# Calc-ulus

## Volume 1

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Student PROJECT

## The Grand Canyon Skywalk

The Grand Canyon Skywalk opened to the public on March 28, 2007. This engineering marvel is a horseshoe-shaped observation platform suspended 4000 ft above the Colorado River on the West Rim of the Grand Canyon. Its crystal-clear glass floor allows stunning views of the canyon below (see the following figure).

**Figure 6.72** The Grand Canyon Skywalk offers magnificent views of the canyon. (credit: 10da_ralta, Wikimedia Commons)

The Skywalk is a cantilever design, meaning that the observation platform extends over the rim of the canyon, with no visible means of support below it. Despite the lack of visible support posts or struts, cantilever structures are engineered to be very stable and the Skywalk is no exception. The observation platform is attached firmly to support posts that extend 46 ft down into bedrock. The structure was built to withstand 100-mph winds and an 8.0-magnitude earthquake within 50 mi, and is capable of supporting more than 70,000,000 lb.

One factor affecting the stability of the Skywalk is the center of gravity of the structure. We are going to calculate the center of gravity of the Skywalk, and examine how the center of gravity changes when tourists walk out onto the observation platform.

The observation platform is U-shaped. The legs of the U are 10 ft wide and begin on land, under the visitors' center, 48 ft from the edge of the canyon. The platform extends 70 ft over the edge of the canyon.

To calculate the center of mass of the structure, we treat it as a lamina and use a two-dimensional region in the *xy*-plane to represent the platform. We begin by dividing the region into three subregions so we can consider each subregion separately. The first region, denoted $R_1$, consists of the curved part of the U. We model $R_1$ as a semicircular annulus, with inner radius 25 ft and outer radius 35 ft, centered at the origin (see the following figure).

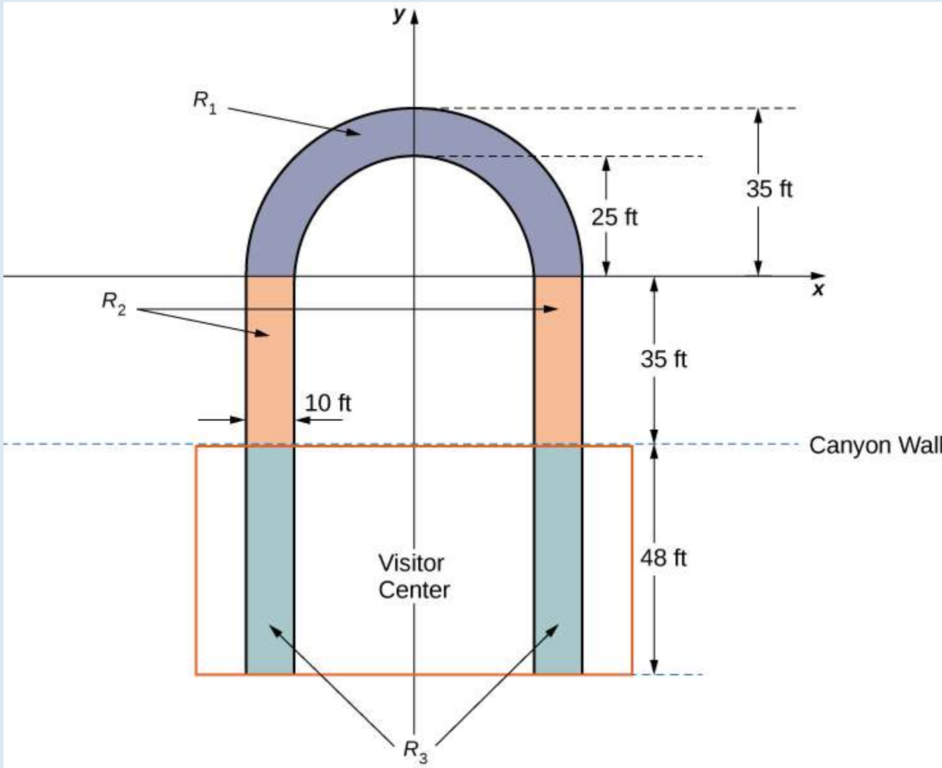



**Figure 6.73** We model the Skywalk with three sub-regions.

The legs of the platform, extending 35 ft between $R_1$ and the canyon wall, comprise the second sub-region, $R_2$. Last, the ends of the legs, which extend 48 ft under the visitor center, comprise the third sub-region, $R_3$. Assume the density of the lamina is constant and assume the total weight of the platform is 1,200,000 lb (not including the weight of the visitor center; we will consider that later). Use $g = 32\ \text{ft/sec}^2$.

1. Compute the area of each of the three sub-regions. Note that the areas of regions $R_2$ and $R_3$ should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.
2. Determine the mass associated with each of the three sub-regions.
3. Calculate the center of mass of each of the three sub-regions.
4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of the entire platform.
5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of $R_3$. Treating the visitor center as a point mass, recalculate the center of mass of the system. How does the center of mass change?
6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform is capable of supporting up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the center of gravity of the system be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

## Theorem of Pappus

This section ends with a discussion of the **theorem of Pappus for volume**, which allows us to find the volume of particular kinds of solids by using the centroid. (There is also a theorem of Pappus for surface area, but it is much less useful than the theorem for volume.)

### Theorem 6.14: Theorem of Pappus for Volume

Let $R$ be a region in the plane and let $l$ be a line in the plane that does not intersect $R$. Then the volume of the solid of revolution formed by revolving $R$ around $l$ is equal to the area of $R$ multiplied by the distance $d$ traveled by the centroid of $R$.

### Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a, b]$, and for which the axis of revolution is the $y$-axis. In this case, the area of the region is $A = \int_a^b [f(x) - g(x)]dx$. Since the axis of rotation is the $y$-axis, the distance traveled by the centroid of the region depends only on the $x$-coordinate of the centroid, $\bar{x}$, which is

$$\bar{x} = \frac{M_y}{m},$$

where

$$m = \rho \int_a^b [f(x) - g(x)]dx \text{ and } M_y = \rho \int_a^b x[f(x) - g(x)]dx.$$

Then,

$$d = 2\pi \frac{\rho \int_a^b x[f(x) - g(x)]dx}{\rho \int_a^b [f(x) - g(x)]dx}$$

and thus

$$d \cdot A = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

However, using the method of cylindrical shells, we have

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

So,

$$V = d \cdot A$$

and the proof is complete.

□

## Example 6.34

### Using the Theorem of Pappus for Volume

Let $R$ be a circle of radius 2 centered at $(4, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving $R$ around the $y$-axis.

### Solution

The region and torus are depicted in the following figure.

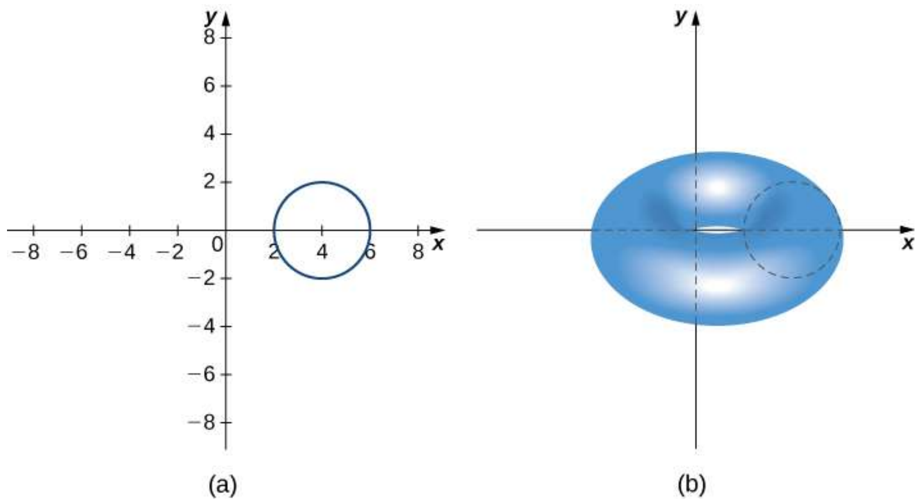



**Figure 6.74** Determining the volume of a torus by using the theorem of Pappus. (a) A circular region $R$ in the plane; (b) the torus generated by revolving $R$ about the $y$-axis.

The region $R$ is a circle of radius 2, so the area of $R$ is $A = 4\pi$ units$^2$. By the symmetry principle, the centroid of $R$ is the center of the circle. The centroid travels around the $y$-axis in a circular path of radius 4, so the centroid travels $d = 8\pi$ units. Then, the volume of the torus is $A \cdot d = 32\pi^2$ units$^3$.

**6.34** Let $R$ be a circle of radius 1 centered at $(3, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving $R$ around the $y$-axis.

# 6.6 EXERCISES

For the following exercises, calculate the center of mass for the collection of masses given.

254. $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

255. $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

256. $m = 3$ at $x = 0, 1, 2, 6$

257. Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

258. $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

259. $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

For the following exercises, compute the center of mass $\bar{x}$.

260. $\rho = 1$ for $x \in (-1, 3)$

261. $\rho = x^2$ for $x \in (0, L)$

262. $\rho = 1$ for $x \in (0, 1)$ and $\rho = 2$ for $x \in (1, 2)$

263. $\rho = \sin x$ for $x \in (0, \pi)$

264. $\rho = \cos x$ for $x \in \left(0, \frac{\pi}{2}\right)$

265. $\rho = e^x$ for $x \in (0, 2)$

266. $\rho = x^3 + xe^{-x}$ for $x \in (0, 1)$

267. $\rho = x \sin x$ for $x \in (0, \pi)$

268. $\rho = \sqrt{x}$ for $x \in (1, 4)$

269. $\rho = \ln x$ for $x \in (1, e)$

For the following exercises, compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

270. $\rho = 7$ in the square $0 \le x \le 1, \quad 0 \le y \le 1$

271. $\rho = 3$ in the triangle with vertices $(0, 0), \quad (a, 0),$ and $(0, b)$

272. $\rho = 2$ for the region bounded by $y = \cos(x),$ $y = -\cos(x), \quad x = -\frac{\pi}{2},$ and $x = \frac{\pi}{2}$

For the following exercises, use a calculator to draw the region, then compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

273. **[T]** The region bounded by $y = \cos(2x),$ $x = -\frac{\pi}{4},$ and $x = \frac{\pi}{4}$

274. **[T]** The region between $y = 2x^2, \quad y = 0, \quad x = 0,$ and $x = 1$

275. **[T]** The region between $y = \frac{5}{4}x^2$ and $y = 5$

276. **[T]** Region between $y = \sqrt{x}, \quad y = \ln(x), \quad x = 1,$ and $x = 4$

277. **[T]** The region bounded by $y = 0, \quad \frac{x^2}{4} + \frac{y^2}{9} = 1$

278. **[T]** The region bounded by $y = 0, \quad x = 0,$ and $\frac{x^2}{4} + \frac{y^2}{9} = 1$

279. **[T]** The region bounded by $y = x^2$ and $y = x^4$ in the first quadrant

For the following exercises, use the theorem of Pappus to determine the volume of the shape.

280. Rotating $y = mx$ around the $x$-axis between $x = 0$ and $x = 1$

281. Rotating $y = mx$ around the $y$-axis between $x = 0$ and $x = 1$

282. A general cone created by rotating a triangle with vertices $(0, 0), \quad (a, 0),$ and $(0, b)$ around the $y$-axis. Does your answer agree with the volume of a cone?

283. A general cylinder created by rotating a rectangle with vertices $(0, 0), \quad (a, 0), (0, b),$ and $(a, b)$ around the $y$-axis. Does your answer agree with the volume of a cylinder?

284. A sphere created by rotating a semicircle with radius $a$ around the $y$-axis. Does your answer agree with the volume of a sphere?

For the following exercises, use a calculator to draw the region enclosed by the curve. Find the area $M$ and the centroid $(\overline{x}, \overline{y})$ for the given shapes. Use symmetry to help locate the center of mass whenever possible.

285. **[T]** Quarter-circle: $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$

286. **[T]** Triangle: $y = x$, $y = 2 - x$, and $y = 0$

287. **[T]** Lens: $y = x^2$ and $y = x$

288. **[T]** Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

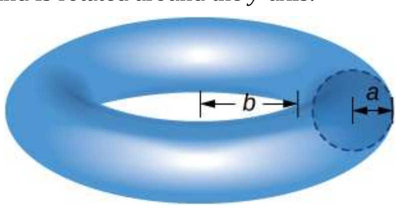
289. **[T]** Half-ring: $y^2 + x^2 = 1$, $y^2 + x^2 = 4$, and $y = 0$

290. Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

291. Find the generalized center of mass between $y = a^2 - x^2$, $x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

292. Find the generalized center of mass between $y = b\sin(ax)$, $x = 0$, and $x = \frac{\pi}{a}$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

293. Use the theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius $a$ is positioned with the left end of the circle at $x = b$, $b > 0$, and is rotated around the *y*-axis.




294. Find the center of mass $(\overline{x}, \overline{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (*Hint:* Use the theorem of Pappus.)

# 6.7 | Integrals, Exponential Functions, and Logarithms

**Learning Objectives**

- **6.7.1** Write the definition of the natural logarithm as an integral.
- **6.7.2** Recognize the derivative of the natural logarithm.
- **6.7.3** Integrate functions involving the natural logarithmic function.
- **6.7.4** Define the number $e$ through an integral.
- **6.7.5** Recognize the derivative and integral of the exponential function.
- **6.7.6** Prove properties of logarithms and exponential functions using integrals.
- **6.7.7** Express general logarithmic and exponential functions in terms of natural logarithms and exponentials.

We already examined exponential functions and logarithms in earlier chapters. However, we glossed over some key details in the previous discussions. For example, we did not study how to treat exponential functions with exponents that are irrational. The definition of the number *e* is another area where the previous development was somewhat incomplete. We now have the tools to deal with these concepts in a more mathematically rigorous way, and we do so in this section.

For purposes of this section, assume we have not yet defined the natural logarithm, the number *e*, or any of the integration and differentiation formulas associated with these functions. By the end of the section, we will have studied these concepts in a mathematically rigorous way (and we will see they are consistent with the concepts we learned earlier).

We begin the section by defining the natural logarithm in terms of an integral. This definition forms the foundation for the section. From this definition, we derive differentiation formulas, define the number $e$, and expand these concepts to logarithms and exponential functions of any base.

## The Natural Logarithm as an Integral

Recall the power rule for integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ \ n \neq -1.$$

Clearly, this does not work when $n = -1$, as it would force us to divide by zero. So, what do we do with $\int \frac{1}{x}dx$? Recall from the Fundamental Theorem of Calculus that $\int_1^x \frac{1}{t}dt$ is an antiderivative of $1/x$. Therefore, we can make the following definition.

### Definition

For $x > 0$, define the natural logarithm function by

$$\ln x = \int_1^x \frac{1}{t}dt. \tag{6.24}$$

For $x > 1$, this is just the area under the curve $y = 1/t$ from $1$ to $x$. For $x < 1$, we have $\int_1^x \frac{1}{t}dt = -\int_x^1 \frac{1}{t}dt$, so in this case it is the negative of the area under the curve from $x$ to $1$ (see the following figure).

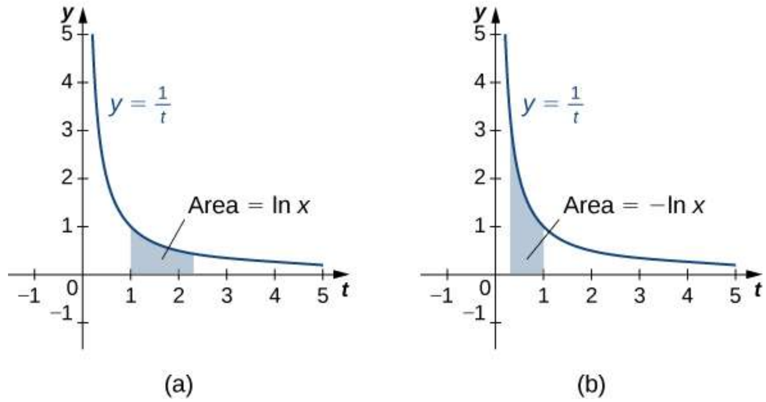



**Figure 6.75** (a) When $x > 1$, the natural logarithm is the area under the curve $y = 1/t$ from $1$ to $x$. (b) When $x < 1$, the natural logarithm is the negative of the area under the curve from $x$ to $1$.

Notice that $\ln 1 = 0$. Furthermore, the function $y = 1/t > 0$ for $x > 0$. Therefore, by the properties of integrals, it is clear that $\ln x$ is increasing for $x > 0$.

## Properties of the Natural Logarithm

Because of the way we defined the natural logarithm, the following differentiation formula falls out immediately as a result of to the Fundamental Theorem of Calculus.

### Theorem 6.15: Derivative of the Natural Logarithm

For $x > 0$, the derivative of the natural logarithm is given by

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

### Theorem 6.16: Corollary to the Derivative of the Natural Logarithm

The function $\ln x$ is differentiable; therefore, it is continuous.

A graph of $\ln x$ is shown in **Figure 6.76**. Notice that it is continuous throughout its domain of $(0, \infty)$.

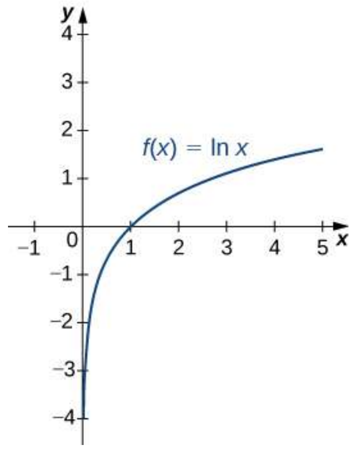



**Figure 6.76** The graph of $f(x) = \ln x$ shows that it is a continuous function.

## Example 6.35

### Calculating Derivatives of Natural Logarithms

Calculate the following derivatives:

a. $\frac{d}{dx}\ln\left(5x^3 - 2\right)$

b. $\frac{d}{dx}(\ln(3x))^2$

### Solution

We need to apply the chain rule in both cases.

a. $\frac{d}{dx}\ln\left(5x^3 - 2\right) = \frac{15x^2}{5x^3 - 2}$

b. $\frac{d}{dx}(\ln(3x))^2 = \frac{2(\ln(3x)) \cdot 3}{3x} = \frac{2(\ln(3x))}{x}$

**6.35** Calculate the following derivatives:

a. $\frac{d}{dx}\ln\left(2x^2 + x\right)$

b. $\frac{d}{dx}\left(\ln\left(x^3\right)\right)^2$

Note that if we use the absolute value function and create a new function $\ln|x|$, we can extend the domain of the natural logarithm to include $x < 0$. Then $(d/(dx))\ln|x| = 1/x$. This gives rise to the familiar integration formula.

### Theorem 6.17: Integral of (1/*u*) *du*

The natural logarithm is the antiderivative of the function $f(u) = 1/u$:

$$\int \frac{1}{u}du = \ln|u| + C.$$

## Example 6.36

### Calculating Integrals Involving Natural Logarithms

Calculate the integral $\int \frac{x}{x^2+4}dx.$

#### Solution

Using $u$ -substitution, let $u = x^2 + 4.$ Then $du = 2x\,dx$ and we have

$$\int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+4\right| + C = \frac{1}{2}\ln\left(x^2+4\right) + C.$$

**6.36** Calculate the integral $\int \frac{x^2}{x^3+6}dx.$

Although we have called our function a "logarithm," we have not actually proved that any of the properties of logarithms hold for this function. We do so here.

### Theorem 6.18: Properties of the Natural Logarithm

If $a, b > 0$ and $r$ is a rational number, then

i. $\ln 1 = 0$

ii. $\ln(ab) = \ln a + \ln b$

iii. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

iv. $\ln(a^r) = r \ln a$

### Proof

i. By definition, $\ln 1 = \int_1^1 \frac{1}{t}dt = 0.$

ii. We have

$$() \ln(ab) = \int_1^{ab} \frac{1}{t}dt = \int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{1}{t}dt.$$

Use $u$-substitution on the last integral in this expression. Let $u = t/a.$ Then $du = (1/a)dt.$ Furthermore, when $t = a,\ u = 1,$ and when $t = ab,\ u = b.$ So we get

$$() \ln(ab) = \int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{1}{t}dt = \int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{a}{t}\cdot\frac{1}{a}dt = \int_1^a \frac{1}{t}dt + \int_1^b \frac{1}{u}du = \ln a + \ln b.$$

iv. Note that

$$\frac{d}{dx}\ln(x^r) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}.$$

Furthermore,

$$\frac{d}{dx}(r\ln x) = \frac{r}{x}.$$

Since the derivatives of these two functions are the same, by the Fundamental Theorem of Calculus, they must differ by a constant. So we have

$$\ln(x^r) = r\ln x + C$$

for some constant $C$. Taking $x = 1,$ we get

$$\begin{aligned} \ln(1^r) &= r\ln(1) + C \\ 0 &= r(0) + C \\ C &= 0. \end{aligned}$$

Thus $\ln(x^r) = r\ln x$ and the proof is complete. Note that we can extend this property to irrational values of $r$ later in this section.
Part iii. follows from parts ii. and iv. and the proof is left to you.

□

## Example 6.37

### Using Properties of Logarithms

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 9 - 2\ln 3 + \ln\left(\frac{1}{3}\right).$$

#### Solution

We have

$$\ln 9 - 2\ln 3 + \ln\left(\frac{1}{3}\right) = \ln\left(3^2\right) - 2\ln 3 + \ln\left(3^{-1}\right) = 2\ln 3 - 2\ln 3 - \ln 3 = -\ln 3.$$

**6.37** Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 8 - \ln 2 - \ln\left(\frac{1}{4}\right).$$

## Defining the Number *e*

Now that we have the natural logarithm defined, we can use that function to define the number $e$.

**Definition**

The number $e$ is defined to be the real number such that

$$\ln e = 1.$$

To put it another way, the area under the curve $y = 1/t$ between $t = 1$ and $t = e$ is $1$ (**Figure 6.77**). The proof that such a number exists and is unique is left to you. (*Hint*: Use the Intermediate Value Theorem to prove existence and the fact that $\ln x$ is increasing to prove uniqueness.)

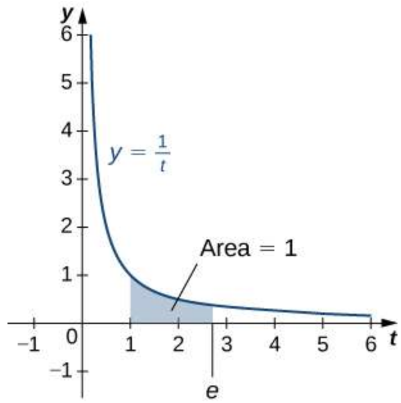



**Figure 6.77** The area under the curve from $1$ to $e$ is equal to one.

The number $e$ can be shown to be irrational, although we won't do so here (see the Student Project in **Taylor and Maclaurin Series (http://cnx.org/content/m53817/latest/)** ). Its approximate value is given by

$$e \approx 2.71828182846.$$

## The Exponential Function

We now turn our attention to the function $e^x$. Note that the natural logarithm is one-to-one and therefore has an inverse function. For now, we denote this inverse function by $\exp x$. Then,

$$\exp(\ln x) = x \text{ for } x > 0 \text{ and } \ln(\exp x) = x \text{ for all } x.$$

The following figure shows the graphs of $\exp x$ and $\ln x$.

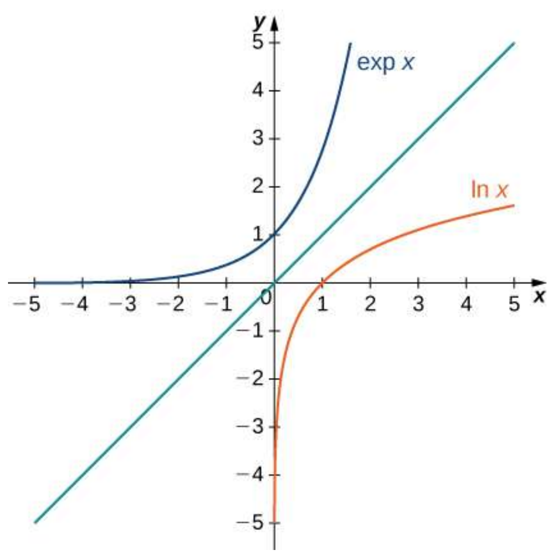



**Figure 6.78** The graphs of $\ln x$ and $\exp x$.

We hypothesize that $\exp x = e^x$. For rational values of $x$, this is easy to show. If $x$ is rational, then we have $\ln(e^x) = x \ln e = x$. Thus, when $x$ is rational, $e^x = \exp x$. For irrational values of $x$, we simply define $e^x$ as the inverse function of $\ln x$.

### Definition

For any real number $x$, define $y = e^x$ to be the number for which

$$\ln y = \ln(e^x) = x. \tag{6.25}$$

Then we have $e^x = \exp(x)$ for all $x$, and thus

$$e^{\ln x} = x \text{ for } x > 0 \text{ and } \ln(e^x) = x \tag{6.26}$$

for all $x$.

## Properties of the Exponential Function

Since the exponential function was defined in terms of an inverse function, and not in terms of a power of $e$, we must verify that the usual laws of exponents hold for the function $e^x$.

### Theorem 6.19: Properties of the Exponential Function

If $p$ and $q$ are any real numbers and $r$ is a rational number, then

i. $e^p e^q = e^{p+q}$

ii. $\dfrac{e^p}{e^q} = e^{p-q}$

iii. $(e^p)^r = e^{pr}$

### Proof

Note that if $p$ and $q$ are rational, the properties hold. However, if $p$ or $q$ are irrational, we must apply the inverse function definition of $e^x$ and verify the properties. Only the first property is verified here; the other two are left to you. We have

$$\ln(e^p e^q) = \ln(e^p) + \ln(e^q) = p + q = \ln\left(e^{p+q}\right).$$

Since $\ln x$ is one-to-one, then

$$e^p e^q = e^{p+q}.$$

□

As with part iv. of the logarithm properties, we can extend property iii. to irrational values of $r$, and we do so by the end of the section.

We also want to verify the differentiation formula for the function $y = e^x$. To do this, we need to use implicit differentiation. Let $y = e^x$. Then

$$\begin{aligned} \ln y &= x \\ \frac{d}{dx}\ln y &= \frac{d}{dx}x \\ \frac{1}{y}\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y. \end{aligned}$$

Thus, we see

$$\frac{d}{dx}e^x = e^x$$

as desired, which leads immediately to the integration formula

$$\int e^x\,dx = e^x + C.$$

We apply these formulas in the following examples.

## Example 6.38

### Using Properties of Exponential Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}e^{3t}e^{t^2}$

b. $\frac{d}{dx}e^{3x^2}$

#### Solution

We apply the chain rule as necessary.

a. $\frac{d}{dt}e^{3t}e^{t^2} = \frac{d}{dt}e^{3t+t^2} = e^{3t+t^2}(3+2t)$

b. $\frac{d}{dx}e^{3x^2} = e^{3x^2}6x$

**6.38** Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right)$

b. $\frac{d}{dt}\left(e^{2t}\right)^3$

## Example 6.39

### Using Properties of Exponential Functions

Evaluate the following integral: $\int 2xe^{-x^2}\,dx.$

#### Solution

Using $u$ -substitution, let $u = -x^2.$ Then $du = -2x\,dx,$ and we have

$$\int 2xe^{-x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{-x^2} + C.$$

**6.39** Evaluate the following integral: $\int \frac{4}{e^{3x}} dx.$

## General Logarithmic and Exponential Functions

We close this section by looking at exponential functions and logarithms with bases other than $e$. Exponential functions are functions of the form $f(x) = a^x$. Note that unless $a = e$, we still do not have a mathematically rigorous definition of these functions for irrational exponents. Let's rectify that here by defining the function $f(x) = a^x$ in terms of the exponential function $e^x$. We then examine logarithms with bases other than $e$ as inverse functions of exponential functions.

### Definition

For any $a > 0$, and for any real number $x$, define $y = a^x$ as follows:

$$y = a^x = e^{x \ln a}.$$

Now $a^x$ is defined rigorously for all values of $x$. This definition also allows us to generalize property iv. of logarithms and property iii. of exponential functions to apply to both rational and irrational values of $r$. It is straightforward to show that properties of exponents hold for general exponential functions defined in this way.

Let's now apply this definition to calculate a differentiation formula for $a^x$. We have

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a.$$

The corresponding integration formula follows immediately.

### Theorem 6.20: Derivatives and Integrals Involving General Exponential Functions

Let $a > 0$. Then,

$$\frac{d}{dx}a^x = a^x \ln a$$

and

$$\int a^x \, dx = \frac{1}{\ln a}a^x + C.$$

If $a \neq 1$, then the function $a^x$ is one-to-one and has a well-defined inverse. Its inverse is denoted by $\log_a x$. Then,

$$y = \log_a x \text{ if and only if } x = a^y.$$

Note that general logarithm functions can be written in terms of the natural logarithm. Let $y = \log_a x$. Then, $x = a^y$. Taking the natural logarithm of both sides of this second equation, we get

$$\begin{aligned} \ln x &= \ln(a^y) \\ \ln x &= y \ln a \\ y &= \frac{\ln x}{\ln a} \\ \log x &= \frac{\ln x}{\ln a}. \end{aligned}$$

Thus, we see that all logarithmic functions are constant multiples of one another. Next, we use this formula to find a differentiation formula for a logarithm with base $a$. Again, let $y = \log_a x$. Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\log_a x)\\ &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \left(\frac{1}{\ln a}\right)\frac{d}{dx}(\ln x)\\ &= \frac{1}{\ln a}\cdot\frac{1}{x}\\ &= \frac{1}{x\ln a}.\end{aligned}$$

### Theorem 6.21: Derivatives of General Logarithm Functions

Let $a > 0$. Then,

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

## Example 6.40

### Calculating Derivatives of General Exponential and Logarithm Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right)$

b. $\frac{d}{dx}\log_8\left(7x^2 + 4\right)$

### Solution

We need to apply the chain rule as necessary.

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t} \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t + t^2}\right) = 2^{2t + t^2}\ln(2)(2 + 2t)$

b. $\frac{d}{dx}\log_8\left(7x^2 + 4\right) = \frac{1}{\left(7x^2 + 4\right)(\ln 8)}(14x)$

**6.40** Evaluate the following derivatives:

a. $\frac{d}{dt}4^{t^4}$

b. $\frac{d}{dx}\log_3\left(\sqrt{x^2 + 1}\right)$

## Example 6.41

### Integrating General Exponential Functions

Evaluate the following integral: $\int \frac{3}{2^{3x}} dx.$

### Solution

Use $u$-substitution and let $u = -3x.$ Then $du = -3dx$ and we have

$$\int \frac{3}{2^{3x}} dx = \int 3 \cdot 2^{-3x} dx = -\int 2^u \, du = -\frac{1}{\ln 2} 2^u + C = -\frac{1}{\ln 2} 2^{-3x} + C.$$

**6.41** Evaluate the following integral: $\int x^2 2^{x^3} dx.$

# 6.7 EXERCISES

For the following exercises, find the derivative $\frac{dy}{dx}$.

295. $y = \ln(2x)$

296. $y = \ln(2x + 1)$

297. $y = \frac{1}{\ln x}$

For the following exercises, find the indefinite integral.

298. $\int \frac{dt}{3t}$

299. $\int \frac{dx}{1 + x}$

For the following exercises, find the derivative $dy/dx$. (You can use a calculator to plot the function and the derivative to confirm that it is correct.)

300. **[T]** $y = \frac{\ln(x)}{x}$

301. **[T]** $y = x\ln(x)$

302. **[T]** $y = \log_{10} x$

303. **[T]** $y = \ln(\sin x)$

304. **[T]** $y = \ln(\ln x)$

305. **[T]** $y = 7\ln(4x)$

306. **[T]** $y = \ln\left((4x)^7\right)$

307. **[T]** $y = \ln(\tan x)$

308. **[T]** $y = \ln(\tan(3x))$

309. **[T]** $y = \ln\left(\cos^2 x\right)$

For the following exercises, find the definite or indefinite integral.

310. $\int_0^1 \frac{dx}{3 + x}$

311. $\int_0^1 \frac{dt}{3 + 2t}$

312. $\int_0^2 \frac{x\,dx}{x^2 + 1}$

313. $\int_0^2 \frac{x^3\,dx}{x^2 + 1}$

314. $\int_2^e \frac{dx}{x\ln x}$

315. $\int_2^e \frac{dx}{x(\ln x)^2}$

316. $\int \frac{\cos x\,dx}{\sin x}$

317. $\int_0^{\pi/4} \tan x\,dx$

318. $\int \cot(3x)dx$

319. $\int \frac{(\ln x)^2\,dx}{x}$

For the following exercises, compute $dy/dx$ by differentiating $\ln y$.

320. $y = \sqrt{x^2 + 1}$

321. $y = \sqrt{x^2 + 1}\sqrt{x^2 - 1}$

322. $y = e^{\sin x}$

323. $y = x^{-1/x}$

324. $y = e^{(ex)}$

325. $y = x^e$

326. $y = x^{(ex)}$

327. $y = \sqrt{x}\sqrt[3]{x}\sqrt[6]{x}$

328. $y = x^{-1/\ln x}$

329. $y = e^{-\ln x}$

For the following exercises, evaluate by any method.

330. $\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

331. $\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

332. $\frac{d}{dx}\int_x^1 \frac{dt}{t}$

333. $\frac{d}{dx}\int_x^{x^2} \frac{dt}{t}$

334. $\frac{d}{dx}\ln(\sec x + \tan x)$

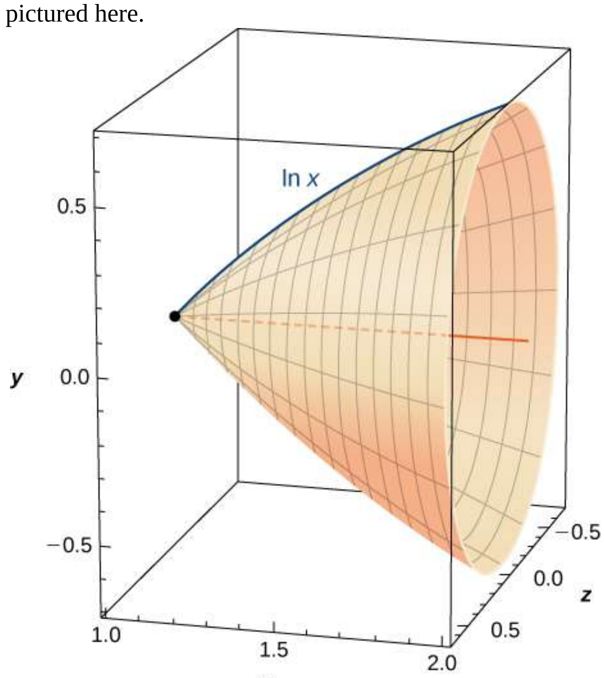
For the following exercises, use the function $\ln x$. If you are unable to find intersection points analytically, use a calculator.

335. Find the area of the region enclosed by $x = 1$ and $y = 5$ above $y = \ln x$.

336. **[T]** Find the arc length of $\ln x$ from $x = 1$ to $x = 2$.

337. Find the area between $\ln x$ and the $x$-axis from $x = 1$ to $x = 2$.

338. Find the volume of the shape created when rotating this curve from $x = 1$ to $x = 2$ around the $x$-axis, as pictured here.




339. **[T]** Find the surface area of the shape created when rotating the curve in the previous exercise from $x = 1$ to $x = 2$ around the $x$-axis.

If you are unable to find intersection points analytically in the following exercises, use a calculator.

340. Find the area of the hyperbolic quarter-circle enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

341. **[T]** Find the arc length of $y = 1/x$ from $x = 1$ to $x = 4$.

342. Find the area under $y = 1/x$ and above the $x$-axis from $x = 1$ to $x = 4$.

For the following exercises, verify the derivatives and antiderivatives.

343. $\frac{d}{dx}\ln\left(x + \sqrt{x^2 + 1}\right) = \frac{1}{\sqrt{1 + x^2}}$

344. $\frac{d}{dx}\ln\left(\frac{x - a}{x + a}\right) = \frac{2a}{\left(x^2 - a^2\right)}$

345. $\frac{d}{dx}\ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) = -\frac{1}{x\sqrt{1 - x^2}}$

346. $\frac{d}{dx}\ln\left(x + \sqrt{x^2 - a^2}\right) = \frac{1}{\sqrt{x^2 - a^2}}$

347. $\int \frac{dx}{x\ln(x)\ln(\ln x)} = \ln(\ln(\ln x)) + C$

# 6.8 | Exponential Growth and Decay

## Learning Objectives

**6.8.1** Use the exponential growth model in applications, including population growth and compound interest.
**6.8.2** Explain the concept of doubling time.
**6.8.3** Use the exponential decay model in applications, including radioactive decay and Newton's law of cooling.
**6.8.4** Explain the concept of half-life.

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of these applications.

## Exponential Growth Model

Many systems exhibit exponential growth. These systems follow a model of the form $y = y_0 e^{kt}$, where $y_0$ represents the initial state of the system and $k$ is a positive constant, called the *growth constant*. Notice that in an exponential growth model, we have

$$y' = ky_0 e^{kt} = ky. \tag{6.27}$$

That is, the rate of growth is proportional to the current function value. This is a key feature of exponential growth. **Equation 6.27** involves derivatives and is called a *differential equation*. We learn more about differential equations in **Introduction to Differential Equations (http://cnx.org/content/m53696/latest/)** .

### Rule: Exponential Growth Model

Systems that exhibit **exponential growth** increase according to the mathematical model

$$y = y_0 e^{kt},$$

where $y_0$ represents the initial state of the system and $k > 0$ is a constant, called the *growth constant*.

Population growth is a common example of exponential growth. Consider a population of bacteria, for instance. It seems plausible that the rate of population growth would be proportional to the size of the population. After all, the more bacteria there are to reproduce, the faster the population grows. **Figure 6.79** and **Table 6.1** represent the growth of a population of bacteria with an initial population of $200$ bacteria and a growth constant of $0.02.$ Notice that after only $2$ hours $(120$ minutes), the population is $10$ times its original size!

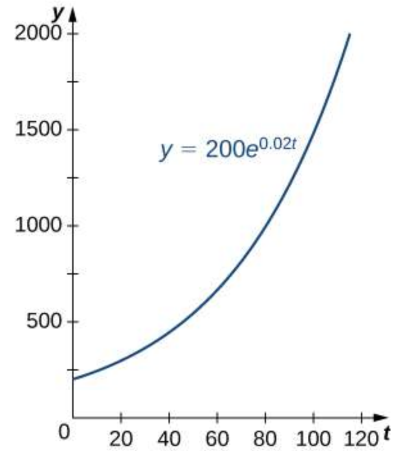



**Figure 6.79** An example of exponential growth for bacteria.

| Time (min) | Population Size (no. of bacteria) |
|---|---|
| 10 | 244 |
| 20 | 298 |
| 30 | 364 |
| 40 | 445 |
| 50 | 544 |
| 60 | 664 |
| 70 | 811 |
| 80 | 991 |
| 90 | 1210 |
| 100 | 1478 |
| 110 | 1805 |
| 120 | 2205 |

**Table 6.1** Exponential Growth of a Bacterial Population

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

## Example 6.42

### Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where $t$ is measured in minutes. How many bacteria are present in the population after $5$ hours $(300$ minutes)? When does the population reach $100,000$ bacteria?

### Solution

We have $f(t) = 200e^{0.02t}$. Then

$$f(300) = 200e^{0.02(300)} \approx 80,686.$$

There are $80,686$ bacteria in the population after $5$ hours.

To find when the population reaches $100,000$ bacteria, we solve the equation

$$\begin{aligned} 100,000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73. \end{aligned}$$

The population reaches $100,000$ bacteria after $310.73$ minutes.

**6.42** Consider a population of bacteria that grows according to the function $f(t) = 500e^{0.05t}$, where $t$ is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach $100$ million bacteria?

Let’s now turn our attention to a financial application: compound interest. Interest that is not compounded is called *simple interest*. Simple interest is paid once, at the end of the specified time period (usually $1$ year). So, if we put $\$1000$ in a savings account earning $2\%$ simple interest per year, then at the end of the year we have

$$1000(1 + 0.02) = \$1020.$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every $6$ months, it credits half of the year’s interest to the account after $6$ months. During the second half of the year, the account earns interest not only on the initial $\$1000$, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

$$1000\left(1 + \frac{0.02}{2}\right)^2 = \$1020.10.$$

Similarly, if the interest is compounded every $4$ months, we have

$$1000\left(1 + \frac{0.02}{3}\right)^3 = \$1020.13,$$

and if the interest is compounded daily $(365$ times per year), we have $\$1020.20$. If we extend this concept, so that the interest is compounded continuously, after $t$ years we have

$$1000 \lim_{n \to \infty} \left(1 + \frac{0.02}{n}\right)^{nt}.$$

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number $e$ can be expressed as a limit:

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m}.$$

Based on this, we want the expression inside the parentheses to have the form $(1 + 1/m)$. Let $n = 0.02m$. Note that as $n \to \infty$, $m \to \infty$ as well. Then we get

$$1000 \lim_{n \to \infty} \left(1 + \frac{0.02}{n}\right)^{nt} = 1000 \lim_{m \to \infty} \left(1 + \frac{0.02}{0.02m}\right)^{0.02mt} = 1000 \left[\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m}\right]^{0.02t}.$$

We recognize the limit inside the brackets as the number $e$. So, the balance in our bank account after $t$ years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of $r\%$, compounded continuously, then the balance of the account after $t$ years is

$$\text{Balance} = Pe^{rt}.$$

## Example 6.43

### Compound Interest

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays $5\%$ annual interest compounded continuously. How much does the student need to invest today to have $\$1$ million when she retires at age $65$? What if she could earn $6\%$ annual interest compounded continuously instead?

