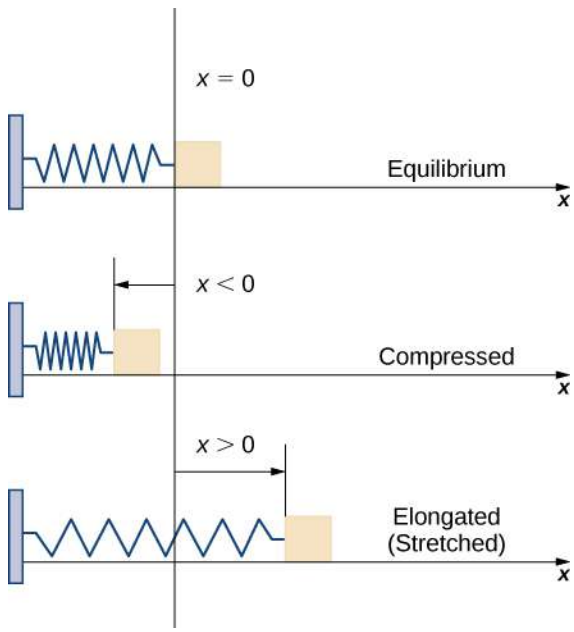


Figure 6.51 A block attached to a horizontal spring at equilibrium, compressed, and elongated.

According to **Hooke's law**, the force required to compress or stretch a spring from an equilibrium position is given by $F(x) = kx,$ for some constant $k.$ The value of k depends on the physical characteristics of the spring. The constant k is called the *spring constant* and is always positive. We can use this information to calculate the work done to compress or elongate a spring, as shown in the following example.

Example 6.25

The Work Required to Stretch or Compress a Spring

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to stretch the spring 0.5 m from the equilibrium position?

Solution

First find the spring constant, k. When $x = -0.2,$ we know $F(x) = -10,$ so

$$\begin{aligned} F(x) &= kx \\ -10 &= k(-0.2) \\ k &= 50 \end{aligned}$$

and $F(x) = 50x$. Then, to calculate work, we integrate the force function, obtaining

$$W = \int_a^b F(x)dx = \int_0^{0.5} 50x\,dx = 25x^2\Big|_0^{0.5} = 6.25.$$

The work done to stretch the spring is 6.25 J.

6.25 Suppose it takes a force of 8 lb to stretch a spring 6 in. from the equilibrium position. How much work is done to stretch the spring 1 ft from the equilibrium position?

Work Done in Pumping

Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are a little more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water, and it takes more work to move the water from the bottom of the tank than it does to move the water from the top of the tank.

We examine the process in the context of a cylindrical tank, then look at a couple of examples using tanks of different shapes. Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

The first thing we need to do is define a frame of reference. We let x represent the vertical distance below the top of the tank. That is, we orient the $x\text{-axis}$ vertically, with the origin at the top of the tank and the downward direction being positive (see the following figure).

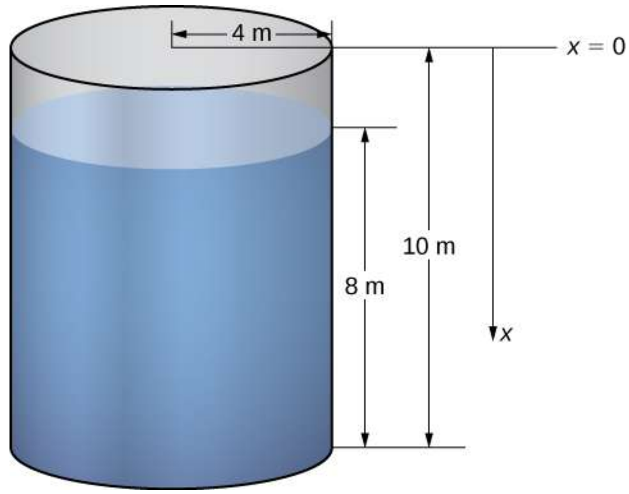


Figure 6.52 How much work is needed to empty a tank partially filled with water?

Using this coordinate system, the water extends from $x = 2$ to $x = 10$. Therefore, we partition the interval $[2,\ 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[2,\ 10],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure 6.53 shows a representative layer.

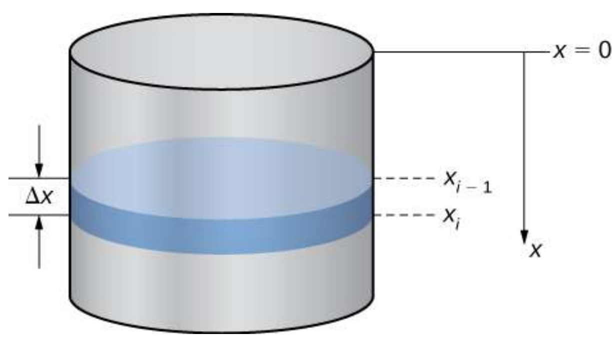


Figure 6.53 A representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is $9800\ \text{N/m}^3,$ or $62.4\ \text{lb/ft}^3,$ calculating the volume of each layer gives us the weight. In this case, we have

$$V = \pi(4)^2 \Delta x = 16\pi\Delta x.$$

Then, the force needed to lift each layer is

$$F = 9800 \cdot 16\pi\Delta x = 156{,}800\pi\Delta x.$$

Note that this step becomes a little more difficult if we have a noncylindrical tank. We look at a noncylindrical tank in the next example.

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use x_i^* as an approximation of the distance the layer must be lifted. Then the work to lift the ith layer of water W_i is approximately

$$W_i \approx 156{,}800\pi x_i^* \ \Delta x.$$

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 156{,}800\pi x_i^* \ \Delta x.$$

This is a Riemann sum, so taking the limit as $n \to \infty$, we get

$$\begin{aligned} W &= \lim_{n \to \infty} \sum_{i=1}^{n} 156{,}800\pi x_i^* \ \Delta x \\ &= 156{,}800\pi \int_2^{10} x dx \\ &= 156{,}800\pi \left[\frac{x^2}{2}\right]\Big|_2^{10} = 7{,}526{,}400\pi \approx 23{,}644{,}883. \end{aligned}$$

The work required to empty the tank is approximately 23,650,000 J.

For pumping problems, the calculations vary depending on the shape of the tank or container. The following problem-solving strategy lays out a step-by-step process for solving pumping problems.

Problem-Solving Strategy: Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.
6. Sum the work required to lift all the layers. This expression is an estimate of the work required to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

We now apply this problem-solving strategy in an example with a noncylindrical tank.

Example 6.26

A Pumping Problem with a Noncylindrical Tank

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft. How much work is required to pump out that amount of water?

Solution

The tank is depicted in **Figure 6.54**. As we did in the example with the cylindrical tank, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (step 1).

