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Calc-ulus

Volume 2

2 | APPLICATIONS OF INTEGRATION

Figure 2.1 Hoover Dam is one of the United States' iconic landmarks, and provides irrigation and hydroelectric power for millions of people in the southwest United States. (credit: modification of work by Lynn Betts, Wikimedia)

Chapter Outline

Introduction

The Hoover Dam is an engineering marvel. When Lake Mead, the reservoir behind the dam, is full, the dam withstands a great deal of force. However, water levels in the lake vary considerably as a result of droughts and varying water demands. Later in this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force (see **Example 2.28**).

Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

2.1 | Areas between Curves

Learning Objectives

2.1.1 Determine the area of a region between two curves by integrating with respect to the independent variable.

2.1.2 Find the area of a compound region.

2.1.3 Determine the area of a region between two curves by integrating with respect to the dependent variable.

In **Introduction to Integration**, we developed the concept of the definite integral to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of x, beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of y.

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in the following figure.

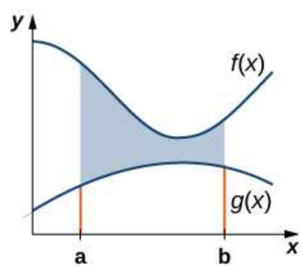


Figure 2.2 The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$.

As we did before, we are going to partition the interval on the x-axis and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2,\ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2,\ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. **Figure 2.3**(a) shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$. **Figure 2.3**(b) shows a representative rectangle in detail.

Use this **calculator (http://www.openstaxcollege.org/l/20_CurveCalc)** to learn more about the areas between two curves.

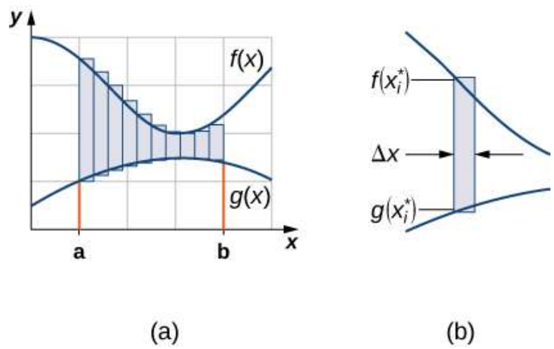


Figure 2.3 (a)We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*)\right]\Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*)\right]\Delta x = \int_a^b [f(x) - g(x)]dx.$$

These findings are summarized in the following theorem.

Theorem 2.1: Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b [f(x) - g(x)]dx. \quad (2.1)$$

We apply this theorem in the following example.

Example 2.1

Finding the Area of a Region between Two Curves 1

If R is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region R.

Solution

The region is depicted in the following figure.

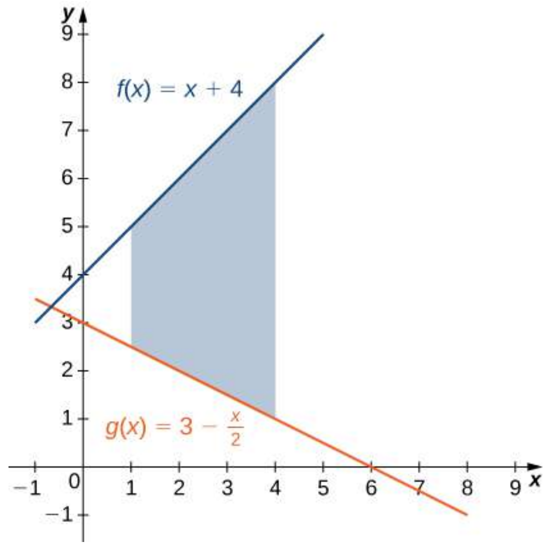


Figure 2.4 A region between two curves is shown where one curve is always greater than the other.

We have

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx \\ &= \int_1^4 \left[(x+4) - \left(3 - \frac{x}{2}\right)\right]dx = \int_1^4 \left[\frac{3x}{2} + 1\right]dx \\ &= \left[\frac{3x^2}{4} + x\right]\Big|_1^4 = \left(16 - \frac{7}{4}\right) = \frac{57}{4}. \end{aligned}$$

The area of the region is $\frac{57}{4}$ units^2.

2.1 If R is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region R.

In **Example 2.1**, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

Example 2.2

Finding the Area of a Region between Two Curves 2

If R is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R.

Solution

The region is depicted in the following figure.

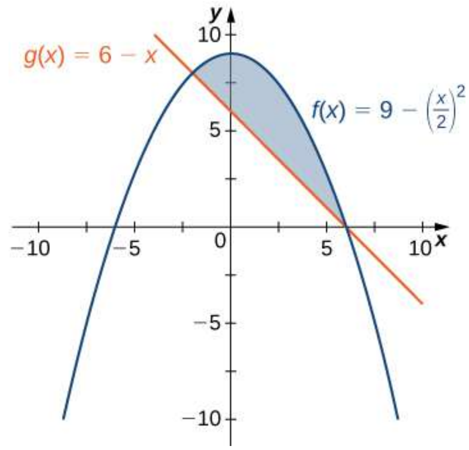


Figure 2.5 This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x),$ we get

$$\begin{aligned} f(x) &= g(x) \\ 9 - \left(\frac{x}{2}\right)^2 &= 6 - x \\ 9 - \frac{x^2}{4} &= 6 - x \\ 36 - x^2 &= 24 - 4x \\ x^2 - 4x - 12 &= 0 \\ (x - 6)(x + 2) &= 0. \end{aligned}$$

The graphs of the functions intersect when $x = 6$ or $x = -2,$ so we want to integrate from -2 to $6.$ Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6,$ we obtain

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^{6} \left[9 - \left(\frac{x}{2}\right)^2 - (6 - x)\right]dx = \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right]dx \\ &= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Bigg|_{-2}^{6} = \frac{64}{3}. \end{aligned}$$

The area of the region is $64/3$ units^2.

2.2 If R is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4,$ find the area of region $R.$

Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

Theorem 2.2: Finding the Area of a Region between Curves That Cross

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let R denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b |f(x) - g(x)| dx.$$

In practice, applying this theorem requires us to break up the interval $[a, b]$ and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

Example 2.3

Finding the Area of a Region Bounded by Functions That Cross

If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region R.

Solution

The region is depicted in the following figure.

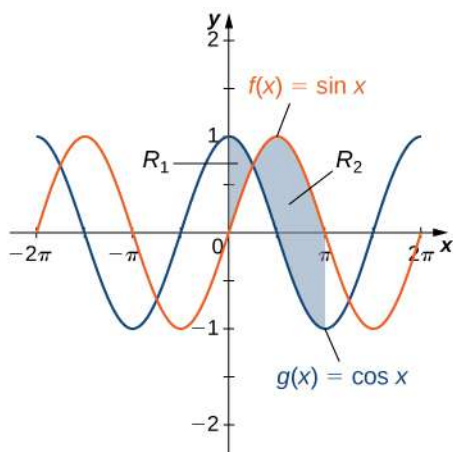


Figure 2.6 The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4]$, $\cos x \geq \sin x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x.$$

On the other hand, for $x \in [\pi/4, \pi]$, $\sin x \geq \cos x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then

$$\begin{aligned} A &= \int_a^b |f(x) - g(x)| dx \\ &= \int_0^\pi |\sin x - \cos x| dx = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\ &= [\sin x + \cos x] \Big|_0^{\pi/4} + [-\cos x - \sin x] \Big|_{\pi/4}^{\pi} \\ &= (\sqrt{2} - 1) + \left(1 + \sqrt{2}\right) = 2\sqrt{2}. \end{aligned}$$

The area of the region is $2\sqrt{2}$ units^2.

2.3 If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[\pi/2, 2\pi]$, find the area of region R.

Example 2.4

Finding the Area of a Complex Region

Consider the region depicted in **Figure 2.7**. Find the area of R.

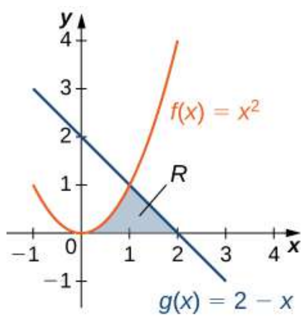


Figure 2.7 Two integrals are required to calculate the area of this region.

Solution

As with **Example 2.3**, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for *x*), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the *x*-axis, so we have

$$A_1 = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the x-axis, so we have

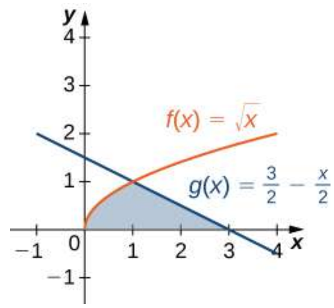
$$A_2 = \int_1^2 (2 - x)dx = \left[2x - \frac{x^2}{2}\right]\Big|_1^2 = \frac{1}{2}.$$

Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $5/6$ units^2.

2.4 Consider the region depicted in the following figure. Find the area of R.



Regions Defined with Respect to *y*

In **Example 2.4**, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y, instead of as functions of x? Review **Figure 2.7**. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of y. However, based on the graph, it is clear we are interested in the positive square root.) Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of y, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y, we need to evaluate one integral only. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in the following figure.

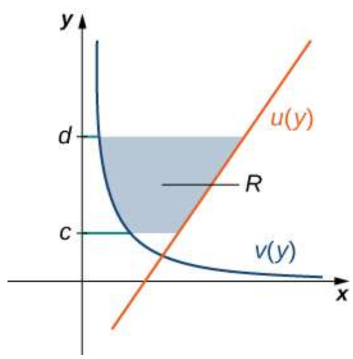


Figure 2.8 We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the y-axis and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2,\ldots, n,$ let $Q = \{y_i\}$ be a regular partition of $[c, d]$. Then, for $i = 1, 2,\ldots, n,$ choose a point $y_i^* \in [y_{i-1}, y_i],$ then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y_i^*)$ to $u(y_i^*)$. **Figure 2.9**(a) shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10$. **Figure 2.9**(b) shows a representative rectangle in detail.

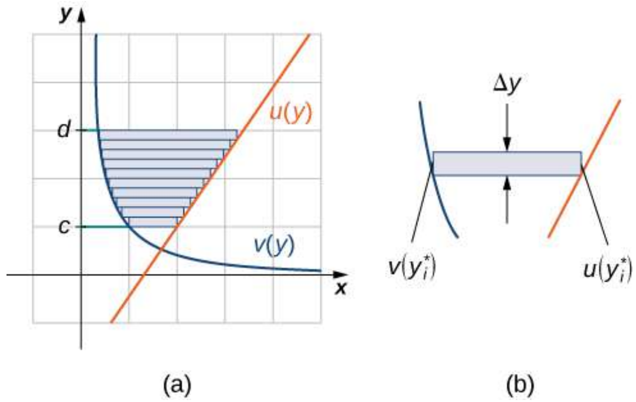


Figure 2.9 (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y)$, with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is Δy and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y.$$

This is a Riemann sum, so we take the limit as $n \to \infty,$ obtaining

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y = \int_c^d [u(y) - v(y)]dy.$$

These findings are summarized in the following theorem.

Theorem 2.3: Finding the Area between Two Curves, Integrating along the *y*-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let R denote the region bounded on the right by the graph of $u(y),$ on the left by the graph of $v(y),$ and above and below by the lines $y = d$ and $y = c,$ respectively. Then, the area of R is given by

$$A = \int_c^d [u(y) - v(y)]dy. \quad (2.2)$$

Example 2.5

Integrating with Respect to *y*

Let's revisit **Example 2.4**, only this time let's integrate with respect to y. Let R be the region depicted in **Figure 2.10**. Find the area of R by integrating with respect to y.

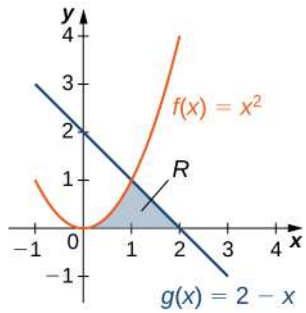


Figure 2.10 The area of region R can be calculated using one integral only when the curves are treated as functions of y.

Solution

We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

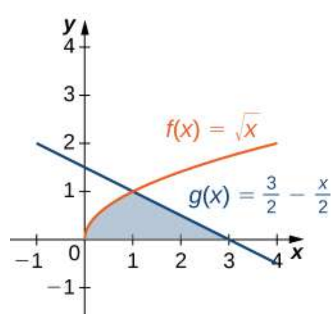
Now we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.

Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y)]dy \\ &= \int_0^1 [(2 - y) - \sqrt{y}]dy = \left[2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\right]\Bigg|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

The area of the region is $5/6$ units^2.

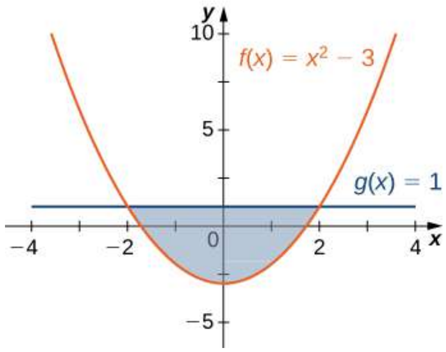
2.5 Let's revisit the checkpoint associated with **Example 2.4**, only this time, let's integrate with respect to y. Let be the region depicted in the following figure. Find the area of R by integrating with respect to y.



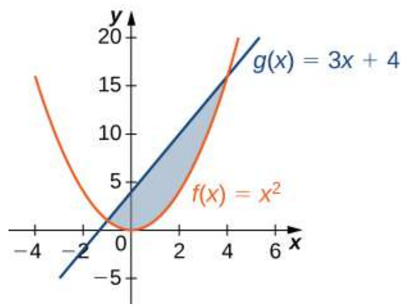
2.1 EXERCISES

For the following exercises, determine the area of the region between the two curves in the given figure by integrating over the x-axis.

1. $y = x^2 - 3$ and $y = 1$

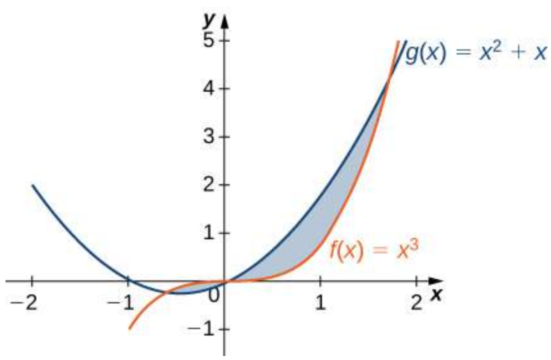


2. $y = x^2$ and $y = 3x + 4$

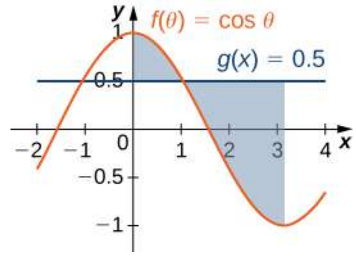


For the following exercises, split the region between the two curves into two smaller regions, then determine the area by integrating over the x-axis. Note that you will have two integrals to solve.

3. $y = x^3$ and $y = x^2 + x$

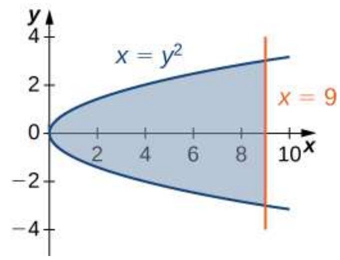


4. $y = \cos\theta$ and $y = 0.5$, for $0 \leq \theta \leq \pi$

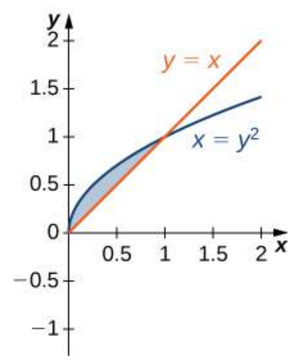


For the following exercises, determine the area of the region between the two curves by integrating over the y-axis.

5. $x = y^2$ and $x = 9$



6. $y = x$ and $x = y^2$



For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis.

7. $y = x^2$ and $y = -x^2 + 18x$

8. $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and $x = 3$

9. $y = \cos x$ and $y = \cos^2 x$ on $x = [-\pi, \pi]$

10. $y = e^x$, $y = e^{2x-1}$, and $x = 0$

11. $y = e^x$, $y = e^{-x}$, $x = -1$ and $x = 1$

12. $y = e,\ y = e^x,\ \text{and } y = e^{-x}$

13. $y = |x|$ and $y = x^2$

For the following exercises, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

14. $y = \sin(\pi x),\ y = 2x,\ \text{and } x > 0$

15. $y = 12 - x,\ y = \sqrt{x},\ \text{and } y = 1$

16. $y = \sin x$ and $y = \cos x$ over $x = [-\pi, \pi]$

17. $y = x^3$ and $y = x^2 - 2x$ over $x = [-1, 1]$

18. $y = x^2 + 9$ and $y = 10 + 2x$ over $x = [-1, 3]$

19. $y = x^3 + 3x$ and $y = 4x$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the y-axis.

20. $x = y^3$ and $x = 3y - 2$

21. $x = 2y$ and $x = y^3 - y$

22. $x = -3 + y^2$ and $x = y - y^2$

23. $y^2 = x$ and $x = y + 2$

24. $x = |y|$ and $2x = -y^2 + 2$

25. $x = \sin y,\ x = \cos(2y),\ y = \pi/2,\ \text{and } y = -\pi/2$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis or y-axis, whichever seems more convenient.

26. $x = y^4$ and $x = y^5$

27. $y = xe^x,\ y = e^x,\ x = 0,\ \text{and } x = 1$

28. $y = x^6$ and $y = x^4$

29. $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$

30. $y = |x|$ and $y = x^2 - 1$

31. $y = 4 - 3x$ and $y = \frac{1}{x}$

32. $y = \sin x,\ x = -\pi/6,\ x = \pi/6,\ \text{and } y = \cos^3 x$

33. $y = x^2 - 3x + 2$ and $y = x^3 - 2x^2 - x + 2$

34. $y = 2\cos^3(3x),\ y = -1,\ x = \frac{\pi}{4},\ \text{and } x = -\frac{\pi}{4}$

35. $y + y^3 = x$ and $2y = x$

36. $y = \sqrt{1 - x^2}$ and $y = x^2 - 1$

37. $y = \cos^{-1} x,\ y = \sin^{-1} x,\ x = -1,\ \text{and } x = 1$

For the following exercises, find the exact area of the region bounded by the given equations if possible. If you are unable to determine the intersection points analytically, use a calculator to approximate the intersection points with three decimal places and determine the approximate area of the region.

38. **[T]** $x = e^y$ and $y = x - 2$

39. **[T]** $y = x^2$ and $y = \sqrt{1 - x^2}$

40. **[T]** $y = 3x^2 + 8x + 9$ and $3y = x + 24$

41. **[T]** $x = \sqrt{4 - y^2}$ and $y^2 = 1 + x^2$

42. **[T]** $x^2 = y^3$ and $x = 3y$

43. **[T]** $y = \sin^3 x + 2,\ y = \tan x,\ x = -1.5,\ \text{and } x = 1.5$

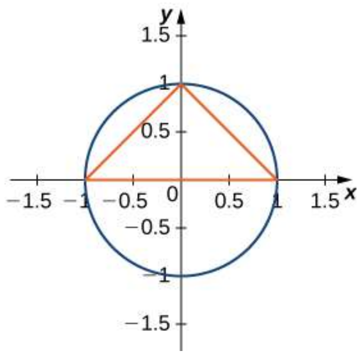
44. **[T]** $y = \sqrt{1 - x^2}$ and $y^2 = x^2$

45. **[T]** $y = \sqrt{1 - x^2}$ and $y = x^2 + 2x + 1$

46. **[T]** $x = 4 - y^2$ and $x = 1 + 3y + y^2$

47. **[T]** $y = \cos x,\ y = e^x,\ x = -\pi,\ \text{and } x = 0$

48. The largest triangle with a base on the x-axis that fits inside the upper half of the unit circle $y^2 + x^2 = 1$ is given by $y = 1 + x$ and $y = 1 - x$. See the following figure. What is the area inside the semicircle but outside the triangle?



49. A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

50. An amusement park has a marginal cost function $C(x) = 1000e^{-x} + 5$, where x represents the number of tickets sold, and a marginal revenue function given by $R(x) = 60 - 0.1x$. Find the total profit generated when selling 550 tickets. Use a calculator to determine intersection points, if necessary, to two decimal places.

51. The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = 1 - \cos((\pi t)/2)$ whereas the speed of the tortoise is $T(t) = (1/2)\tan^{-1}(t/4)$, where t is time measured in hours and the speed is measured in miles per hour. Find the area between the curves from time $t = 0$ to the first time after one hour when the tortoise and hare are traveling at the same speed. What does it represent? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

52. The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = (1/2) - (1/2)\cos(2\pi t)$ whereas the speed of the tortoise is $T(t) = \sqrt{t}$, where t is time measured in hours and speed is measured in kilometers per hour. If the race is over in 1 hour, who won the race and by how much? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

For the following exercises, find the area between the curves by integrating with respect to x and then with respect to y. Is one method easier than the other? Do you obtain the same answer?

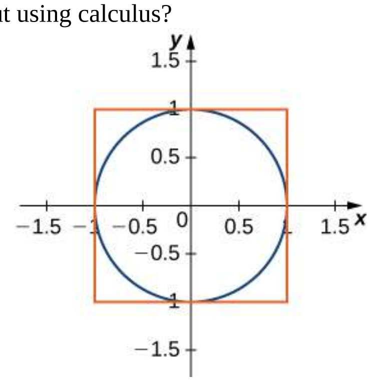
53. $y = x^2 + 2x + 1$ and $y = -x^2 - 3x + 4$

54. $y = x^4$ and $x = y^5$

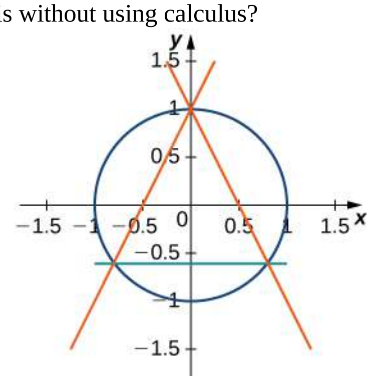
55. $x = y^2 - 2$ and $x = 2y$

For the following exercises, solve using calculus, then check your answer with geometry.

56. Determine the equations for the sides of the square that touches the unit circle on all four sides, as seen in the following figure. Find the area between the perimeter of this square and the unit circle. Is there another way to solve this without using calculus?



57. Find the area between the perimeter of the unit circle and the triangle created from $y = 2x + 1$, $y = 1 - 2x$ and $y = -\frac{3}{5}$, as seen in the following figure. Is there a way to solve this without using calculus?



2.2 | Determining Volumes by Slicing

Learning Objectives

- **2.2.1** Determine the volume of a solid by integrating a cross-section (the slicing method).
- **2.2.2** Find the volume of a solid of revolution using the disk method.
- **2.2.3** Find the volume of a solid of revolution with a cavity using the washer method.

In the preceding section, we used definite integrals to find the area between two curves. In this section, we use definite integrals to find volumes of three-dimensional solids. We consider three approaches—slicing, disks, and washers—for finding these volumes, depending on the characteristics of the solid.

Volume and the Slicing Method

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas. The volume of a rectangular solid, for example, can be computed by multiplying length, width, and height: $V = lwh.$ The formulas for the volume of a sphere $\left(V = \frac{4}{3}\pi r^3\right),$ a cone $\left(V = \frac{1}{3}\pi r^2 h\right),$ and a pyramid $\left(V = \frac{1}{3}Ah\right)$ have also been introduced. Although some of these formulas were derived using geometry alone, all these formulas can be obtained by using integration.

We can also calculate the volume of a cylinder. Although most of us think of a cylinder as having a circular base, such as a soup can or a metal rod, in mathematics the word *cylinder* has a more general meaning. To discuss cylinders in this more general context, we first need to define some vocabulary.

We define the **cross-section** of a solid to be the intersection of a plane with the solid. A *cylinder* is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the *axis* of the cylinder. Thus, all cross-sections perpendicular to the axis of a cylinder are identical. The solid shown in **Figure 2.11** is an example of a cylinder with a noncircular base. To calculate the volume of a cylinder, then, we simply multiply the area of the cross-section by the height of the cylinder: $V = A \cdot h.$ In the case of a right circular cylinder (soup can), this becomes $V = \pi r^2 h.$

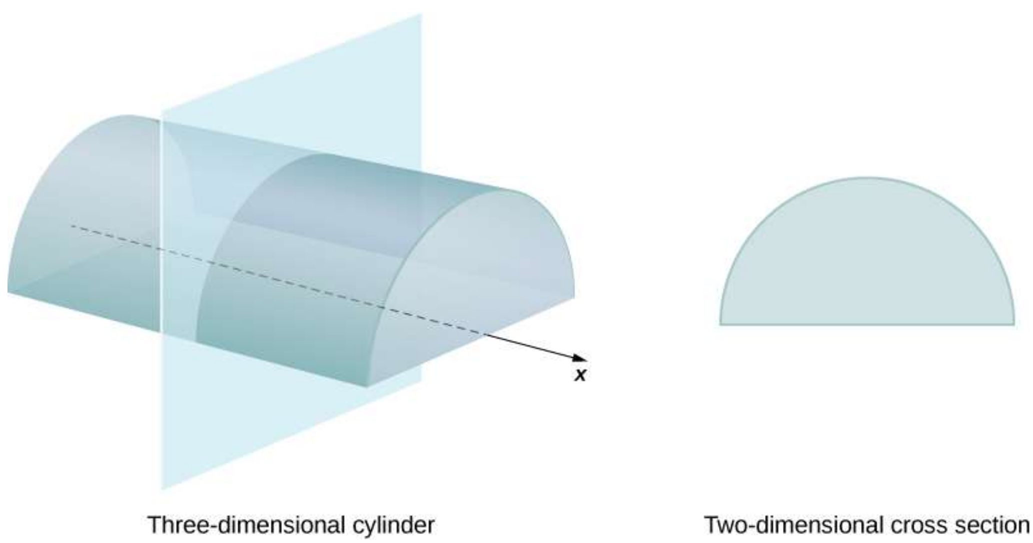


Figure 2.11 Each cross-section of a particular cylinder is identical to the others.

If a solid does not have a constant cross-section (and it is not one of the other basic solids), we may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid. We do this by slicing the solid into pieces, estimating the volume of each slice, and then adding those estimated volumes together. The slices should all be parallel to one another, and when we put all the slices together, we should get the whole solid. Consider, for example, the solid S shown in **Figure 2.12**, extending along the x-axis.

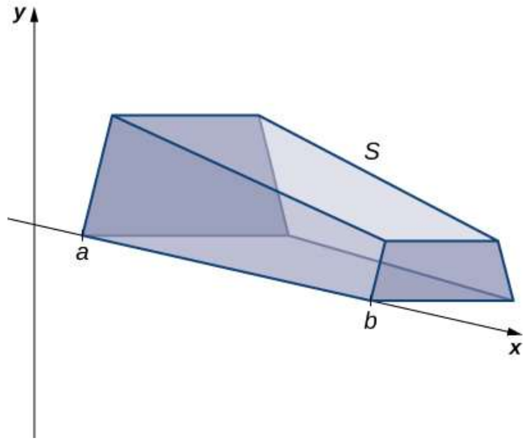


Figure 2.12 A solid with a varying cross-section.

We want to divide S into slices perpendicular to the x-axis. As we see later in the chapter, there may be times when we want to slice the solid in some other direction—say, with slices perpendicular to the y-axis. The decision of which way to slice the solid is very important. If we make the wrong choice, the computations can get quite messy. Later in the chapter, we examine some of these situations in detail and look at how to decide which way to slice the solid. For the purposes of this section, however, we use slices perpendicular to the x-axis.

Because the cross-sectional area is not constant, we let $A(x)$ represent the area of the cross-section at point x. Now let $P = \{x_0, x_1 \ldots, X_n\}$ be a regular partition of $[a, b]$, and for $i = 1, 2, \ldots n$, let S_i represent the slice of S stretching from x_{i-1} to x_i. The following figure shows the sliced solid with $n = 3$.

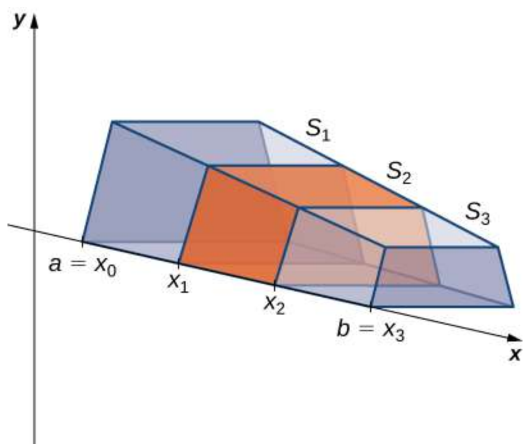


Figure 2.13 The solid S has been divided into three slices perpendicular to the x-axis.

Finally, for $i = 1, 2, \ldots n$, let x_i^* be an arbitrary point in $[x_{i-1}, x_i]$. Then the volume of slice S_i can be estimated by $V(S_i) \approx A(x_i^*)\Delta x$. Adding these approximations together, we see the volume of the entire solid S can be approximated by

$$V(S) \approx \sum_{i=1}^{n} A(x_i^*)\Delta x.$$

By now, we can recognize this as a Riemann sum, and our next step is to take the limit as $n \to \infty$. Then we have

$$V(S) = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx.$$

The technique we have just described is called the **slicing method**. To apply it, we use the following strategy.

Problem-Solving Strategy: Finding Volumes by the Slicing Method

1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
2. Determine a formula for the area of the cross-section.
3. Integrate the area formula over the appropriate interval to get the volume.

Recall that in this section, we assume the slices are perpendicular to the x-axis. Therefore, the area formula is in terms of x and the limits of integration lie on the x-axis. However, the problem-solving strategy shown here is valid regardless of how we choose to slice the solid.

Example 2.6

Deriving the Formula for the Volume of a Pyramid

We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah.$ If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2 h,$ where a denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

Solution

We want to apply the slicing method to a pyramid with a square base. To set up the integral, consider the pyramid shown in **Figure 2.14**, oriented along the x-axis.

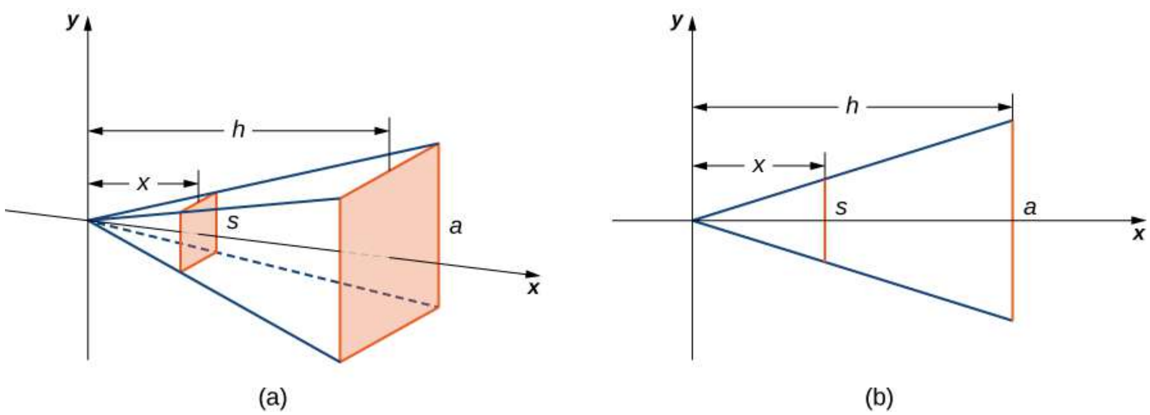


Figure 2.14 (a) A pyramid with a square base is oriented along the x-axis. (b) A two-dimensional view of the pyramid is seen from the side.

We first want to determine the shape of a cross-section of the pyramid. We are know the base is a square, so the cross-sections are squares as well (step 1). Now we want to determine a formula for the area of one of these cross-sectional squares. Looking at **Figure 2.14**(b), and using a proportion, since these are similar triangles, we have

$$\frac{s}{a} = \frac{x}{h} \text{ or } s = \frac{ax}{h}.$$

Therefore, the area of one of the cross-sectional squares is

$$A(x) = s^2 = \left(\frac{ax}{h}\right)^2 \text{(step 2)}.$$

Then we find the volume of the pyramid by integrating from 0 to h (step 3):

$$\begin{aligned} V &= \int_0^h A(x)dx \\ &= \int_0^h \left(\frac{ax}{h}\right)^2 dx = \frac{a^2}{h^2}\int_0^h x^2 dx \\ &= \left[\frac{a^2}{h^2}\left(\frac{1}{3}x^3\right)\right]\Bigg|_0^h = \frac{1}{3}a^2 h. \end{aligned}$$

This is the formula we were looking for.

2.6 Use the slicing method to derive the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a circular cone.

Solids of Revolution

If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**, as shown in the following figure.

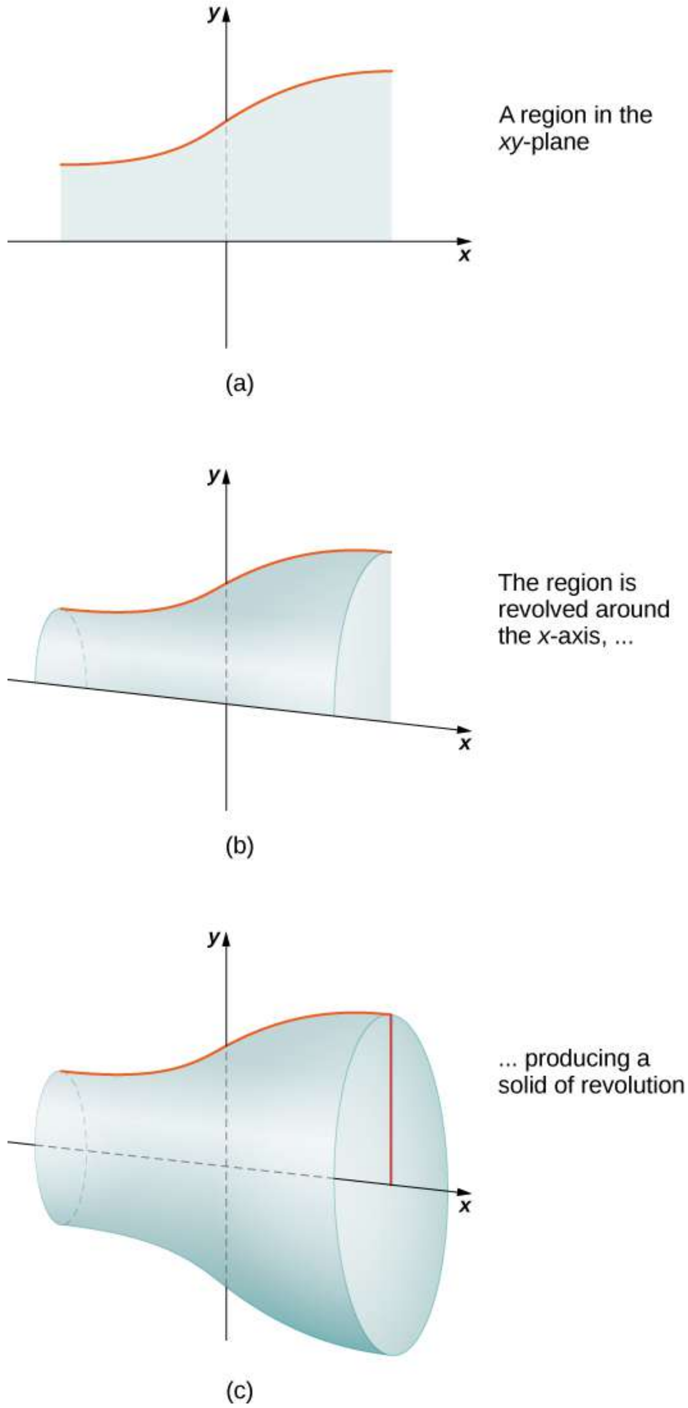


Figure 2.15 (a) This is the region that is revolved around the *x*-axis. (b) As the region begins to revolve around the axis, it sweeps out a solid of revolution. (c) This is the solid that results when the revolution is complete.

Solids of revolution are common in mechanical applications, such as machine parts produced by a lathe. We spend the rest of this section looking at solids of this type. The next example uses the slicing method to calculate the volume of a solid of revolution.

Use an online **integral calculator (http://www.openstaxcollege.org/l/20_IntCalc2)** to learn more.

Example 2.7

Using the Slicing Method to find the Volume of a Solid of Revolution

Use the slicing method to find the volume of the solid of revolution bounded by the graphs of $f(x) = x^2 - 4x + 5,\ x = 1,$ and $x = 4,$ and rotated about the x-axis.

Solution

Using the problem-solving strategy, we first sketch the graph of the quadratic function over the interval $[1, 4]$ as shown in the following figure.

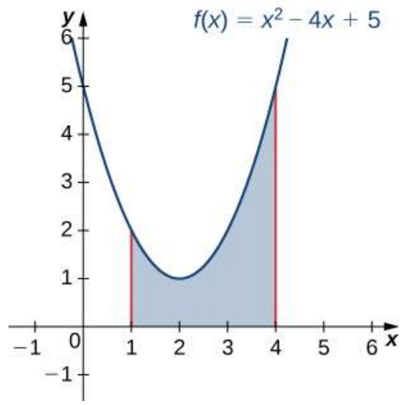


Figure 2.16 A region used to produce a solid of revolution.

Next, revolve the region around the x-axis, as shown in the following figure.

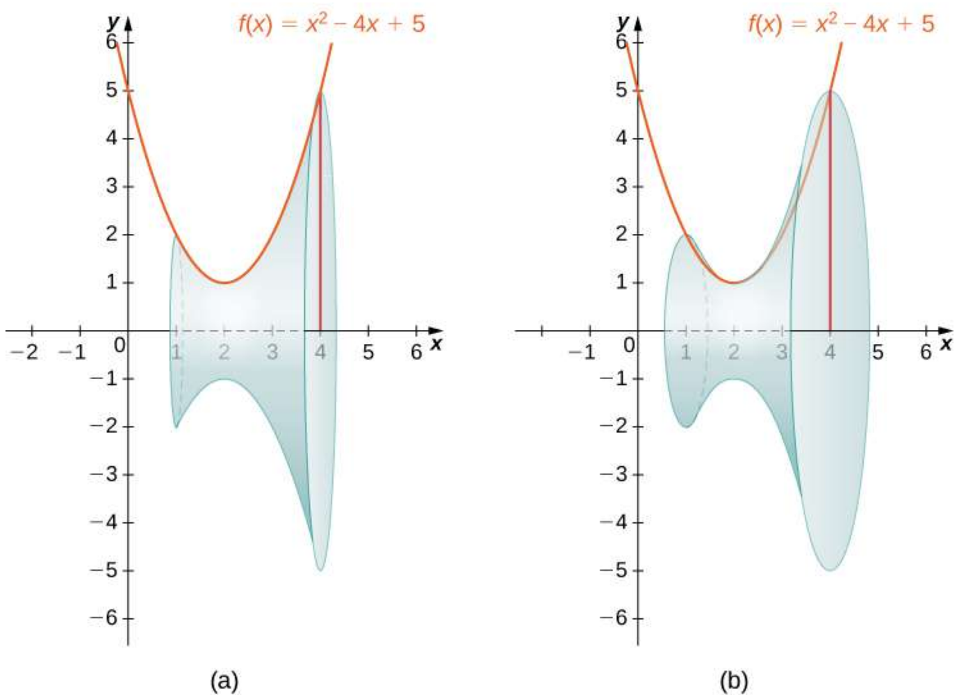


Figure 2.17 Two views, (a) and (b), of the solid of revolution produced by revolving the region in **Figure 2.16** about the x-axis.

Since the solid was formed by revolving the region around the x-axis, the cross-sections are circles (step 1). The area of the cross-section, then, is the area of a circle, and the radius of the circle is given by $f(x)$. Use the formula for the area of the circle:

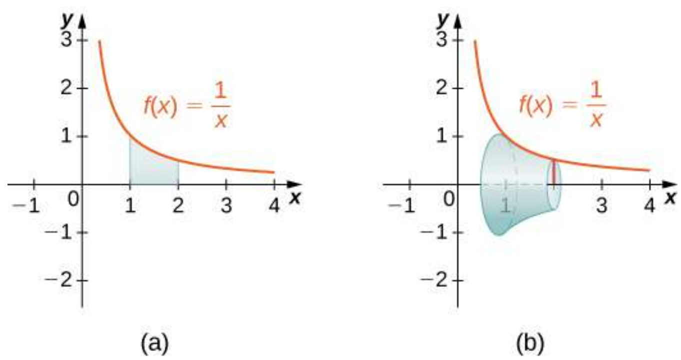
$$A(x) = \pi r^2 = \pi[f(x)]^2 = \pi\left(x^2 - 4x + 5\right)^2 \text{ (step 2).}$$

The volume, then, is (step 3)

$$\begin{aligned} V &= \int_a^h A(x)dx \\ &= \int_1^4 \pi\left(x^2 - 4x + 5\right)^2 dx = \pi \int_1^4 \left(x^4 - 8x^3 + 26x^2 - 40x + 25\right)dx \\ &= \pi\left(\frac{x^5}{5} - 2x^4 + \frac{26x^3}{3} - 20x^2 + 25x\right)\Bigg|_1^4 = \frac{78}{5}\pi. \end{aligned}$$

The volume is $78\pi/5$.

2.7 Use the method of slicing to find the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = 1/x$ and the x-axis over the interval $[1, 2]$ around the x-axis. See the following figure.



The Disk Method

When we use the slicing method with solids of revolution, it is often called the **disk method** because, for solids of revolution, the slices used to over approximate the volume of the solid are disks. To see this, consider the solid of revolution generated by revolving the region between the graph of the function $f(x) = (x - 1)^2 + 1$ and the x-axis over the interval $[-1, 3]$ around the x-axis. The graph of the function and a representative disk are shown in **Figure 2.18**(a) and (b). The region of revolution and the resulting solid are shown in **Figure 2.18**(c) and (d).