### Solution

We have

$$\begin{aligned} 1{,}000{,}000 &= Pe^{0.05(40)} \\ P &= 135{,}335.28. \end{aligned}$$

She must invest $\$135{,}335.28$ at $5\%$ interest.

If, instead, she is able to earn $6\%$, then the equation becomes

$$\begin{aligned} 1{,}000{,}000 &= Pe^{0.06(40)} \\ P &= 90{,}717.95. \end{aligned}$$

In this case, she needs to invest only $\$90{,}717.95$. This is roughly two-thirds the amount she needs to invest at $5\%$. The fact that the interest is compounded continuously greatly magnifies the effect of the $1\%$ increase in interest rate.

**6.43** Suppose instead of investing at age $25$, the student waits until age $35$. How much would she have to invest at $5\%$? At $6\%$?

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from $100$ to $200$ bacteria as it does to grow from $10{,}000$ to $20{,}000$ bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

$$\begin{aligned} 2y_0 &= y_0 e^{kt} \\ 2 &= e^{kt} \\ \ln 2 &= kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

**Definition**

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

$$\text{Doubling time } = \frac{\ln 2}{k}.$$

## Example 6.44

### Using the Doubling Time

Assume a population of fish grows exponentially. A pond is stocked initially with $500$ fish. After $6$ months, there are $1000$ fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches $10{,}000.$ When will the owner's friends be allowed to fish?

### Solution

We know it takes the population of fish $6$ months to double in size. So, if $t$ represents time in months, by the doubling-time formula, we have $6 = (\ln 2)/k.$ Then, $k = (\ln 2)/6.$ Thus, the population is given by $y = 500e^{((\ln 2)/6)t}.$ To figure out when the population reaches $10{,}000$ fish, we must solve the following equation:

$$\begin{aligned} 10{,}000 &= 500e^{(\ln 2/6)t} \\ 20 &= e^{(\ln 2/6)t} \\ \ln 20 &= \left(\frac{\ln 2}{6}\right)t \\ t &= \frac{6(\ln 20)}{\ln 2} \approx 25.93. \end{aligned}$$

The owner's friends have to wait $25.93$ months (a little more than $2$ years) to fish in the pond.

**6.44** Suppose it takes $9$ months for the fish population in **Example 6.44** to reach $1000$ fish. Under these circumstances, how long do the owner's friends have to wait?

## Exponential Decay Model

Exponential functions can also be used to model populations that shrink (from disease, for example), or chemical compounds that break down over time. We say that such systems exhibit exponential decay, rather than exponential growth. The model is nearly the same, except there is a negative sign in the exponent. Thus, for some positive constant $k,$ we have $y = y_0 e^{-kt}.$

As with exponential growth, there is a differential equation associated with exponential decay. We have

$$y' = -ky_0 e^{-kt} = -ky.$$

### Rule: Exponential Decay Model

Systems that exhibit **exponential decay** behave according to the model

$$y = y_0 e^{-kt},$$

where $y_0$ represents the initial state of the system and $k > 0$ is a constant, called the *decay constant*.

The following figure shows a graph of a representative exponential decay function.

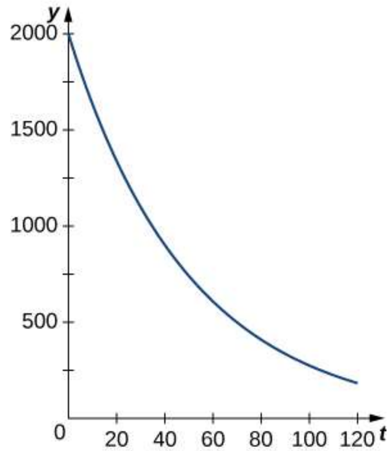



**Figure 6.80** An example of exponential decay.

Let's look at a physical application of exponential decay. Newton's law of cooling says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if $T$ represents the temperature of the object and $T_a$ represents the ambient temperature in a room, then

$$T' = -k(T - T_a).$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional $T_a$ term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a$. Then $y'(t) = T'(t) - 0 = T'(t)$, and our equation becomes

$$y' = -ky.$$

From our previous work, we know this relationship between *y* and its derivative leads to exponential decay. Thus,

$$y = y_0 e^{-kt},$$

and we see that

$$\begin{aligned} T - T_a &= (T_0 - T_a)e^{-kt} \\ T &= (T_0 - T_a)e^{-kt} + T_a \end{aligned}$$

where $T_0$ represents the initial temperature. Let's apply this formula in the following example.

## Example 6.45

### Newton's Law of Cooling

According to experienced baristas, the optimal temperature to serve coffee is between $155°\text{F}$ and $175°\text{F}$. Suppose coffee is poured at a temperature of $200°\text{F}$, and after $2$ minutes in a $70°\text{F}$ room it has cooled to

$180°\text{F}.$ When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

### Solution

We have

$$\begin{aligned} T &= (T_0 - T_a)e^{-kt} + T_a \\ 180 &= (200 - 70)e^{-k(2)} + 70 \\ 110 &= 130e^{-2k} \\ \frac{11}{13} &= e^{-2k} \\ \ln\frac{11}{13} &= -2k \\ \ln 11 - \ln 13 &= -2k \\ k &= \frac{\ln 13 - \ln 11}{2}. \end{aligned}$$

Then, the model is

$$T = 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} + 70.$$

The coffee reaches $175°\text{F}$ when

$$\begin{aligned} 175 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} + 70 \\ 105 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\ \frac{21}{26} &= e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\ \ln\frac{21}{26} &= \frac{\ln 11 - \ln 13}{2}t \\ \ln 21 - \ln 26 &= \frac{\ln 11 - \ln 13}{2}t \\ t &= \frac{2(\ln 21 - \ln 26)}{\ln 11 - \ln 13} \approx 2.56. \end{aligned}$$

The coffee can be served about $2.5$ minutes after it is poured. The coffee reaches $155°\text{F}$ at

$$\begin{aligned} 155 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} + 70 \\ 85 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\ \frac{17}{26} &= e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\ \ln 17 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\ t &= \frac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13} \approx 5.09. \end{aligned}$$

The coffee is too cold to be served about $5$ minutes after it is poured.

**6.45** Suppose the room is warmer $(75°\text{F})$ and, after $2$ minutes, the coffee has cooled only to $185°\text{F}.$ When is the coffee first cool enough to serve? When is the coffee be too cold to serve? Round answers to the nearest half minute.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

*Note*: This is the same expression we came up with for doubling time.

### Definition

If a quantity decays exponentially, the **half-life** is the amount of time it takes the quantity to be reduced by half. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}.$$

## Example 6.46

### Radiocarbon Dating

One of the most common applications of an exponential decay model is carbon dating. $\text{Carbon-14}$ decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon was originally present in an object and how much carbon remains, we can determine the age of the object. The half-life of $\text{carbon-14}$ is approximately $5730$ years—meaning, after that many years, half the material has converted from the original $\text{carbon-14}$ to the new nonradioactive $\text{nitrogen-14}.$ If we have $100$ g $\text{carbon-14}$ today, how much is left in $50$ years? If an artifact that originally contained $100$ g of carbon now contains $10$ g of carbon, how old is it? Round the answer to the nearest hundred years.

### Solution

We have

$$\begin{aligned} 5730 &= \frac{\ln 2}{k} \\ k &= \frac{\ln 2}{5730}. \end{aligned}$$

So, the model says

$$y = 100e^{-(\ln 2/5730)t}.$$

In $50$ years, we have

$$\begin{aligned} y &= 100e^{-(\ln 2/5730)(50)} \\ &\approx 99.40. \end{aligned}$$

Therefore, in $50$ years, $99.40$ g of $\text{carbon-14}$ remains.

To determine the age of the artifact, we must solve

$$\begin{aligned} 10 &= 100e^{-(\ln 2/5730)t} \\ \frac{1}{10} &= e^{-(\ln 2/5730)t} \\ t &\approx 19035. \end{aligned}$$

The artifact is about $19,000$ years old.

**6.46** If we have $100$ g of carbon-14, how much is left after $t$ years? If an artifact that originally contained $100$ g of carbon now contains $20g$ of carbon, how old is it? Round the answer to the nearest hundred years.

# 6.8 EXERCISES

*True or False*? If true, prove it. If false, find the true answer.

348. The doubling time for $y = e^{ct}$ is $(\ln(2))/(\ln(c))$.

349. If you invest $\$500$, an annual rate of interest of $3\%$ yields more money in the first year than a $2.5\%$ continuous rate of interest.

350. If you leave a $100°C$ pot of tea at room temperature $(25°C)$ and an identical pot in the refrigerator $(5°C)$, with $k = 0.02$, the tea in the refrigerator reaches a drinkable temperature $(70°C)$ more than $5$ minutes before the tea at room temperature.

351. If given a half-life of $t$ years, the constant $k$ for $y = e^{kt}$ is calculated by $k = \ln(1/2)/t$.

For the following exercises, use $y = y_0 e^{kt}$.

352. If a culture of bacteria doubles in $3$ hours, how many hours does it take to multiply by $10$?

353. If bacteria increase by a factor of $10$ in $10$ hours, how many hours does it take to increase by $100$?

354. How old is a skull that contains one-fifth as much radiocarbon as a modern skull? Note that the half-life of radiocarbon is $5730$ years.

355. If a relic contains $90\%$ as much radiocarbon as new material, can it have come from the time of Christ (approximately $2000$ years ago)? Note that the half-life of radiocarbon is $5730$ years.

356. The population of Cairo grew from $5$ million to $10$ million in $20$ years. Use an exponential model to find when the population was $8$ million.

357. The populations of New York and Los Angeles are growing at $1\%$ and $1.4\%$ a year, respectively. Starting from $8$ million (New York) and $6$ million (Los Angeles), when are the populations equal?

358. Suppose the value of $\$1$ in Japanese yen decreases at $2\%$ per year. Starting from $\$1 = ¥250$, when will $\$1 = ¥1$?

359. The effect of advertising decays exponentially. If $40\%$ of the population remembers a new product after $3$ days, how long will $20\%$ remember it?

360. If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4$, what was $y_0$ at $t = 0$?

361. If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8$, when does $y = 1$?

362. If a bank offers annual interest of $7.5\%$ or continuous interest of $7.25\%$, which has a better annual yield?

363. What continuous interest rate has the same yield as an annual rate of $9\%$?

364. If you deposit $\$5000$ at $8\%$ annual interest, how many years can you withdraw $\$500$ (starting after the first year) without running out of money?

365. You are trying to save $\$50,000$ in $20$ years for college tuition for your child. If interest is a continuous $10\%$, how much do you need to invest initially?

366. You are cooling a turkey that was taken out of the oven with an internal temperature of $165°F$. After $10$ minutes of resting the turkey in a $70°F$ apartment, the temperature has reached $155°F$. What is the temperature of the turkey $20$ minutes after taking it out of the oven?

367. You are trying to thaw some vegetables that are at a temperature of $1°F$. To thaw vegetables safely, you must put them in the refrigerator, which has an ambient temperature of $44°F$. You check on your vegetables $2$ hours after putting them in the refrigerator to find that they are now $12°F$. Plot the resulting temperature curve and use it to determine when the vegetables reach $33°F$.

368. You are an archaeologist and are given a bone that is claimed to be from a Tyrannosaurus Rex. You know these dinosaurs lived during the Cretaceous Era $(146$ million years to $65$ million years ago), and you find by radiocarbon dating that there is $0.000001\%$ the amount of radiocarbon. Is this bone from the Cretaceous?

369. The spent fuel of a nuclear reactor contains plutonium-239, which has a half-life of $24,000$ years. If $1$ barrel containing $10\,\text{kg}$ of plutonium-239 is sealed, how many years must pass until only $10g$ of plutonium-239 is left?

For the next set of exercises, use the following table, which features the world population by decade.

| Years since 1950 | Population (millions) |
|---|---|
| 0 | 2,556 |
| 10 | 3,039 |
| 20 | 3,706 |
| 30 | 4,453 |
| 40 | 5,279 |
| 50 | 6,083 |
| 60 | 6,849 |

*Source*: http://www.factmonster.com/ipka/A0762181.html.

370. **[T]** The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 2686e^{0.01604t}$. Use a graphing calculator to graph the data and the exponential curve together.

371. **[T]** Find and graph the derivative $y'$ of your equation. Where is it increasing and what is the meaning of this increase?

372. **[T]** Find and graph the second derivative of your equation. Where is it increasing and what is the meaning of this increase?

373. **[T]** Find the predicted date when the population reaches $10$ billion. Using your previous answers about the first and second derivatives, explain why exponential growth is unsuccessful in predicting the future.

For the next set of exercises, use the following table, which shows the population of San Francisco during the 19th century.

| Years since 1850 | Population (thousands) |
|---|---|
| 0 | 21.00 |
| 10 | 56.80 |
| 20 | 149.5 |
| 30 | 234.0 |

*Source*: http://www.sfgenealogy.com/sf/history/hgpop.htm.

374. **[T]** The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 35.26e^{0.06407t}$. Use a graphing calculator to graph the data and the exponential curve together.

375. **[T]** Find and graph the derivative $y'$ of your equation. Where is it increasing? What is the meaning of this increase? Is there a value where the increase is maximal?

376. **[T]** Find and graph the second derivative of your equation. Where is it increasing? What is the meaning of this increase?

# 6.9 | Calculus of the Hyperbolic Functions

**Learning Objectives**

**6.9.1** Apply the formulas for derivatives and integrals of the hyperbolic functions.

**6.9.2** Apply the formulas for the derivatives of the inverse hyperbolic functions and their associated integrals.

**6.9.3** Describe the common applied conditions of a catenary curve.

We were introduced to hyperbolic functions in **Introduction to Functions and Graphs**, along with some of their basic properties. In this section, we look at differentiation and integration formulas for the hyperbolic functions and their inverses.

## Derivatives and Integrals of the Hyperbolic Functions

Recall that the hyperbolic sine and hyperbolic cosine are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in the following figure.

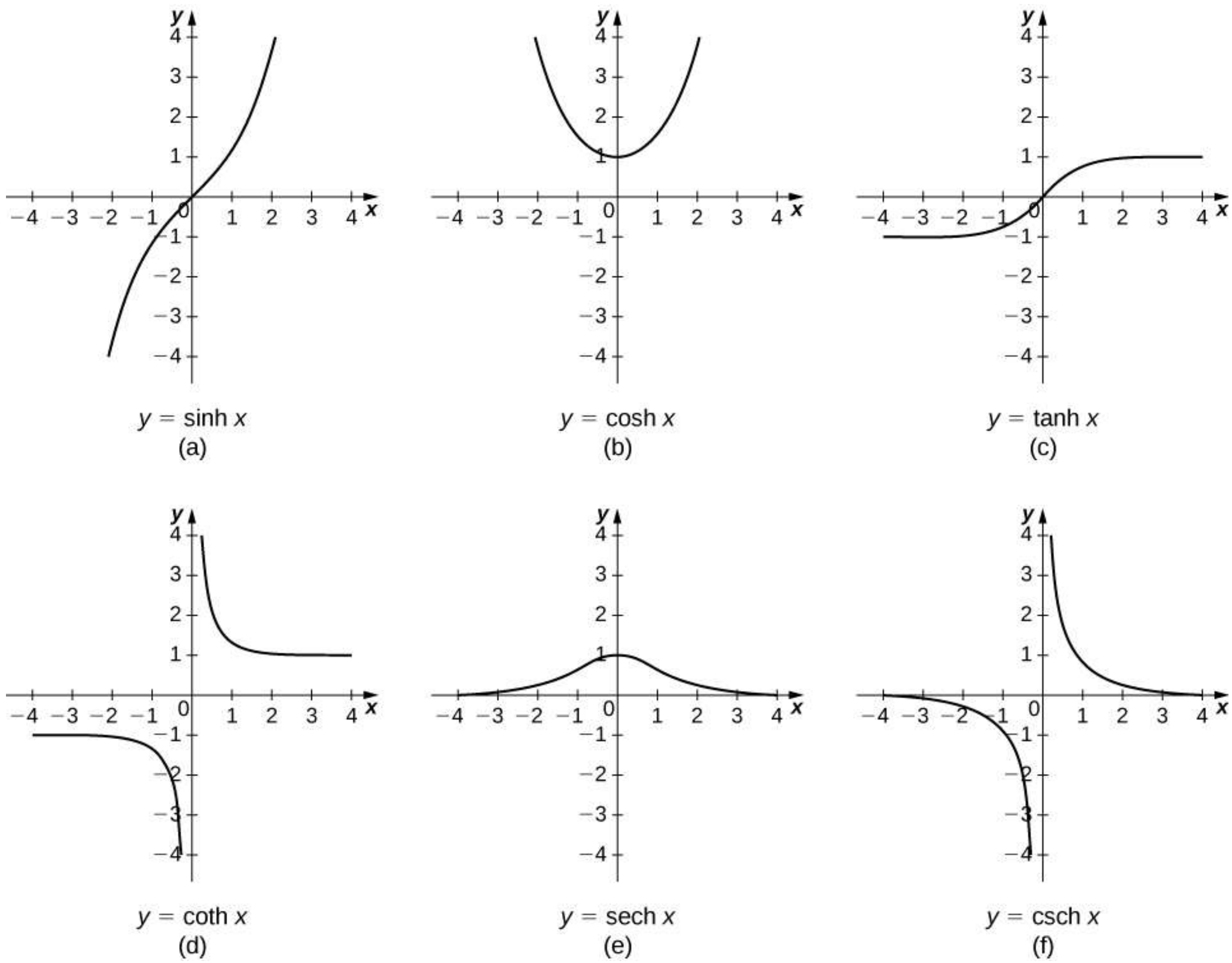



**Figure 6.81** Graphs of the hyperbolic functions.

It is easy to develop differentiation formulas for the hyperbolic functions. For example, looking at $\sinh x$ we have

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right] \\ &= \frac{1}{2}[e^x + e^{-x}] = \cosh x.\end{aligned}$$

Similarly, $(d/dx)\cosh x = \sinh x$. We summarize the differentiation formulas for the hyperbolic functions in the following table.

| $f(x)$ | $\frac{d}{dx}f(x)$ |
|---|---|
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\text{sech}^2\, x$ |
| $\coth x$ | $-\text{csch}^2\, x$ |
| $\text{sech}\, x$ | $-\text{sech}\, x \tanh x$ |
| $\text{csch}\, x$ | $-\text{csch}\, x \coth x$ |

**Table 6.2** Derivatives of the Hyperbolic Functions

Let's take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities, but differences as well. For example, the derivatives of the sine functions match: $(d/dx)\sin x = \cos x$ and $(d/dx)\sinh x = \cosh x$. The derivatives of the cosine functions, however, differ in sign: $(d/dx)\cos x = -\sin x$, but $(d/dx)\cosh x = \sinh x$. As we continue our examination of the hyperbolic functions, we must be mindful of their similarities and differences to the standard trigonometric functions.

These differentiation formulas for the hyperbolic functions lead directly to the following integral formulas.

$$\begin{aligned}\int \sinh u\, du &= \cosh u + C & \int \text{csch}^2\, u\, du &= -\coth u + C \\ \int \cosh u\, du &= \sinh u + C & \int \text{sech}\, u \tanh u\, du &= -\text{sech}\, u + C \\ \int \text{sech}^2 u\, du &= \tanh u + C & \int \text{csch}\, u \coth u\, du &= -\text{csch}\, u + C\end{aligned}$$

## Example 6.47

### Differentiating Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right)$

b. $\frac{d}{dx}(\cosh x)^2$

### Solution

Using the formulas in **Table 6.2** and the chain rule, we get

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right) = \cosh\left(x^2\right) \cdot 2x$

b. $\frac{d}{dx}(\cosh x)^2 = 2 \cosh x \sinh x$

**6.47** Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\tanh\left(x^2 + 3x\right)\right)$

b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)$

## Example 6.48

### Integrals Involving Hyperbolic Functions

Evaluate the following integrals:

a. $\int x \cosh\left(x^2\right) dx$

b. $\int \tanh x \, dx$

### Solution

We can use *u*-substitution in both cases.

a. Let $u = x^2$. Then, $du = 2x\,dx$ and

$$\int x \cosh\left(x^2\right) dx = \int \frac{1}{2} \cosh u \, du = \frac{1}{2} \sinh u + C = \frac{1}{2} \sinh\left(x^2\right) + C.$$

b. Let $u = \cosh x$. Then, $du = \sinh x \, dx$ and

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|\cosh x| + C.$$

Note that $\cosh x > 0$ for all $x$, so we can eliminate the absolute value signs and obtain

$$\int \tanh x \, dx = \ln(\cosh x) + C.$$

**6.48** Evaluate the following integrals:

a. $\int \sinh^3 x \cosh x \, dx$

b. $\int \text{sech}^2 (3x) dx$

## Calculus of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in the following table.

| Function | Domain | Range |
|---|---|---|
| $\sinh^{-1} x$ | $(-\infty, \infty)$ | $(-\infty, \infty)$ |
| $\cosh^{-1} x$ | $[1, \infty)$ | $[0, \infty)$ |
| $\tanh^{-1} x$ | $(-1, 1)$ | $(-\infty, \infty)$ |
| $\coth^{-1} x$ | $(-\infty, -1) \cup (1, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ |
| $\text{sech}^{-1} x$ | $(0, 1]$ | $[0, \infty)$ |
| $\text{csch}^{-1} x$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ |

**Table 6.3** Domains and Ranges of the Inverse Hyperbolic Functions

The graphs of the inverse hyperbolic functions are shown in the following figure.

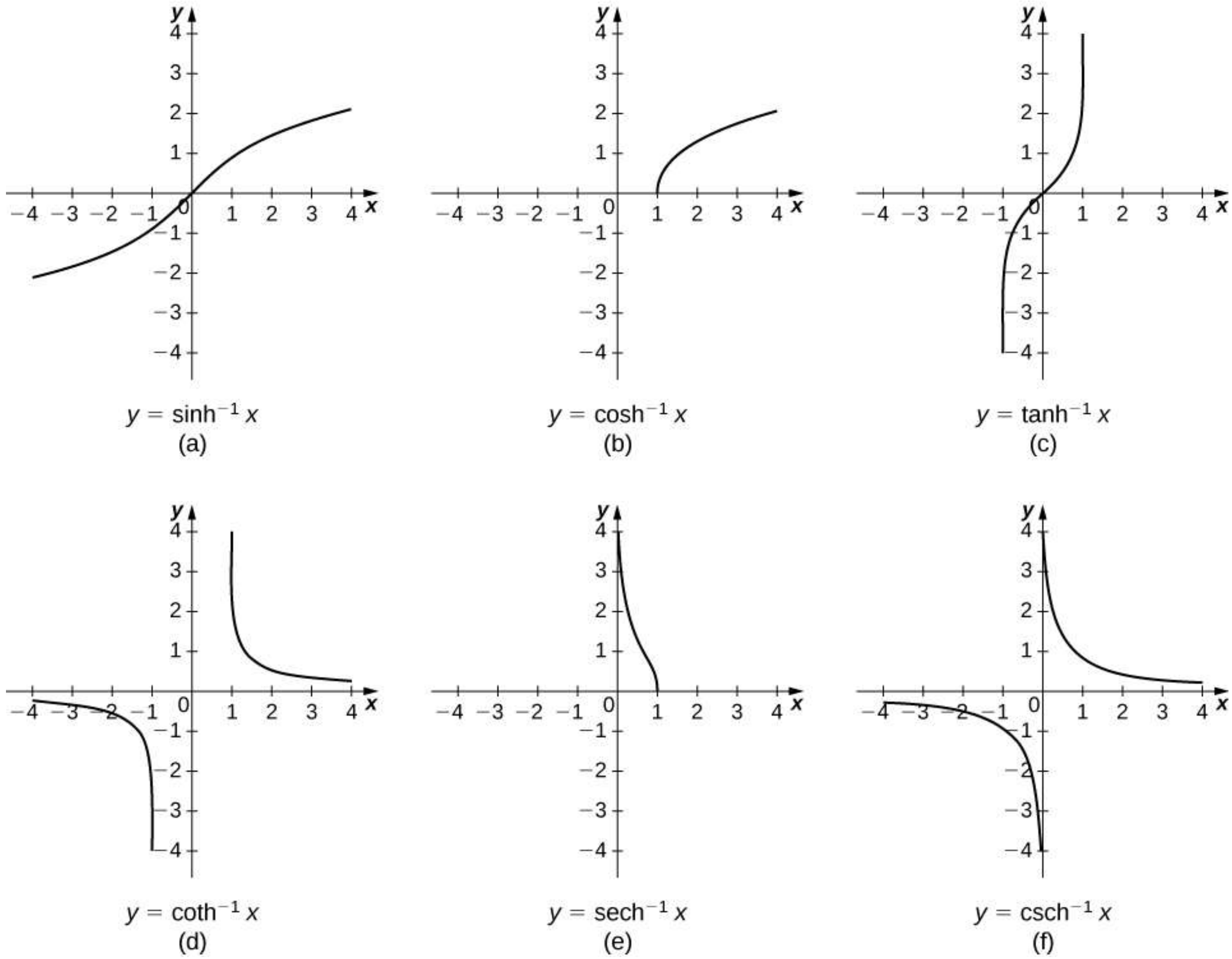



**Figure 6.82** Graphs of the inverse hyperbolic functions.

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$\begin{aligned} y &= \sinh^{-1} x \\ \sinh y &= x \\ \frac{d}{dx}\sinh y &= \frac{d}{dx}x \\ \cosh y\frac{dy}{dx} &= 1. \end{aligned}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1 + \sinh^2 y}$. Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

We can derive differentiation formulas for the other inverse hyperbolic functions in a similar fashion. These differentiation formulas are summarized in the following table.

| $f(x)$ | $\frac{d}{dx}f(x)$ |
|---|---|
| $\sinh^{-1} x$ | $\frac{1}{\sqrt{1+x^2}}$ |
| $\cosh^{-1} x$ | $\frac{1}{\sqrt{x^2-1}}$ |
| $\tanh^{-1} x$ | $\frac{1}{1-x^2}$ |
| $\coth^{-1} x$ | $\frac{1}{1-x^2}$ |
| $\operatorname{sech}^{-1} x$ | $\frac{-1}{x\sqrt{1-x^2}}$ |
| $\operatorname{csch}^{-1} x$ | $\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$ |

**Table 6.4** Derivatives of the Inverse Hyperbolic Functions

Note that the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ are the same. Thus, when we integrate $1/\left(1-x^2\right)$, we need to select the proper antiderivative based on the domain of the functions and the values of $x$. Integration formulas involving the inverse hyperbolic functions are summarized as follows.

$$\begin{aligned}
\int \frac{1}{\sqrt{1+u^2}}du &= \sinh^{-1} u + C & \int \frac{1}{u\sqrt{1-u^2}}du &= -\operatorname{sech}^{-1}|u| + C \\
\int \frac{1}{\sqrt{u^2-1}}du &= \cosh^{-1} u + C & \int \frac{1}{u\sqrt{1+u^2}}du &= -\operatorname{csch}^{-1}|u| + C \\
\int \frac{1}{1-u^2}du &= \begin{cases} \tanh^{-1} u + C \text{ if } |u| < 1 \\ \coth^{-1} u + C \text{ if } |u| > 1 \end{cases}
\end{aligned}$$

## Example 6.49

### Differentiating Inverse Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$

b. $\frac{d}{dx}\left(\tanh^{-1} x\right)^2$

### Solution

Using the formulas in **Table 6.4** and the chain rule, we obtain the following results:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right) = \frac{1}{3\sqrt{1+\frac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$

b. $\frac{d}{dx}\left(\tanh^{-1} x\right)^2 = \frac{2\left(\tanh^{-1} x\right)}{1-x^2}$

**6.49** Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\cosh^{-1}(3x)\right)$

b. $\frac{d}{dx}\left(\coth^{-1} x\right)^3$

## Example 6.50

### Integrals Involving Inverse Hyperbolic Functions

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{4x^2-1}}dx$

b. $\int \frac{1}{2x\sqrt{1-9x^2}}dx$

### Solution

We can use $u$-substitution in both cases.

a. Let $u = 2x$. Then, $du = 2dx$ and we have

$$\int \frac{1}{\sqrt{4x^2-1}}dx = \int \frac{1}{2\sqrt{u^2-1}}du = \frac{1}{2}\cosh^{-1} u + C = \frac{1}{2}\cosh^{-1}(2x) + C.$$

b. Let $u = 3x$. Then, $du = 3dx$ and we obtain

$$\int \frac{1}{2x\sqrt{1-9x^2}}dx = \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}du = -\frac{1}{2}\operatorname{sech}^{-1}|u| + C = -\frac{1}{2}\operatorname{sech}^{-1}|3x| + C.$$

**6.50** Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{x^2-4}}dx, \quad x > 2$

b. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$

## Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a **catenary**. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries. The following figure shows chains hanging from a row of posts.

Figure 6.83 Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a\cosh(x/a)$ are catenaries. Figure 6.84 shows the graph of $y = 2\cosh(x/2)$.

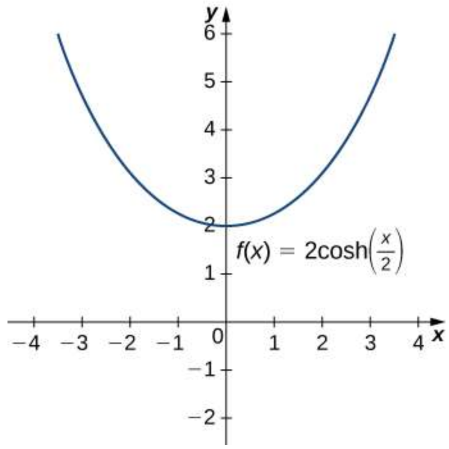



**Figure 6.84** A hyperbolic cosine function forms the shape of a catenary.

## Example 6.51

### Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10\cosh(x/10)$ for $-15 \le x \le 15$, where $x$ is measured in feet. Determine the length of the cable (in feet).

### Solution

Recall from Section 2.4 that the formula for arc length is

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We have $f(x) = 10\cosh(x/10)$, so $f'(x) = \sinh(x/10)$. Then

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx.\end{aligned}$$

Now recall that $1 + \sinh^2 x = \cosh^2 x$, so we have

$$\begin{aligned}\text{Arc Length} &= \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx \\ &= \int_{-15}^{15} \cosh\left(\frac{x}{10}\right) dx \\ &= 10\sinh\left(\frac{x}{10}\right)\Big|_{-15}^{15} = 10\left[\sinh\left(\frac{3}{2}\right) - \sinh\left(-\frac{3}{2}\right)\right] = 20\sinh\left(\frac{3}{2}\right) \\ &\approx 42.586 \text{ ft.}\end{aligned}$$

**6.51** Assume a hanging cable has the shape $15\cosh(x/15)$ for $-20 \le x \le 20$. Determine the length of the cable (in feet).

# 6.9 EXERCISES

377. **[T]** Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$. Use a calculator to graph these functions and ensure your expression is correct.

378. From the definitions of $\cosh(x)$ and $\sinh(x)$, find their antiderivatives.

379. Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

380. Use the quotient rule to verify that $\tanh(x)' = \text{sech}^2(x)$.

381. Derive $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$ from the definition.

382. Take the derivative of the previous expression to find an expression for $\sinh(2x)$.

383. Prove $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ by changing the expression to exponentials.

384. Take the derivative of the previous expression to find an expression for $\cosh(x + y)$.

For the following exercises, find the derivatives of the given functions and graph along with the function to ensure your answer is correct.

385. **[T]** $\cosh(3x + 1)$

386. **[T]** $\sinh\left(x^2\right)$

387. **[T]** $\dfrac{1}{\cosh(x)}$

388. **[T]** $\sinh(\ln(x))$

389. **[T]** $\cosh^2(x) + \sinh^2(x)$

390. **[T]** $\cosh^2(x) - \sinh^2(x)$

391. **[T]** $\tanh\left(\sqrt{x^2 + 1}\right)$

392. **[T]** $\dfrac{1 + \tanh(x)}{1 - \tanh(x)}$

393. **[T]** $\sinh^6(x)$

394. **[T]** $\ln(\text{sech}(x) + \tanh(x))$

For the following exercises, find the antiderivatives for the given functions.

395. $\cosh(2x + 1)$

396. $\tanh(3x + 2)$

397. $x\cosh\left(x^2\right)$

398. $3x^3\tanh\left(x^4\right)$

399. $\cosh^2(x)\sinh(x)$

400. $\tanh^2(x)\text{sech}^2(x)$

401. $\dfrac{\sinh(x)}{1 + \cosh(x)}$

402. $\coth(x)$

403. $\cosh(x) + \sinh(x)$

404. $(\cosh(x) + \sinh(x))^n$

For the following exercises, find the derivatives for the functions.

405. $\tanh^{-1}(4x)$

406. $\sinh^{-1}\left(x^2\right)$

407. $\sinh^{-1}(\cosh(x))$

408. $\cosh^{-1}\left(x^3\right)$

409. $\tanh^{-1}(\cos(x))$

410. $e^{\sinh^{-1}(x)}$

411. $\ln\left(\tanh^{-1}(x)\right)$

For the following exercises, find the antiderivatives for the functions.

412. $\displaystyle\int \frac{dx}{4 - x^2}$

413. $\displaystyle\int \frac{dx}{a^2 - x^2}$

414. $\int \frac{dx}{\sqrt{x^2+1}}$

415. $\int \frac{x\,dx}{\sqrt{x^2+1}}$

416. $\int -\frac{dx}{x\sqrt{1-x^2}}$

417. $\int \frac{e^x}{\sqrt{e^{2x}-1}}$

418. $\int -\frac{2x}{x^4-1}$

For the following exercises, use the fact that a falling body with friction equal to velocity squared obeys the equation $dv/dt = g - v^2$.

419. Show that $v(t) = \sqrt{g}\tanh((\sqrt{g})t)$ satisfies this equation.

420. Derive the previous expression for $v(t)$ by integrating $\frac{dv}{g - v^2} = dt$.

421. **[T]** Estimate how far a body has fallen in $12$ seconds by finding the area underneath the curve of $v(t)$.

For the following exercises, use this scenario: A cable hanging under its own weight has a slope $S = dy/dx$ that satisfies $dS/dx = c\sqrt{1+S^2}$. The constant $c$ is the ratio of cable density to tension.

422. Show that $S = \sinh(cx)$ satisfies this equation.

423. Integrate $dy/dx = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

424. Sketch the cable and determine how far down it sags at $x = 0$.

For the following exercises, solve each problem.

425. **[T]** A chain hangs from two posts $2$ m apart to form a catenary described by the equation $y = 2\cosh(x/2) - 1$. Find the slope of the catenary at the left fence post.

426. **[T]** A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh(x/4) - 3$. Find the total length of the catenary (arc length).

427. **[T]** A high-voltage power line is a catenary described by $y = 10\cosh(x/10)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

428. A telephone line is a catenary described by $y = a\cosh(x/a)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

429. Prove the formula for the derivative of $y = \sinh^{-1}(x)$ by differentiating $x = \sinh(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

430. Prove the formula for the derivative of $y = \cosh^{-1}(x)$ by differentiating $x = \cosh(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

431. Prove the formula for the derivative of $y = \operatorname{sech}^{-1}(x)$ by differentiating $x = \operatorname{sech}(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

432. Prove that $(\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$.

433. Prove the expression for $\sinh^{-1}(x)$. Multiply $x = \sinh(y) = (1/2)\left(e^y - e^{-y}\right)$ by $2e^y$ and solve for $y$. Does your expression match the textbook?

434. Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = (1/2)\left(e^y - e^{-y}\right)$ by $2e^y$ and solve for $y$. Does your expression match the textbook?

# CHAPTER 6 REVIEW

## KEY TERMS

**arc length** the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

**catenary** a curve in the shape of the function $y = a\cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

**center of mass** the point at which the total mass of the system could be concentrated without changing the moment

**centroid** the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

**cross-section** the intersection of a plane and a solid object

**density function** a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

**disk method** a special case of the slicing method used with solids of revolution when the slices are disks

**doubling time** if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

**exponential decay** systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$

**exponential growth** systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$

**frustum** a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

**half-life** if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

**Hooke's law** this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F = kx$, where $k$ is a constant

**hydrostatic pressure** the pressure exerted by water on a submerged object

**lamina** a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

**method of cylindrical shells** a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

**moment** if $n$ masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b]$, then the moments of the region with respect to the $x$- and $y$-axes are given by $M_x = \rho\int_a^b \frac{[f(x)]^2}{2}dx$ and $M_y = \rho\int_a^b x f(x)dx$, respectively

**slicing method** a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

**solid of revolution** a solid generated by revolving a region in a plane around a line in that plane

**surface area** the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

**symmetry principle** the symmetry principle states that if a region $R$ is symmetric about a line $l$, then the centroid of $R$ lies on $l$

**theorem of Pappus for volume** this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

**washer method** a special case of the slicing method used with solids of revolution when the slices are washers

**work** the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

## KEY EQUATIONS

- **Area between two curves, integrating on the $x$-axis**
  $$A = \int_a^b [f(x) - g(x)]dx$$
- **Area between two curves, integrating on the $y$-axis**
  $$A = \int_c^d [u(y) - v(y)]dy$$
- **Disk Method along the $x$-axis**
  $$V = \int_a^b \pi[f(x)]^2\, dx$$
- **Disk Method along the $y$-axis**
  $$V = \int_c^d \pi[g(y)]^2\, dy$$
- **Washer Method**
  $$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$$
- **Method of Cylindrical Shells**
  $$V = \int_a^b (2\pi x f(x))dx$$
- **Arc Length of a Function of $x$**
  $$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$
- **Arc Length of a Function of $y$**
  $$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$
- **Surface Area of a Function of $x$**
  $$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + (f'(x))^2}\right)dx$$
- **Mass of a one-dimensional object**
  $$m = \int_a^b \rho(x)dx$$
- **Mass of a circular object**
  $$m = \int_0^r 2\pi x\rho(x)dx$$

- **Work done on an object**

  $W = \int_a^b F(x)dx$

- **Hydrostatic force on a plate**

  $F = \int_a^b \rho w(x)s(x)dx$

- **Mass of a lamina**

  $m = \rho \int_a^b f(x)dx$

- **Moments of a lamina**

  $M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx$ and $M_y = \rho \int_a^b x f(x)dx$

- **Center of mass of a lamina**

  $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$

- **Natural logarithm function**
- $\ln x = \int_1^x \frac{1}{t} dt$ Z
- **Exponential function** $y = e^x$
- $\ln y = \ln(e^x) = x$ Z

# KEY CONCEPTS

## 6.1 Areas between Curves

- Just as definite integrals can be used to find the area under a curve, they can also be used to find the area between two curves.
- To find the area between two curves defined by functions, integrate the difference of the functions.
- If the graphs of the functions cross, or if the region is complex, use the absolute value of the difference of the functions. In this case, it may be necessary to evaluate two or more integrals and add the results to find the area of the region.
- Sometimes it can be easier to integrate with respect to $y$ to find the area. The principles are the same regardless of which variable is used as the variable of integration.

## 6.2 Determining Volumes by Slicing

- Definite integrals can be used to find the volumes of solids. Using the slicing method, we can find a volume by integrating the cross-sectional area.
- For solids of revolution, the volume slices are often disks and the cross-sections are circles. The method of disks involves applying the method of slicing in the particular case in which the cross-sections are circles, and using the formula for the area of a circle.
- If a solid of revolution has a cavity in the center, the volume slices are washers. With the method of washers, the area of the inner circle is subtracted from the area of the outer circle before integrating.

## 6.3 Volumes of Revolution: Cylindrical Shells

- The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other.

- The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

## 6.4 Arc Length of a Curve and Surface Area

- The arc length of a curve can be calculated using a definite integral.
- The arc length is first approximated using line segments, which generates a Riemann sum. Taking a limit then gives us the definite integral formula. The same process can be applied to functions of $y$.
- The concepts used to calculate the arc length can be generalized to find the surface area of a surface of revolution.
- The integrals generated by both the arc length and surface area formulas are often difficult to evaluate. It may be necessary to use a computer or calculator to approximate the values of the integrals.

## 6.5 Physical Applications

- Several physical applications of the definite integral are common in engineering and physics.
- Definite integrals can be used to determine the mass of an object if its density function is known.
- Work can also be calculated from integrating a force function, or when counteracting the force of gravity, as in a pumping problem.
- Definite integrals can also be used to calculate the force exerted on an object submerged in a liquid.