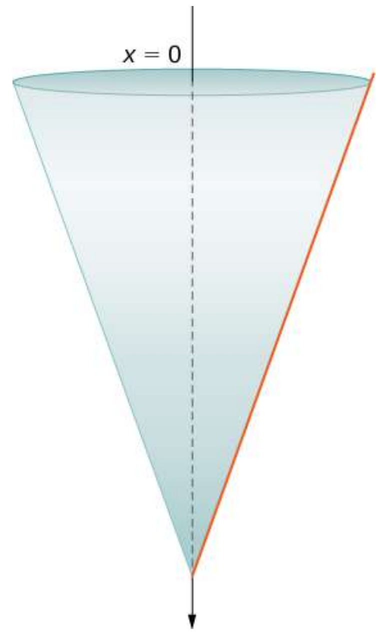


Figure 6.54 A water tank in the shape of an inverted cone.

The tank starts out full and ends with 4 ft of water left, so, based on our chosen frame of reference, we need to partition the interval $[0, 8]$. Then, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[0, 8],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. We can approximate the volume of a layer by using a disk, then use similar triangles to find the radius of the disk (see the following figure).

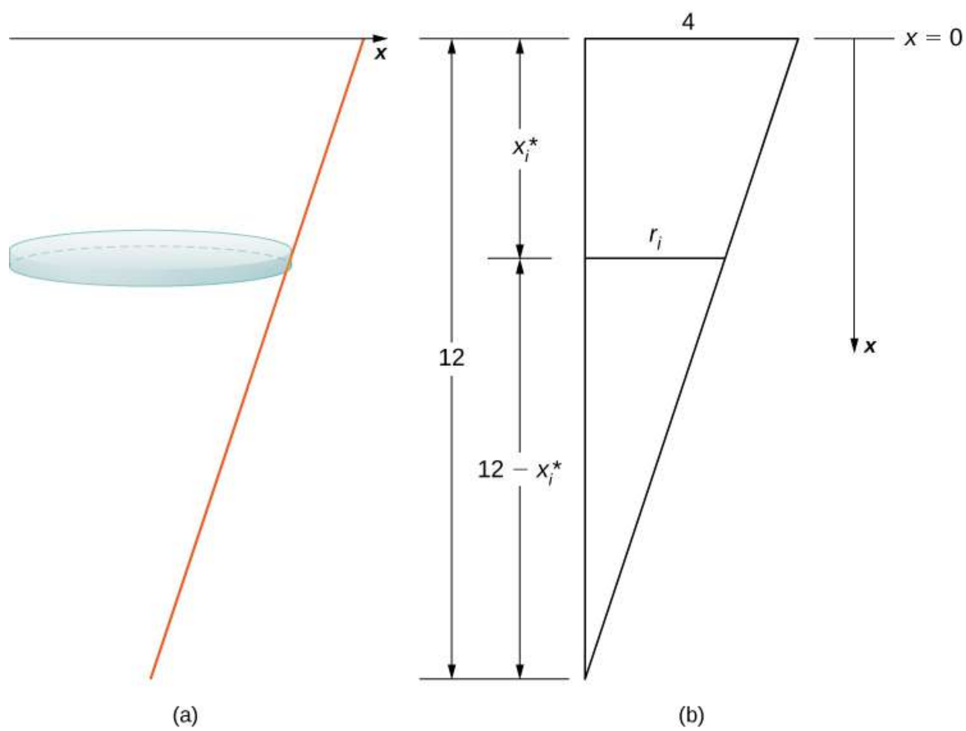


Figure 6.55 Using similar triangles to express the radius of a disk of water.

From properties of similar triangles, we have

$$\begin{aligned} \frac{r_i}{12 - x_i^*} &= \frac{4}{12} = \frac{1}{3} \\ 3r_i &= 12 - x_i^* \\ r_i &= \frac{12 - x_i^*}{3} \\ &= 4 - \frac{x_i^*}{3}. \end{aligned}$$

Then the volume of the disk is

$$V_i = \pi\left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 2).}$$

The weight-density of water is 62.4 lb/ft^3, so the force needed to lift each layer is approximately

$$F_i \approx 62.4\pi\left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 3).}$$

Based on the diagram, the distance the water must be lifted is approximately x_i^* feet (step 4), so the approximate work needed to lift the layer is

$$W_i \approx 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 5).}$$

Summing the work required to lift all the layers, we get an approximate value of the total work:

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 6).}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} W &= \lim_{n \to \infty} \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \\ &= \int_0^8 62.4\pi x\left(4 - \frac{x}{3}\right)^2 dx \\ &= 62.4\pi\int_0^8 x\left(16 - \frac{8x}{3} + \frac{x^2}{9}\right)dx = 62.4\pi\int_0^8 \left(16x - \frac{8x^2}{3} + \frac{x^3}{9}\right)dx \\ &= 62.4\pi\left[8x^2 - \frac{8x^3}{9} + \frac{x^4}{36}\right]\Bigg|_0^8 = 10{,}649.6\pi \approx 33{,}456.7. \end{aligned}$$

It takes approximately $33{,}450$ ft-lb of work to empty the tank to the desired level.

6.26 A tank is in the shape of an inverted cone, with height 10 ft and base radius 6 ft. The tank is filled to a depth of 8 ft to start with, and water is pumped over the upper edge of the tank until 3 ft of water remain in the tank. How much work is required to pump out that amount of water?

Hydrostatic Force and Pressure

In this last section, we look at the force and pressure exerted on an object submerged in a liquid. In the English system, force is measured in pounds. In the metric system, it is measured in newtons. Pressure is force per unit area, so in the English system we have pounds per square foot (or, perhaps more commonly, pounds per square inch, denoted psi). In the metric system we have newtons per square meter, also called *pascals*.

Let's begin with the simple case of a plate of area A submerged horizontally in water at a depth *s* (**Figure 6.56**). Then, the force exerted on the plate is simply the weight of the water above it, which is given by $F = \rho As,$ where ρ is the weight density of water (weight per unit volume). To find the **hydrostatic pressure**—that is, the pressure exerted by water on a submerged object—we divide the force by the area. So the pressure is $p = F/A = \rho s.$

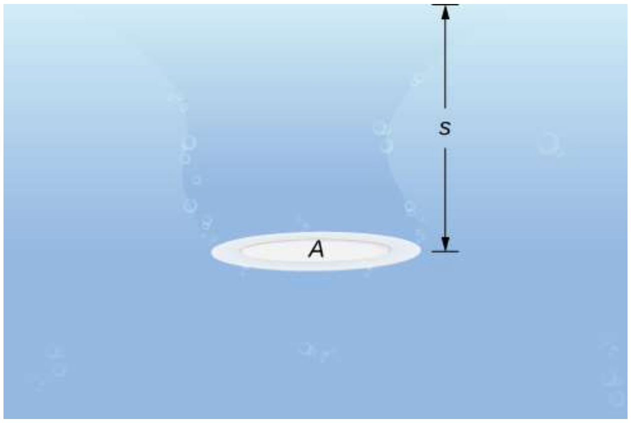


Figure 6.56 A plate submerged horizontally in water.

By Pascal's principle, the pressure at a given depth is the same in all directions, so it does not matter if the plate is submerged horizontally or vertically. So, as long as we know the depth, we know the pressure. We can apply Pascal's principle to find the force exerted on surfaces, such as dams, that are oriented vertically. We cannot apply the formula $F = \rho As$ directly, because the depth varies from point to point on a vertically oriented surface. So, as we have done many times before, we form a partition, a Riemann sum, and, ultimately, a definite integral to calculate the force.