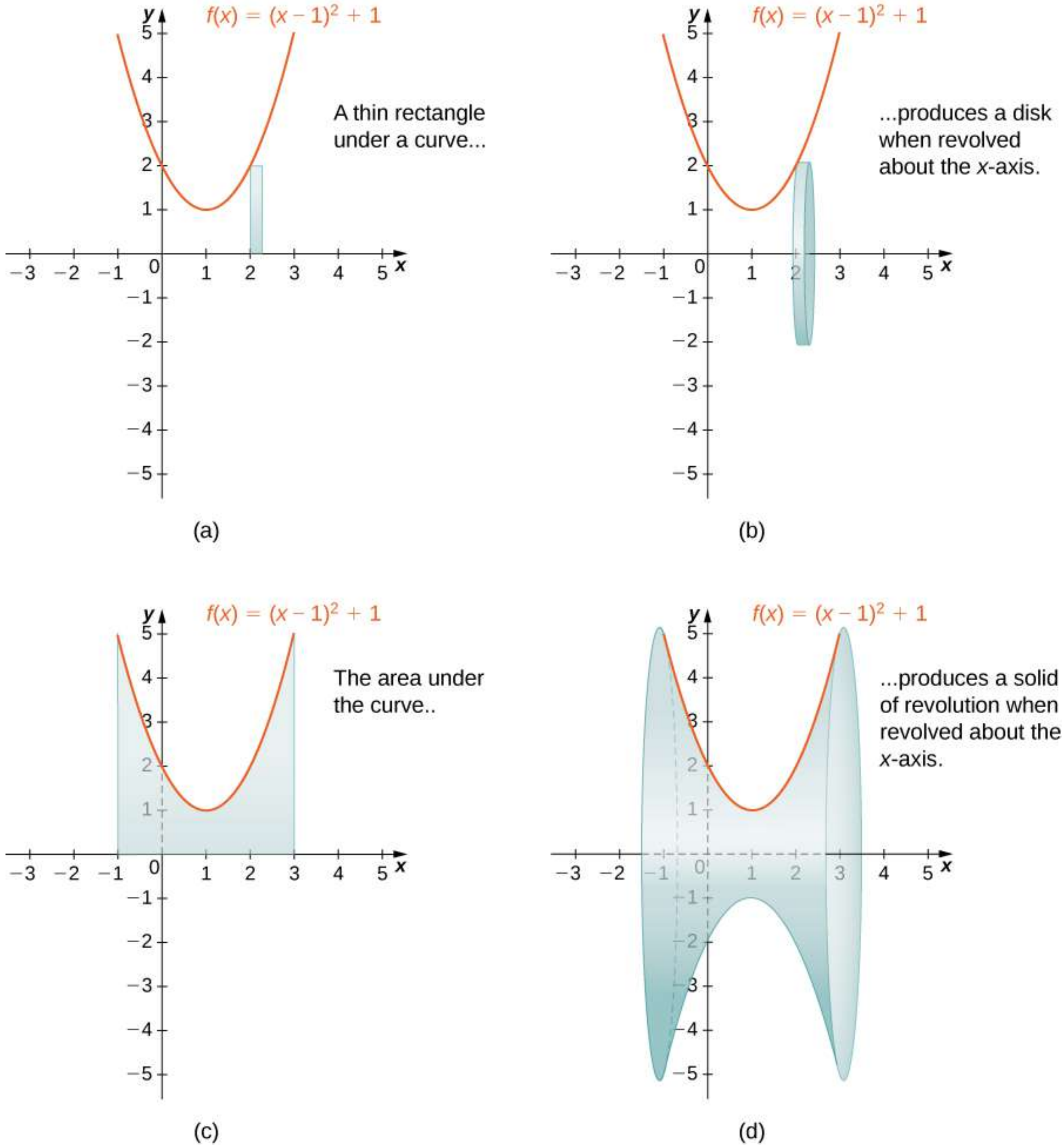


Figure 2.18 (a) A thin rectangle for approximating the area under a curve. (b) A representative disk formed by revolving the rectangle about the x-axis. (c) The region under the curve is revolved about the x-axis, resulting in (d) the solid of revolution.

We already used the formal Riemann sum development of the volume formula when we developed the slicing method. We know that

$$V = \int_a^b A(x)dx.$$

The only difference with the disk method is that we know the formula for the cross-sectional area ahead of time; it is the area of a circle. This gives the following rule.

Rule: The Disk Method

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the

x-axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi[f(x)]^2 dx. \quad (2.3)$$

The volume of the solid we have been studying (Figure 2.18) is given by

$$\begin{aligned} V &= \int_a^b \pi[f(x)]^2 dx \\ &= \int_{-1}^3 \pi\left[(x-1)^2+1\right]^2 dx = \pi\int_{-1}^3 \left[(x-1)^4 + 2(x-1)^2 + 1\right]^2 dx \\ &= \pi\left[\frac{1}{5}(x-1)^5 + \frac{2}{3}(x-1)^3 + x\right]\Big|_{-1}^3 = \pi\left[\left(\frac{32}{5}+\frac{16}{3}+3\right) - \left(-\frac{32}{5}-\frac{16}{3}-1\right)\right] = \frac{412\pi}{15} \text{ units}^3. \end{aligned}$$

Let's look at some examples.

Example 2.8

Using the Disk Method to Find the Volume of a Solid of Revolution 1

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{x}$ and the x-axis over the interval $[1, 4]$ around the x-axis.

Solution

The graphs of the function and the solid of revolution are shown in the following figure.

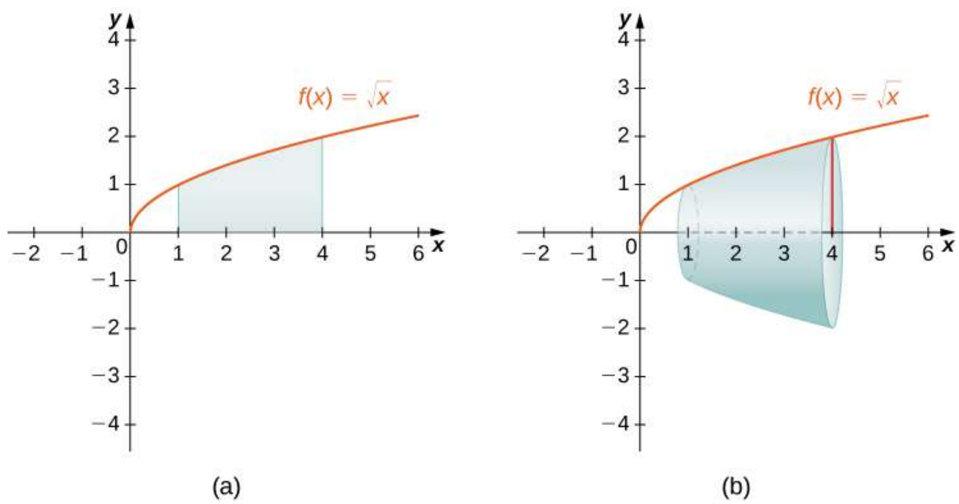


Figure 2.19 (a) The function $f(x) = \sqrt{x}$ over the interval $[1, 4]$. (b) The solid of revolution obtained by revolving the region under the graph of $f(x)$ about the x-axis.

We have

$$
\begin{aligned}
V &= \int_a^b \pi[f(x)]^2\, dx \\
&= \int_1^4 \pi[\sqrt{x}]^2 dx = \pi \int_1^4 x\, dx \\
&= \frac{\pi}{2}x^2\Big|_1^4 = \frac{15\pi}{2}.
\end{aligned}
$$

The volume is $(15\pi)/2$ units^3.

2.8 Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{4 - x}$ and the x-axis over the interval $[0,\ 4]$ around the x-axis.

So far, our examples have all concerned regions revolved around the x-axis, but we can generate a solid of revolution by revolving a plane region around any horizontal or vertical line. In the next example, we look at a solid of revolution that has been generated by revolving a region around the y-axis. The mechanics of the disk method are nearly the same as when the x-axis is the axis of revolution, but we express the function in terms of y and we integrate with respect to y as well. This is summarized in the following rule.

Rule: The Disk Method for Solids of Revolution around the *y*-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y-axis is given by

$$V = \int_c^d \pi[g(y)]^2\, dy. \tag{2.4}$$

The next example shows how this rule works in practice.

Example 2.9

Using the Disk Method to Find the Volume of a Solid of Revolution 2

Let R be the region bounded by the graph of $g(y) = \sqrt{4 - y}$ and the y-axis over the y-axis interval $[0,\ 4]$. Use the disk method to find the volume of the solid of revolution generated by rotating R around the y-axis.

Solution

Figure 2.20 shows the function and a representative disk that can be used to estimate the volume. Notice that since we are revolving the function around the y-axis, the disks are horizontal, rather than vertical.

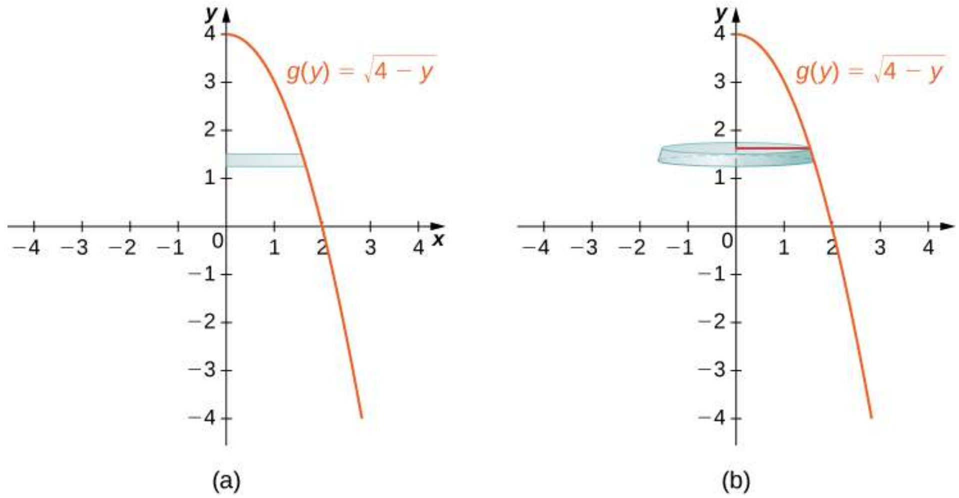


Figure 2.20 (a) Shown is a thin rectangle between the curve of the function $g(y)=\sqrt{4-y}$ and the $y\text{-axis}$. (b) The rectangle forms a representative disk after revolution around the $y\text{-axis}$.

The region to be revolved and the full solid of revolution are depicted in the following figure.

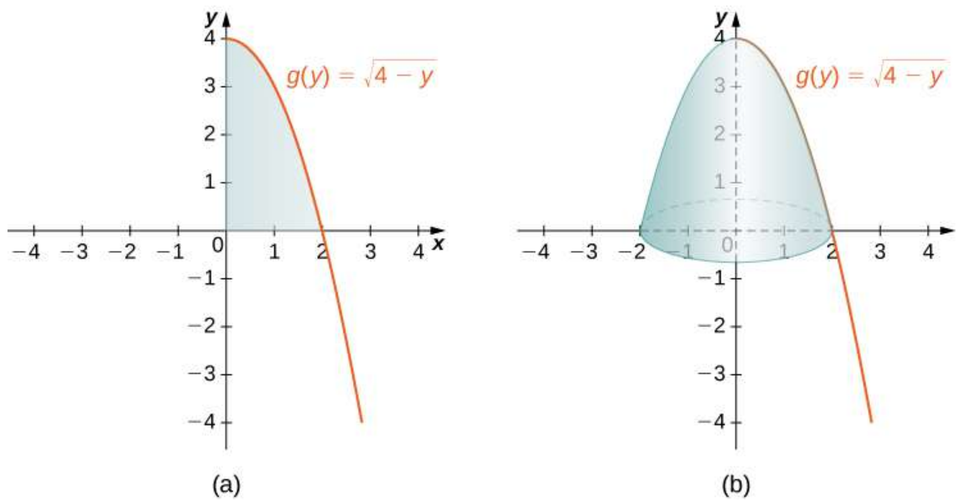


Figure 2.21 (a) The region to the left of the function $g(y)=\sqrt{4-y}$ over the $y\text{-axis}$ interval $[0,\ 4]$. (b) The solid of revolution formed by revolving the region about the $y\text{-axis}$.

To find the volume, we integrate with respect to y. We obtain

$$\begin{aligned}V &= \int_c^d \pi[g(y)]^2\,dy \\ &= \int_0^4 \pi\left[\sqrt{4-y}\right]^2 dy = \pi\int_0^4 (4-y)dy \\ &= \pi\left[4y-\frac{y^2}{2}\right]\Bigg|_0^4 = 8\pi.\end{aligned}$$

The volume is 8π units^3.

2.9 Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $g(y) = y$ and the y-axis over the interval $[1, 4]$ around the y-axis.

The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happen is when an axis of revolution other than the x-axis or y-axis is selected.

When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but washers (disks with holes in the center). For example, consider the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1$ over the interval $[1, 4]$. When this region is revolved around the x-axis, the result is a solid with a cavity in the middle, and the slices are washers. The graph of the function and a representative washer are shown in **Figure 2.22**(a) and (b). The region of revolution and the resulting solid are shown in **Figure 2.22**(c) and (d).

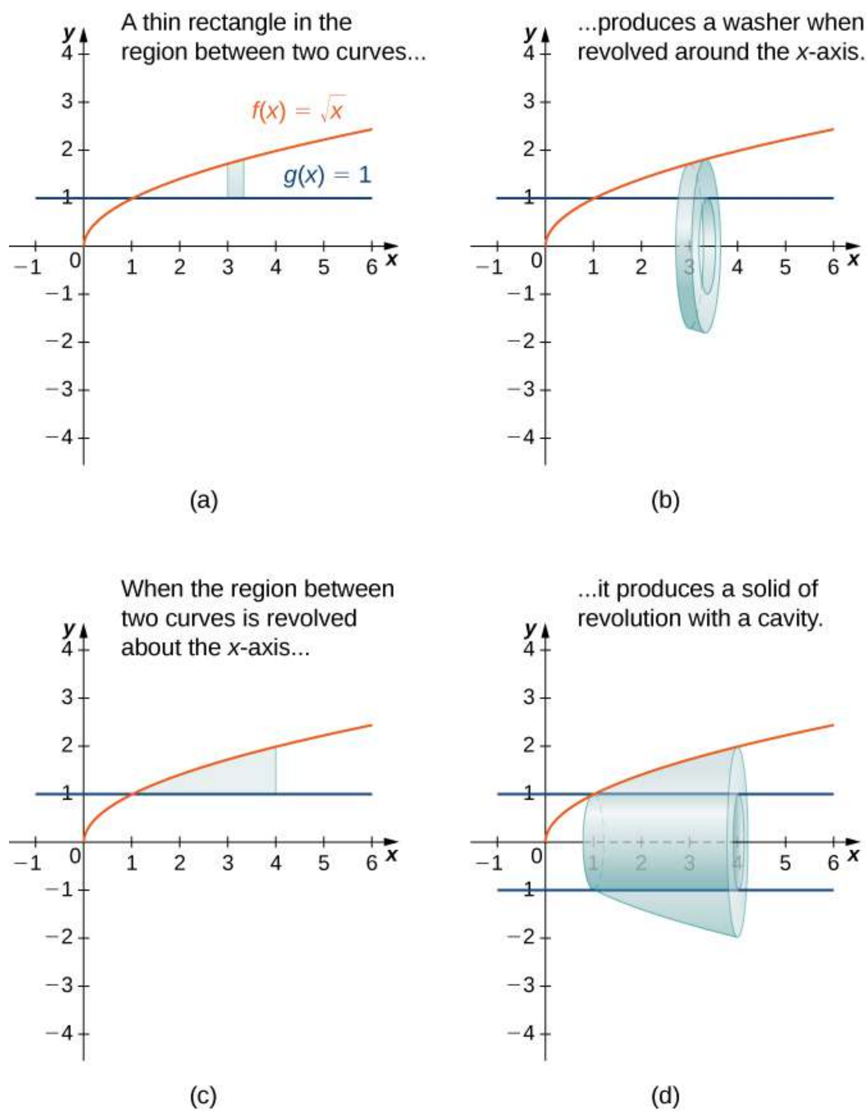


Figure 2.22 (a) A thin rectangle in the region between two curves. (b) A representative disk formed by revolving the rectangle about the x-axis. (c) The region between the curves over the given interval. (d) The resulting solid of revolution.

The cross-sectional area, then, is the area of the outer circle less the area of the inner circle. In this case,

$$A(x) = \pi(\sqrt{x})^2 - \pi(1)^2 = \pi(x-1).$$

Then the volume of the solid is

$$\begin{aligned} V &= \int_a^b A(x)dx \\ &= \int_1^4 \pi(x-1)dx = \pi\left[\frac{x^2}{2} - x\right]\Bigg|_1^4 = \frac{9}{2}\pi \text{ units}^3. \end{aligned}$$

Generalizing this process gives the **washer method**.

Rule: The Washer Method

Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx. \tag{2.5}$$

Example 2.10

Using the Washer Method

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = 1/x$ over the interval $[1, 4]$ around the x-axis.

Solution

The graphs of the functions and the solid of revolution are shown in the following figure.

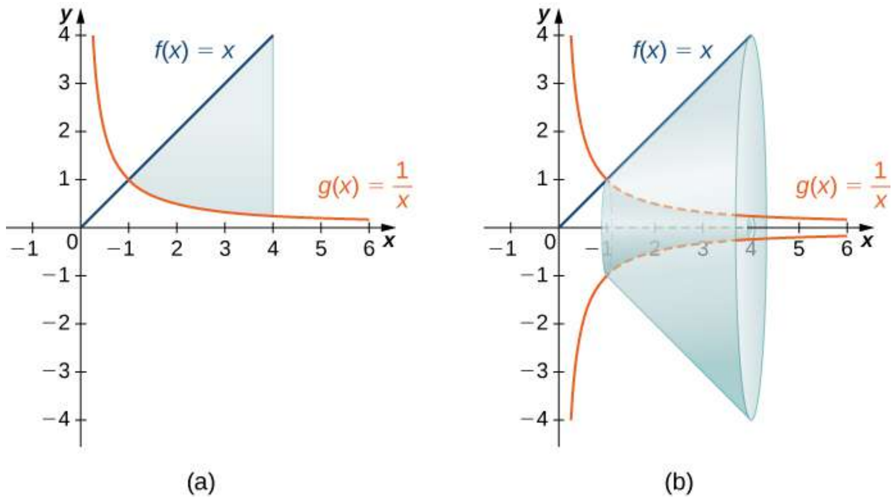


Figure 2.23 (a) The region between the graphs of the functions $f(x) = x$ and $g(x) = 1/x$ over the interval $[1, 4]$. (b) Revolving the region about the x-axis generates a solid of revolution with a cavity in the middle.

We have

$$\begin{aligned} V &= \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx \\ &= \pi\int_1^4 \left[x^2 - \left(\frac{1}{x}\right)^2\right]dx \;= \pi\left[\frac{x^3}{3} + \frac{1}{x}\right]\Bigg|_1^4 = \frac{81\pi}{4}\text{ units}^3. \end{aligned}$$

2.10 Find the volume of a solid of revolution formed by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = 1/x$ over the interval $[1, 3]$ around the x-axis.

As with the disk method, we can also apply the washer method to solids of revolution that result from revolving a region around the y-axis. In this case, the following rule applies.

Rule: The Washer Method for Solids of Revolution around the *y*-axis

Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \le u(y)$ for $y \in [c, d]$. Let Q denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y-axis is given by

$$V = \int_c^d \pi\left[(u(y))^2 - (v(y))^2\right]dy.$$

Rather than looking at an example of the washer method with the y-axis as the axis of revolution, we now consider an example in which the axis of revolution is a line other than one of the two coordinate axes. The same general method applies, but you may have to visualize just how to describe the cross-sectional area of the volume.

Example 2.11

The Washer Method with a Different Axis of Revolution

Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the x-axis over the interval $[0, 4]$ around the line $y = -2$.

Solution

The graph of the region and the solid of revolution are shown in the following figure.

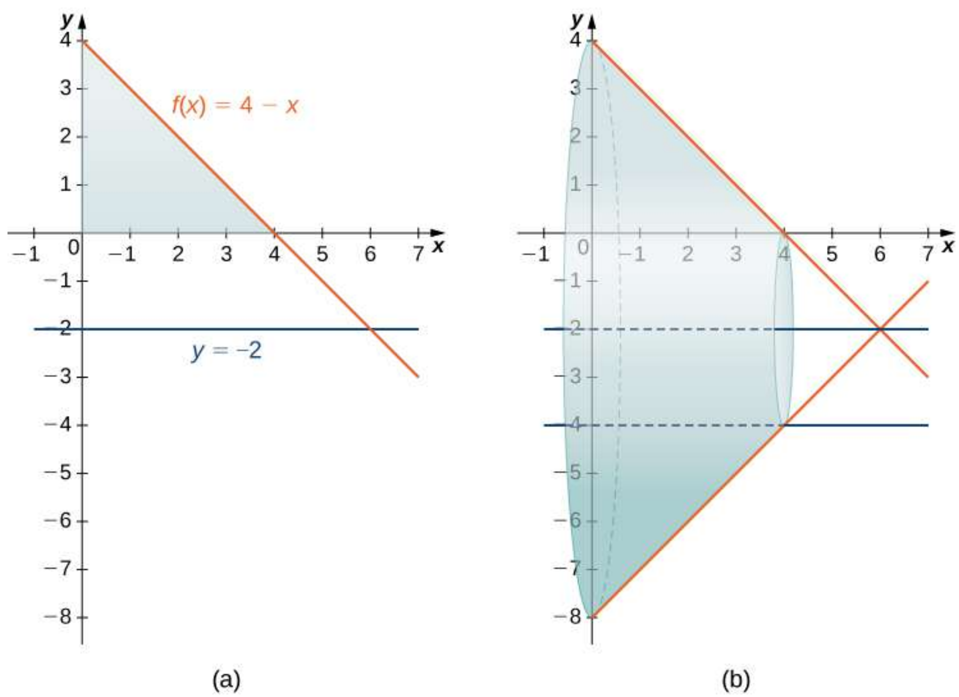


Figure 2.24 (a) The region between the graph of the function $f(x) = 4 - x$ and the x-axis over the interval $[0, 4]$. (b) Revolving the region about the line $y = -2$ generates a solid of revolution with a cylindrical hole through its middle.

We can’t apply the volume formula to this problem directly because the axis of revolution is not one of the coordinate axes. However, we still know that the area of the cross-section is the area of the outer circle less the area of the inner circle. Looking at the graph of the function, we see the radius of the outer circle is given by $f(x) + 2,$ which simplifies to

$$f(x) + 2 = (4 - x) + 2 = 6 - x.$$

The radius of the inner circle is $g(x) = 2.$ Therefore, we have

$$\begin{aligned} V &= \int_0^4 \pi\left[(6 - x)^2 - (2)^2\right]dx \\ &= \pi\int_0^4 \left(x^2 - 12x + 32\right)dx = \pi\left[\frac{x^3}{3} - 6x^2 + 32x\right]\Bigg|_0^4 = \frac{160\pi}{3} \text{ units}^3. \end{aligned}$$

2.11 Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x + 2$ and below by the x-axis over the interval $[0, 3]$ around the line $y = -1.$

2.2 EXERCISES

58. Derive the formula for the volume of a sphere using the slicing method.

59. Use the slicing method to derive the formula for the volume of a cone.

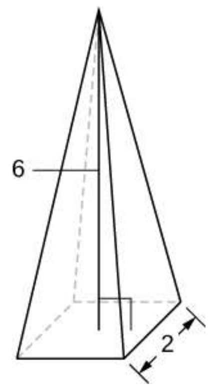
60. Use the slicing method to derive the formula for the volume of a tetrahedron with side length a.

61. Use the disk method to derive the formula for the volume of a trapezoidal cylinder.

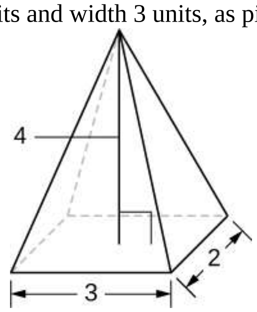
62. Explain when you would use the disk method versus the washer method. When are they interchangeable?

For the following exercises, draw a typical slice and find the volume using the slicing method for the given volume.

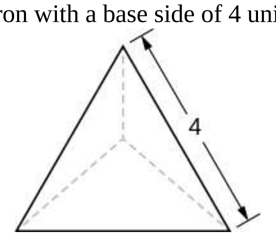
63. A pyramid with height 6 units and square base of side 2 units, as pictured here.



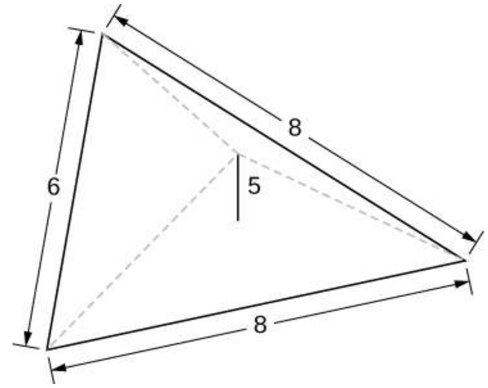
64. A pyramid with height 4 units and a rectangular base with length 2 units and width 3 units, as pictured here.



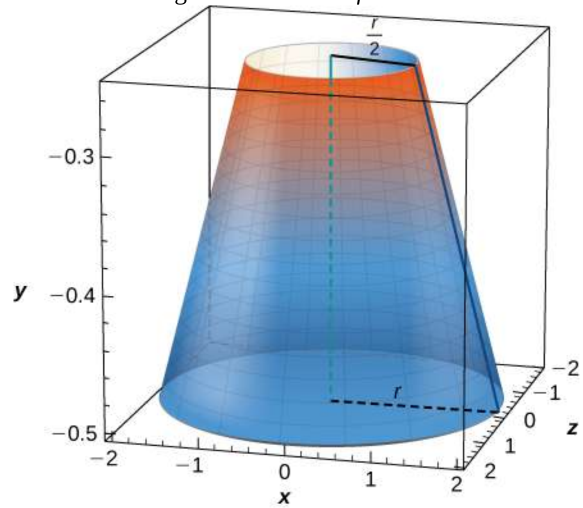
65. A tetrahedron with a base side of 4 units, as seen here.



66. A pyramid with height 5 units, and an isosceles triangular base with lengths of 6 units and 8 units, as seen here.



67. A cone of radius r and height h has a smaller cone of radius $r/2$ and height $h/2$ removed from the top, as seen here. The resulting solid is called a *frustum*.



For the following exercises, draw an outline of the solid and find the volume using the slicing method.

68. The base is a circle of radius a. The slices perpendicular to the base are squares.

69. The base is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Slices perpendicular to the xy-plane are semicircles.

70. The base is the region under the parabola $y = 1 - x^2$ in the first quadrant. Slices perpendicular to the xy-plane are squares.

71. The base is the region under the parabola $y = 1 - x^2$ and above the x-axis. Slices perpendicular to the y-axis are squares.

72. The base is the region enclosed by $y = x^2$ and $y = 9$. Slices perpendicular to the x-axis are right isosceles triangles.

73. The base is the area between $y = x$ and $y = x^2$. Slices perpendicular to the x-axis are semicircles.

For the following exercises, draw the region bounded by the curves. Then, use the disk method to find the volume when the region is rotated around the x-axis.

74. $x + y = 8,\ x = 0,$ and $y = 0$

75. $y = 2x^2,\ x = 0,\ x = 4,$ and $y = 0$

76. $y = e^x + 1,\ x = 0,\ x = 1,$ and $y = 0$

77. $y = x^4,\ x = 0,$ and $y = 1$

78. $y = \sqrt{x},\ x = 0,\ x = 4,$ and $y = 0$

79. $y = \sin x,\ y = \cos x,$ and $x = 0$

80. $y = \frac{1}{x},\ x = 2,$ and $y = 3$

81. $x^2 - y^2 = 9$ and $x + y = 9,\ y = 0$ and $x = 0$

For the following exercises, draw the region bounded by the curves. Then, find the volume when the region is rotated around the y-axis.

82. $y = 4 - \frac{1}{2}x,\ x = 0,$ and $y = 0$

83. $y = 2x^3,\ x = 0,\ x = 1,$ and $y = 0$

84. $y = 3x^2,\ x = 0,$ and $y = 3$

85. $y = \sqrt{4 - x^2},\ y = 0,$ and $x = 0$

86. $y = \frac{1}{\sqrt{x + 1}},\ x = 0,$ and $x = 3$

87. $x = \sec(y)$ and $y = \frac{\pi}{4},\ y = 0$ and $x = 0$

88. $y = \frac{1}{x + 1},\ x = 0,$ and $x = 2$

89. $y = 4 - x,\ y = x,$ and $x = 0$

For the following exercises, draw the region bounded by the curves. Then, find the volume when the region is rotated around the x-axis.

90. $y = x + 2,\ y = x + 6,\ x = 0,$ and $x = 5$

91. $y = x^2$ and $y = x + 2$

92. $x^2 = y^3$ and $x^3 = y^2$

93. $y = 4 - x^2$ and $y = 2 - x$

94. **[T]** $y = \cos x,\ y = e^{-x},\ x = 0,$ and $x = 1.2927$

95. $y = \sqrt{x}$ and $y = x^2$

96. $y = \sin x, y = 5 \sin x,\ x = 0$ and $x = \pi$

97. $y = \sqrt{1 + x^2}$ and $y = \sqrt{4 - x^2}$

For the following exercises, draw the region bounded by the curves. Then, use the washer method to find the volume when the region is revolved around the y-axis.

98. $y = \sqrt{x},\ x = 4,$ and $y = 0$

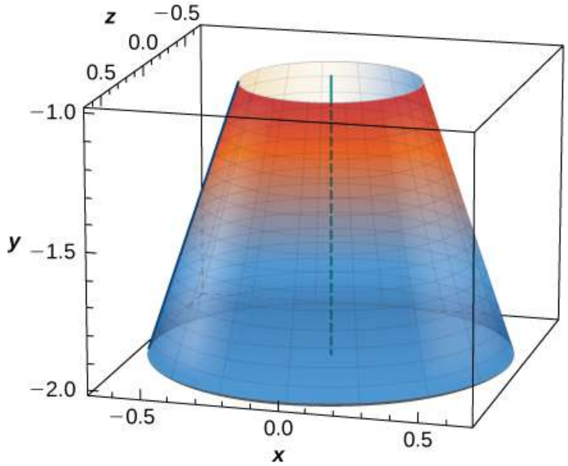
99. $y = x + 2,\ y = 2x - 1,$ and $x = 0$

100. $y = \sqrt[3]{x}$ and $y = x^3$

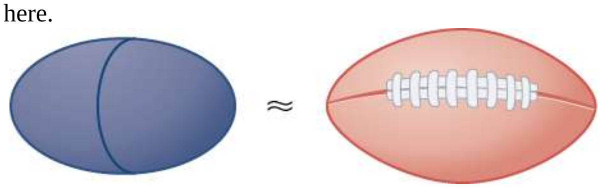
101. $x = e^{2y},\ x = y^2,\ y = 0,$ and $y = \ln(2)$

102. $x = \sqrt{9 - y^2},\ x = e^{-y},\ y = 0,$ and $y = 3$

103. Yogurt containers can be shaped like frustums. Rotate the line $y = \frac{1}{m}x$ around the y-axis to find the volume between $y = a$ and $y = b$.

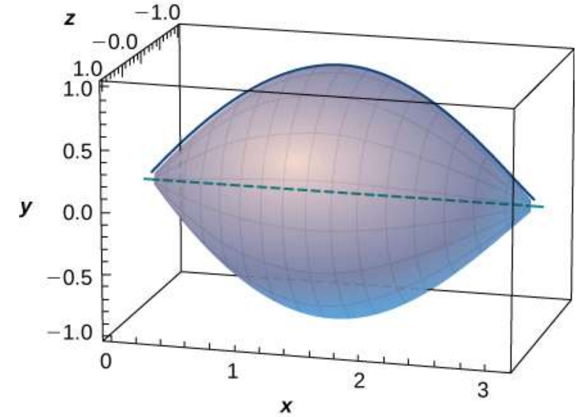


104. Rotate the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ around the x-axis to approximate the volume of a football, as seen here.

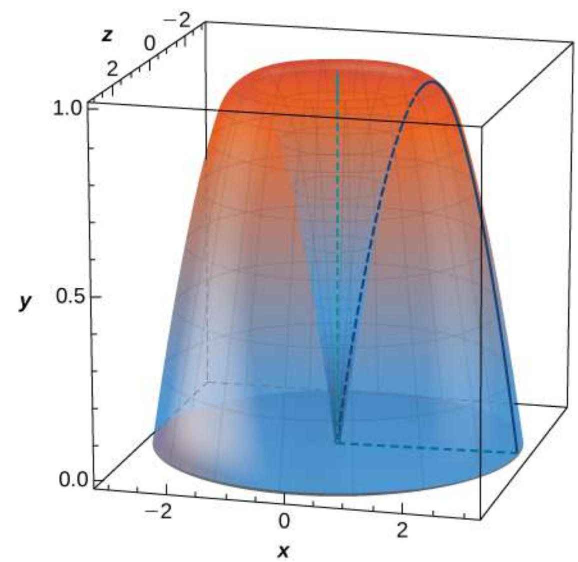


105. Rotate the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ around the y-axis to approximate the volume of a football.

106. A better approximation of the volume of a football is given by the solid that comes from rotating $y = \sin x$ around the x-axis from $x = 0$ to $x = \pi$. What is the volume of this football approximation, as seen here?



107. What is the volume of the Bundt cake that comes from rotating $y = \sin x$ around the y-axis from $x = 0$ to $x = \pi$?

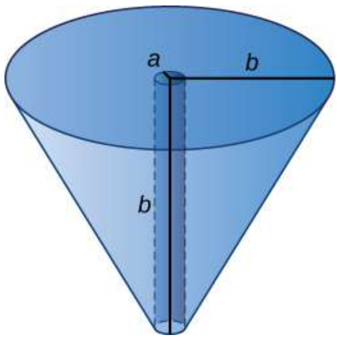


For the following exercises, find the volume of the solid described.

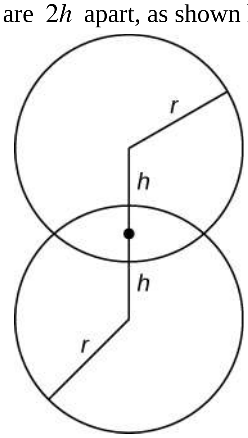
108. The base is the region between $y = x$ and $y = x^2$. Slices perpendicular to the *x*-axis are semicircles.

109. The base is the region enclosed by the generic ellipse $\left(x^2/a^2\right) + \left(y^2/b^2\right) = 1$. Slices perpendicular to the *x*-axis are semicircles.

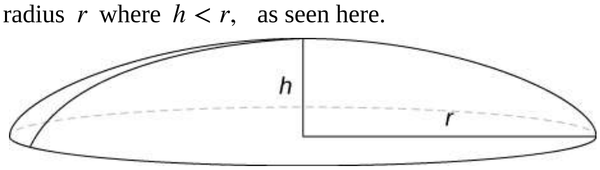
110. Bore a hole of radius a down the axis of a right cone and through the base of radius b, as seen here.



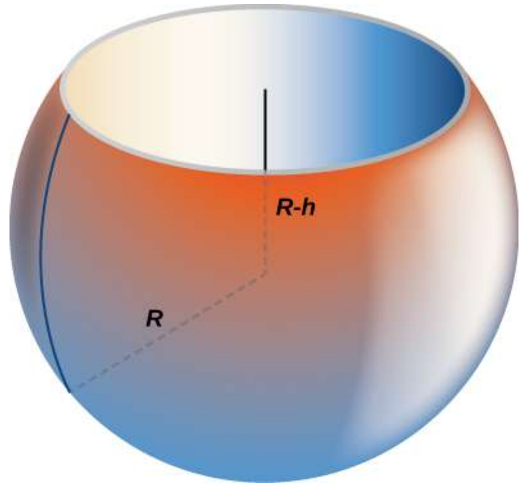
111. Find the volume common to two spheres of radius r with centers that are $2h$ apart, as shown here.



112. Find the volume of a spherical cap of height h and radius r where $h < r$, as seen here.



113. Find the volume of a sphere of radius R with a cap of height h removed from the top, as seen here.



2.3 | Volumes of Revolution: Cylindrical Shells

Learning Objectives

2.3.1 Calculate the volume of a solid of revolution by using the method of cylindrical shells.

2.3.2 Compare the different methods for calculating a volume of revolution.

In this section, we examine the method of cylindrical shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method; however, with the disk and washer methods, we integrate along the coordinate axis parallel to the axis of revolution. With the method of cylindrical shells, we integrate along the coordinate axis *perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the washer method. In the last part of this section, we review all the methods for finding volume that we have studied and lay out some guidelines to help you determine which method to use in a given situation.

The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region $R,$ bounded above by the graph of a function $y = f(x),$ below by the x-axis, and on the left and right by the lines $x = a$ and $x = b,$ respectively, as shown in **Figure 2.25**(a). We then revolve this region around the y-axis, as shown in **Figure 2.25**(b). Note that this is different from what we have done before. Previously, regions defined in terms of functions of x were revolved around the x-axis or a line parallel to it.

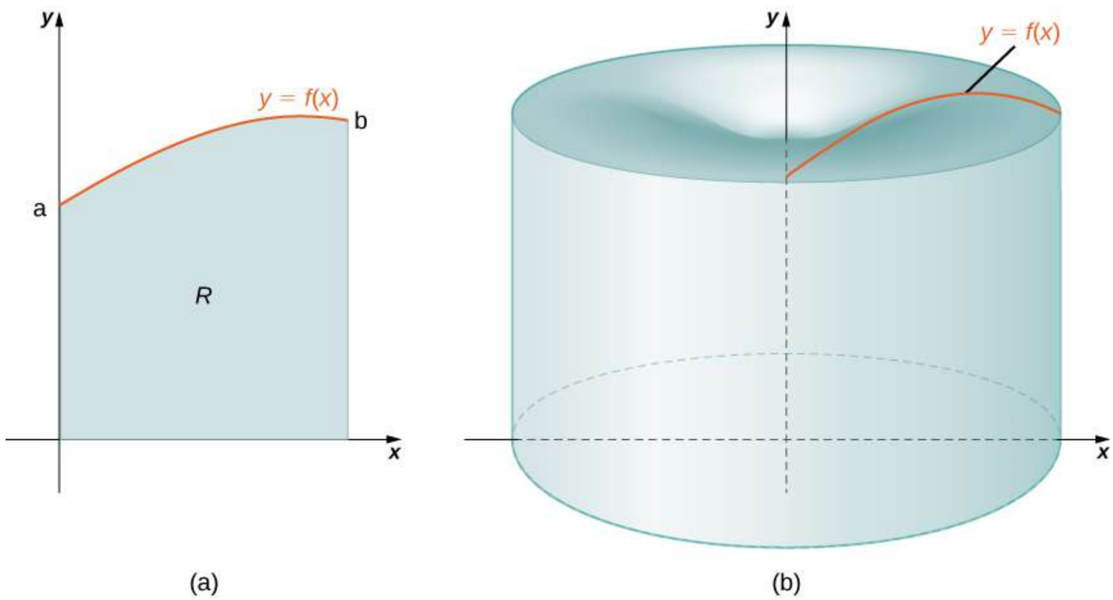


Figure 2.25 (a) A region bounded by the graph of a function of $x.$ (b) The solid of revolution formed when the region is revolved around the y-axis.

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = \{x_0, x_1, \ldots, x_n\}$ and, for $i = 1, 2, \ldots, n,$ choose a point $x_i^* \in [x_{i-1}, x_i].$ Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width $\Delta x.$ A representative rectangle is shown in **Figure 2.26**(a). When that rectangle is revolved around the y-axis, instead of a disk or a washer, we get a cylindrical shell, as shown in the following figure.

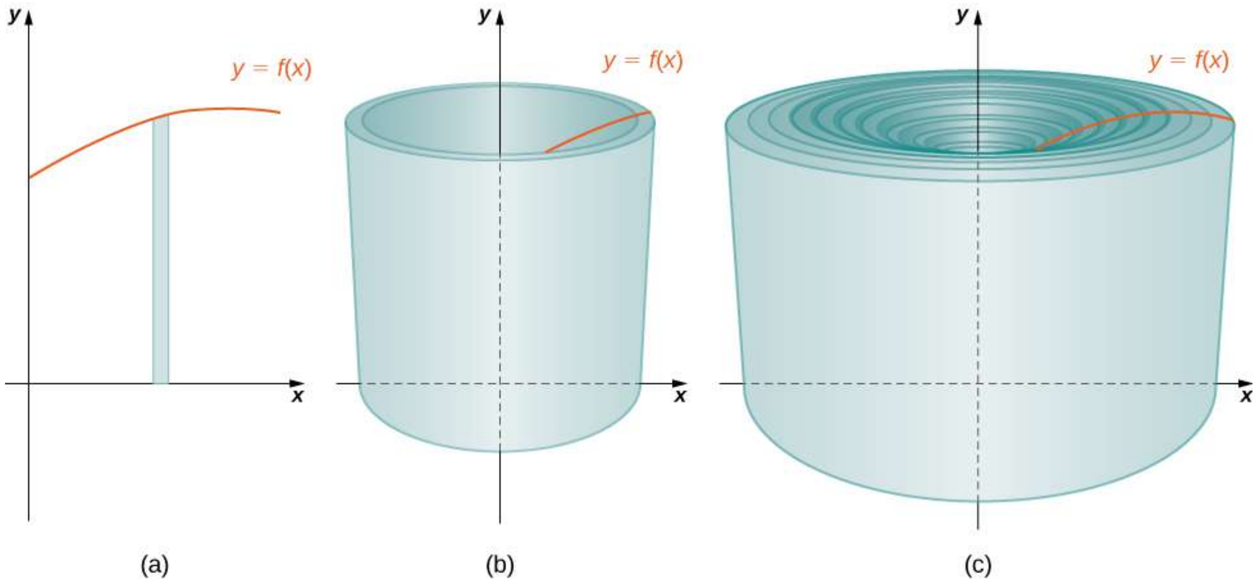


Figure 2.26 (a) A representative rectangle. (b) When this rectangle is revolved around the y-axis, the result is a cylindrical shell. (c) When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider **Figure 2.27**.

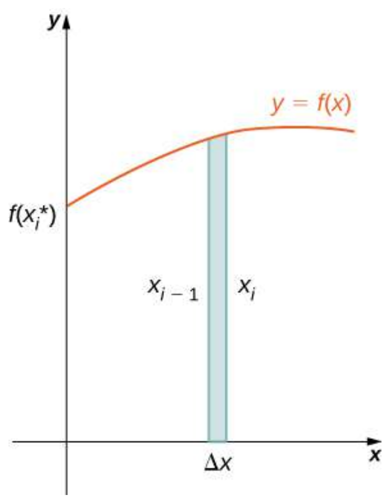


Figure 2.27 Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are annuli (ring-shaped regions—essentially, circles with a hole in the center), with outer radius x_i and inner radius x_{i-1}. Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)\left(x_i^2 - x_{i-1}^2\right) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x$, so we have

$$V_{\text{shell}} = 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x.$$

Furthermore, $\frac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by x_i^*. We then have

$$V_{\text{shell}} \approx 2\pi f(x_i^*)x_i^* \Delta x.$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate (**Figure 2.28**).

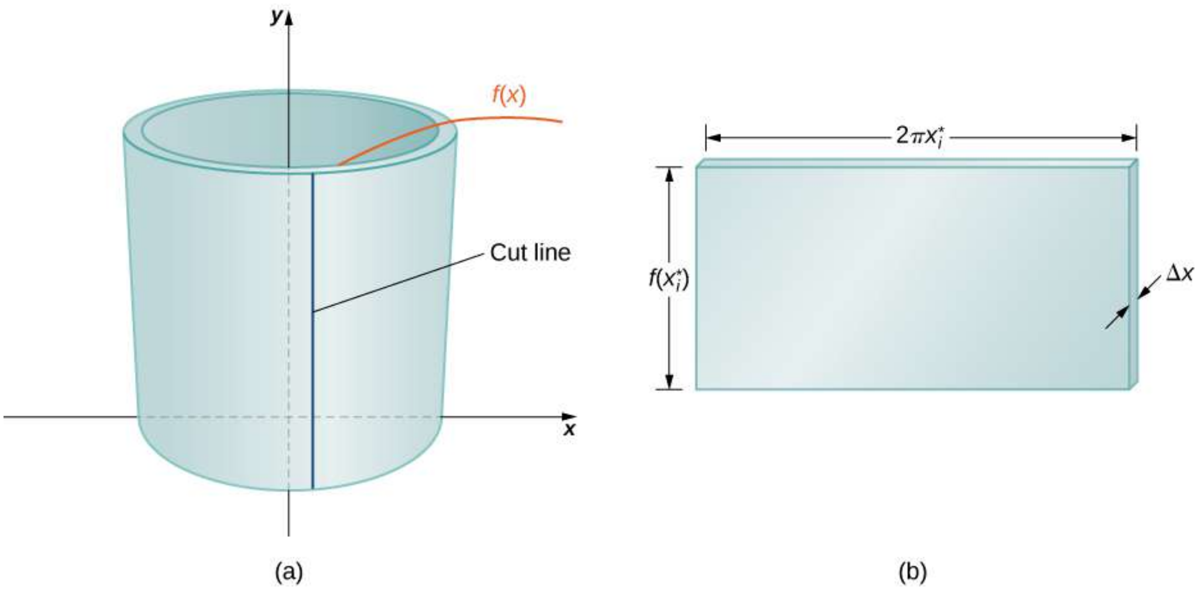


Figure 2.28 (a) Make a vertical cut in a representative shell. (b) Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius, and hence the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness Δx (**Figure 2.28**). The volume of the shell, then, is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

$$V_{\text{shell}} \approx f(x_i^*)(2\pi x_i^*)\Delta x,$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n}\left(2\pi x_i^* f(x_i^*)\Delta x\right).$$

Here we have another Riemann sum, this time for the function $2\pi x f(x)$. Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n \to \infty}\sum_{i=1}^{n}\left(2\pi x_i^* f(x_i^*)\Delta x\right) = \int_a^b \left(2\pi x f(x)\right)dx.$$

This leads to the following rule for the **method of cylindrical shells.**

Rule: The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x-axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving R around the y-axis is given by

$$V = \int_a^b (2\pi x f(x))dx. \quad (2.6)$$

Now let's consider an example.