## 6.6 Moments and Centers of Mass

- Mathematically, the center of mass of a system is the point at which the total mass of the system could be concentrated without changing the moment. Loosely speaking, the center of mass can be thought of as the balancing point of the system.
- For point masses distributed along a number line, the moment of the system with respect to the origin is $M = \sum_{i=1}^{n} m_i x_i$. For point masses distributed in a plane, the moments of the system with respect to the *x*- and *y*-axes, respectively, are $M_x = \sum_{i=1}^{n} m_i y_i$ and $M_y = \sum_{i=1}^{n} m_i x_i$, respectively.
- For a lamina bounded above by a function $f(x)$, the moments of the system with respect to the *x*- and *y*-axes, respectively, are $M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx$ and $M_y = \rho \int_a^b x f(x) dx$.
- The *x*- and *y*-coordinates of the center of mass can be found by dividing the moments around the *y*-axis and around the *x*-axis, respectively, by the total mass. The symmetry principle says that if a region is symmetric with respect to a line, then the centroid of the region lies on the line.
- The theorem of Pappus for volume says that if a region is revolved around an external axis, the volume of the resulting solid is equal to the area of the region multiplied by the distance traveled by the centroid of the region.

## 6.7 Integrals, Exponential Functions, and Logarithms

- The earlier treatment of logarithms and exponential functions did not define the functions precisely and formally. This section develops the concepts in a mathematically rigorous way.
- The cornerstone of the development is the definition of the natural logarithm in terms of an integral.
- The function $e^x$ is then defined as the inverse of the natural logarithm.
- General exponential functions are defined in terms of $e^x$, and the corresponding inverse functions are general logarithms.

- Familiar properties of logarithms and exponents still hold in this more rigorous context.

### 6.8 Exponential Growth and Decay

- Exponential growth and exponential decay are two of the most common applications of exponential functions.
- Systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$.
- In exponential growth, the rate of growth is proportional to the quantity present. In other words, $y' = ky$.
- Systems that exhibit exponential growth have a constant doubling time, which is given by $(\ln 2)/k$.
- Systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$.
- Systems that exhibit exponential decay have a constant half-life, which is given by $(\ln 2)/k$.

### 6.9 Calculus of the Hyperbolic Functions

- Hyperbolic functions are defined in terms of exponential functions.
- Term-by-term differentiation yields differentiation formulas for the hyperbolic functions. These differentiation formulas give rise, in turn, to integration formulas.
- With appropriate range restrictions, the hyperbolic functions all have inverses.
- Implicit differentiation yields differentiation formulas for the inverse hyperbolic functions, which in turn give rise to integration formulas.
- The most common physical applications of hyperbolic functions are calculations involving catenaries.

## CHAPTER 6 REVIEW EXERCISES

*True or False?* Justify your answer with a proof or a counterexample.

**435.** The amount of work to pump the water out of a half-full cylinder is half the amount of work to pump the water out of the full cylinder.

**436.** If the force is constant, the amount of work to move an object from $x = a$ to $x = b$ is $F(b - a)$.

**437.** The disk method can be used in any situation in which the washer method is successful at finding the volume of a solid of revolution.

**438.** If the half-life of seaborgium-266 is 360 ms, then $k = (\ln(2))/360$.

For the following exercises, use the requested method to determine the volume of the solid.

**439.** The volume that has a base of the ellipse $x^2/4 + y^2/9 = 1$ and cross-sections of an equilateral triangle perpendicular to the $y$-axis. Use the method of slicing.

**440.** $y = x^2 - x$, from $x = 1$ to $x = 4$, rotated around the$y$-axis using the washer method

**441.** $x = y^2$ and $x = 3y$ rotated around the $y$-axis using the washer method

**442.** $x = 2y^2 - y^3$, $x = 0$, and $y = 0$ rotated around the $x$-axis using cylindrical shells

For the following exercises, find

a. the area of the region,
b. the volume of the solid when rotated around the $x$-axis, and
c. the volume of the solid when rotated around the $y$-axis. Use whichever method seems most appropriate to you.

**443.** $y = x^3$, $x = 0$, $y = 0$, and $x = 2$

**444.** $y = x^2 - x$ and $x = 0$

**445.** **[T]** $y = \ln(x) + 2$ and $y = x$

**446.** $y = x^2$ and $y = \sqrt{x}$

**447.** $y = 5 + x$, $y = x^2$, $x = 0$, and $x = 1$

**448.** Below $x^2 + y^2 = 1$ and above $y = 1 - x$

**449.** Find the mass of $\rho = e^{-x}$ on a disk centered at the origin with radius $4$.

**450.** Find the center of mass for $\rho = \tan^2 x$ on $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

**451.** Find the mass and the center of mass of $\rho = 1$ on the region bounded by $y = x^5$ and $y = \sqrt{x}$.

For the following exercises, find the requested arc lengths.

**452.** The length of $x$ for $y = \cosh(x)$ from $x = 0$ to $x = 2$.

**453.** The length of $y$ for $x = 3 - \sqrt{y}$ from $y = 0$ to $y = 4$

For the following exercises, find the surface area and volume when the given curves are revolved around the specified axis.

**454.** The shape created by revolving the region between $y = 4 + x$, $y = 3 - x$, $x = 0$, and $x = 2$ rotated around the $y$-axis.

**455.** The loudspeaker created by revolving $y = 1/x$ from $x = 1$ to $x = 4$ around the $x$-axis.

For the following exercises, consider the Karun-3 dam in Iran. Its shape can be approximated as an isosceles triangle with height $205$ m and width $388$ m. Assume the current depth of the water is $180$ m. The density of water is $1000$ kg/m $^3$.

**456.** Find the total force on the wall of the dam.

**457.** You are a crime scene investigator attempting to determine the time of death of a victim. It is noon and $45°\text{F}$ outside and the temperature of the body is $78°\text{F}$. You know the cooling constant is $k = 0.00824°\text{F/min}$. When did the victim die, assuming that a human's temperature is $98°\text{F}$ ?

For the following exercise, consider the stock market crash in $1929$ in the United States. The table lists the Dow Jones industrial average per year leading up to the crash.

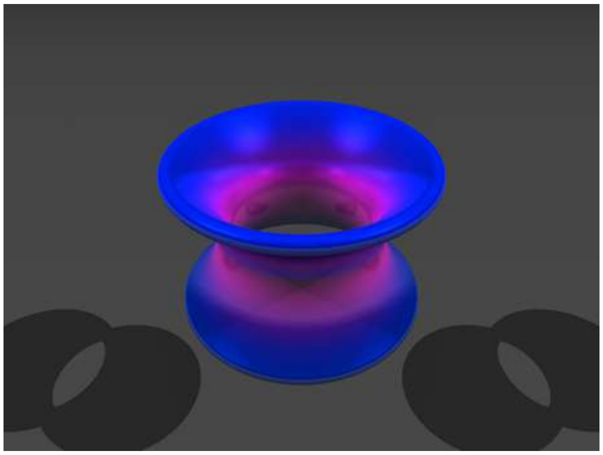
| **Years after 1920** | **Value ($)** |
|---|---|
| 1 | 63.90 |
| 3 | 100 |
| 5 | 110 |
| 7 | 160 |
| 9 | 381.17 |

*Source*: http://stockcharts.com/freecharts/historical/djia19201940.html

**458.** **[T]** The best-fit exponential curve to these data is given by $y = 40.71 + 1.224^x$. Why do you think the gains of the market were unsustainable? Use first and second derivatives to help justify your answer. What would this model predict the Dow Jones industrial average to be in $2014$ ?

For the following exercises, consider the catenoid, the only solid of revolution that has a minimal surface, or zero mean curvature. A catenoid in nature can be found when stretching soap between two rings.

**459.** Find the volume of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the $x$-axis, as shown here.



**460.** Find surface area of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the $x$-axis.

# APPENDIX A | TABLE OF INTEGRALS

## Basic Integrals

1. $\int u^n\, du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$

2. $\int \frac{du}{u} = \ln|u| + C$

3. $\int e^u\, du = e^u + C$

4. $\int a^u\, du = \frac{a^u}{\ln a} + C$

5. $\int \sin u\, du = -\cos u + C$

6. $\int \cos u\, du = \sin u + C$

7. $\int \sec^2 u\, du = \tan u + C$

8. $\int \csc^2 u\, du = -\cot u + C$

9. $\int \sec u \tan u\, du = \sec u + C$

10. $\int \csc u \cot u\, du = -\csc u + C$

11. $\int \tan u\, du = \ln|\sec u| + C$

12. $\int \cot u\, du = \ln|\sin u| + C$

13. $\int \sec u\, du = \ln|\sec u + \tan u| + C$

14. $\int \csc u\, du = \ln|\csc u - \cot u| + C$

15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

## Trigonometric Integrals

18. $\int \sin^2 u\, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

19. $\int \cos^2 u\, du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

20. $\int \tan^2 u\, du = \tan u - u + C$

21. $\int \cot^2 u\, du = -\cot u - u + C$

22. $\int \sin^3 u\, du = -\frac{1}{3}\left(2 + \sin^2 u\right)\cos u + C$

23. $\int \cos^3 u\, du = \frac{1}{3}\left(2 + \cos^2 u\right)\sin u + C$

24. $\int \tan^3 u\, du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u\, du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\, du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\, du$

29. $\int \cos^n u\, du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\, du$

30. $\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du$

31. $\int \cot^n u\, du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\, du$

32. $\int \sec^n u\, du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\, du$

33. $\int \csc^n u\, du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\, du$

34. $\int \sin au \sin bu\, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu\, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu\, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u \sin u\, du = \sin u - u \cos u + C$

38. $\int u \cos u\, du = \cos u + u \sin u + C$

39. $\int u^n \sin u\, du = -u^n \cos u + n\int u^{n-1} \cos u\, du$

40. $\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1} \sin u\, du$

41. $$\begin{aligned}\int \sin^n u \cos^m u\, du &= -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u\, du \\ &= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u\, du\end{aligned}$$

# Exponential and Logarithmic Integrals

42. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au}+C$

43. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1} e^{au}\,du$

44. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu)+C$

45. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu)+C$

46. $\int \ln u\,du = u\ln u - u + C$

47. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1]+C$

48. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

# Hyperbolic Integrals

49. $\int \sinh u\,du = \cosh u + C$

50. $\int \cosh u\,du = \sinh u + C$

51. $\int \tanh u\,du = \ln\cosh u + C$

52. $\int \coth u\,du = \ln|\sinh u| + C$

53. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

54. $\int \operatorname{csch} u\,du = \ln\left|\tanh\frac{1}{2}u\right| + C$

55. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

56. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

57. $\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$

58. $\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

# Inverse Trigonometric Integrals

59. $\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

60. $\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

61. $\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln\left(1+u^2\right) + C$

62. $\int u\sin^{-1} u\,du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u \cos^{-1} u \, du = \frac{2u^2 - 1}{4} \cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u \tan^{-1} u \, du = \frac{u^2 + 1}{2} \tan^{-1} u - \frac{u}{2} + C$

65. $\int u^n \sin^{-1} u \, du = \frac{1}{n+1}\left[u^{n+1} \sin^{-1} u - \int \frac{u^{n+1} \, du}{\sqrt{1-u^2}}\right], \; n \neq -1$

66. $\int u^n \cos^{-1} u \, du = \frac{1}{n+1}\left[u^{n+1} \cos^{-1} u + \int \frac{u^{n+1} \, du}{\sqrt{1-u^2}}\right], \; n \neq -1$

67. $\int u^n \tan^{-1} u \, du = \frac{1}{n+1}\left[u^{n+1} \tan^{-1} u - \int \frac{u^{n+1} \, du}{1+u^2}\right], \; n \neq -1$

## Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2 + u^2} \, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2} \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

69. $\int u^2 \sqrt{a^2 + u^2} \, du = \frac{u}{8}\left(a^2 + 2u^2\right)\sqrt{a^2 + u^2} - \frac{a^4}{8} \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

70. $\int \frac{\sqrt{a^2 + u^2}}{u} \, du = \sqrt{a^2 + u^2} - a \ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2 + u^2}}{u^2} \, du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2 + u^2}} = \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

73. $\int \frac{u^2 \, du}{\sqrt{a^2 + u^2}} = \frac{u}{2}\left(\sqrt{a^2 + u^2}\right) - \frac{a^2}{2} \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \ln\left|\frac{\sqrt{a^2 + u^2} + a}{u}\right| + C$

75. $\int \frac{du}{u^2 \sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C$

76. $\int \frac{du}{\left(a^2 + u^2\right)^{3/2}} = \frac{u}{a^2 \sqrt{a^2 + u^2}} + C$

## Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2 - a^2} \, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2} \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

78. $\int u^2 \sqrt{u^2 - a^2} \, du = \frac{u}{8}\left(2u^2 - a^2\right)\sqrt{u^2 - a^2} - \frac{a^4}{8} \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

79. $\int \frac{\sqrt{u^2 - a^2}}{u} \, du = \sqrt{u^2 - a^2} - a\cos^{-1} \frac{a}{|u|} + C$

80. $\int \frac{\sqrt{u^2 - a^2}}{u^2} \, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

81. $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

82. $\int \frac{u^2\,du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

83. $\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

84. $\int \frac{du}{\left(u^2 - a^2\right)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

## Integrals Involving $a^2 - u^2$, $a > 0$

85. $\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

86. $\int u^2\sqrt{a^2 - u^2}\,du = \frac{u}{8}\left(2u^2 - a^2\right)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

87. $\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

88. $\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

89. $\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{u}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

90. $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

91. $\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

92. $\int \left(a^2 - u^2\right)^{3/2}\,du = -\frac{u}{8}\left(2u^2 - 5a^2\right)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

93. $\int \frac{du}{\left(a^2 - u^2\right)^{3/2}} = -\frac{u}{a^2\sqrt{a^2 - u^2}} + C$

## Integrals Involving $2au - u^2$, $a > 0$

94. $\int \sqrt{2au - u^2}\,du = \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

95. $\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\left(\frac{a - u}{a}\right) + C$

96. $\int u\sqrt{2au - u^2}\,du = \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

97. $\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$

# Integrals Involving $a + bu$, $a \neq 0$

98. $\int \frac{u\,du}{a+bu} = \frac{1}{b^2}(a + bu - a\ln|a + bu|) + C$

99. $\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2 \ln|a+bu|\right] + C$

100. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

101. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

102. $\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

103. $\int \frac{u\,du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

104. $\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

105. $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a+bu)^{3/2} + C$

106. $\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a+bu} + C$

107. $\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}\left(8a^2 + 3b^2u^2 - 4abu\right)\sqrt{a+bu} + C$

108.
$$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$$
$$= \frac{2}{\sqrt{-a}}\tan-1\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$$

109. $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

110. $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

111. $\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

112. $\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

113. $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

# APPENDIX B | TABLE OF DERIVATIVES

## General Formulas

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

4. $\frac{d}{dx}(x^n) = nx^{n-1}$, for real numbers $n$

5. $\frac{d}{dx}(cf(x)) = cf'(x)$

6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

## Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$

10. $\frac{d}{dx}(\tan x) = \sec^2 x$

11. $\frac{d}{dx}(\sec x) = \sec x \tan x$

12. $\frac{d}{dx}(\cos x) = -\sin x$

13. $\frac{d}{dx}(\cot x) = -\csc^2 x$

14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

## Inverse Trigonometric Functions

15. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$

16. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2}$

17. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2 - 1}}$

18. $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1 - x^2}}$

19. $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1 + x^2}$

20. $\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2 - 1}}$

# Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$

22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

23. $\frac{d}{dx}(b^x) = b^x \ln b$

24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

# Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$

26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

28. $\frac{d}{dx}(\cosh x) = \sinh x$

29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

# Inverse Hyperbolic Functions

31. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2 + 1}}$

32. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1 - x^2}(|x| < 1)$

33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1 - x^2}} \quad (0 < x < 1)$

34. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2 - 1}} \quad (x > 1)$

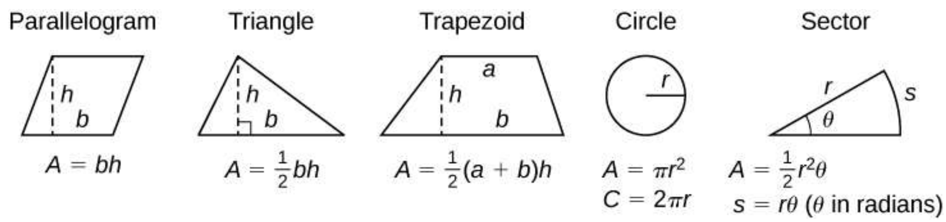
35. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1 - x^2} \quad (|x| > 1)$

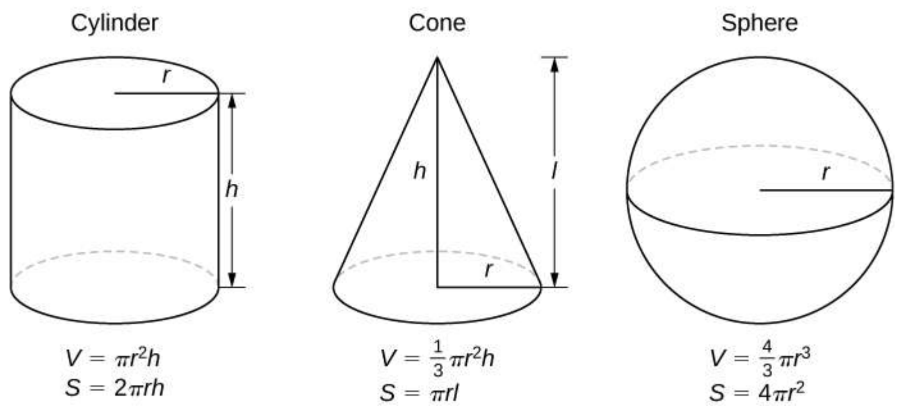
36. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{|x|\sqrt{1 + x^2}} (x \neq 0)$

# APPENDIX C | REVIEW OF PRE-CALCULUS

## Formulas from Geometry

$A = \text{area}, \quad V = \text{Volume}, \quad \text{and} \quad S = \text{lateral surface area}$




## Formulas from Algebra

### Laws of Exponents

$$
\begin{aligned}
x^m x^n &= x^{m+n} & \frac{x^m}{x^n} &= x^{m-n} & (x^m)^n &= x^{mn} \\
x^{-n} &= \frac{1}{x^n} & (xy)^n &= x^n y^n & \left(\frac{x}{y}\right)^n &= \frac{x^n}{y^n} \\
x^{1/n} &= \sqrt[n]{x} & \sqrt[n]{xy} &= \sqrt[n]{x}\sqrt[n]{y} & \sqrt[n]{\frac{x}{y}} &= \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \\
x^{m/n} &= \sqrt[n]{x^m} = (\sqrt[n]{x})^m
\end{aligned}
$$

### Special Factorizations

$$
\begin{aligned}
x^2 - y^2 &= (x + y)(x - y) \\
x^3 + y^3 &= (x + y)\left(x^2 - xy + y^2\right) \\
x^3 - y^3 &= (x - y)\left(x^2 + xy + y^2\right)
\end{aligned}
$$

### Quadratic Formula

$$\text{If } ax^2 + bx + c = 0, \quad \text{then } x = \frac{-b \pm \sqrt{b^2 - 4ca}}{2a}.$$

## Binomial Theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ \cdots \ + \binom{n}{n-1}ab^{n-1} + b^n,$$

where $\binom{n}{k} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 3 \cdot 2 \cdot 1} = \dfrac{n!}{k!(n-k)!}$

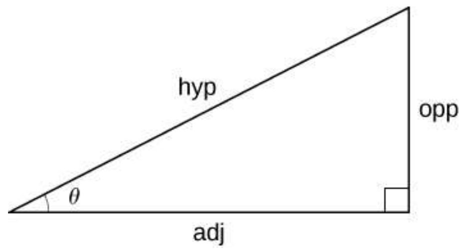
# Formulas from Trigonometry

## Right-Angle Trigonometry

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} \qquad \cot\theta = \dfrac{\text{adj}}{\text{opp}}$




## Trigonometric Functions of Important Angles

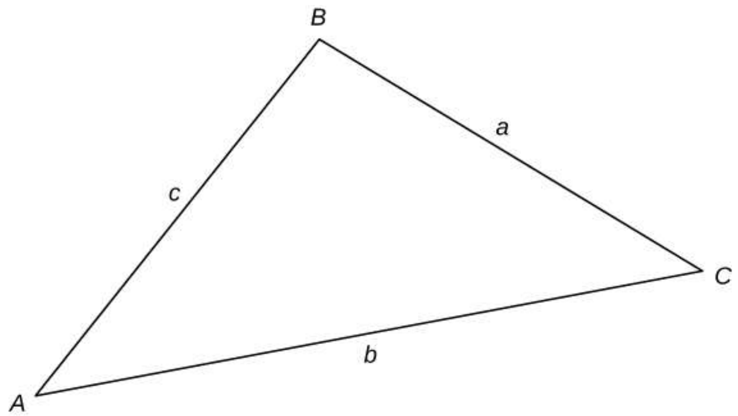
| $\theta$ | Radians | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|---|
| 0° | 0 | 0 | 1 | 0 |
| 30° | $\pi/6$ | 1/2 | $\sqrt{3}/2$ | $\sqrt{3}/3$ |
| 45° | $\pi/4$ | $\sqrt{2}/2$ | $\sqrt{2}/2$ | 1 |
| 60° | $\pi/3$ | $\sqrt{3}/2$ | 1/2 | $\sqrt{3}$ |
| 90° | $\pi/2$ | 1 | 0 | — |

## Fundamental Identities

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 & \sin(-\theta) &= -\sin\theta \\
1 + \tan^2\theta &= \sec^2\theta & \cos(-\theta) &= \cos\theta \\
1 + \cot^2\theta &= \csc^2\theta & \tan(-\theta) &= -\tan\theta \\
\sin\left(\frac{\pi}{2} - \theta\right) &= \cos\theta & \sin(\theta + 2\pi) &= \sin\theta \\
\cos\left(\frac{\pi}{2} - \theta\right) &= \sin\theta & \cos(\theta + 2\pi) &= \cos\theta \\
\tan\left(\frac{\pi}{2} - \theta\right) &= \cot\theta & \tan(\theta + \pi) &= \tan\theta
\end{aligned}$$

## Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$




## Law of Cosines

$$\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \\
b^2 &= a^2 + c^2 - 2ac\cos B \\
c^2 &= a^2 + b^2 - 2ab\cos C
\end{aligned}$$

## Addition and Subtraction Formulas

$$\begin{aligned}
\sin(x + y) &= \sin x\cos y + \cos x\sin y \\
\sin(x - y) &= \sin x\cos y - \cos x\sin y \\
\cos(x + y) &= \cos x\cos y - \sin x\sin y \\
\cos(x - y) &= \cos x\cos y + \sin x\sin y \\
\tan(x + y) &= \frac{\tan x + \tan y}{1 - \tan x\tan y} \\
\tan(x - y) &= \frac{\tan x - \tan y}{1 + \tan x\tan y}
\end{aligned}$$

## Double-Angle Formulas

$$\begin{aligned}
\sin 2x &= 2\sin x\cos x \\
\cos 2x &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \\
\tan 2x &= \frac{2\tan x}{1 - \tan^2 x}
\end{aligned}$$

## Half-Angle Formulas

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2}\\ \cos^2 x &= \frac{1+\cos 2x}{2}\end{aligned}$$

# ANSWER KEY

## Chapter 1

### Checkpoint

**1.1**. $f(1) = 3$ and $f(a+h) = a^2 + 2ah + h^2 - 3a - 3h + 5$

**1.2**. Domain = $\{x|x \leq 2\}$, range = $\{y|y \geq 5\}$

**1.3**. $x = 0,\ 2,\ 3$

**1.4**. $\left(\frac{f}{g}\right)(x) = \frac{x^2+3}{2x-5}$. The domain is $\left\{x|x \neq \frac{5}{2}\right\}$.

**1.5**. $(f \circ g)(x) = 2 - 5\sqrt{x}$.

**1.6**. $(g \circ f)(x) = 0.63x$

**1.7**. $f(x)$ is odd.

**1.8**. Domain = $(-\infty,\ \infty)$, range = $\{y|y \geq -4\}$.

**1.9**. $m = 1/2$. The point-slope form is $y - 4 = \frac{1}{2}(x-1)$. The slope-intercept form is $y = \frac{1}{2}x + \frac{7}{2}$.

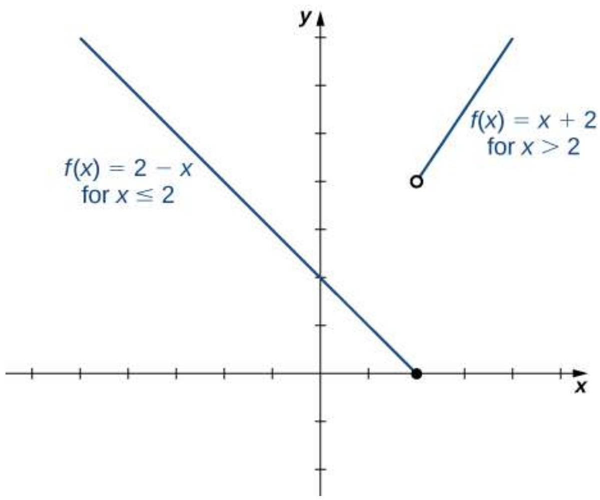
**1.10**. The zeros are $x = 1 \pm \sqrt{3}/3$. The parabola opens upward.

**1.11**. The domain is the set of real numbers $x$ such that $x \neq 1/2$. The range is the set $\{y|y \neq 5/2\}$.

**1.12**. The domain of $f$ is $(-\infty,\ \infty)$. The domain of $g$ is $\{x|x \geq 1/5\}$.

**1.13**. Algebraic

**1.14**.




**1.15**. $C(x) = \begin{cases} 49, & 0 < x \leq 1 \\ 70, & 1 < x \leq 2 \\ 91, & 2 < x \leq 3 \end{cases}$

**1.16**. Shift the graph $y = x^2$ to the left 1 unit, reflect about the $x$-axis, then shift down 4 units.

**1.17**. $7\pi/6$; $330°$

**1.18**. $\cos(3\pi/4) = -\sqrt{2}/2$; $\sin(-\pi/6) = -1/2$

**1.19**. $10\ \text{ft}$

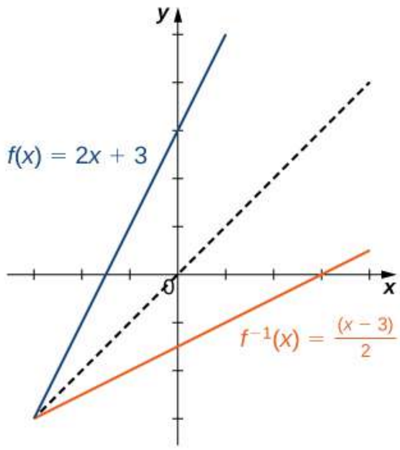
**1.20**. $\theta = \frac{3\pi}{2} + 2n\pi,\ \frac{\pi}{6} + 2n\pi,\ \frac{5\pi}{6} + 2n\pi$ for $n = 0,\ \pm 1,\ \pm 2, \ldots$

**1.22**. To graph $f(x) = 3\sin(4x) - 5$, the graph of $y = \sin(x)$ needs to be compressed horizontally by a factor of 4, then stretched vertically by a factor of 3, then shifted down 5 units. The function $f$ will have a period of $\pi/2$ and an amplitude of 3.

**1.23**. No.

**1.24**. $f^{-1}(x) = \dfrac{2x}{x-3}$. The domain of $f^{-1}$ is $\{x|x \neq 3\}$. The range of $f^{-1}$ is $\{y|y \neq 2\}$.

**1.25**.




**1.26**. The domain of $f^{-1}$ is $(0, \infty)$. The range of $f^{-1}$ is $(-\infty, 0)$. The inverse function is given by the formula $f^{-1}(x) = -1/\sqrt{x}$.

**1.27**. $f(4) = 900$; $f(10) = 24,\ 300$.

**1.28**. $x/\left(2y^3\right)$

**1.29**. $A(t) = 750e^{0.04t}$. After $30$ years, there will be approximately $\$2,\ 490.09$.

**1.30**. $x = \dfrac{\ln 3}{2}$

**1.31**. $x = \dfrac{1}{e}$

**1.32**. $1.29248$

**1.33**. The magnitude $8.4$ earthquake is roughly $10$ times as severe as the magnitude $7.4$ earthquake.

**1.34**. $(x^2 + x^{-2})/2$

**1.35**. $\dfrac{1}{2}\ln(3) \approx 0.5493$.

## Section Exercises

**1**. a. Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, range = $\{0, 1, 4, 9\}$ b. Yes, a function

**3**. a. Domain = $\{0, 1, 2, 3\}$, range = $\{-3, -2, -1, 0, 1, 2, 3\}$ b. No, not a function

**5**. a. Domain = $\{3, 5, 8, 10, 15, 21, 33\}$, range = $\{0, 1, 2, 3\}$ b. Yes, a function

**7**. a. $-2$ b. 3 c. 13 d. $-5x - 2$ e. $5a - 2$ f. $5a + 5h - 2$

**9**. a. Undefined b. 2 c. $\dfrac{2}{3}$ d. $-\dfrac{2}{x}$ e $\dfrac{2}{a}$ f. $\dfrac{2}{a+h}$

**11**. a. $\sqrt{5}$ b. $\sqrt{11}$ c. $\sqrt{23}$ d. $\sqrt{-6x+5}$ e. $\sqrt{6a+5}$ f. $\sqrt{6a+6h+5}$

**13**. a. 9 b. 9 c. 9 d. 9 e. 9 f. 9

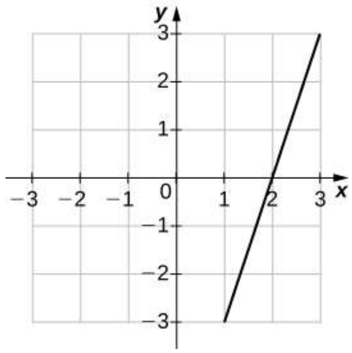
**15**. $x \geq \dfrac{1}{8}$; $y \geq 0$; $x = \dfrac{1}{8}$; no $y$-intercept

**17**. $x \geq -2$; $y \geq -1$; $x = -1$; $y = -1 + \sqrt{2}$

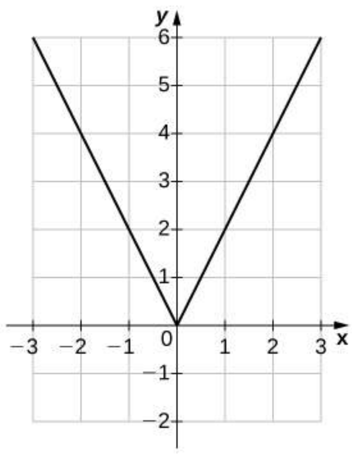
**19**. $x \neq 4$; $y \neq 0$; no $x$-intercept; $y = -\dfrac{3}{4}$

**21**. $x > 5$; $y > 0$; no intercepts

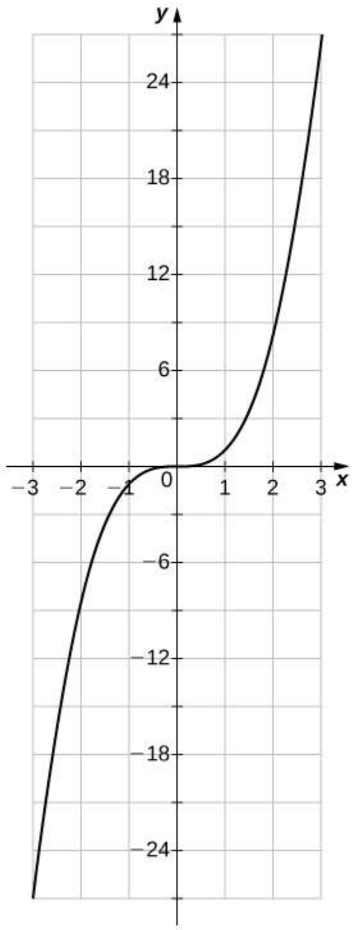
**23**.




**25**.




**27**.




**29**. Function; a. Domain: all real numbers, range: $y \geq 0$ b. $x = \pm 1$ c. $y = 1$ d. $-1 < x < 0$ and $1 < x < \infty$ e. $-\infty < x < -1$ and $0 < x < 1$ f. Not constant g. $y$-axis h. Even

**31**. Function; a. Domain: all real numbers, range: $-1.5 \leq y \leq 1.5$ b. $x = 0$ c. $y = 0$ d. all real numbers e. None f. Not constant g. Origin h. Odd

**33**. Function; a. Domain: $-\infty < x < \infty,$ range: $-2 \leq y \leq 2$ b. $x = 0$ c. $y = 0$ d. $-2 < x < 2$ e. Not decreasing f. $-\infty < x < -2$ and $2 < x < \infty$ g. Origin h. Odd

**35**. Function; a. Domain: $-4 \leq x \leq 4,$ range: $-4 \leq y \leq 4$ b. $x = 1.2$ c. $y = 4$ d. Not increasing e. $0 < x < 4$ f. $-4 < x < 0$ g. No Symmetry h. Neither

**37**. a. $5x^2 + x - 8;$ all real numbers b. $-5x^2 + x - 8;$ all real numbers c. $5x^3 - 40x^2;$ all real numbers d. $\frac{x-8}{5x^2};\ x \neq 0$

**39**. a. $-2x + 6;$ all real numbers b. $-2x^2 + 2x + 12;$ all real numbers c. $-x^4 + 2x^3 + 12x^2 - 18x - 27;$ all real numbers d. $-\frac{x+3}{x+1};\ x \neq -1,\ 3$

**41**. a. $6 + \frac{2}{x};\ x \neq 0$ b. 6; $x \neq 0$ c. $\frac{6}{x} + \frac{1}{x^2};\ x \neq 0$ d. $6x + 1;\ x \neq 0$

**43**. a. $4x + 3;$ all real numbers b. $4x + 15;$ all real numbers

**45**. a. $x^4 - 6x^2 + 16;$ all real numbers b. $x^4 + 14x^2 + 46;$ all real numbers

**47**. a. $\frac{3x}{4+x};\ x \neq 0,\ -4$ b. $\frac{4x+2}{3};\ x \neq -\frac{1}{2}$

**49**. a. Yes, because there is only one winner for each year. b. No, because there are three teams that won more than once during the years 2001 to 2012.

**51**. a. $V(s) = s^3$ b. $V(11.8) \approx 1643;$ a cube of side length 11.8 each has a volume of approximately 1643 cubic units.

**53**. a. $N(x) = 15x$ b. i. $N(20) = 15(20) = 300;$ therefore, the vehicle can travel 300 mi on a full tank of gas. Ii. $N(15) = 225;$ therefore, the vehicle can travel 225 mi on 3/4 of a tank of gas. c. Domain: $0 \leq x \leq 20;$ range: $[0,\ 300]$ d. The driver had to stop at least once, given that it takes approximately 39 gal of gas to drive a total of 578 mi.

**55**. a. $A(t) = A(r(t)) = \pi \cdot \left(6 - \frac{5}{t^2 + 1}\right)^2$ b. Exact: $\frac{121\pi}{4};$ approximately 95 cm$^2$ c. $C(t) = C(r(t)) = 2\pi\left(6 - \frac{5}{t^2 + 1}\right)$ d. Exact: $11\pi;$ approximately 35 cm

**57**. a. $S(x) = 8.5x + 750$ b. \$962.50, \$1090, \$1217.50 c. 77 skateboards

**59**. a. −1 b. Decreasing

**61**. a. 3/4 b. Increasing

**63**. a. 4/3 b. Increasing

**65**. a. 0 b. Horizontal

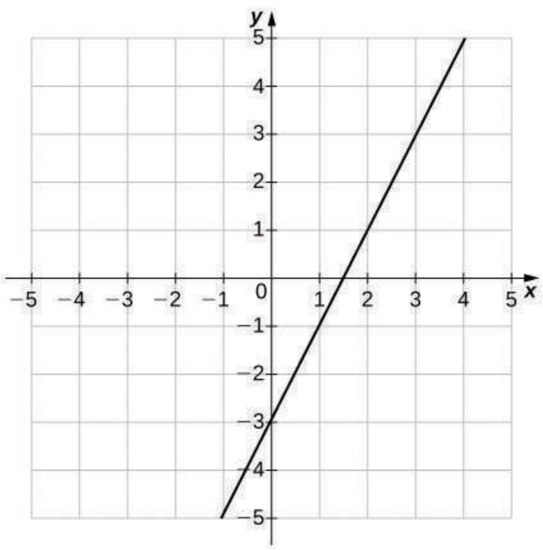
**67**. $y = -6x + 9$

**69**. $y = \frac{1}{3}x + 4$

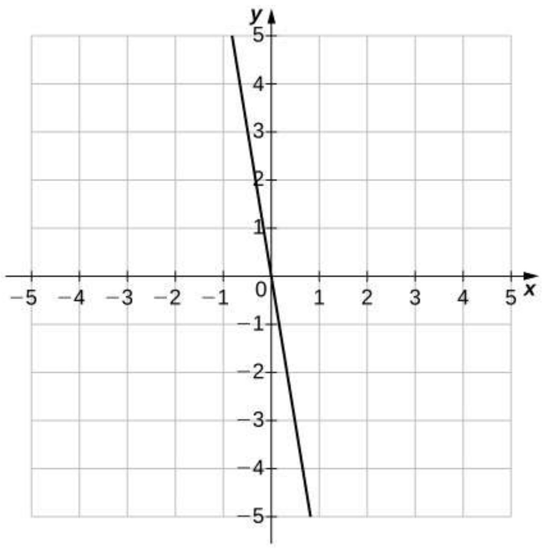
**71**. $y = \frac{1}{2}x$

**73**. $y = \frac{3}{5}x - 3$

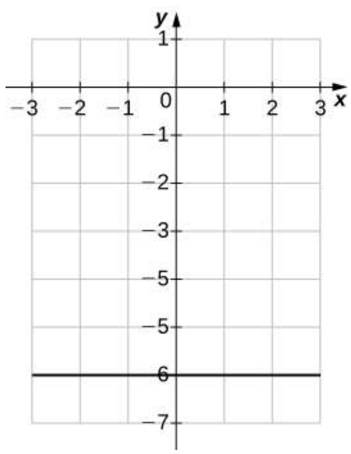
**75**. a. $(m = 2,\ b = -3)$ b.



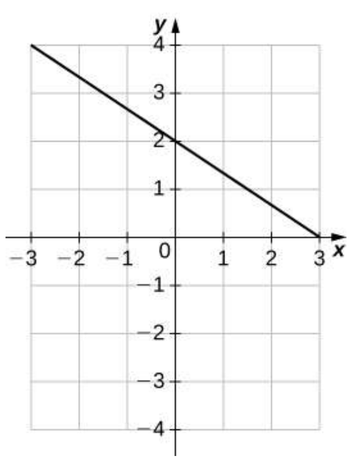

**77**. a. $(m = -6,\ b = 0)$ b.




**79**. a. $(m = 0, \, b = -6)$ b.




**81**. a. $\left(m = -\frac{2}{3}, \, b = 2\right)$ b.

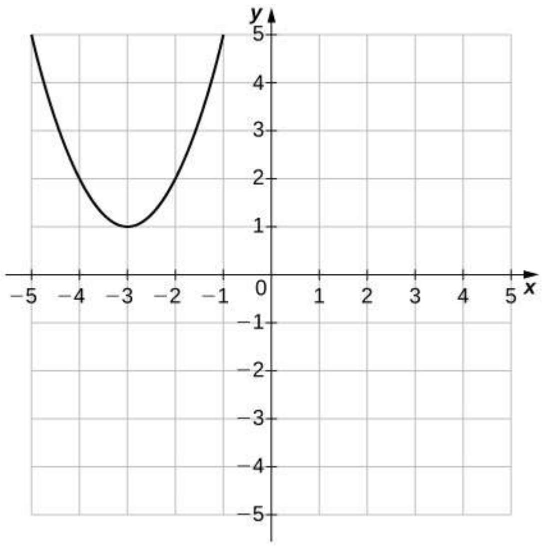



**83**. a. 2 b. $\frac{5}{2}, \, -1;$ c. −5 d. Both ends rise e. Neither

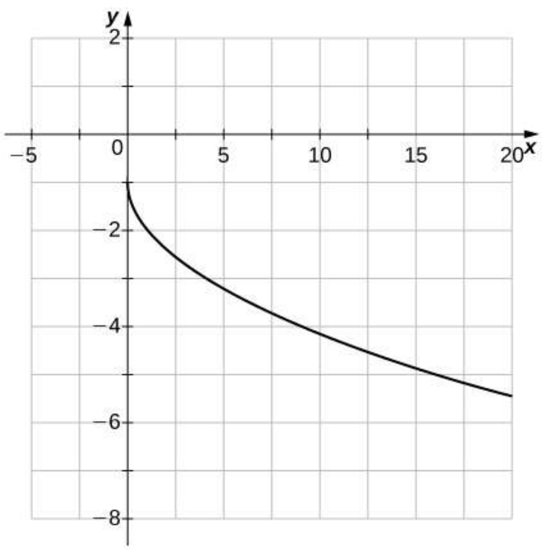
**85**. a. 2 b. $\pm\sqrt{2}$ c. −1 d. Both ends rise e. Even

**87**. a. 3 b. 0, $\pm\sqrt{3}$ c. 0 d. Left end rises, right end falls e. Odd

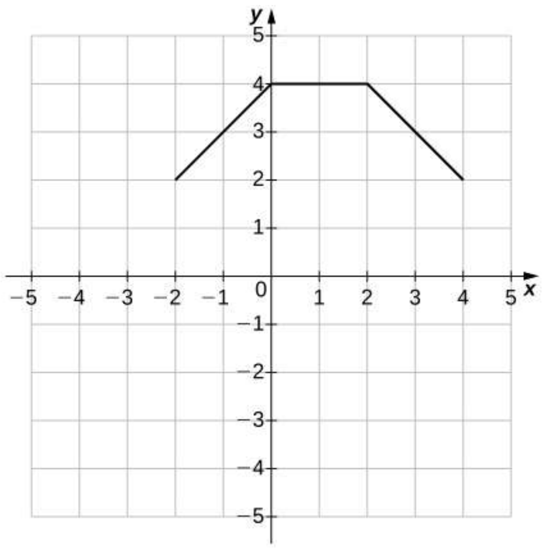
**89**.