Suppose a thin plate is submerged in water. We choose our frame of reference such that the *x*-axis is oriented vertically, with the downward direction being positive, and point $x = 0$ corresponding to a logical reference point. Let $s(x)$ denote the depth at point *x*. Note we often let $x = 0$ correspond to the surface of the water. In this case, depth at any point is simply given by $s(x) = x.$ However, in some cases we may want to select a different reference point for $x = 0,$ so we proceed with the development in the more general case. Last, let $w(x)$ denote the width of the plate at the point $x.$

Assume the top edge of the plate is at point $x = a$ and the bottom edge of the plate is at point $x = b.$ Then, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i].$ The partition divides the plate into several thin, rectangular strips (see the following figure).

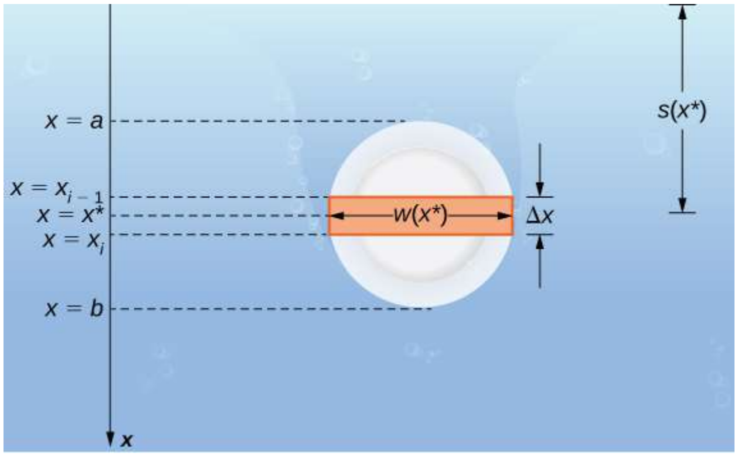


Figure 6.57 A thin plate submerged vertically in water.

Let’s now estimate the force on a representative strip. If the strip is thin enough, we can treat it as if it is at a constant depth, $s(x_i^*)$. We then have

$$F_i = \rho A s = \rho\left[w(x_i^*)\Delta x\right]s(x_i^*).$$

Adding the forces, we get an estimate for the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho\left[w(x_i^*)\Delta x\right]s(x_i^*).$$

This is a Riemann sum, so taking the limit gives us the exact force. We obtain

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} \rho\left[w(x_i^*)\Delta x\right]s(x_i^*) = \int_a^b \rho w(x)s(x)dx. \tag{6.13}$$

Evaluating this integral gives us the force on the plate. We summarize this in the following problem-solving strategy.

Problem-Solving Strategy: Finding Hydrostatic Force

1. Sketch a picture and select an appropriate frame of reference. (Note that if we select a frame of reference other than the one used earlier, we may have to adjust **Equation 6.13** accordingly.)
2. Determine the depth and width functions, $s(x)$ and $w(x)$.
3. Determine the weight-density of whatever liquid with which you are working. The weight-density of water is 62.4 lb/ft^3, or 9800 N/m^3.
4. Use the equation to calculate the total force.

Example 6.27

Finding Hydrostatic Force

A water trough 15 ft long has ends shaped like inverted isosceles triangles, with base 8 ft and height 3 ft. Find the force on one end of the trough if the trough is full of water.

Solution

Figure 6.58 shows the trough and a more detailed view of one end.

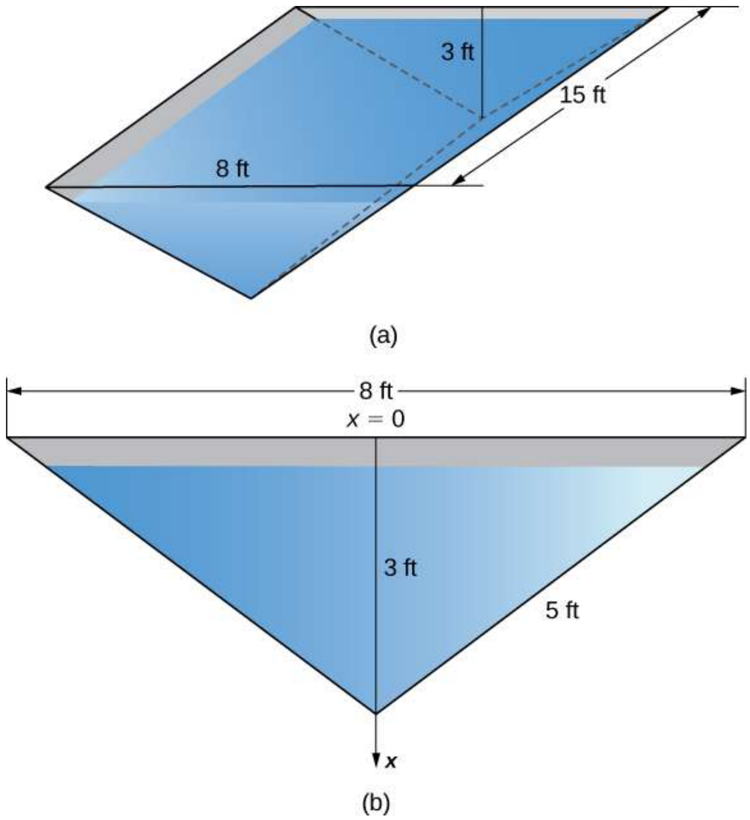


Figure 6.58 (a) A water trough with a triangular cross-section. (b) Dimensions of one end of the water trough.

Select a frame of reference with the x-axis oriented vertically and the downward direction being positive. Select the top of the trough as the point corresponding to $x = 0$ (step 1). The depth function, then, is $s(x) = x.$ Using similar triangles, we see that $w(x) = 8 - (8/3)x$ (step 2). Now, the weight density of water is 62.4 lb/ft^3 (step 3), so applying **Equation 6.13**, we obtain

$$\begin{aligned} F &= \int_a^b \rho w(x) s(x) dx \\ &= \int_0^3 62.4\left(8 - \frac{8}{3}x\right)x\,dx = 62.4\int_0^3 \left(8x - \frac{8}{3}x^2\right)dx \\ &= 62.4\left[4x^2 - \frac{8}{9}x^3\right]\Bigg|_0^3 = 748.8. \end{aligned}$$

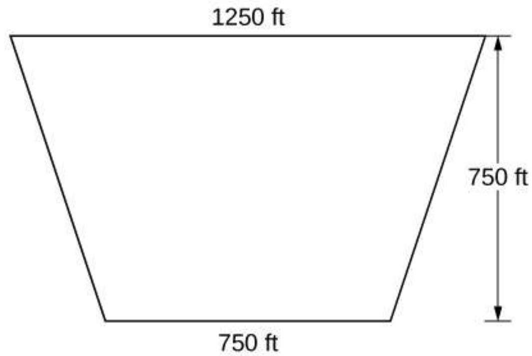
The water exerts a force of 748.8 lb on the end of the trough (step 4).

6.27 A water trough 12 m long has ends shaped like inverted isosceles triangles, with base 6 m and height 4 m. Find the force on one end of the trough if the trough is full of water.