Example 2.12

The Method of Cylindrical Shells 1

Define R as the region bounded above by the graph of $f(x) = 1/x$ and below by the x-axis over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

First we must graph the region R and the associated solid of revolution, as shown in the following figure.

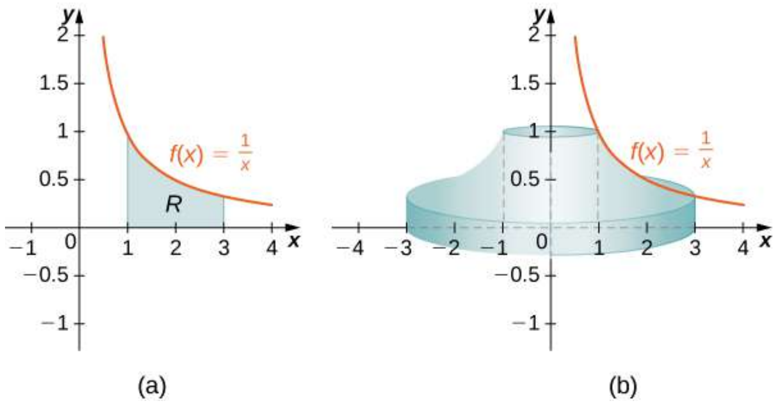


Figure 2.29 (a) The region R under the graph of $f(x) = 1/x$ over the interval $[1, 3]$. (b) The solid of revolution generated by revolving R about the y-axis.

Then the volume of the solid is given by

$$\begin{aligned} V &= \int_a^b (2\pi x f(x))dx \\ &= \int_1^3 \left(2\pi x\left(\frac{1}{x}\right)\right)dx \\ &= \int_1^3 2\pi\, dx = 2\pi x|_1^3 = 4\pi \text{ units}^3. \end{aligned}$$

2.12 Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Example 2.13

The Method of Cylindrical Shells 2

Define R as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

First graph the region R and the associated solid of revolution, as shown in the following figure.

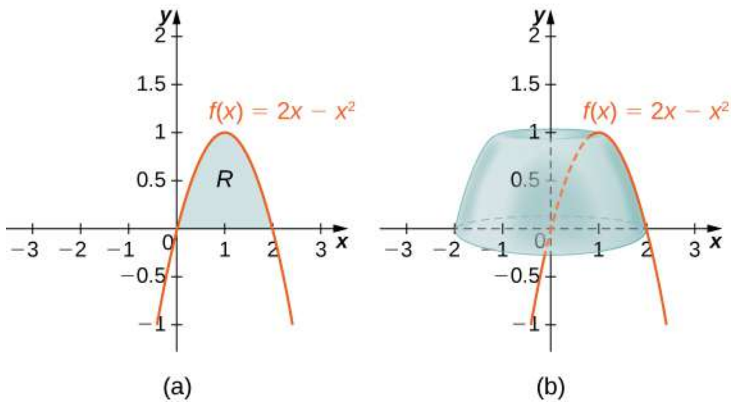


Figure 2.30 (a) The region R under the graph of $f(x) = 2x - x^2$ over the interval $[0, 2]$. (b) The volume of revolution obtained by revolving R about the y-axis.

Then the volume of the solid is given by

$$\begin{aligned} V &= \int_a^b (2\pi x f(x))dx \\ &= \int_0^2 \left(2\pi x\left(2x - x^2\right)\right)dx = 2\pi \int_0^2 \left(2x^2 - x^3\right)dx \\ &= 2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]\Big|_0^2 = \frac{8\pi}{3} \text{ units}^3. \end{aligned}$$

2.13 Define R as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the x-axis, when we want to integrate with respect to y. The analogous rule for this type of solid is given here.

Rule: The Method of Cylindrical Shells for Solids of Revolution around the *x*-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the x-axis is given by

$$V = \int_c^d (2\pi y g(y))dy.$$

Example 2.14

The Method of Cylindrical Shells for a Solid Revolved around the *x*-axis

Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Solution

First, we need to graph the region Q and the associated solid of revolution, as shown in the following figure.

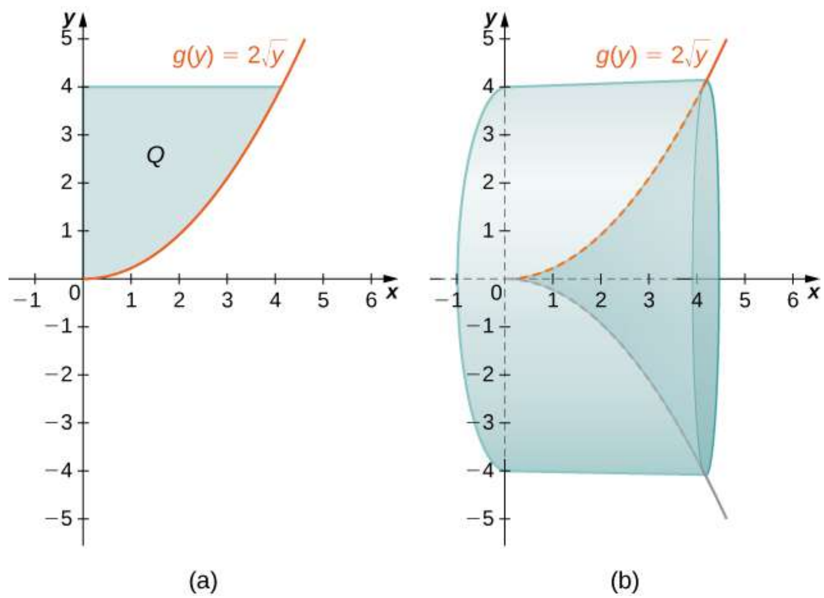


Figure 2.31 (a) The region Q to the left of the function $g(y)$ over the interval $[0, 4]$. (b) The solid of revolution generated by revolving Q around the x-axis.

Label the shaded region Q. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_c^d (2\pi y g(y))dy \\ &= \int_0^4 (2\pi y(2\sqrt{y}))dy = 4\pi \int_0^4 y^{3/2}dy \\ &= 4\pi \left[\frac{2y^{5/2}}{5} \right] \Bigg|_0^4 = \frac{256\pi}{5} \text{ units}^3. \end{aligned}$$

2.14 Define Q as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the y-axis for $y \in [1, 3]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

For the next example, we look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate axes. To set this up, we need to revisit the development of the method of cylindrical shells. Recall that we found the volume of one of the shells to be given by

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)\left(x_i^2 - x_{i-1}^2\right) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

This was based on a shell with an outer radius of x_i and an inner radius of x_{i-1}. If, however, we rotate the region around a line other than the y-axis, we have a different outer and inner radius. Suppose, for example, that we rotate the region around the line $x = -k$, where k is some positive constant. Then, the outer radius of the shell is $x_i + k$ and the inner radius of the shell is $x_{i-1} + k$. Substituting these terms into the expression for volume, we see that when a plane region is rotated around the line $x = -k$, the volume of a shell is given by

$$\begin{aligned} V_{\text{shell}} &= 2\pi f(x_i^*)\left(\frac{(x_i + k) + (x_{i-1} + k)}{2}\right)\left((x_i + k) - (x_{i-1} + k)\right) \\ &= 2\pi f(x_i^*)\left(\left(\frac{x_i + x_{i-2}}{2}\right) + k\right)\Delta x. \end{aligned}$$

As before, we notice that $\frac{x_i + x_{i-1}}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$ and can be approximated by x_i^*. Then, the approximate volume of the shell is

$$V_{\text{shell}} \approx 2\pi\left(x_i^* + k\right)f(x_i^*)\Delta x.$$

The remainder of the development proceeds as before, and we see that

$$V = \int_a^b (2\pi(x + k)f(x))dx.$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the x-term in the integral must be replaced with an expression representing the radius of a shell. To see how this works, consider the following example.

Example 2.15

A Region of Revolution Revolved around a Line

Define R as the region bounded above by the graph of $f(x) = x$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -1$.

Solution

First, graph the region R and the associated solid of revolution, as shown in the following figure.

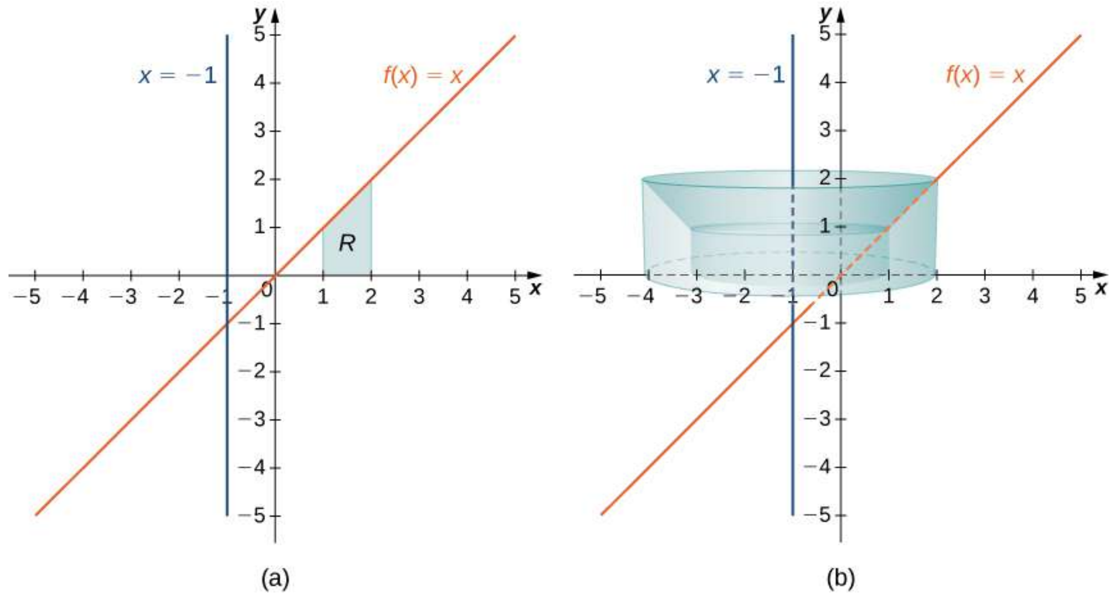


Figure 2.32 (a) The region R between the graph of $f(x)$ and the x-axis over the interval $[1, 2]$. (b) The solid of revolution generated by revolving R around the line $x = -1$.

Note that the radius of a shell is given by $x + 1$. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^2 (2\pi(x+1)f(x))dx \\ &= \int_1^2 (2\pi(x+1)x)dx = 2\pi\int_1^2 \left(x^2 + x\right)dx \\ &= 2\pi\left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Big|_1^2 = \frac{23\pi}{3}\,\text{units}^3. \end{aligned}$$

2.15 Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -2$.

For our final example in this section, let’s look at the volume of a solid of revolution for which the region of revolution is bounded by the graphs of two functions.

Example 2.16

A Region of Revolution Bounded by the Graphs of Two Functions

Define R as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1/x$ over the interval $[1, 4]$. Find the volume of the solid of revolution generated by revolving R around the y-axis.

Solution

First, graph the region R and the associated solid of revolution, as shown in the following figure.

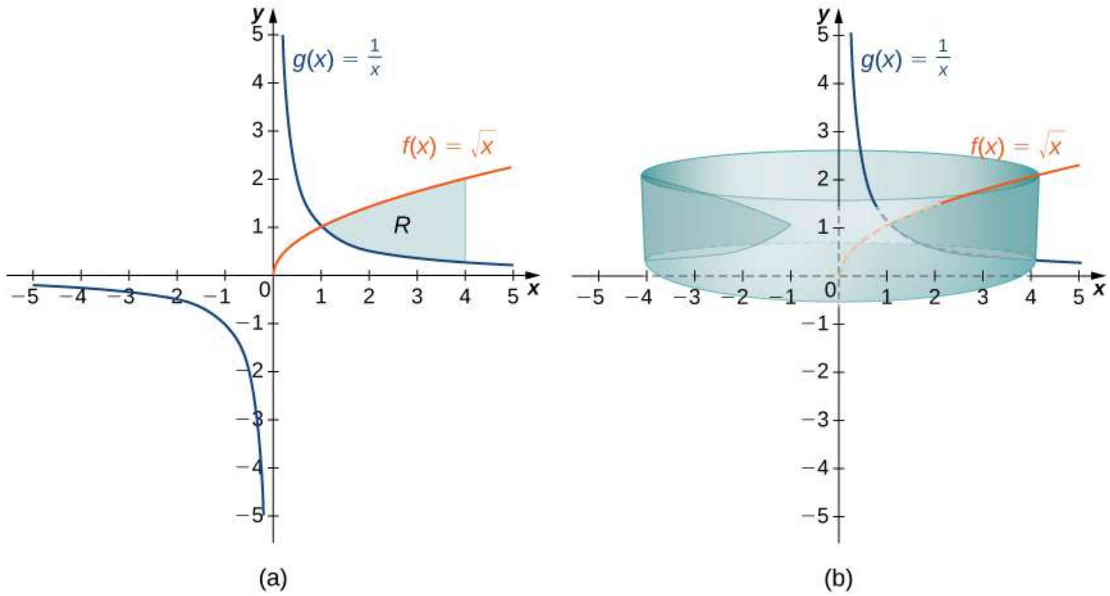


Figure 2.33 (a) The region R between the graph of $f(x)$ and the graph of $g(x)$ over the interval $[1, 4]$. (b) The solid of revolution generated by revolving R around the y-axis.

Note that the axis of revolution is the y-axis, so the radius of a shell is given simply by x. We don't need to make any adjustments to the x-term of our integrand. The height of a shell, though, is given by $f(x) - g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^4 (2\pi x(f(x) - g(x)))dx \\ &= \int_1^4 \left(2\pi x\left(\sqrt{x} - \frac{1}{x}\right)\right)dx = 2\pi \int_1^4 \left(x^{3/2} - 1\right)dx \\ &= 2\pi \left[\frac{2x^{5/2}}{5} - x\right]\Bigg|_1^4 = \frac{94\pi}{5} \text{ units}^3. \end{aligned}$$

2.16 Define R as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Which Method Should We Use?

We have studied several methods for finding the volume of a solid of revolution, but how do we know which method to use? It often comes down to a choice of which integral is easiest to evaluate. Figure 2.34 describes the different approaches for solids of revolution around the x-axis. It's up to you to develop the analogous table for solids of revolution around the y-axis.

Comparing the Methods for Finding the Volume of a Solid Revolution around the *x*-axis

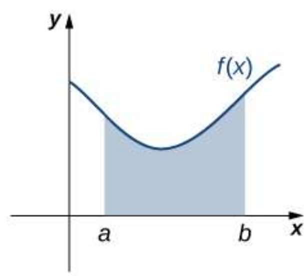
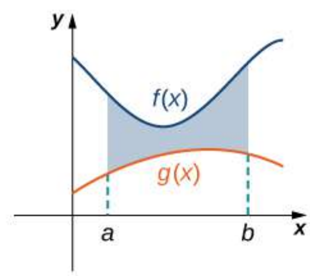
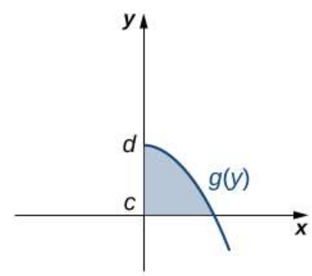
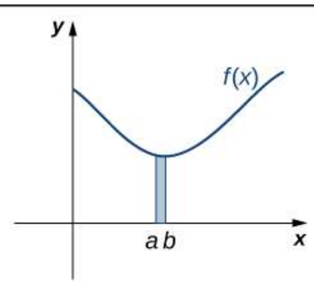
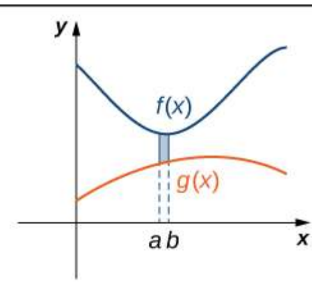
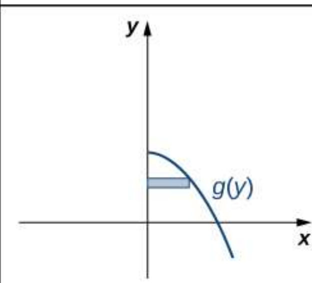
Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V = \int_a^b \pi[f(x)]^2\,dx$	$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\,dx$	$V = \int_c^d 2\pi y\,g(y)\,dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on *x*-axis	$[a, b]$ on *x*-axis	$[c, d]$ on *y*-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

Figure 2.34

Let’s take a look at a couple of additional problems and decide on the best approach to take for solving them.

Example 2.17

Selecting the Best Method

For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral).

a. The region bounded by the graphs of $y = x$, $y = 2 - x$, and the x-axis.

b. The region bounded by the graphs of $y = 4x - x^2$ and the x-axis.

Solution

a. First, sketch the region and the solid of revolution as shown.

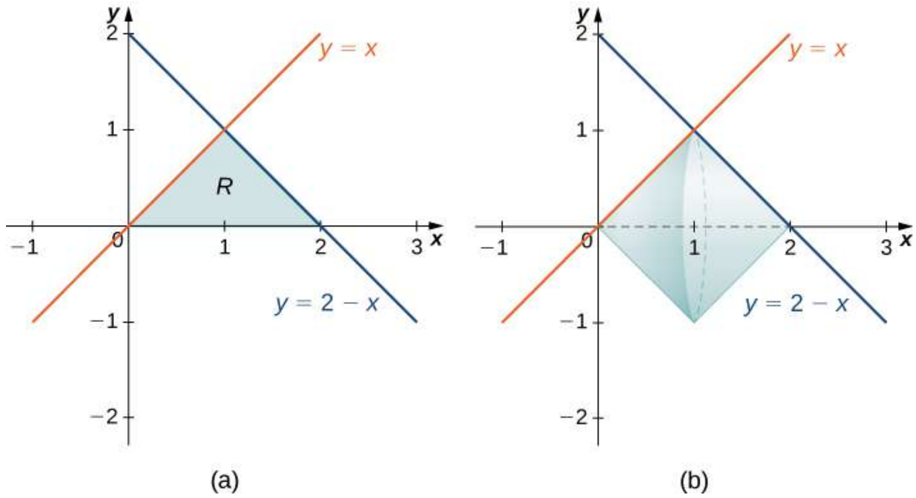


Figure 2.35 (a) The region R bounded by two lines and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, if we want to integrate with respect to x, we would have to break the integral into two pieces, because we have different functions bounding the region over $[0, 1]$ and $[1, 2]$. In this case, using the disk method, we would have

$$V = \int_0^1 \left(\pi x^2\right) dx + \int_1^2 \left(\pi(2 - x)^2\right) dx.$$

If we used the shell method instead, we would use functions of y to represent the curves, producing

$$\begin{aligned} V &= \int_0^1 (2\pi y[(2 - y) - y]) dy \\ &= \int_0^1 (2\pi y[2 - 2y]) dy. \end{aligned}$$

Neither of these integrals is particularly onerous, but since the shell method requires only one integral, and the integrand requires less simplification, we should probably go with the shell method in this case.

b. First, sketch the region and the solid of revolution as shown.

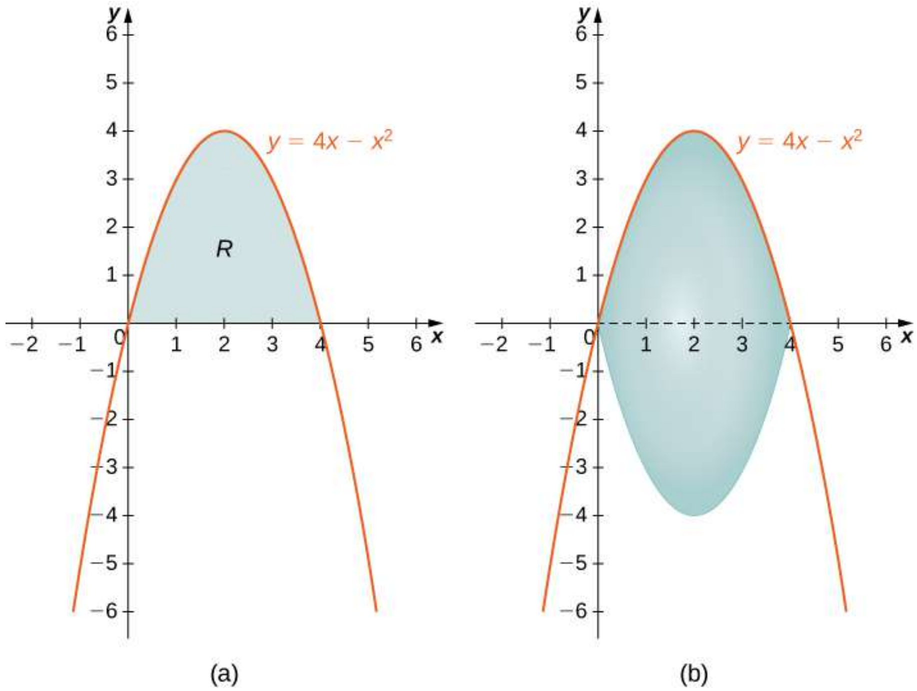


Figure 2.36 (a) The region R between the curve and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, it would be problematic to define a horizontal rectangle; the region is bounded on the left and right by the same function. Therefore, we can dismiss the method of shells. The solid has no cavity in the middle, so we can use the method of disks. Then

$$V = \int_0^4 \pi\left(4x - x^2\right)^2 dx.$$

2.17 Select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral): the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

2.3 EXERCISES

For the following exercise, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the washer method. Use technology to graph the functions and draw a typical slice by hand.

114. **[T]** Over the curve of $y = 3x,\ x = 0,$ and $y = 3$ rotated around the y-axis.

115. **[T]** Under the curve of $y = 3x,\ x = 0,$ and $x = 3$ rotated around the y-axis.

116. **[T]** Over the curve of $y = 3x,\ x = 0,$ and $y = 3$ rotated around the x-axis.

117. **[T]** Under the curve of $y = 3x,\ x = 0,$ and $x = 3$ rotated around the x-axis.

118. **[T]** Under the curve of $y = 2x^3,\ x = 0,$ and $x = 2$ rotated around the y-axis.

119. **[T]** Under the curve of $y = 2x^3,\ x = 0,$ and $x = 2$ rotated around the x-axis.

For the following exercises, use shells to find the volumes of the given solids. Note that the rotated regions lie between the curve and the x-axis and are rotated around the y-axis.

120. $y = 1 - x^2,\ x = 0,$ and $x = 1$

121. $y = 5x^3,\ x = 0,$ and $x = 1$

122. $y = \frac{1}{x},\ x = 1,$ and $x = 100$

123. $y = \sqrt{1 - x^2},\ x = 0,$ and $x = 1$

124. $y = \dfrac{1}{1 + x^2},\ x = 0,$ and $x = 3$

125. $y = \sin x^2,\ x = 0,$ and $x = \sqrt{\pi}$

126. $y = \dfrac{1}{\sqrt{1 - x^2}},\ x = 0,$ and $x = \dfrac{1}{2}$

127. $y = \sqrt{x},\ x = 0,$ and $x = 1$

128. $y = \left(1 + x^2\right)^3,\ x = 0,$ and $x = 1$

129. $y = 5x^3 - 2x^4,\ x = 0,$ and $x = 2$

For the following exercises, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the x-axis.

130. $y = \sqrt{1 - x^2},\ x = 0,$ and $x = 1$

131. $y = x^2,\ x = 0,$ and $x = 2$

132. $y = e^x,\ x = 0,$ and $x = 1$

133. $y = \ln(x),\ x = 1,$ and $x = e$

134. $x = \dfrac{1}{1 + y^2},\ y = 1,$ and $y = 4$

135. $x = \dfrac{1 + y^2}{y},\ y = 0,$ and $y = 2$

136. $x = \cos y,\ y = 0,$ and $y = \pi$

137. $x = y^3 - 4y^2,\ x = -1,$ and $x = 2$

138. $x = ye^y,\ x = -1,$ and $x = 2$

139. $x = \cos ye^y,\ x = 0,$ and $x = \pi$

For the following exercises, find the volume generated when the region between the curves is rotated around the given axis.

140. $y = 3 - x,\ y = 0,\ x = 0,$ and $x = 2$ rotated around the y-axis.

141. $y = x^3,\ y = 0,$ and $y = 8$ rotated around the y-axis.

142. $y = x^2,\ y = x,$ rotated around the y-axis.

143. $y = \sqrt{x},\ x = 0,$ and $x = 1$ rotated around the line $x = 2$.

144. $y = \dfrac{1}{4 - x},\ x = 1,$ and $x = 2$ rotated around the line $x = 4$.

145. $y = \sqrt{x}$ and $y = x^2$ rotated around the y-axis.

146. $y = \sqrt{x}$ and $y = x^2$ rotated around the line $x = 2$.

147. $x = y^3,\ y = \frac{1}{x},\ x = 1,$ and $y = 2$ rotated around the x-axis.

148. $x = y^2$ and $y = x$ rotated around the line $y = 2.$

149. **[T]** Left of $x = \sin(\pi y),$ right of $y = x,$ around the y-axis.

For the following exercises, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find the volume.

150. **[T]** $y = x^2$ and $y = 4x$ rotated around the y-axis.

151. **[T]** $y = \cos(\pi x),\ y = \sin(\pi x),\ x = \frac{1}{4},$ and $x = \frac{5}{4}$ rotated around the y-axis.

152. **[T]** $y = x^2 - 2x,\ x = 2,$ and $x = 4$ rotated around the y-axis.

153. **[T]** $y = x^2 - 2x,\ x = 2,$ and $x = 4$ rotated around the x-axis.

154. **[T]** $y = 3x^3 - 2,\ y = x,$ and $x = 2$ rotated around the x-axis.

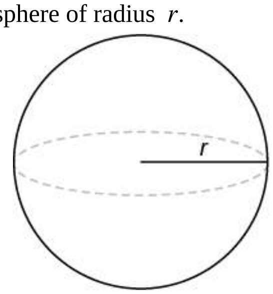
155. **[T]** $y = 3x^3 - 2,\ y = x,$ and $x = 2$ rotated around the y-axis.

156. **[T]** $x = \sin\left(\pi y^2\right)$ and $x = \sqrt{2}y$ rotated around the x-axis.

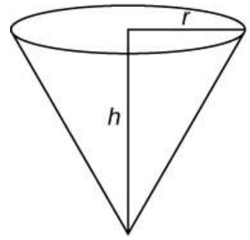
157. **[T]** $x = y^2,\ x = y^2 - 2y + 1,$ and $x = 2$ rotated around the y-axis.

For the following exercises, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.

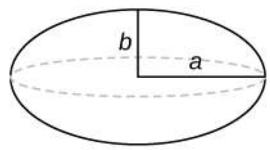
158. Use the method of shells to find the volume of a sphere of radius $r.$

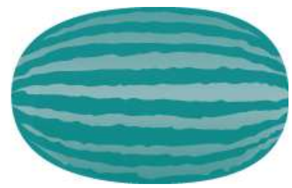


159. Use the method of shells to find the volume of a cone with radius r and height $h.$

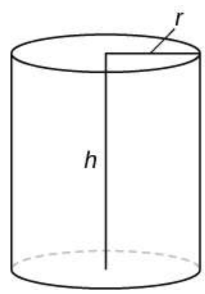


160. Use the method of shells to find the volume of an ellipse $\left(x^2/a^2\right) + \left(y^2/b^2\right) = 1$ rotated around the x-axis.

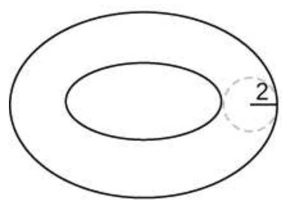




161. Use the method of shells to find the volume of a cylinder with radius r and height $h.$



162. Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4.$



163. Consider the region enclosed by the graphs of $y = f(x),\ y = 1 + f(x),\ x = 0,\ y = 0,$ and $x = a > 0.$ What is the volume of the solid generated when this region is rotated around the y-axis? Assume that the function is defined over the interval $[0,\ a].$

164. Consider the function $y = f(x)$, which decreases from $f(0) = b$ to $f(1) = 0$. Set up the integrals for determining the volume, using both the shell method and the disk method, of the solid generated when this region, with $x = 0$ and $y = 0$, is rotated around the y-axis. Prove that both methods approximate the same volume. Which method is easier to apply? (*Hint:* Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y)$.)

2.4 | Arc Length of a Curve and Surface Area

Learning Objectives

2.4.1 Determine the length of a curve, $y = f(x)$, between two points.

2.4.2 Determine the length of a curve, $x = g(y)$, between two points.

2.4.3 Find the surface area of a solid of revolution.

In this section, we use definite integrals to find the arc length of a curve. We can think of **arc length** as the distance you would travel if you were walking along the path of the curve. Many real-world applications involve arc length. If a rocket is launched along a parabolic path, we might want to know how far the rocket travels. Or, if a curve on a map represents a road, we might want to know how far we have to drive to reach our destination.

We begin by calculating the arc length of curves defined as functions of x, then we examine the same process for curves defined as functions of y. (The process is identical, with the roles of x and y reversed.) The techniques we use to find arc length can be extended to find the surface area of a surface of revolution, and we close the section with an examination of this concept.

Arc Length of the Curve *y* = *f*(*x*)

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and furthermore we require its derivative, $f'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called *smooth*. (This property comes up again in later chapters.)

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0, 1, 2,\ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2,\ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure 2.37 depicts this construct for $n = 5$.

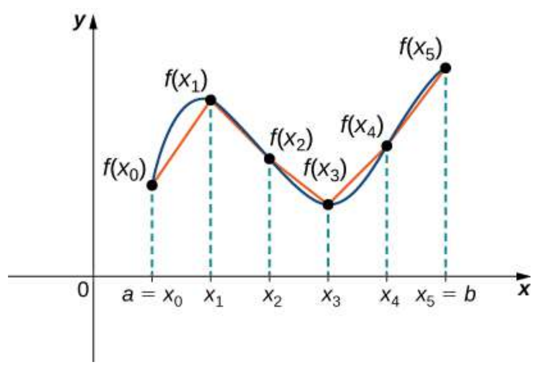


Figure 2.37 We can approximate the length of a curve by adding line segments.

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by Δx. The change in vertical distance varies from interval to interval, though, so we use $\Delta y_i = f(x_i) - f(x_{i-1})$ to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure 2.38. Note that some (or all) Δy_i may be negative.

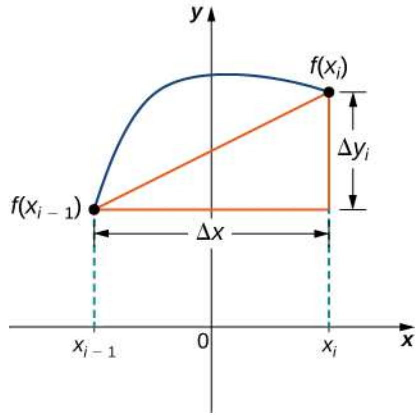


Figure 2.38 A representative line segment approximates the curve over the interval $[x_{i-1}, x_i]$.

By the Pythagorean theorem, the length of the line segment is $\sqrt{(\Delta x)^2 + (\Delta y_i)^2}$. We can also write this as $\Delta x\sqrt{1 + ((\Delta y_i)/(\Delta x))^2}$. Now, by the Mean Value Theorem, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/(\Delta x)$. Then the length of the line segment is given by $\Delta x\sqrt{1 + [f'(x_i^*)]^2}$. Adding up the lengths of all the line segments, we get

$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

We summarize these findings in the following theorem.

Theorem 2.4: Arc Length for *y* = *f*(*x*)

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx. \tag{2.7}$$

Note that we are integrating an expression involving $f'(x)$, so we need to be sure $f'(x)$ is integrable. This is why we require $f(x)$ to be smooth. The following example shows how to apply the theorem.

Example 2.18

Calculating the Arc Length of a Function of *x*

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Solution

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \\ &= \int_0^1 \sqrt{1 + 9x}\,dx.\end{aligned}$$

Substitute $u = 1 + 9x$. Then, $du = 9\,dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 10$. Thus,

$$\begin{aligned}\text{Arc Length} &= \int_0^1 \sqrt{1 + 9x}\,dx \\ &= \frac{1}{9}\int_0^1 \sqrt{1 + 9x} 9dx = \frac{1}{9}\int_1^{10} \sqrt{u}\,du \\ &= \frac{1}{9} \cdot \frac{2}{3} u^{3/2}\Big|_1^{10} = \frac{2}{27}\left[10\sqrt{10} - 1\right] \approx 2.268 \text{ units.}\end{aligned}$$

2.18 Let $f(x) = (4/3)x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Although it is nice to have a formula for calculating arc length, this particular theorem can generate expressions that are difficult to integrate. We study some techniques for integration in **Introduction to Techniques of Integration**. In some cases, we may have to use a computer or calculator to approximate the value of the integral.

Example 2.19

Using a Computer or Calculator to Determine the Arc Length of a Function of *x*

Let $f(x) = x^2$. Calculate the arc length of the graph of $f(x)$ over the interval $[1, 3]$.

Solution

We have $f'(x) = 2x$, so $[f'(x)]^2 = 4x^2$. Then the arc length is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx = \int_1^3 \sqrt{1 + 4x^2}\,dx.$$

Using a computer to approximate the value of this integral, we get

$$\int_1^3 \sqrt{1 + 4x^2}\,dx \approx 8.26815.$$

2.19 Let $f(x) = \sin x$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, \pi]$. Use a computer or calculator to approximate the value of the integral.

Arc Length of the Curve *x* = *g*(*y*)

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of $y,$ we can repeat the same process, except we partition the y-axis instead of the x-axis. **Figure 2.39** shows a representative line segment.

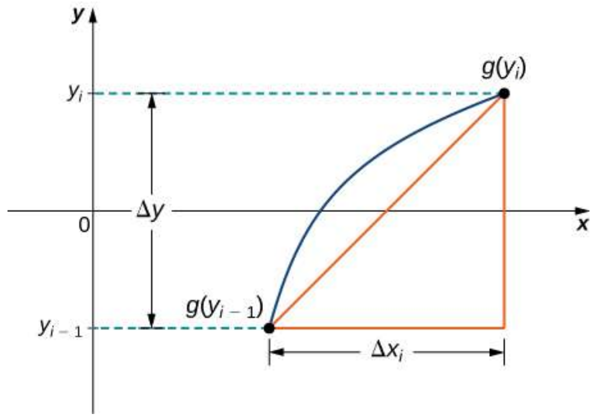


Figure 2.39 A representative line segment over the interval $[y_{i-1}, y_i]$.

Then the length of the line segment is $\sqrt{(\Delta y)^2 + (\Delta x_i)^2},$ which can also be written as $\Delta y\sqrt{1 + ((\Delta x_i)/(\Delta y))^2}.$ If we now follow the same development we did earlier, we get a formula for arc length of a function $x = g(y).$

Theorem 2.5: Arc Length for *x* = *g*(*y*)

Let $g(y)$ be a smooth function over an interval $[c, d].$ Then, the arc length of the graph of $g(y)$ from the point $(c, g(c))$ to the point $(d, g(d))$ is given by

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \tag{2.8}$$

Example 2.20

Calculating the Arc Length of a Function of *y*

Let $g(y) = 3y^3.$ Calculate the arc length of the graph of $g(y)$ over the interval $[1, 2].$

Solution

We have $g'(y) = 9y^2,$ so $[g'(y)]^2 = 81y^4.$ Then the arc length is

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_1^2 \sqrt{1 + 81y^4}\, dy.$$

Using a computer to approximate the value of this integral, we obtain

$$\int_1^2 \sqrt{1 + 81y^4}\, dy \approx 21.0277.$$

2.20 Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 4]$. Use a computer or calculator to approximate the value of the integral.

Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following figure.

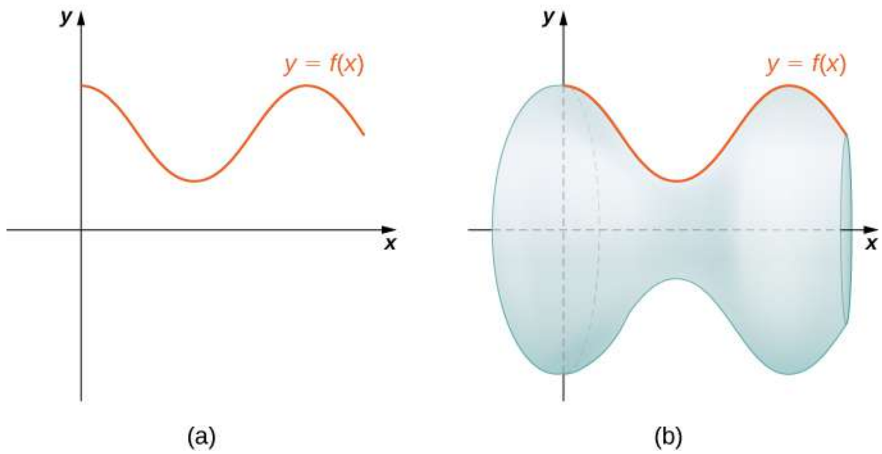


Figure 2.40 (a) A curve representing the function $f(x)$. (b) The surface of revolution formed by revolving the graph of $f(x)$ around the x-axis.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2,\ldots, n,$ construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the x-axis to generate an approximation of the surface of revolution as shown in the following figure.

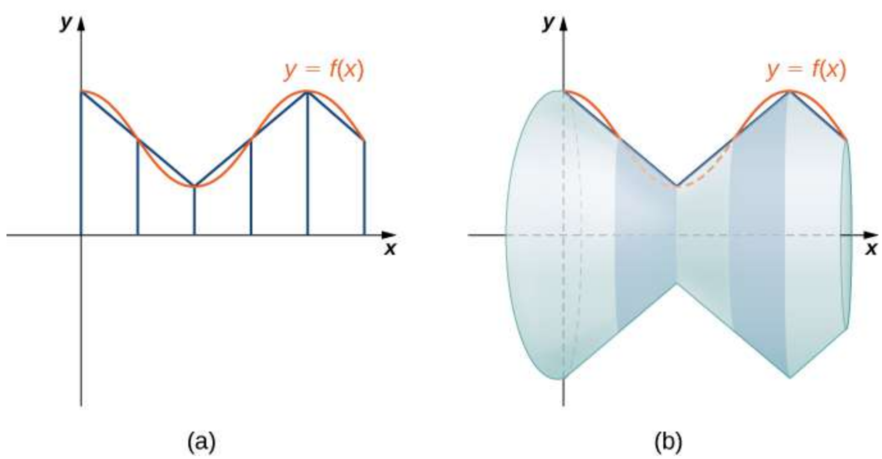


Figure 2.41 (a) Approximating $f(x)$ with line segments. (b) The surface of revolution formed by revolving the line segments around the x-axis.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off). A piece of a cone like this is called a **frustum** of a cone.

To find the surface area of the band, we need to find the lateral surface area, $S,$ of the frustum (the area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces). Let r_1 and r_2 be the radii of the wide end and the narrow end of the frustum, respectively, and let l be the slant height of the frustum as shown in the following figure.

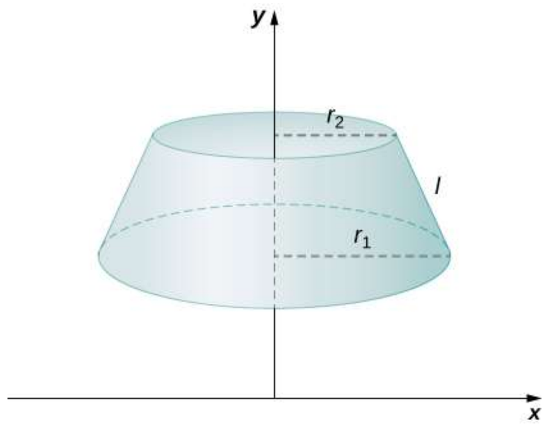


Figure 2.42 A frustum of a cone can approximate a small part of surface area.

We know the lateral surface area of a cone is given by

$$\text{Lateral Surface Area} = \pi rs,$$

where r is the radius of the base of the cone and s is the slant height (see the following figure).

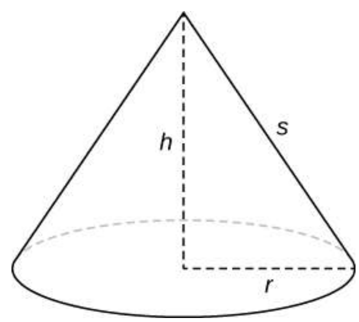


Figure 2.43 The lateral surface area of the cone is given by $\pi rs.$

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (see the following figure).

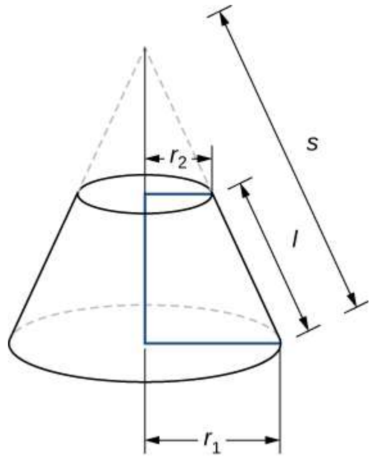


Figure 2.44 Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s}.$$

Solving for s, we get

$$\begin{aligned}\frac{r_2}{r_1} &= \frac{s-l}{s}\\ r_2 s &= r_1(s-l)\\ r_2 s &= r_1 s - r_1 l\\ r_1 l &= r_1 s - r_2 s\\ r_1 l &= (r_1 - r_2)s\\ \frac{r_1 l}{r_1 - r_2} &= s.\end{aligned}$$

Then the lateral surface area (SA) of the frustum is

$$\begin{aligned}S &= \text{(Lateral SA of large cone)} - \text{(Lateral SA of small cone)}\\ &= \pi r_1 s - \pi r_2 (s-l)\\ &= \pi r_1\left(\frac{r_1 l}{r_1 - r_2}\right) - \pi r_2\left(\frac{r_1 l}{r_1 - r_2} - l\right)\\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \pi r_2 l\\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_2 l(r_1 - r_2)}{r_1 - r_2}\\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_1 r_2 l}{r_1 - r_2} - \frac{\pi r_2{}^2 l}{r_1 - r_2}\\ &= \frac{\pi\left(r_1^2 - r_2^2\right)l}{r_1 - r_2} = \frac{\pi(r_1 - r_2)(r_1 + r_2)l}{r_1 - r_2} = \pi(r_1 + r_2)l.\end{aligned}$$

Let's now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the x-axis. A representative band is shown in the following figure.