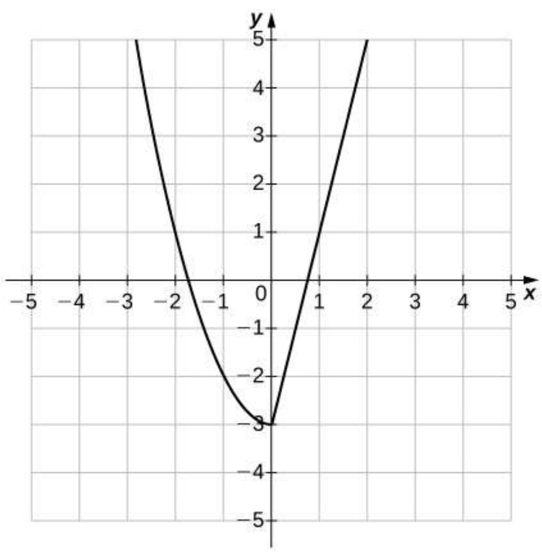

**91**.



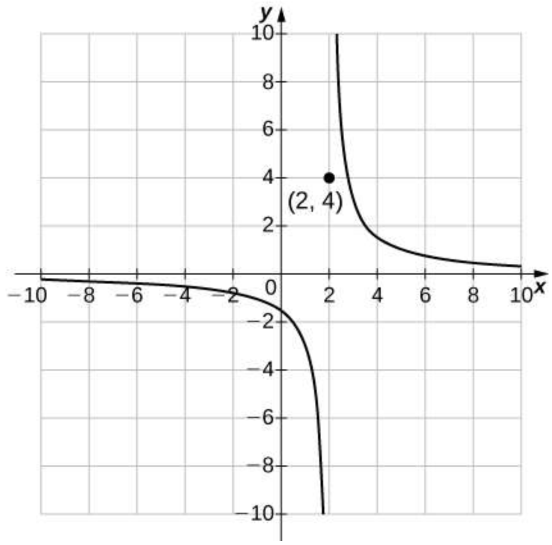

**93**.




**95**. a. $13, -3, 5$ b.




**97**. a. $\frac{-3}{2}, \frac{-1}{2}, 4$ b.




**99**. True; $n = 3$

**101**. False; $f(x) = x^b$, where $b$ is a real-valued constant, is a power function

**103**. a. $V(t) = -2733t + 20500$ b. $(0,\ 20,\ 500)$ means that the initial purchase price of the equipment is $20,500; $(7.5,\ 0)$ means that in 7.5 years the computer equipment has no value. c. $6835 d. In approximately 6.4 years

**105**. a. $C = 0.75x + 125$ b. $245 c. 167 cupcakes

**107**. a. $V(t) = -1500t + 26{,}000$ b. In 4 years, the value of the car is $20,000.

**109**. $30,337.50

**111**. 96% of the total capacity

**113**. $\frac{4\pi}{3}$ rad

**115**. $\frac{-\pi}{3}$

**117**. $\frac{11\pi}{6}$ rad

**119**. $210°$

**121**. $-540°$

**123**. $-0.5$

**125**. $-\frac{\sqrt{2}}{2}$

**127**. $\frac{\sqrt{3}-1}{2\sqrt{2}}$

**129**. a. $b = 5.7$ b. $\sin A = \frac{4}{7}, \cos A = \frac{5.7}{7}, \tan A = \frac{4}{5.7}, \csc A = \frac{7}{4}, \sec A = \frac{7}{5.7}, \cot A = \frac{5.7}{4}$

**131**. a. $c = 151.7$ b. $\sin A = 0.5623, \cos A = 0.8273, \tan A = 0.6797, \csc A = 1.778, \sec A = 1.209, \cot A = 1.471$

**133**. a. $c = 85$ b. $\sin A = \frac{84}{85}, \cos A = \frac{13}{85}, \tan A = \frac{84}{13}, \csc A = \frac{85}{84}, \sec A = \frac{85}{13}, \cot A = \frac{13}{84}$

**135**. a. $y = \frac{24}{25}$ b. $\sin\theta = \frac{24}{25}, \cos\theta = \frac{7}{25}, \tan\theta = \frac{24}{7}, \csc\theta = \frac{25}{24}, \sec\theta = \frac{25}{7}, \cot\theta = \frac{7}{24}$

**137**. a. $x = \frac{-\sqrt{2}}{3}$ b. $\sin\theta = \frac{\sqrt{7}}{3}, \cos\theta = \frac{-\sqrt{2}}{3}, \tan\theta = \frac{-\sqrt{14}}{2}, \csc\theta = \frac{3\sqrt{7}}{7}, \sec\theta = \frac{-3\sqrt{2}}{2}, \cot\theta = \frac{-\sqrt{14}}{7}$

**139**. $\sec^2 x$

**141**. $\sin^2 x$

**143**. $\sec^2\theta$

**145**. $\frac{1}{\sin t} = \csc t$

**155**. $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

**157**. $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

**159**. $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

**161**. $\left\{0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$

**163**. $y = 4\sin\left(\frac{\pi}{4}x\right)$

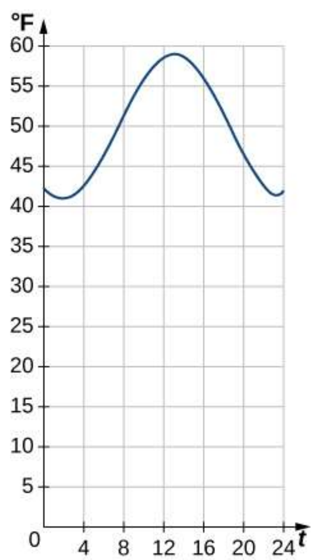
**165**. $y = \cos(2\pi x)$

**167**. a. 1 b. $2\pi$ c. $\frac{\pi}{4}$ units to the right

**169**. a. $\frac{1}{2}$ b. $8\pi$ c. No phase shift

**171**. a. 3 b. $2$ c. $\frac{2}{\pi}$ units to the left

**173**. Approximately 42 in.
**175**. a. 0.550 rad/sec b. 0.236 rad/sec c. 0.698 rad/min d. 1.697 rad/min
**177**. $\approx 30.9\text{ in}^2$
**179**. a. π/184; the voltage repeats every π/184 sec b. Approximately 59 periods
**181**. a. Amplitude = $10$; period $= 24$ b. $47.4\ °F$ c. 14 hours later, or 2 p.m. d.

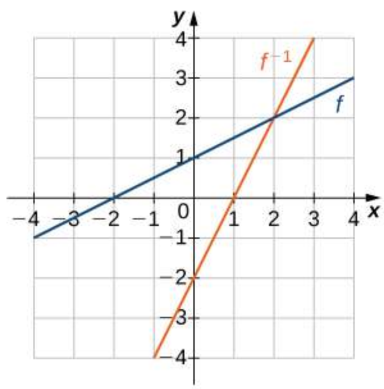



**183**. Not one-to-one
**185**. Not one-to-one
**187**. One-to-one
**189**. a. $f^{-1}(x) = \sqrt{x+4}$ b. Domain $: x \geq -4,$ range: $y \geq 0$

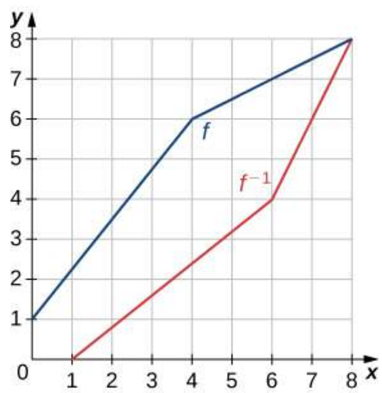
**191**. a. $f^{-1}(x) = \sqrt[3]{x-1}$ b. Domain: all real numbers, range: all real numbers

**193**. a. $f^{-1}(x) = x^2 + 1,$ b. Domain: $x \geq 0,$ range: $y \geq 1$

**195**.




**197**.




**199**. These are inverses.
**201**. These are not inverses.
**203**. These are inverses.
**205**. These are inverses.
**207**. $\frac{\pi}{6}$

**209**. $\frac{\pi}{4}$

**211**. $\frac{\pi}{6}$

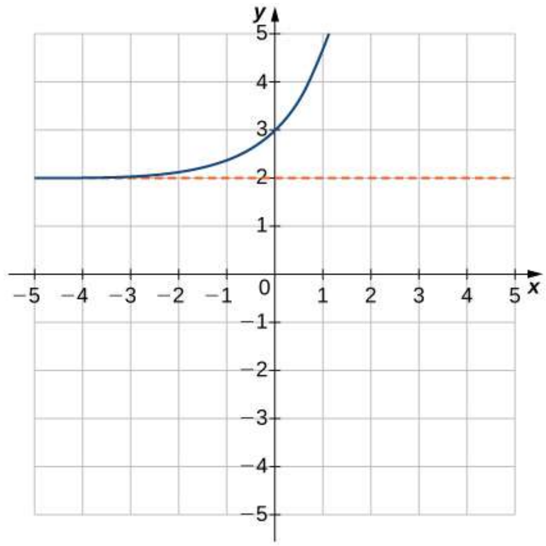
**213**. $\frac{\sqrt{2}}{2}$

**215**. $-\frac{\pi}{6}$

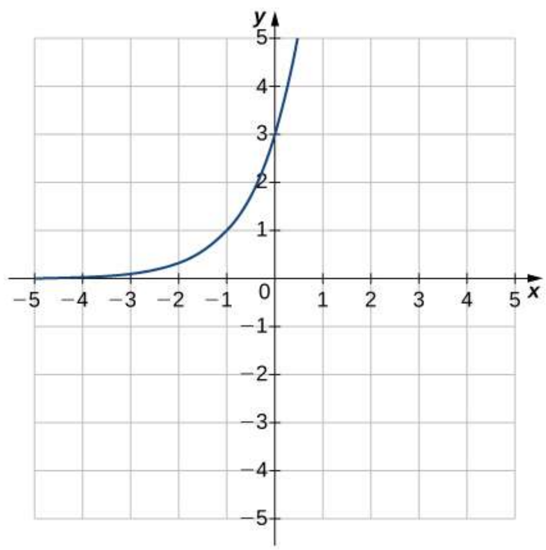
**217**. a. $x = f^{-1}(V) = \sqrt{0.04 - \frac{V}{500}}$ b. The inverse function determines the distance from the center of the artery at which blood is flowing with velocity $V$. c. 0.1 cm; 0.14 cm; 0.17 cm
**219**. a. $31,250, $66,667, $107,143 b. $\left(p = \frac{85C}{C + 75}\right)$ c. 34 ppb
**221**. a. $\sim 92°$ b. $\sim 42°$ c. $\sim 27°$
**223**. $x \approx 6.69,\ 8.51;$ so, the temperature occurs on June 21 and August 15
**225**. $\sim 1.5$ sec
**227**. $\tan^{-1}(\tan(2.1)) \approx -1.0416;$ the expression does not equal 2.1 since $2.1 > 1.57 = \frac{\pi}{2}$—in other words, it is not in the restricted domain of $\tan x$. $\cos^{-1}(\cos(2.1)) = 2.1,$ since 2.1 is in the restricted domain of $\cos x$.
**229**. a. 125 b. 2.24 c. 9.74
**231**. a. 0.01 b. 10,000 c. 46.42
**233**. d
**235**. b

**237**. e

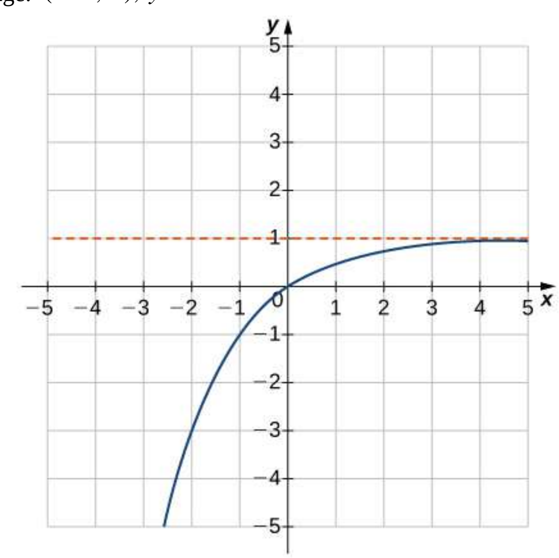
**239**. Domain: all real numbers, range: $(2, \infty)$, $y = 2$



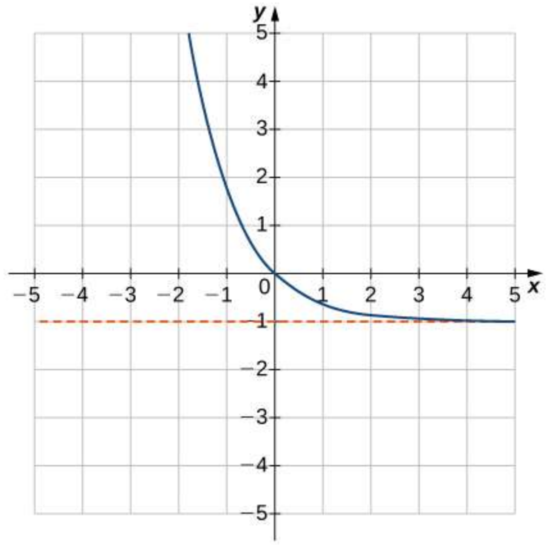
**241**. Domain: all real numbers, range: $(0, \infty)$, $y = 0$



**243**. Domain: all real numbers, range: $(-\infty, 1)$, $y = 1$



**245**. Domain: all real numbers, range: $(-1, \infty)$, $y = -1$




**247**. $8^{1/3} = 2$

**249**. $5^2 = 25$

**251**. $e^{-3} = \frac{1}{e^3}$

**253**. $e^0 = 1$

**255**. $\log_4\left(\frac{1}{16}\right) = -2$

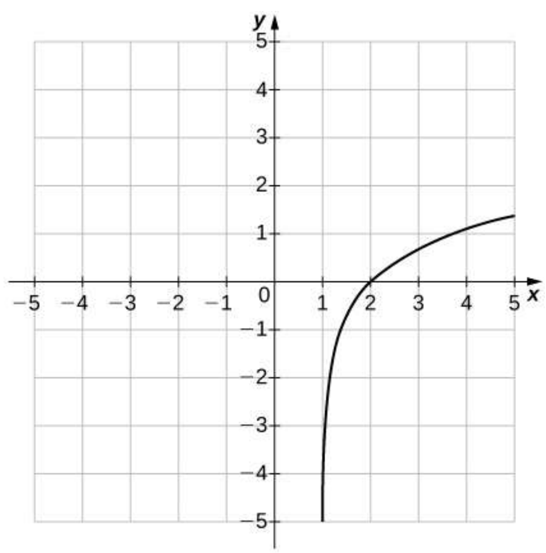
**257**. $\log_9 1 = 0$

**259**. $\log_{64} 4 = \frac{1}{3}$

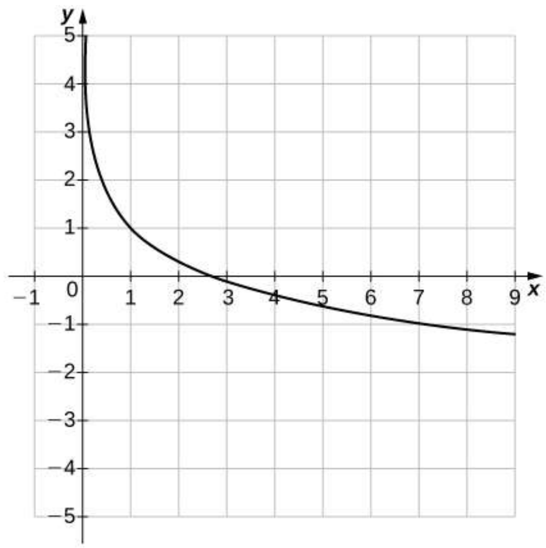
**261**. $\log_9 150 = y$

**263**. $\log_4 0.125 = -\frac{3}{2}$

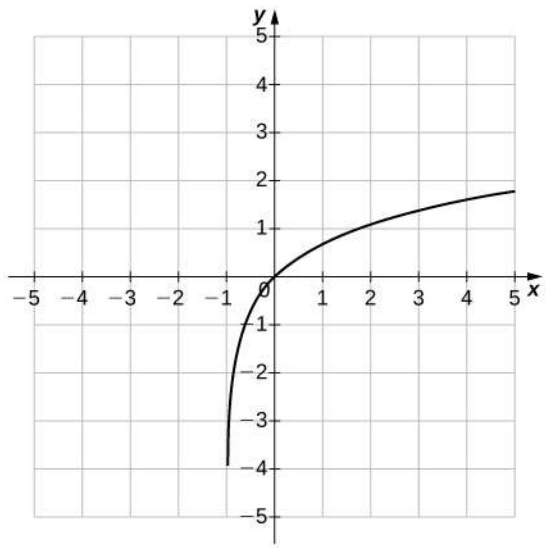
**265**. Domain: $(1, \infty)$, range: $(-\infty, \infty)$, $x = 1$




**267**. Domain: $(0, \infty)$, range: $(-\infty, \infty)$, $x = 0$




**269**. Domain: $(-1,\ \infty)$, range: $(-\infty,\ \infty)$, $x = -1$




**271**. $2 + 3\log_3 a - \log_3 b$

**273**. $\frac{3}{2} + \frac{1}{2}\log_5 x + \frac{3}{2}\log_5 y$

**275**. $-\frac{3}{2} + \ln 6$

**277**. $\frac{\ln 15}{3}$

**279**. $\frac{3}{2}$

**281**. $\log 7.21$

**283**. $\frac{2}{3} + \frac{\log 11}{3\log 7}$

**285**. $x = \frac{1}{25}$

**287**. $x = 4$

**289**. $x = 3$

**291**. $1 + \sqrt{5}$

**293**. $\left(\frac{\log 82}{\log 7} \approx 2.2646\right)$

**295**. $\left(\frac{\log 211}{\log 0.5} \approx -7.7211\right)$

**297**. $\left(\frac{\log 0.452}{\log 0.2} \approx 0.4934\right)$

**299**. $\sim 17,\ 491$

**301**. Approximately $131,653 is accumulated in 5 years.
**303**. i. a. pH = 8 b. Base ii. a. pH = 3 b. Acid iii. a. pH = 4 b. Acid
**305**. a. $\sim 333$ million b. 94 years from 2013, or in 2107
**307**. a. $k \approx 0.0578$ b. $\approx 92$ hours
**309**. The San Francisco earthquake had $10^{3.4}$ or $\sim 2512$ times more energy than the Japan earthquake.

## Review Exercises

**311**. False
**313**. False
**315**. Domain: $x > 5,$ range: all real numbers
**317**. Domain: $x > 2$ and $x < -4,$ range: all real numbers
**319**. Degree of 3, $y$ -intercept: 0, zeros: 0, $\sqrt{3} - 1,\ -1 - \sqrt{3}$
**321**. $\cos^2 x - \sin^2 x = \cos 2x$ or $= \frac{1 - 2\sin^2 x}{2}$ or $= \frac{2\cos^2 x - 1}{2}$
**323**. $0,\ \pm 2\pi$
**325**. 4
**327**. One-to-one; yes, the function has an inverse; inverse: $f^{-1}(x) = \frac{1}{y}$
**329**. $x \geq -\frac{3}{2},\ f^{-1}(x) = -\frac{3}{2} + \frac{1}{2}\sqrt{4y - 7}$
**331**. a. $C(x) = 300 + 7x$ b. 100 shirts
**333**. The population is less than 20,000 from December 8 through January 23 and more than 140,000 from May 29 through August 2
**335**. 78.51%

# Chapter 2

## Checkpoint

**2.1**. 2.25
**2.2**. 12.006001
**2.3**. 16 unit$^2$
**2.4**. $\lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1} = -1$
**2.5**. $\lim_{x \to 2} h(x) = -1.$
**2.6**. $\lim_{x \to 2} \frac{|x^2 - 4|}{x - 2}$ does not exist.
**2.7**. a. $\lim_{x \to 2^-} \frac{|x^2 - 4|}{x - 2} = -4;$ b. $\lim_{x \to 2^+} \frac{|x^2 - 4|}{x - 2} = 4$
**2.8**. a. $\lim_{x \to 0^-} \frac{1}{x^2} = +\infty;$ b. $\lim_{x \to 0^+} \frac{1}{x^2} = +\infty;$ c. $\lim_{x \to 0} \frac{1}{x^2} = +\infty$
**2.9**. a. $\lim_{x \to 2^-} \frac{1}{(x - 2)^3} = -\infty;$ b. $\lim_{x \to 2^+} \frac{1}{(x - 2)^3} = +\infty;$ c. $\lim_{x \to 2} \frac{1}{(x - 2)^3}$ DNE. The line $x = 2$ is the vertical asymptote of $f(x) = 1/(x - 2)^3.$
**2.10**. Does not exist.
**2.11**. $11\sqrt{10}$
**2.12**. −13;

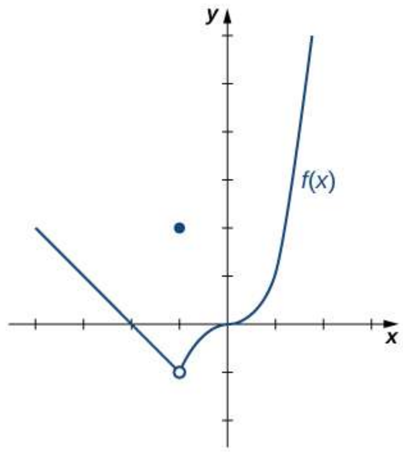
**2.13**. $\frac{1}{3}$

**2.14**. $\frac{1}{4}$

**2.15**. −1;

**2.16**. $\frac{1}{4}$

**2.17**.




$\lim_{x \to -1^-} f(x) = -1$

**2.18**. +∞

**2.19**. 0

**2.20**. 0

**2.21**. *f* is not continuous at 1 because $f(1) = 2 \neq 3 = \lim_{x \to 1} f(x)$.

**2.22**. $f(x)$ is continuous at every real number.

**2.23**. Discontinuous at 1; removable

**2.24**. $[-3, +\infty)$

**2.25**. 0

**2.26**. $f(0) = 1 > 0$, $f(1) = -2 < 0$; $f(x)$ is continuous over $[0, 1]$. It must have a zero on this interval.

**2.27**. Let $\varepsilon > 0$; choose $\delta = \frac{\varepsilon}{3}$; assume $0 < |x - 2| < \delta$. Thus, $|(3x - 2) - 4| = |3x - 6| = |3| \cdot |x - 2| < 3 \cdot \delta = 3 \cdot (\varepsilon/3) = \varepsilon$. Therefore, $\lim_{x \to 2} 3x - 2 = 4$.

**2.28**. Choose $\delta = \min\left\{9 - (3 - \varepsilon)^2, (3 + \varepsilon)^2 - 9\right\}$.

**2.29**. $\left|x^2 - 1\right| = |x - 1| \cdot |x + 1| < \varepsilon/3 \cdot 3 = \varepsilon$

**2.30**. $\delta = \varepsilon^2$

## Section Exercises

**1**. a. 2.2100000; b. 2.0201000; c. 2.0020010; d. 2.0002000; e. (1.1000000, 2.2100000); f. (1.0100000, 2.0201000); g. (1.0010000, 2.0020010); h. (1.0001000, 2.0002000); i. 2.1000000; j. 2.0100000; k. 2.0010000; l. 2.0001000

**3**. $y = 2x$

**5**. 3

**7**. a. 2.0248457; b. 2.0024984; c. 2.0002500; d. 2.0000250; e. (4.1000000,2.0248457); f. (4.0100000,2.0024984); g. (4.0010000,2.0002500); h. (4.00010000,2.0000250); i. 0.24845673; j. 0.24984395; k. 0.24998438; l. 0.24999844

**9**. $y = \frac{x}{4} + 1$

**11**. $\pi$

**13**. a. −0.95238095; b. −0.99009901; c. −0.99502488; d. −0.99900100; e. (−1;.0500000,−0;.95238095); f. (−1;.0100000,−0;.9909901); g. (−1;.0050000,−0;.99502488); h. (1.0010000,−0;.99900100); i. −0.95238095; j. −0.99009901; k. −0.99502488; l. −0.99900100

**15**. $y = -x - 2$

**17**. −49 m/sec (velocity of the ball is 49 m/sec downward)
**19**. 5.2 m/sec
**21**. −9.8 m/sec
**23**. 6 m/sec

**25**. Under, 1 unit$^2$; over: 4 unit$^2$. The exact area of the two triangles is $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = 2.5\text{ units}^2$.

**27**. Under, 0.96 unit$^2$; over, 1.92 unit$^2$. The exact area of the semicircle with radius 1 is $\frac{\pi(1)^2}{2} = \frac{\pi}{2}$ unit$^2$.

**29**. Approximately 1.3333333 unit$^2$

**31**. $\lim\limits_{x \to 1} f(x)$ does not exist because $\lim\limits_{x \to 1^-} f(x) = -2 \neq \lim\limits_{x \to 1^+} f(x) = 2.$

**33**. $\lim\limits_{x \to 0}(1 + x)^{1/x} = 2.7183$

**35**. a. 1.98669331; b. 1.99986667; c. 1.99999867; d. 1.99999999; e. 1.98669331; f. 1.99986667; g. 1.99999867; h. 1.99999999; $\lim\limits_{x \to 0} \frac{\sin 2x}{x} = 2$

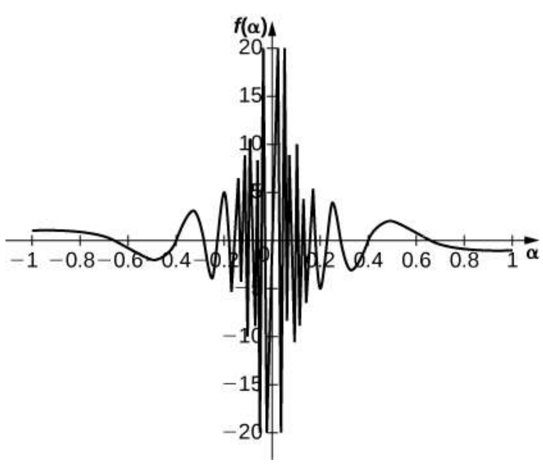
**37**. $\lim\limits_{x \to 0} \frac{\sin ax}{x} = a$

**39**. a. −0.80000000; b. −0.98000000; c. −0.99800000; d. −0.99980000; e. −1.2000000; f. −1.0200000; g. −1.0020000; h. −1.0002000; $\lim\limits_{x \to 1}(1 - 2x) = -1$

**41**. a. −37.931934; b. −3377.9264; c. −333,777.93; d. −33,337,778; e. −29.032258; f. −3289.0365; g. −332,889.04; h. −33,328,889 $\lim\limits_{x \to 0} \frac{z - 1}{z^2(z + 3)} = -\infty$

**43**. a. 0.13495277; b. 0.12594300; c. 0.12509381; d. 0.12500938; e. 0.11614402; f. 0.12406794; g. 0.12490631; h. 0.12499063; $\therefore \lim\limits_{x \to 2} \frac{1 - \frac{2}{x}}{x^2 - 4} = 0.1250 = \frac{1}{8}$

**45**. a. −10.00000; b. −100.00000; c. −1000.0000; d. −10,000.000; Guess: $\lim\limits_{\alpha \to 0^+} \frac{1}{\alpha}\cos\left(\frac{\pi}{\alpha}\right) = \infty,$ actual: DNE

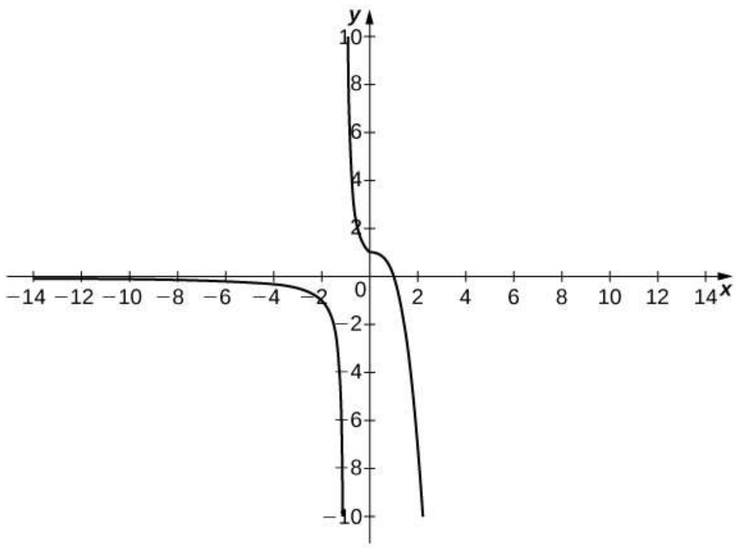



**47**. False; $\lim\limits_{x \to -2^+} f(x) = +\infty$

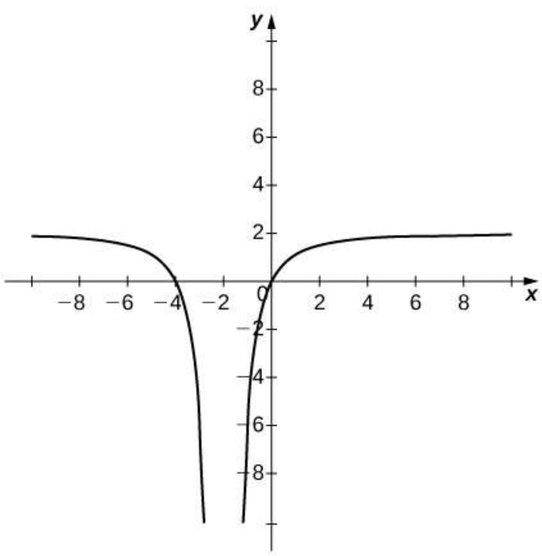
**49**. False; $\lim\limits_{x \to 6} f(x)$ DNE since $\lim\limits_{x \to 6^-} f(x) = 2$ and $\lim\limits_{x \to 6^+} f(x) = 5.$

**51**. 2
**53**. 1
**55**. 1
**57**. DNE
**59**. 0
**61**. DNE
**63**. 2
**65**. 3

**67**. DNE
**69**. 0
**71**. −2
**73**. DNE
**75**. 0
**77**. Answers may vary.




**79**. Answers may vary.




**81**. a. $\rho_2$ b. $\rho_1$ c. DNE unless $\rho_1 = \rho_2$. As you approach $x_{\text{SF}}$ from the right, you are in the high-density area of the shock. When you approach from the left, you have not experienced the "shock" yet and are at a lower density.
**83**. Use constant multiple law and difference law: $\lim\limits_{x \to 0}\left(4x^2 - 2x + 3\right) = 4\lim\limits_{x \to 0} x^2 - 2\lim\limits_{x \to 0} x + \lim\limits_{x \to 0} 3 = 3$
**85**. Use root law: $\lim\limits_{x \to -2}\sqrt{x^2 - 6x + 3} = \sqrt{\lim\limits_{x \to -2}\left(x^2 - 6x + 3\right)} = \sqrt{19}$
**87**. 49
**89**. 1
**91**. $-\frac{5}{7}$
**93**. $\lim\limits_{x \to 4}\frac{x^2 - 16}{x - 4} = \frac{16 - 16}{4 - 4} = \frac{0}{0}$; then, $\lim\limits_{x \to 4}\frac{x^2 - 16}{x - 4} = \lim\limits_{x \to 4}\frac{(x + 4)(x - 4)}{x - 4} = 8$

**95**. $\lim\limits_{x \to 6} \frac{3x - 18}{2x - 12} = \frac{18 - 18}{12 - 12} = \frac{0}{0}$; then, $\lim\limits_{x \to 6} \frac{3x - 18}{2x - 12} = \lim\limits_{x \to 6} \frac{3(x - 6)}{2(x - 6)} = \frac{3}{2}$

**97**. $\lim\limits_{x \to 9} \frac{t - 9}{\sqrt{t} - 3} = \frac{9 - 9}{3 - 3} = \frac{0}{0}$; then, $\lim\limits_{t \to 9} \frac{t - 9}{\sqrt{t} - 3} = \lim\limits_{t \to 9} \frac{t - 9}{\sqrt{t} - 3} \frac{\sqrt{t} + 3}{\sqrt{t} + 3} = \lim\limits_{t \to 9} (\sqrt{t} + 3) = 6$

**99**. $\lim\limits_{\theta \to \pi} \frac{\sin\theta}{\tan\theta} = \frac{\sin\pi}{\tan\pi} = \frac{0}{0}$; then, $\lim\limits_{\theta \to \pi} \frac{\sin\theta}{\tan\theta} = \lim\limits_{\theta \to \pi} \frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} = \lim\limits_{\theta \to \pi} \cos\theta = -1$

**101**. $\lim\limits_{x \to 1/2} \frac{2x^2 + 3x - 2}{2x - 1} = \frac{\frac{1}{2} + \frac{3}{2} - 2}{1 - 1} = \frac{0}{0}$; then, $\lim\limits_{x \to 1/2} \frac{2x^2 + 3x - 2}{2x - 1} = \lim\limits_{x \to 1/2} \frac{(2x - 1)(x + 2)}{2x - 1} = \frac{5}{2}$

**103**. $-\infty$

**105**. $-\infty$

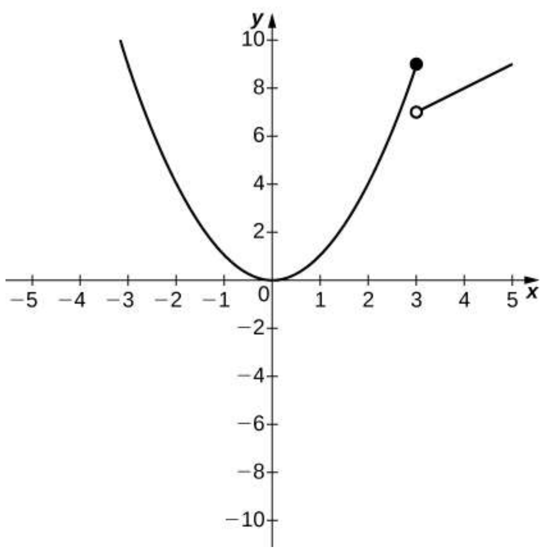
**107**. $\lim\limits_{x \to 6} 2f(x)g(x) = 2 \lim\limits_{x \to 6} f(x) \lim\limits_{x \to 6} g(x) = 72$

**109**. $\lim\limits_{x \to 6} \left(f(x) + \frac{1}{3}g(x)\right) = \lim\limits_{x \to 6} f(x) + \frac{1}{3} \lim\limits_{x \to 6} g(x) = 7$

**111**. $\lim\limits_{x \to 6} \sqrt{g(x) - f(x)} = \sqrt{\lim\limits_{x \to 6} g(x) - \lim\limits_{x \to 6} f(x)} = \sqrt{5}$

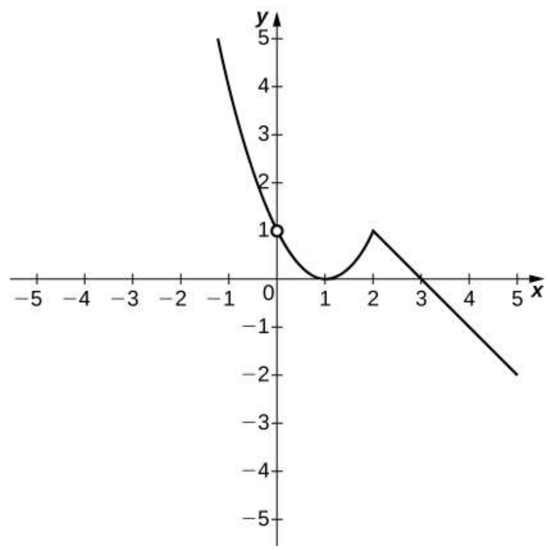
**113**. $\lim\limits_{x \to 6} [(x + 1)f(x)] = \left(\lim\limits_{x \to 6} (x + 1)\right)\left(\lim\limits_{x \to 6} f(x)\right) = 28$

**115**.




a. 9; b. 7

**117**.




a. 1; b. 1

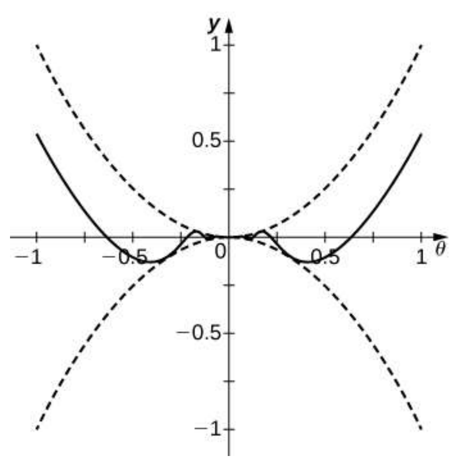
**119**. $\lim_{x \to -3^-}(f(x) - 3g(x)) = \lim_{x \to -3^-} f(x) - 3\lim_{x \to -3^-} g(x) = 0 + 6 = 6$

**121**. $\lim_{x \to -5} \frac{2 + g(x)}{f(x)} = \frac{2 + \left(\lim_{x \to -5} g(x)\right)}{\lim_{x \to -5} f(x)} = \frac{2 + 0}{2} = 1$

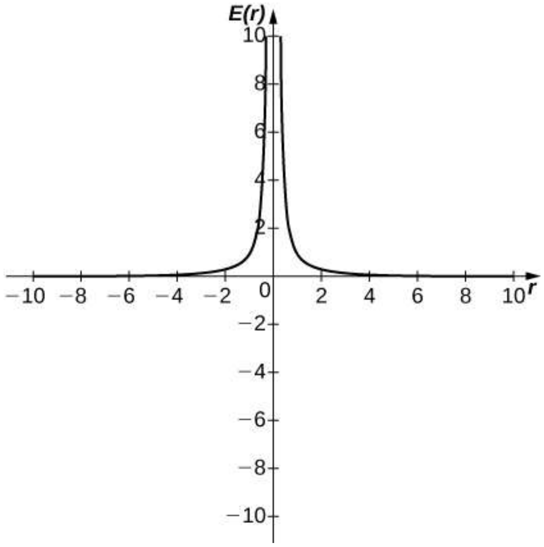
**123**. $\lim_{x \to 1} \sqrt[3]{f(x) - g(x)} = \sqrt[3]{\lim_{x \to 1} f(x) - \lim_{x \to 1} g(x)} = \sqrt[3]{2 + 5} = \sqrt[3]{7}$

**125**. $\lim_{x \to -9}(xf(x) + 2g(x)) = \left(\lim_{x \to -9} x\right)\left(\lim_{x \to -9} f(x)\right) + 2\lim_{x \to -9}(g(x)) = (-9)(6) + 2(4) = -46$

**127**. The limit is zero.




**129**. a.




b. ∞. The magnitude of the electric field as you approach the particle $q$ becomes infinite. It does not make physical sense to evaluate negative distance.

**131**. The function is defined for all $x$ in the interval $(0, \infty)$.

**133**. Removable discontinuity at $x = 0;$ infinite discontinuity at $x = 1$

**135**. Infinite discontinuity at $x = \ln 2$

**137**. Infinite discontinuities at $x = \dfrac{(2k+1)\pi}{4},$ for $k = 0, \pm 1, \pm 2, \pm 3, \ldots$

**139**. No. It is a removable discontinuity.

**141**. Yes. It is continuous.

**143**. Yes. It is continuous.

**145**. $k = -5$

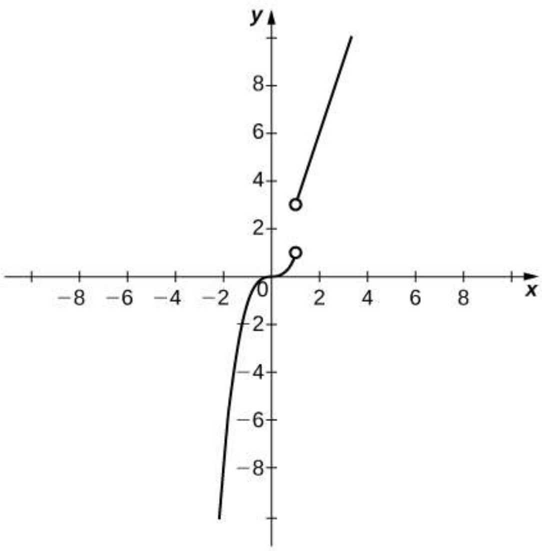
**147**. $k = -1$

**149**. $k = \dfrac{16}{3}$

**151**. Since both $s$ and $y = t$ are continuous everywhere, then $h(t) = s(t) - t$ is continuous everywhere and, in particular, it is continuous over the closed interval $[2, 5]$. Also, $h(2) = 3 > 0$ and $h(5) = -3 < 0$. Therefore, by the IVT, there is a value $x = c$ such that $h(c) = 0$.