Example 6.28

Chapter Opener: Finding Hydrostatic Force

We now return our attention to the Hoover Dam, mentioned at the beginning of this chapter. The actual dam is arched, rather than flat, but we are going to make some simplifying assumptions to help us with the calculations. Assume the face of the Hoover Dam is shaped like an isosceles trapezoid with lower base 750 ft, upper base 1250 ft, and height 750 ft (see the following figure).



When the reservoir is full, Lake Mead's maximum depth is about 530 ft, and the surface of the lake is about 10 ft below the top of the dam (see the following figure).

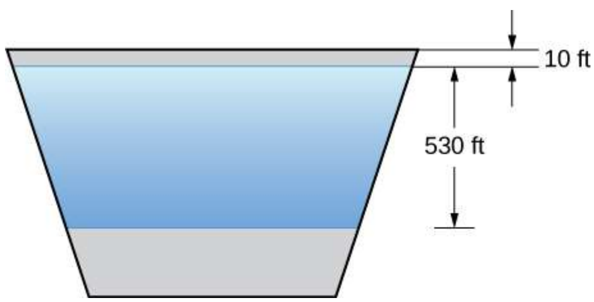


Figure 6.59 A simplified model of the Hoover Dam with assumed dimensions.

a. Find the force on the face of the dam when the reservoir is full.

b. The southwest United States has been experiencing a drought, and the surface of Lake Mead is about 125 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Solution

a. We begin by establishing a frame of reference. As usual, we choose to orient the x-axis vertically, with the downward direction being positive. This time, however, we are going to let $x = 0$ represent the top of the dam, rather than the surface of the water. When the reservoir is full, the surface of the water is 10 ft below the top of the dam, so $s(x) = x - 10$ (see the following figure).

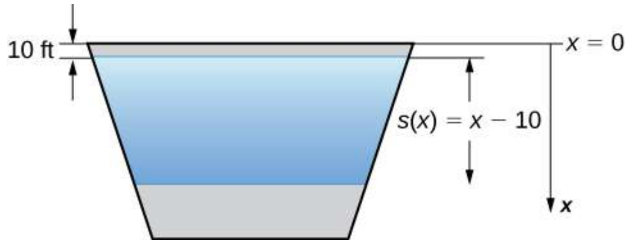


Figure 6.60 We first choose a frame of reference.

To find the width function, we again turn to similar triangles as shown in the figure below.

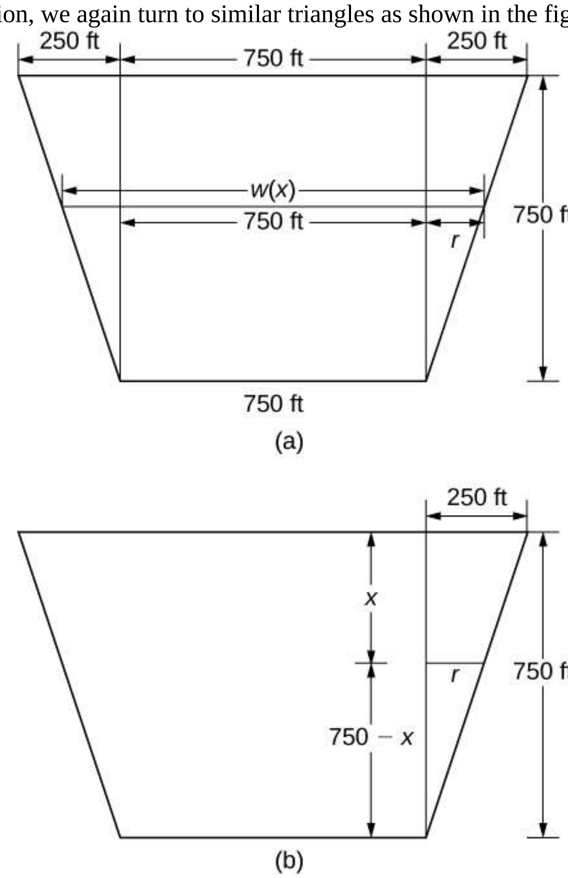


Figure 6.61 We use similar triangles to determine a function for the width of the dam. (a) Assumed dimensions of the dam; (b) highlighting the similar triangles.

From the figure, we see that $w(x) = 750 + 2r$. Using properties of similar triangles, we get $r = 250 - (1/3)x$. Thus,

$$w(x) = 1250 - \frac{2}{3}x \text{ (step 2)}.$$

Using a weight-density of 62.4 lb/ft^3 (step 3) and applying **Equation 6.13**, we get

$$\begin{aligned} F &= \int_a^b \rho w(x)s(x)dx \\ &= \int_{10}^{540} 62.4\left(1250 - \frac{2}{3}x\right)(x - 10)dx = 62.4\int_{10}^{540} -\frac{2}{3}\left[x^2 - 1885x + 18750\right]dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - \frac{1885x^2}{2} + 18750x\right]\Bigg|_{10}^{540} \approx 8{,}832{,}245{,}000 \text{ lb} = 4{,}416{,}122.5 \text{ t.} \end{aligned}$$

Note the change from pounds to tons $(2000 \text{ lb} = 1 \text{ ton})$ (step 4).

b. Notice that the drought changes our depth function, $s(x)$, and our limits of integration. We have $s(x) = x - 135.$ The lower limit of integration is $135.$ The upper limit remains $540.$ Evaluating the integral, we get

$$\begin{aligned} F &= \int_a^b \rho w(x)s(x)dx \\ &= \int_{135}^{540} 62.4\left(1250 - \frac{2}{3}x\right)(x - 135)dx \\ &= -62.4\left(\frac{2}{3}\right)\int_{135}^{540}(x - 1875)(x - 135)dx = -62.4\left(\frac{2}{3}\right)\int_{135}^{540}\left(x^2 - 2010x + 253125\right)dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - 1005x^2 + 253125x\right]\Bigg|_{135}^{540} \approx 5{,}015{,}230{,}000 \text{ lb} = 2{,}507{,}615 \text{ t.} \end{aligned}$$

6.28 When the reservoir is at its average level, the surface of the water is about 50 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

To learn more about Hoover Dam, see this **article (http://www.openstax.org/l/20_HooverDam)** published by the History Channel.

6.5 EXERCISES

For the following exercises, find the work done.

218. Find the work done when a constant force $F = 12$ lb moves a chair from $x = 0.9$ to $x = 1.1$ ft.

219. How much work is done when a person lifts a 50 lb box of comics onto a truck that is 3 ft off the ground?

220. What is the work done lifting a 20 kg child from the floor to a height of 2 m? (Note that 1 kg equates to 9.8 N)

221. Find the work done when you push a box along the floor 2 m, when you apply a constant force of $F = 100\,\text{N}$.

222. Compute the work done for a force $F = 12/x^2$ N from $x = 1$ to $x = 2$ m.

223. What is the work done moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

For the following exercises, find the mass of the one-dimensional object.