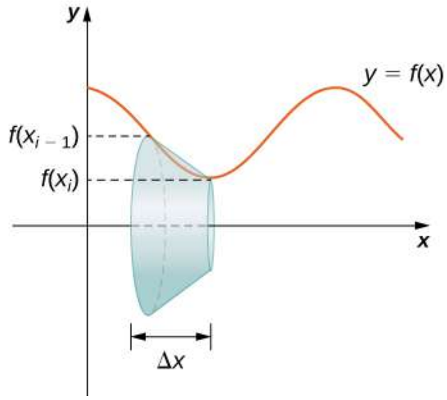


Figure 2.45 A representative band used for determining surface area.

Note that the slant height of this frustum is just the length of the line segment used to generate it. So, applying the surface area formula, we have

$$\begin{aligned} S &= \pi(r_1 + r_2)l \\ &= \pi\left(f(x_{i-1}) + f(x_i)\right)\sqrt{\Delta x^2 + (\Delta y_i)^2} \\ &= \pi\left(f(x_{i-1}) + f(x_i)\right)\Delta x\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}. \end{aligned}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/\Delta x.$ This gives us

$$S = \pi\left(f(x_{i-1}) + f(x_i)\right)\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2}.$$

Furthermore, since $f(x)$ is continuous, by the Intermediate Value Theorem, there is a point $x_i^{**} \in [x_{i-1}, x_i]$ such that $f(x_i^{**}) = (1/2)\left[f(x_{i-1}) + f(x_i)\right],$ so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2}.$$

This *almost* looks like a Riemann sum, except we have functions evaluated at two different points, x_i^* and $x_i^{**},$ over the interval $[x_{i-1}, x_i].$ Although we do not examine the details here, it turns out that because $f(x)$ is smooth, if we let $n \to \infty,$ the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both x_i^* and x_i^{**} are in the interval $[x_{i-1}, x_i],$ so it makes sense that as $n \to \infty,$ both x_i^* and x_i^{**} approach $x.$ Those of you who are interested in the details should consult an advanced calculus text.

Taking the limit as $n \to \infty,$ we get

$$\text{Surface Area} = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2} = \int_a^b \left(2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\right)dx.$$

As with arc length, we can conduct a similar development for functions of y to get a formula for the surface area of surfaces of revolution about the y-axis. These findings are summarized in the following theorem.

Theorem 2.6: Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\right)dx. \quad (2.9)$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c, d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the y-axis is given by

$$\text{Surface Area} = \int_c^d \left(2\pi g(y)\sqrt{1 + \left(g'(y)\right)^2}\right)dy.$$

Example 2.21

Calculating the Surface Area of a Surface of Revolution 1

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Solution

The graph of $f(x)$ and the surface of rotation are shown in the following figure.

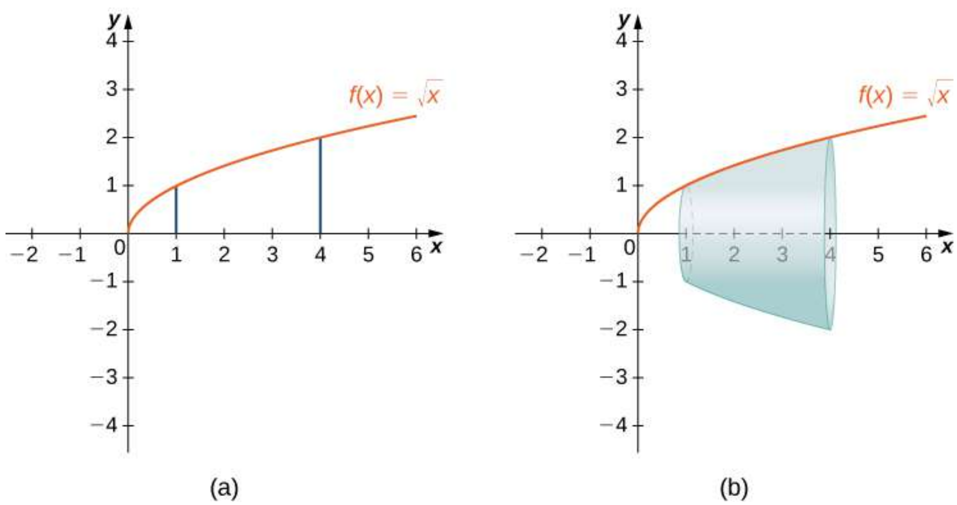


Figure 2.46 (a) The graph of $f(x)$. (b) The surface of revolution.

We have $f(x) = \sqrt{x}$. Then, $f'(x) = 1/(2\sqrt{x})$ and $\left(f'(x)\right)^2 = 1/(4x)$. Then,

$$\begin{aligned}\text{Surface Area} &= \int_a^b \left(2\pi f(x)\sqrt{1+(f'(x))^2}\right)dx \\ &= \int_1^4 \left(2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\right)dx \\ &= \int_1^4 \left(2\pi\sqrt{x+\frac{1}{4}}\right)dx.\end{aligned}$$

Let $u = x + 1/4$. Then, $du = dx$. When $x = 1$, $u = 5/4$, and when $x = 4$, $u = 17/4$. This gives us

$$\begin{aligned}\int_0^1 \left(2\pi\sqrt{x+\frac{1}{4}}\right)dx &= \int_{5/4}^{17/4} 2\pi\sqrt{u}\,du \\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right]\Bigg|_{5/4}^{17/4} = \frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right] \approx 30.846.\end{aligned}$$

2.21 Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Example 2.22

Calculating the Surface Area of a Surface of Revolution 2

Let $f(x) = y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \le y \le 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.

Solution

Notice that we are revolving the curve around the y-axis, and the interval is in terms of y, so we want to rewrite the function as a function of y. We get $x = g(y) = (1/3)y^3$. The graph of $g(y)$ and the surface of rotation are shown in the following figure.

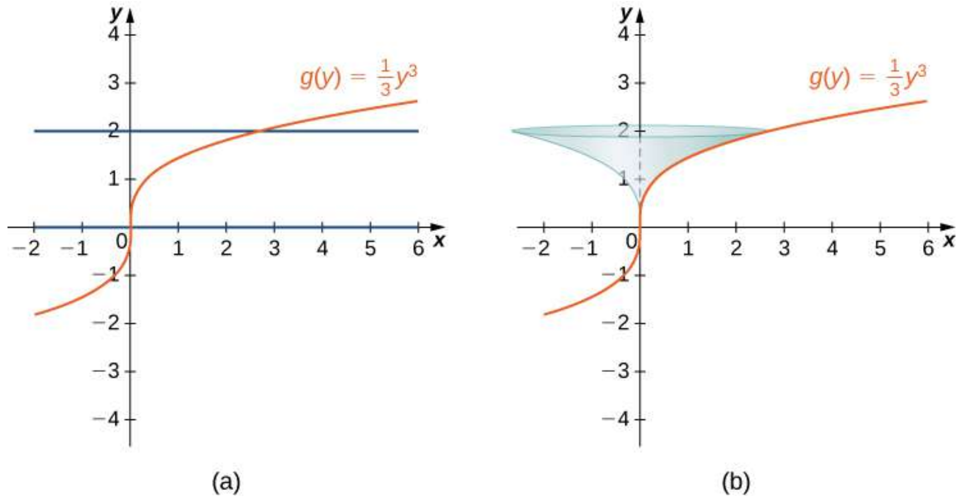


Figure 2.47 (a) The graph of $g(y)$. (b) The surface of revolution.

We have $g(y) = (1/3)y^3$, so $g'(y) = y^2$ and $(g'(y))^2 = y^4$. Then

$$\begin{aligned}\text{Surface Area} &= \int_c^d \left(2\pi g(y)\sqrt{1 + (g'(y))^2}\right)dy \\ &= \int_0^2 \left(2\pi\left(\frac{1}{3}y^3\right)\sqrt{1 + y^4}\right)dy \\ &= \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1 + y^4}\right)dy.\end{aligned}$$

Let $u = y^4 + 1$. Then $du = 4y^3\,dy$. When $y = 0$, $u = 1$, and when $y = 2$, $u = 17$. Then

$$\begin{aligned}\frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1 + y^4}\right)dy &= \frac{2\pi}{3}\int_1^{17} \frac{1}{4}\sqrt{u}\,du \\ &= \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]\Bigg|_1^{17} = \frac{\pi}{9}\left[(17)^{3/2} - 1\right] \approx 24.118.\end{aligned}$$

2.22 Let $g(y) = \sqrt{9 - y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the y-axis.

2.4 EXERCISES

For the following exercises, find the length of the functions over the given interval.

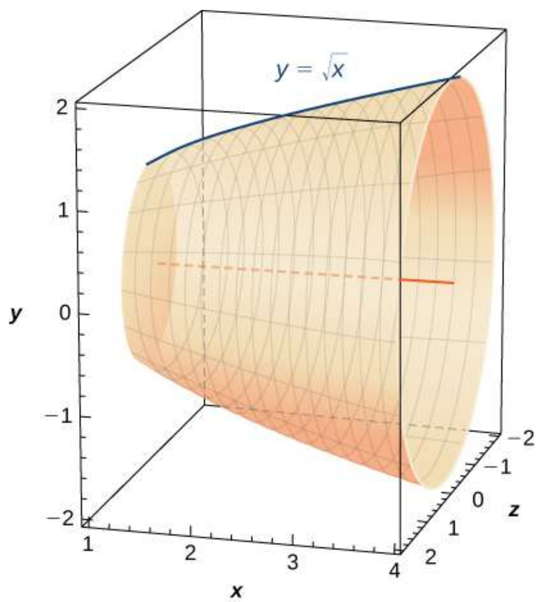
165. $y = 5x$ from $x = 0$ to $x = 2$

166. $y = -\frac{1}{2}x + 25$ from $x = 1$ to $x = 4$

167. $x = 4y$ from $y = -1$ to $y = 1$

168. Pick an arbitrary linear function $x = g(y)$ over any interval of your choice (y_1, y_2). Determine the length of the function and then prove the length is correct by using geometry.

169. Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolves around the x-axis from $(1, 1)$ to $(4, 2)$, as seen here.



170. Find the surface area of the volume generated when the curve $y = x^2$ revolves around the y-axis from $(1, 1)$ to $(3, 9)$.

For the following exercises, find the lengths of the functions of x over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

171. $y = x^{3/2}$ from $(0, 0)$ to $(1, 1)$

172. $y = x^{2/3}$ from $(1, 1)$ to $(8, 4)$

173. $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 1$

174. $y = \frac{1}{3}(x^2 - 2)^{3/2}$ from $x = 2$ to $x = 4$

175. **[T]** $y = e^x$ on $x = 0$ to $x = 1$

176. $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 3$

177. $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$

178. $y = \frac{2x^{3/2}}{3} - \frac{x^{1/2}}{2}$ from $x = 1$ to $x = 4$

179. $y = \frac{1}{27}(9x^2 + 6)^{3/2}$ from $x = 0$ to $x = 2$

180. **[T]** $y = \sin x$ on $x = 0$ to $x = \pi$

For the following exercises, find the lengths of the functions of y over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

181. $y = \frac{5 - 3x}{4}$ from $y = 0$ to $y = 4$

182. $x = \frac{1}{2}(e^y + e^{-y})$ from $y = -1$ to $y = 1$

183. $x = 5y^{3/2}$ from $y = 0$ to $y = 1$

184. **[T]** $x = y^2$ from $y = 0$ to $y = 1$

185. $x = \sqrt{y}$ from $y = 0$ to $y = 1$

186. $x = \frac{2}{3}(y^2 + 1)^{3/2}$ from $y = 1$ to $y = 3$

187. **[T]** $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$

188. **[T]** $x = \cos^2 y$ from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$

189. **[T]** $x = 4^y$ from $y = 0$ to $y = 2$

190. **[T]** $x = \ln(y)$ on $y = \frac{1}{e}$ to $y = e$

For the following exercises, find the surface area of the volume generated when the following curves revolve around the x-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

191. $y = \sqrt{x}$ from $x = 2$ to $x = 6$

192. $y = x^3$ from $x = 0$ to $x = 1$

193. $y = 7x$ from $x = -1$ to $x = 1$

194. **[T]** $y = \frac{1}{x^2}$ from $x = 1$ to $x = 3$

195. $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

196. $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$

197. $y = 5x$ from $x = 1$ to $x = 5$

198. **[T]** $y = \tan x$ from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$

For the following exercises, find the surface area of the volume generated when the following curves revolve around the y-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

199. $y = x^2$ from $x = 0$ to $x = 2$

200. $y = \frac{1}{2}x^2 + \frac{1}{2}$ from $x = 0$ to $x = 1$

201. $y = x + 1$ from $x = 0$ to $x = 3$

202. **[T]** $y = \frac{1}{x}$ from $x = \frac{1}{2}$ to $x = 1$

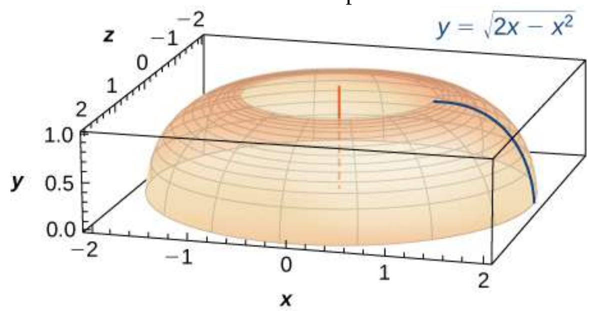
203. $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$

204. **[T]** $y = 3x^4$ from $x = 0$ to $x = 1$

205. **[T]** $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

206. **[T]** $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$

207. The base of a lamp is constructed by revolving a quarter circle $y = \sqrt{2x - x^2}$ around the y-axis from $x = 1$ to $x = 2$, as seen here. Create an integral for the surface area of this curve and compute it.



208. A light bulb is a sphere with radius $1/2$ in. with the bottom sliced off to fit exactly onto a cylinder of radius $1/4$ in. and length $1/3$ in., as seen here. The sphere is cut off at the bottom to fit exactly onto the cylinder, so the radius of the cut is $1/4$ in. Find the surface area (not including the top or bottom of the cylinder).

209. **[T]** A lampshade is constructed by rotating $y = 1/x$ around the x-axis from $y = 1$ to $y = 2$, as seen here. Determine how much material you would need to construct this lampshade—that is, the surface area—accurate to four decimal places.

210. **[T]** An anchor drags behind a boat according to the function $y = 24e^{-x/2} - 24$, where y represents the depth beneath the boat and x is the horizontal distance of the anchor from the back of the boat. If the anchor is 23 ft below the boat, how much rope do you have to pull to reach the anchor? Round your answer to three decimal places.

211. **[T]** You are building a bridge that will span 10 ft. You intend to add decorative rope in the shape of $y = 5|\sin((x\pi)/5)|$, where x is the distance in feet from one end of the bridge. Find out how much rope you need to buy, rounded to the nearest foot.

For the following exercises, find the exact arc length for the following problems over the given interval.

212. $y = \ln(\sin x)$ from $x = \pi/4$ to $x = (3\pi)/4$. (*Hint:* Recall trigonometric identities.)

213. Draw graphs of $y = x^2$, $y = x^6$, and $y = x^{10}$. For $y = x^n$, as n increases, formulate a prediction on the arc length from $(0, 0)$ to $(1, 1)$. Now, compute the lengths of these three functions and determine whether your prediction is correct.

214. Compare the lengths of the parabola $x = y^2$ and the line $x = by$ from $(0, 0)$ to $\left(b^2, b\right)$ as b increases. What do you notice?

215. Solve for the length of $x = y^2$ from $(0, 0)$ to $(1, 1)$. Show that $x = (1/2)y^2$ from $(0, 0)$ to $(2, 2)$ is twice as long. Graph both functions and explain why this is so.

216. **[T]** Which is longer between $(1, 1)$ and $(2, 1/2)$: the hyperbola $y = 1/x$ or the graph of $x + 2y = 3$?

217. Explain why the surface area is infinite when $y = 1/x$ is rotated around the x-axis for $1 \le x < \infty$, but the volume is finite.

2.5 | Physical Applications

Learning Objectives

2.5.1 Determine the mass of a one-dimensional object from its linear density function.
2.5.2 Determine the mass of a two-dimensional circular object from its radial density function.
2.5.3 Calculate the work done by a variable force acting along a line.
2.5.4 Calculate the work done in pumping a liquid from one height to another.
2.5.5 Find the hydrostatic force against a submerged vertical plate.

In this section, we examine some physical applications of integration. Let's begin with a look at calculating mass from a density function. We then turn our attention to work, and close the section with a study of hydrostatic force.

Mass and Density

We can use integration to develop a formula for calculating mass based on a density function. First we consider a thin rod or wire. Orient the rod so it aligns with the x-axis, with the left end of the rod at $x = a$ and the right end of the rod at $x = b$ (Figure 2.48). Note that although we depict the rod with some thickness in the figures, for mathematical purposes we assume the rod is thin enough to be treated as a one-dimensional object.

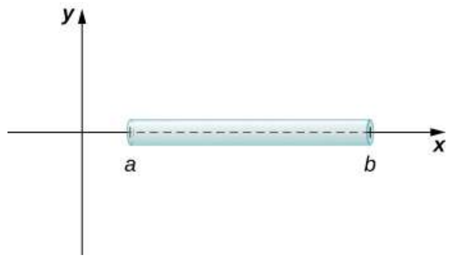


Figure 2.48 We can calculate the mass of a thin rod oriented along the x-axis by integrating its density function.

If the rod has constant density $\rho,$ given in terms of mass per unit length, then the mass of the rod is just the product of the density and the length of the rod: $(b - a)\rho.$ If the density of the rod is not constant, however, the problem becomes a little more challenging. When the density of the rod varies from point to point, we use a linear **density function**, $\rho(x),$ to denote the density of the rod at any point, $x.$ Let $\rho(x)$ be an integrable linear density function. Now, for $i = 0, 1, 2,\ldots, n$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b],$ and for $i = 1, 2,\ldots, n$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i].$ Figure 2.49 shows a representative segment of the rod.

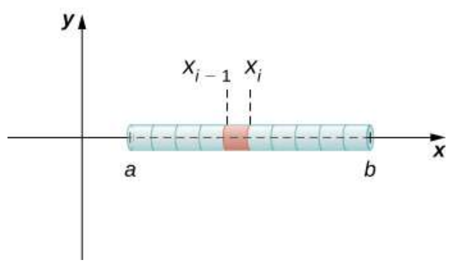


Figure 2.49 A representative segment of the rod.

The mass m_i of the segment of the rod from x_{i-1} to x_i is approximated by

$$m_i \approx \rho(x_i^*)(x_i - x_{i-1}) = \rho(x_i^*)\Delta x.$$

Adding the masses of all the segments gives us an approximation for the mass of the entire rod:

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we get an expression for the exact mass of the rod:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x = \int_a^b \rho(x)dx.$$

We state this result in the following theorem.

Theorem 2.7: Mass–Density Formula of a One-Dimensional Object

Given a thin rod oriented along the x-axis over the interval $[a, b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point *x* in the interval. Then the mass of the rod is given by

$$m = \int_a^b \rho(x)dx. \tag{2.10}$$

We apply this theorem in the next example.

Example 2.23

Calculating Mass from Linear Density

Consider a thin rod oriented on the x-axis over the interval $[\pi/2, \pi]$. If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

Solution

Applying **Equation 2.10** directly, we have

$$m = \int_a^b \rho(x)dx = \int_{\pi/2}^{\pi} \sin x\, dx = -\cos x\Big|_{\pi/2}^{\pi} = 1.$$

2.23 Consider a thin rod oriented on the x-axis over the interval $[1, 3]$. If the density of the rod is given by $\rho(x) = 2x^2 + 3$, what is the mass of the rod?

We now extend this concept to find the mass of a two-dimensional disk of radius r. As with the rod we looked at in the one-dimensional case, here we assume the disk is thin enough that, for mathematical purposes, we can treat it as a two-dimensional object. We assume the density is given in terms of mass per unit area (called *area density*), and further assume the density varies only along the disk's radius (called *radial density*). We orient the disk in the xy-plane, with the center at the origin. Then, the density of the disk can be treated as a function of x, denoted $\rho(x)$. We assume $\rho(x)$ is integrable. Because density is a function of x, we partition the interval from $[0, r]$ along the x-axis. For $i = 0, 1, 2,\ldots, n$, let $P = \{x_i\}$ be a regular partition of the interval $[0, r]$, and for $i = 1, 2,\ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Now, use the partition to break up the disk into thin (two-dimensional) washers. A disk and a representative washer are depicted in the following figure.

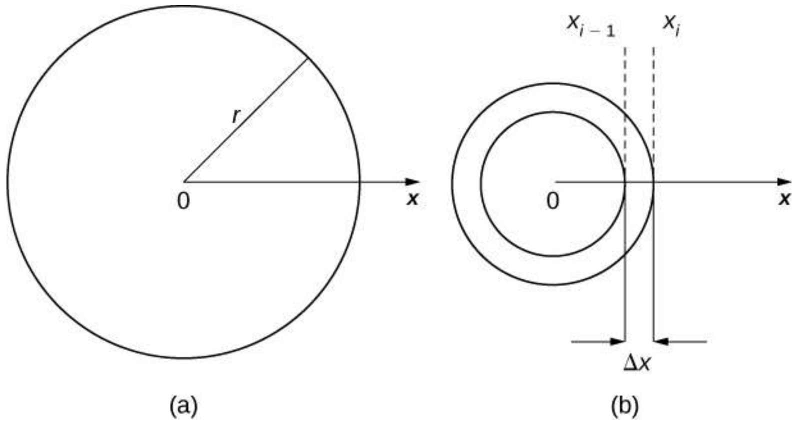


Figure 2.50 (a) A thin disk in the *xy*-plane. (b) A representative washer.

We now approximate the density and area of the washer to calculate an approximate mass, m_i. Note that the area of the washer is given by

$$\begin{aligned} A_i &= \pi(x_i)^2 - \pi(x_{i-1})^2 \\ &= \pi\left[x_i^2 - x_{i-1}^2\right] \\ &= \pi(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \pi(x_i + x_{i-1})\Delta x. \end{aligned}$$

You may recall that we had an expression similar to this when we were computing volumes by shells. As we did there, we use $x_i^* \approx (x_i + x_{i-1})/2$ to approximate the average radius of the washer. We obtain

$$A_i = \pi(x_i + x_{i-1})\Delta x \approx 2\pi x_i^* \Delta x.$$

Using $\rho(x_i^*)$ to approximate the density of the washer, we approximate the mass of the washer by

$$m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x.$$

Adding up the masses of the washers, we see the mass m of the entire disk is approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x.$$

We again recognize this as a Riemann sum, and take the limit as $n \to \infty$. This gives us

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x = \int_0^r 2\pi x \rho(x) dx.$$

We summarize these findings in the following theorem.

Theorem 2.8: Mass–Density Formula of a Circular Object

Let $\rho(x)$ be an integrable function representing the radial density of a disk of radius r. Then the mass of the disk is given by

$$m = \int_0^r 2\pi x \rho(x) dx. \quad (2.11)$$

Example 2.24

Calculating Mass from Radial Density

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

Solution

Applying the formula, we find

$$\begin{aligned} m &= \int_0^r 2\pi x \rho(x) dx \\ &= \int_0^4 2\pi x \sqrt{x} dx = 2\pi \int_0^4 x^{3/2} dx \\ &= 2\pi \frac{2}{5} x^{5/2} \Big|_0^4 = \frac{4\pi}{5}[32] = \frac{128\pi}{5}. \end{aligned}$$

2.24 Let $\rho(x) = 3x + 2$ represent the radial density of a disk. Calculate the mass of a disk of radius 2.

Work Done by a Force

We now consider work. In physics, work is related to force, which is often intuitively defined as a push or pull on an object. When a force moves an object, we say the force does work on the object. In other words, work can be thought of as the amount of energy it takes to move an object. According to physics, when we have a constant force, work can be expressed as the product of force and distance.

In the English system, the unit of force is the pound and the unit of distance is the foot, so work is given in foot-pounds. In the metric system, kilograms and meters are used. One newton is the force needed to accelerate 1 kilogram of mass at the rate of 1 m/sec^2. Thus, the most common unit of work is the newton-meter. This same unit is also called the *joule*. Both are defined as kilograms times meters squared over seconds squared $\left(\text{kg} \cdot \text{m}^2/\text{s}^2\right)$.

When we have a constant force, things are pretty easy. It is rare, however, for a force to be constant. The work done to compress (or elongate) a spring, for example, varies depending on how far the spring has already been compressed (or stretched). We look at springs in more detail later in this section.

Suppose we have a variable force $F(x)$ that moves an object in a positive direction along the *x*-axis from point a to point b. To calculate the work done, we partition the interval $[a, b]$ and estimate the work done over each subinterval. So, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. To calculate the work done to move an object from point x_{i-1} to point $x_i,$ we assume the force is roughly constant over the interval, and use $F(x_i^*)$ to approximate the force. The work done over the interval $[x_{i-1}, x_i],$ then, is given by

$$W_i \approx F(x_i^*)(x_i - x_{i-1}) = F(x_i^*)\Delta x.$$

Therefore, the work done over the interval $[a, b]$ is approximately

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Taking the limit of this expression as $n \to \infty$ gives us the exact value for work:

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i^*)\Delta x = \int_a^b F(x)dx.$$

Thus, we can define work as follows.

Definition

If a variable force $F(x)$ moves an object in a positive direction along the *x*-axis from point *a* to point *b*, then the **work** done on the object is

$$W = \int_a^b F(x)dx. \tag{2.12}$$

Note that if *F* is constant, the integral evaluates to $F \cdot (b - a) = F \cdot d,$ which is the formula we stated at the beginning of this section.

Now let's look at the specific example of the work done to compress or elongate a spring. Consider a block attached to a horizontal spring. The block moves back and forth as the spring stretches and compresses. Although in the real world we would have to account for the force of friction between the block and the surface on which it is resting, we ignore friction here and assume the block is resting on a frictionless surface. When the spring is at its natural length (at rest), the system is said to be at equilibrium. In this state, the spring is neither elongated nor compressed, and in this equilibrium position the block does not move until some force is introduced. We orient the system such that $x = 0$ corresponds to the equilibrium position (see the following figure).

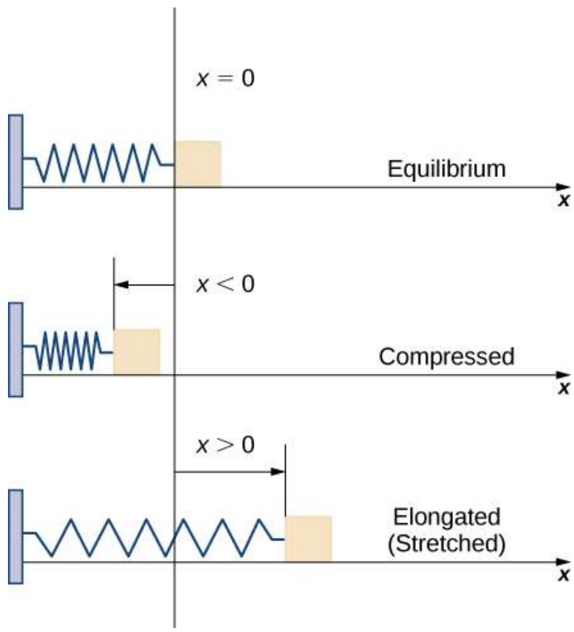


Figure 2.51 A block attached to a horizontal spring at equilibrium, compressed, and elongated.

According to **Hooke's law**, the force required to compress or stretch a spring from an equilibrium position is given by $F(x) = kx,$ for some constant $k.$ The value of k depends on the physical characteristics of the spring. The constant k is called the *spring constant* and is always positive. We can use this information to calculate the work done to compress or elongate a spring, as shown in the following example.

Example 2.25

The Work Required to Stretch or Compress a Spring

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to stretch the spring 0.5 m from the equilibrium position?

Solution

First find the spring constant, k. When $x = -0.2$, we know $F(x) = -10$, so

$$\begin{aligned} F(x) &= kx \\ -10 &= k(-0.2) \\ k &= 50 \end{aligned}$$

and $F(x) = 50x$. Then, to calculate work, we integrate the force function, obtaining

$$W = \int_a^b F(x)dx = \int_0^{0.5} 50x\,dx = 25x^2\Big|_0^{0.5} = 6.25.$$

The work done to stretch the spring is 6.25 J.

2.25 Suppose it takes a force of 8 lb to stretch a spring 6 in. from the equilibrium position. How much work is done to stretch the spring 1 ft from the equilibrium position?

Work Done in Pumping

Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are a little more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water, and it takes more work to move the water from the bottom of the tank than it does to move the water from the top of the tank.

We examine the process in the context of a cylindrical tank, then look at a couple of examples using tanks of different shapes. Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

The first thing we need to do is define a frame of reference. We let x represent the vertical distance below the top of the tank. That is, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (see the following figure).

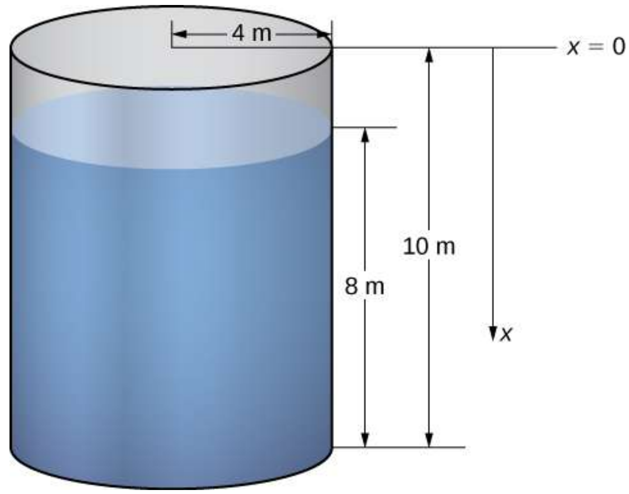


Figure 2.52 How much work is needed to empty a tank partially filled with water?

Using this coordinate system, the water extends from $x = 2$ to $x = 10$. Therefore, we partition the interval $[2, 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[2, 10],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure 2.53 shows a representative layer.

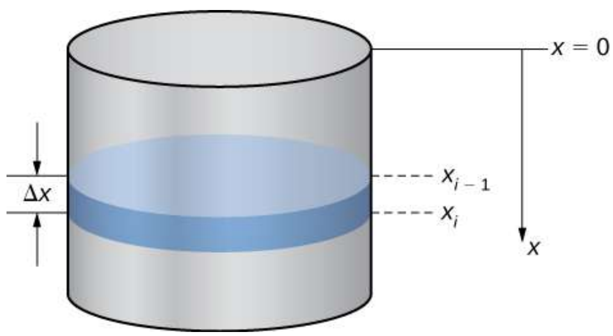


Figure 2.53 A representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is 9800 N/m^3, or 62.4 lb/ft^3, calculating the volume of each layer gives us the weight. In this case, we have

$$V = \pi(4)^2 \Delta x = 16\pi\Delta x.$$

Then, the force needed to lift each layer is

$$F = 9800 \cdot 16\pi\Delta x = 156{,}800\pi\Delta x.$$

Note that this step becomes a little more difficult if we have a noncylindrical tank. We look at a noncylindrical tank in the next example.

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use x_i^* as an approximation of the distance the layer must be lifted. Then the work to lift the ith layer of water W_i is approximately

$$W_i \approx 156{,}800\pi x_i^* \Delta x.$$

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 156{,}800\pi x_i^* \ \Delta x.$$

This is a Riemann sum, so taking the limit as $n \to \infty,$ we get

$$\begin{aligned} W &= \lim_{n \to \infty} \sum_{i=1}^{n} 156{,}800\pi x_i^* \ \Delta x \\ &= 156{,}800\pi \int_2^{10} x dx \\ &= 156{,}800\pi \left[\frac{x^2}{2}\right]\Big|_2^{10} = 7{,}526{,}400\pi \approx 23{,}644{,}883. \end{aligned}$$

The work required to empty the tank is approximately 23,650,000 J.

For pumping problems, the calculations vary depending on the shape of the tank or container. The following problem-solving strategy lays out a step-by-step process for solving pumping problems.

Problem-Solving Strategy: Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.
6. Sum the work required to lift all the layers. This expression is an estimate of the work required to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

We now apply this problem-solving strategy in an example with a noncylindrical tank.

Example 2.26

A Pumping Problem with a Noncylindrical Tank

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft. How much work is required to pump out that amount of water?

Solution

The tank is depicted in **Figure 2.54**. As we did in the example with the cylindrical tank, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (step 1).

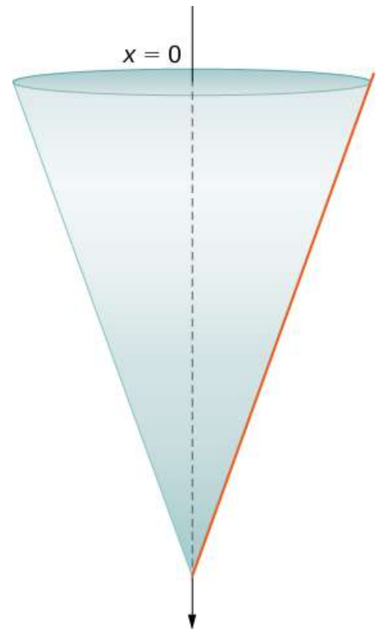


Figure 2.54 A water tank in the shape of an inverted cone.

The tank starts out full and ends with 4 ft of water left, so, based on our chosen frame of reference, we need to partition the interval $[0, 8]$. Then, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[0, 8],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i].$ We can approximate the volume of a layer by using a disk, then use similar triangles to find the radius of the disk (see the following figure).

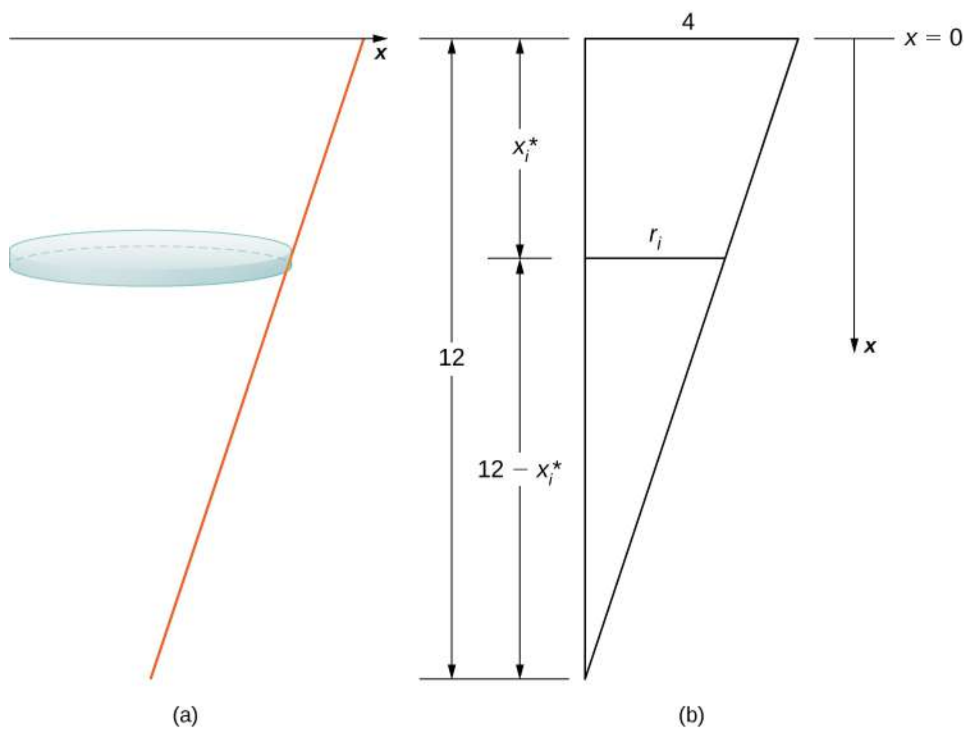


Figure 2.55 Using similar triangles to express the radius of a disk of water.

From properties of similar triangles, we have

$$\begin{aligned} \frac{r_i}{12 - x_i^*} &= \frac{4}{12} = \frac{1}{3} \\ 3r_i &= 12 - x_i^* \\ r_i &= \frac{12 - x_i^*}{3} \\ &= 4 - \frac{x_i^*}{3}. \end{aligned}$$

Then the volume of the disk is

$$V_i = \pi\left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 2).}$$

The weight-density of water is 62.4 lb/ft^3, so the force needed to lift each layer is approximately

$$F_i \approx 62.4\pi\left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 3).}$$

Based on the diagram, the distance the water must be lifted is approximately x_i^* feet (step 4), so the approximate work needed to lift the layer is

$$W_i \approx 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 5).}$$

Summing the work required to lift all the layers, we get an approximate value of the total work:

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 6).}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} W &= \lim_{n \to \infty} \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \\ &= \int_0^8 62.4\pi x\left(4 - \frac{x}{3}\right)^2 dx \\ &= 62.4\pi \int_0^8 x\left(16 - \frac{8x}{3} + \frac{x^2}{9}\right)dx = 62.4\pi \int_0^8 \left(16x - \frac{8x^2}{3} + \frac{x^3}{9}\right)dx \\ &= 62.4\pi\left[8x^2 - \frac{8x^3}{9} + \frac{x^4}{36}\right]\Big|_0^8 = 10{,}649.6\pi \approx 33{,}456.7. \end{aligned}$$

It takes approximately $33{,}450$ ft-lb of work to empty the tank to the desired level.

2.26 A tank is in the shape of an inverted cone, with height 10 ft and base radius 6 ft. The tank is filled to a depth of 8 ft to start with, and water is pumped over the upper edge of the tank until 3 ft of water remain in the tank. How much work is required to pump out that amount of water?

Hydrostatic Force and Pressure

In this last section, we look at the force and pressure exerted on an object submerged in a liquid. In the English system, force is measured in pounds. In the metric system, it is measured in newtons. Pressure is force per unit area, so in the English system we have pounds per square foot (or, perhaps more commonly, pounds per square inch, denoted psi). In the metric system we have newtons per square meter, also called *pascals*.

Let's begin with the simple case of a plate of area A submerged horizontally in water at a depth s (**Figure 2.56**). Then, the force exerted on the plate is simply the weight of the water above it, which is given by $F = \rho As,$ where ρ is the weight density of water (weight per unit volume). To find the **hydrostatic pressure**—that is, the pressure exerted by water on a submerged object—we divide the force by the area. So the pressure is $p = F/A = \rho s.$

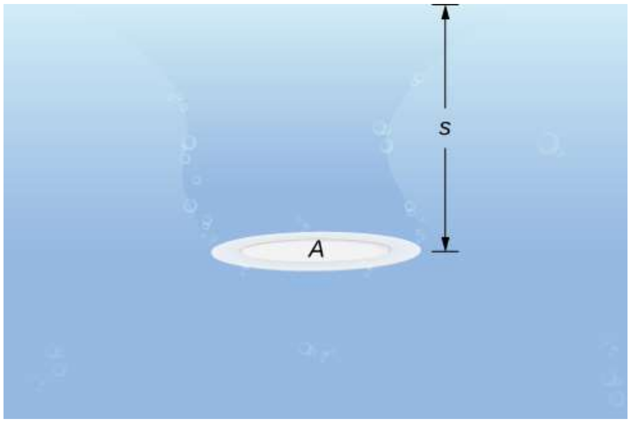


Figure 2.56 A plate submerged horizontally in water.

By Pascal's principle, the pressure at a given depth is the same in all directions, so it does not matter if the plate is submerged horizontally or vertically. So, as long as we know the depth, we know the pressure. We can apply Pascal's principle to find the force exerted on surfaces, such as dams, that are oriented vertically. We cannot apply the formula $F = \rho As$ directly, because the depth varies from point to point on a vertically oriented surface. So, as we have done many times before, we form a partition, a Riemann sum, and, ultimately, a definite integral to calculate the force.

Suppose a thin plate is submerged in water. We choose our frame of reference such that the x-axis is oriented vertically, with the downward direction being positive, and point $x = 0$ corresponding to a logical reference point. Let $s(x)$ denote the depth at point x. Note we often let $x = 0$ correspond to the surface of the water. In this case, depth at any point is simply given by $s(x) = x.$ However, in some cases we may want to select a different reference point for $x = 0,$ so we proceed with the development in the more general case. Last, let $w(x)$ denote the width of the plate at the point $x.$

Assume the top edge of the plate is at point $x = a$ and the bottom edge of the plate is at point $x = b.$ Then, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i].$ The partition divides the plate into several thin, rectangular strips (see the following figure).

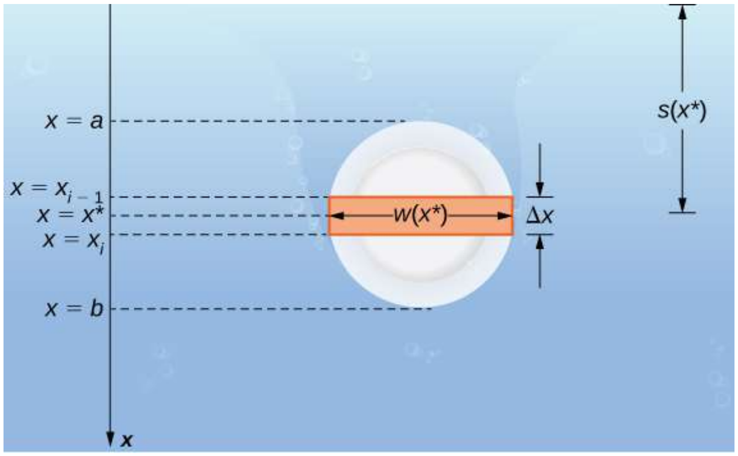


Figure 2.57 A thin plate submerged vertically in water.

Let's now estimate the force on a representative strip. If the strip is thin enough, we can treat it as if it is at a constant depth, $s(x_i^*)$. We then have

$$F_i = \rho As = \rho\left[w(x_i^*)\Delta x\right]s(x_i^*).$$

Adding the forces, we get an estimate for the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho\left[w(x_i^*)\Delta x\right]s(x_i^*).$$

This is a Riemann sum, so taking the limit gives us the exact force. We obtain

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} \rho\left[w(x_i^*)\Delta x\right]s(x_i^*) = \int_a^b \rho w(x)s(x)dx. \tag{2.13}$$

Evaluating this integral gives us the force on the plate. We summarize this in the following problem-solving strategy.

Problem-Solving Strategy: Finding Hydrostatic Force

1. Sketch a picture and select an appropriate frame of reference. (Note that if we select a frame of reference other than the one used earlier, we may have to adjust **Equation 2.13** accordingly.)
2. Determine the depth and width functions, $s(x)$ and $w(x)$.
3. Determine the weight-density of whatever liquid with which you are working. The weight-density of water is $62.4\ \text{lb/ft}^3$, or 9800 N/m^3.
4. Use the equation to calculate the total force.

Example 2.27

Finding Hydrostatic Force

A water trough 15 ft long has ends shaped like inverted isosceles triangles, with base 8 ft and height 3 ft. Find the force on one end of the trough if the trough is full of water.

Solution

Figure 2.58 shows the trough and a more detailed view of one end.