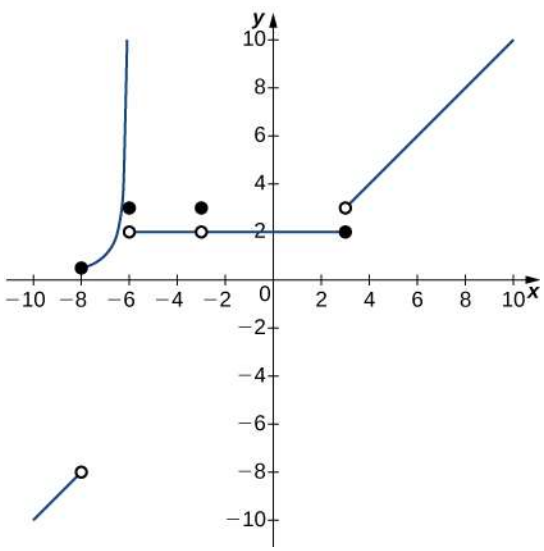
**153**. The function $f(x) = 2^x - x^3$ is continuous over the interval $[1.25, 1.375]$ and has opposite signs at the endpoints.

**155**. a.

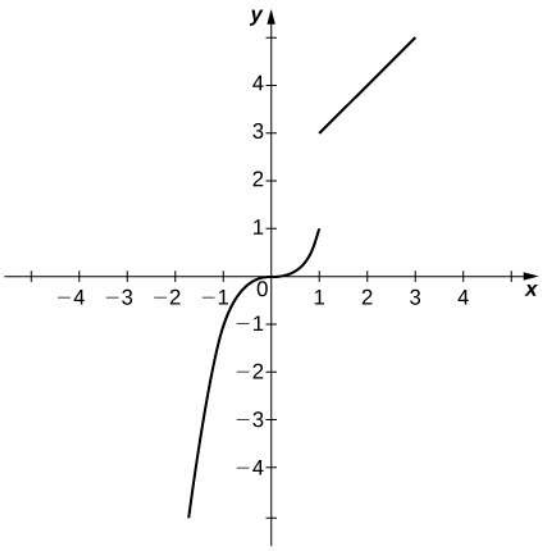



b. It is not possible to redefine $f(1)$ since the discontinuity is a jump discontinuity.

**157**. Answers may vary; see the following example:




**159**. Answers may vary; see the following example:




**161**. False. It is continuous over $(-\infty, 0) \cup (0, \infty)$.

**163**. False. Consider $f(x) = \begin{cases} x \text{ if } x \neq 0 \\ 4 \text{ if } x = 0 \end{cases}$.

**165**. False. IVT only says that there is at least one solution; it does not guarantee that there is exactly one. Consider $f(x) = \cos(x)$ on $[-\pi, 2\pi]$.

**167**. False. The IVT does *not* work in reverse! Consider $(x-1)^2$ over the interval $[-2, 2]$.

**169**. $R = 0.0001519 \text{ m}$

**171**. $D = 345{,}826 \text{ km}$

**173**. For all values of $a$, $f(a)$ is defined, $\lim_{\theta \to a} f(\theta)$ exists, and $\lim_{\theta \to a} f(\theta) = f(a)$. Therefore, $f(\theta)$ is continuous everywhere.

**175**. Nowhere

**177**. For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |t - b| < \delta$, then $|g(t) - M| < \varepsilon$

**179**. For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |x - a| < \delta$, then $|\varphi(x) - A| < \varepsilon$

**181**. $\delta \leq 0.25$

**183**. $\delta \leq 2$

**185**. $\delta \leq 1$

**187**. $\delta < 0.3900$

**189**. Let $\delta = \varepsilon$. If $0 < |x - 3| < \varepsilon$, then $|x + 3 - 6| = |x - 3| < \varepsilon$.

**191**. Let $\delta = \sqrt[4]{\varepsilon}$. If $0 < |x| < \sqrt[4]{\varepsilon}$, then $\left|x^4\right| = x^4 < \varepsilon$.

**193**. Let $\delta = \varepsilon^2$. If $5 - \varepsilon^2 < x < 5$, then $\left|\sqrt{5 - x}\right| = \sqrt{5 - x} < \varepsilon$.

**195**. Let $\delta = \varepsilon/5$. If $1 - \varepsilon/5 < x < 1$, then $|f(x) - 3| = 5x - 5 < \varepsilon$.

**197**. Let $\delta = \sqrt{\frac{3}{M}}$. If $0 < |x + 1| < \sqrt{\frac{3}{M}}$, then $f(x) = \frac{3}{(x+1)^2} > M$.

**199**. 0.328 cm, $\varepsilon = 8$, $\delta = 0.33$, $a = 12$, $L = 144$

**201**. Answers may vary.

**203**. 0

**205**. $f(x) - g(x) = f(x) + (-1)g(x)$

**207**. Answers may vary.

## Review Exercises

**209**. False

**211**. False. A removable discontinuity is possible.

**213**. 5
**215**. 8/7
**217**. DNE
**219**. 2/3
**221**. −4;
**223**. Since $-1 \leq \cos(2\pi x) \leq 1$, then $-x^2 \leq x^2 \cos(2\pi x) \leq x^2$. Since $\lim_{x \to 0} x^2 = 0 = \lim_{x \to 0} -x^2$, it follows that $\lim_{x \to 0} x^2 \cos(2\pi x) = 0$.
**225**. $[2, \infty]$
**227**. $c = -1$
**229**. $\delta = \sqrt[3]{\varepsilon}$
**231**. 0 m/sec

# Chapter 3

## Checkpoint

**3.1**. $\frac{1}{4}$
**3.2**. 6
**3.3**. $f'(1) = 5$
**3.4**. $-32$ ft/s
**3.5**. $P'(3.25) = 20 > 0$; raise prices
**3.6**. $f'(x) = 2x$
**3.7**. $(0, +\infty)$
**3.8**. $a = 6$ and $b = -9$
**3.9**. $f''(x) = 2$
**3.10**. $a(t) = 6t$
**3.11**. 0
**3.12**. $4x^3$
**3.13**. $f'(x) = 7x^6$
**3.14**. $f'(x) = 6x^2 - 12x$.
**3.15**. $y = 12x - 23$
**3.16**. $j'(x) = 10x^4\left(4x^2 + x\right) + (8x + 1)\left(2x^5\right) = 56x^6 + 12x^5$.
**3.17**. $k'(x) = -\frac{13}{(4x - 3)^2}$.
**3.18**. $g'(x) = -7x^{-8}$.
**3.19**. $3f'(x) - 2g'(x)$.
**3.20**. $\frac{5}{8}$
**3.21**. $-4.4$
**3.22**. left to right
**3.23**. 3,300
**3.24**. $2
**3.25**. $f'(x) = \cos^2 x - \sin^2 x$
**3.26**. $\frac{\cos x + x \sin x}{\cos^2 x}$
**3.27**. $t = \frac{\pi}{3}$, $t = \frac{2\pi}{3}$
**3.28**. $f'(x) = -\csc^2 x$

**3.29**. $f'(x) = 2\sec^2 x + 3\csc^2 x$

**3.30**. $\frac{4}{3}$

**3.31**. $\cos x$

**3.32**. $-\cos x$

**3.33**. $v\left(\frac{5\pi}{6}\right) = -\sqrt{3} < 0$ and $a\left(\frac{5\pi}{6}\right) = -1 < 0.$ The block is speeding up.

**3.34**. $h'(x) = 4\left(2x^3 + 2x - 1\right)^3\left(6x^2 + 2\right) = 8\left(3x^2 + 1\right)\left(2x^3 + 2x - 1\right)^3$

**3.35**. $y = -48x - 88$

**3.36**. $h'(x) = 7\cos(7x + 2)$

**3.37**. $h'(x) = \frac{3 - 4x}{(2x + 3)^4}$

**3.38**. $h'(x) = 18x^2 \sin^5\left(x^3\right)\cos\left(x^3\right)$

**3.39**. $a(t) = -16\sin(4t)$

**3.40**. $28$

**3.41**. $\frac{dy}{dx} = -3x^2 \sin\left(x^3\right)$

**3.42**. $g'(x) = -\frac{1}{(x + 2)^2}$

**3.43**. $g(x) = \frac{1}{5}x^{-4/5}$

**3.44**. $s'(t) = (2t + 1)^{-1/2}$

**3.45**. $g'(x) = \frac{1}{1 + x^2}$

**3.46**. $h'(x) = \frac{-3}{\sqrt{6x - 9x^2}}$

**3.47**. $y = x$

**3.48**. $\frac{dy}{dx} = \frac{5 - 20x^4}{\sec^2 y - 2y}$

**3.49**. $y = \frac{5}{3}x - \frac{16}{3}$

**3.50**. $h'(x) = e^{2x} + 2xe^{2x}$

**3.51**. 996

**3.52**. $f'(x) = \frac{15}{3x + 2}$

**3.53**. $9\ln(3)$

**3.54**. $\frac{dy}{dx} = x^x(1 + \ln x)$

**3.55**. $y' = \pi(\tan x)^{\pi - 1}\sec^2 x$

## Section Exercises

**1**. $4$

**3**. $8.5$

**5**. $-\frac{3}{4}$

**7**. $0.2$

**9**. $0.25$

**11**. a. $-4$ b. $y = 3 - 4x$

**13**. a. $3$ b. $y = 3x - 1$

**15**. a. $\frac{-7}{9}$ b. $y = \frac{-7}{9}x + \frac{14}{3}$

**17**. a. $12$ b. $y = 12x + 14$

**19**. a. $-2$ b. $y = -2x - 10$

**21**. $5$

**23**. $13$

**25**. $\frac{1}{4}$

**27**. $-\frac{1}{4}$

**29**. $-3$

**31**. a. (i) $5.100000,$ (ii) $5.010000,$ (iii) $5.001000,$ (iv) $5.000100,$ (v) $5.000010,$ (vi) $5.000001,$ (vii) $4.900000,$ (viii) $4.990000,$ (ix) $4.999000,$ (x) $4.999900,$ (xi) $4.999990,$ (x) $4.999999$ b. $m_{\tan} = 5$ c. $y = 5x + 3$

**33**. a. (i) $4.8771,$ (ii) $4.9875$ (iii) $4.9988,$ (iv) $4.9999,$ (v) $4.9999,$ (vi) $4.9999$ b. $m_{\tan} = 5$ c. $y = 5x + 10$

**35**. a. $\frac{1}{3}$; b. (i) $0.\bar{3}$ m/s, (ii) $0.\bar{3}$ m/s, (iii) $0.\bar{3}$ m/s, (iv) $0.\bar{3}$ m/s; c. $0.\bar{3} = \frac{1}{3}$ m/s

**37**. a. $2\left(h^2 + 6h + 12\right)$; b. (i) $25.22$ m/s, (ii) $24.12$ m/s, (iii) $24.01$ m/s, (iv) $24$ m/s; c. $24$ m/s

**39**. a. $1.25$; b. $0.5$

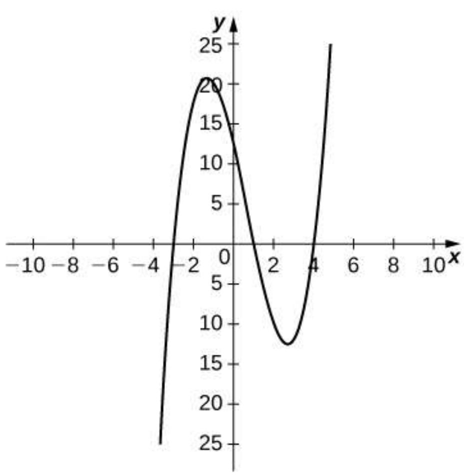
**41**. $\lim_{x \to 0^-} \frac{x^{1/3} - 0}{x - 0} = \lim_{x \to 0^-} \frac{1}{x^{2/3}} = \infty$

**43**. $\lim_{x \to 1^-} \frac{1 - 1}{x - 1} = 0 \neq 1 = \lim_{x \to 1^+} \frac{x - 1}{x - 1}$

**45**. a. (i) $61.7244$ ft/s, (ii) $61.0725$ ft/s (iii) $61.0072$ ft/s (iv) $61.0007$ ft/s b. At $4$ seconds the race car is traveling at a rate/velocity of $61$ ft/s.

**47**. a. The vehicle represented by $f(t),$ because it has traveled $2$ feet, whereas $g(t)$ has traveled $1$ foot. b. The velocity of $f(t)$ is constant at $1$ ft/s, while the velocity of $g(t)$ is approximately $2$ ft/s. c. The vehicle represented by $g(t),$ with a velocity of approximately $4$ ft/s. d. Both have traveled $4$ feet in $4$ seconds.

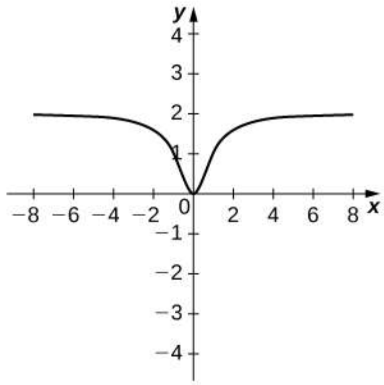
**49**. a.




b. $a \approx -1.361, 2.694$

**51**. a. $N(x) = \frac{x}{30}$ b. $\sim 3.3$ gallons. When the vehicle travels $100$ miles, it has used $3.3$ gallons of gas. c. $\frac{1}{30}$. The rate of gas consumption in gallons per mile that the vehicle is achieving after having traveled $100$ miles.

**53**. a.




b. $-0.028, -0.16, 0.16, 0.028$

**55**. $-3$

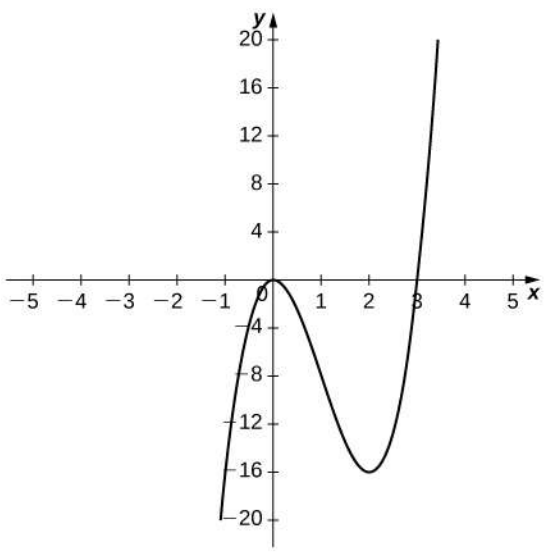
**57**. $8x$

**59**. $\frac{1}{\sqrt{2x}}$

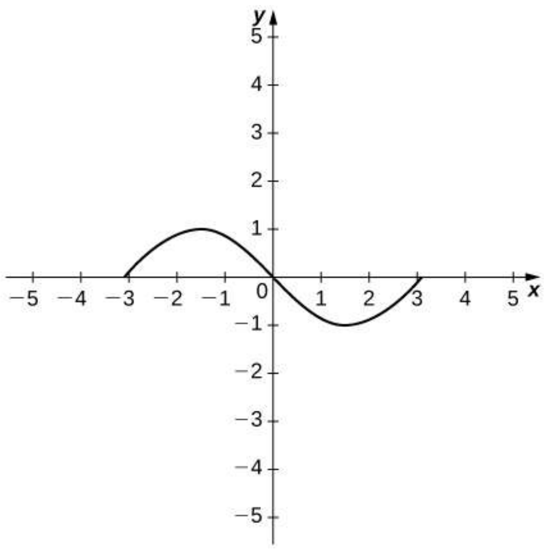
**61**. $\frac{-9}{x^2}$

**63**. $\frac{-1}{2x^{3/2}}$

**65**.




**67**.

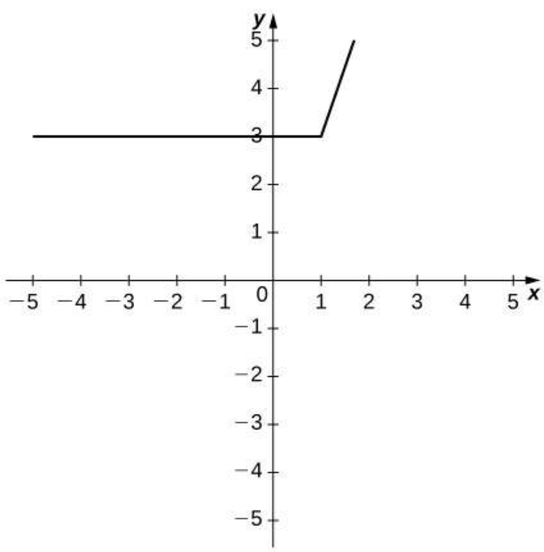



**69**. $f(x) = 3x^2 + 2,\ a = 2$

**71**. $f(x) = x^4,\ a = 2$

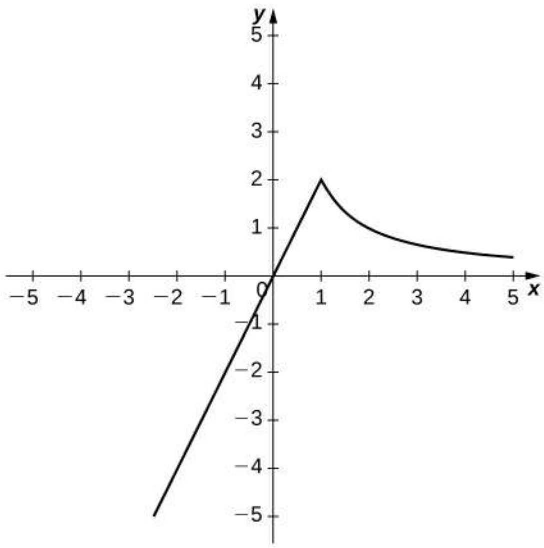
**73**. $f(x) = e^x,\ a = 0$

**75**. a.




b. $\lim_{h \to 1^-} \frac{3-3}{h} \neq \lim_{h \to 1^+} \frac{3h}{h}$

**77**. a.



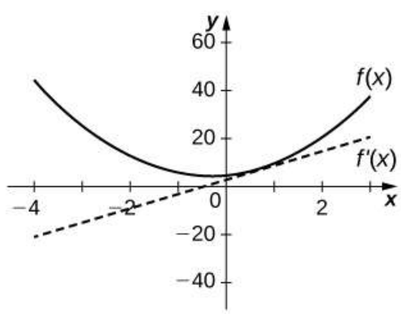

b. $\lim_{h \to 1^-} \frac{2h}{h} \neq \lim_{h \to 1^+} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$.

**79**. a. $x = 1$, b. $x = 2$

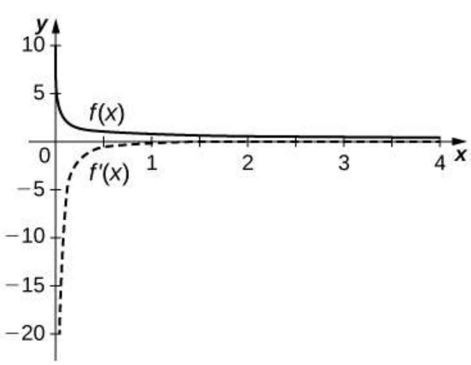
**81**. $0$

**83**. $\frac{2}{x^3}$

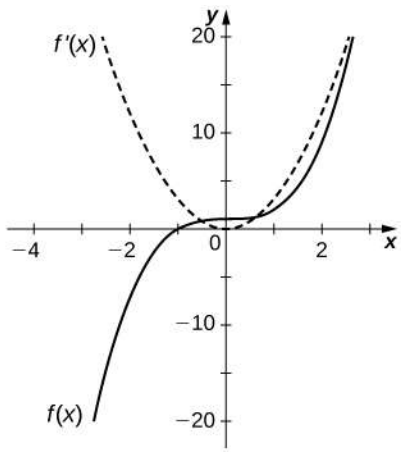
**85**. $f'(x) = 6x + 2$




**87**. $f'(x) = -\frac{1}{(2x)^{3/2}}$




**89**. $f'(x) = 3x^2$




**91**. a. Average rate at which customers spent on concessions in thousands per customer. b. Rate (in thousands per customer) at which $x$ customers spent money on concessions in thousands per customer.

**93**. a. Average grade received on the test with an average study time between two values. b. Rate (in percentage points per hour) at which the grade on the test increased or decreased for a given average study time of $x$ hours.

**95**. a. Average change of atmospheric pressure between two different altitudes. b. Rate (torr per foot) at which atmospheric pressure is increasing or decreasing at $x$ feet.

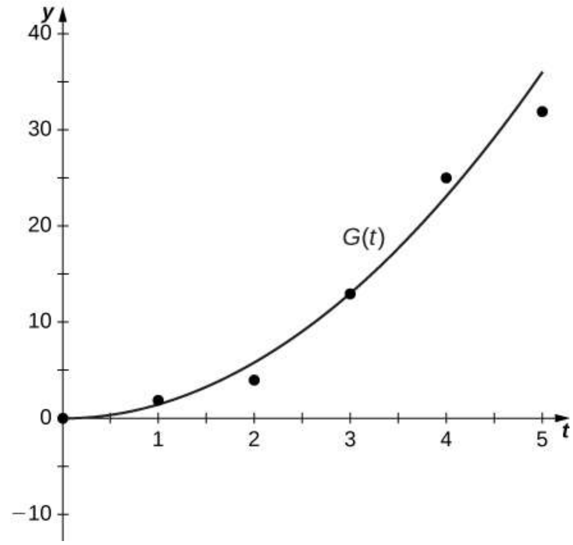
**97**. a. The rate (in degrees per foot) at which temperature is increasing or decreasing for a given height $x.$ b. The rate of change of temperature as altitude changes at $1000$ feet is $-0.1$ degrees per foot.

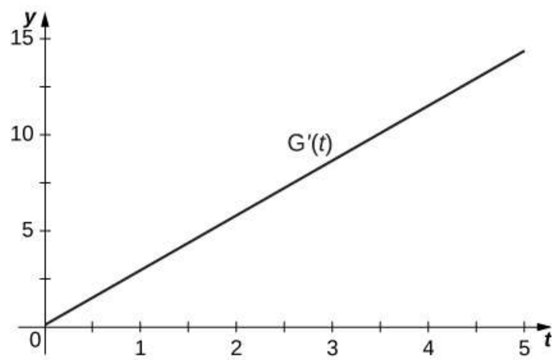
**99**. a. The rate at which the number of people who have come down with the flu is changing $t$ weeks after the initial outbreak. b. The rate is increasing sharply up to the third week, at which point it slows down and then becomes constant.

**101**.

| Time (seconds) | $h'(t)$ (m/s) |
|---|---|
| 0 | 2 |
| 1 | 2 |
| 2 | 5.5 |
| 3 | 10.5 |
| 4 | 9.5 |
| 5 | 7 |

**103**. $G'(t) = 2.858t + 0.0857$




**105**. $H''(t) = 0,\ G''(t) = 2.858$ and $f''(t) = 1.222t + 5.912$ represent the acceleration of the rocket, with units of meters per second squared $\left(\mathrm{m/s}^2\right)$.

**107**. $f'(x) = 15x^2 - 1$

**109**. $f'(x) = 32x^3 + 18x$

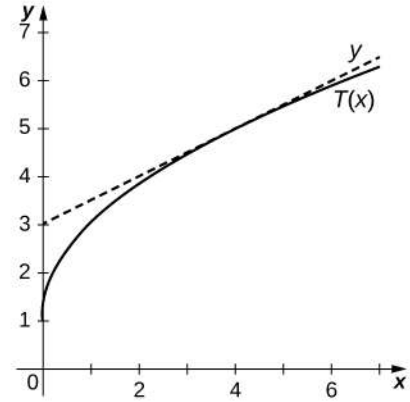
**111**. $f'(x) = 270x^4 + \dfrac{39}{(x+1)^2}$

**113**. $f'(x) = \dfrac{-5}{x^2}$

**115**. $f'(x) = \dfrac{4x^4 + 2x^2 - 2x}{x^4}$

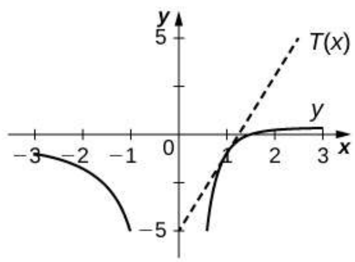
**117**. $f'(x) = \dfrac{-x^2 - 18x + 64}{\left(x^2 - 7x + 1\right)^2}$

**119**.




$T(x) = \frac{1}{2}x + 3$

**121**.




$T(x) = 4x - 5$

**123**. $h'(x) = 3x^2 f(x) + x^3 f'(x)$

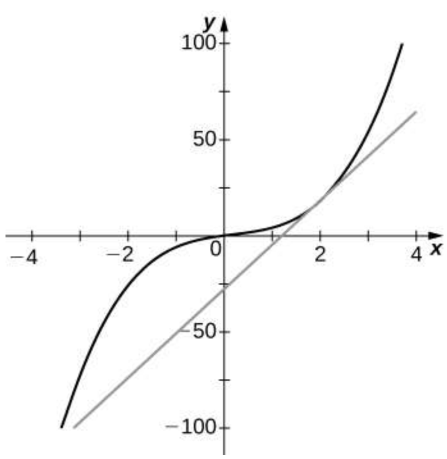
**125**. $h'(x) = \dfrac{3f'(x)(g(x) + 2) - 3f(x)g'(x)}{(g(x) + 2)^2}$

**127**. $\frac{16}{9}$

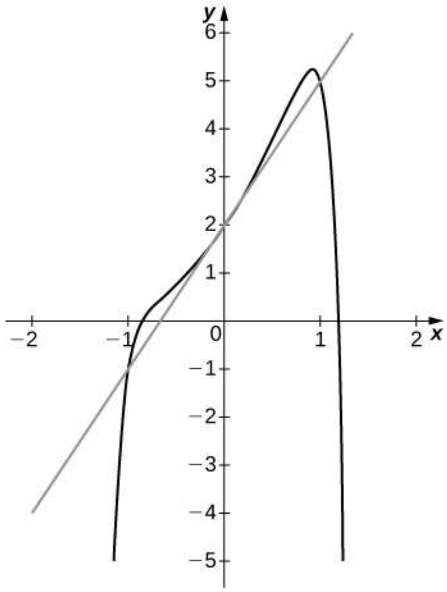
**129**. Undefined

**131**. a. $2,$ b. does not exist, c. $2.5$

**133**. a. 23, b. $y = 23x - 28$




**135**. a. 3, b. $y = 3x + 2$




**137**. $y = -7x - 3$

**139**. $y = -5x + 7$

**141**. $y = -\frac{3}{2}x + \frac{15}{2}$

**143**. $y = -3x^2 + 9x - 1$

**145**. $\frac{12}{121}$ or 0.0992 ft/s

**147**. a. $\frac{-2t^4 - 2t^3 + 200t + 50}{\left(t^3 + 50\right)^2}$ b. $-0.02395$ mg/L-hr, −0.01344 mg/L-hr, −0.003566 mg/L-hr, −0.001579 mg/L-hr c. The rate at which the concentration of drug in the bloodstream decreases is slowing to 0 as time increases.

**149**. a. $F'(d) = \frac{-2Gm_1 m_2}{d^3}$ b. $-1.33 \times 10^{-7}$ N/m

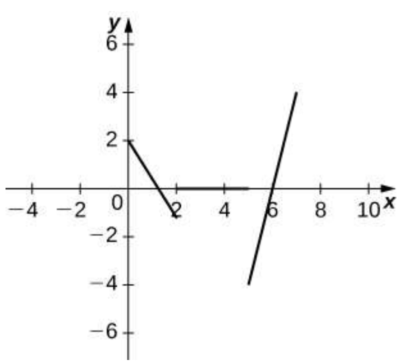
**151**. a. $v(t) = 6t^2 - 30t + 36,\ a(t) = 12t - 30;$ b. speeds up $(2, 2.5) \cup (3, \infty),$ slows down $(0, 2) \cup (2.5, 3)$

**153**. a. $464\ \text{ft/s}^2$ b. $-32\ \text{ft/s}^2$

**155**. a. 5 ft/s b. 9 ft/s

**157**. a. 84 ft/s, −84 ft/s b. 84 ft/s c. $\frac{25}{8}$ s d. $-32\ \text{ft/s}^2$ in both cases e. $\frac{1}{8}\left(25 + \sqrt{965}\right)$ s f. $-4\sqrt{965}$ ft/s

**159**. a. Velocity is positive on $(0, 1.5) \cup (6, 7),$ negative on $(1.5, 2) \cup (5, 6),$ and zero on $(2, 5).$ b.




c. Acceleration is positive on $(5, 7),$ negative on $(0, 2),$ and zero on $(2, 5).$ d. The object is speeding up on $(6, 7) \cup (1.5, 2)$ and slowing down on $(0, 1.5) \cup (5, 6).$

**161**. a. $R(x) = 10x - 0.001x^2$ b. $R'(x) = 10 - 0.002x$ c. \$6 per item, \$0 per item

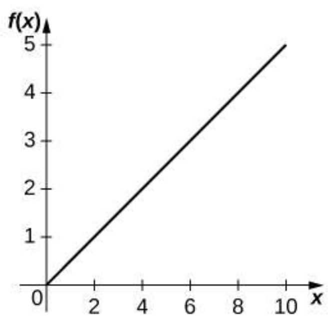
**163**. a. $C'(x) = 65$ b. $R(x) = 143x - 0.03x^2$, $R'(x) = 143 - 0.06x$ c. $83, -97.$ At a production level of 1000 cordless drills, revenue is increasing at a rate of \$83 per drill; at a production level of 4000 cordless drills, revenue is decreasing at a rate of \$97 per drill. d. $P(x) = -0.03x^2 + 78x - 75000$, $P'(x) = -0.06x + 78$ e. $18, -162.$ At a production level of 1000 cordless drills, profit is increasing at a rate of \$18 per drill; at a production level of 4000 cordless drills, profit is decreasing at a rate of \$162 per drill.

**165**. a. $N'(t) = 3000\left(\dfrac{-4t^2 + 400}{\left(t^2 + 100\right)^2}\right)$ b. $120, 0, -14.4, -9.6$ c. The bacteria population increases from time 0 to 10 hours; afterwards, the bacteria population decreases. d. $0, -6, 0.384, 0.432.$ The rate at which the bacteria is increasing is decreasing during the first 10 hours. Afterwards, the bacteria population is decreasing at a decreasing rate.

**167**. a. $P(t) = 0.03983 + 0.4280$ b. $P'(t) = 0.03983.$ The population is increasing. c. $P''(t) = 0.$ The rate at which the population is increasing is constant.

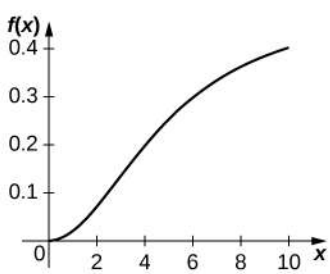
**169**. a. $p(t) = -0.6071x^2 + 0.4357x - 0.3571$ b. $p'(t) = -1.214x + 0.4357.$ This is the velocity of the sensor. c. $p''(t) = -1.214.$ This is the acceleration of the sensor; it is a constant acceleration downward.

**171**. a.




b. $f'(x) = a.$ The more increase in prey, the more growth for predators. c. $f''(x) = 0.$ As the amount of prey increases, the rate at which the predator population growth increases is constant. d. This equation assumes that if there is more prey, the predator is able to increase consumption linearly. This assumption is unphysical because we would expect there to be some saturation point at which there is too much prey for the predator to consume adequately.

**173**. a.




b. $f'(x) = \dfrac{2axn^2}{\left(n^2 + x^2\right)^2}.$ When the amount of prey increases, the predator growth increases. c. $f''(x) = \dfrac{2an^2\left(n^2 - 3x^2\right)}{\left(n^2 + x^2\right)^3}.$ When the amount of prey is extremely small, the rate at which predator growth is increasing is increasing, but when the amount of prey reaches above a certain threshold, the rate at which predator growth is increasing begins to decrease. d. At lower levels of prey, the prey is more easily able to avoid detection by the predator, so fewer prey individuals are consumed, resulting in less predator growth.

**175**. $\dfrac{dy}{dx} = 2x - \sec x \tan x$

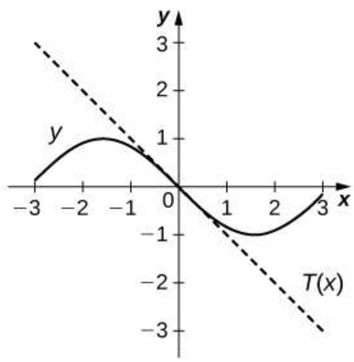
**177**. $\dfrac{dy}{dx} = 2x\cot x - x^2 \csc^2 x$

**179**. $\frac{dy}{dx} = \frac{x\sec x\tan x - \sec x}{x^2}$

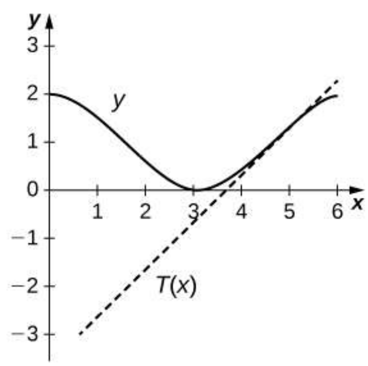
**181**. $\frac{dy}{dx} = (1 - \sin x)(1 - \sin x) - \cos x(x + \cos x)$

**183**. $\frac{dy}{dx} = \frac{2\csc^2 x}{(1 + \cot x)^2}$

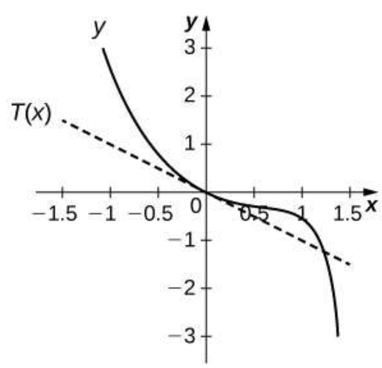
**185**. $y = -x$




**187**. $y = x + \frac{2 - 3\pi}{2}$




**189**. $y = -x$




**191**. $3\cos x - x\sin x$

**193**. $\frac{1}{2}\sin x$

**195**. $2\csc x\left(\csc^2 x + \cot^2 x\right)$

**197**. $\frac{(2n+1)\pi}{4}$, where $n$ is an integer

**199**. $\left(\frac{\pi}{4},\ 1\right), \left(\frac{3\pi}{4},\ -1\right)$

**201**. $a = 0,\ b = 3$

**203**. $y' = 5\cos(x)$, increasing on $\left(0,\ \frac{\pi}{2}\right), \left(\frac{3\pi}{2},\ \frac{5\pi}{2}\right)$, and $\left(\frac{7\pi}{2},\ 12\right)$

**209**. $3\sin x$

**211**. $5\cos x$

**213**. $720x^7 - 5\tan(x)\sec^3(x) - \tan^3(x)\sec(x)$

**215**. $18u^2 \cdot 7 = 18(7x-4)^2 \cdot 7$

**217**. $-\sin u \cdot \frac{-1}{8} = -\sin\left(\frac{-x}{8}\right) \cdot \frac{-1}{8}$

**219**. $\frac{8x-24}{2\sqrt{4u+3}} = \frac{4x-12}{\sqrt{4x^2-24x+3}}$

**221**. a. $u = 3x^2 + 1$; b. $18x\left(3x^2+1\right)^2$

**223**. a. $f(u) = u^7,\ u = \frac{x}{7} + \frac{7}{x}$; b. $7\left(\frac{x}{7} + \frac{7}{x}\right)^6 \cdot \left(\frac{1}{7} - \frac{7}{x^2}\right)$

**225**. a. $f(u) = \csc u,\ u = \pi x + 1$; b. $-\pi\csc(\pi x+1)\cdot\cot(\pi x+1)$

**227**. a. $f(u) = -6u^{-3},\ u = \sin x$, b. $18\sin^{-4} x \cdot \cos x$

**229**. $\frac{4}{(5-2x)^3}$

**231**. $6\left(2x^3 - x^2 + 6x + 1\right)^2\left(3x^2 - x + 3\right)$

**233**. $-3(\tan x + \sin x)^{-4} \cdot \left(\sec^2 x + \cos x\right)$

**235**. $-7\cos(\cos 7x)\cdot\sin 7x$

**237**. $-12\cot^2(4x+1)\cdot\csc^2(4x+1)$

**239**. $10\frac{3}{4}$

**241**. $y = \frac{-1}{2}x$

**243**. $x = \pm\sqrt{6}$

**245**. 10

**247**. $-\frac{1}{8}$

**249**. $-4$

**251**. $-12$

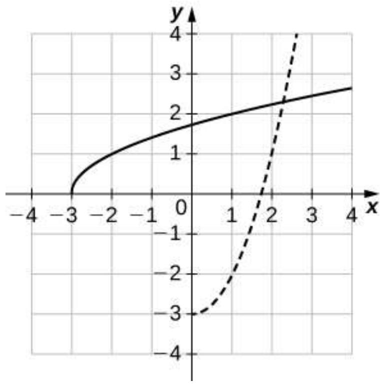
**253**. a. $-\frac{200}{343}$ m/s, b. $\frac{600}{2401}$ m/s$^2$, c. The train is slowing down since velocity and acceleration have opposite signs.

**255**. a. $C'(x) = 0.0003x^2 - 0.04x + 3$ b. $\frac{dC}{dt} = 100 \cdot \left(0.0003x^2 - 0.04x + 3\right)$ c. Approximately $90,300 per week

**257**. a. $\frac{dS}{dt} = -\frac{8\pi r^2}{(t+1)^3}$ b. The volume is decreasing at a rate of $-\frac{\pi}{36}$ ft$^3$/min.

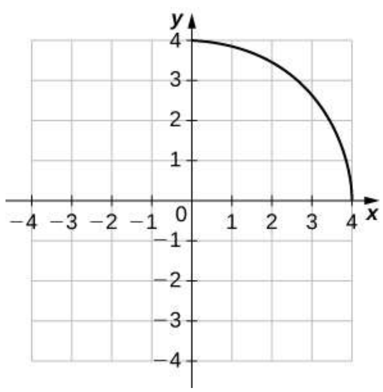
**259**. $\sim 2.3$ ft/hr

**261**. a.




b. $\left(f^{-1}\right)'(1) \sim 2$

**263**. a.




b. $\left(f^{-1}\right)'(1) \sim -1/\sqrt{3}$

**265**. a. 6, b. $x = f^{-1}(y) = \left(\dfrac{y+3}{2}\right)^{1/3}$, c. $\dfrac{1}{6}$

**267**. a. $1$, b. $x = f^{-1}(y) = \sin^{-1} y$, c. $1$

**269**. $\dfrac{1}{5}$

**271**. $\dfrac{1}{3}$

**273**. $1$

**275**. a. $4$, b. $y = 4x$

**277**. a. $-\dfrac{1}{96}$, b. $y = -\dfrac{1}{13}x + \dfrac{18}{13}$

**279**. $\dfrac{2x}{\sqrt{1-x^4}}$

**281**. $\dfrac{-1}{\sqrt{1-x^2}}$

**283**. $\dfrac{3\left(1+\tan^{-1} x\right)^2}{1+x^2}$

**285**. $\dfrac{-1}{\left(1+x^2\right)\left(\tan^{-1} x\right)^2}$

**287**. $\dfrac{x}{\left(5-x^2\right)\sqrt{4-x^2}}$

**289**. $-1$

**291**. $\frac{1}{2}$

**293**. $\frac{1}{10}$

**295**. a. $v(t)=\frac{1}{1+t^2}$ b. $a(t)=\frac{-2t}{\left(1+t^2\right)^2}$ c. $(a)0.2,\ 0.06,\ 0.03;\ (b)-0.16,\ -0.028,\ -0.0088$ d. The hockey puck is decelerating/slowing down at 2, 4, and 6 seconds.

**297**. $-0.0168$ radians per foot

**299**. a. $\frac{d\theta}{dx}=\frac{10}{100+x^2}-\frac{40}{1600+x^2}$ b. $\frac{18}{325},\ \frac{9}{340},\ \frac{42}{4745},\ 0$ c. As a person moves farther away from the screen, the viewing angle is increasing, which implies that as he or she moves farther away, his or her screen vision is widening. d. $-\frac{54}{12905},\ -\frac{3}{500},\ -\frac{198}{29945},\ -\frac{9}{1360}$ e. As the person moves beyond 20 feet from the screen, the viewing angle is decreasing. The optimal distance the person should stand for maximizing the viewing angle is 20 feet.