224. A wire that is 2 ft long (starting at $x = 0$) and has a density function of $\rho(x) = x^2 + 2x$ lb/ft

225. A car antenna that is 3 ft long (starting at $x = 0$) and has a density function of $\rho(x) = 3x + 2$ lb/ft

226. A metal rod that is 8 in. long (starting at $x = 0$) and has a density function of $\rho(x) = e^{1/2x}$ lb/in.

227. A pencil that is 4 in. long (starting at $x = 2$) and has a density function of $\rho(x) = 5/x$ oz/in.

228. A ruler that is 12 in. long (starting at $x = 5$) and has a density function of $\rho(x) = \ln(x) + (1/2)x^2$ oz/in.

For the following exercises, find the mass of the two-dimensional object that is centered at the origin.

229. An oversized hockey puck of radius 2 in. with density function $\rho(x) = x^3 - 2x + 5$

230. A frisbee of radius 6 in. with density function $\rho(x) = e^{-x}$

231. A plate of radius 10 in. with density function $\rho(x) = 1 + \cos(\pi x)$

232. A jar lid of radius 3 in. with density function $\rho(x) = \ln(x + 1)$

233. A disk of radius 5 cm with density function $\rho(x) = \sqrt{3x}$

234. A 12-in. spring is stretched to 15 in. by a force of 75 lb. What is the spring constant?

235. A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

236. A 1-m spring requires 10 J to stretch the spring to 1.1 m. How much work would it take to stretch the spring from 1 m to 1.2 m?

237. A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length of the spring?

238. A shock absorber is compressed 1 in. by a weight of 1 t. What is the spring constant?

239. A force of $F = 20x - x^3$ N stretches a nonlinear spring by x meters. What work is required to stretch the spring from $x = 0$ to $x = 2$ m?

240. Find the work done by winding up a hanging cable of length 100 ft and weight-density 5 lb/ft.

241. For the cable in the preceding exercise, how much work is done to lift the cable 50 ft?

242. For the cable in the preceding exercise, how much additional work is done by hanging a 200 lb weight at the end of the cable?

243. **[T]** A pyramid of height 500 ft has a square base 800 ft by 800 ft. Find the area A at height h. If the rock used to build the pyramid weighs approximately $w = 100\,\text{lb/ft}^3$, how much work did it take to lift all the rock?

244. **[T]** For the pyramid in the preceding exercise, assume there were 1000 workers each working 10 hours a day, 5 days a week, 50 weeks a year. If the workers, on average, lifted 10 100 lb rocks 2 ft/hr, how long did it take to build the pyramid?

245. **[T]** The force of gravity on a mass m is $F = -\left((GMm)/x^2\right)$ newtons. For a rocket of mass $m = 1000\,\text{kg},$ compute the work to lift the rocket from $x = 6400$ to $x = 6500$ m. State your answers with three significant figures. (*Note*: $G = 6.67 \times 10^{-17}$ N m^2/kg^2 and $M = 6 \times 10^{24}$ kg.)

246. **[T]** For the rocket in the preceding exercise, find the work to lift the rocket from $x = 6400$ to $x = \infty$.

247. **[T]** A rectangular dam is 40 ft high and 60 ft wide. Compute the total force F on the dam when

a. the surface of the water is at the top of the dam and
b. the surface of the water is halfway down the dam.

248. **[T]** Find the work required to pump all the water out of a cylinder that has a circular base of radius 5 ft and height 200 ft. Use the fact that the density of water is 62 lb/ft^3.

249. **[T]** Find the work required to pump all the water out of the cylinder in the preceding exercise if the cylinder is only half full.

250. **[T]** How much work is required to pump out a swimming pool if the area of the base is 800 ft^2, the water is 4 ft deep, and the top is 1 ft above the water level? Assume that the density of water is 62 lb/ft^3.

251. A cylinder of depth H and cross-sectional area A stands full of water at density ρ. Compute the work to pump all the water to the top.

252. For the cylinder in the preceding exercise, compute the work to pump all the water to the top if the cylinder is only half full.

253. A cone-shaped tank has a cross-sectional area that increases with its depth: $A = \left(\pi r^2 h^2\right)/H^3$. Show that the work to empty it is half the work for a cylinder with the same height and base.

6.6 | Moments and Centers of Mass

Learning Objectives

6.6.1 Find the center of mass of objects distributed along a line.
6.6.2 Locate the center of mass of a thin plate.
6.6.3 Use symmetry to help locate the centroid of a thin plate.
6.6.4 Apply the theorem of Pappus for volume.

In this section, we consider centers of mass (also called *centroids*, under certain conditions) and moments. The basic idea of the center of mass is the notion of a balancing point. Many of us have seen performers who spin plates on the ends of sticks. The performers try to keep several of them spinning without allowing any of them to drop. If we look at a single plate (without spinning it), there is a sweet spot on the plate where it balances perfectly on the stick. If we put the stick anywhere other than that sweet spot, the plate does not balance and it falls to the ground. (That is why performers spin the plates; the spin helps keep the plates from falling even if the stick is not exactly in the right place.) Mathematically, that sweet spot is called the *center of mass of the plate*.

In this section, we first examine these concepts in a one-dimensional context, then expand our development to consider centers of mass of two-dimensional regions and symmetry. Last, we use centroids to find the volume of certain solids by applying the theorem of Pappus.

Center of Mass and Moments

Let's begin by looking at the center of mass in a one-dimensional context. Consider a long, thin wire or rod of negligible mass resting on a fulcrum, as shown in **Figure 6.62**(a). Now suppose we place objects having masses m_1 and m_2 at distances d_1 and d_2 from the fulcrum, respectively, as shown in **Figure 6.62**(b).

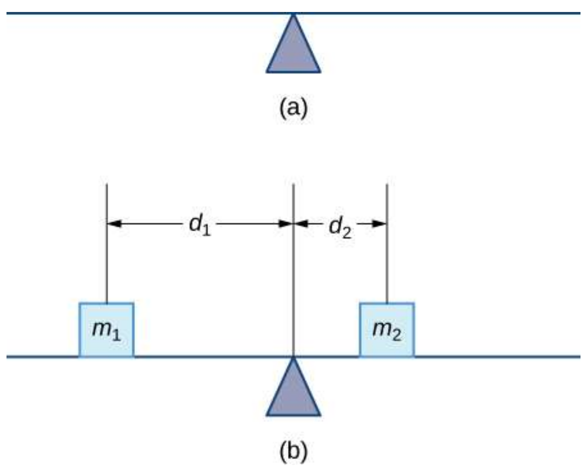


Figure 6.62 (a) A thin rod rests on a fulcrum. (b) Masses are placed on the rod.

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks down and the lighter child is lifted into the air. If the heavier child slides in toward the center, though, the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if $m_1 d_1 = m_2 d_2$.

In the seesaw example, we balanced the system by moving the masses (children) with respect to the fulcrum. However, we are really interested in systems in which the masses are not allowed to move, and instead we balance the system by moving the fulcrum. Suppose we have two point masses, m_1 and m_2, located on a number line at points x_1 and x_2, respectively (**Figure 6.63**). The center of mass, $\bar{x}$, is the point where the fulcrum should be placed to make the system balance.