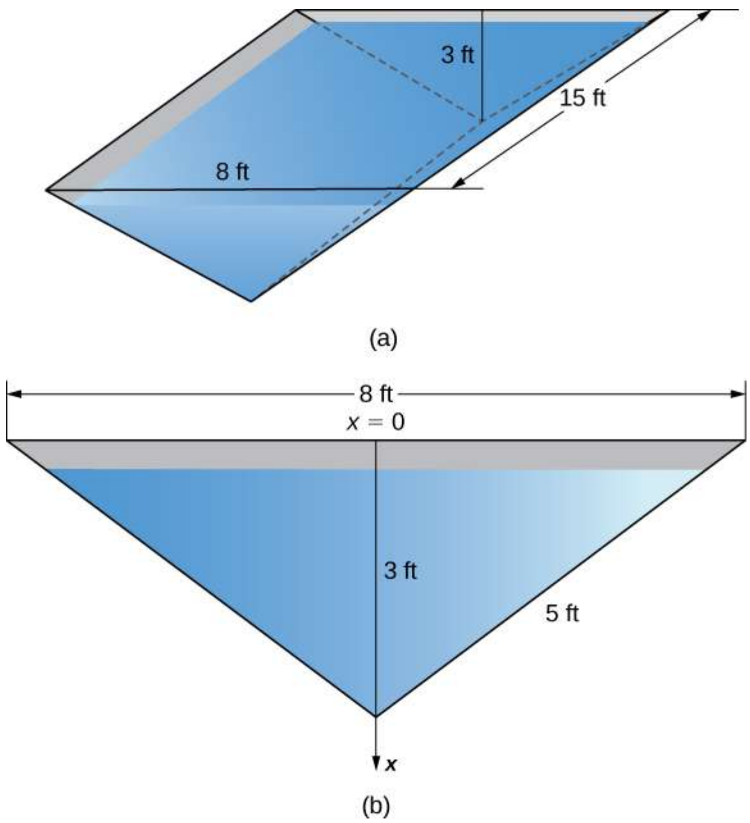


Figure 2.58 (a) A water trough with a triangular cross-section. (b) Dimensions of one end of the water trough.

Select a frame of reference with the x-axis oriented vertically and the downward direction being positive. Select the top of the trough as the point corresponding to $x = 0$ (step 1). The depth function, then, is $s(x) = x$. Using similar triangles, we see that $w(x) = 8 - (8/3)x$ (step 2). Now, the weight density of water is 62.4 lb/ft^3 (step 3), so applying **Equation 2.13**, we obtain

$$\begin{aligned} F &= \int_a^b \rho w(x)s(x)dx \\ &= \int_0^3 62.4\left(8 - \frac{8}{3}x\right)x\,dx = 62.4\int_0^3 \left(8x - \frac{8}{3}x^2\right)dx \\ &= 62.4\left[4x^2 - \frac{8}{9}x^3\right]\Big|_0^3 = 748.8. \end{aligned}$$

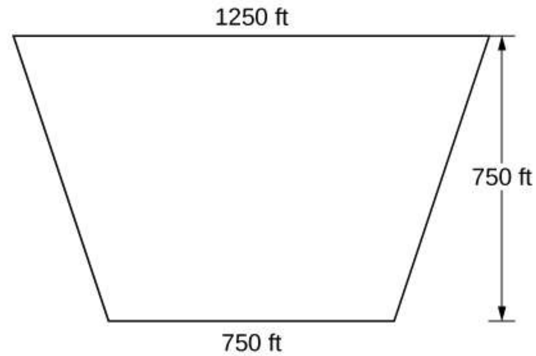
The water exerts a force of 748.8 lb on the end of the trough (step 4).

2.27 A water trough 12 m long has ends shaped like inverted isosceles triangles, with base 6 m and height 4 m. Find the force on one end of the trough if the trough is full of water.

Example 2.28

Chapter Opener: Finding Hydrostatic Force

We now return our attention to the Hoover Dam, mentioned at the beginning of this chapter. The actual dam is arched, rather than flat, but we are going to make some simplifying assumptions to help us with the calculations. Assume the face of the Hoover Dam is shaped like an isosceles trapezoid with lower base 750 ft, upper base 1250 ft, and height 750 ft (see the following figure).

When the reservoir is full, Lake Mead's maximum depth is about 530 ft, and the surface of the lake is about 10 ft below the top of the dam (see the following figure).

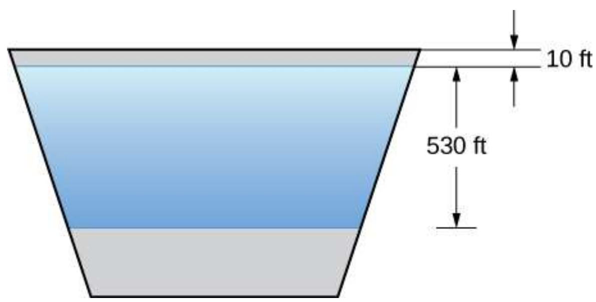


Figure 2.59 A simplified model of the Hoover Dam with assumed dimensions.

a. Find the force on the face of the dam when the reservoir is full.

b. The southwest United States has been experiencing a drought, and the surface of Lake Mead is about 125 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Solution

a. We begin by establishing a frame of reference. As usual, we choose to orient the x-axis vertically, with the downward direction being positive. This time, however, we are going to let $x = 0$ represent the top of the dam, rather than the surface of the water. When the reservoir is full, the surface of the water is 10 ft below the top of the dam, so $s(x) = x - 10$ (see the following figure).

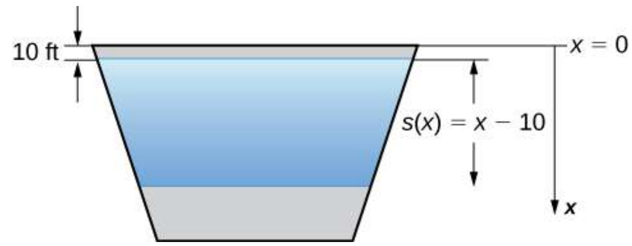


Figure 2.60 We first choose a frame of reference.

To find the width function, we again turn to similar triangles as shown in the figure below.

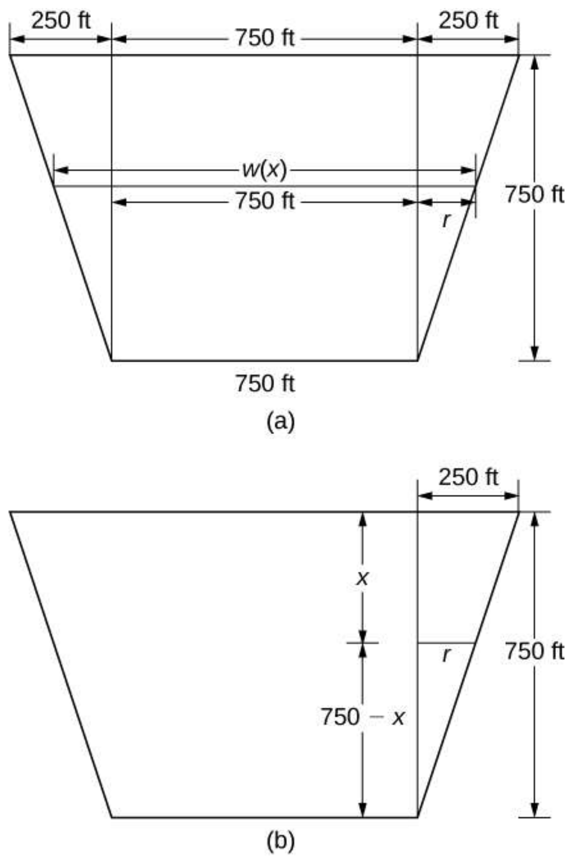


Figure 2.61 We use similar triangles to determine a function for the width of the dam. (a) Assumed dimensions of the dam; (b) highlighting the similar triangles.

From the figure, we see that $w(x) = 750 + 2r$. Using properties of similar triangles, we get $r = 250 - (1/3)x$. Thus,

$$w(x) = 1250 - \frac{2}{3}x \text{ (step 2).}$$

Using a weight-density of 62.4 lb/ft^3 (step 3) and applying **Equation 2.13**, we get

$$\begin{aligned} F &= \int_a^b \rho w(x)s(x)dx \\ &= \int_{10}^{540} 62.4\left(1250 - \frac{2}{3}x\right)(x - 10)dx = 62.4\int_{10}^{540} -\frac{2}{3}\left[x^2 - 1885x + 18750\right]dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - \frac{1885x^2}{2} + 18750x\right]\Bigg|_{10}^{540} \approx 8{,}832{,}245{,}000 \text{ lb} = 4{,}416{,}122.5 \text{ t.} \end{aligned}$$

Note the change from pounds to tons $(2000 \text{ lb} = 1 \text{ ton})$ (step 4). This changes our depth function, $s(x)$, and our limits of integration. We have $s(x) = x - 135$. The lower limit of integration is 135. The upper limit remains 540. Evaluating the integral, we get

$$\begin{aligned} F &= \int_a^b \rho w(x)s(x)dx \\ &= \int_{135}^{540} 62.4\left(1250 - \frac{2}{3}x\right)(x - 135)dx \\ &= -62.4\left(\frac{2}{3}\right)\int_{135}^{540} (x - 1875)(x - 135)dx = -62.4\left(\frac{2}{3}\right)\int_{135}^{540} \left(x^2 - 2010x + 253125\right)dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - 1005x^2 + 253125x\right]\Bigg|_{135}^{540} \approx 5{,}015{,}230{,}000 \text{ lb} = 2{,}507{,}615 \text{ t.} \end{aligned}$$

2.28 When the reservoir is at its average level, the surface of the water is about 50 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

To learn more about Hoover Dam, see this **article (http://www.openstaxcollege.org/l/20_HooverDam)** published by the History Channel.

2.5 EXERCISES

For the following exercises, find the work done.

218. Find the work done when a constant force $F = 12$ lb moves a chair from $x = 0.9$ to $x = 1.1$ ft.

219. How much work is done when a person lifts a 50 lb box of comics onto a truck that is 3 ft off the ground?

220. What is the work done lifting a 20 kg child from the floor to a height of 2 m? (Note that 1 kg equates to 9.8 N)

221. Find the work done when you push a box along the floor 2 m, when you apply a constant force of $F = 100$ N.

222. Compute the work done for a force $F = 12/x^2$ N from $x = 1$ to $x = 2$ m.

223. What is the work done moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

For the following exercises, find the mass of the one-dimensional object.

224. A wire that is 2 ft long (starting at $x = 0$) and has a density function of $\rho(x) = x^2 + 2x$ lb/ft

225. A car antenna that is 3 ft long (starting at $x = 0$) and has a density function of $\rho(x) = 3x + 2$ lb/ft

226. A metal rod that is 8 in. long (starting at $x = 0$) and has a density function of $\rho(x) = e^{1/2x}$ lb/in.

227. A pencil that is 4 in. long (starting at $x = 2$) and has a density function of $\rho(x) = 5/x$ oz/in.

228. A ruler that is 12 in. long (starting at $x = 5$) and has a density function of $\rho(x) = \ln(x) + (1/2)x^2$ oz/in.

For the following exercises, find the mass of the two-dimensional object that is centered at the origin.

229. An oversized hockey puck of radius 2 in. with density function $\rho(x) = x^3 - 2x + 5$

230. A frisbee of radius 6 in. with density function $\rho(x) = e^{-x}$

231. A plate of radius 10 in. with density function $\rho(x) = 1 + \cos(\pi x)$

232. A jar lid of radius 3 in. with density function $\rho(x) = \ln(x + 1)$

233. A disk of radius 5 cm with density function $\rho(x) = \sqrt{3x}$

234. A 12-in. spring is stretched to 15 in. by a force of 75 lb. What is the spring constant?

235. A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

236. A 1-m spring requires 10 J to stretch the spring to 1.1 m. How much work would it take to stretch the spring from 1 m to 1.2 m?

237. A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length of the spring?

238. A shock absorber is compressed 1 in. by a weight of 1 t. What is the spring constant?

239. A force of $F = 20x - x^3$ N stretches a nonlinear spring by x meters. What work is required to stretch the spring from $x = 0$ to $x = 2$ m?

240. Find the work done by winding up a hanging cable of length 100 ft and weight-density 5 lb/ft.

241. For the cable in the preceding exercise, how much work is done to lift the cable 50 ft?

242. For the cable in the preceding exercise, how much additional work is done by hanging a 200 lb weight at the end of the cable?

243. **[T]** A pyramid of height 500 ft has a square base 800 ft by 800 ft. Find the area A at height h. If the rock used to build the pyramid weighs approximately $w = 100\ \text{lb/ft}^3$, how much work did it take to lift all the rock?

244. **[T]** For the pyramid in the preceding exercise, assume there were 1000 workers each working 10 hours a day, 5 days a week, 50 weeks a year. If the workers, on average, lifted 10 100 lb rocks 2 ft/hr, how long did it take to build the pyramid?

245. **[T]** The force of gravity on a mass m is $F = -\left((GMm)/x^2\right)$ newtons. For a rocket of mass $m = 1000\,\text{kg}$, compute the work to lift the rocket from $x = 6400$ to $x = 6500$ km. (*Note*: $G = 6 \times 10^{-17}$ N m^2/kg^2 and $M = 6 \times 10^{24}$ kg.)

246. **[T]** For the rocket in the preceding exercise, find the work to lift the rocket from $x = 6400$ to $x = \infty$.

247. **[T]** A rectangular dam is 40 ft high and 60 ft wide. Compute the total force F on the dam when

a. the surface of the water is at the top of the dam and
b. the surface of the water is halfway down the dam.

248. **[T]** Find the work required to pump all the water out of a cylinder that has a circular base of radius 5 ft and height 200 ft. Use the fact that the density of water is 62 lb/ft^3.

249. **[T]** Find the work required to pump all the water out of the cylinder in the preceding exercise if the cylinder is only half full.

250. **[T]** How much work is required to pump out a swimming pool if the area of the base is 800 ft^2, the water is 4 ft deep, and the top is 1 ft above the water level? Assume that the density of water is 62 lb/ft^3.

251. A cylinder of depth H and cross-sectional area A stands full of water at density ρ. Compute the work to pump all the water to the top.

252. For the cylinder in the preceding exercise, compute the work to pump all the water to the top if the cylinder is only half full.

253. A cone-shaped tank has a cross-sectional area that increases with its depth: $A = \left(\pi r^2 h^2\right)/H^3$. Show that the work to empty it is half the work for a cylinder with the same height and base.

2.6 | Moments and Centers of Mass

Learning Objectives

- **2.6.1** Find the center of mass of objects distributed along a line.
- **2.6.2** Locate the center of mass of a thin plate.
- **2.6.3** Use symmetry to help locate the centroid of a thin plate.
- **2.6.4** Apply the theorem of Pappus for volume.

In this section, we consider centers of mass (also called *centroids*, under certain conditions) and moments. The basic idea of the center of mass is the notion of a balancing point. Many of us have seen performers who spin plates on the ends of sticks. The performers try to keep several of them spinning without allowing any of them to drop. If we look at a single plate (without spinning it), there is a sweet spot on the plate where it balances perfectly on the stick. If we put the stick anywhere other than that sweet spot, the plate does not balance and it falls to the ground. (That is why performers spin the plates; the spin helps keep the plates from falling even if the stick is not exactly in the right place.) Mathematically, that sweet spot is called the *center of mass of the plate*.

In this section, we first examine these concepts in a one-dimensional context, then expand our development to consider centers of mass of two-dimensional regions and symmetry. Last, we use centroids to find the volume of certain solids by applying the theorem of Pappus.

Center of Mass and Moments

Let's begin by looking at the center of mass in a one-dimensional context. Consider a long, thin wire or rod of negligible mass resting on a fulcrum, as shown in **Figure 2.62**(a). Now suppose we place objects having masses m_1 and m_2 at distances d_1 and d_2 from the fulcrum, respectively, as shown in **Figure 2.62**(b).

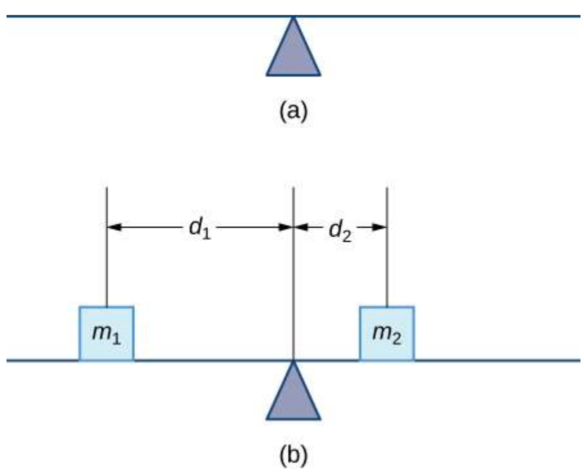


Figure 2.62 (a) A thin rod rests on a fulcrum. (b) Masses are placed on the rod.

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks down and the lighter child is lifted into the air. If the heavier child slides in toward the center, though, the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if $m_1 d_1 = m_2 d_2$.

In the seesaw example, we balanced the system by moving the masses (children) with respect to the fulcrum. However, we are really interested in systems in which the masses are not allowed to move, and instead we balance the system by moving the fulcrum. Suppose we have two point masses, m_1 and m_2, located on a number line at points x_1 and x_2, respectively (**Figure 2.63**). The center of mass, $\bar{x}$, is the point where the fulcrum should be placed to make the system balance.

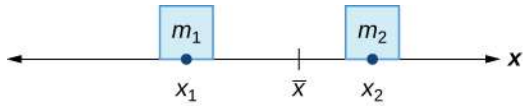


Figure 2.63 The center of mass $\bar{x}$ is the balance point of the system.

Thus, we have

$$\begin{aligned} m_1 |x_1 - \bar{x}| &= m_2 |x_2 - \bar{x}| \\ m_1 (\bar{x} - x_1) &= m_2 (x_2 - \bar{x}) \\ m_1 \bar{x} - m_1 x_1 &= m_2 x_2 - m_2 \bar{x} \\ \bar{x}(m_1 + m_2) &= m_1 x_1 + m_2 x_2 \\ \bar{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \end{aligned}$$

The expression in the numerator, $m_1 x_1 + m_2 x_2$, is called the *first moment of the system with respect to the origin.* If the context is clear, we often drop the word *first* and just refer to this expression as the **moment** of the system. The expression in the denominator, $m_1 + m_2$, is the total mass of the system. Thus, the **center of mass** of the system is the point at which the total mass of the system could be concentrated without changing the moment.

This idea is not limited just to two point masses. In general, if *n* masses, $m_1, m_2, \ldots, m_n$, are placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, then the center of mass of the system is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$

Theorem 2.9: Center of Mass of Objects on a Line

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then, the moment of the system with respect to the origin is given by

$$M = \sum_{i=1}^{n} m_i x_i \qquad (2.14)$$

and the center of mass of the system is given by

$$\bar{x} = \frac{M}{m}. \qquad (2.15)$$

We apply this theorem in the following example.

Example 2.29

Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

$$m_1 = 30\text{ kg, placed at } x_1 = -2\text{ m} \qquad m_2 = 5\text{ kg, placed at } x_2 = 3\text{ m}$$
$$m_3 = 10\text{ kg, placed at } x_3 = 6\text{ m} \qquad m_4 = 15\text{ kg, placed at } x_4 = -3\text{ m}.$$

Find the moment of the system with respect to the origin and find the center of mass of the system.

Solution

First, we need to calculate the moment of the system:

$$\begin{aligned} M &= \sum_{i=1}^{4} m_i x_i \\ &= -60 + 15 + 60 - 45 = -30. \end{aligned}$$

Now, to find the center of mass, we need the total mass of the system:

$$\begin{aligned} m &= \sum_{i=1}^{4} m_i \\ &= 30 + 5 + 10 + 15 = 60\text{ kg}. \end{aligned}$$

Then we have

$$\bar{x} = \frac{M}{m} = \frac{-30}{60} = -\frac{1}{2}.$$

The center of mass is located 1/2 m to the left of the origin.

2.29 Suppose four point masses are placed on a number line as follows:

$$m_1 = 12\text{ kg, placed at } x_1 = -4\text{ m} \qquad m_2 = 12\text{ kg, placed at } x_2 = 4\text{ m}$$
$$m_3 = 30\text{ kg, placed at } x_3 = 2\text{ m} \qquad m_4 = 6\text{ kg, placed at } x_4 = -6\text{ m}.$$

Find the moment of the system with respect to the origin and find the center of mass of the system.

We can generalize this concept to find the center of mass of a system of point masses in a plane. Let m_1 be a point mass located at point (x_1, y_1) in the plane. Then the moment M_x of the mass with respect to the *x*-axis is given by $M_x = m_1 y_1$. Similarly, the moment M_y with respect to the *y*-axis is given by $M_y = m_1 x_1$. Notice that the *x*-coordinate of the point is used to calculate the moment with respect to the *y*-axis, and vice versa. The reason is that the *x*-coordinate gives the distance from the point mass to the *y*-axis, and the *y*-coordinate gives the distance to the *x*-axis (see the following figure).

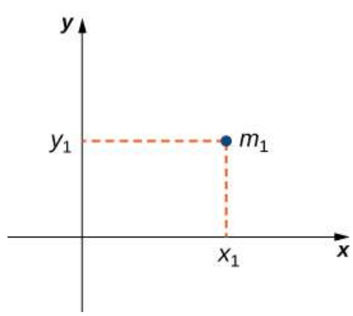


Figure 2.64 Point mass m_1 is located at point (x_1, y_1) in the plane.

If we have several point masses in the *xy*-plane, we can use the moments with respect to the *x*- and *y*-axes to calculate the *x*- and *y*-coordinates of the center of mass of the system.

Theorem 2.10: Center of Mass of Objects in a Plane

Let $m_1, m_2, \ldots, m_n$ be point masses located in the *xy*-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$ respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the moments M_x and M_y of the system with respect to the *x*- and *y*-axes, respectively, are given by

$$M_x = \sum_{i=1}^{n} m_i y_i \quad \text{and} \quad M_y = \sum_{i=1}^{n} m_i x_i. \tag{2.16}$$

Also, the coordinates of the center of mass $(\bar{x}, \bar{y})$ of the system are

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}. \tag{2.17}$$

The next example demonstrates how to apply this theorem.

Example 2.30

Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the *xy*-plane as follows (assume coordinates are given in meters):

$$m_1 = 2\text{ kg, placed at } (-1, 3),$$
$$m_2 = 6\text{ kg, placed at } (1, 1),$$
$$m_3 = 4\text{ kg, placed at } (2, -2).$$

Find the center of mass of the system.

Solution

First we calculate the total mass of the system:

$$m = \sum_{i=1}^{3} m_i = 2 + 6 + 4 = 12\text{ kg}.$$

Next we find the moments with respect to the *x*- and *y*-axes:

$$M_y = \sum_{i=1}^{3} m_i x_i = -2 + 6 + 8 = 12,$$
$$M_x = \sum_{i=1}^{3} m_i y_i = 6 + 6 - 8 = 4.$$

Then we have

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1 \text{ and } \bar{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}.$$

The center of mass of the system is $(1, 1/3),$ in meters.

2.30 Suppose three point masses are placed on a number line as follows (assume coordinates are given in meters):

$$m_1 = 5\text{ kg, placed at } (-2, -3),$$
$$m_2 = 3\text{ kg, placed at } (2, 3),$$
$$m_3 = 2\text{ kg, placed at } (-3, -2).$$

Find the center of mass of the system.

Center of Mass of Thin Plates

So far we have looked at systems of point masses on a line and in a plane. Now, instead of having the mass of a system concentrated at discrete points, we want to look at systems in which the mass of the system is distributed continuously across a thin sheet of material. For our purposes, we assume the sheet is thin enough that it can be treated as if it is two-dimensional. Such a sheet is called a **lamina**. Next we develop techniques to find the center of mass of a lamina. In this section, we also assume the density of the lamina is constant.

Laminas are often represented by a two-dimensional region in a plane. The geometric center of such a region is called its **centroid**. Since we have assumed the density of the lamina is constant, the center of mass of the lamina depends only on the shape of the corresponding region in the plane; it does not depend on the density. In this case, the center of mass of the lamina corresponds to the centroid of the delineated region in the plane. As with systems of point masses, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes.

We first consider a lamina in the shape of a rectangle. Recall that the center of mass of a lamina is the point where the lamina balances. For a rectangle, that point is both the horizontal and vertical center of the rectangle. Based on this understanding, it is clear that the center of mass of a rectangular lamina is the point where the diagonals intersect, which is a result of the **symmetry principle**, and it is stated here without proof.

Theorem 2.11: The Symmetry Principle

If a region R is symmetric about a line l, then the centroid of R lies on l.

Let's turn to more general laminas. Suppose we have a lamina bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in the following figure.

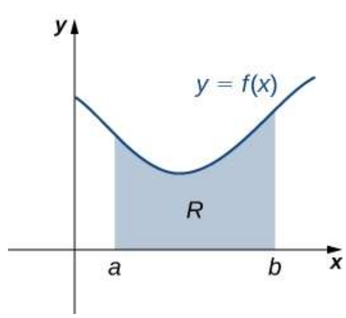


Figure 2.65 A region in the plane representing a lamina.

As with systems of point masses, to find the center of mass of the lamina, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes. As we have done many times before, we approximate these quantities by partitioning the interval $[a, b]$ and constructing rectangles.

For $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of $[a, b]$. Recall that we can choose any point within the interval $[x_{i-1}, x_i]$ as our x_i^*. In this case, we want x_i^* to be the x-coordinate of the centroid of our rectangles. Thus, for $i = 1, 2,\ldots, n,$ we select $x_i^* \in [x_{i-1}, x_i]$ such that x_i^* is the midpoint of the interval. That is, $x_i^* = (x_{i-1} + x_i)/2.$ Now, for $i = 1, 2,\ldots, n,$ construct a rectangle of height $f(x_i^*)$ on $[x_{i-1}, x_i]$. The center of mass of this rectangle is

$\left(x_i^*, \left(f(x_i^*)\right)/2\right),$ as shown in the following figure.

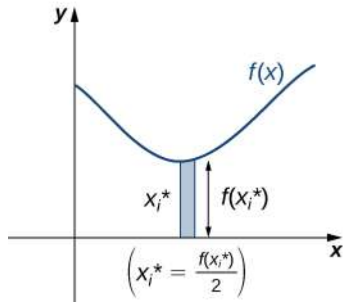


Figure 2.66 A representative rectangle of the lamina.

Next, we need to find the total mass of the rectangle. Let ρ represent the density of the lamina (note that ρ is a constant). In this case, ρ is expressed in terms of mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by ρ. Then, the mass of the rectangle is given by $\rho f(x_i^*)\Delta x.$

To get the approximate mass of the lamina, we add the masses of all the rectangles to get

$$m \approx \sum_{i=1}^{n} \rho f(x_i^*)\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$ gives the exact mass of the lamina:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x = \rho \int_a^b f(x)dx.$$

Next, we calculate the moment of the lamina with respect to the *x*-axis. Returning to the representative rectangle, recall its center of mass is $\left(x_i^*, \left(f(x_i^*)\right)/2\right)$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change the moment. Thus, the moment of the rectangle with respect to the *x*-axis is given by the mass of the rectangle, $\rho f(x_i^*)\Delta x,$ multiplied by the distance from the center of mass to the *x*-axis: $\left(f(x_i^*)\right)/2.$ Therefore, the moment with respect to the *x*-axis of the rectangle is $\rho\left(\left[f(x_i^*)\right]^2/2\right)\Delta x.$ Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the *x*-axis is

$$M_x = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \frac{\left[f(x_i^*)\right]^2}{2}\Delta x = \rho \int_a^b \frac{\left[f(x)\right]^2}{2}dx.$$

We derive the moment with respect to the *y*-axis similarly, noting that the distance from the center of mass of the rectangle to the *y*-axis is x_i^*. Then the moment of the lamina with respect to the *y*-axis is given by

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{n} \rho x_i^* f(x_i^*)\Delta x = \rho \int_a^b x f(x)dx.$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\bar{x} = M_y/m$ and $\bar{y} = M_x/m.$ If we look closely at the expressions for $M_x, M_y,$ and $m,$ we notice that the constant ρ cancels out when $\bar{x}$ and $\bar{y}$ are calculated.

We summarize these findings in the following theorem.

Theorem 2.12: Center of Mass of a Thin Plate in the *xy*-Plane

Let *R* denote a region bounded above by the graph of a continuous function $f(x),$ below by the *x*-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b f(x)dx. \tag{2.18}$$

ii. The moments M_x and M_y of the lamina with respect to the *x*- and *y*-axes, respectively, are

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx \text{ and } M_y = \rho \int_a^b x f(x)dx. \tag{2.19}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}. \tag{2.20}$$

In the next example, we use this theorem to find the center of mass of a lamina.

Example 2.31

Finding the Center of Mass of a Lamina

Let *R* be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the *x*-axis over the interval $[0, 4]$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

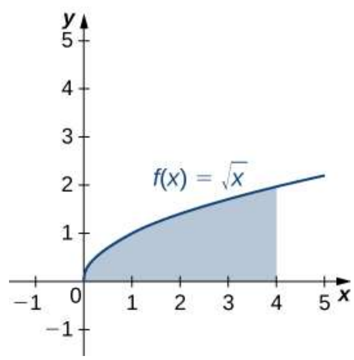


Figure 2.67 Finding the center of mass of a lamina.

Since we are only asked for the centroid of the region, rather than the mass or moments of the associated lamina, we know the density constant ρ cancels out of the calculations eventually. Therefore, for the sake of convenience, let's assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)dx = \int_0^4 \sqrt{x}\, dx \\ &= \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{2}{3}[8 - 0] = \frac{16}{3}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \int_0^4 \frac{x}{2} dx = \frac{1}{4} x^2 \Big|_0^4 = 4 \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho \int_a^b x f(x) dx \\ &= \int_0^4 x\sqrt{x} dx = \int_0^4 x^{3/2} dx \\ &= \frac{2}{5} x^{5/2} \Big|_0^4 = \frac{2}{5}[32 - 0] = \frac{64}{5}. \end{aligned}$$

Thus, we have

$$\bar{x} = \frac{M_y}{m} = \frac{64/5}{16/3} = \frac{64}{5} \cdot \frac{3}{16} = \frac{12}{5} \text{ and } \bar{y} = \frac{M_x}{y} = \frac{4}{16/3} = 4 \cdot \frac{3}{16} = \frac{3}{4}.$$

The centroid of the region is $(12/5, 3/4)$.

2.31 Let R be the region bounded above by the graph of the function $f(x) = x^2$ and below by the x-axis over the interval $[0, 2]$. Find the centroid of the region.

We can adapt this approach to find centroids of more complex regions as well. Suppose our region is bounded above by the graph of a continuous function $f(x)$, as before, but now, instead of having the lower bound for the region be the x-axis, suppose the region is bounded below by the graph of a second continuous function, $g(x)$, as shown in the following figure.

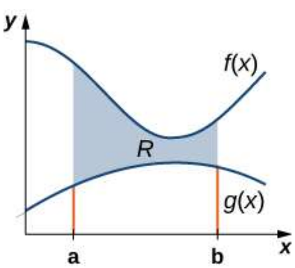


Figure 2.68 A region between two functions.

Again, we partition the interval $[a, b]$ and construct rectangles. A representative rectangle is shown in the following figure.

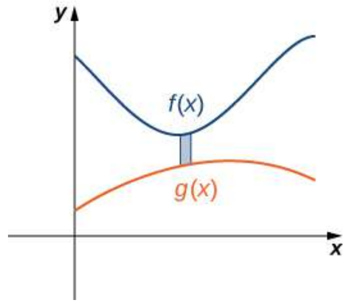


Figure 2.69 A representative rectangle of the region between two functions.

Note that the centroid of this rectangle is $\left(x_i^*, \left(f(x_i^*) + g(x_i^*)\right)/2\right)$. We won't go through all the details of the Riemann sum development, but let's look at some of the key steps. In the development of the formulas for the mass of the lamina and the moment with respect to the *y*-axis, the height of each rectangle is given by $f(x_i^*) - g(x_i^*)$, which leads to the expression $f(x) - g(x)$ in the integrands.

In the development of the formula for the moment with respect to the *x*-axis, the moment of each rectangle is found by multiplying the area of the rectangle, $\rho\left[f(x_i^*) - g(x_i^*)\right]\Delta x$, by the distance of the centroid from the *x*-axis, $\left(f(x_i^*) + g(x_i^*)\right)/2$, which gives $\rho(1/2)\left\{\left[f(x_i^*)\right]^2 - \left[g(x_i^*)\right]^2\right\}\Delta x$. Summarizing these findings, we arrive at the following theorem.

Theorem 2.13: Center of Mass of a Lamina Bounded by Two Functions

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the graph of the continuous function $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho\int_a^b \left[f(x) - g(x)\right]dx. \tag{2.21}$$

ii. The moments M_x and M_y of the lamina with respect to the *x*- and *y*-axes, respectively, are

$$M_x = \rho\int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx \text{ and } M_y = \rho\int_a^b x\left[f(x) - g(x)\right]dx. \tag{2.22}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}. \tag{2.23}$$

We illustrate this theorem in the following example.

Example 2.32

Finding the Centroid of a Region Bounded by Two Functions

Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by the graph of the function $g(x) = x - 1$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

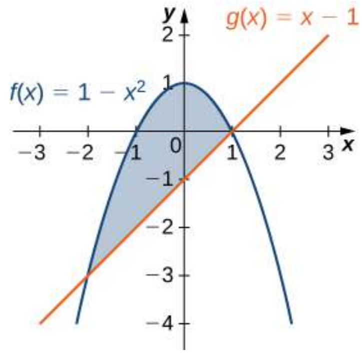


Figure 2.70 Finding the centroid of a region between two curves.

The graphs of the functions intersect at $(-2,\ -3)$ and $(1,\ 0),$ so we integrate from −2 to 1. Once again, for the sake of convenience, assume $\rho = 1.$

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^{1} \left[1 - x^2 - (x-1)\right]dx = \int_{-2}^{1} (2 - x^2 - x)dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]\Big|_{-2}^{1} = \left[2 - \frac{1}{3} - \frac{1}{2}\right] - \left[-4 + \frac{8}{3} - 2\right] = \frac{9}{2}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)dx \\ &= \frac{1}{2}\int_{-2}^{1}\left(\left(1 - x^2\right)^2 - (x-1)^2\right)dx = \frac{1}{2}\int_{-2}^{1}\left(x^4 - 3x^2 + 2x\right)dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - x^3 + x^2\right]\Big|_{-2}^{1} = -\frac{27}{10} \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho \int_a^b x[f(x) - g(x)]dx \\ &= \int_{-2}^{1} x\left[(1 - x^2) - (x-1)\right]dx = \int_{-2}^{1} x\left[2 - x^2 - x\right]dx = \int_{-2}^{1}\left(2x - x^4 - x^2\right)dx \\ &= \left[x^2 - \frac{x^5}{5} - \frac{x^3}{3}\right]\Big|_{-2}^{1} = -\frac{9}{4}. \end{aligned}$$

Therefore, we have

$$\bar{x} = \frac{M_y}{m} = -\frac{9}{4} \cdot \frac{2}{9} = -\frac{1}{2} \text{ and } \bar{y} = \frac{M_x}{y} = -\frac{27}{10} \cdot \frac{2}{9} = -\frac{3}{5}.$$

The centroid of the region is $(-(1/2), -(3/5))$.

2.32 Let R be the region bounded above by the graph of the function $f(x) = 6 - x^2$ and below by the graph of the function $g(x) = 3 - 2x$. Find the centroid of the region.

The Symmetry Principle

We stated the symmetry principle earlier, when we were looking at the centroid of a rectangle. The symmetry principle can be a great help when finding centroids of regions that are symmetric. Consider the following example.

Example 2.33

Finding the Centroid of a Symmetric Region

Let R be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the x-axis. Find the centroid of the region.

Solution

The region is depicted in the following figure.

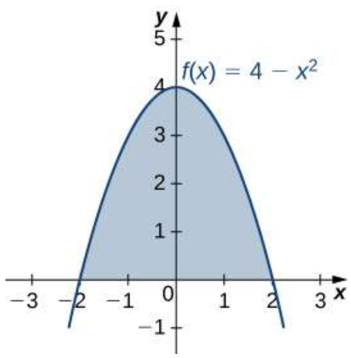


Figure 2.71 We can use the symmetry principle to help find the centroid of a symmetric region.

The region is symmetric with respect to the y-axis. Therefore, the x-coordinate of the centroid is zero. We need only calculate $\bar{y}$. Once again, for the sake of convenience, assume $\rho = 1$.

First, we calculate the total mass:

$$\begin{aligned} m &= \rho\int_a^b f(x)dx \\ &= \int_{-2}^{2}\left(4 - x^2\right)dx \\ &= \left[4x - \frac{x^3}{3}\right]\Big|_{-2}^{2} = \frac{32}{3}. \end{aligned}$$

Next, we calculate the moments. We only need M_x:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \frac{1}{2}\int_{-2}^{2} \left[4 - x^2\right]^2 dx = \frac{1}{2}\int_{-2}^{2} \left(16 - 8x^2 + x^4\right) dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]\Bigg|_{-2}^{2} = \frac{256}{15}. \end{aligned}$$

Then we have

$$\bar{y} = \frac{M_x}{y} = \frac{256}{15} \cdot \frac{3}{32} = \frac{8}{5}.$$

The centroid of the region is $(0, 8/5)$.

2.33 Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by x-axis. Find the centroid of the region.

Student PROJECT

The Grand Canyon Skywalk

The Grand Canyon Skywalk opened to the public on March 28, 2007. This engineering marvel is a horseshoe-shaped observation platform suspended 4000 ft above the Colorado River on the West Rim of the Grand Canyon. Its crystal-clear glass floor allows stunning views of the canyon below (see the following figure).

Figure 2.72 The Grand Canyon Skywalk offers magnificent views of the canyon. (credit: 10da_ralta, Wikimedia Commons)

The Skywalk is a cantilever design, meaning that the observation platform extends over the rim of the canyon, with no visible means of support below it. Despite the lack of visible support posts or struts, cantilever structures are engineered to be very stable and the Skywalk is no exception. The observation platform is attached firmly to support posts that extend 46 ft down into bedrock. The structure was built to withstand 100-mph winds and an 8.0-magnitude earthquake within 50 mi, and is capable of supporting more than 70,000,000 lb.

One factor affecting the stability of the Skywalk is the center of gravity of the structure. We are going to calculate the center of gravity of the Skywalk, and examine how the center of gravity changes when tourists walk out onto the observation platform.

The observation platform is U-shaped. The legs of the U are 10 ft wide and begin on land, under the visitors' center, 48 ft from the edge of the canyon. The platform extends 70 ft over the edge of the canyon.

To calculate the center of mass of the structure, we treat it as a lamina and use a two-dimensional region in the *xy*-plane to represent the platform. We begin by dividing the region into three subregions so we can consider each subregion separately. The first region, denoted R_1, consists of the curved part of the U. We model R_1 as a semicircular annulus, with inner radius 25 ft and outer radius 35 ft, centered at the origin (see the following figure).

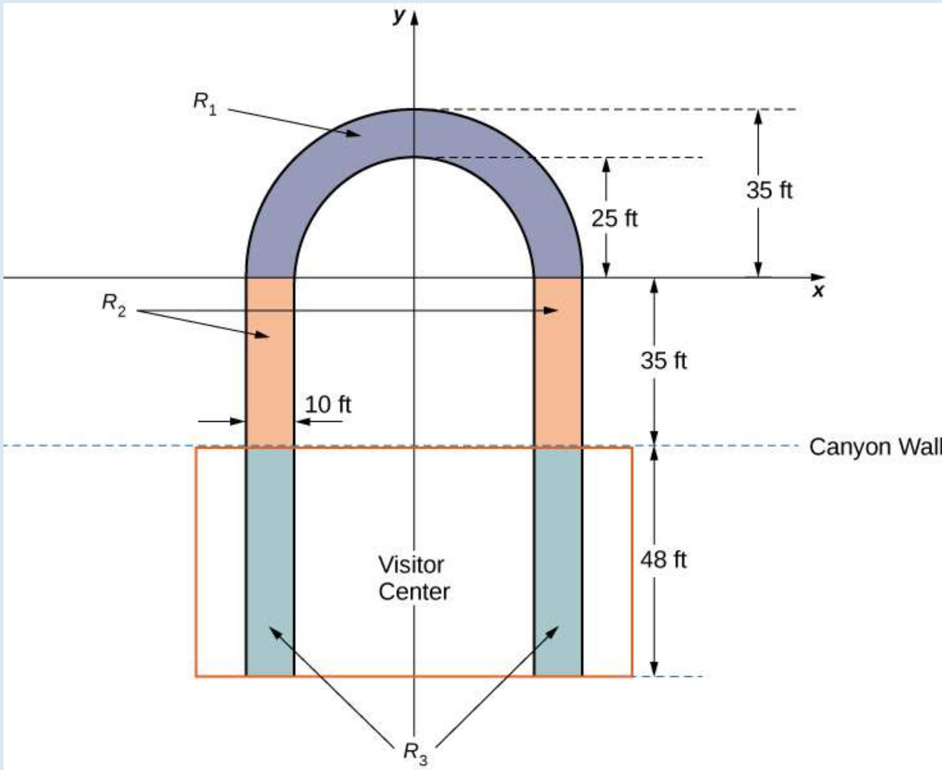


Figure 2.73 We model the Skywalk with three sub-regions.

The legs of the platform, extending 35 ft between R_1 and the canyon wall, comprise the second sub-region, R_2. Last, the ends of the legs, which extend 48 ft under the visitor center, comprise the third sub-region, R_3. Assume the density of the lamina is constant and assume the total weight of the platform is 1,200,000 lb (not including the weight of the visitor center; we will consider that later). Use $g = 32 \text{ ft/sec}^2$.

1. Compute the area of each of the three sub-regions. Note that the areas of regions R_2 and R_3 should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.
2. Determine the mass associated with each of the three sub-regions.
3. Calculate the center of mass of each of the three sub-regions.
4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of the entire platform.
5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of R_3. Treating the visitor center as a point mass, recalculate the center of mass of the system. How does the center of mass change?
6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform is capable of supporting up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the center of gravity of the system be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

Theorem of Pappus

This section ends with a discussion of the **theorem of Pappus for volume**, which allows us to find the volume of particular kinds of solids by using the centroid. (There is also a theorem of Pappus for surface area, but it is much less useful than the theorem for volume.)

Theorem 2.14: Theorem of Pappus for Volume

Let R be a region in the plane and let l be a line in the plane that does not intersect R. Then the volume of the solid of revolution formed by revolving R around l is equal to the area of R multiplied by the distance d traveled by the centroid of R.

Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a, b]$, and for which the axis of revolution is the y-axis. In this case, the area of the region is $A = \int_a^b [f(x) - g(x)]dx.$ Since the axis of rotation is the y-axis, the distance traveled by the centroid of the region depends only on the x-coordinate of the centroid, $\bar{x}$, which is

$$\bar{x} = \frac{M_y}{m},$$

where

$$m = \rho \int_a^b [f(x) - g(x)]dx \text{ and } M_y = \rho \int_a^b x[f(x) - g(x)]dx.$$

Then,

$$d = 2\pi \frac{\rho \int_a^b x[f(x) - g(x)]dx}{\rho \int_a^b [f(x) - g(x)]dx}$$

and thus

$$d \cdot A = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

However, using the method of cylindrical shells, we have

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

So,

$$V = d \cdot A$$

and the proof is complete.

□

Example 2.34

Using the Theorem of Pappus for Volume

Let R be a circle of radius 2 centered at $(4, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

Solution

The region and torus are depicted in the following figure.