**301**. $\frac{dy}{dx}=\frac{-2x}{y}$

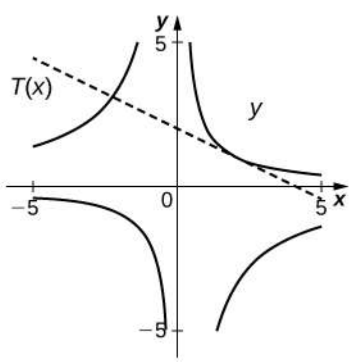
**303**. $\frac{dy}{dx}=\frac{x}{3y}-\frac{y}{2x}$

**305**. $\frac{dy}{dx}=\frac{y-\frac{y}{2\sqrt{x+4}}}{\sqrt{x+4}-x}$

**307**. $\frac{dy}{dx}=\frac{y^2\cos(xy)}{2y-\sin(xy)-xy\cos xy}$

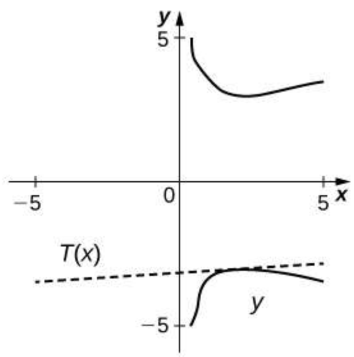
**309**. $\frac{dy}{dx}=\frac{-3x^2y-y^3}{x^3+3xy^2}$

**311**.



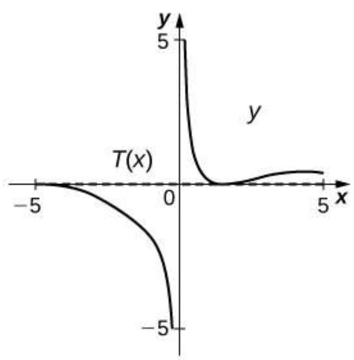

$y=\frac{-1}{2}x+2$

**313**.




$y = \frac{1}{\pi + 12}x - \frac{3\pi + 38}{\pi + 12}$

**315**.




$y = 0$

**317**. a. $y = -x + 2$ b. $(3, -1)$

**319**. a. $\left(\pm\sqrt{7}, 0\right)$ b. $-2$ c. They are parallel since the slope is the same at both intercepts.

**321**. $y = -x + 1$

**323**. a. $-0.5926$ b. When $81 is spent on labor and $16 is spent on capital, the amount spent on capital is decreasing by $0.5926 per $1 spent on labor.

**325**. $-8$

**327**. $-2.67$

**329**. $y' = -\frac{1}{\sqrt{1 - x^2}}$

**331**. $2xe^x + x^2 e^x$

**333**. $e^{x^3 \ln x}\left(3x^2 \ln x + x^2\right)$

**335**. $\frac{4}{(e^x + e^{-x})^2}$

**337**. $2^{4x + 2} \cdot \ln 2 + 8x$

**339**. $\pi x^{\pi - 1} \cdot \pi^x + x^\pi \cdot \pi^x \ln \pi$

**341**. $\frac{5}{2(5x - 7)}$

**343**. $\frac{\tan x}{\ln 10}$

**345**. $2^x \cdot \ln 2 \cdot \log_3 7^{x^2 - 4} + 2^x \cdot \frac{2x \ln 7}{\ln 3}$

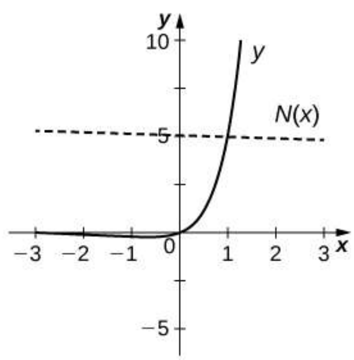
**347**. $(\sin 2x)^{4x}[4 \cdot \ln(\sin 2x) + 8x \cdot \cot 2x]$

**349**. $x^{\log_2 x} \cdot \frac{2\ln x}{x\ln 2}$

**351**. $x^{\cot x} \cdot \left[-\csc^2 x \cdot \ln x + \frac{\cot x}{x}\right]$

**353**. $x^{-1/2}\left(x^2+3\right)^{2/3}(3x-4)^4 \cdot \left[\frac{-1}{2x} + \frac{4x}{3\left(x^2+3\right)} + \frac{12}{3x-4}\right]$

**355**.




$y = \frac{-1}{5+5\ln 5}x + \left(5 + \frac{1}{5+5\ln 5}\right)$

**357**. a. $x = e \sim 2.718$ b. $(e, \infty)$, $(0, e)$

**359**. a. $P = 500{,}000(1.05)^t$ individuals b. $P'(t) = 24395 \cdot (1.05)^t$ individuals per year c. $39{,}737$ individuals per year

**361**. a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1963 there were approximately 723 cases of the disease in the United States. b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of $-4.611$ thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of $-0.2808$ thousand per year.

**363**. $p = 35741(1.045)^t$

**365**.

| Years since 1790 | $P''$ |
|---|---|
| 0 | 69.25 |
| 10 | 107.5 |
| 20 | 167.0 |
| 30 | 259.4 |
| 40 | 402.8 |
| 50 | 625.5 |
| 60 | 971.4 |
| 70 | 1508.5 |

## Review Exercises

**367**. False.

**369**. False

**371**. $\frac{1}{2\sqrt{x+4}}$

**373**. $9x^2 + \frac{8}{x^3}$

**375**. $e^{\sin x} \cos x$

**377**. $x \sec^2(x) + 2x \cos(x) + \tan(x) - x^2 \sin(x)$

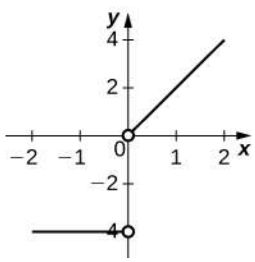
**379**. $\frac{1}{4}\left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1}(x)\right)$

**381**. $\cos x \cdot (\ln x + 1) - x \ln(x) \sin x$

**383**. $4^x (\ln 4)^2 + 2\sin x + 4x \cos x - x^2 \sin x$

**385**. $T = (2+e)x - 2$

**387**.




**389**. $w'(3) = -\frac{2.9\pi}{6}$. At 3 a.m. the tide is decreasing at a rate of 1.514 ft/hr.

**391**. $-7.5$. The wind speed is decreasing at a rate of 7.5 mph/hr

# Chapter 4

## Checkpoint

**4.1**. $\frac{1}{72\pi}$ cm/sec, or approximately 0.0044 cm/sec

**4.2**. $500$ ft/sec

**4.3**. $\frac{1}{10}$ rad/sec

**4.4**. $-0.61$ ft/sec

**4.5**. $L(x) = 2 + \frac{1}{12}(x-8)$; 2.00833

**4.6**. $L(x) = -x + \frac{\pi}{2}$

**4.7**. $L(x) = 1 + 4x$

**4.8**. $dy = 2xe^{x^2} dx$

**4.9**. $dy = 1.6, \quad \Delta y = 1.64$

**4.10**. The volume measurement is accurate to within $21.6 \text{ cm}^3$.

**4.11**. 7.6%

**4.12**. $x = -\frac{2}{3}, \quad x = 1$

**4.13**. The absolute maximum is $3$ and it occurs at $x = 4$. The absolute minimum is $-1$ and it occurs at $x = 2$.

**4.14**. $c = 2$

**4.15**. $\frac{5}{2\sqrt{2}}$ sec

**4.16**. $f$ has a local minimum at $-2$ and a local maximum at $3.$

**4.17**. $f$ has no local extrema because $f'$ does not change sign at $x = 1.$

**4.18**. $f$ is concave up over the interval $\left(-\infty, \frac{1}{2}\right)$ and concave down over the interval $\left(\frac{1}{2}, \infty\right)$

**4.19**. $f$ has a local maximum at $-2$ and a local minimum at $3.$

**4.20**. Both limits are $3.$ The line $y = 3$ is a horizontal asymptote.

**4.21**. Let $\varepsilon > 0.$ Let $N = \frac{1}{\sqrt{\varepsilon}}.$ Therefore, for all $x > N,$ we have $\left|3 - \frac{1}{x^2} - 3\right| = \frac{1}{x^2} < \frac{1}{N^2} = \varepsilon$ Therefore, $\lim_{x \to \infty}\left(3 - 1/x^2\right) = 3.$

**4.22**. Let $M > 0.$ Let $N = \sqrt{\frac{M}{3}}.$ Then, for all $x > N,$ we have $3x^2 > 3N^2 = 3\left(\sqrt{\frac{M}{3}}\right)^2 2 = \frac{3M}{3} = M$

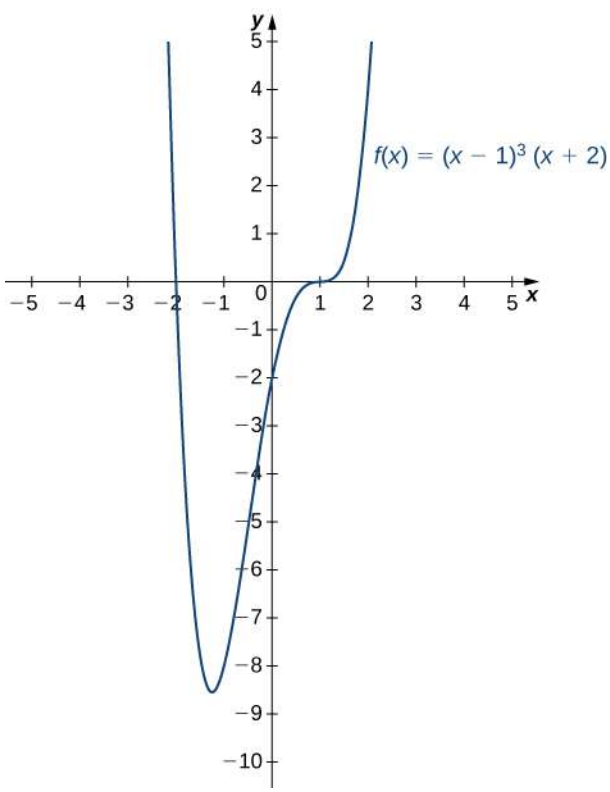
**4.23**. $-\infty$

**4.24**. $\frac{3}{5}$

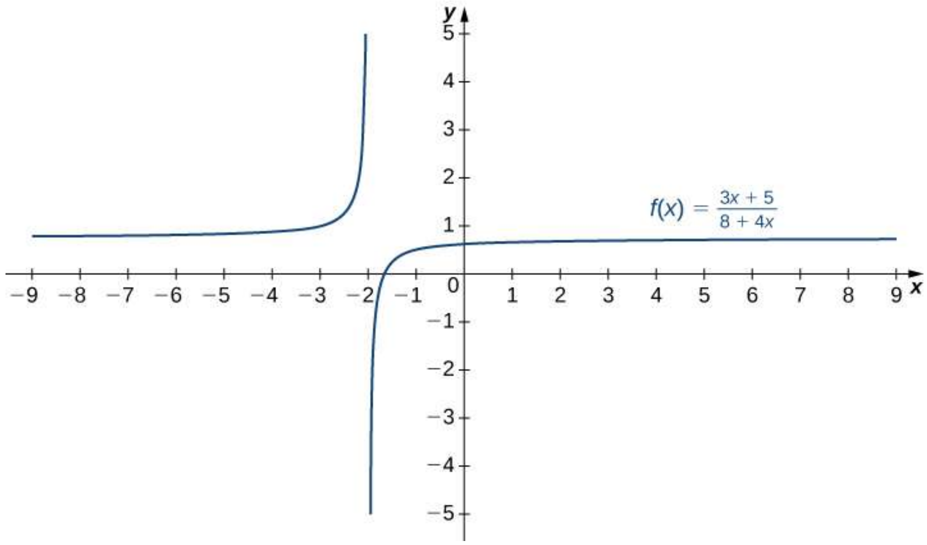
**4.25**. $\pm\sqrt{3}$

**4.26**. $\lim_{x \to \infty} f(x) = \frac{3}{5},\quad \lim_{x \to -\infty} f(x) = -2$

**4.27**.




**4.28**.




**4.29**. $y = \frac{3}{2}x$

**4.30**. The function $f$ has a cusp at $(0, 5)$ $\lim_{x \to 0^-} f'(x) = \infty$, $\lim_{x \to 0^+} f'(x) = -\infty$. For end behavior, $\lim_{x \to \pm\infty} f(x) = -\infty$.

**4.31**. The maximum area is $5000 \text{ ft}^2$.

**4.32**. $V(x) = x(20 - 2x)(30 - 2x)$. The domain is $[0, 10]$.

**4.33**. $T(x) = \frac{x}{6} + \frac{\sqrt{(15 - x)^2 + 1}}{2.5}$

**4.34**. The company should charge $\$75$ per car per day.

**4.35**. $A(x) = 4x\sqrt{1 - x^2}$. The domain of consideration is $[0, 1]$.

**4.36**. $c(x) = \frac{259.2}{x} + 0.2x^2$ dollars

**4.37**. $1$

**4.38**. $0$

**4.39**. $\lim_{x \to 0^+} \cos x = 1$. Therefore, we cannot apply L'Hôpital's rule. The limit of the quotient is $\infty$

**4.40**. $1$

**4.41**. $0$

**4.42**. $e$

**4.43**. $1$

**4.44**. The function $2^x$ grows faster than $x^{100}$.

**4.45**. $x_1 \approx 0.33333333,\ x_2 \approx 0.347222222$

**4.46**. $x_1 = 2,\ x_2 = 1.75$

**4.47**. $x_1 \approx -1.842105263,\ x_2 \approx -1.772826920$

**4.48**. $x_1 = 6,\ x_2 = 8,\ x_3 = \frac{26}{3},\ x_4 = \frac{80}{9},\ x_5 = \frac{242}{27};\ x* = 9$

**4.49**. $-\cos x + C$

**4.50**. $\frac{d}{dx}(x \sin x + \cos x + C) = \sin x + x\cos x - \sin x = x\cos x$

**4.51**. $x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 - 7x + C$

**4.52**. $y = -\frac{3}{x} + 5$

**4.53**. $2.93$ sec, $64.5$ ft

## Section Exercises

**1**. $8$

**3**. $\frac{13}{\sqrt{10}}$

**5**. $2\sqrt{3}$ ft/sec

**7**. The distance is decreasing at $390$ mi/h.

**9**. The distance between them shrinks at a rate of $\frac{1320}{13} \approx 101.5$ mph.

**11**. $\frac{9}{2}$ ft/sec

**13**. It grows at a rate $\frac{4}{9}$ ft/sec

**15**. The distance is increasing at $\frac{(135\sqrt{26})}{26}$ ft/sec

**17**. $-\frac{5}{6}$ m/sec

**19**. $240\pi$ m$^2$/sec

**21**. $\frac{1}{2\sqrt{\pi}}$ cm

**23**. The area is increasing at a rate $\frac{(3\sqrt{3})}{8}$ft /sec.

**25**. The depth of the water decreases at $\frac{128}{125\pi}$ ft/min.

**27**. The volume is decreasing at a rate of $\frac{(25\pi)}{16}\text{ft}^3$/min.

**29**. The water flows out at rate $\frac{(2\pi)}{5}$m /min.

**31**. $\frac{3}{2}$ m/sec

**33**. $\frac{25}{19\pi}$ ft/min

**35**. $\frac{2}{45\pi}$ ft/min

**37**. The angle decreases at $\frac{400}{1681}$ rad/sec.

**39**. $100\pi$mi/min

**41**. The angle is changing at a rate of $\frac{11}{25}$ rad/sec.

**43**. The distance is increasing at a rate of $62.50$ ft/sec.

**45**. The distance is decreasing at a rate of $11.99$ ft/sec.

**47**. $f'(a) = 0$

**49**. The linear approximation exact when $y = f(x)$ is linear or constant.

**51**. $L(x) = \frac{1}{2} - \frac{1}{4}(x-2)$

**53**. $L(x) = 1$

**55**. $L(x) = 0$

**57**. 0.02

**59**. $1.9996875$

**61**. $0.001593$

**63**. $1$; error, ~$0.00005$

**65**. $0.97$; error, ~$0.0006$

**67**. $3 - \frac{1}{600}$; error, $\sim 4.632 \times 10^{-7}$
**69**. $dy = (\cos x - x \sin x) dx$
**71**. $dy = \left( \frac{x^2 - 2x - 2}{(x-1)^2} \right) dx$
**73**. $dy = -\frac{1}{(x+1)^2} dx, \quad -\frac{1}{16}$
**75**. $dy = \frac{9x^2 + 12x - 2}{2(x+1)^{3/2}} dx, \quad -0.1$
**77**. $dy = \left( 3x^2 + 2 - \frac{1}{x^2} \right) dx, \quad 0.2$
**79**. $12x\,dx$
**81**. $4\pi r^2\,dr$
**83**. $-1.2\pi\ \text{cm}^3$
**85**. $-100\ \text{ft}^3$
**91**. Answers may vary
**93**. Answers will vary
**95**. No; answers will vary
**97**. Since the absolute maximum is the function (output) value rather than the *x* value, the answer is no; answers will vary
**99**. When $a = 0$
**101**. Absolute minimum at 3; Absolute maximum at −2.2; local minima at −2, 1; local maxima at −1, 2
**103**. Absolute minima at −2, 2; absolute maxima at −2.5, 2.5; local minimum at 0; local maxima at −1, 1
**105**. Answers may vary.
**107**. Answers may vary.
**109**. $x = 1$
**111**. None
**113**. $x = 0$
**115**. None
**117**. $x = -1,\ 1$
**119**. Absolute maximum: $x = 4, \quad y = \frac{33}{2}$; absolute minimum: $x = 1, \quad y = 3$
**121**. Absolute minimum: $x = \frac{1}{2}, \quad y = 4$
**123**. Absolute maximum: $x = 2\pi, \quad y = 2\pi$; absolute minimum: $x = 0, \quad y = 0$
**125**. Absolute maximum: $x = -3$; absolute minimum: $-1 \le x \le 1, \quad y = 2$
**127**. Absolute maximum: $x = \frac{\pi}{4}, \quad y = \sqrt{2}$; absolute minimum: $x = \frac{5\pi}{4}, \quad y = -\sqrt{2}$
**129**. Absolute minimum: $x = -2, \quad y = 1$
**131**. Absolute minimum: $x = -3, \quad y = -135$; local maximum: $x = 0, \quad y = 0$; local minimum: $x = 1, \quad y = -7$
**133**. Local maximum: $x = 1 - 2\sqrt{2}, \quad y = 3 - 4\sqrt{2}$; local minimum: $x = 1 + 2\sqrt{2}, \quad y = 3 + 4\sqrt{2}$
**135**. Absolute maximum: $x = \frac{\sqrt{2}}{2}, \quad y = \frac{3}{2}$; absolute minimum: $x = -\frac{\sqrt{2}}{2}, \quad y = -\frac{3}{2}$
**137**. Local maximum: $x = -2, \quad y = 59$; local minimum: $x = 1, \quad y = -130$
**139**. Absolute maximum: $x = 0, \quad y = 1$; absolute minimum: $x = -2, 2, \quad y = 0$
**141**. $h = \frac{9245}{49}$ m, $t = \frac{300}{49}$ s
**143**. The global minimum was in 1848, when no gold was produced.
**145**. Absolute minima: $x = 0, \quad x = 2, \quad y = 1$; local maximum at $x = 1, \quad y = 2$
**147**. No maxima/minima if $a$ is odd, minimum at $x = 1$ if $a$ is even
**149**. One example is $f(x) = |x| + 3, \ -2 \le x \le 2$
**151**. Yes, but the Mean Value Theorem still does not apply

**153**. $(-\infty, 0), (0, \infty)$

**155**. $(-\infty, -2), (2, \infty)$

**157**. 2 points

**159**. 5 points

**161**. $c = \frac{2\sqrt{3}}{3}$

**163**. $c = \frac{1}{2}, 1, \frac{3}{2}$

**165**. $c = 1$

**167**. Not differentiable

**169**. Not differentiable

**171**. Yes

**173**. The Mean Value Theorem does not apply since the function is discontinuous at $x = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}.$

**175**. Yes

**177**. The Mean Value Theorem does not apply; discontinuous at $x = 0.$

**179**. Yes

**181**. The Mean Value Theorem does not apply; not differentiable at $x = 0.$

**183**. $b = \pm 2\sqrt{c}$

**185**. $c = \pm\frac{1}{\pi}\cos^{-1}\left(\frac{\sqrt{\pi}}{2}\right), \quad c = \pm 0.1533$

**187**. The Mean Value Theorem does not apply.

**189**. $\frac{1}{2\sqrt{c+1}} - \frac{2}{c^3} = \frac{521}{2880}; \; c = 3.133, 5.867$

**191**. Yes

**193**. It is constant.

**195**. It is not a local maximum/minimum because $f'$ does not change sign

**197**. No

**199**. False; for example, $y = \sqrt{x}.$

**201**. Increasing for $-2 < x < -1$ and $x > 2;$ decreasing for $x < -2$ and $-1 < x < 2$

**203**. Decreasing for $x < 1,$ increasing for $x > 1$

**205**. Decreasing for $-2 < x < -1$ and $1 < x < 2;$ increasing for $-1 < x < 1$ and $x < -2$ and $x > 2$

**207**. a. Increasing over $-2 < x < -1, 0 < x < 1, x > 2,$ decreasing over $x < -2, \quad -1 < x < 0, 1 < x < 2;$ b. maxima at $x = -1$ and $x = 1,$ minima at $x = -2$ and $x = 0$ and $x = 2$

**209**. a. Increasing over $x > 0,$ decreasing over $x < 0;$ b. Minimum at $x = 0$

**211**. Concave up on all $x,$ no inflection points

**213**. Concave up on all $x,$ no inflection points

**215**. Concave up for $x < 0$ and $x > 1,$ concave down for $0 < x < 1,$ inflection points at $x = 0$ and $x = 1$

**217**. Answers will vary

**219**. Answers will vary

**221**. a. Increasing over $-\frac{\pi}{2} < x < \frac{\pi}{2},$ decreasing over $x < -\frac{\pi}{2}, x > \frac{\pi}{2}$ b. Local maximum at $x = \frac{\pi}{2};$ local minimum at $x = -\frac{\pi}{2}$

**223**. a. Concave up for $x > \frac{4}{3},$ concave down for $x < \frac{4}{3}$ b. Inflection point at $x = \frac{4}{3}$

**225**. a. Increasing over $x < 0$ and $x > 4,$ decreasing over $0 < x < 4$ b. Maximum at $x = 0,$ minimum at $x = 4$ c. Concave up for $x > 2,$ concave down for $x < 2$ d. Infection point at $x = 2$

**227**. a. Increasing over $x < 0$ and $x > \frac{60}{11},$ decreasing over $0 < x < \frac{60}{11}$ b. Minimum at $x = \frac{60}{11}$ c. Concave down for $x < \frac{54}{11},$ concave up for $x > \frac{54}{11}$ d. Inflection point at $x = \frac{54}{11}$

**229**. a. Increasing over $x > -\frac{1}{2},$ decreasing over $x < -\frac{1}{2}$ b. Minimum at $x = -\frac{1}{2}$ c. Concave up for all $x$ d. No inflection points

**231**. a. Increases over $-\frac{1}{4} < x < \frac{3}{4},$ decreases over $x > \frac{3}{4}$ and $x < -\frac{1}{4}$ b. Minimum at $x = -\frac{1}{4},$ maximum at $x = \frac{3}{4}$ c. Concave up for $-\frac{3}{4} < x < \frac{1}{4},$ concave down for $x < -\frac{3}{4}$ and $x > \frac{1}{4}$ d. Inflection points at $x = -\frac{3}{4}, x = \frac{1}{4}$

**233**. a. Increasing for all $x$ b. No local minimum or maximum c. Concave up for $x > 0,$ concave down for $x < 0$ d. Inflection point at $x = 0$

**235**. a. Increasing for all $x$ where defined b. No local minima or maxima c. Concave up for $x < 1;$ concave down for $x > 1$ d. No inflection points in domain

**237**. a. Increasing over $-\frac{\pi}{4} < x < \frac{3\pi}{4},$ decreasing over $x > \frac{3\pi}{4}, x < -\frac{\pi}{4}$ b. Minimum at $x = -\frac{\pi}{4},$ maximum at $x = \frac{3\pi}{4}$ c. Concave up for $-\frac{\pi}{2} < x < \frac{\pi}{2},$ concave down for $x < -\frac{\pi}{2}, x > \frac{\pi}{2}$ d. Infection points at $x = \pm\frac{\pi}{2}$

**239**. a. Increasing over $x > 4,$ decreasing over $0 < x < 4$ b. Minimum at $x = 4$ c. Concave up for $0 < x < 8\sqrt[3]{2},$ concave down for $x > 8\sqrt[3]{2}$ d. Inflection point at $x = 8\sqrt[3]{2}$

**241**. $f > 0, f' > 0, f'' < 0$

**243**. $f > 0, f' < 0, f'' < 0$

**245**. $f > 0, f' > 0, f'' > 0$

**247**. True, by the Mean Value Theorem

**249**. True, examine derivative

**251**. $x = 1$

**253**. $x = -1, x = 2$

**255**. $x = 0$

**257**. Yes, there is a vertical asymptote

**259**. Yes, there is vertical asymptote

**261**. $0$

**263**. $\infty$

**265**. $-\frac{1}{7}$

**267**. $-2$

**269**. $-4$

**271**. Horizontal: none, vertical: $x = 0$

**273**. Horizontal: none, vertical: $x = \pm 2$

**275**. Horizontal: none, vertical: none

**277**. Horizontal: $y = 0,$ vertical: $x = \pm 1$

**279**. Horizontal: $y = 0,$ vertical: $x = 0$ and $x = -1$

**281**. Horizontal: $y = 1,$ vertical: $x = 1$

**283**. Horizontal: none, vertical: none

**285**. Answers will vary, for example: $y = \frac{2x}{x - 1}$

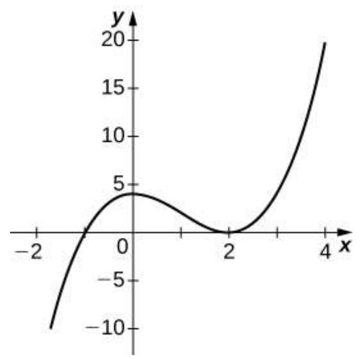
**287**. Answers will vary, for example: $y = \frac{4x}{x + 1}$

**289**. $y = 0$

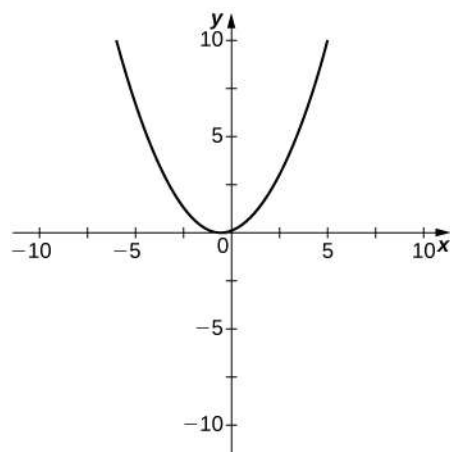
**291**. $\infty$

**293**. $y = 3$

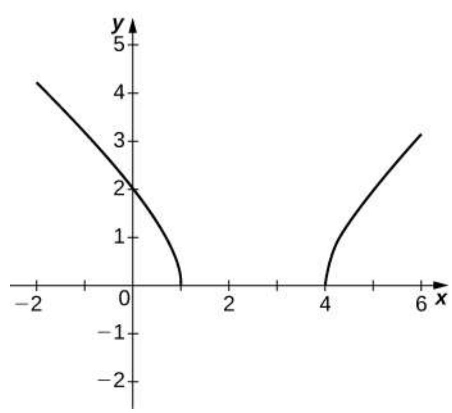
**295**.



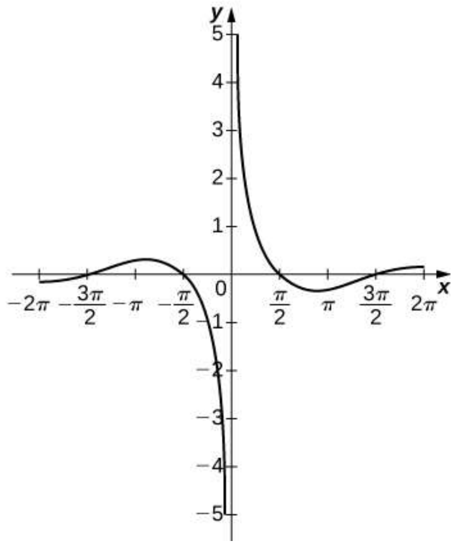

**297**.



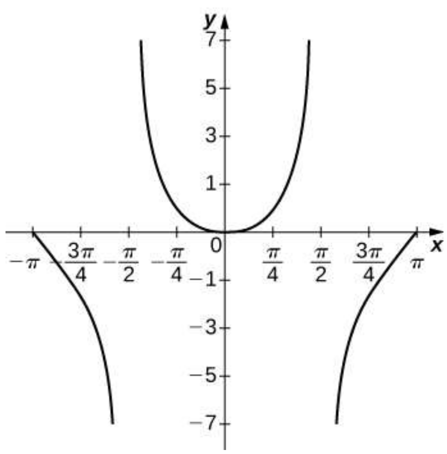

**299**.



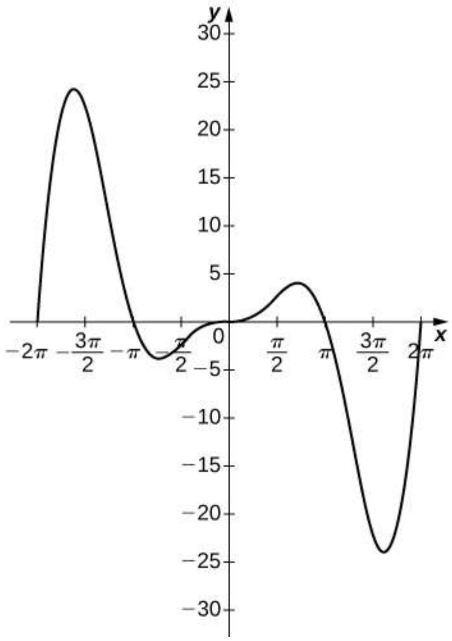

**301**.


y
5
4
3
2
1
0
−1
−2
−3
−4
−5
x
$-2\pi$
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$\pi$
$\frac{3\pi}{2}$
$2\pi$

**303**.


y
7
5
3
1
0
−1
−3
−5
−7
x
$-\pi$
$-\frac{3\pi}{4}$
$-\frac{\pi}{2}$
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
$\pi$

**305**.




**307**. The degree of $Q(x)$ must be greater than the degree of $P(x)$.

**309**. $\lim_{x \to 1^-} f(x) = -\infty$ and $\lim_{x \to 1^-} g(x) = \infty$

**311**. The critical points can be the minima, maxima, or neither.

**313**. False; $y = -x^2$ has a minimum only

**315**. $h = \frac{62}{3}$ in.

**317**. $1$

**319**. $100$ ft by $100$ ft

**321**. $40$ ft by $40$ ft

**323**. $19.73$ ft.

**325**. $84$ bpm

**327**. $T(\theta) = \frac{40\theta}{3v} + \frac{40\cos\theta}{v}$

**329**. $v = \sqrt{\frac{b}{a}}$

**331**. approximately $34.02$ mph

**333**. $4$

**335**. $0$

**337**. Maximal: $x = 5,\ y = 5$; minimal: $x = 0,\ y = 10$ and $y = 0,\ x = 10$

**339**. Maximal: $x = 1,\ y = 9$; minimal: none

**341**. $\frac{4\pi}{3\sqrt{3}}$

**343**. $6$

**345**. $r = 2,\ h = 4$

**347**. $(2,\ 1)$

**349**. $(0.8351,\ 0.6974)$

**351**. $A = 20r - 2r^2 - \frac{1}{2}\pi r^2$

**353**. $C(x) = 5x^2 + \frac{32}{x}$ Differentiating, setting the derivative equal to zero and solving, we obtain $x = \sqrt[3]{\frac{2}{5}}$ and $h = \sqrt[3]{\frac{25}{4}}$.

**355**. $P(x) = (50 - x)(800 + 25x - 50)$
**357**. $\infty$
**359**. $\frac{1}{2a}$
**361**. $\frac{1}{na^{n-1}}$
**363**. Cannot apply directly; use logarithms
**365**. Cannot apply directly; rewrite as $\lim_{x \to 0} x^3$
**367**. $6$
**369**. $-2$
**371**. $-1$
**373**. $n$
**375**. $-\frac{1}{2}$
**377**. $\frac{1}{2}$
**379**. $1$
**381**. $\frac{1}{6}$
**383**. $1$
**385**. $0$
**387**. $0$
**389**. $-1$
**391**. $\infty$
**393**. $0$
**395**. $\frac{1}{e}$
**397**. $0$
**399**. $1$
**401**. $0$
**403**. $\tan(1)$
**405**. $2$
**407**. $F(x_n) = x_n - \frac{x_n{}^3 + 2x_n + 1}{3x_n{}^2 + 2}$
**409**. $F(x_n) = x_n - \frac{e^{x_n}}{e^{x_n}}$
**411**. $|c| > 0.5$ fails, $|c| \le 0.5$ works
**413**. $c = \frac{1}{f'(x_n)}$
**415**. a. $x_1 = \frac{12}{25}, x_2 = \frac{312}{625}$; b. $x_1 = -4, x_2 = -40$
**417**. a. $x_1 = 1.291, x_2 = 0.8801$; b. $x_1 = 0.7071, x_2 = 1.189$
**419**. a. $x_1 = -\frac{26}{25}, x_2 = -\frac{1224}{625}$; b. $x_1 = 4, x_2 = 18$
**421**. a. $x_1 = \frac{6}{10}, x_2 = \frac{6}{10}$; b. $x_1 = 2, x_2 = 2$
**423**. $3.1623 \text{ or } -3.1623$
**425**. $0, -1 \text{ or } 1$
**427**. $0$
**429**. $0.5188 \text{ or } -1.2906$
**431**. $0$
**433**. $4.493$

**435**. $0.159,\ 3.146$

**437**. We need $f$ to be twice continuously differentiable.

**439**. $x = 0$

**441**. $x = -1$

**443**. $x = 5.619$

**445**. $x = -1.326$

**447**. There is no solution to the equation.

**449**. It enters a cycle.

**451**. $0$

**453**. $-0.3513$

**455**. Newton: $11$ iterations, secant: $16$ iterations

**457**. Newton: three iterations, secant: six iterations

**459**. Newton: five iterations, secant: eight iterations

**461**. $E = 4.071$

**463**. $4.394\%$

**465**. $F'(x) = 15x^2 + 4x + 3$

**467**. $F'(x) = 2xe^x + x^2 e^x$

**469**. $F'(x) = e^x$

**471**. $F(x) = e^x - x^3 - \cos(x) + C$

**473**. $F(x) = \frac{x^2}{2} - x - 2\cos(2x) + C$

**475**. $F(x) = \frac{1}{2}x^2 + 4x^3 + C$

**477**. $F(x) = \frac{2}{5}(\sqrt{x})^5 + C$

**479**. $F(x) = \frac{3}{2}x^{2/3} + C$

**481**. $F(x) = x + \tan(x) + C$

**483**. $F(x) = \frac{1}{3}\sin^3(x) + C$

**485**. $F(x) = -\frac{1}{2}\cot(x) - \frac{1}{x} + C$

**487**. $F(x) = -\sec x - 4\csc x + C$

**489**. $F(x) = -\frac{1}{8}e^{-4x} - \cos x + C$

**491**. $-\cos x + C$

**493**. $3x - \frac{2}{x} + C$

**495**. $\frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$

**497**. $14x - \frac{2}{x} - \frac{1}{2x^2} + C$

**499**. $f(x) = -\frac{1}{2x^2} + \frac{3}{2}$

**501**. $f(x) = \sin x + \tan x + 1$

**503**. $f(x) = -\frac{1}{6}x^3 - \frac{2}{x} + \frac{13}{6}$

**505**. Answers may vary; one possible answer is $f(x) = e^{-x}$

**507**. Answers may vary; one possible answer is $f(x) = -\sin x$

**509**. $5.867$ sec

**511**. $7.333$ sec

**513**. $13.75$ ft/sec$^2$

**515**. $F(x) = \frac{1}{3}x^3 + 2x$

**517**. $F(x) = x^2 - \cos x + 1$

**519**. $F(x) = -\frac{1}{(x+1)} + 1$

**521**. True
**523**. False

## Review Exercises

**525**. True, by Mean Value Theorem
**527**. True
**529**. Increasing: $(-2, 0) \cup (4, \infty)$, decreasing: $(-\infty, -2) \cup (0, 4)$

**531**. $L(x) = \frac{17}{16} + \frac{1}{2}(1 + 4\pi)\left(x - \frac{1}{4}\right)$

**533**. Critical point: $x = \frac{3\pi}{4}$, absolute minimum: $x = 0$, absolute maximum: $x = \pi$

**535**. Increasing: $(-1, 0) \cup (3, \infty)$, decreasing: $(-\infty, -1) \cup (0, 3)$, concave up: $\left(-\infty, \frac{1}{3}\left(2 - \sqrt{13}\right)\right) \cup \left(\frac{1}{3}\left(2 + \sqrt{13}\right), \infty\right)$, concave down: $\left(\frac{1}{3}\left(2 - \sqrt{13}\right), \frac{1}{3}\left(2 + \sqrt{13}\right)\right)$

**537**. Increasing: $\left(\frac{1}{4}, \infty\right)$, decreasing: $\left(0, \frac{1}{4}\right)$, concave up: $(0, \infty)$, concave down: nowhere

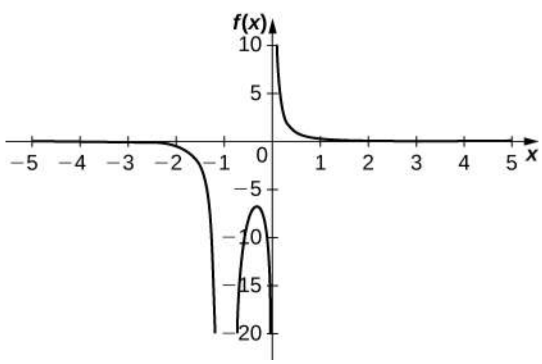
**539**. $3$

**541**. $-\frac{1}{\pi}$

**543**. $x_1 = -1, x_2 = -1$

**545**. $F(x) = \frac{2x^{3/2}}{3} + \frac{1}{x} + C$

**547**.




Inflection points: none; critical points: $x = -\frac{1}{3}$; zeros: none; vertical asymptotes: $x = -1$, $x = 0$; horizontal asymptote: $y = 0$

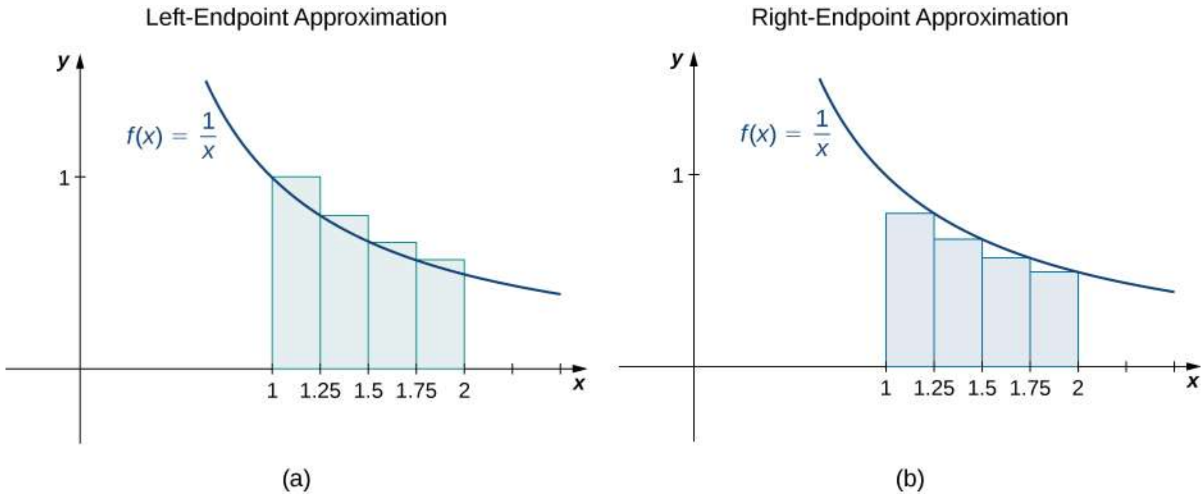
**549**. The height is decreasing at a rate of $0.125$ m/sec
**551**. $x = \sqrt{ab}$ feet

# Chapter 5

## Checkpoint

**5.1**. $\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$

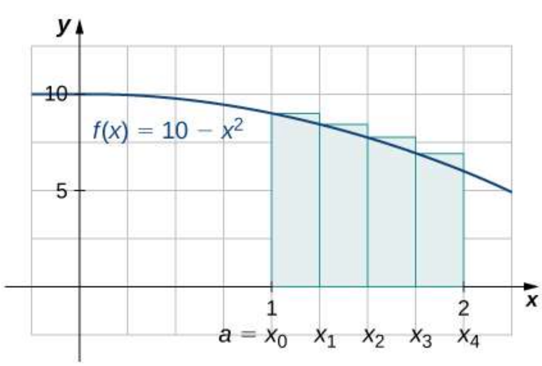
**5.2**. 15,550
**5.3**. 440
**5.4**. The left-endpoint approximation is 0.7595. The right-endpoint approximation is 0.6345. See the below **image**.