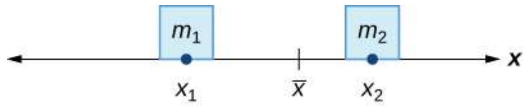


Figure 6.63 The center of mass $\bar{x}$ is the balance point of the system.

Thus, we have

$$\begin{aligned} m_1|x_1 - \bar{x}| &= m_2|x_2 - \bar{x}| \\ m_1(\bar{x} - x_1) &= m_2(x_2 - \bar{x}) \\ m_1\bar{x} - m_1x_1 &= m_2x_2 - m_2\bar{x} \\ \bar{x}(m_1 + m_2) &= m_1x_1 + m_2x_2 \\ \bar{x} &= \frac{m_1x_1 + m_2x_2}{m_1 + m_2}. \end{aligned}$$

The expression in the numerator, $m_1x_1 + m_2x_2,$ is called the *first moment of the system with respect to the origin.* If the context is clear, we often drop the word *first* and just refer to this expression as the **moment** of the system. The expression in the denominator, $m_1 + m_2,$ is the total mass of the system. Thus, the **center of mass** of the system is the point at which the total mass of the system could be concentrated without changing the moment.

This idea is not limited just to two point masses. In general, if *n* masses, $m_1, m_2, \ldots, m_n,$ are placed on a number line at points $x_1, x_2, \ldots, x_n,$ respectively, then the center of mass of the system is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$

Theorem 6.9: Center of Mass of Objects on a Line

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n,$ respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then, the moment of the system with respect to the origin is given by

$$M = \sum_{i=1}^{n} m_i x_i \quad (6.14)$$

and the center of mass of the system is given by

$$\bar{x} = \frac{M}{m}. \quad (6.15)$$

We apply this theorem in the following example.

Example 6.29

Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

$$m_1 = 30 \text{ kg, placed at } x_1 = -2 \text{ m} \qquad m_2 = 5 \text{ kg, placed at } x_2 = 3 \text{ m}$$
$$m_3 = 10 \text{ kg, placed at } x_3 = 6 \text{ m} \qquad m_4 = 15 \text{ kg, placed at } x_4 = -3 \text{ m.}$$

Find the moment of the system with respect to the origin and find the center of mass of the system.

Solution

First, we need to calculate the moment of the system:

$$\begin{aligned} M &= \sum_{i=1}^{4} m_i x_i \\ &= -60 + 15 + 60 - 45 = -30. \end{aligned}$$

Now, to find the center of mass, we need the total mass of the system:

$$\begin{aligned} m &= \sum_{i=1}^{4} m_i \\ &= 30 + 5 + 10 + 15 = 60 \text{ kg.} \end{aligned}$$

Then we have

$$\bar{x} = \frac{M}{m} = \frac{-30}{60} = -\frac{1}{2}.$$

The center of mass is located 1/2 m to the left of the origin.

6.29 Suppose four point masses are placed on a number line as follows:

$$m_1 = 12 \text{ kg, placed at } x_1 = -4 \text{ m} \qquad m_2 = 12 \text{ kg, placed at } x_2 = 4 \text{ m}$$
$$m_3 = 30 \text{ kg, placed at } x_3 = 2 \text{ m} \qquad m_4 = 6 \text{ kg, placed at } x_4 = -6 \text{ m.}$$

Find the moment of the system with respect to the origin and find the center of mass of the system.

We can generalize this concept to find the center of mass of a system of point masses in a plane. Let m_1 be a point mass located at point (x_1, y_1) in the plane. Then the moment M_x of the mass with respect to the *x*-axis is given by $M_x = m_1 y_1$. Similarly, the moment M_y with respect to the *y*-axis is given by $M_y = m_1 x_1$. Notice that the *x*-coordinate of the point is used to calculate the moment with respect to the *y*-axis, and vice versa. The reason is that the *x*-coordinate gives the distance from the point mass to the *y*-axis, and the *y*-coordinate gives the distance to the *x*-axis (see the following figure).

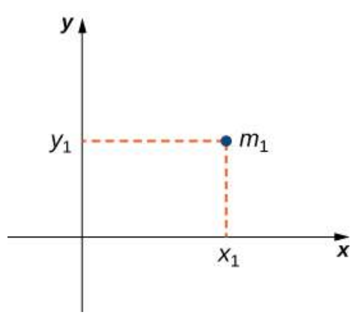


Figure 6.64 Point mass m_1 is located at point (x_1, y_1) in the plane.

If we have several point masses in the *xy*-plane, we can use the moments with respect to the *x*- and *y*-axes to calculate the

x- and y-coordinates of the center of mass of the system.

Theorem 6.10: Center of Mass of Objects in a Plane

Let $m_1, m_2, \ldots, m_n$ be point masses located in the xy-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$ respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the moments M_x and M_y of the system with respect to the x- and y-axes, respectively, are given by

$$M_x = \sum_{i=1}^{n} m_i y_i \quad \text{and} \quad M_y = \sum_{i=1}^{n} m_i x_i. \tag{6.16}$$

Also, the coordinates of the center of mass $(\bar{x}, \bar{y})$ of the system are

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}. \tag{6.17}$$

The next example demonstrates how to apply this theorem.

Example 6.30

Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the xy-plane as follows (assume coordinates are given in meters):

$$m_1 = 2\text{ kg, placed at } (-1, 3),$$
$$m_2 = 6\text{ kg, placed at } (1, 1),$$
$$m_3 = 4\text{ kg, placed at } (2, -2).$$

Find the center of mass of the system.

Solution

First we calculate the total mass of the system:

$$m = \sum_{i=1}^{3} m_i = 2 + 6 + 4 = 12\text{ kg}.$$

Next we find the moments with respect to the x- and y-axes:

$$M_y = \sum_{i=1}^{3} m_i x_i = -2 + 6 + 8 = 12,$$
$$M_x = \sum_{i=1}^{3} m_i y_i = 6 + 6 - 8 = 4.$$

Then we have

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1 \text{ and } \bar{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}.$$

The center of mass of the system is $(1, 1/3),$ in meters.

6.30 Suppose three point masses are placed on a number line as follows (assume coordinates are given in meters):

$$m_1 = 5\text{ kg, placed at }(-2,\ -3),$$
$$m_2 = 3\text{ kg, placed at }(2,\ 3),$$
$$m_3 = 2\text{ kg, placed at }(-3,\ -2).$$

Find the center of mass of the system.

Center of Mass of Thin Plates

So far we have looked at systems of point masses on a line and in a plane. Now, instead of having the mass of a system concentrated at discrete points, we want to look at systems in which the mass of the system is distributed continuously across a thin sheet of material. For our purposes, we assume the sheet is thin enough that it can be treated as if it is two-dimensional. Such a sheet is called a **lamina**. Next we develop techniques to find the center of mass of a lamina. In this section, we also assume the density of the lamina is constant.

Laminas are often represented by a two-dimensional region in a plane. The geometric center of such a region is called its **centroid**. Since we have assumed the density of the lamina is constant, the center of mass of the lamina depends only on the shape of the corresponding region in the plane; it does not depend on the density. In this case, the center of mass of the lamina corresponds to the centroid of the delineated region in the plane. As with systems of point masses, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the *x*- and *y*-axes.