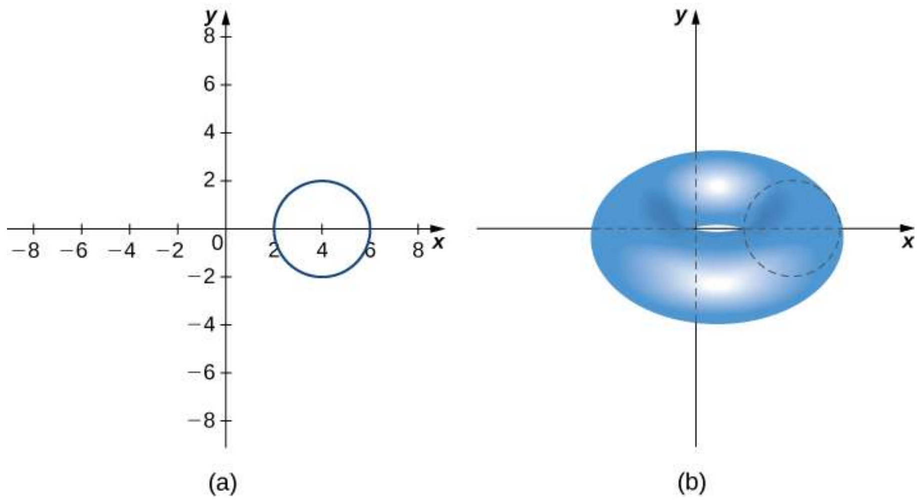


Figure 2.74 Determining the volume of a torus by using the theorem of Pappus. (a) A circular region R in the plane; (b) the torus generated by revolving R about the y-axis.

The region R is a circle of radius 2, so the area of R is $A = 4\pi$ units^2. By the symmetry principle, the centroid of R is the center of the circle. The centroid travels around the y-axis in a circular path of radius 4, so the centroid travels $d = 8\pi$ units. Then, the volume of the torus is $A \cdot d = 32\pi^2$ units^3.

2.34 Let R be a circle of radius 1 centered at $(3, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

2.6 EXERCISES

For the following exercises, calculate the center of mass for the collection of masses given.

254. $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

255. $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

256. $m = 3$ at $x = 0, 1, 2, 6$

257. Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

258. $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

259. $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

For the following exercises, compute the center of mass $\bar{x}$.

260. $\rho = 1$ for $x \in (-1, 3)$

261. $\rho = x^2$ for $x \in (0, L)$

262. $\rho = 1$ for $x \in (0, 1)$ and $\rho = 2$ for $x \in (1, 2)$

263. $\rho = \sin x$ for $x \in (0, \pi)$

264. $\rho = \cos x$ for $x \in \left(0, \frac{\pi}{2}\right)$

265. $\rho = e^x$ for $x \in (0, 2)$

266. $\rho = x^3 + xe^{-x}$ for $x \in (0, 1)$

267. $\rho = x \sin x$ for $x \in (0, \pi)$

268. $\rho = \sqrt{x}$ for $x \in (1, 4)$

269. $\rho = \ln x$ for $x \in (1, e)$

For the following exercises, compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

270. $\rho = 7$ in the square $0 \le x \le 1, \quad 0 \le y \le 1$

271. $\rho = 3$ in the triangle with vertices $(0, 0), \quad (a, 0)$, and $(0, b)$

272. $\rho = 2$ for the region bounded by $y = \cos(x)$, $y = -\cos(x), \quad x = -\frac{\pi}{2}$, and $x = \frac{\pi}{2}$

For the following exercises, use a calculator to draw the region, then compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

273. **[T]** The region bounded by $y = \cos(2x)$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$

274. **[T]** The region between $y = 2x^2, \quad y = 0, \quad x = 0$, and $x = 1$

275. **[T]** The region between $y = \frac{5}{4}x^2$ and $y = 5$

276. **[T]** Region between $y = \sqrt{x}, \quad y = \ln(x), \quad x = 1$, and $x = 4$

277. **[T]** The region bounded by $y = 0, \quad \frac{x^2}{4} + \frac{y^2}{9} = 1$

278. **[T]** The region bounded by $y = 0, \quad x = 0$, and $\frac{x^2}{4} + \frac{y^2}{9} = 1$

279. **[T]** The region bounded by $y = x^2$ and $y = x^4$ in the first quadrant

For the following exercises, use the theorem of Pappus to determine the volume of the shape.

280. Rotating $y = mx$ around the x-axis between $x = 0$ and $x = 1$

281. Rotating $y = mx$ around the y-axis between $x = 0$ and $x = 1$

282. A general cone created by rotating a triangle with vertices $(0, 0), \quad (a, 0)$, and $(0, b)$ around the y-axis. Does your answer agree with the volume of a cone?

283. A general cylinder created by rotating a rectangle with vertices $(0, 0), \quad (a, 0), (0, b)$, and (a, b) around the y-axis. Does your answer agree with the volume of a cylinder?

284. A sphere created by rotating a semicircle with radius a around the y-axis. Does your answer agree with the volume of a sphere?

For the following exercises, use a calculator to draw the region enclosed by the curve. Find the area M and the centroid $(\overline{x}, \overline{y})$ for the given shapes. Use symmetry to help locate the center of mass whenever possible.

285. **[T]** Quarter-circle: $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$

286. **[T]** Triangle: $y = x$, $y = 2 - x$, and $y = 0$

287. **[T]** Lens: $y = x^2$ and $y = x$

288. **[T]** Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

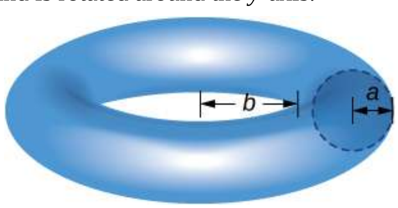
289. **[T]** Half-ring: $y^2 + x^2 = 1$, $y^2 + x^2 = 4$, and $y = 0$

290. Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

291. Find the generalized center of mass between $y = a^2 - x^2$, $x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

292. Find the generalized center of mass between $y = b\sin(ax)$, $x = 0$, and $x = \frac{\pi}{a}$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

293. Use the theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius a is positioned with the left end of the circle at $x = b$, $b > 0$, and is rotated around the *y*-axis.



294. Find the center of mass $(\overline{x}, \overline{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (*Hint:* Use the theorem of Pappus.)

2.7 | Integrals, Exponential Functions, and Logarithms

Learning Objectives

2.7.1 Write the definition of the natural logarithm as an integral.
2.7.2 Recognize the derivative of the natural logarithm.
2.7.3 Integrate functions involving the natural logarithmic function.
2.7.4 Define the number e through an integral.
2.7.5 Recognize the derivative and integral of the exponential function.
2.7.6 Prove properties of logarithms and exponential functions using integrals.
2.7.7 Express general logarithmic and exponential functions in terms of natural logarithms and exponentials.

We already examined exponential functions and logarithms in earlier chapters. However, we glossed over some key details in the previous discussions. For example, we did not study how to treat exponential functions with exponents that are irrational. The definition of the number *e* is another area where the previous development was somewhat incomplete. We now have the tools to deal with these concepts in a more mathematically rigorous way, and we do so in this section.

For purposes of this section, assume we have not yet defined the natural logarithm, the number *e*, or any of the integration and differentiation formulas associated with these functions. By the end of the section, we will have studied these concepts in a mathematically rigorous way (and we will see they are consistent with the concepts we learned earlier).

We begin the section by defining the natural logarithm in terms of an integral. This definition forms the foundation for the section. From this definition, we derive differentiation formulas, define the number e, and expand these concepts to logarithms and exponential functions of any base.

The Natural Logarithm as an Integral

Recall the power rule for integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ \ n \neq -1.$$

Clearly, this does not work when $n = -1$, as it would force us to divide by zero. So, what do we do with $\int \frac{1}{x}dx$? Recall from the Fundamental Theorem of Calculus that $\int_1^x \frac{1}{t}dt$ is an antiderivative of $1/x$. Therefore, we can make the following definition.

Definition

For $x > 0$, define the natural logarithm function by

$$\ln x = \int_1^x \frac{1}{t}dt. \tag{2.24}$$

For $x > 1$, this is just the area under the curve $y = 1/t$ from 1 to x. For $x < 1$, we have $\int_1^x \frac{1}{t}dt = -\int_x^1 \frac{1}{t}dt$, so in this case it is the negative of the area under the curve from x to 1 (see the following figure).

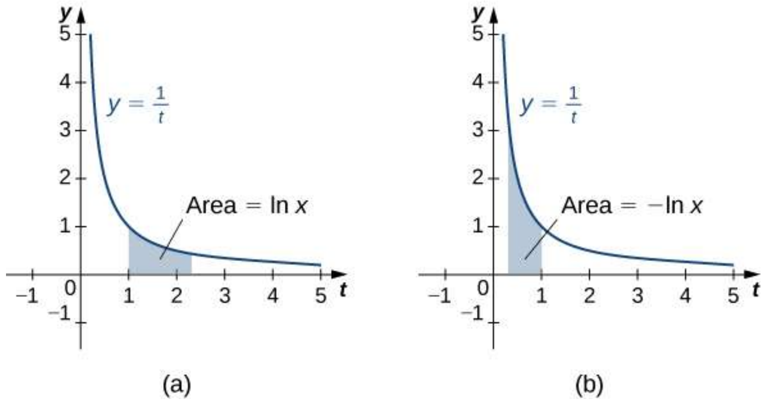


Figure 2.75 (a) When $x > 1,$ the natural logarithm is the area under the curve $y = 1/t$ from 1 to x. (b) When $x < 1,$ the natural logarithm is the negative of the area under the curve from x to 1.

Notice that $\ln 1 = 0$. Furthermore, the function $y = 1/t > 0$ for $x > 0$. Therefore, by the properties of integrals, it is clear that $\ln x$ is increasing for $x > 0$.

Properties of the Natural Logarithm

Because of the way we defined the natural logarithm, the following differentiation formula falls out immediately as a result of to the Fundamental Theorem of Calculus.

Theorem 2.15: Derivative of the Natural Logarithm

For $x > 0,$ the derivative of the natural logarithm is given by

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Theorem 2.16: Corollary to the Derivative of the Natural Logarithm

The function $\ln x$ is differentiable; therefore, it is continuous.

A graph of $\ln x$ is shown in **Figure 2.76**. Notice that it is continuous throughout its domain of $(0, \infty)$.

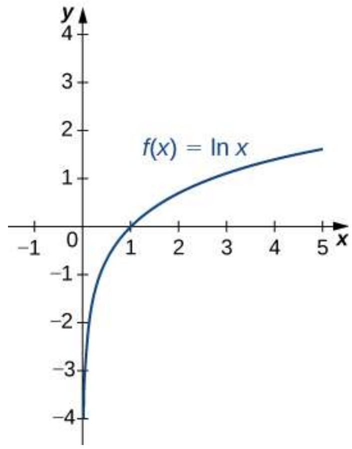


Figure 2.76 The graph of $f(x) = \ln x$ shows that it is a continuous function.

Example 2.35

Calculating Derivatives of Natural Logarithms

Calculate the following derivatives:

a. $\frac{d}{dx}\ln\left(5x^3 - 2\right)$

b. $\frac{d}{dx}(\ln(3x))^2$

Solution

We need to apply the chain rule in both cases.

a. $\frac{d}{dx}\ln\left(5x^3 - 2\right) = \frac{15x^2}{5x^3 - 2}$

b. $\frac{d}{dx}(\ln(3x))^2 = \frac{2(\ln(3x)) \cdot 3}{3x} = \frac{2(\ln(3x))}{x}$

2.35 Calculate the following derivatives:

a. $\frac{d}{dx}\ln\left(2x^2 + x\right)$

b. $\frac{d}{dx}\left(\ln\left(x^3\right)\right)^2$

Note that if we use the absolute value function and create a new function $\ln|x|$, we can extend the domain of the natural logarithm to include $x < 0$. Then $(d/(dx))\ln|x| = 1/x$. This gives rise to the familiar integration formula.

Theorem 2.17: Integral of (1/*u*) *du*

The natural logarithm is the antiderivative of the function $f(u) = 1/u$:

$$\int \frac{1}{u}du = \ln|u| + C.$$

Example 2.36

Calculating Integrals Involving Natural Logarithms

Calculate the integral $\int \frac{x}{x^2+4}dx.$

Solution

Using u -substitution, let $u = x^2 + 4.$ Then $du = 2x\,dx$ and we have

$$\int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{1}{u}du\frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+4\right| + C = \frac{1}{2}\ln\left(x^2+4\right) + C.$$

2.36 Calculate the integral $\int \frac{x^2}{x^3+6}dx.$

Although we have called our function a “logarithm,” we have not actually proved that any of the properties of logarithms hold for this function. We do so here.

Theorem 2.18: Properties of the Natural Logarithm

If $a, b > 0$ and r is a rational number, then

i. $\ln 1 = 0$

ii. $\ln(ab) = \ln a + \ln b$

iii. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

iv. $\ln(a^r) = r\ln a$

Proof

i. By definition, $\ln 1 = \int_1^1 \frac{1}{t}dt = 0.$

ii. We have

$$\ln(ab) = \int_1^{ab}\frac{1}{t}dt = \int_1^a \frac{1}{t}dt + \int_a^{ab}\frac{1}{t}dt.$$

Use u-substitution on the last integral in this expression. Let $u = t/a.$ Then $du = (1/a)dt.$ Furthermore, when $t = a,\ u = 1,$ and when $t = ab,\ u = b.$ So we get

$$\ln(ab) = \int_1^a \frac{1}{t}dt + \int_a^{ab}\frac{1}{t}dt = \int_1^a \frac{1}{t}dt + \int_1^{ab}\frac{a}{t}\cdot\frac{1}{a}dt = \int_1^a \frac{1}{t}dt + \int_1^b \frac{1}{u}du = \ln a + \ln b.$$

iii. Note that

$$\frac{d}{dx}\ln(x^r) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}.$$

Furthermore,

$$\frac{d}{dx}(r \ln x) = \frac{r}{x}.$$

Since the derivatives of these two functions are the same, by the Fundamental Theorem of Calculus, they must differ by a constant. So we have

$$\ln(x^r) = r \ln x + C$$

for some constant C. Taking $x = 1,$ we get

$$\begin{aligned} \ln(1^r) &= r\ln(1) + C \\ 0 &= r(0) + C \\ C &= 0. \end{aligned}$$

Thus $\ln(x^r) = r \ln x$ and the proof is complete. Note that we can extend this property to irrational values of r later in this section.

Part iii. follows from parts ii. and iv. and the proof is left to you.

□

Example 2.37

Using Properties of Logarithms

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 9 - 2\ln 3 + \ln\left(\frac{1}{3}\right).$$

Solution

We have

$$\ln 9 - 2\ln 3 + \ln\left(\frac{1}{3}\right) = \ln\left(3^2\right) - 2\ln 3 + \ln\left(3^{-1}\right) = 2\ln 3 - 2\ln 3 - \ln 3 = -\ln 3.$$

2.37 Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 8 - \ln 2 - \ln\left(\frac{1}{4}\right).$$

Defining the Number *e*

Now that we have the natural logarithm defined, we can use that function to define the number e.

Definition

The number e is defined to be the real number such that

$$\ln e = 1.$$

To put it another way, the area under the curve $y = 1/t$ between $t = 1$ and $t = e$ is 1 (**Figure 2.77**). The proof that such a number exists and is unique is left to you. (*Hint*: Use the Intermediate Value Theorem to prove existence and the fact that

$\ln x$ is increasing to prove uniqueness.)

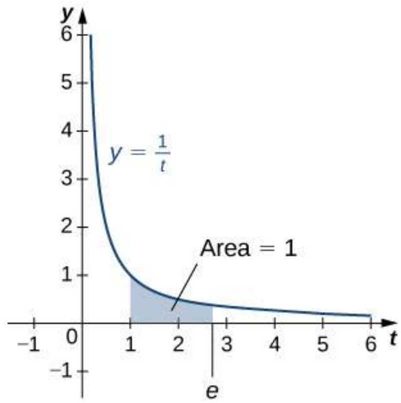


Figure 2.77 The area under the curve from 1 to e is equal to one.

The number e can be shown to be irrational, although we won't do so here (see the Student Project in **Taylor and Maclaurin Series**). Its approximate value is given by

$$e \approx 2.71828182846.$$

The Exponential Function

We now turn our attention to the function e^x. Note that the natural logarithm is one-to-one and therefore has an inverse function. For now, we denote this inverse function by $\exp x$. Then,

$$\exp(\ln x) = x \text{ for } x > 0 \text{ and } \ln(\exp x) = x \text{ for all } x.$$

The following figure shows the graphs of $\exp x$ and $\ln x$.

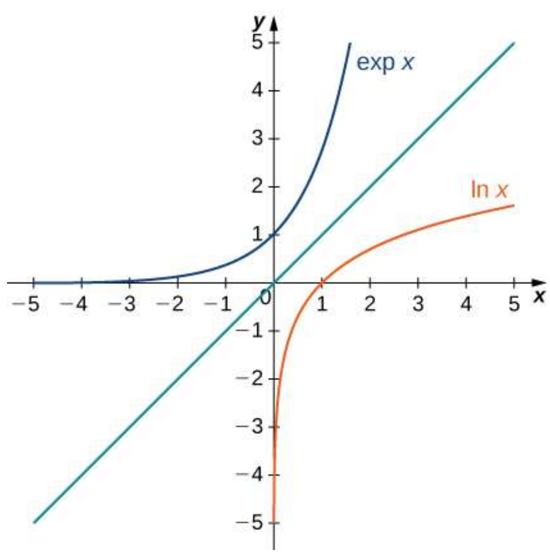


Figure 2.78 The graphs of $\ln x$ and $\exp x$.

We hypothesize that $\exp x = e^x$. For rational values of x, this is easy to show. If x is rational, then we have $\ln(e^x) = x \ln e = x$. Thus, when x is rational, $e^x = \exp x$. For irrational values of x, we simply define e^x as the inverse function of $\ln x$.

Definition

For any real number x, define $y = e^x$ to be the number for which

$$\ln y = \ln(e^x) = x. \tag{2.25}$$

Then we have $e^x = \exp(x)$ for all x, and thus

$$e^{\ln x} = x \text{ for } x > 0 \text{ and } \ln(e^x) = x \tag{2.26}$$

for all x.

Properties of the Exponential Function

Since the exponential function was defined in terms of an inverse function, and not in terms of a power of e, we must verify that the usual laws of exponents hold for the function e^x.

Theorem 2.19: Properties of the Exponential Function

If p and q are any real numbers and r is a rational number, then

i. $e^p e^q = e^{p+q}$

ii. $\dfrac{e^p}{e^q} = e^{p-q}$

iii. $(e^p)^r = e^{pr}$

Proof

Note that if p and q are rational, the properties hold. However, if p or q are irrational, we must apply the inverse function definition of e^x and verify the properties. Only the first property is verified here; the other two are left to you. We have

$$\ln(e^p e^q) = \ln(e^p) + \ln(e^q) = p + q = \ln\left(e^{p+q}\right).$$

Since $\ln x$ is one-to-one, then

$$e^p e^q = e^{p+q}.$$

□

As with part iv. of the logarithm properties, we can extend property iii. to irrational values of r, and we do so by the end of the section.

We also want to verify the differentiation formula for the function $y = e^x$. To do this, we need to use implicit differentiation. Let $y = e^x$. Then

$$\begin{aligned} \ln y &= x \\ \frac{d}{dx}\ln y &= \frac{d}{dx}x \\ \frac{1}{y}\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y. \end{aligned}$$

Thus, we see

$$\frac{d}{dx}e^x = e^x$$

as desired, which leads immediately to the integration formula

$$\int e^x\,dx = e^x + C.$$

We apply these formulas in the following examples.

Example 2.38

Using Properties of Exponential Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}e^{3t}e^{t^2}$

b. $\frac{d}{dx}e^{3x^2}$

Solution

We apply the chain rule as necessary.

a. $\frac{d}{dt}e^{3t}e^{t^2} = \frac{d}{dt}e^{3t+t^2} = e^{3t+t^2}(3+2t)$

b. $\frac{d}{dx}e^{3x^2} = e^{3x^2}6x$

2.38 Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right)$

b. $\frac{d}{dt}\left(e^{2t}\right)^3$

Example 2.39

Using Properties of Exponential Functions

Evaluate the following integral: $\int 2xe^{-x^2}\,dx.$

Solution

Using u-substitution, let $u = -x^2.$ Then $du = -2x\,dx,$ and we have

$$\int 2xe^{-x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{-x^2} + C.$$

2.39 Evaluate the following integral: $\int \frac{4}{e^{3x}}dx.$

General Logarithmic and Exponential Functions

We close this section by looking at exponential functions and logarithms with bases other than e. Exponential functions are functions of the form $f(x) = a^x$. Note that unless $a = e$, we still do not have a mathematically rigorous definition of these functions for irrational exponents. Let's rectify that here by defining the function $f(x) = a^x$ in terms of the exponential function e^x. We then examine logarithms with bases other than e as inverse functions of exponential functions.

Definition

For any $a > 0$, and for any real number x, define $y = a^x$ as follows:

$$y = a^x = e^{x \ln a}.$$

Now a^x is defined rigorously for all values of x. This definition also allows us to generalize property iv. of logarithms and property iii. of exponential functions to apply to both rational and irrational values of r. It is straightforward to show that properties of exponents hold for general exponential functions defined in this way.

Let's now apply this definition to calculate a differentiation formula for a^x. We have

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a.$$

The corresponding integration formula follows immediately.

Theorem 2.20: Derivatives and Integrals Involving General Exponential Functions

Let $a > 0$. Then,

$$\frac{d}{dx}a^x = a^x \ln a$$

and

$$\int a^x\, dx = \frac{1}{\ln a}a^x + C.$$

If $a \neq 1$, then the function a^x is one-to-one and has a well-defined inverse. Its inverse is denoted by $\log_a x$. Then,

$$y = \log_a x \text{ if and only if } x = a^y.$$

Note that general logarithm functions can be written in terms of the natural logarithm. Let $y = \log_a x$. Then, $x = a^y$. Taking the natural logarithm of both sides of this second equation, we get

$$\begin{aligned} \ln x &= \ln(a^y) \\ \ln x &= y \ln a \\ y &= \frac{\ln x}{\ln a} \\ \log\ x &= \frac{\ln x}{\ln a}. \end{aligned}$$

Thus, we see that all logarithmic functions are constant multiples of one another. Next, we use this formula to find a differentiation formula for a logarithm with base a. Again, let $y = \log_a x$. Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\log_a x)\\ &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \left(\frac{1}{\ln a}\right)\frac{d}{dx}(\ln x)\\ &= \frac{1}{\ln a}\cdot\frac{1}{x}\\ &= \frac{1}{x\ln a}.\end{aligned}$$

Theorem 2.21: Derivatives of General Logarithm Functions

Let $a > 0$. Then,

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

Example 2.40

Calculating Derivatives of General Exponential and Logarithm Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right)$

b. $\frac{d}{dx}\log_8\left(7x^2 + 4\right)$

Solution

We need to apply the chain rule as necessary.

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t} \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t + t^2}\right) = 2^{2t + t^2}\ln(2)(2 + 2t)$

b. $\frac{d}{dx}\log_8\left(7x^2 + 4\right) = \frac{1}{\left(7x^2 + 4\right)(\ln 8)}(14x)$

2.40 Evaluate the following derivatives:

a. $\frac{d}{dt}4^{t^4}$

b. $\frac{d}{dx}\log_3\left(\sqrt{x^2 + 1}\right)$

Example 2.41

Integrating General Exponential Functions

Evaluate the following integral: $\int \frac{3}{2^{3x}} dx$.

Solution

Use u-substitution and let $u = -3x$. Then $du = -3dx$ and we have

$$\int \frac{3}{2^{3x}} dx = \int 3 \cdot 2^{-3x} dx = -\int 2^u \, du = -\frac{1}{\ln 2} 2^u + C = -\frac{1}{\ln 2} 2^{-3x} + C.$$

2.41 Evaluate the following integral: $\int x^2 2^{x^3} dx$.

2.7 EXERCISES

For the following exercises, find the derivative $\frac{dy}{dx}$.

295. $y = \ln(2x)$

296. $y = \ln(2x + 1)$

297. $y = \frac{1}{\ln x}$

For the following exercises, find the indefinite integral.

298. $\int \frac{dt}{3t}$

299. $\int \frac{dx}{1 + x}$

For the following exercises, find the derivative dy/dx. (You can use a calculator to plot the function and the derivative to confirm that it is correct.)

300. **[T]** $y = \frac{\ln(x)}{x}$

301. **[T]** $y = x\ln(x)$

302. **[T]** $y = \log_{10} x$

303. **[T]** $y = \ln(\sin x)$

304. **[T]** $y = \ln(\ln x)$

305. **[T]** $y = 7\ln(4x)$

306. **[T]** $y = \ln\left((4x)^7\right)$

307. **[T]** $y = \ln(\tan x)$

308. **[T]** $y = \ln(\tan(3x))$

309. **[T]** $y = \ln\left(\cos^2 x\right)$

For the following exercises, find the definite or indefinite integral.

310. $\int_0^1 \frac{dx}{3 + x}$

311. $\int_0^1 \frac{dt}{3 + 2t}$

312. $\int_0^2 \frac{x\,dx}{x^2 + 1}$

313. $\int_0^2 \frac{x^3\,dx}{x^2 + 1}$

314. $\int_2^e \frac{dx}{x\ln x}$

315. $\int_2^e \frac{dx}{(x\ln(x))^2}$

316. $\int \frac{\cos x\,dx}{\sin x}$

317. $\int_0^{\pi/4} \tan x\,dx$

318. $\int \cot(3x)dx$

319. $\int \frac{(\ln x)^2\,dx}{x}$

For the following exercises, compute dy/dx by differentiating $\ln y$.

320. $y = \sqrt{x^2 + 1}$

321. $y = \sqrt{x^2 + 1}\sqrt{x^2 - 1}$

322. $y = e^{\sin x}$

323. $y = x^{-1/x}$

324. $y = e^{(ex)}$

325. $y = x^e$

326. $y = x^{(ex)}$

327. $y = \sqrt{x}\sqrt[3]{x}\sqrt[6]{x}$

328. $y = x^{-1/\ln x}$

329. $y = e^{-\ln x}$

For the following exercises, evaluate by any method.

330. $\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

331. $\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

332. $\frac{d}{dx}\int_x^1 \frac{dt}{t}$

333. $\frac{d}{dx}\int_x^{x^2} \frac{dt}{t}$

334. $\frac{d}{dx}\ln(\sec x + \tan x)$

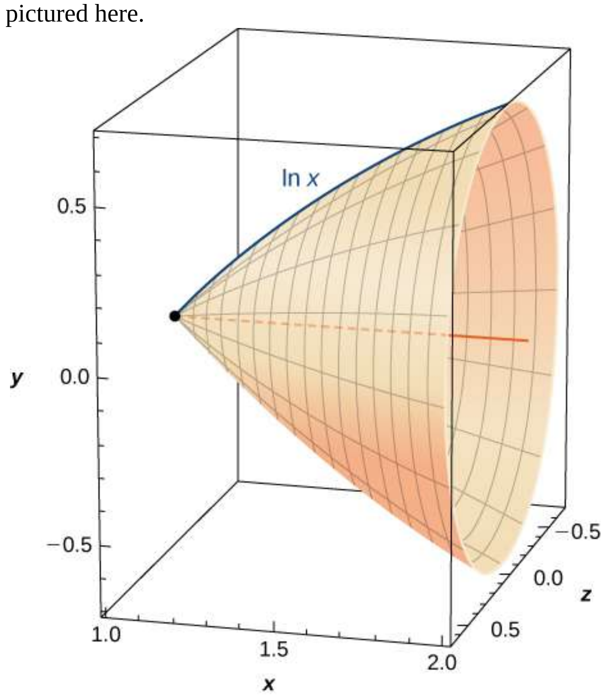
For the following exercises, use the function $\ln x$. If you are unable to find intersection points analytically, use a calculator.

335. Find the area of the region enclosed by $x = 1$ and $y = 5$ above $y = \ln x$.

336. **[T]** Find the arc length of $\ln x$ from $x = 1$ to $x = 2$.

337. Find the area between $\ln x$ and the x-axis from $x = 1$ to $x = 2$.

338. Find the volume of the shape created when rotating this curve from $x = 1$ to $x = 2$ around the x-axis, as pictured here.



339. **[T]** Find the surface area of the shape created when rotating the curve in the previous exercise from $x = 1$ to $x = 2$ around the x-axis.

If you are unable to find intersection points analytically in the following exercises, use a calculator.

340. Find the area of the hyperbolic quarter-circle enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

341. **[T]** Find the arc length of $y = 1/x$ from $x = 1$ to $x = 4$.

342. Find the area under $y = 1/x$ and above the x-axis from $x = 1$ to $x = 4$.

For the following exercises, verify the derivatives and antiderivatives.

343. $\frac{d}{dx}\ln\left(x + \sqrt{x^2 + 1}\right) = \frac{1}{\sqrt{1 + x^2}}$

344. $\frac{d}{dx}\ln\left(\frac{x - a}{x + a}\right) = \frac{2a}{\left(x^2 - a^2\right)}$

345. $\frac{d}{dx}\ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) = -\frac{1}{x\sqrt{1 - x^2}}$

346. $\frac{d}{dx}\ln\left(x + \sqrt{x^2 - a^2}\right) = \frac{1}{\sqrt{x^2 - a^2}}$

347. $\int \frac{dx}{x\ln(x)\ln(\ln x)} = \ln(\ln(\ln x)) + C$

2.8 | Exponential Growth and Decay

Learning Objectives

2.8.1 Use the exponential growth model in applications, including population growth and compound interest.

2.8.2 Explain the concept of doubling time.

2.8.3 Use the exponential decay model in applications, including radioactive decay and Newton's law of cooling.

2.8.4 Explain the concept of half-life.

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of these applications.

Exponential Growth Model

Many systems exhibit exponential growth. These systems follow a model of the form $y = y_0 e^{kt}$, where y_0 represents the initial state of the system and k is a positive constant, called the *growth constant*. Notice that in an exponential growth model, we have

$$y' = ky_0 e^{kt} = ky. \tag{2.27}$$

That is, the rate of growth is proportional to the current function value. This is a key feature of exponential growth. **Equation 2.27** involves derivatives and is called a *differential equation*. We learn more about differential equations in **Introduction to Differential Equations**.

Rule: Exponential Growth Model

Systems that exhibit **exponential growth** increase according to the mathematical model

$$y = y_0 e^{kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *growth constant*.

Population growth is a common example of exponential growth. Consider a population of bacteria, for instance. It seems plausible that the rate of population growth would be proportional to the size of the population. After all, the more bacteria there are to reproduce, the faster the population grows. **Figure 2.79** and **Table 2.1** represent the growth of a population of bacteria with an initial population of 200 bacteria and a growth constant of 0.02. Notice that after only 2 hours $(120$ minutes), the population is 10 times its original size!

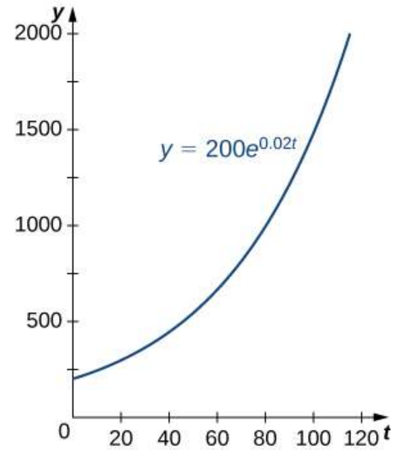


Figure 2.79 An example of exponential growth for bacteria.

Time (min)	Population Size (no. of bacteria)
10	244
20	298
30	364
40	445
50	544
60	664
70	811
80	991
90	1210
100	1478
110	1805
120	2205

Table 2.1 Exponential Growth of a Bacterial Population

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

Example 2.42

Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where t is measured in minutes. How many bacteria are present in the population after 5 hours $(300$ minutes)? When does the population reach $100,000$ bacteria?

Solution

We have $f(t) = 200e^{0.02t}$. Then

$$f(300) = 200e^{0.02(300)} \approx 80,686.$$

There are $80,686$ bacteria in the population after 5 hours.

To find when the population reaches $100,000$ bacteria, we solve the equation

$$\begin{aligned} 100,000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73. \end{aligned}$$

The population reaches $100,000$ bacteria after 310.73 minutes.

2.42 Consider a population of bacteria that grows according to the function $f(t) = 500e^{0.05t}$, where t is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach 100 million bacteria?

Let's now turn our attention to a financial application: compound interest. Interest that is not compounded is called *simple interest*. Simple interest is paid once, at the end of the specified time period (usually 1 year). So, if we put $\$1000$ in a savings account earning 2% simple interest per year, then at the end of the year we have

$$1000(1 + 0.02) = \$1020.$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every 6 months, it credits half of the year's interest to the account after 6 months. During the second half of the year, the account earns interest not only on the initial $\$1000$, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

$$1000\left(1 + \frac{0.02}{2}\right)^2 = \$1020.10.$$

Similarly, if the interest is compounded every 4 months, we have

$$1000\left(1 + \frac{0.02}{3}\right)^3 = \$1020.13,$$

and if the interest is compounded daily $(365$ times per year), we have $\$1020.20$. If we extend this concept, so that the interest is compounded continuously, after t years we have

$$1000 \lim_{n \to \infty} \left(1 + \frac{0.02}{n}\right)^{nt}.$$

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number e can be expressed as a limit:

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m}.$$

Based on this, we want the expression inside the parentheses to have the form $(1 + 1/m)$. Let $n = 0.02m$. Note that as $n \to \infty$, $m \to \infty$ as well. Then we get

$$1000 \lim_{n \to \infty} \left(1 + \frac{0.02}{n}\right)^{nt} = 1000 \lim_{m \to \infty} \left(1 + \frac{0.02}{0.02m}\right)^{0.02mt} = 1000 \left[\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m}\right]^{0.02t}.$$

We recognize the limit inside the brackets as the number e. So, the balance in our bank account after t years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of $r\%$, compounded continuously, then the balance of the account after t years is

$$\text{Balance} = Pe^{rt}.$$

Example 2.43

Compound Interest

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays 5% annual interest compounded continuously. How much does the student need to invest today to have $\$1$ million when she retires at age 65? What if she could earn 6% annual interest compounded continuously instead?

Solution

We have

$$\begin{aligned} 1{,}000{,}000 &= Pe^{0.05(40)} \\ P &= 135{,}335.28. \end{aligned}$$

She must invest $\$135{,}335.28$ at 5% interest.

If, instead, she is able to earn 6%, then the equation becomes

$$\begin{aligned} 1{,}000{,}000 &= Pe^{0.06(40)} \\ P &= 90{,}717.95. \end{aligned}$$

In this case, she needs to invest only $\$90{,}717.95$. This is roughly two-thirds the amount she needs to invest at 5%. The fact that the interest is compounded continuously greatly magnifies the effect of the 1% increase in interest rate.

2.43 Suppose instead of investing at age $25\sqrt{b^2 - 4ac}$, the student waits until age 35. How much would she have to invest at 5%? At 6%?

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from 100 to 200 bacteria as it does to grow from $10{,}000$ to $20{,}000$ bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

$$\begin{aligned} 2y_0 &= y_0 e^{kt} \\ 2 &= e^{kt} \\ \ln 2 &= kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Definition

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

$$\text{Doubling time} = \frac{\ln 2}{k}.$$

Example 2.44

Using the Doubling Time

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches $10{,}000.$ When will the owner's friends be allowed to fish?

Solution

We know it takes the population of fish 6 months to double in size. So, if t represents time in months, by the doubling-time formula, we have $6 = (\ln 2)/k.$ Then, $k = (\ln 2)/6.$ Thus, the population is given by $y = 500e^{((\ln 2)/6)t}.$ To figure out when the population reaches $10{,}000$ fish, we must solve the following equation:

$$\begin{aligned} 10{,}000 &= 500e^{(\ln 2/6)t} \\ 20 &= e^{(\ln 2/6)t} \\ \ln 20 &= \left(\frac{\ln 2}{6}\right)t \\ t &= \frac{6(\ln 20)}{\ln 2} \approx 25.93. \end{aligned}$$

The owner's friends have to wait 25.93 months (a little more than 2 years) to fish in the pond.

2.44 Suppose it takes 9 months for the fish population in **Example 2.44** to reach 1000 fish. Under these circumstances, how long do the owner's friends have to wait?

Exponential Decay Model

Exponential functions can also be used to model populations that shrink (from disease, for example), or chemical compounds that break down over time. We say that such systems exhibit exponential decay, rather than exponential growth. The model is nearly the same, except there is a negative sign in the exponent. Thus, for some positive constant $k,$ we have $y = y_0 e^{-kt}.$

As with exponential growth, there is a differential equation associated with exponential decay. We have

$$y' = -ky_0 e^{-kt} = -ky.$$

Rule: Exponential Decay Model

Systems that exhibit **exponential decay** behave according to the model

$$y = y_0 e^{-kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *decay constant*.

The following figure shows a graph of a representative exponential decay function.

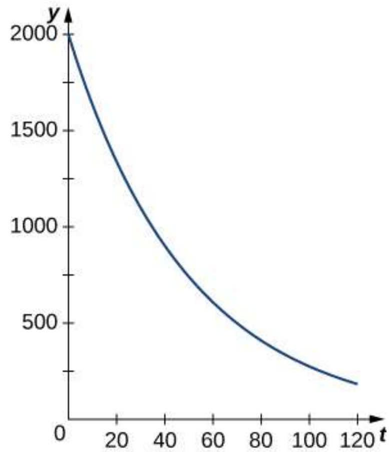


Figure 2.80 An example of exponential decay.

Let's look at a physical application of exponential decay. Newton's law of cooling says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if T represents the temperature of the object and T_a represents the ambient temperature in a room, then

$$T' = -k(T - T_a).$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional T_a term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a.$ Then $y'(t) = T'(t) - 0 = T'(t),$ and our equation becomes

$$y' = -ky.$$

From our previous work, we know this relationship between y and its derivative leads to exponential decay. Thus,

$$y = y_0 e^{-kt},$$

and we see that

$$\begin{aligned} T - T_a &= (T_0 - T_a)e^{-kt} \\ T &= (T_0 - T_a)e^{-kt} + T_a \end{aligned}$$

where T_0 represents the initial temperature. Let's apply this formula in the following example.

Example 2.45

Newton's Law of Cooling

According to experienced baristas, the optimal temperature to serve coffee is between $155°F$ and $175°F.$ Suppose coffee is poured at a temperature of $200°F,$ and after 2 minutes in a $70°F$ room it has cooled to $180°F.$ When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

Solution

We have

$$\begin{aligned} T &= (T_0 - T_a)e^{-kt} + T_a \\ 180 &= (200 - 70)e^{-k(2)} + 70 \\ 110 &= 130e^{-2k} \\ \frac{11}{13} &= e^{-2k} \\ \ln\frac{11}{13} &= -2k \\ \ln 11 - \ln 13 &= -2k \\ k &= \frac{\ln 13 - \ln 11}{2}. \end{aligned}$$

Then, the model is

$$T = 130e^{(\ln 11 - \ln 13/2)t} + 70.$$

The coffee reaches $175°F$ when

$$\begin{aligned} 175 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 105 &= 130e^{(\ln 11 - \ln 13/2)t} \\ \frac{21}{26} &= e^{(\ln 11 - \ln 13/2)t} \\ \ln\frac{21}{26} &= \frac{\ln 11 - \ln 13}{2}t \\ \ln 21 - \ln 26 &= \frac{\ln 11 - \ln 13}{2}t \\ t &= \frac{2(\ln 21 - \ln 26)}{\ln 11 - \ln 13} \approx 2.56. \end{aligned}$$

The coffee can be served about 2.5 minutes after it is poured. The coffee reaches $155°F$ at

$$\begin{aligned} 155 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 85 &= 130e^{(\ln 11 - \ln 13)t} \\ \frac{17}{26} &= e^{(\ln 11 - \ln 13)t} \\ \ln 17 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\ t &= \frac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13} \approx 5.09. \end{aligned}$$

The coffee is too cold to be served about 5 minutes after it is poured.

2.45 Suppose the room is warmer $(75°F)$ and, after 2 minutes, the coffee has cooled only to $185°F.$ When is the coffee first cool enough to serve? When is the coffee be too cold to serve? Round answers to the nearest half minute.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Note: This is the same expression we came up with for doubling time.

Definition

If a quantity decays exponentially, the **half-life** is the amount of time it takes the quantity to be reduced by half. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}.$$

Example 2.46

Radiocarbon Dating

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon was originally present in an object and how much carbon remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to the new nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon now contains 10 g of carbon, how old is it? Round the answer to the nearest hundred years.

Solution

We have

$$\begin{aligned} 5730 &= \frac{\ln 2}{k} \\ k &= \frac{\ln 2}{5730}. \end{aligned}$$

So, the model says

$$y = 100e^{-(\ln 2/5730)t}.$$

In 50 years, we have

$$\begin{aligned} y &= 100e^{-(\ln 2/5730)(50)} \\ &\approx 99.40. \end{aligned}$$

Therefore, in 50 years, 99.40 g of carbon-14 remains.

To determine the age of the artifact, we must solve

$$\begin{aligned} 10 &= 100e^{-(\ln 2/5730)t} \\ \frac{1}{10} &= e^{-(\ln 2/5730)t} \\ t &\approx 19035. \end{aligned}$$

The artifact is about $19,000$ years old.

2.46 If we have 100 g of $\text{carbon-14},$ how much is left after. years? If an artifact that originally contained 100 g of carbon now contains $20g$ of carbon, how old is it? Round the answer to the nearest hundred years.

2.8 EXERCISES

True or False? If true, prove it. If false, find the true answer.

348. The doubling time for $y = e^{ct}$ is $(\ln(2))/(\ln(c))$.

349. If you invest $\$500,$ an annual rate of interest of 3% yields more money in the first year than a 2.5% continuous rate of interest.

350. If you leave a $100°\text{C}$ pot of tea at room temperature $(25°\text{C})$ and an identical pot in the refrigerator $(5°\text{C}),$ with $k = 0.02,$ the tea in the refrigerator reaches a drinkable temperature $(70°\text{C})$ more than 5 minutes before the tea at room temperature.

351. If given a half-life of t years, the constant k for $y = e^{kt}$ is calculated by $k = \ln(1/2)/t$.

For the following exercises, use $y = y_0 e^{kt}$.

352. If a culture of bacteria doubles in 3 hours, how many hours does it take to multiply by 10?

353. If bacteria increase by a factor of 10 in 10 hours, how many hours does it take to increase by 100?

354. How old is a skull that contains one-fifth as much radiocarbon as a modern skull? Note that the half-life of radiocarbon is 5730 years.

355. If a relic contains 90% as much radiocarbon as new material, can it have come from the time of Christ (approximately 2000 years ago)? Note that the half-life of radiocarbon is 5730 years.

356. The population of Cairo grew from 5 million to 10 million in 20 years. Use an exponential model to find when the population was 8 million.

357. The populations of New York and Los Angeles are growing at 1% and 1.4% a year, respectively. Starting from 8 million (New York) and 6 million (Los Angeles), when are the populations equal?

358. Suppose the value of $\$1$ in Japanese yen decreases at 2% per year. Starting from $\$1 = ¥250,$ when will $\$1 = ¥1$?

359. The effect of advertising decays exponentially. If 40% of the population remembers a new product after 3 days, how long will 20% remember it?

360. If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4,$ what was y_0 at $t = 0$?

361. If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8,$ when does $y = 1$?

362. If a bank offers annual interest of 7.5% or continuous interest of $7.25\%,$ which has a better annual yield?

363. What continuous interest rate has the same yield as an annual rate of 9%?

364. If you deposit $\$5000$ at 8% annual interest, how many years can you withdraw $\$500$ (starting after the first year) without running out of money?

365. You are trying to save $\$50{,}000$ in 20 years for college tuition for your child. If interest is a continuous $10\%,$ how much do you need to invest initially?

366. You are cooling a turkey that was taken out of the oven with an internal temperature of $165°\text{F}.$ After 10 minutes of resting the turkey in a $70°\text{F}$ apartment, the temperature has reached $155°\text{F}.$ What is the temperature of the turkey 20 minutes after taking it out of the oven?