**5.5**.

a. Upper sum $= 8.0313.$

b.




**5.6**. $A \approx 1.125$

**5.7**. 6

**5.8**. 18 square units

**5.9**. 6

**5.10**. 18

**5.11**. $6\int_1^3 x^3\,dx - 4\int_1^3 x^2\,dx + 2\int_1^3 x\,dx - \int_1^3 3\,dx$

**5.12**. −7

**5.13**. 3

**5.14**. Average value $= 1.5$; $c = 3$

**5.15**. $c = \sqrt{3}$

**5.16**. $g'(r) = \sqrt{r^2 + 4}$

**5.17**. $F'(x) = 3x^2 \cos x^3$

**5.18**. $F'(x) = 2x \cos x^2 - \cos x$

**5.19**. $\frac{7}{24}$

**5.20**. Kathy still wins, but by a much larger margin: James skates 24 ft in 3 sec, but Kathy skates 29.3634 ft in 3 sec.

**5.21**. $-\frac{10}{3}$

**5.22**. Net displacement: $\frac{e^2-9}{2} \approx -0.8055$ m; total distance traveled: $4\ln 4 - 7.5 + \frac{e^2}{2} \approx 1.740$ m

**5.23**. 17.5 mi

**5.24**. $\frac{64}{5}$

**5.25**. $\int 3x^2\left(x^3-3\right)^2 dx = \frac{1}{3}\left(x^3-3\right)^3 + C$

**5.26**. $\frac{\left(x^3+5\right)^{10}}{30} + C$

**5.27**. $-\frac{1}{\sin t} + C$

**5.28**. $-\frac{\cos^4 t}{4} + C$

**5.29**. $\frac{91}{3}$

**5.30**. $\frac{2}{3\pi} \approx 0.2122$

**5.31**. $\int x^2 e^{-2x^3} dx = -\frac{1}{6}e^{-2x^3} + C$

**5.32**. $\int e^x (3e^x-2)^2 dx = \frac{1}{9}(3e^x-2)^3$

**5.33**. $\int 2x^3 e^{x^4} dx = \frac{1}{2}e^{x^4}$

**5.34**. $\frac{1}{2}\int_0^4 e^u du = \frac{1}{2}\left(e^4-1\right)$

**5.35**. $Q(t) = \frac{2^t}{\ln 2} + 8.557.$ There are 20,099 bacteria in the dish after 3 hours.

**5.36**. There are 116 flies.

**5.37**. $\int_1^2 \frac{1}{x^3}e^{4x^{-2}} dx = \frac{1}{8}\left[e^4 - e\right]$

**5.38**. $\ln|x+2| + C$

**5.39**. $\frac{x}{\ln 3}(\ln x - 1) + C$

**5.40**. $\frac{1}{4}\sin^{-1}(4x) + C$

**5.41**. $\sin^{-1}\left(\frac{x}{3}\right) + C$

**5.42**. $\frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right) + C$

**5.43**. $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$

**5.44**. $\frac{\pi}{8}$

## Section Exercises

**1**. a. They are equal; both represent the sum of the first 10 whole numbers. b. They are equal; both represent the sum of the first 10 whole numbers. c. They are equal by substituting $j = i - 1.$ d. They are equal; the first sum factors the terms of the second.

**3**. $385 - 30 = 355$

**5**. $15 - (-12) = 27$

**7**. $5(15) + 4(-12) = 27$

**9**. $\sum_{j=1}^{50} j^2 - 2\sum_{j=1}^{50} j = \frac{(50)(51)(101)}{6} - \frac{2(50)(51)}{2} = 40,\ 375$

**11**. $4\sum_{k=1}^{25} k^2 - 100\sum_{k=1}^{25} k = \frac{4(25)(26)(51)}{6} - 50(25)(26) = -10,\ 400$

**13**. $R_4 =$ -0.25

**15**. $R_6 = 0.372$

**17**. $L_4 = 2.20$

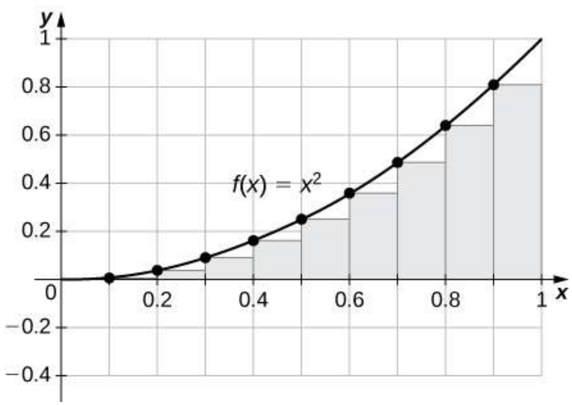
**19**. $L_8 = 0.6875$

**21**. $L_6 = 9.000 = R_6.$ The graph of *f* is a triangle with area 9.

**23**. $L_6 = 13.12899 = R_6.$ They are equal.

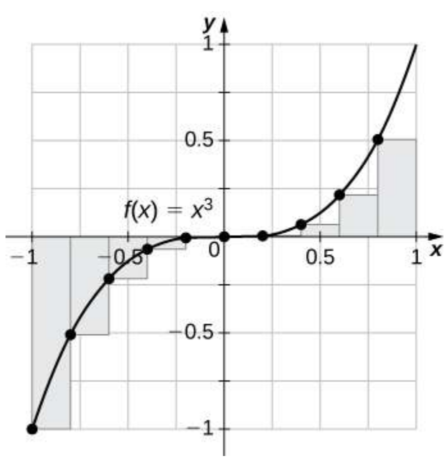
**25**. $L_{10} = \frac{4}{10}\sum_{i=1}^{10} \sqrt{4 - \left(-2 + 4\frac{(i-1)}{10}\right)}$

**27**. $R_{100} = \frac{e-1}{100}\sum_{i=1}^{100} \ln\left(1 + (e-1)\frac{i}{100}\right)$



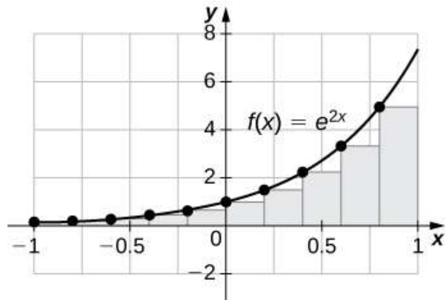

**29**.

$R_{100} = 0.33835,\ L_{100} = 0.32835.$ The plot shows that the left Riemann sum is an underestimate because the function is increasing. Similarly, the right Riemann sum is an overestimate. The area lies between the left and right Riemann sums. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.




**31**.

$L_{100} = -0.02,\ R_{100} = 0.02.$ The left endpoint sum is an underestimate because the function is increasing. Similarly, a right endpoint approximation is an overestimate. The area lies between the left and right endpoint estimates.




**33**. $L_{100} = 3.555,\ R_{100} = 3.670.$ The plot shows that the left Riemann sum is an underestimate because the function is increasing. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.
**35**. The sum represents the cumulative rainfall in January 2009.
**37**. The total mileage is $7 \times \sum_{i=1}^{25} \left(1 + \frac{(i-1)}{10}\right) = 7 \times 25 + \frac{7}{10} \times 12 \times 25 = 385$ mi.
**39**. Add the numbers to get 8.1-in. net increase.
**41**. 309,389,957
**43**. $L_8 = 3 + 2 + 1 + 2 + 3 + 4 + 5 + 4 = 24$
**45**. $L_8 = 3 + 5 + 7 + 6 + 8 + 6 + 5 + 4 = 44$
**47**. $L_{10} \approx 1.7604,\ L_{30} \approx 1.7625,\ L_{50} \approx 1.76265$
**49**. $R_1 = -1,\ L_1 = 1,\ R_{10} = -0.1,\ L_{10} = 0.1,\ L_{100} = 0.01,$ and $R_{100} = -0.1.$ By symmetry of the graph, the exact area is zero.
**51**. $R_1 = 0,\ L_1 = 0,\ R_{10} = 2.4499,\ L_{10} = 2.4499,\ R_{100} = 2.1365,\ L_{100} = 2.1365$
**53**. If $[c, d]$ is a subinterval of $[a, b]$ under one of the left-endpoint sum rectangles, then the area of the rectangle contributing to the left-endpoint estimate is $f(c)(d - c).$ But, $f(c) \le f(x)$ for $c \le x \le d,$ so the area under the graph of *f* between *c* and *d* is $f(c)(d - c)$ plus the area below the graph of *f* but above the horizontal line segment at height $f(c),$ which is positive. As this is true for each left-endpoint sum interval, it follows that the left Riemann sum is less than or equal to the area below the graph of *f* on $[a, b]$.
**55**. $L_N = \frac{b-a}{N} \sum_{i=1}^{N} f\left(a + (b-a)\frac{i-1}{N}\right) = \frac{b-a}{N} \sum_{i=0}^{N-1} f\left(a + (b-a)\frac{i}{N}\right)$ and $R_N = \frac{b-a}{N} \sum_{i=1}^{N} f\left(a + (b-a)\frac{i}{N}\right).$ The left sum has a term corresponding to $i = 0$ and the right sum has a term corresponding to $i = N.$ In $R_N - L_N,$ any term corresponding to $i = 1, 2, \ldots, N-1$ occurs once with a plus sign and once with a minus sign, so each such term cancels and one is left with $R_N - L_N = \frac{b-a}{N}\left(f(a + (b-a))\frac{N}{N}\right) - \left(f(a) + (b-a)\frac{0}{N}\right) = \frac{b-a}{N}(f(b) - f(a)).$
**57**. Graph 1: a. $L(A) = 0,\ B(A) = 20;$ b. $U(A) = 20.$ Graph 2: a. $L(A) = 9;$ b. $B(A) = 11,\ U(A) = 20.$ Graph 3: a. $L(A) = 11.0;$ b. $B(A) = 4.5,\ U(A) = 15.5.$
**59**. Let *A* be the area of the unit circle. The circle encloses *n* congruent triangles each of area $\frac{\sin\left(\frac{2\pi}{n}\right)}{2},$ so $\frac{n}{2}\sin\left(\frac{2\pi}{n}\right) \le A.$ Similarly, the circle is contained inside *n* congruent triangles each of area $\frac{BH}{2} = \frac{1}{2}\left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)\right)\sin\left(\frac{2\pi}{n}\right),$ so $A \le \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\left(\cos\left(\frac{\pi}{n}\right)\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right).$ As $n \to \infty,\ \frac{n}{2}\sin\left(\frac{2\pi}{n}\right) = \frac{\pi \sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)} \to \pi,$ so we conclude $\pi \le A.$ Also, as $n \to \infty,\ \cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right) \to 1,$ so we also have $A \le \pi.$ By the squeeze theorem for limits, we conclude that $A = \pi.$
**61**. $\int_0^2 \left(5x^2 - 3x^3\right) dx$
**63**. $\int_0^1 \cos^2(2\pi x)dx$

**65**. $\int_0^1 x dx$

**67**. $\int_3^6 x dx$

**69**. $\int_1^2 x \log(x^2) dx$

**71**. $1 + 2 \cdot 2 + 3 \cdot 3 = 14$

**73**. $1 - 4 + 9 = 6$

**75**. $1 - 2\pi + 9 = 10 - 2\pi$

**77**. The integral is the area of the triangle, $\frac{1}{2}$

**79**. The integral is the area of the triangle, 9.

**81**. The integral is the area $\frac{1}{2}\pi r^2 = 2\pi$.

**83**. The integral is the area of the "big" triangle less the "missing" triangle, $9 - \frac{1}{2}$.

**85**. $L = 2 + 0 + 10 + 5 + 4 = 21,\ R = 0 + 10 + 10 + 2 + 0 = 22,\ \frac{L + R}{2} = 21.5$

**87**. $L = 0 + 4 + 0 + 4 + 2 = 10,\ R = 4 + 0 + 2 + 4 + 0 = 10,\ \frac{L + R}{2} = 10$

**89**. $\int_2^4 f(x)dx + \int_2^4 g(x)dx = 8 - 3 = 5$

**91**. $\int_2^4 f(x)dx - \int_2^4 g(x)dx = 8 + 3 = 11$

**93**. $4\int_2^4 f(x)dx - 3\int_2^4 g(x)dx = 32 + 9 = 41$

**95**. The integrand is odd; the integral is zero.

**97**. The integrand is antisymmetric with respect to $x = 3$. The integral is zero.

**99**. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$

**101**. $\int_0^1 (1 - 6x + 12x^2 - 8x^3)dx = \left(x - 3x^2 + 4x^3 - 2x^4\right) = \left(1 - 3 + 4 - 2\right)\left(0 - 0 + 0 - 0\right) = 0$

**103**. $7 - \frac{5}{4} = \frac{23}{4}$

**105**. The integrand is negative over $[-2, 3]$.

**107**. $x \le x^2$ over $[1, 2]$, so $\sqrt{1 + x} \le \sqrt{1 + x^2}$ over $[1, 2]$.

**109**. $\cos(t) \ge \frac{\sqrt{2}}{2}$. Multiply by the length of the interval to get the inequality.

**111**. $f_{\text{ave}} = 0;\ c = 0$

**113**. $\frac{3}{2}$ when $c = \pm\frac{3}{2}$

**115**. $f_{\text{ave}} = 0;\ c = \frac{\pi}{2}, \frac{3\pi}{2}$

**117**. $L_{100} = 1.294,\ R_{100} = 1.301;$ the exact average is between these values.

**119**. $L_{100} \times \left(\frac{1}{2}\right) = 0.5178,\ R_{100} \times \left(\frac{1}{2}\right) = 0.5294$

**121**. $L_1 = 0,\ L_{10} \times \left(\frac{1}{2}\right) = 8.743493,\ L_{100} \times \left(\frac{1}{2}\right) = 12.861728.$ The exact answer $\approx 26.799,$ so $L_{100}$ is not accurate.

**123**. $L_1 \times \left(\frac{1}{\pi}\right) = 1.352,\ L_{10} \times \left(\frac{1}{\pi}\right) = -0.1837,\ L_{100} \times \left(\frac{1}{\pi}\right) = -0.2956.$ The exact answer $\approx -0.303,$ so $L_{100}$ is not accurate to first decimal.

**125**. Use $\tan^2\theta + 1 = \sec^2\theta.$ Then, $B - A = \int_{-\pi/4}^{\pi/4} 1dx = \frac{\pi}{2}.$

**127**. $\int_0^{2\pi} \cos^2 tdt = \pi,$ so divide by the length $2\pi$ of the interval. $\cos^2 t$ has period $\pi$, so yes, it is true.

**129**. The integral is maximized when one uses the largest interval on which $p$ is nonnegative. Thus, $A = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $B = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$

**131**. If $f(t_0) > g(t_0)$ for some $t_0 \in [a, b],$ then since $f - g$ is continuous, there is an interval containing $t_0$ such that $f(t) > g(t)$ over the interval $[c, d],$ and then $\int_d^d f(t)dt > \int_c^d g(t)dt$ over this interval.

**133**. The integral of $f$ over an interval is the same as the integral of the average of $f$ over that interval. Thus, $\int_a^b f(t)dt = \int_{a_0}^{a_1} f(t)dt + \int_{a_1}^{a_2} f(t)dt + \cdots + \int_{a_{N+1}}^{a_N} f(t)dt = \int_{a_0}^{a_1} 1dt + \int_{a_1}^{a_2} 1dt + \cdots + \int_{a_{N+1}}^{a_N} 1dt$ $= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_N - a_{N-1}) = a_N - a_0 = b - a.$ Dividing through by $b - a$ gives the desired identity.

**135**. $\int_0^N f(t)dt = \sum_{i=1}^{N} \int_{i-1}^{i} f(t)dt = \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

**137**. $L_{10} = 1.815,\ R_{10} = 1.515,\ \frac{L_{10} + R_{10}}{2} = 1.665,$ so the estimate is accurate to two decimal places.

**139**. The average is $1/2,$ which is equal to the integral in this case.

**141**. a. The graph is antisymmetric with respect to $t = \frac{1}{2}$ over $[0, 1],$ so the average value is zero. b. For any value of $a$, the graph between $[a, a+1]$ is a shift of the graph over $[0, 1],$ so the net areas above and below the axis do not change and the average remains zero.

**143**. Yes, the integral over any interval of length 1 is the same.

**145**. Yes. It is implied by the Mean Value Theorem for Integrals.

**147**. $F'(2) = -1;$ average value of $F'$ over $[1, 2]$ is $-1/2.$

**149**. $e^{\cos t}$

**151**. $\frac{1}{\sqrt{16 - x^2}}$

**153**. $\sqrt{x}\frac{d}{dx}\sqrt{x} = \frac{1}{2}$

**155**. $-\sqrt{1 - \cos^2 x}\frac{d}{dx}\cos x = |\sin x|\sin x$

**157**. $2x\frac{|x|}{1 + x^2}$

**159**. $\ln(e^{2x})\frac{d}{dx}e^x = 2xe^x$

**161**. a. $f$ is positive over $[1, 2]$ and $[5, 6],$ negative over $[0, 1]$ and $[3, 4],$ and zero over $[2, 3]$ and $[4, 5].$ b. The maximum value is 2 and the minimum is −3. c. The average value is 0.

**163**. a. $\ell$ is positive over $[0, 1]$ and $[3, 6],$ and negative over $[1, 3].$ b. It is increasing over $[0, 1]$ and $[3, 5],$ and it is constant over $[1, 3]$ and $[5, 6].$ c. Its average value is $\frac{1}{3}.$

**165**. $T_{10} = 49.08,\ \int_{-2}^{3} x^3 + 6x^2 + x - 5dx = 48$

**167**. $T_{10} = 260.836,\ \int_1^9 \left(\sqrt{x} + x^2\right)dx = 260$

**169**. $T_{10} = 3.058,\ \int_1^4 \frac{4}{x^2}dx = 3$

**171**. $F(x) = \frac{x^3}{3} + \frac{3x^2}{2} - 5x,\ F(3) - F(-2) = -\frac{35}{6}$

**173**. $F(x) = -\frac{t^5}{5} + \frac{13t^3}{3} - 36t,\ F(3) - F(2) = \frac{62}{15}$

**175**. $F(x) = \frac{x^{100}}{100},\ F(1) - F(0) = \frac{1}{100}$

**177**. $F(x) = \frac{x^3}{3} + \frac{1}{x},\ F(4) - F\left(\frac{1}{4}\right) = \frac{1125}{64}$

**179**. $F(x) = \sqrt{x},\ F(4) - F(1) = 1$

**181**. $F(x) = \frac{4}{3}t^{3/4},\ F(16) - F(1) = \frac{28}{3}$

**183**. $F(x) = -\cos x,\ F\left(\frac{\pi}{2}\right) - F(0) = 1$

**185**. $F(x) = \sec x,\ F\left(\frac{\pi}{4}\right) - F(0) = \sqrt{2} - 1$

**187**. $F(x) = -\cot(x),\ F\left(\frac{\pi}{2}\right) - F\left(\frac{\pi}{4}\right) = 1$

**189**. $F(x) = -\frac{1}{x} + \frac{1}{2x^2},\ F(-1) - F(-2) = \frac{7}{8}$

**191**. $F(x) = e^x - e$

**193**. $F(x) = 0$

**195**. $\int_{-2}^{-1}\left(t^2 - 2t - 3\right)dt - \int_{-1}^{3}\left(t^2 - 2t - 3\right)dt + \int_{3}^{4}\left(t^2 - 2t - 3\right)dt = \frac{46}{3}$

**197**. $-\int_{-\pi/2}^{0} \sin t dt + \int_0^{\pi/2} \sin t dt = 2$

**199**. a. The average is $11.21 \times 10^9$ since $\cos\left(\frac{\pi t}{6}\right)$ has period 12 and integral 0 over any period. Consumption is equal to the average when $\cos\left(\frac{\pi t}{6}\right) = 0,$ when $t = 3,$ and when $t = 9.$ b. Total consumption is the average rate times duration: $11.21 \times 12 \times 10^9 = 1.35 \times 10^{11}$ c. $10^9\left(11.21 - \frac{1}{6}\int_3^9 \cos\left(\frac{\pi t}{6}\right)dt\right) = 10^9\left(11.21 + \frac{2}{\pi}\right) = 11.84x10^9$

**201**. If $f$ is not constant, then its average is strictly smaller than the maximum and larger than the minimum, which are attained over $[a, b]$ by the extreme value theorem.

**203**. a. $d^2\theta = (a\cos\theta + c)^2 + b^2\sin^2\theta = a^2 + c^2\cos^2\theta + 2ac\cos\theta = (a + c\cos\theta)^2;$ b. $\bar{d} = \frac{1}{2\pi}\int_0^{2\pi}(a + 2c\cos\theta)d\theta = a$

**205**. Mean gravitational force = $\frac{GmM}{2}\int_0^{2\pi} \frac{1}{\left(a + 2\sqrt{a^2 - b^2}\cos\theta\right)^2}d\theta.$

**207**. $\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx = \int x^{1/2}dx - \int x^{-1/2}dx = \frac{2}{3}x^{3/2} + C_1 - 2x^{1/2} + C_2 = \frac{2}{3}x^{3/2} - 2x^{1/2} + C$

**209**. $\int\frac{dx}{2x} = \frac{1}{2}\ln|x| + C$

**211**. $\int_0^{\pi} \sin x dx - \int_0^{\pi} \cos x dx = -\cos x|_0^{\pi} - (\sin x)|_0^{\pi} = (-(-1) + 1) - (0 - 0) = 2$

**213**. $P(s) = 4s$, so $\frac{dP}{ds} = 4$ and $\int_2^4 4ds = 8.$

**215**. $\int_1^2 Nds = N$

**217**. With $p$ as in the previous exercise, each of the 12 pentagons increases in area from $2p$ to $4p$ units so the net increase in the area of the dodecahedron is $36p$ units.

**219**. $18s^2 = 6\int_s^{2s} 2xdx$

**221**. $12\pi R^2 = 8\pi\int_R^{2R} rdr$

**223**. $d(t) = \int_0^t v(s)ds = 4t - t^2.$ The total distance is $d(2) = 4$ m.

**225**. $d(t) = \int_0^t v(s)ds.$ For $t < 3,\ d(t) = \int_0^t (6 - 2t)dt = 6t - t^2.$ For $t > 3,\ d(t) = d(3) + \int_3^t (2t - 6)dt = 9 + (t^2 - 6t)\Big|_3^6.$ The total distance is $d(6) = 18$ m.

**227**. $v(t) = 40 - 9.8t$m/sec; $h(t) = 1.5 + 40t - 4.9t^2$ m/s

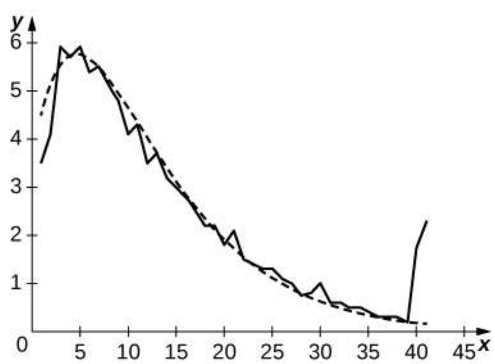
**229**. The net increase is 1 unit.

**231**. At $t = 5,$ the height of water is $x = \left(\frac{15}{\pi}\right)^{1/3}$ m.. The net change in height from $t = 5$ to $t = 10$ is $\left(\frac{30}{\pi}\right)^{1/3} - \left(\frac{15}{\pi}\right)^{1/3}$ m.

**233**. The total daily power consumption is estimated as the sum of the hourly power rates, or 911 gW-h.

**235**. 17 kJ

**237**. a. 54.3%; b. 27.00%; c. The curve in the following plot is $2.35(t + 3)e^{-0.15(t + 3)}$.

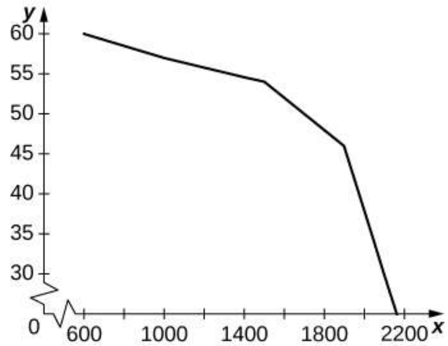



**239**. In dry conditions, with initial velocity $v_0 = 30$ m/s, $D = 64.3$ and, if $v_0 = 25,\ D = 44.64.$ In wet conditions, if $v_0 = 30,$ and $D = 180$ and if $v_0 = 25,\ D = 125.$

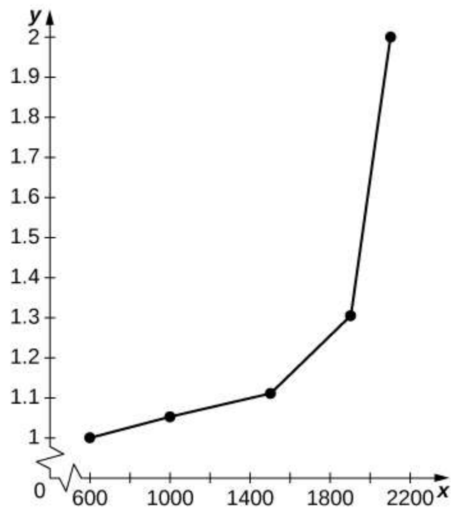
**241**. 225 cal

**243**. $E(150) = 28,\ E(300) = 22,\ E(450) = 16$

**245**. a.




b. Between 600 and 1000 the average decrease in vehicles per hour per lane is −0.0075. Between 1000 and 1500 it is −0.006 per vehicles per hour per lane, and between 1500 and 2100 it is −0.04 vehicles per hour per lane. c.




The graph is nonlinear, with minutes per mile increasing dramatically as vehicles per hour per lane reach 2000.

**247**. $\frac{1}{37}\int_0^{37} p(t)dt = \frac{0.07(37)^3}{4} + \frac{2.42(37)^2}{3} - \frac{25.63(37)}{2} + 521.23 \approx 2037$

**249**. Average acceleration is $A = \frac{1}{5}\int_0^5 a(t)dt = -\frac{0.7\left(5^2\right)}{3} + \frac{1.44(5)}{2} + 10.44 \approx 8.2$ mph/s

**251**. $d(t) = \int_0^1 |v(t)|dt = \int_0^t \left(\frac{7}{30}t^3 - 0.72t^2 - 10.44t + 41.033\right)dt = \frac{7}{120}t^4 - 0.24t^3 - 5.22t^3 + 41.033t.$ Then, $d(5) \approx 81.12$ mph $\times$ sec $\approx 119$ feet.

**253**. $\frac{1}{40}\int_0^{40} (-0.068t + 5.14)dt = -\frac{0.068(40)}{2} + 5.14 = 3.78\text{m/sec}$

**255**. $u = h(x)$

**257**. $f(u) = \frac{(u+1)^2}{\sqrt{u}}$

**259**. $du = 8xdx;\ f(u) = \frac{1}{8\sqrt{u}}$

**261**. $\frac{1}{5}(x+1)^5 + C$

**263**. $-\frac{1}{12(3-2x)^6} + C$

**265**. $\sqrt{x^2+1} + C$

**267**. $\frac{1}{8}\left(x^2 - 2x\right)^4 + C$

**269**. $\sin\theta - \frac{\sin^3\theta}{3} + C$

**271**. $\frac{(1-x)^{101}}{101} - \frac{(1-x)^{100}}{100} + C$

**273**. $\int (11x - 7)^{-2}\,\mathrm{dx} = -\frac{1}{22(11x-7)^2} + C$

**275**. $-\frac{\cos^4\theta}{4} + C$

**277**. $-\frac{\cos^3(\pi t)}{3\pi} + C$

**279**. $-\frac{1}{4}\cos^2\left(t^2\right) + C$

**281**. $-\frac{1}{3(x^3-3)} + C$

**283**. $-\frac{2\left(y^3-2\right)}{3\sqrt{1-y^3}}$

**285**. $\frac{1}{33}\left(1-\cos^3\theta\right)^{11} + C$

**287**. $\frac{1}{12}\left(\sin^3\theta - 3\sin^2\theta\right)^4 + C$

**289**. $L_{50} = -8.5779.$ The exact area is $\frac{-81}{8}$

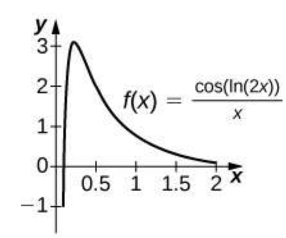
**291**. $L_{50} = -0.006399$ … The exact area is 0.

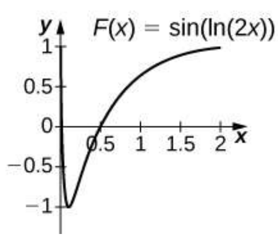
**293**. $u = 1 + x^2,\ du = 2x\,dx,\ \frac{1}{2}\int_1^2 u^{-1/2}\,du = \sqrt{2} - 1$

**295**. $u = 1 + t^3,\ du = 3t^2\,dt,\ \frac{1}{3}\int_1^2 u^{-1/2}\,du = \frac{2}{3}(\sqrt{2} - 1)$

**297**. $u = \cos\theta,\ du = -\sin\theta\,d\theta,\ \int_{1/\sqrt{2}}^{1} u^{-4}\,du = \frac{1}{3}(2\sqrt{2} - 1)$

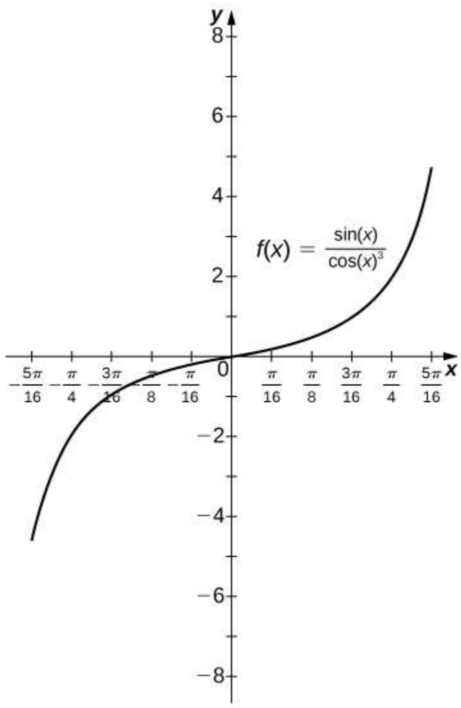
**299**.

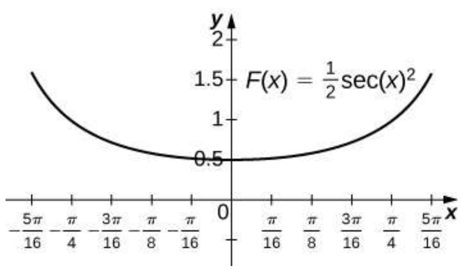



The antiderivative is $y = \sin(\ln(2x))$. Since the antiderivative is not continuous at $x = 0$, one cannot find a value of $C$ that would make $y = \sin(\ln(2x)) - C$ work as a definite integral.

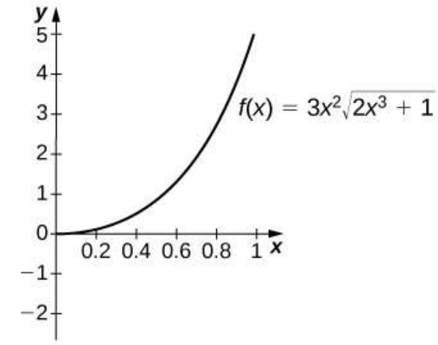
**301**.

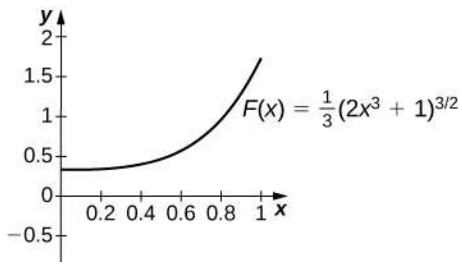



The antiderivative is $y = \frac{1}{2}\sec^2 x$. You should take $C = -2$ so that $F\left(-\frac{\pi}{3}\right) = 0$.

**303**.




The antiderivative is $y = \frac{1}{3}\left(2x^3 + 1\right)^{3/2}$. One should take $C = -\frac{1}{3}$.

**305**. No, because the integrand is discontinuous at $x = 1$.

**307**. $u = \sin\left(t^2\right)$; the integral becomes $\frac{1}{2}\int_0^0 u\,du$.

**309**. $u = \left(1 + \left(t - \frac{1}{2}\right)^2\right)$; the integral becomes $-\int_{5/4}^{5/4} \frac{1}{u}du$.

**311**. $u = 1 - t$; the integral becomes

$$\begin{aligned}&\int_1^{-1} u\cos(\pi(1-u))du\\ &= \int_1^{-1} u[\cos\pi\cos\pi u - \sin\pi\sin\pi u]du\\ &= -\int_1^{-1} u\cos\pi u du\\ &= \int_{-1}^{1} u\cos\pi u du = 0\end{aligned}$$

since the integrand is odd.

**313**. Setting $u = cx$ and $du = cdx$ gets you $\frac{1}{\frac{b}{c} - \frac{a}{c}}\int_{a/c}^{b/c} f(cx)dx = \frac{c}{b-a}\int_{u=a}^{u=b} f(u)\frac{du}{c} = \frac{1}{b-a}\int_a^b f(u)du$.

**315**. $\int_0^x g(t)dt = \frac{1}{2}\int_{u=1-x^2}^{1} \frac{du}{u^a} = \frac{1}{2(1-a)}u^{1-a}\Big|_{u=1-x^2}^{1} = \frac{1}{2(1-a)}\left(1 - \left(1 - x^2\right)^{1-a}\right)$. As $x \to 1$ the limit is $\frac{1}{2(1-a)}$ if $a < 1$, and the limit diverges to $+\infty$ if $a > 1$.

**317**. $\int_{t=\pi}^{0} b\sqrt{1 - \cos^2 t} \times (-a\sin t)dt = \int_{t=0}^{\pi} ab\sin^2 t dt$

**319**. $f(t) = 2\cos(3t) - \cos(2t)$; $\int_0^{\pi/2} (2\cos(3t) - \cos(2t)) = -\frac{2}{3}$

**321**. $\frac{-1}{3}e^{-3x} + C$

**323**. $-\frac{3^{-x}}{\ln 3} + C$

**325**. $\ln\left(x^2\right) + C$

**327**. $2\sqrt{x} + C$

**329**. $-\frac{1}{\ln x} + C$

**331**. $\ln(\ln(\ln x)) + C$

**333**. $\ln(x\cos x) + C$

**335**. $-\frac{1}{2}(\ln(\cos(x)))^2 + C$

**337**. $\frac{-e^{-x^3}}{3} + C$

**339**. $e^{\tan x} + C$

**341**. $t + C$

**343**. $\frac{1}{9}x^3\left(\ln\left(x^3\right) - 1\right) + C$

**345**. $2\sqrt{x}(\ln x - 2) + C$

**347**. $\int_0^{\ln x} e^t\, dt = e^t\Big|_0^{\ln x} = e^{\ln x} - e^0 = x - 1$

**349**. $-\frac{1}{3}\ln(\sin(3x) + \cos(3x))$

**351**. $-\frac{1}{2}\ln\left|\csc\left(x^2\right) + \cot\left(x^2\right)\right| + C$

**353**. $-\frac{1}{2}(\ln(\csc x))^2 + C$

**355**. $\frac{1}{3}\ln\left(\frac{26}{7}\right)$

**357**. $\ln\left(\sqrt{3} - 1\right)$

**359**. $\frac{1}{2}\ln\frac{3}{2}$

**361**. $y - 2\ln|y + 1| + C$

**363**. $\ln|\sin x - \cos x| + C$

**365**. $-\frac{1}{3}\left(1 - \left(\ln x^2\right)\right)^{3/2} + C$

**367**. Exact solution: $\frac{e-1}{e}$, $R_{50} = 0.6258$. Since *f* is decreasing, the right endpoint estimate underestimates the area.

**369**. Exact solution: $\frac{2\ln(3) - \ln(6)}{2}$, $R_{50} = 0.2033$. Since *f* is increasing, the right endpoint estimate overestimates the area.

**371**. Exact solution: $-\frac{1}{\ln(4)}$, $R_{50} = -0.7164$. Since *f* is increasing, the right endpoint estimate overestimates the area (the actual area is a larger negative number).

**373**. $\frac{11}{2}\ln 2$

**375**. $\frac{1}{\ln(65,\ 536)}$

**377**. $\int_N^{N+1} xe^{-x^2}\, dx = \frac{1}{2}\left(e^{-N^2} - e^{-(N+1)^2}\right)$. The quantity is less than 0.01 when $N = 2$.

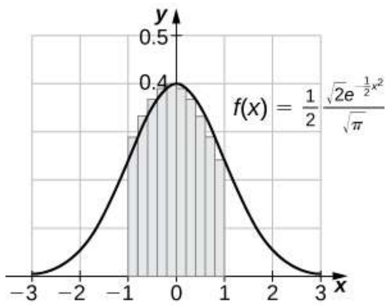
**379**. $\int_a^b \frac{dx}{x} = \ln(b) - \ln(a) = \ln\left(\frac{1}{a}\right) - \ln\left(\frac{1}{b}\right) = \int_{1/b}^{1/a} \frac{dx}{x}$

**381**. 23

**383**. We may assume that $x > 1$, so $\frac{1}{x} < 1$. Then, $\int_1^{1/x} \frac{dt}{t}$. Now make the substitution $u = \frac{1}{t}$, so $du = -\frac{dt}{t^2}$ and $\frac{du}{u} = -\frac{dt}{t}$, and change endpoints: $\int_1^{1/x} \frac{dt}{t} = -\int_1^x \frac{du}{u} = -\ln x$.

**387**. $x = E(\ln(x))$. Then, $1 = \frac{E'(\ln x)}{x}$ or $x = E'(\ln x)$. Since any number $t$ can be written $t = \ln x$ for some $x$, and for such $t$ we have $x = E(t)$, it follows that for any $t$, $E'(t) = E(t)$.

**389**. $R_{10} = 0.6811$, $R_{100} = 0.6827$




**391**. $\sin^{-1} x \Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$

**393**. $\tan^{-1} x \Big|_{\sqrt{3}}^{1} = -\frac{\pi}{12}$

**395**. $\sec^{-1} x \Big|_1^{\sqrt{2}} = \frac{\pi}{4}$

**397**. $\sin^{-1}\left(\frac{x}{3}\right) + C$

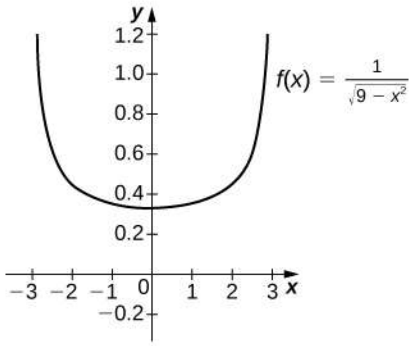
**399**. $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

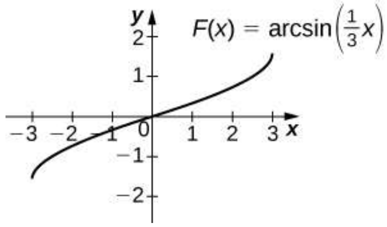
**401**. $\frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right) + C$

**403**. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$. So, $\sin^{-1} t = \frac{\pi}{2} - \cos^{-1} t$. They differ by a constant.

**405**. $\sqrt{1 - t^2}$ is not defined as a real number when $t > 1$.

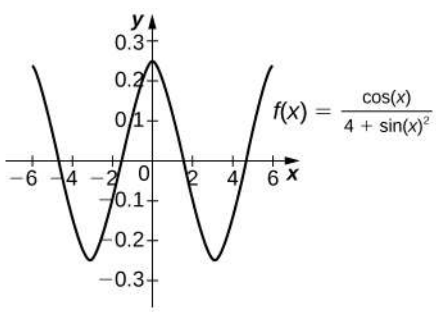
**407**.

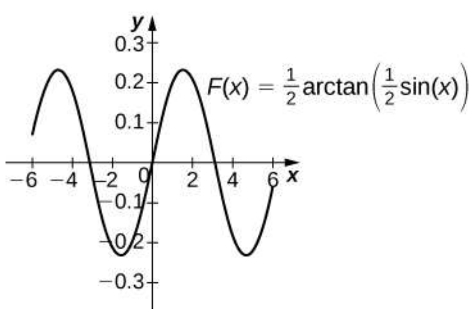



The antiderivative is $\sin^{-1}\left(\frac{x}{3}\right) + C$. Taking $C = \frac{\pi}{2}$ recovers the definite integral.