We first consider a lamina in the shape of a rectangle. Recall that the center of mass of a lamina is the point where the lamina balances. For a rectangle, that point is both the horizontal and vertical center of the rectangle. Based on this understanding, it is clear that the center of mass of a rectangular lamina is the point where the diagonals intersect, which is a result of the **symmetry principle**, and it is stated here without proof.

Theorem 6.11: The Symmetry Principle

If a region R is symmetric about a line l, then the centroid of R lies on l.

Let's turn to more general laminas. Suppose we have a lamina bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in the following figure.

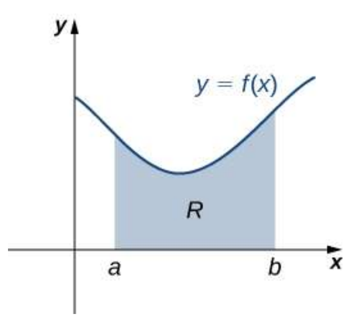


Figure 6.65 A region in the plane representing a lamina.

As with systems of point masses, to find the center of mass of the lamina, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the *x*- and *y*-axes. As we have done many times before, we approximate these quantities by partitioning the interval $[a, b]$ and constructing rectangles.

For $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of $[a, b]$. Recall that we can choose any point within the interval $[x_{i-1}, x_i]$ as our x_i^*. In this case, we want x_i^* to be the x-coordinate of the centroid of our rectangles. Thus, for $i = 1, 2,\ldots, n,$ we select $x_i^* \in [x_{i-1}, x_i]$ such that x_i^* is the midpoint of the interval. That is, $x_i^* = (x_{i-1} + x_i)/2.$

Now, for $i = 1, 2,\ldots, n,$ construct a rectangle of height $f(x_i^*)$ on $[x_{i-1}, x_i]$. The center of mass of this rectangle is

$\left(x_i^*, \left(f(x_i^*)\right)/2\right),$ as shown in the following figure.

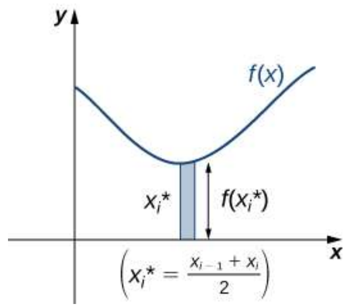


Figure 6.66 A representative rectangle of the lamina.

Next, we need to find the total mass of the rectangle. Let ρ represent the density of the lamina (note that ρ is a constant). In this case, ρ is expressed in terms of mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by ρ. Then, the mass of the rectangle is given by $\rho f(x_i^*)\Delta x$.

To get the approximate mass of the lamina, we add the masses of all the rectangles to get

$$m \approx \sum_{i=1}^{n} \rho f(x_i^*)\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$ gives the exact mass of the lamina:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x = \rho \int_a^b f(x)dx.$$

Next, we calculate the moment of the lamina with respect to the *x*-axis. Returning to the representative rectangle, recall its center of mass is $\left(x_i^*, \left(f(x_i^*)\right)/2\right)$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change the moment. Thus, the moment of the rectangle with respect to the *x*-axis is given by the mass of the rectangle, $\rho f(x_i^*)\Delta x,$ multiplied by the distance from the center of mass to the *x*-axis: $\left(f(x_i^*)\right)/2$. Therefore, the moment with respect to the *x*-axis of the rectangle is $\rho\left(\left[f(x_i^*)\right]^2/2\right)\Delta x$. Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the *x*-axis is

$$M_x = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \frac{\left[f(x_i^*)\right]^2}{2}\Delta x = \rho \int_a^b \frac{\left[f(x)\right]^2}{2}dx.$$

We derive the moment with respect to the *y*-axis similarly, noting that the distance from the center of mass of the rectangle to the *y*-axis is x_i^*. Then the moment of the lamina with respect to the *y*-axis is given by

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{n} \rho x_i^* f(x_i^*)\Delta x = \rho \int_a^b x f(x)dx.$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$. If we look closely at the expressions for $M_x, M_y,$ and $m,$ we notice that the constant ρ cancels out when $\bar{x}$ and $\bar{y}$ are calculated.

We summarize these findings in the following theorem.

Theorem 6.12: Center of Mass of a Thin Plate in the *xy*-Plane

Let *R* denote a region bounded above by the graph of a continuous function $f(x),$ below by the *x*-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b f(x)dx. \tag{6.18}$$

ii. The moments M_x and M_y of the lamina with respect to the *x*- and *y*-axes, respectively, are

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx \text{ and } M_y = \rho \int_a^b x f(x)dx. \tag{6.19}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}. \tag{6.20}$$

In the next example, we use this theorem to find the center of mass of a lamina.

Example 6.31

Finding the Center of Mass of a Lamina

Let R be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the *x*-axis over the interval $[0, 4]$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

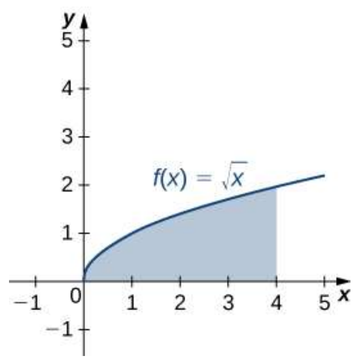


Figure 6.67 Finding the center of mass of a lamina.

Since we are only asked for the centroid of the region, rather than the mass or moments of the associated lamina, we know the density constant ρ cancels out of the calculations eventually. Therefore, for the sake of convenience, let's assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)dx = \int_0^4 \sqrt{x}\, dx \\ &= \frac{2}{3} x^{3/2} \Big|_0^4 = \frac{2}{3}[8 - 0] = \frac{16}{3}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \int_0^4 \frac{x}{2} dx = \frac{1}{4} x^2 \Big|_0^4 = 4 \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho \int_a^b x f(x) dx \\ &= \int_0^4 x\sqrt{x} dx = \int_0^4 x^{3/2} dx \\ &= \frac{2}{5} x^{5/2} \Big|_0^4 = \frac{2}{5}[32 - 0] = \frac{64}{5}. \end{aligned}$$

Thus, we have

$$\bar{x} = \frac{M_y}{m} = \frac{64/5}{16/3} = \frac{64}{5} \cdot \frac{3}{16} = \frac{12}{5} \text{ and } \bar{y} = \frac{M_x}{y} = \frac{4}{16/3} = 4 \cdot \frac{3}{16} = \frac{3}{4}.$$

The centroid of the region is $(12/5, 3/4)$.

6.31 Let R be the region bounded above by the graph of the function $f(x) = x^2$ and below by the x-axis over the interval $[0, 2]$. Find the centroid of the region.

We can adapt this approach to find centroids of more complex regions as well. Suppose our region is bounded above by the graph of a continuous function $f(x)$, as before, but now, instead of having the lower bound for the region be the x-axis, suppose the region is bounded below by the graph of a second continuous function, $g(x)$, as shown in the following figure.

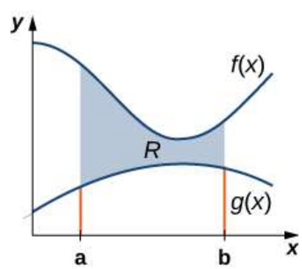


Figure 6.68 A region between two functions.