367. You are trying to thaw some vegetables that are at a temperature of $1°\text{F}.$ To thaw vegetables safely, you must put them in the refrigerator, which has an ambient temperature of $44°\text{F}.$ You check on your vegetables 2 hours after putting them in the refrigerator to find that they are now $12°\text{F}.$ Plot the resulting temperature curve and use it to determine when the vegetables reach $33°\text{F}.$

368. You are an archaeologist and are given a bone that is claimed to be from a Tyrannosaurus Rex. You know these dinosaurs lived during the Cretaceous Era $(146$ million years to 65 million years ago), and you find by radiocarbon dating that there is 0.000001% the amount of radiocarbon. Is this bone from the Cretaceous?

369. The spent fuel of a nuclear reactor contains plutonium-239, which has a half-life of $24{,}000$ years. If 1 barrel containing $10\,\text{kg}$ of plutonium-239 is sealed, how many years must pass until only $10g$ of plutonium-239 is left?

For the next set of exercises, use the following table, which features the world population by decade.

Years since 1950	Population (millions)
0	2,556
10	3,039
20	3,706
30	4,453
40	5,279
50	6,083
60	6,849

Source: http://www.factmonster.com/ipka/A0762181.html.

370. **[T]** The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 2686e^{0.01604t}$. Use a graphing calculator to graph the data and the exponential curve together.

371. **[T]** Find and graph the derivative y' of your equation. Where is it increasing and what is the meaning of this increase?

372. **[T]** Find and graph the second derivative of your equation. Where is it increasing and what is the meaning of this increase?

373. **[T]** Find the predicted date when the population reaches 10 billion. Using your previous answers about the first and second derivatives, explain why exponential growth is unsuccessful in predicting the future.

For the next set of exercises, use the following table, which shows the population of San Francisco during the 19th century.

Years since 1850	Population (thousands)
0	21.00
10	56.80
20	149.5
30	234.0

Source: http://www.sfgenealogy.com/sf/history/hgpop.htm.

374. **[T]** The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 35.26e^{0.06407t}$. Use a graphing calculator to graph the data and the exponential curve together.

375. **[T]** Find and graph the derivative y' of your equation. Where is it increasing? What is the meaning of this increase? Is there a value where the increase is maximal?

376. **[T]** Find and graph the second derivative of your equation. Where is it increasing? What is the meaning of this increase?

2.9 | Calculus of the Hyperbolic Functions

Learning Objectives

2.9.1 Apply the formulas for derivatives and integrals of the hyperbolic functions.

2.9.2 Apply the formulas for the derivatives of the inverse hyperbolic functions and their associated integrals.

2.9.3 Describe the common applied conditions of a catenary curve.

We were introduced to hyperbolic functions in **Introduction to Functions and Graphs (http://cnx.org/content/m53472/latest/)** , along with some of their basic properties. In this section, we look at differentiation and integration formulas for the hyperbolic functions and their inverses.

Derivatives and Integrals of the Hyperbolic Functions

Recall that the hyperbolic sine and hyperbolic cosine are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in the following figure.

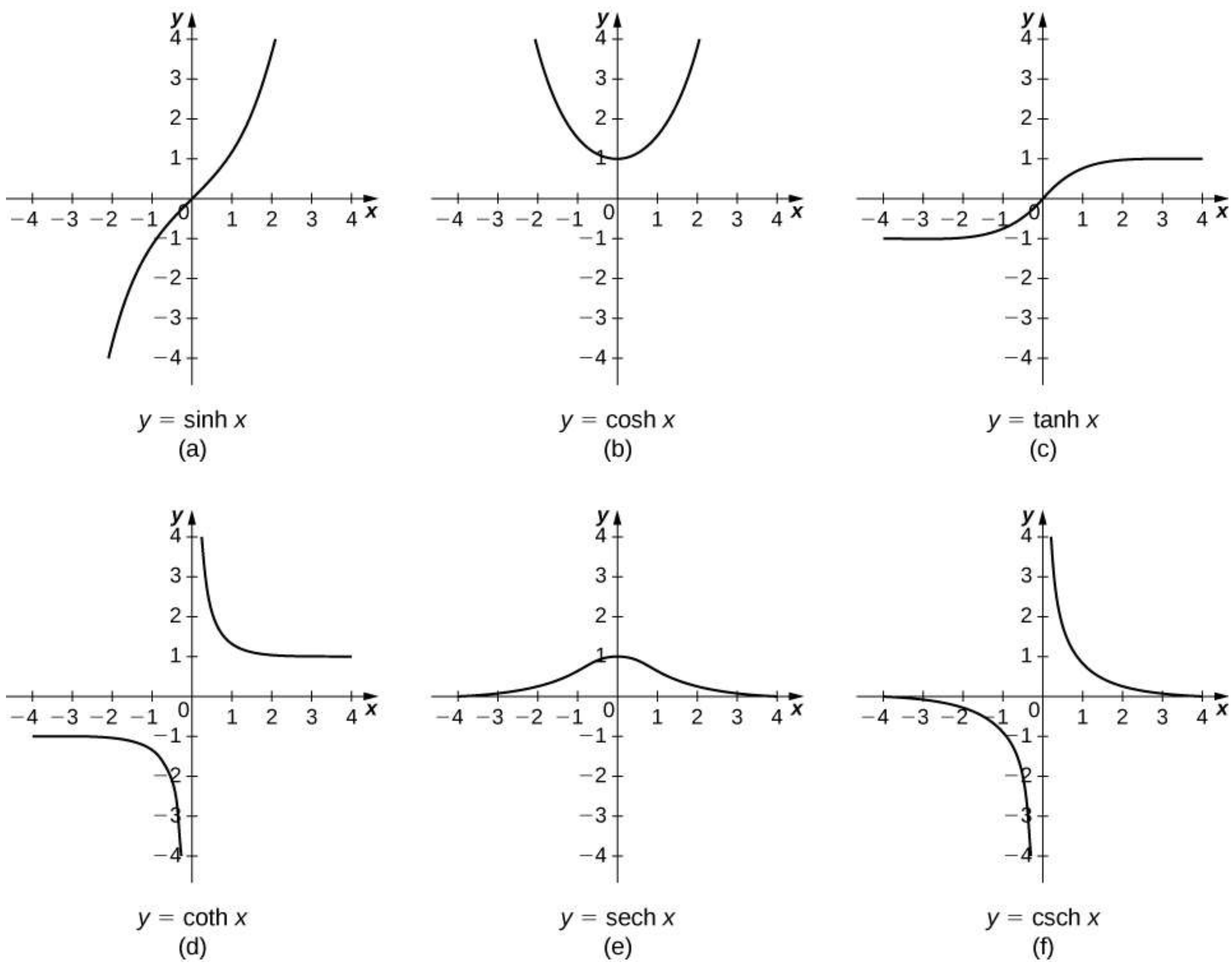


Figure 2.81 Graphs of the hyperbolic functions.

It is easy to develop differentiation formulas for the hyperbolic functions. For example, looking at $\sinh x$ we have

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)\\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right]\\ &= \frac{1}{2}[e^x + e^{-x}] = \cosh x.\end{aligned}$$

Similarly, $(d/dx)\cosh x = \sinh x$. We summarize the differentiation formulas for the hyperbolic functions in the following table.

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2\, x$
$\coth x$	$-\text{csch}^2\, x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{csch}\, x$	$-\text{csch}\, x \coth x$

Table 2.2 Derivatives of the Hyperbolic Functions

Let's take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities, but differences as well. For example, the derivatives of the sine functions match: $(d/dx)\sin x = \cos x$ and $(d/dx)\sinh x = \cosh x$. The derivatives of the cosine functions, however, differ in sign: $(d/dx)\cos x = -\sin x$, but $(d/dx)\cosh x = \sinh x$. As we continue our examination of the hyperbolic functions, we must be mindful of their similarities and differences to the standard trigonometric functions.

These differentiation formulas for the hyperbolic functions lead directly to the following integral formulas.

$$\begin{aligned}\int \sinh u\, du &= \cosh u + C & \int \text{csch}^2\, u\, du &= -\coth u + C\\ \int \cosh u\, du &= \sinh u + C & \int \text{sech}\, u \tanh u\, du &= -\text{sech}\, u + C\\ \int \text{sech}^2 u\, du &= \tanh u + C & \int \text{csch}\, u \coth u\, du &= -\text{csch}\, u + C\end{aligned}$$

Example 2.47

Differentiating Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right)$

b. $\frac{d}{dx}(\cosh x)^2$

Solution

Using the formulas in **Table 2.2** and the chain rule, we get

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right) = \cosh\left(x^2\right) \cdot 2x$

b. $\frac{d}{dx}(\cosh x)^2 = 2 \cosh x \sinh x$

2.47 Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\tanh\left(x^2 + 3x\right)\right)$

b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)$

Example 2.48

Integrals Involving Hyperbolic Functions

Evaluate the following integrals:

a. $\int x \cosh\left(x^2\right) dx$

b. $\int \tanh x \, dx$

Solution

We can use *u*-substitution in both cases.

a. Let $u = x^2.$ Then, $du = 2x \, dx$ and

$$\int x \cosh\left(x^2\right) dx = \int \frac{1}{2} \cosh u \, du = \frac{1}{2} \sinh u + C = \frac{1}{2} \sinh\left(x^2\right) + C.$$

b. Let $u = \cosh x.$ Then, $du = \sinh x \, dx$ and

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|\cosh x| + C.$$

Note that $\cosh x > 0$ for all $x,$ so we can eliminate the absolute value signs and obtain

$$\int \tanh x \, dx = \ln(\cosh x) + C.$$

2.48 Evaluate the following integrals:

a. $\int \sinh^3 x \cosh x \, dx$

b. $\int \operatorname{sech}^2 (3x) dx$

Calculus of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in the following table.

Function	Domain	Range
$\sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh^{-1} x$	$(1, \infty)$	$[0, \infty)$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\operatorname{sech}^{-1} x$	$(0, 1)$	$[0, \infty)$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Table 2.3 Domains and Ranges of the Inverse Hyperbolic Functions

The graphs of the inverse hyperbolic functions are shown in the following figure.

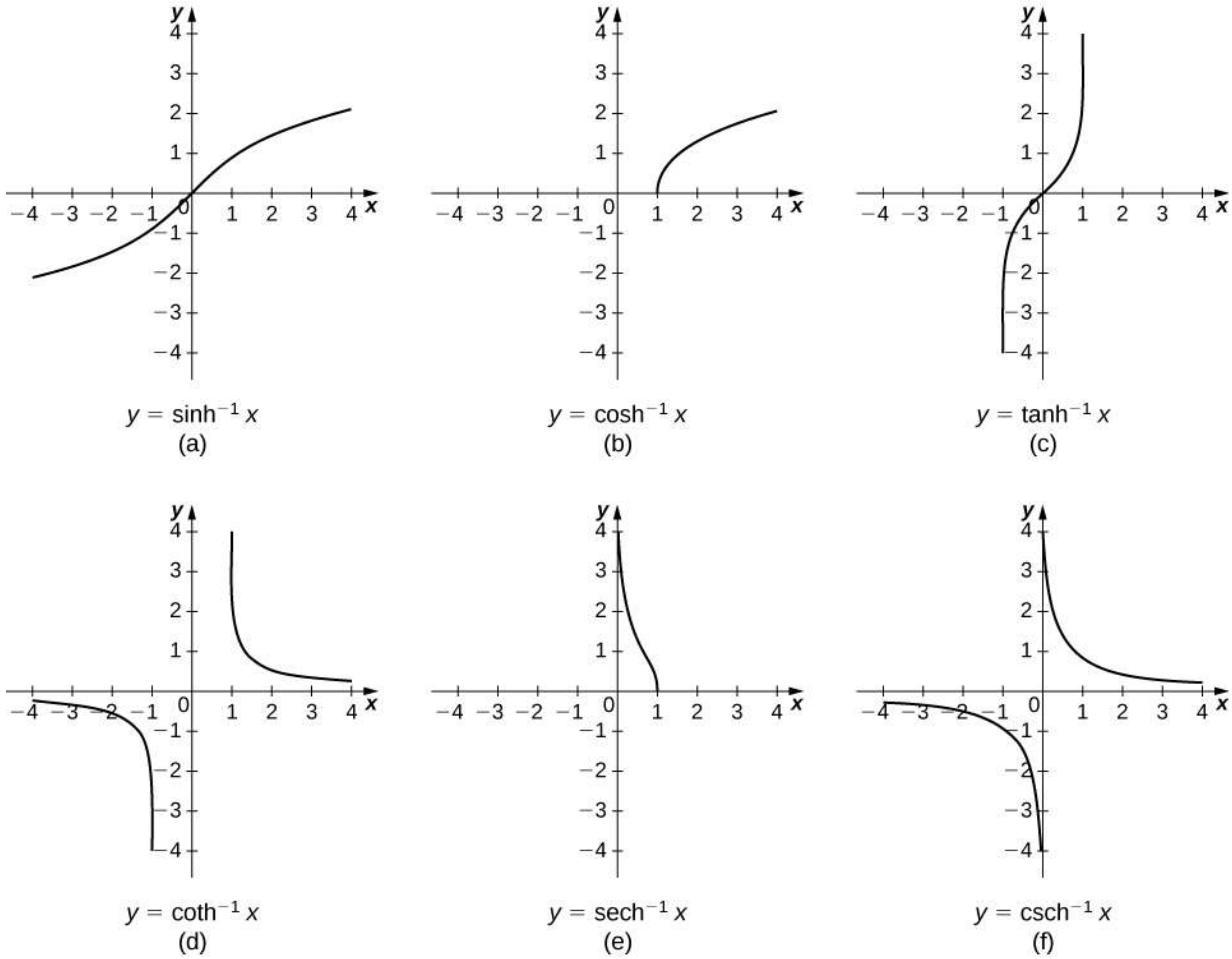


Figure 2.82 Graphs of the inverse hyperbolic functions.

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$\begin{aligned} y &= \sinh^{-1} x \\ \sinh y &= x \\ \frac{d}{dx}\sinh y &= \frac{d}{dx}x \\ \cosh y\frac{dy}{dx} &= 1. \end{aligned}$$

Recall that $\cosh^2 y - \sinh^2 y = 1,$ so $\cosh y = \sqrt{1 + \sinh^2 y}.$ Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

We can derive differentiation formulas for the other inverse hyperbolic functions in a similar fashion. These differentiation formulas are summarized in the following table.

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\coth^{-1} x$	$\frac{1}{1-x^2}$
$\text{sech}^{-1} x$	$\frac{-1}{x\sqrt{1-x^2}}$
$\text{csch}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Table 2.4 Derivatives of the Inverse Hyperbolic Functions

Note that the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ are the same. Thus, when we integrate $1/\left(1-x^2\right)$, we need to select the proper antiderivative based on the domain of the functions and the values of x. Integration formulas involving the inverse hyperbolic functions are summarized as follows.

$$\int \frac{1}{\sqrt{1+u^2}}du = \sinh^{-1} u + C \qquad \int \frac{1}{u\sqrt{1-u^2}}du = -\text{sech}^{-1}\lvert u\rvert + C$$

$$\int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1} u + C \qquad \int \frac{1}{u\sqrt{1+u^2}}du = -\text{csch}^{-1}\lvert u\rvert + C$$

$$\int \frac{1}{1-u^2}du = \begin{cases} \tanh^{-1} u + C \text{ if } \lvert u\rvert < 1 \\ \coth^{-1} u + C \text{ if } \lvert u\rvert > 1 \end{cases}$$

Example 2.49

Differentiating Inverse Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$

b. $\frac{d}{dx}\left(\tanh^{-1} x\right)^2$

Solution

Using the formulas in **Table 2.4** and the chain rule, we obtain the following results:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right) = \frac{1}{3\sqrt{1+\frac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$

b. $\frac{d}{dx}\left(\tanh^{-1} x\right)^2 = \frac{2\left(\tanh^{-1} x\right)}{1-x^2}$

2.49 Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\cosh^{-1}(3x)\right)$

b. $\frac{d}{dx}\left(\coth^{-1} x\right)^3$

Example 2.50

Integrals Involving Inverse Hyperbolic Functions

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{4x^2-1}}dx$

b. $\int \frac{1}{2x\sqrt{1-9x^2}}dx$

Solution

We can use u-substitution in both cases.

a. Let $u = 2x.$ Then, $du = 2dx$ and we have

$$\int \frac{1}{\sqrt{4x^2-1}}dx = \int \frac{1}{2\sqrt{u^2-1}}du = \frac{1}{2}\cosh^{-1} u + C = \frac{1}{2}\cosh^{-1}(2x) + C.$$

b. Let $u = 3x.$ Then, $du = 3dx$ and we obtain

$$\int \frac{1}{2x\sqrt{1-9x^2}}dx = \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}du = -\frac{1}{2}\text{sech}^{-1}|u| + C = -\frac{1}{2}\text{sech}^{-1}|3x| + C.$$

2.50 Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{x^2-4}}dx, \quad x > 2$

b. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$

Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a **catenary**. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries. The following figure shows chains hanging from a row of posts.

Figure 2.83 Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a\cosh(x/a)$ are catenaries. **Figure 2.84** shows the graph of $y = 2\cosh(x/2)$.

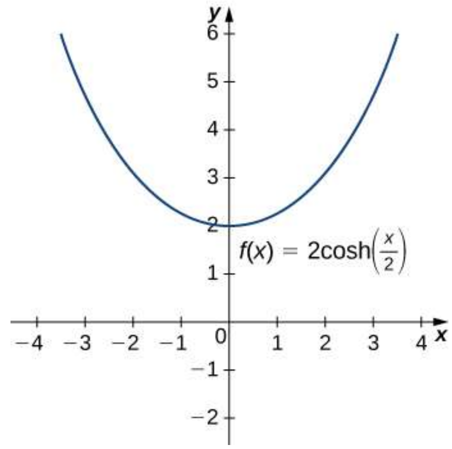


Figure 2.84 A hyperbolic cosine function forms the shape of a catenary.

Example 2.51

Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10\cosh(x/10)$ for $-15 \le x \le 15$, where x is measured in feet. Determine the length of the cable (in feet).

Solution

Recall from Section 6.4 that the formula for arc length is

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We have $f(x) = 10\cosh(x/10)$, so $f'(x) = \sinh(x/10)$. Then

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx.\end{aligned}$$

Now recall that $1 + \sinh^2 x = \cosh^2 x$, so we have

$$\begin{aligned}\text{Arc Length} &= \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx \\ &= \int_{-15}^{15} \cosh\left(\frac{x}{10}\right) dx \\ &= 10\sinh\left(\frac{x}{10}\right)\Big|_{-15}^{15} = 10\left[\sinh\left(\frac{3}{2}\right) - \sinh\left(-\frac{3}{2}\right)\right] = 20\sinh\left(\frac{3}{2}\right) \\ &\approx 42.586\text{ ft.}\end{aligned}$$

2.51 Assume a hanging cable has the shape $15\cosh(x/15)$ for $-20 \le x \le 20$. Determine the length of the cable (in feet).

2.9 EXERCISES

377. **[T]** Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$. Use a calculator to graph these functions and ensure your expression is correct.

378. From the definitions of $\cosh(x)$ and $\sinh(x)$, find their antiderivatives.

379. Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

380. Use the quotient rule to verify that $\tanh(x)' = \text{sech}^2(x)$.

381. Derive $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$ from the definition.

382. Take the derivative of the previous expression to find an expression for $\sinh(2x)$.

383. Prove $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ by changing the expression to exponentials.

384. Take the derivative of the previous expression to find an expression for $\cosh(x+y)$.

For the following exercises, find the derivatives of the given functions and graph along with the function to ensure your answer is correct.

385. **[T]** $\cosh(3x+1)$

386. **[T]** $\sinh(x^2)$

387. **[T]** $\dfrac{1}{\cosh(x)}$

388. **[T]** $\sinh(\ln(x))$

389. **[T]** $\cosh^2(x) + \sinh^2(x)$

390. **[T]** $\cosh^2(x) - \sinh^2(x)$

391. **[T]** $\tanh\left(\sqrt{x^2+1}\right)$

392. **[T]** $\dfrac{1+\tanh(x)}{1-\tanh(x)}$

393. **[T]** $\sinh^6(x)$

394. **[T]** $\ln(\text{sech}(x) + \tanh(x))$

For the following exercises, find the antiderivatives for the given functions.

395. $\cosh(2x+1)$

396. $\tanh(3x+2)$

397. $x\cosh(x^2)$

398. $3x^3\tanh(x^4)$

399. $\cosh^2(x)\sinh(x)$

400. $\tanh^2(x)\text{sech}^2(x)$

401. $\dfrac{\sinh(x)}{1+\cosh(x)}$

402. $\coth(x)$

403. $\cosh(x) + \sinh(x)$

404. $(\cosh(x) + \sinh(x))^n$

For the following exercises, find the derivatives for the functions.

405. $\tanh^{-1}(4x)$

406. $\sinh^{-1}(x^2)$

407. $\sinh^{-1}(\cosh(x))$

408. $\cosh^{-1}(x^3)$

409. $\tanh^{-1}(\cos(x))$

410. $e^{\sinh^{-1}(x)}$

411. $\ln(\tanh^{-1}(x))$

For the following exercises, find the antiderivatives for the functions.

412. $\displaystyle\int \frac{dx}{4-x^2}$

413. $\displaystyle\int \frac{dx}{a^2-x^2}$

414. $\int \frac{dx}{\sqrt{x^2+1}}$

415. $\int \frac{x\,dx}{\sqrt{x^2+1}}$

416. $\int -\frac{dx}{x\sqrt{1-x^2}}$

417. $\int \frac{e^x}{\sqrt{e^{2x}-1}}$

418. $\int -\frac{2x}{x^4-1}$

For the following exercises, use the fact that a falling body with friction equal to velocity squared obeys the equation $dv/dt = g - v^2$.

419. Show that $v(t) = \sqrt{g}\tanh\left(\sqrt{g}t\right)$ satisfies this equation.

420. Derive the previous expression for $v(t)$ by integrating $\frac{dv}{g - v^2} = dt$.

421. **[T]** Estimate how far a body has fallen in 12 seconds by finding the area underneath the curve of $v(t)$.

For the following exercises, use this scenario: A cable hanging under its own weight has a slope $S = dy/dx$ that satisfies $dS/dx = c\sqrt{1 + S^2}$. The constant c is the ratio of cable density to tension.

422. Show that $S = \sinh(cx)$ satisfies this equation.

423. Integrate $dy/dx = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

424. Sketch the cable and determine how far down it sags at $x = 0$.

For the following exercises, solve each problem.

425. **[T]** A chain hangs from two posts 2 m apart to form a catenary described by the equation $y = 2\cosh(x/2) - 1$. Find the slope of the catenary at the left fence post.

426. **[T]** A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh(x/4) - 3$. Find the total length of the catenary (arc length).

427. **[T]** A high-voltage power line is a catenary described by $y = 10\cosh(x/10)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

428. A telephone line is a catenary described by $y = a\cosh(x/a)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

429. Prove the formula for the derivative of $y = \sinh^{-1}(x)$ by differentiating $x = \sinh(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

430. Prove the formula for the derivative of $y = \cosh^{-1}(x)$ by differentiating $x = \cosh(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

431. Prove the formula for the derivative of $y = \operatorname{sech}^{-1}(x)$ by differentiating $x = \operatorname{sech}(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

432. Prove that $(\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$.

433. Prove the expression for $\sinh^{-1}(x)$. Multiply $x = \sinh(y) = (1/2)\left(e^y - e^{-y}\right)$ by $2e^y$ and solve for y. Does your expression match the textbook?

434. Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = (1/2)\left(e^y - e^{-y}\right)$ by $2e^y$ and solve for y. Does your expression match the textbook?

CHAPTER 2 REVIEW

KEY TERMS

arc length the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

catenary a curve in the shape of the function $y = a\cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

center of mass the point at which the total mass of the system could be concentrated without changing the moment

centroid the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

cross-section the intersection of a plane and a solid object

density function a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

disk method a special case of the slicing method used with solids of revolution when the slices are disks

doubling time if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

exponential decay systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$

exponential growth systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$

frustum a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

half-life if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

Hooke's law this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F = kx,$ where k is a constant

hydrostatic pressure the pressure exerted by water on a submerged object

lamina a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

method of cylindrical shells a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

moment if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b],$ then the moments of the region with respect to the x- and y-axes are given by $M_x = \rho\int_a^b \frac{[f(x)]^2}{2}dx$ and $M_y = \rho\int_a^b x f(x)dx,$ respectively

slicing method a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

solid of revolution a solid generated by revolving a region in a plane around a line in that plane

surface area the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

symmetry principle the symmetry principle states that if a region R is symmetric about a line l, then the centroid of R lies on l

theorem of Pappus for volume this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

washer method a special case of the slicing method used with solids of revolution when the slices are washers

work the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

KEY EQUATIONS

- **Area between two curves, integrating on the *x*-axis**

  $$A = \int_a^b [f(x) - g(x)]dx$$

- **Area between two curves, integrating on the *y*-axis**

  $$A = \int_c^d [u(y) - v(y)]dy$$

- **Disk Method along the *x*-axis**

  $$V = \int_a^b \pi[f(x)]^2\, dx$$

- **Disk Method along the *y*-axis**

  $$V = \int_c^d \pi[g(y)]^2\, dy$$

- **Washer Method**

  $$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$$

- **Method of Cylindrical Shells**

  $$V = \int_a^b (2\pi x f(x))dx$$

- **Arc Length of a Function of *x***

  $$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

- **Arc Length of a Function of *y***

  $$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

- **Surface Area of a Function of *x***

  $$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + (f'(x))^2}\right)dx$$

- **Mass of a one-dimensional object**

  $$m = \int_a^b \rho(x)dx$$

- **Mass of a circular object**

  $$m = \int_0^r 2\pi x \rho(x)dx$$

- **Work done on an object**
  $W = \int_a^b F(x)dx$
- **Hydrostatic force on a plate**
  $F = \int_a^b \rho w(x)s(x)dx$
- **Mass of a lamina**
  $m = \rho \int_a^b f(x)dx$
- **Moments of a lamina**
  $M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx$ and $M_y = \rho \int_a^b x f(x)dx$
- **Center of mass of a lamina**
  $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$
- **Natural logarithm function**
- $\ln x = \int_1^x \frac{1}{t} dt$ Z
- **Exponential function** $y = e^x$
- $\ln y = \ln(e^x) = x$ Z

KEY CONCEPTS

2.1 Areas between Curves

- Just as definite integrals can be used to find the area under a curve, they can also be used to find the area between two curves.
- To find the area between two curves defined by functions, integrate the difference of the functions.
- If the graphs of the functions cross, or if the region is complex, use the absolute value of the difference of the functions. In this case, it may be necessary to evaluate two or more integrals and add the results to find the area of the region.
- Sometimes it can be easier to integrate with respect to *y* to find the area. The principles are the same regardless of which variable is used as the variable of integration.

2.2 Determining Volumes by Slicing

- Definite integrals can be used to find the volumes of solids. Using the slicing method, we can find a volume by integrating the cross-sectional area.
- For solids of revolution, the volume slices are often disks and the cross-sections are circles. The method of disks involves applying the method of slicing in the particular case in which the cross-sections are circles, and using the formula for the area of a circle.
- If a solid of revolution has a cavity in the center, the volume slices are washers. With the method of washers, the area of the inner circle is subtracted from the area of the outer circle before integrating.

2.3 Volumes of Revolution: Cylindrical Shells

- The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other.

- The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

2.4 Arc Length of a Curve and Surface Area

- The arc length of a curve can be calculated using a definite integral.
- The arc length is first approximated using line segments, which generates a Riemann sum. Taking a limit then gives us the definite integral formula. The same process can be applied to functions of y.
- The concepts used to calculate the arc length can be generalized to find the surface area of a surface of revolution.
- The integrals generated by both the arc length and surface area formulas are often difficult to evaluate. It may be necessary to use a computer or calculator to approximate the values of the integrals.

2.5 Physical Applications

- Several physical applications of the definite integral are common in engineering and physics.
- Definite integrals can be used to determine the mass of an object if its density function is known.
- Work can also be calculated from integrating a force function, or when counteracting the force of gravity, as in a pumping problem.
- Definite integrals can also be used to calculate the force exerted on an object submerged in a liquid.

2.6 Moments and Centers of Mass

- Mathematically, the center of mass of a system is the point at which the total mass of the system could be concentrated without changing the moment. Loosely speaking, the center of mass can be thought of as the balancing point of the system.
- For point masses distributed along a number line, the moment of the system with respect to the origin is $M = \sum_{i=1}^{n} m_i x_i$. For point masses distributed in a plane, the moments of the system with respect to the *x*- and *y*-axes, respectively, are $M_x = \sum_{i=1}^{n} m_i y_i$ and $M_y = \sum_{i=1}^{n} m_i x_i$, respectively.
- For a lamina bounded above by a function $f(x)$, the moments of the system with respect to the *x*- and *y*-axes, respectively, are $M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx$ and $M_y = \rho \int_a^b x f(x) dx$.
- The *x*- and *y*-coordinates of the center of mass can be found by dividing the moments around the *y*-axis and around the *x*-axis, respectively, by the total mass. The symmetry principle says that if a region is symmetric with respect to a line, then the centroid of the region lies on the line.
- The theorem of Pappus for volume says that if a region is revolved around an external axis, the volume of the resulting solid is equal to the area of the region multiplied by the distance traveled by the centroid of the region.

2.7 Integrals, Exponential Functions, and Logarithms

- The earlier treatment of logarithms and exponential functions did not define the functions precisely and formally. This section develops the concepts in a mathematically rigorous way.
- The cornerstone of the development is the definition of the natural logarithm in terms of an integral.
- The function e^x is then defined as the inverse of the natural logarithm.
- General exponential functions are defined in terms of e^x, and the corresponding inverse functions are general logarithms.

- Familiar properties of logarithms and exponents still hold in this more rigorous context.

2.8 Exponential Growth and Decay

- Exponential growth and exponential decay are two of the most common applications of exponential functions.
- Systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$.
- In exponential growth, the rate of growth is proportional to the quantity present. In other words, $y' = ky$.
- Systems that exhibit exponential growth have a constant doubling time, which is given by $(\ln 2)/k$.
- Systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$.
- Systems that exhibit exponential decay have a constant half-life, which is given by $(\ln 2)/k$.

2.9 Calculus of the Hyperbolic Functions

- Hyperbolic functions are defined in terms of exponential functions.
- Term-by-term differentiation yields differentiation formulas for the hyperbolic functions. These differentiation formulas give rise, in turn, to integration formulas.
- With appropriate range restrictions, the hyperbolic functions all have inverses.
- Implicit differentiation yields differentiation formulas for the inverse hyperbolic functions, which in turn give rise to integration formulas.
- The most common physical applications of hyperbolic functions are calculations involving catenaries.

CHAPTER 2 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample.

435. The amount of work to pump the water out of a half-full cylinder is half the amount of work to pump the water out of the full cylinder.

436. If the force is constant, the amount of work to move an object from $x = a$ to $x = b$ is $F(b - a)$.

437. The disk method can be used in any situation in which the washer method is successful at finding the volume of a solid of revolution.

438. If the half-life of seaborgium-266 is 360 ms, then $k = (\ln(2))/360$.

For the following exercises, use the requested method to determine the volume of the solid.

439. The volume that has a base of the ellipse $x^2/4 + y^2/9 = 1$ and cross-sections of an equilateral triangle perpendicular to the y-axis. Use the method of slicing.

440. $y = x^2 - x$, from $x = 1$ to $x = 4$, rotated around the$y$-axis using the washer method

441. $x = y^2$ and $x = 3y$ rotated around the y-axis using the washer method

442. $x = 2y^2 - y^3$, $x = 0$, and $y = 0$ rotated around the x-axis using cylindrical shells

For the following exercises, find

a. the area of the region,
b. the volume of the solid when rotated around the x-axis, and
c. the volume of the solid when rotated around the y-axis. Use whichever method seems most appropriate to you.

443. $y = x^3$, $x = 0$, $y = 0$, and $x = 2$

444. $y = x^2 - x$ and $x = 0$

445. **[T]** $y = \ln(x) + 2$ and $y = x$

446. $y = x^2$ and $y = \sqrt{x}$

447. $y = 5 + x$, $y = x^2$, $x = 0$, and $x = 1$

448. Below $x^2 + y^2 = 1$ and above $y = 1 - x$

449. Find the mass of $\rho = e^{-x}$ on a disk centered at the origin with radius 4.

450. Find the center of mass for $\rho = \tan^2 x$ on $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

451. Find the mass and the center of mass of $\rho = 1$ on the region bounded by $y = x^5$ and $y = \sqrt{x}$.

For the following exercises, find the requested arc lengths.

452. The length of x for $y = \cosh(x)$ from $x = 0$ to $x = 2$.

453. The length of y for $x = 3 - \sqrt{y}$ from $y = 0$ to $y = 4$

For the following exercises, find the surface area and volume when the given curves are revolved around the specified axis.

454. The shape created by revolving the region between $y = 4 + x$, $y = 3 - x$, $x = 0$, and $x = 2$ rotated around the y-axis.

455. The loudspeaker created by revolving $y = 1/x$ from $x = 1$ to $x = 4$ around the x-axis.

For the following exercises, consider the Karun-3 dam in Iran. Its shape can be approximated as an isosceles triangle with height 205 m and width 388 m. Assume the current depth of the water is 180 m. The density of water is 1000 kg/m ^{3}.

456. Find the total force on the wall of the dam.

457. You are a crime scene investigator attempting to determine the time of death of a victim. It is noon and $45°\text{F}$ outside and the temperature of the body is $78°\text{F}$. You know the cooling constant is $k = 0.00824°\text{F/min}$. When did the victim die, assuming that a human's temperature is $98°\text{F}$?

For the following exercise, consider the stock market crash in 1929 in the United States. The table lists the Dow Jones industrial average per year leading up to the crash.

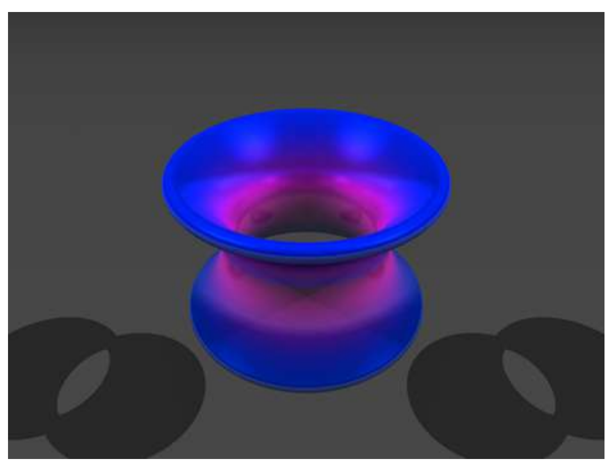
Years after 1920	Value ($)
1	63.90
3	100
5	110
7	160
9	381.17

Source: http://stockcharts.com/freecharts/historical/djia19201940.html

458. **[T]** The best-fit exponential curve to these data is given by $y = 40.71 + 1.224^x$. Why do you think the gains of the market were unsustainable? Use first and second derivatives to help justify your answer. What would this model predict the Dow Jones industrial average to be in 2014 ?

For the following exercises, consider the catenoid, the only solid of revolution that has a minimal surface, or zero mean curvature. A catenoid in nature can be found when stretching soap between two rings.

459. Find the volume of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-axis, as shown here.



460. Find surface area of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-axis.

3 | TECHNIQUES OF INTEGRATION

Figure 3.1 Careful planning of traffic signals can prevent or reduce the number of accidents at busy intersections. (credit: modification of work by David McKelvey, Flickr)

Chapter Outline

Introduction

In a large city, accidents occurred at an average rate of one every three months at a particularly busy intersection. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the eight-month interval without an accident a result of chance? We explore this question later in this chapter and see that integration is an essential part of determining the answer (see **Example 3.49**).

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

3.1 | Integration by Parts

Learning Objectives

3.1.1 Recognize when to use integration by parts.
3.1.2 Use the integration-by-parts formula to solve integration problems.
3.1.3 Use the integration-by-parts formula for definite integrals.

By now we have a fairly thorough procedure for how to evaluate many basic integrals. However, although we can integrate $\int x\sin(x^2)dx$ by using the substitution, $u = x^2$, something as simple looking as $\int x\sin x\, dx$ defies us. Many students want to know whether there is a product rule for integration. There isn't, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique **integration by parts**.

The Integration-by-Parts Formula

If, $h(x) = f(x)g(x)$, then by using the product rule, we obtain $h'(x) = f'(x)g(x) + g'(x)f(x)$. Although at first it may seem counterproductive, let's now integrate both sides of this equation: $\int h'(x)dx = \int (g(x)f'(x) + f(x)g'(x))dx$.

This gives us

$$h(x) = f(x)g(x) = \int g(x)f'(x)dx + \int f(x)g'(x)dx.$$

Now we solve for $\int f(x)g'(x)dx$:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x)dx$ and $dv = g'(x)dx$, we have the more compact form

$$\int u\, dv = uv - \int v\, du.$$

Theorem 3.1: Integration by Parts

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u\, dv = uv - \int v\, du. \tag{3.1}$$

The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. The following example illustrates its use.

Example 3.1

Using Integration by Parts

Use integration by parts with $u = x$ and $dv = \sin x\, dx$ to evaluate $\int x\sin x\, dx$.

Solution

By choosing $u = x$, we have $du = 1dx$. Since $dv = \sin x\,dx$, we get $v = \int \sin x\,dx = -\cos x$. It is handy to keep track of these values as follows:

$$\begin{aligned} u &= x & dv &= \sin x\,dx \\ du &= 1dx & v &= \int \sin x\,dx = -\cos x. \end{aligned}$$

Applying the integration-by-parts formula results in

$$\begin{aligned} \int x\sin x\,dx &= (x)(-\cos x) - \int(-\cos x)(1dx) && \text{Substitute.} \\ &= -x\cos x + \int \cos x\,dx && \text{Simplify.} \\ &= -x\cos x + \sin x + C. && \text{Use } \int \cos x\,dx = \sin x + C. \end{aligned}$$

Analysis

At this point, there are probably a few items that need clarification. First of all, you may be curious about what would have happened if we had chosen $u = \sin x$ and $dv = x$. If we had done so, then we would have $du = \cos x$ and $v = \frac{1}{2}x^2$. Thus, after applying integration by parts, we have $\int x\sin x\,dx = \frac{1}{2}x^2\sin x - \int \frac{1}{2}x^2\cos x\,dx$. Unfortunately, with the new integral, we are in no better position than before. It is important to keep in mind that when we apply integration by parts, we may need to try several choices for u and dv before finding a choice that works.

Second, you may wonder why, when we find $v = \int \sin x\,dx = -\cos x$, we do not use $v = -\cos x + K$. To see that it makes no difference, we can rework the problem using $v = -\cos x + K$:

$$\begin{aligned} \int x\sin x\,dx &= (x)(-\cos x + K) - \int(-\cos x + K)(1dx) \\ &= -x\cos x + Kx + \int \cos x\,dx - \int Kdx \\ &= -x\cos x + Kx + \sin x - Kx + C \\ &= -x\cos x + \sin x + C. \end{aligned}$$

As you can see, it makes no difference in the final solution.

Last, we can check to make sure that our antiderivative is correct by differentiating $-x\cos x + \sin x + C$:

$$\begin{aligned} \frac{d}{dx}(-x\cos x + \sin x + C) &= (-1)\cos x + (-x)(-\sin x) + \cos x \\ &= x\sin x. \end{aligned}$$

Therefore, the antiderivative checks out.

Watch this **video (http://www.openstaxcollege.org/l/20_intbyparts1)** and visit this **website (http://www.openstaxcollege.org/l/20_intbyparts2)** for examples of integration by parts.

3.1 Evaluate $\int xe^{2x}\,dx$ using the integration-by-parts formula with $u = x$ and $dv = e^{2x}\,dx$.

The natural question to ask at this point is: How do we know how to choose u and dv? Sometimes it is a matter of trial and error; however, the acronym LIATE can often help to take some of the guesswork out of our choices. This acronym stands for **L**ogarithmic Functions, **I**nverse Trigonometric Functions, **A**lgebraic Functions, **T**rigonometric Functions, and **E**xponential Functions. This mnemonic serves as an aid in determining an appropriate choice for u.

The type of function in the integral that appears first in the list should be our first choice of u. For example, if an integral contains a logarithmic function and an algebraic function, we should choose u to be the logarithmic function, because L comes before A in LIATE. The integral in **Example 3.1** has a trigonometric function $(\sin x)$ and an algebraic function (x). Because A comes before T in LIATE, we chose u to be the algebraic function. When we have chosen u, dv is selected to be the remaining part of the function to be integrated, together with dx.

Why does this mnemonic work? Remember that whatever we pick to be dv must be something we can integrate. Since we do not have integration formulas that allow us to integrate simple logarithmic functions and inverse trigonometric functions, it makes sense that they should not be chosen as values for dv. Consequently, they should be at the head of the list as choices for u. Thus, we put LI at the beginning of the mnemonic. (We could just as easily have started with IL, since these two types of functions won't appear together in an integration-by-parts problem.) The exponential and trigonometric functions are at the end of our list because they are fairly easy to integrate and make good choices for dv. Thus, we have TE at the end of our mnemonic. (We could just as easily have used ET at the end, since when these types of functions appear together it usually doesn't really matter which one is u and which one is dv.) Algebraic functions are generally easy both to integrate and to differentiate, and they come in the middle of the mnemonic.

Example 3.2

Using Integration by Parts

Evaluate $\int \frac{\ln x}{x^3} dx$.

Solution

Begin by rewriting the integral:

$$\int \frac{\ln x}{x^3} dx = \int x^{-3} \ln x \, dx.$$

Since this integral contains the algebraic function x^{-3} and the logarithmic function $\ln x$, choose $u = \ln x$, since L comes before A in LIATE. After we have chosen $u = \ln x$, we must choose $dv = x^{-3} \, dx$.

Next, since $u = \ln x$, we have $du = \frac{1}{x} dx$. Also, $v = \int x^{-3} \, dx = -\frac{1}{2} x^{-2}$. Summarizing,

$$\begin{aligned} u &= \ln x & dv &= x^{-3} \, dx \\ du &= \frac{1}{x} dx & v &= \int x^{-3} \, dx = -\frac{1}{2} x^{-2}. \end{aligned}$$

Substituting into the integration-by-parts formula (**Equation 3.1**) gives

$$\begin{aligned} \int \frac{\ln x}{x^3} dx &= \int x^{-3} \ln x \, dx = (\ln x)\left(-\frac{1}{2} x^{-2}\right) - \int \left(-\frac{1}{2} x^{-2}\right)\left(\frac{1}{x} dx\right) \\ &= -\frac{1}{2} x^{-2} \ln x + \int \frac{1}{2} x^{-3} \, dx && \text{Simplify.} \\ &= -\frac{1}{2} x^{-2} \ln x - \frac{1}{4} x^{-2} + C && \text{Integrate.} \\ &= -\frac{1}{2x^2} \ln x - \frac{1}{4x^2} + C. && \text{Rewrite with positive integers.} \end{aligned}$$

3.2 Evaluate $\int x \ln x \, dx.$

In some cases, as in the next two examples, it may be necessary to apply integration by parts more than once.