**409**.



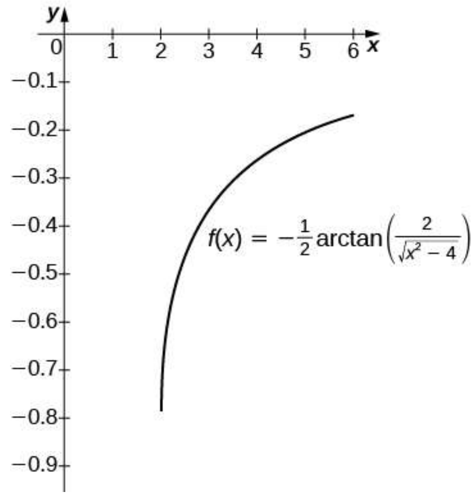

The antiderivative is $\frac{1}{2}\tan^{-1}\left(\frac{\sin x}{2}\right) + C$. Taking $C = \frac{1}{2}\tan^{-1}\left(\frac{\sin(6)}{2}\right)$ recovers the definite integral.

**411**. $\frac{1}{2}\left(\sin^{-1} t\right)^2 + C$

**413**. $\frac{1}{4}\left(\tan^{-1}(2t)\right)^2$

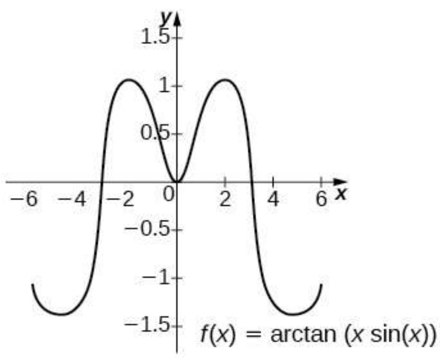
**415**. $\frac{1}{4}\left(\sec^{-1}\left(\frac{t}{2}\right)^2\right) + C$

**417**.



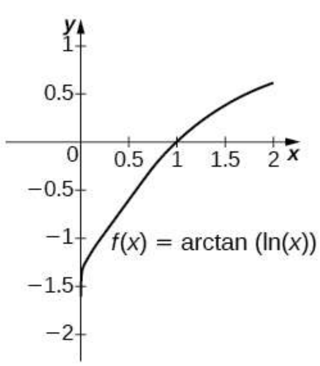

The antiderivative is $\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C$. Taking $C = 0$ recovers the definite integral over $[2, 6]$.

**419**.




The general antiderivative is $\tan^{-1}(x\sin x) + C$. Taking $C = -\tan^{-1}(6\sin(6))$ recovers the definite integral.

**421**.




The general antiderivative is $\tan^{-1}(\ln x) + C$. Taking $C = \frac{\pi}{2} = \tan^{-1}\infty$ recovers the definite integral.

**423**. $\sin^{-1}(e^t) + C$

**425**. $\sin^{-1}(\ln t) + C$

**427**. $-\frac{1}{2}\left(\cos^{-1}(2t)\right)^2 + C$

**429**. $\frac{1}{2}\ln\left(\frac{4}{3}\right)$

**431**. $1-\frac{2}{\sqrt{5}}$

**433**. $2\tan^{-1}(A) \to \pi$ as $A \to \infty$

**435**. Using the hint, one has $\int \frac{\csc^2 x}{\csc^2 x + \cot^2 x}dx = \int \frac{\csc^2 x}{1+2\cot^2 x}dx.$ Set $u = \sqrt{2}\cot x.$ Then, $du = -\sqrt{2}\csc^2 x$ and the integral is $-\frac{1}{\sqrt{2}}\int \frac{du}{1+u^2} = -\frac{1}{\sqrt{2}}\tan^{-1} u + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\cot x\right) + C.$ If one uses the identity $\tan^{-1} s + \tan^{-1}\left(\frac{1}{s}\right) = \frac{\pi}{2},$ then this can also be written $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C.$

**437**. $x \approx \pm 1.7321.$ The left endpoint estimate with $N = 100$ is 4.781 and these decimals persist for $N = 500.$

## Review Exercises

**439**. False
**441**. True
**443**. $L_4 = 5.25,\ R_4 = 3.25,$ exact answer: 4

**445**. $L_4 = 5.364,\ R_4 = 5.364,$ exact answer: 5.870

**447**. $-\frac{4}{3}$

**449**. 1

**451**. $-\frac{1}{2(x+4)^2} + C$

**453**. $\frac{4}{3}\sin^{-1}\left(x^3\right) + C$

**455**. $\frac{\sin t}{\sqrt{1+t^2}}$

**457**. $4\frac{\ln x}{x} + 1$

**459**. $6,328,113
**461**. $73.36
**463**. $\frac{19117}{12}$ft/sec, or 1593 ft/sec

# Chapter 6

## Checkpoint

**6.1**. $12$ units$^2$

**6.2**. $\frac{3}{10}$ unit$^2$

**6.3**. $2 + 2\sqrt{2}$ units$^2$

**6.4**. $\frac{5}{3}$ units$^2$

**6.5**. $\frac{5}{3}$ units$^2$

**6.7**. $\frac{\pi}{2}$

**6.8**. $8\pi$ units$^3$
**6.9**. $21\pi$ units$^3$

**6.10**. $\frac{10\pi}{3}$ units$^3$

**6.11**. $60\pi$ units$^3$

**6.12**. $\frac{15\pi}{2}$ units$^3$

**6.13**. $8\pi$ units$^3$

**6.14**. $12\pi$ units$^3$

**6.15**. $\frac{11\pi}{6}$ units$^3$

**6.16**. $\frac{\pi}{6}$ units$^3$

**6.17**. Use the method of washers; $V = \int_{-1}^{1} \pi\left[\left(2 - x^2\right)^2 - \left(x^2\right)^2\right] dx$

**6.18**. $\frac{1}{6}\left(5\sqrt{5} - 1\right) \approx 1.697$

**6.19**. $\text{Arc Length} \approx 3.8202$

**6.20**. $\text{Arc Length} = 3.15018$

**6.21**. $\frac{\pi}{6}\left(5\sqrt{5} - 3\sqrt{3}\right) \approx 3.133$

**6.22**. $12\pi$

**6.23**. $70/3$

**6.24**. $24\pi$

**6.25**. $8$ ft-lb

**6.26**. Approximately $43{,}255.2$ ft-lb

**6.27**. $156{,}800$ N

**6.28**. Approximately 7,164,520,000 lb or 3,582,260 t

**6.29**. $M = 24,\ \bar{x} = \frac{2}{5}$ m

**6.30**. $(-1,\ -1)$ m

**6.31**. The centroid of the region is $(3/2,\ 6/5)$.

**6.32**. The centroid of the region is $(1,\ 13/5)$.

**6.33**. The centroid of the region is $(0,\ 2/5)$.

**6.34**. $6\pi^2$ units$^3$

**6.35**.

a. $\frac{d}{dx}\ln\left(2x^2 + x\right) = \frac{4x + 1}{2x^2 + x}$

b. $\frac{d}{dx}\left(\ln\left(x^3\right)\right)^2 = \frac{6\ln\left(x^3\right)}{x}$

**6.36**. $\int \frac{x^2}{x^3 + 6} dx = \frac{1}{3}\ln\left|x^3 + 6\right| + C$

**6.37**. $4\ln 2$

**6.38**.

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right) = e^{x^2 - 5x}(2x - 5)$

b. $\frac{d}{dt}\left(e^{2t}\right)^3 = 6e^{6t}$

**6.39**. $\int \frac{4}{e^{3x}} dx = -\frac{4}{3}e^{-3x} + C$

**6.40**.

a. $\frac{d}{dt}4^{t^4} = 4^{t^4}(\ln 4)\left(4t^3\right)$

b. $\frac{d}{dx}\log_3\left(\sqrt{x^2 + 1}\right) = \frac{x}{(\ln 3)\left(x^2 + 1\right)}$

**6.41**. $\int x^2 2^{x^3} dx = \frac{1}{3\ln 2}2^{x^3} + C$

**6.42**. There are $81{,}377{,}396$ bacteria in the population after $4$ hours. The population reaches $100$ million bacteria after

$244.12$ minutes.

**6.43**. At $5\%$ interest, she must invest $\$223{,}130.16$. At $6\%$ interest, she must invest $\$165{,}298.89$.

**6.44**. $38.90$ months

**6.45**. The coffee is first cool enough to serve about $3.5$ minutes after it is poured. The coffee is too cold to serve about $7$ minutes after it is poured.

**6.46**. A total of $94.13$ g of carbon remains. The artifact is approximately $13{,}300$ years old.

**6.47**.

a. $\frac{d}{dx}\left(\tanh\left(x^2+3x\right)\right)=\left(\operatorname{sech}^2\left(x^2+3x\right)\right)(2x+3)$

b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)=\frac{d}{dx}(\sinh x)^{-2}=-2(\sinh x)^{-3}\cosh x$

**6.48**.

a. $\int\sinh^3 x\cosh x\,dx=\frac{\sinh^4 x}{4}+C$

b. $\int\operatorname{sech}^2(3x)dx=\frac{\tanh(3x)}{3}+C$

**6.49**.

a. $\frac{d}{dx}\left(\cosh^{-1}(3x)\right)=\frac{3}{\sqrt{9x^2-1}}$

b. $\frac{d}{dx}\left(\coth^{-1}x\right)^3=\frac{3\left(\coth^{-1}x\right)^2}{1-x^2}$

**6.50**.

a. $\int\frac{1}{\sqrt{x^2-4}}dx=\cosh^{-1}\left(\frac{x}{2}\right)+C$

b. $\int\frac{1}{\sqrt{1-e^{2x}}}dx=-\operatorname{sech}^{-1}(e^x)+C$

**6.51**. $52.95$ ft

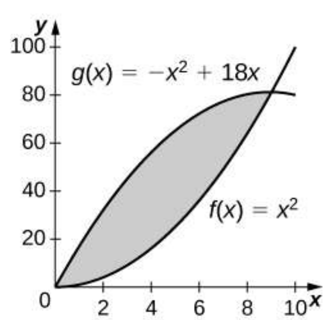
## Section Exercises

**1**. $\frac{32}{3}$

**3**. $\frac{13}{12}$

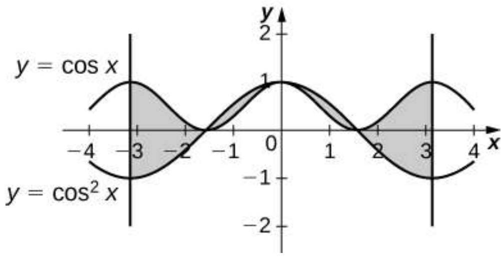
**5**. $36$

**7**.



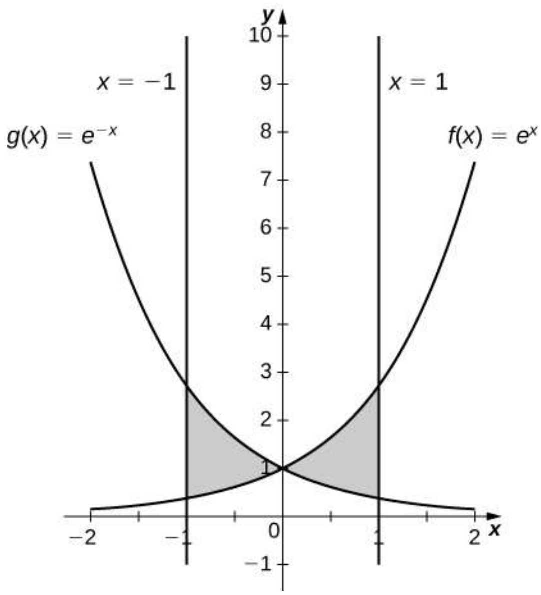

243 square units

**9**.



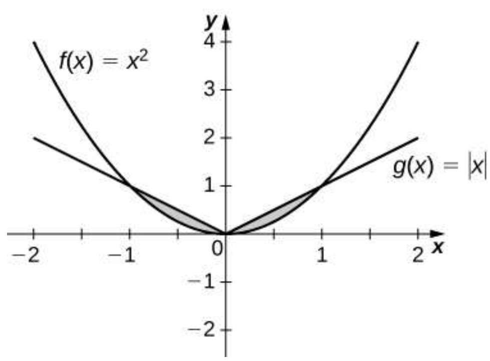

4

**11**.



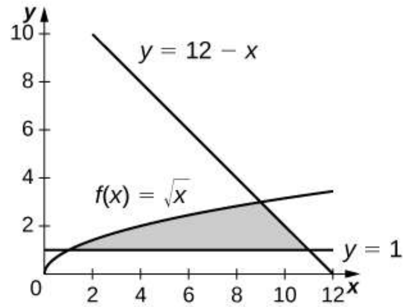

$$\frac{2(e-1)^2}{e}$$

**13**.



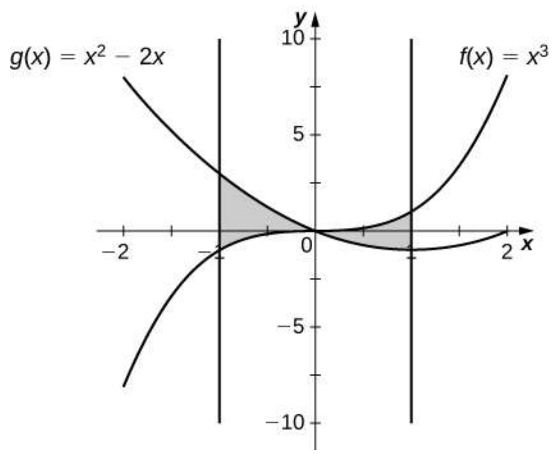

$$\frac{1}{3}$$

**15**.



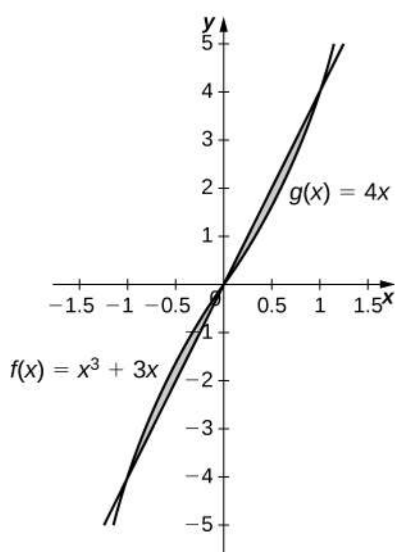

$\frac{34}{3}$

**17**.



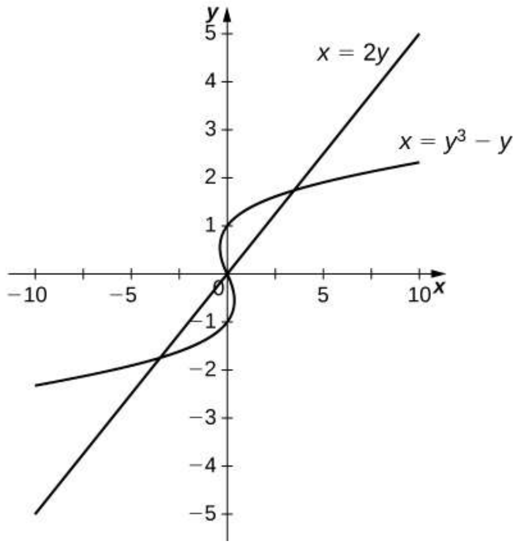

$\frac{5}{2}$

**19**.



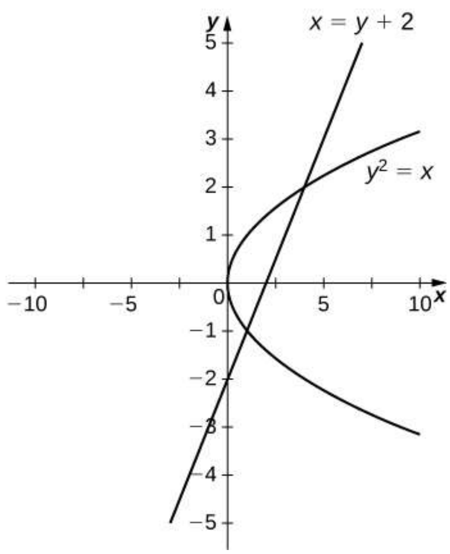

$\frac{1}{2}$

**21**.



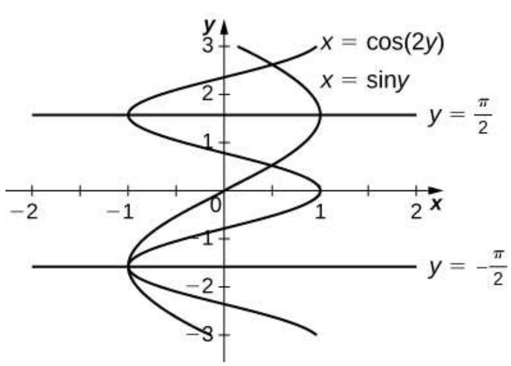

$\frac{9}{2}$

**23**.



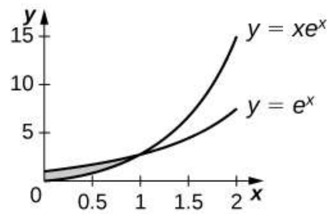

$\frac{9}{2}$

**25**.



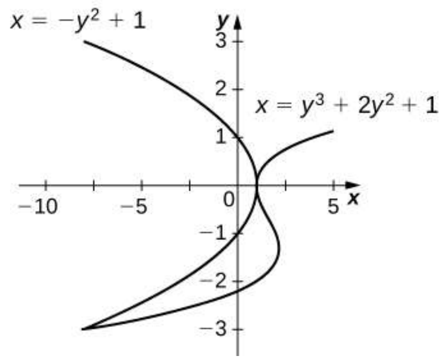
$\frac{3\sqrt{3}}{2}$

**27**.



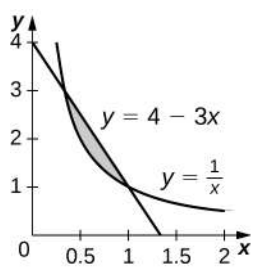

$e-2$

**29**.



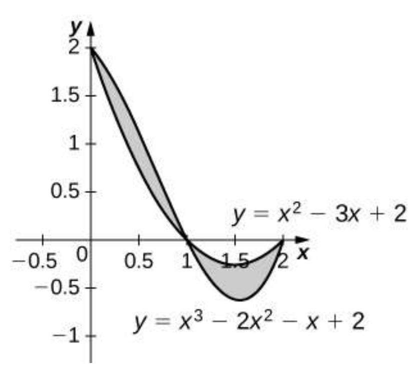

$\frac{27}{4}$

**31**.



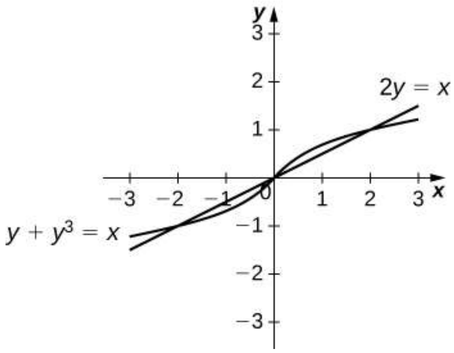

$\frac{4}{3} - \ln(3)$

**33**.



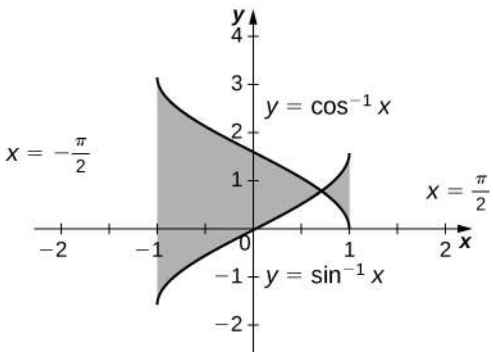

$\frac{1}{2}$

**35**.




$\frac{1}{2}$

**37**.




$-2\left(\sqrt{2}-\pi\right)$

**39**. $1.067$

**41**. $0.852$

**43**. $7.523$

**45**. $\frac{3\pi-4}{12}$

**47**. $1.429$

**49**. $\$33,333.33$ total profit for $200$ cell phones sold

**51**. $3.263$ mi represents how far ahead the hare is from the tortoise

**53**. $\frac{343}{24}$

**55**. $4\sqrt{3}$

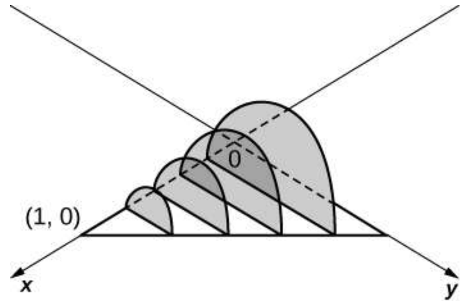
**57**. $\pi-\frac{32}{25}$

**63**. 8 units$^3$

**65**. $\frac{32}{3\sqrt{2}}$ units$^3$

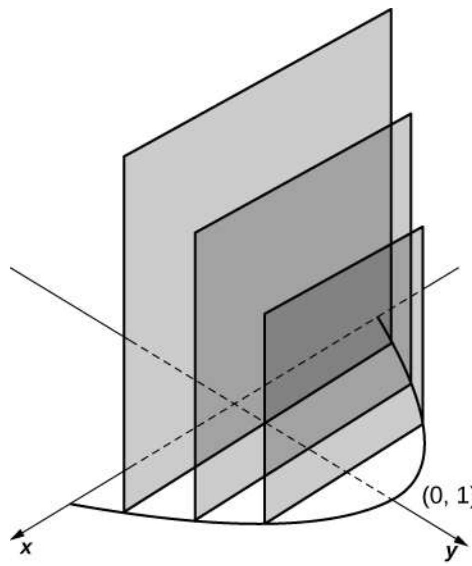
**67**. $\frac{7\pi}{12}hr^2$ units$^3$

**69**.



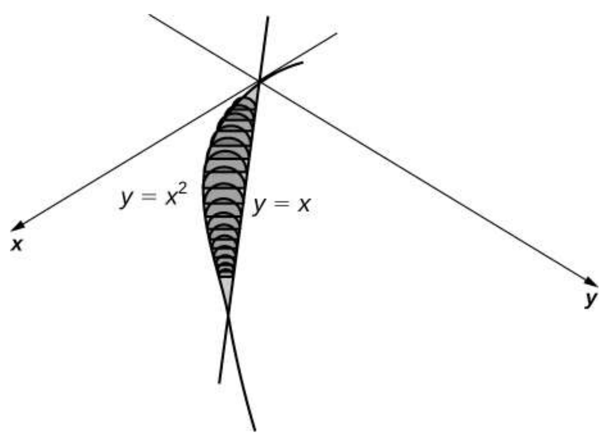

$\frac{\pi}{24}$ units$^3$

**71**.



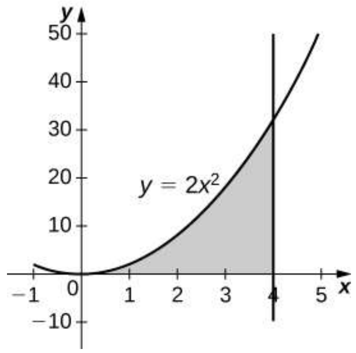

$2$ units$^3$

**73**.



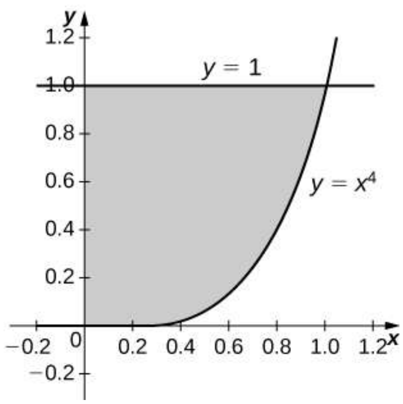

$\frac{\pi}{240}$ units$^3$

**75**.



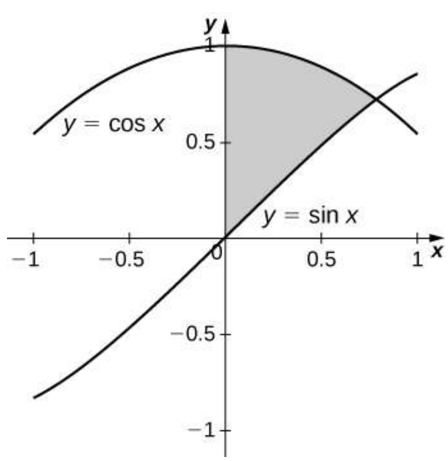

$\frac{4096\pi}{5}$ units$^3$

**77**.



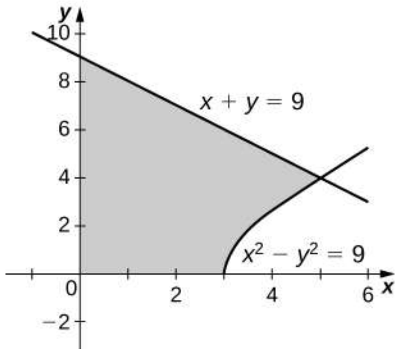

$\frac{8\pi}{9}$ units$^3$

**79**.



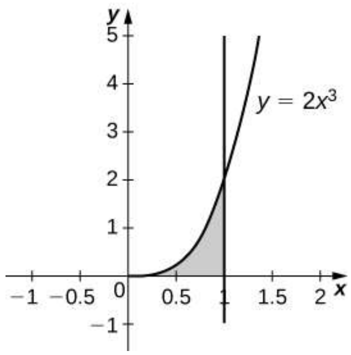

$\frac{\pi}{2}$ units$^3$

**81**.



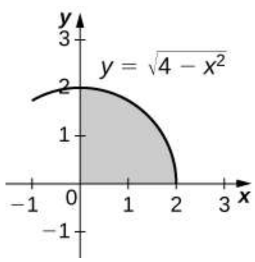

$207\pi$ units$^3$

**83**.



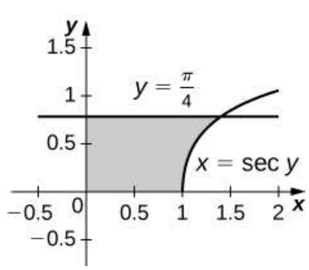

$\frac{4\pi}{5}$ units$^3$

**85**.



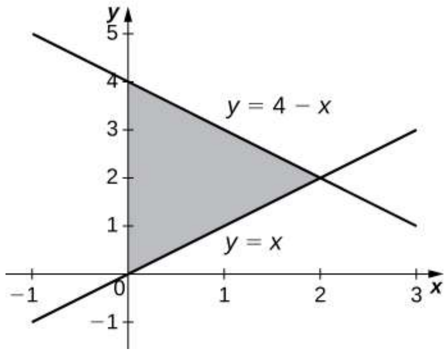

$\frac{16\pi}{3}$ units$^3$

**87**.



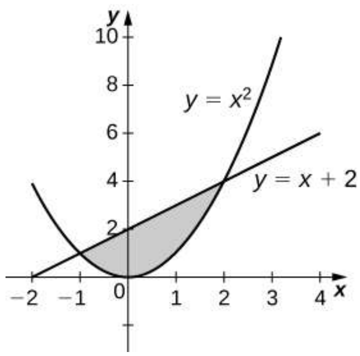

$\pi$ units$^3$

**89**.



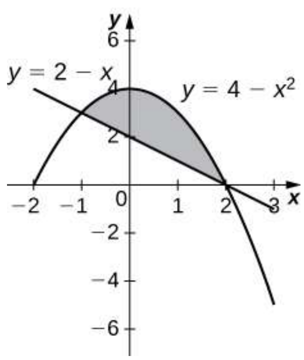

$\frac{16\pi}{3}$ units$^3$

**91**.



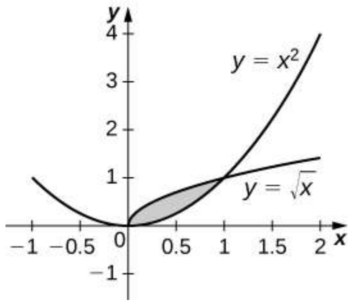

$\frac{72\pi}{5}$ units$^3$

**93**.



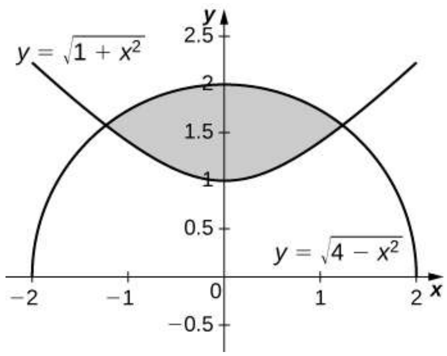

$\frac{108\pi}{5}$ units$^3$

**95**.



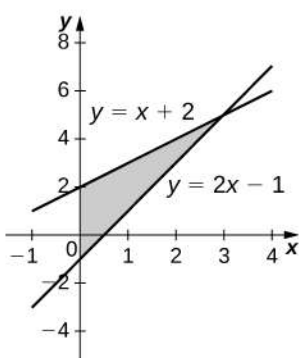

$\frac{3\pi}{10}$ units$^3$

**97**.



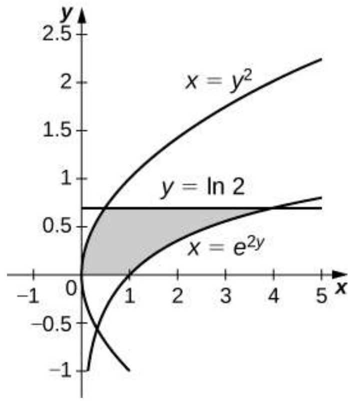

$2\sqrt{6}\pi$ units$^3$

**99**.




$9\pi$ units$^3$

**101**.




$\frac{\pi}{20}\left(75 - 4\ln^{5}(2)\right)$ units$^3$

**103**. $\frac{m^2\pi}{3}\left(b^3 - a^3\right)$ units$^3$

**105**. $\frac{4a^2 b\pi}{3}$ units$^3$

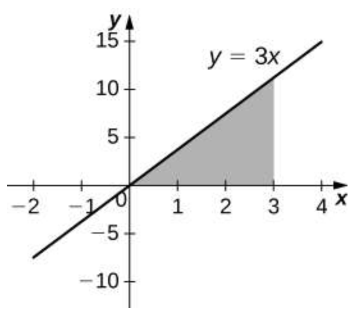
**107**. $2\pi^2$ units$^3$

**109**. $\frac{2ab^2\pi}{3}$ units$^3$

**111**. $\frac{\pi}{12}(r+h)^2(6r-h)$ units$^3$

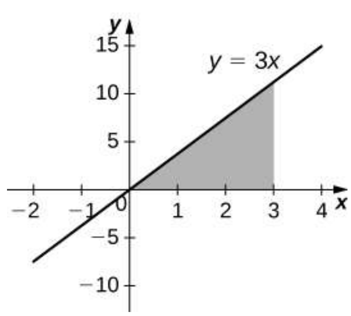
**113**. $\frac{\pi}{3}(h+R)(h-2R)^2$ units$^3$

**115**.



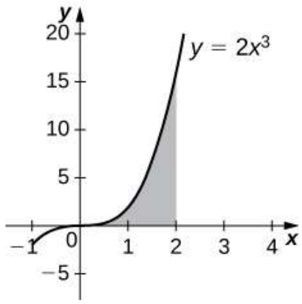

$54\pi$ units$^3$

**117**.




$81\pi$ units$^3$

**119**.




$\frac{512\pi}{7}$ units$^3$

**121**. $2\pi$ units$^3$

**123**. $\frac{2\pi}{3}$ units$^3$

**125**. $2\pi$ units$^3$

**127**. $\frac{4\pi}{5}$ units$^3$

**129**. $\frac{64\pi}{3}$ units$^3$

**131**. $\frac{32\pi}{5}$ units$^3$

**133**. $\frac{7\pi}{6}$

**135**. $48\pi$

**137**. $\frac{97\pi}{5}$

**139**. $\frac{512\pi}{7}$

**141**. $\frac{64\pi}{5}$ units$^3$

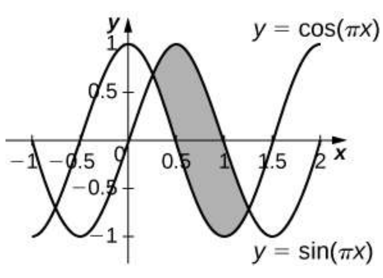
**143**. $\frac{28\pi}{15}$ units$^3$

**145**. $\frac{3\pi}{10}$ units$^3$

**147**. $\frac{52\pi}{5}$ units$^3$

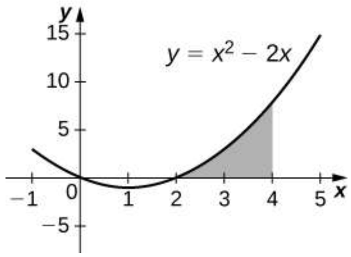
**149**. $0.9876$ units$^3$

**151**.



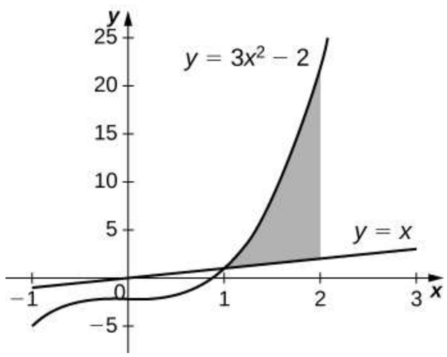

$3\sqrt{2}$ units$^3$

**153**.



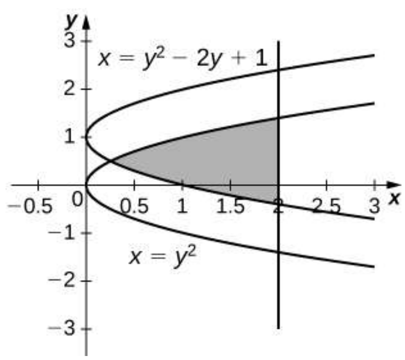

$\frac{496\pi}{15}$ units$^3$

**155**.




$\frac{398\pi}{15}$ units$^3$

**157**.




$15.9074$ units$^3$

**159**. $\frac{1}{3}\pi r^2 h$ units$^3$

**161**. $\pi r^2 h$ units$^3$

**163**. $\pi a^2$ units$^3$

**165**. $2\sqrt{26}$

**167**. $2\sqrt{17}$

**169**. $\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$

**171**. $\frac{13\sqrt{13} - 8}{27}$

**173**. $\frac{4}{3}$

**175**. $2.0035$

**177**. $\frac{123}{32}$

**179**. $10$

**181**. $\frac{20}{3}$

**183**. $\frac{1}{675}\left(229\sqrt{229}-8\right)$

**185**. $\frac{1}{8}\left(4\sqrt{5}+\ln\left(9+4\sqrt{5}\right)\right)$

**187**. $1.201$

**189**. $15.2341$

**191**. $\frac{49\pi}{3}$

**193**. $70\pi\sqrt{2}$

**195**. $8\pi$

**197**. $120\pi\sqrt{26}$

**199**. $\frac{\pi}{6}(17\sqrt{17}-1)$

**201**. $9\sqrt{2}\pi$

**203**. $\frac{10\sqrt{10}\pi}{27}\left(73\sqrt{73}-1\right)$

**205**. $25.645$

**207**. $2\pi$

**209**. $10.5017$

**211**. $23\text{ ft}$

**213**. $2$

**215**. Answers may vary

**217**. For more information, look up Gabriel's Horn.

**219**. $150\text{ ft-lb}$

**221**. $200\,\text{J}$

**223**. $1\text{ J}$

**225**. $\frac{39}{2}$

**227**. $\ln(243)$

**229**. $\frac{332\pi}{15}$

**231**. $100\pi$

**233**. $20\pi\sqrt{15}$

**235**. $6\text{ J}$

**237**. $5\text{ cm}$

**239**. $36\text{ J}$

**241**. $18{,}750\text{ ft-lb}$

**243**. $\frac{32}{3}\times 10^9\text{ ft-lb}$

**245**. $9.71\times 10^2\text{ N m}$

**247**. a. $3{,}000{,}000\text{ lb}$, b. $749{,}000\text{ lb}$

**249**. $23.25\pi\text{ million ft-lb}$

**251**. $\frac{A\rho H^2}{2}$

**253**. Answers may vary

**255**. $\frac{5}{4}$

**257**. $\left(\frac{2}{3}, \frac{2}{3}\right)$

**259**. $\left(\frac{7}{4}, \frac{3}{2}\right)$

**261**. $\frac{3L}{4}$

**263**. $\frac{\pi}{2}$

**265**. $\frac{e^2+1}{e^2-1}$

**267**. $\frac{\pi^2-4}{\pi}$

**269**. $\frac{1}{4}\left(1+e^2\right)$

**271**. $\left(\frac{a}{3}, \frac{b}{3}\right)$

**273**. $\left(0, \frac{\pi}{8}\right)$

**275**. $(0, 3)$

**277**. $\left(0, \frac{4}{\pi}\right)$

**279**. $\left(\frac{5}{8}, \frac{1}{3}\right)$

**281**. $\frac{m\pi}{3}$

**283**. $\pi a^2 b$

**285**. $\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$

**287**. $\left(\frac{1}{2}, \frac{2}{5}\right)$

**289**. $\left(0, \frac{28}{9\pi}\right)$

**291**. Center of mass: $\left(\frac{a}{6}, \frac{4a^2}{5}\right)$, volume: $\frac{2\pi a^4}{9}$

**293**. Volume: $2\pi^2 a^2 (b+a)$

**295**. $\frac{1}{x}$

**297**. $-\frac{1}{x(\ln x)^2}$

**299**. $\ln(x+1)+C$

**301**. $\ln(x)+1$

**303**. $\cot(x)$

**305**. $\frac{7}{x}$

**307**. $\csc(x)\sec x$

**309**. $-2\tan x$

**311**. $\frac{1}{2}\ln\left(\frac{5}{3}\right)$

**313**. $2-\frac{1}{2}\ln(5)$

**315**. $\frac{1}{\ln(2)}-1$

**317**. $\frac{1}{2}\ln(2)$

**319**. $\frac{1}{3}(\ln x)^3$

**321**. $\frac{2x^3}{\sqrt{x^2+1}\sqrt{x^2-1}}$

**323**. $x^{-2-(1/x)}(\ln x - 1)$

**325**. $ex^{e-1}$

**327**. $1$

**329**. $-\frac{1}{x^2}$

**331**. $\pi - \ln(2)$

**333**. $\frac{1}{x}$

**335**. $e^5 - 6 \text{ units}^2$

**337**. $\ln(4) - 1 \text{ units}^2$

**339**. $2.8656$

**341**. $3.1502$

**349**. True

**351**. False; $k = \frac{\ln(2)}{t}$

**353**. $20$ hours

**355**. No. The relic is approximately $871$ years old.

**357**. $71.92$ years

**359**. $5$ days $6$ hours $27$ minutes

**361**. $12$

**363**. $8.618\%$

**365**. $\$6766.76$

**367**. $9$ hours $13$ minutes

**369**. $239,179$ years

**371**. $P'(t) = 43e^{0.01604t}$. The population is always increasing.

**373**. The population reaches $10$ billion people in $2027$.

**375**. $P'(t) = 2.259e^{0.06407t}$. The population is always increasing.

**377**. $e^x$ and $e^{-x}$

**379**. Answers may vary

**381**. Answers may vary

**383**. Answers may vary

**385**. $3\sinh(3x+1)$

**387**. $-\tanh(x)\text{sech}(x)$

**389**. $4\cosh(x)\sinh(x)$

**391**. $\frac{x\,\text{sech}^2\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$

**393**. $6\sinh^5(x)\cosh(x)$

**395**. $\frac{1}{2}\sinh(2x+1)+C$

**397**. $\frac{1}{2}\sinh^2\left(x^2\right)+C$

**399**. $\frac{1}{3}\cosh^3(x)+C$

**401**. $\ln(1+\cosh(x))+C$

**403**. $\cosh(x)+\sinh(x)+C$

**405**. $\frac{4}{1 - 16x^2}$

**407**. $\frac{\sinh(x)}{\sqrt{\cosh^2(x) + 1}}$

**409**. $-\csc(x)$

**411**. $-\frac{1}{\left(x^2 - 1\right)\tanh^{-1}(x)}$

**413**. $\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C$

**415**. $\sqrt{x^2 + 1} + C$

**417**. $\cosh^{-1}(e^x) + C$

**419**. Answers may vary

**421**. $37.30$

**423**. $y = \frac{1}{c}\cosh(cx)$

**425**. $-0.521095$

**427**. $10$

## Review Exercises

**435**. False

**437**. False

**439**. $32\sqrt{3}$

**441**. $\frac{162\pi}{5}$

**443**. a. $4$, b. $\frac{128\pi}{7}$, c. $\frac{64\pi}{5}$

**445**. a. $1.949$, b. $21.952$, c. $17.099$

**447**. a. $\frac{31}{6}$, b. $\frac{452\pi}{15}$, c. $\frac{31\pi}{6}$

**449**. $245.282$

**451**. Mass: $\frac{1}{2}$, center of mass: $\left(\frac{18}{35}, \frac{9}{11}\right)$

**453**. $\sqrt{17} + \frac{1}{8}\ln(33 + 8\sqrt{17})$

**455**. Volume: $\frac{3\pi}{4}$, surface area: $\pi\left(\sqrt{2} - \sinh^{-1}(1) + \sinh^{-1}(16) - \frac{\sqrt{257}}{16}\right)$

**457**. 11:02 a.m.

**459**. $\pi(1 + \sinh(1)\cosh(1))$

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