Again, we partition the interval $[a, b]$ and construct rectangles. A representative rectangle is shown in the following figure.

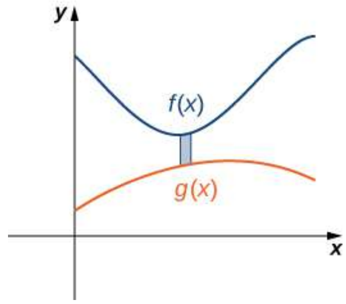


Figure 6.69 A representative rectangle of the region between two functions.

Note that the centroid of this rectangle is $\left(x_i^*,\ \left(f(x_i^*)+g(x_i^*)\right)/2\right).$ We won't go through all the details of the Riemann sum development, but let's look at some of the key steps. In the development of the formulas for the mass of the lamina and the moment with respect to the *y*-axis, the height of each rectangle is given by $f(x_i^*)-g(x_i^*),$ which leads to the expression $f(x)-g(x)$ in the integrands.

In the development of the formula for the moment with respect to the *x*-axis, the moment of each rectangle is found by multiplying the area of the rectangle, $\rho\left[f(x_i^*)-g(x_i^*)\right]\Delta x,$ by the distance of the centroid from the *x*-axis, $\left(f(x_i^*)+g(x_i^*)\right)/2,$ which gives $\rho(1/2)\left\{\left[f(x_i^*)\right]^2-\left[g(x_i^*)\right]^2\right\}\Delta x.$ Summarizing these findings, we arrive at the following theorem.

Theorem 6.13: Center of Mass of a Lamina Bounded by Two Functions

Let R denote a region bounded above by the graph of a continuous function $f(x),$ below by the graph of the continuous function $g(x),$ and on the left and right by the lines $x=a$ and $x=b,$ respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m=\rho\int_a^b\left[f(x)-g(x)\right]dx. \tag{6.21}$$

ii. The moments M_x and M_y of the lamina with respect to the *x*- and *y*-axes, respectively, are

$$M_x=\rho\int_a^b\frac{1}{2}\left(\left[f(x)\right]^2-\left[g(x)\right]^2\right)dx \text{ and } M_y=\rho\int_a^b x\left[f(x)-g(x)\right]dx. \tag{6.22}$$

iii. The coordinates of the center of mass $(\bar{x},\ \bar{y})$ are

$$\bar{x}=\frac{M_y}{m} \text{ and } \bar{y}=\frac{M_x}{m}. \tag{6.23}$$

We illustrate this theorem in the following example.

Example 6.32

Finding the Centroid of a Region Bounded by Two Functions

Let R be the region bounded above by the graph of the function $f(x)=1-x^2$ and below by the graph of the function $g(x)=x-1.$ Find the centroid of the region.

Solution

The region is depicted in the following figure.

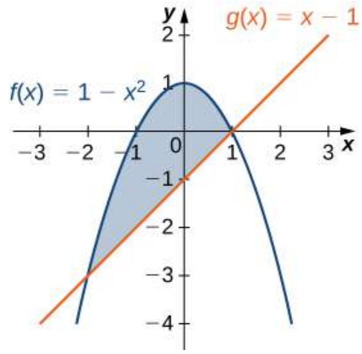


Figure 6.70 Finding the centroid of a region between two curves.

The graphs of the functions intersect at $(-2,\ -3)$ and $(1,\ 0)$, so we integrate from −2 to 1. Once again, for the sake of convenience, assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho\int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^1 \left[1 - x^2 - (x-1)\right]dx = \int_{-2}^1 (2 - x^2 - x)dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]\Big|_{-2}^1 = \left[2 - \frac{1}{3} - \frac{1}{2}\right] - \left[-4 + \frac{8}{3} - 2\right] = \frac{9}{2}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho\int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)dx \\ &= \frac{1}{2}\int_{-2}^1 \left(\left(1 - x^2\right)^2 - (x-1)^2\right)dx = \frac{1}{2}\int_{-2}^1 \left(x^4 - 3x^2 + 2x\right)dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - x^3 + x^2\right]\Big|_{-2}^1 = -\frac{27}{10} \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho\int_a^b x[f(x) - g(x)]dx \\ &= \int_{-2}^1 x\left[(1 - x^2) - (x-1)\right]dx = \int_{-2}^1 x\left[2 - x^2 - x\right]dx \\ &= \int_{-2}^1 \left(2x - x^4 - x^2\right)dx \\ &= \left[x^2 - \frac{x^5}{5} - \frac{x^3}{3}\right]\Big|_{-2}^1 = -\frac{9}{4}. \end{aligned}$$

Therefore, we have

$$\bar{x} = \frac{M_y}{m} = -\frac{9}{4} \cdot \frac{2}{9} = -\frac{1}{2} \text{ and } \bar{y} = \frac{M_x}{m} = -\frac{27}{10} \cdot \frac{2}{9} = -\frac{3}{5}.$$

The centroid of the region is $(-(1/2),\ -(3/5))$.

6.32 Let R be the region bounded above by the graph of the function $f(x) = 6 - x^2$ and below by the graph of the function $g(x) = 3 - 2x$. Find the centroid of the region.

The Symmetry Principle

We stated the symmetry principle earlier, when we were looking at the centroid of a rectangle. The symmetry principle can be a great help when finding centroids of regions that are symmetric. Consider the following example.

Example 6.33

Finding the Centroid of a Symmetric Region

Let R be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the x-axis. Find the centroid of the region.

Solution

The region is depicted in the following figure.

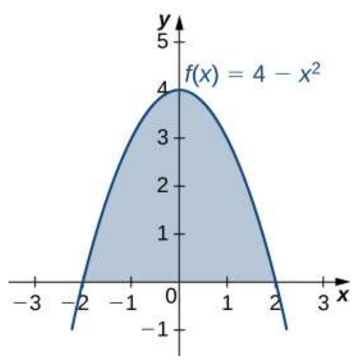


Figure 6.71 We can use the symmetry principle to help find the centroid of a symmetric region.

The region is symmetric with respect to the y-axis. Therefore, the x-coordinate of the centroid is zero. We need only calculate $\bar{y}$. Once again, for the sake of convenience, assume $\rho = 1$.

First, we calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)dx \\ &= \int_{-2}^{2} \left(4 - x^2\right) dx \\ &= \left[4x - \frac{x^3}{3}\right]\Bigg|_{-2}^{2} = \frac{32}{3}. \end{aligned}$$

Next, we calculate the moments. We only need M_x:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \frac{1}{2} \int_{-2}^{2} \left[4 - x^2\right]^2 dx = \frac{1}{2} \int_{-2}^{2} \left(16 - 8x^2 + x^4\right) dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]\Bigg|_{-2}^{2} = \frac{256}{15}. \end{aligned}$$

Then we have

$$\bar{y} = \frac{M_x}{m} = \frac{256}{15} \cdot \frac{3}{32} = \frac{8}{5}.$$

The centroid of the region is $(0, 8/5)$.

6.33 Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by x-axis. Find the centroid of the region.