Example 3.3

Applying Integration by Parts More Than Once

Evaluate $\int x^2 e^{3x} dx.$

Solution

Using LIATE, choose $u = x^2$ and $dv = e^{3x} dx.$ Thus, $du = 2x \, dx$ and $v = \int e^{3x} dx = \left(\frac{1}{3}\right)e^{3x}.$ Therefore,

$$\begin{array}{rclrcl} u & = & x^2 & dv & = & e^{3x} dx \\ du & = & 2x \, dx & v & = & \int e^{3x} dx = \frac{1}{3}e^{3x}. \end{array}$$

Substituting into **Equation 3.1** produces

$$\int x^2 e^{3x} dx = \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}xe^{3x} dx.$$

We still cannot integrate $\int \frac{2}{3}xe^{3x} dx$ directly, but the integral now has a lower power on $x.$ We can evaluate this new integral by using integration by parts again. To do this, choose $u = x$ and $dv = \frac{2}{3}e^{3x} dx.$ Thus, $du = dx$ and $v = \int \left(\frac{2}{3}\right)e^{3x} dx = \left(\frac{2}{9}\right)e^{3x}.$ Now we have

$$\begin{array}{rclrcl} u & = & x & dv & = & \frac{2}{3}e^{3x} dx \\ du & = & dx & v & = & \int \frac{2}{3}e^{3x} dx = \frac{2}{9}e^{3x}. \end{array}$$

Substituting back into the previous equation yields

$$\int x^2 e^{3x} dx = \frac{1}{3}x^2 e^{3x} - \left(\frac{2}{9}xe^{3x} - \int \frac{2}{9}e^{3x} dx\right).$$

After evaluating the last integral and simplifying, we obtain

$$\int x^2 e^{3x} dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C.$$

Example 3.4

Applying Integration by Parts When LIATE Doesn't Quite Work

Evaluate $\int t^3 e^{t^2} dt.$

Solution

If we use a strict interpretation of the mnemonic LIATE to make our choice of $u,$ we end up with $u = t^3$ and $dv = e^{t^2}dt.$ Unfortunately, this choice won't work because we are unable to evaluate $\int e^{t^2}dt.$ However, since we can evaluate $\int te^{t^2}dx,$ we can try choosing $u = t^2$ and $dv = te^{t^2}dt.$ With these choices we have

$$\begin{array}{rclrcl} u & = & t^2 & dv & = & te^{t^2}dt \\ du & = & 2t\,dt & v & = & \int te^{t^2}dt = \frac{1}{2}e^{t^2}. \end{array}$$

Thus, we obtain

$$\begin{aligned} \int t^3 e^{t^2}dt &= \frac{1}{2}t^2 e^{t^2} - \int \frac{1}{2}e^{t^2}2tdt \\ &= \frac{1}{2}t^2 e^{t^2} - \frac{1}{2}e^{t^2} + C. \end{aligned}$$

Example 3.5

Applying Integration by Parts More Than Once

Evaluate $\int \sin(\ln x)dx.$

Solution

This integral appears to have only one function—namely, $\sin(\ln x)$ —however, we can always use the constant function 1 as the other function. In this example, let's choose $u = \sin(\ln x)$ and $dv = 1dx.$ (The decision to use $u = \sin(\ln x)$ is easy. We can't choose $dv = \sin(\ln x)dx$ because if we could integrate it, we wouldn't be using integration by parts in the first place!) Consequently, $du = (1/x)\cos(\ln x)dx$ and $v = \int 1dx = x.$ After applying integration by parts to the integral and simplifying, we have

$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx.$$

Unfortunately, this process leaves us with a new integral that is very similar to the original. However, let's see what happens when we apply integration by parts again. This time let's choose $u = \cos(\ln x)$ and $dv = 1dx,$ making $du = -(1/x)\sin(\ln x)dx$ and $v = \int 1dx = x.$ Substituting, we have

$$\int \sin(\ln x)dx = x\sin(\ln x) - \left(x\cos(\ln x) — \int -\sin(\ln x)dx\right).$$

After simplifying, we obtain

$$\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx.$$

The last integral is now the same as the original. It may seem that we have simply gone in a circle, but now we can actually evaluate the integral. To see how to do this more clearly, substitute $I = \int \sin(\ln x)dx.$ Thus, the equation becomes

$$I = x\sin(\ln x) - x\cos(\ln x) - I.$$

First, add I to both sides of the equation to obtain

$$2I = x\sin(\ln x) - x\cos(\ln x).$$

Next, divide by 2:

$$I = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

Substituting $I = \int \sin(\ln x)dx$ again, we have

$$\int \sin(\ln x)dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

From this we see that $(1/2)x\sin(\ln x) - (1/2)x\cos(\ln x)$ is an antiderivative of $\sin(\ln x)dx$. For the most general antiderivative, add $+C$:

$$\int \sin(\ln x)dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C.$$

Analysis

If this method feels a little strange at first, we can check the answer by differentiation:

$$\begin{aligned}&\frac{d}{dx}\left(\frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x)\right)\\ &= \frac{1}{2}(\sin(\ln x)) + \cos(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x - \left(\frac{1}{2}\cos(\ln x) - \sin(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x\right)\\ &= \sin(\ln x).\end{aligned}$$

3.3 Evaluate $\int x^2 \sin x\, dx.$

Integration by Parts for Definite Integrals

Now that we have used integration by parts successfully to evaluate indefinite integrals, we turn our attention to definite integrals. The integration technique is really the same, only we add a step to evaluate the integral at the upper and lower limits of integration.

Theorem 3.2: Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du. \tag{3.2}$$

Example 3.6

Finding the Area of a Region

Find the area of the region bounded above by the graph of $y = \tan^{-1} x$ and below by the x-axis over the interval $[0,\ 1]$.

Solution

This region is shown in **Figure 3.2**. To find the area, we must evaluate $\int_0^1 \tan^{-1} x\, dx$.

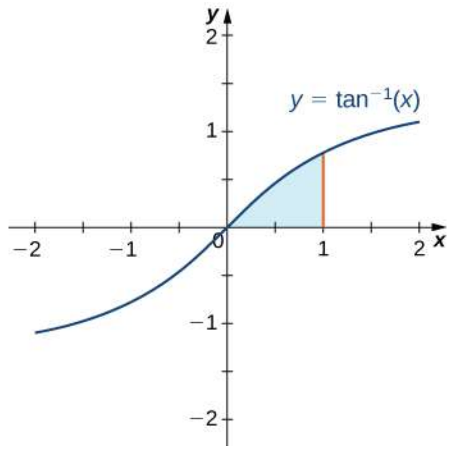


Figure 3.2 To find the area of the shaded region, we have to use integration by parts.

For this integral, let's choose $u = \tan^{-1} x$ and $dv = dx$, thereby making $du = \frac{1}{x^2 + 1} dx$ and $v = x$. After applying the integration-by-parts formula (**Equation 3.2**) we obtain

$$\text{Area} = x \tan^{-1} x|_0^1 - \int_0^1 \frac{x}{x^2 + 1} dx.$$

Use u-substitution to obtain

$$\int_0^1 \frac{x}{x^2 + 1} dx = \frac{1}{2} \ln\left|x^2 + 1\right|\Big|_0^1.$$

Thus,

$$\text{Area} = x \tan^{-1} x\Big|_0^1 - \frac{1}{2} \ln\left|x^2 + 1\right|\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

At this point it might not be a bad idea to do a "reality check" on the reasonableness of our solution. Since $\frac{\pi}{4} - \frac{1}{2} \ln 2 \approx 0.4388$, and from **Figure 3.2** we expect our area to be slightly less than 0.5, this solution appears to be reasonable.

Example 3.7

Finding a Volume of Revolution

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = e^{-x}$, the x-axis, the y-axis, and the line $x = 1$ about the y-axis.

Solution

The best option to solving this problem is to use the shell method. Begin by sketching the region to be revolved, along with a typical rectangle (see the following graph).

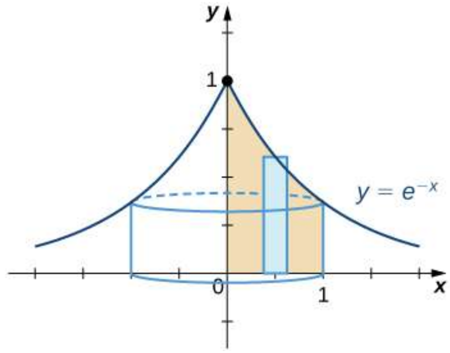


Figure 3.3 We can use the shell method to find a volume of revolution.

To find the volume using shells, we must evaluate $2\pi\int_0^1 xe^{-x}\,dx$. To do this, let $u = x$ and $dv = e^{-x}$. These choices lead to $du = dx$ and $v = \int e^{-x} = -e^{-x}$. Substituting into **Equation 3.2**, we obtain

$$\begin{aligned} \text{Volume} &= 2\pi\int_0^1 xe^{-x}\,dx = 2\pi\left(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx\right) && \text{Use integration by parts.} \\ &= -2\pi xe^{-x}\Big|_0^1 - 2\pi e^{-x}\Big|_0^1 && \text{Evaluate } \int_0^1 e^{-x}\,dx = -e^{-x}\Big|_0^1. \\ &= 2\pi - \frac{4\pi}{e}. && \text{Evaluate and simplify.} \end{aligned}$$

Analysis

Again, it is a good idea to check the reasonableness of our solution. We observe that the solid has a volume slightly less than that of a cylinder of radius 1 and height of $1/e$ added to the volume of a cone of base radius 1 and height of $1 - \frac{1}{3}$. Consequently, the solid should have a volume a bit less than

$$\pi(1)^2\frac{1}{e} + \left(\frac{\pi}{3}\right)(1)^2\left(1 - \frac{1}{e}\right) = \frac{2\pi}{3e} - \frac{\pi}{3} \approx 1.8177.$$

Since $2\pi - \frac{4\pi}{e} \approx 1.6603$, we see that our calculated volume is reasonable.

3.4 Evaluate $\int_0^{\pi/2} x\cos x\,dx$.

3.1 EXERCISES

In using the technique of integration by parts, you must carefully choose which expression is *u*. For each of the following problems, use the guidelines in this section to choose *u*. Do **not** evaluate the integrals.

1. $\int x^3 e^{2x} dx$

2. $\int x^3 \ln(x) dx$

3. $\int y^3 \cos y dx$

4. $\int x^2 \arctan x \, dx$

5. $\int e^{3x} \sin(2x) dx$

Find the integral by using the simplest method. Not all problems require integration by parts.

6. $\int v \sin v dv$

7. $\int \ln x \, dx$ (*Hint:* $\int \ln x \, dx$ is equivalent to $\int 1 \cdot \ln(x) dx$.)

8. $\int x \cos x \, dx$

9. $\int \tan^{-1} x \, dx$

10. $\int x^2 e^x dx$

11. $\int x \sin(2x) dx$

12. $\int x e^{4x} dx$

13. $\int x e^{-x} dx$

14. $\int x \cos 3x \, dx$

15. $\int x^2 \cos x \, dx$

16. $\int x \ln x \, dx$

17. $\int \ln(2x+1) dx$

18. $\int x^2 e^{4x} dx$

19. $\int e^x \sin x \, dx$

20. $\int e^x \cos x \, dx$

21. $\int x e^{-x^2} dx$

22. $\int x^2 e^{-x} dx$

23. $\int \sin(\ln(2x)) dx$

24. $\int cos(\ln x) dx$

25. $\int (\ln x)^2 dx$

26. $\int \ln(x^2) dx$

27. $\int x^2 \ln x \, dx$

28. $\int \sin^{-1} x \, dx$

29. $\int \cos^{-1}(2x) dx$

30. $\int x \arctan x \, dx$

31. $\int x^2 \sin x \, dx$

32. $\int x^3 \cos x \, dx$

33. $\int x^3 \sin x \, dx$

34. $\int x^3 e^x dx$

35. $\int x \sec^{-1} x \, dx$

36. $\int x \sec^2 x \, dx$

37. $\int x \cosh x \, dx$

Compute the definite integrals. Use a graphing utility to confirm your answers.

38. $\int_{1/e}^{1} \ln x \, dx$

39. $\int_0^1 xe^{-2x} \, dx$ (Express the answer in exact form.)

40. $\int_0^1 e^{\sqrt{x}} \, dx (\text{let } u = \sqrt{x})$

41. $\int_1^e \ln(x^2) dx$

42. $\int_0^{\pi} x \cos x \, dx$

43. $\int_{-\pi}^{\pi} x \sin x \, dx$ (Express the answer in exact form.)

44. $\int_0^3 \ln(x^2 + 1) dx$ (Express the answer in exact form.)

45. $\int_0^{\pi/2} x^2 \sin x \, dx$ (Express the answer in exact form.)

46. $\int_0^1 x5^x \, dx$ (Express the answer using five significant digits.)

47. Evaluate $\int \cos x \ln(\sin x) dx$

Derive the following formulas using the technique of integration by parts. Assume that n is a positive integer. These formulas are called *reduction formulas* because the exponent in the x term has been reduced by one in each case. The second integral is simpler than the original integral.

48. $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$

49. $\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$

50. $\int x^n \sin x \, dx =$ ______

51. Integrate $\int 2x\sqrt{2x-3} dx$ using two methods:

a. Using parts, letting $dv = \sqrt{2x-3} \, dx$
b. Substitution, letting $u = 2x - 3$

State whether you would use integration by parts to evaluate the integral. If so, identify u and dv. If not, describe the technique used to perform the integration without actually doing the problem.

52. $\int x \ln x \, dx$

53. $\int \frac{\ln^2 x}{x} dx$

54. $\int xe^x \, dx$

55. $\int xe^{x^2 - 3} \, dx$

56. $\int x^2 \sin x \, dx$

57. $\int x^2 \sin(3x^3 + 2) dx$

Sketch the region bounded above by the curve, the x-axis, and $x = 1$, and find the area of the region. Provide the exact form or round answers to the number of places indicated.

58. $y = 2xe^{-x}$ (Approximate answer to four decimal places.)

59. $y = e^{-x} \sin(\pi x)$ (Approximate answer to five decimal places.)

Find the volume generated by rotating the region bounded by the given curves about the specified line. Express the answers in exact form or approximate to the number of decimal places indicated.

60. $y = \sin x, \ y = 0, \ x = 2\pi, \ x = 3\pi$ about the y-axis (Express the answer in exact form.)

61. $y = e^{-x}$ $y = 0, \ x = -1 x = 0;$ about $x = 1$ (Express the answer in exact form.)

62. A particle moving along a straight line has a velocity of $v(t) = t^2 e^{-t}$ after t sec. How far does it travel in the first 2 sec? (Assume the units are in feet and express the answer in exact form.)

63. Find the area under the graph of $y = \sec^3 x$ from $x = 0$ to $x = 1$. (Round the answer to two significant digits.)

64. Find the area between $y = (x - 2)e^x$ and the x-axis from $x = 2$ to $x = 5$. (Express the answer in exact form.)

65. Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis for $\frac{11\pi}{2} \le x \le \frac{13\pi}{2}$. (Express the answer in exact form.)

66. Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the x-axis, and the vertical line $x = e^2$ about the x-axis. (Express the answer in exact form.)

67. Find the volume of the solid generated by revolving the region bounded by the curve $y = 4\cos x$ and the x-axis, $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$, about the x-axis. (Express the answer in exact form.)

68. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^x$ and the x-axis, from $x = 0$ to $x = \ln(7)$, about the y-axis. (Express the answer in exact form.)

3.2 | Trigonometric Integrals

Learning Objectives

3.2.1 Solve integration problems involving products and powers of $\sin x$ and $\cos x$.

3.2.2 Solve integration problems involving products and powers of $\tan x$ and $\sec x$.

3.2.3 Use reduction formulas to solve trigonometric integrals.

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called **trigonometric integrals**. They are an important part of the integration technique called *trigonometric substitution*, which is featured in **Trigonometric Substitution**. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$.

Integrating Products and Powers of sin*x* and cos*x*

A key idea behind the strategy used to integrate combinations of products and powers of $\sin x$ and $\cos x$ involves rewriting these expressions as sums and differences of integrals of the form $\int \sin^j x \cos x\, dx$ or $\int \cos^j x \sin x\, dx$. After rewriting these integrals, we evaluate them using *u*-substitution. Before describing the general process in detail, let's take a look at the following examples.

Example 3.8

Integrating $\int \cos^j x \sin x\, dx$

Evaluate $\int \cos^3 x \sin x\, dx$.

Solution

Use u-substitution and let $u = \cos x$. In this case, $du = -\sin x\, dx$. Thus,

$$\begin{aligned}\int \cos^3 x \sin x\, dx &= -\int u^3\, du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4 x + C.\end{aligned}$$

3.5 Evaluate $\int \sin^4 x \cos x\, dx$.

Example 3.9

A Preliminary Example: Integrating $\int \cos^j x \sin^k x\, dx$ Where *k* is Odd

Evaluate $\int \cos^2 x \sin^3 x\,dx$.

Solution

To convert this integral to integrals of the form $\int \cos^j x \sin x\,dx$, rewrite $\sin^3 x = \sin^2 x \sin x$ and make the substitution $\sin^2 x = 1 - \cos^2 x$. Thus,

$$\begin{aligned}\int \cos^2 x \sin^3 x\,dx &= \int \cos^2 x(1-\cos^2 x)\sin x\,dx \text{ Let } u=\cos x;\text{ then } du = -\sin x\,dx.\\ &= -\int u^2\left(1-u^2\right)du\\ &= \int\left(u^4-u^2\right)du\\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C\\ &= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.\end{aligned}$$

3.6 Evaluate $\int \cos^3 x \sin^2 x\,dx$.

In the next example, we see the strategy that must be applied when there are only even powers of $\sin x$ and $\cos x$. For integrals of this type, the identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1-\cos(2x)}{2}$$

and

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1+\cos(2x)}{2}$$

are invaluable. These identities are sometimes known as *power-reducing identities* and they may be derived from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

Example 3.10

Integrating an Even Power of $\sin x$

Evaluate $\int \sin^2 x\,dx$.

Solution

To evaluate this integral, let's use the trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Thus,

$$\begin{aligned}\int \sin^2 x\,dx &= \int\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx\\ &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.\end{aligned}$$

3.7 Evaluate $\int \cos^2 x\, dx$.

The general process for integrating products of powers of $\sin x$ and $\cos x$ is summarized in the following set of guidelines.

Problem-Solving Strategy: Integrating Products and Powers of sin *x* and cos *x*

To integrate $\int \cos^j x \sin^k x\, dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x\, dx$.
2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x\, dx$. (*Note*: If both j and k are odd, either strategy 1 or strategy 2 may be used.)
3. If both j and k are even, use $\sin^2 x = (1/2) - (1/2)\cos(2x)$ and $\cos^2 x = (1/2) + (1/2)\cos(2x)$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Example 3.11

Integrating $\int \cos^j x \sin^k x\, dx$ where *k* is Odd

Evaluate $\int \cos^8 x \sin^5 x\, dx$.

Solution

Since the power on $\sin x$ is odd, use strategy 1. Thus,

$$\begin{aligned}
\int \cos^8 x \sin^5 x\, dx &= \int \cos^8 x \sin^4 x \sin x\, dx && \text{Break off } \sin x. \\
&= \int \cos^8 x (\sin^2 x)^2 \sin x\, dx && \text{Rewrite } \sin^4 x = (\sin^2 x)^2. \\
&= \int \cos^8 x (1 - \cos^2 x)^2 \sin x\, dx && \text{Substitute } \sin^2 x = 1 - \cos^2 x. \\
&= \int u^8 (1 - u^2)^2 (-du) && \text{Let } u = \cos x \text{ and } du = -\sin x\, dx. \\
&= \int \left(-u^8 + 2u^{10} - u^{12}\right) du && \text{Expand.} \\
&= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C && \text{Evaluate the integral.} \\
&= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C. && \text{Substitute } u = \cos x.
\end{aligned}$$

Example 3.12

Integrating $\int \cos^j x \sin^k x\,dx$ where k and j are Even

Evaluate $\int \sin^4 x\,dx$.

Solution

Since the power on $\sin x$ is even $(k = 4)$ and the power on $\cos x$ is even $(j = 0)$, we must use strategy 3. Thus,

$$\begin{aligned}
\int \sin^4 x\,dx &= \int \left(\sin^2 x\right)^2 dx && \text{Rewrite } \sin^4 x = \left(\sin^2 x\right)^2. \\
&= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx && \text{Substitute } \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x). \\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right)dx && \text{Expand} \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2. \\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right)dx.
\end{aligned}$$

Since $\cos^2(2x)$ has an even power, substitute $\cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x)$:

$$\begin{aligned}
&= \int \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right)dx && \text{Simplify.} \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C && \text{Evaluate the integral.}
\end{aligned}$$

3.8 Evaluate $\int \cos^3 x\,dx$.

3.9 Evaluate $\int \cos^2(3x)dx$.

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$. These integrals are evaluated by applying trigonometric identities, as outlined in the following rule.

Rule: Integrating Products of Sines and Cosines of Different Angles

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x) \tag{3.3}$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x) \tag{3.4}$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x) \tag{3.5}$$

These formulas may be derived from the sum-of-angle formulas for sine and cosine.

Example 3.13

Evaluating $\int \sin(ax)\cos(bx)dx$

Evaluate $\int \sin(5x)\cos(3x)dx$.

Solution

Apply the identity $\sin(5x)\cos(3x) = \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(8x)$. Thus,

$$\begin{aligned}\int \sin(5x)\cos(3x)dx &= \int \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(8x)dx \\ &= -\frac{1}{4}\cos(2x) - \frac{1}{16}\sin(8x) + C.\end{aligned}$$

3.10 Evaluate $\int \cos(6x)\cos(5x)dx$.

Integrating Products and Powers of tan*x* and sec*x*

Before discussing the integration of products and powers of $\tan x$ and $\sec x$, it is useful to recall the integrals involving $\tan x$ and $\sec x$ we have already learned:

1. $\int \sec^2 x\,dx = \tan x + C$
2. $\int \sec x \tan x\,dx = \sec x + C$
3. $\int \tan x\,dx = \ln|\sec x| + C$
4. $\int \sec x\,dx = \ln|\sec x + \tan x| + C.$

For most integrals of products and powers of $\tan x$ and $\sec x$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j x \sec^2 x\,dx$ or $\int \sec^j x \tan x\,dx$. As we see in the following example, we can evaluate these new integrals by using u-substitution.

Example 3.14

Evaluating $\int \sec^j x \tan x\,dx$

Evaluate $\int \sec^5 x \tan x\,dx$.

Solution

Start by rewriting $\sec^5 x \tan x$ as $\sec^4 x \sec x \tan x$.

$$\begin{aligned}\int \sec^5 x \tan x\,dx &= \int \sec^4 x \sec x \tan x\,dx && \text{Let } u = \sec x; \text{ then, } du = \sec x \tan x\,dx.\\ &= \int u^4\,du && \text{Evaluate the integral.}\\ &= \frac{1}{5}u^5 + C && \text{Substitute } \sec x = u.\\ &= \frac{1}{5}\sec^5 x + C\end{aligned}$$

You can read some interesting information at this **website (http://www.openstaxcollege.org/l/20_intseccube)** to learn about a common integral involving the secant.

3.11 Evaluate $\int \tan^5 x \sec^2 x\,dx.$

We now take a look at the various strategies for integrating products and powers of $\sec x$ and $\tan x$.

Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x\,dx$

To integrate $\int \tan^k x \sec^j x\,dx,$ use the following strategies:

1. If j is even and $j \ge 2,$ rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x.$ Let $u = \tan x$ and $du = \sec^2 x.$
2. If k is odd and $j \ge 1,$ rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x.$ Let $u = \sec x$ and $du = \sec x \tan x\,dx.$ (*Note*: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)
3. If k is odd where $k \ge 3$ and $j = 0,$ rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x(\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x.$ It may be necessary to repeat this process on the $\tan^{k-2} x$ term.
4. If k is even and j is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan^k x$ in terms of $\sec x.$ Use integration by parts to integrate odd powers of $\sec x.$

Example 3.15

Integrating $\int \tan^k x \sec^j x\,dx$ when j is Even

Evaluate $\int \tan^6 x \sec^4 x\,dx.$

Solution

Since the power on $\sec x$ is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite the first $\sec^2 x$ in terms of $\tan x$. Thus,

$$\begin{aligned}
\int \tan^6 x \sec^4 x\,dx &= \int \tan^6 x\left(\tan^2 x + 1\right)\sec^2 x\,dx && \text{Let } u = \tan x \text{ and } du = \sec^2 x. \\
&= \int u^6\left(u^2 + 1\right)du && \text{Expand.} \\
&= \int (u^8 + u^6)du && \text{Evaluate the integral.} \\
&= \frac{1}{9}u^9 + \frac{1}{7}u^7 + C && \text{Substitute } \tan x = u. \\
&= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C.
\end{aligned}$$

Example 3.16

Integrating $\int \tan^k x \sec^j x\,dx$ when k is Odd

Evaluate $\int \tan^5 x \sec^3 x\,dx$.

Solution

Since the power on $\tan x$ is odd, begin by rewriting $\tan^5 x \sec^3 x = \tan^4 x \sec^2 x \sec x \tan x$. Thus,

$$\begin{aligned}
\tan^5 x \sec^3 x &= \tan^4 x \sec^2 x \sec x \tan x. && \text{Write } \tan^4 x = (\tan^2 x)^2. \\
\int \tan^5 x \sec^3 x\,dx &= \int (\tan^2 x)^2 \sec^2 x \sec x \tan x\,dx && \text{Use } \tan^2 x = \sec^2 x - 1. \\
&= \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x\,dx && \text{Let } u = \sec x \text{ and } du = \sec x \tan x\,dx. \\
&= \int (u^2 - 1)^2 u^2\,du && \text{Expand.} \\
&= \int \left(u^6 - 2u^4 + u^2\right)du && \text{Integrate.} \\
&= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C && \text{Substitute } \sec x = u. \\
&= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C.
\end{aligned}$$

Example 3.17

Integrating $\int \tan^k x\,dx$ where k is Odd and $k \geq 3$

Evaluate $\int \tan^3 x\,dx$.

Solution

Begin by rewriting $\tan^3 x = \tan x \tan^2 x = \tan x(\sec^2 x - 1) = \tan x \sec^2 x - \tan x$. Thus,

$$\begin{aligned}\int \tan^3 x\,dx &= \int (\tan x \sec^2 x - \tan x)dx \\ &= \int \tan x \sec^2 x\,dx - \int \tan x\,dx \\ &= \frac{1}{2}\tan^2 x - \ln|\sec x| + C.\end{aligned}$$

For the first integral, use the substitution $u = \tan x$. For the second integral, use the formula.

Example 3.18

Integrating $\int \sec^3 x\,dx$

Integrate $\int \sec^3 x\,dx$.

Solution

This integral requires integration by parts. To begin, let $u = \sec x$ and $dv = \sec^2 x$. These choices make $du = \sec x \tan x$ and $v = \tan x$. Thus,

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\ &= \sec x \tan x - \int \tan^2 x \sec x\,dx && \text{Simplify.} \\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx && \text{Substitute } \tan^2 x = \sec^2 x - 1. \\ &= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx && \text{Rewrite.} \\ &= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx. && \text{Evaluate} \int \sec x\,dx.\end{aligned}$$

We now have

$$\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx.$$

Since the integral $\int \sec^3 x\,dx$ has reappeared on the right-hand side, we can solve for $\int \sec^3 x\,dx$ by adding it to both sides. In doing so, we obtain

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Dividing by 2, we arrive at

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

3.12 Evaluate $\int \tan^3 x \sec^7 x\,dx$.

Reduction Formulas

Evaluating $\int \sec^n x\,dx$ for values of n where n is odd requires integration by parts. In addition, we must also know the value of $\int \sec^{n-2} x\,dx$ to evaluate $\int \sec^n x\,dx$. The evaluation of $\int \tan^n x\,dx$ also requires being able to integrate $\int \tan^{n-2} x\,dx$. To make the process easier, we can derive and apply the following **power reduction formulas**. These rules allow us to replace the integral of a power of $\sec x$ or $\tan x$ with the integral of a lower power of $\sec x$ or $\tan x$.

Rule: Reduction Formulas for $\int \sec^n x\,dx$ and $\int \tan^n x\,dx$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \tag{3.6}$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \tag{3.7}$$

The first power reduction rule may be verified by applying integration by parts. The second may be verified by following the strategy outlined for integrating odd powers of $\tan x$.

Example 3.19

Revisiting $\int \sec^3 x\,dx$

Apply a reduction formula to evaluate $\int \sec^3 x\,dx$.

Solution

By applying the first reduction formula, we obtain

$$\begin{aligned}\int \sec^3 x\,dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

Example 3.20

Using a Reduction Formula

Evaluate $\int \tan^4 x\,dx$.

Solution

Applying the reduction formula for $\int \tan^4 x\,dx$ we have

$$\begin{aligned}
\int \tan^4 x\,dx &= \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx \\
&= \frac{1}{3}\tan^3 x - \left(\tan x - \int \tan^0 x\,dx\right) && \text{Apply the reduction formula to} \int \tan^2 x\,dx. \\
&= \frac{1}{3}\tan^3 x - \tan x + \int 1\,dx && \text{Simplify.} \\
&= \frac{1}{3}\tan^3 x - \tan x + x + C. && \text{Evaluate} \int 1 dx.
\end{aligned}$$

3.13 Apply the reduction formula to $\int \sec^5 x\,dx$.

3.2 EXERCISES

Fill in the blank to make a true statement.

69. $\sin^2 x + \_\_\_\_\_\_\_ = 1$

70. $\sec^2 x - 1 = \_\_\_\_\_\_\_$

Use an identity to reduce the power of the trigonometric function to a trigonometric function raised to the first power.

71. $\sin^2 x = \_\_\_\_\_\_\_$

72. $\cos^2 x = \_\_\_\_\_\_\_$

Evaluate each of the following integrals by *u*-substitution.

73. $\int \sin^3 x \cos x \, dx$

74. $\int \sqrt{\cos x} \sin x \, dx$

75. $\int \tan^5(2x) \sec^2(2x) dx$

76. $\int \sin^7(2x) \cos(2x) dx$

77. $\int \tan\left(\frac{x}{2}\right) \sec^2\left(\frac{x}{2}\right) dx$

78. $\int \tan^2 x \sec^2 x \, dx$

Compute the following integrals using the guidelines for integrating powers of trigonometric functions. Use a CAS to check the solutions. (*Note*: Some of the problems may be done using techniques of integration learned previously.)

79. $\int \sin^3 x \, dx$

80. $\int \cos^3 x \, dx$

81. $\int \sin x \cos x \, dx$

82. $\int \cos^5 x \, dx$

83. $\int \sin^5 x \cos^2 x \, dx$

84. $\int \sin^3 x \cos^3 x \, dx$

85. $\int \sqrt{\sin x} \cos x \, dx$

86. $\int \sqrt{\sin x} \cos^3 x \, dx$

87. $\int \sec x \tan x \, dx$

88. $\int \tan(5x) dx$

89. $\int \tan^2 x \sec x \, dx$

90. $\int \tan x \sec^3 x \, dx$

91. $\int \sec^4 x \, dx$

92. $\int \cot x \, dx$

93. $\int \csc x \, dx$

94. $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$

For the following exercises, find a general formula for the integrals.

95. $\int \sin^2 ax \cos ax \, dx$

96. $\int \sin ax \cos ax \, dx.$

Use the double-angle formulas to evaluate the following integrals.

97. $\int_0^{\pi} \sin^2 x \, dx$

98. $\int_0^{\pi} \sin^4 x \, dx$

99. $\int \cos^2 3x \, dx$

100. $\int \sin^2 x \cos^2 x \, dx$

101. $\int \sin^2 x \, dx + \int \cos^2 x \, dx$

102. $\int \sin^2 x \cos^2(2x) dx$

For the following exercises, evaluate the definite integrals. Express answers in exact form whenever possible.

103. $\int_0^{2\pi} \cos x \sin 2x\, dx$

104. $\int_0^{\pi} \sin 3x \sin 5x\, dx$

105. $\int_0^{\pi} \cos(99x)\sin(101x)dx$

106. $\int_{-\pi}^{\pi} \cos^2(3x)dx$

107. $\int_0^{2\pi} \sin x \sin(2x)\sin(3x)dx$

108. $\int_0^{4\pi} \cos(x/2)\sin(x/2)dx$

109. $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}dx$ (Round this answer to three decimal places.)

110. $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1}\, dx$

111. $\int_0^{\pi/2} \sqrt{1 - \cos(2x)}\, dx$

112. Find the area of the region bounded by the graphs of the equations $y = \sin x$, $y = \sin^3 x$, $x = 0$, and $x = \frac{\pi}{2}$.

113. Find the area of the region bounded by the graphs of the equations $y = \cos^2 x$, $y = \sin^2 x$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$.

114. A particle moves in a straight line with the velocity function $v(t) = \sin(\omega t)\cos^2(\omega t)$. Find its position function $x = f(t)$ if $f(0) = 0$.

115. Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ over the interval $[-\pi, \pi]$.

For the following exercises, solve the differential equations.

116. $\frac{dy}{dx} = \sin^2 x$. The curve passes through point $(0, 0)$.

117. $\frac{dy}{d\theta} = \sin^4(\pi\theta)$

118. Find the length of the curve $y = \ln(\csc x)$, $\frac{\pi}{4} \le x \le \frac{\pi}{2}$.

119. Find the length of the curve $y = \ln(\sin x)$, $\frac{\pi}{3} \le x \le \frac{\pi}{2}$.

120. Find the volume generated by revolving the curve $y = \cos(3x)$ about the x-axis, $0 \le x \le \frac{\pi}{36}$.

For the following exercises, use this information: The inner product of two functions f and g over $[a, b]$ is defined by $f(x) \cdot g(x) = \langle f, g \rangle = \int_a^b f \cdot g dx$. Two distinct functions f and g are said to be orthogonal if $\langle f, g \rangle = 0$.

121. Show that $\{\sin(2x), \cos(3x)\}$ are orthogonal over the interval $[-\pi, \pi]$.

122. Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx$.

123. Integrate $y' = \sqrt{\tan x}\sec^4 x$.

For each pair of integrals, determine which one is more difficult to evaluate. Explain your reasoning.

124. $\int \sin^{456} x \cos x\, dx$ or $\int \sin^2 x \cos^2 x\, dx$

125. $\int \tan^{350} x \sec^2 x\, dx$ or $\int \tan^{350} x \sec x\, dx$

3.3 | Trigonometric Substitution

Learning Objectives

3.3.1 Solve integration problems involving the square root of a sum or difference of two squares.

In this section, we explore integrals containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$, where the values of a are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many still remain inaccessible. The technique of **trigonometric substitution** comes in very handy when evaluating these integrals. This technique uses substitution to rewrite these integrals as trigonometric integrals.

Integrals Involving $\sqrt{a^2 - x^2}$

Before developing a general strategy for integrals containing $\sqrt{a^2 - x^2}$, consider the integral $\int \sqrt{9 - x^2}dx$. This integral cannot be evaluated using any of the techniques we have discussed so far. However, if we make the substitution $x = 3\sin\theta$, we have $dx = 3\cos\theta d\theta$. After substituting into the integral, we have

$$\int \sqrt{9 - x^2}dx = \int \sqrt{9 - (3\sin\theta)^2}3\cos\theta d\theta.$$

After simplifying, we have

$$\int \sqrt{9 - x^2}dx = \int 9\sqrt{1 - \sin^2\theta}\cos\theta d\theta.$$

Letting $1 - \sin^2\theta = \cos^2\theta$, we now have

$$\int \sqrt{9 - x^2}dx = \int 9\sqrt{\cos^2\theta}\cos\theta d\theta.$$

Assuming that $\cos\theta \ge 0$, we have

$$\int \sqrt{9 - x^2}dx = \int 9\cos^2\theta d\theta.$$

At this point, we can evaluate the integral using the techniques developed for integrating powers and products of trigonometric functions. Before completing this example, let's take a look at the general theory behind this idea.

To evaluate integrals involving $\sqrt{a^2 - x^2}$, we make the substitution $x = a\sin\theta$ and $dx = a\cos\theta$. To see that this actually makes sense, consider the following argument: The domain of $\sqrt{a^2 - x^2}$ is $[-a, a]$. Thus, $-a \le x \le a$. Consequently, $-1 \le \frac{x}{a} \le 1$. Since the range of $\sin x$ over $[-(\pi/2), \pi/2]$ is $[-1, 1]$, there is a unique angle θ satisfying $-(\pi/2) \le \theta \le \pi/2$ so that $\sin\theta = x/a$, or equivalently, so that $x = a\sin\theta$. If we substitute $x = a\sin\theta$ into $\sqrt{a^2 - x^2}$, we get

$$\begin{aligned}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a\sin\theta)^2} && \text{Let } x = a\sin\theta \text{ where } -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}. \text{ Simplify.} \\
&= \sqrt{a^2 - a^2\sin^2\theta} && \text{Factor out } a^2. \\
&= \sqrt{a^2(1 - \sin^2\theta)} && \text{Substitute } 1 - \sin^2 x = \cos^2 x. \\
&= \sqrt{a^2\cos^2\theta} && \text{Take the square root.} \\
&= |a\cos\theta| \\
&= a\cos\theta.
\end{aligned}$$

Since $\cos x \ge 0$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $a > 0$, $|a\cos\theta| = a\cos\theta$. We can see, from this discussion, that by making the substitution $x = a\sin\theta$, we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving x. To see how to do this, let's begin by assuming that $0 < x < a$. In this case, $0 < \theta < \frac{\pi}{2}$. Since $\sin\theta = \frac{x}{a}$, we can draw the reference triangle in **Figure 3.4** to assist in expressing the values of $\cos\theta$, $\tan\theta$, and the remaining trigonometric functions in terms of x. It can be shown that this triangle actually produces the correct values of the trigonometric functions evaluated at θ for all θ satisfying $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. It is useful to observe that the expression $\sqrt{a^2 - x^2}$ actually appears as the length of one side of the triangle. Last, should θ appear by itself, we use $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

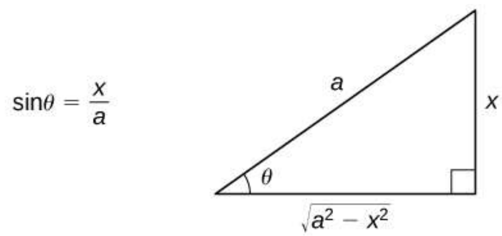


Figure 3.4 A reference triangle can help express the trigonometric functions evaluated at θ in terms of x.

The essential part of this discussion is summarized in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 - x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2 - x^2}}dx$, $\int \frac{x}{\sqrt{a^2 - x^2}}dx$, and $\int x\sqrt{a^2 - x^2}dx$, they can each be integrated directly either by formula or by a simple *u*-substitution.
2. Make the substitution $x = a\sin\theta$ and $dx = a\cos\theta d\theta$. *Note*: This substitution yields $\sqrt{a^2 - x^2} = a\cos\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from **Figure 3.4** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

The following example demonstrates the application of this problem-solving strategy.

Example 3.21

Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate $\int \sqrt{9 - x^2}dx$.

Solution

Begin by making the substitutions $x = 3\sin\theta$ and $dx = 3\cos\theta d\theta$. Since $\sin\theta = \frac{x}{3}$, we can construct the reference triangle shown in the following figure.

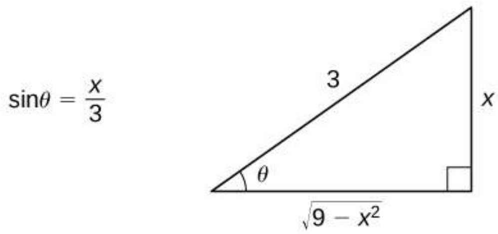


Figure 3.5 A reference triangle can be constructed for **Example 3.21**.

Thus,

$$\begin{aligned}
\int \sqrt{9-x^2}\,dx &= \int \sqrt{9-(3\sin\theta)^2}\,3\cos\theta d\theta && \text{Substitute } x = 3\sin\theta \text{ and } dx = 3\cos\theta d\theta.\\
&= \int \sqrt{9(1-\sin^2\theta)}\,3\cos\theta d\theta && \text{Simplify.}\\
&= \int \sqrt{9\cos^2\theta}\,3\cos\theta d\theta && \text{Substitute } \cos^2\theta = 1-\sin^2\theta.\\
&= \int 3|\cos\theta|3\cos\theta d\theta && \text{Take the square root.}\\
&= \int 9\cos^2\theta d\theta && \text{Simplify. Since } -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2},\ \cos\theta \ge 0 \text{ and } |\cos\theta| = \cos\theta.\\
&= \int 9\left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta && \text{Use the strategy for integrating an even power of } \cos\theta.\\
&= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C && \text{Evaluate the integral.}\\
&= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C && \text{Substitute } \sin(2\theta) = 2\sin\theta\cos\theta.\\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C && \text{Substitute } \sin^{-1}\left(\tfrac{x}{3}\right) = \theta \text{ and } \sin\theta = \tfrac{x}{3}. \text{ Use the reference triangle to see that } \cos\theta = \tfrac{\sqrt{9-x^2}}{3} \text{ and make this substitution.}\\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C. && \text{Simplify.}
\end{aligned}$$

Example 3.22

Integrating an Expression Involving $\sqrt{a^2-x^2}$

Evaluate $\int \frac{\sqrt{4-x^2}}{x}dx.$

Solution

First make the substitutions $x = 2\sin\theta$ and $dx = 2\cos\theta d\theta$. Since $\sin\theta = \frac{x}{2}$, we can construct the reference triangle shown in the following figure.

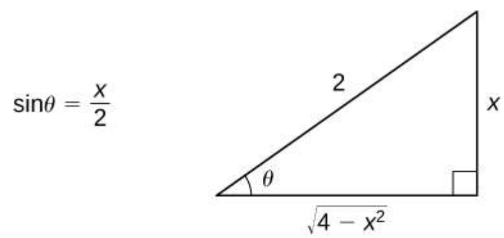


Figure 3.6 A reference triangle can be constructed for **Example 3.22**.

Thus,

$$\begin{aligned}
\int \frac{\sqrt{4-x^2}}{x}dx &= \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta} 2\cos\theta d\theta && \text{Substitute } x = 2\sin\theta \text{ and } = 2\cos\theta d\theta. \\
&= \int \frac{2\cos^2\theta}{\sin\theta} d\theta && \text{Substitute } \cos^2\theta = 1-\sin^2\theta \text{ and simplify.} \\
&= \int \frac{2(1-\sin^2\theta)}{\sin\theta} d\theta && \text{Substitute } \sin^2\theta = 1-\cos^2\theta. \\
&= \int (2\csc\theta - 2\sin\theta) d\theta && \text{Separate the numerator, simplify, and use } \csc\theta = \frac{1}{\sin\theta}. \\
&= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C && \text{Evaluate the integral.} \\
&= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C. && \text{Use the reference triangle to rewrite the expression in terms of } x \text{ and simplify.}
\end{aligned}$$

In the next example, we see that we sometimes have a choice of methods.

Example 3.23

Integrating an Expression Involving $\sqrt{a^2 - x^2}$ Two Ways

Evaluate $\int x^3 \sqrt{1-x^2}dx$ two ways: first by using the substitution $u = 1 - x^2$ and then by using a trigonometric substitution.

Solution

Method 1

Let $u = 1 - x^2$ and hence $x^2 = 1 - u$. Thus, $du = -2x dx$. In this case, the integral becomes

$$\int x^3\sqrt{1-x^2}dx = -\frac{1}{2}\int x^2\sqrt{1-x^2}(-2x\,dx) \quad \text{Make the substitution.}$$
$$= -\frac{1}{2}\int (1-u)\sqrt{u}\,du \quad \text{Expand the expression.}$$
$$= -\frac{1}{2}\int \left(u^{1/2}-u^{3/2}\right)du \quad \text{Evaluate the integral.}$$
$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right)+C \quad \text{Rewrite in terms of } x.$$
$$= -\frac{1}{3}\left(1-x^2\right)^{3/2}+\frac{1}{5}\left(1-x^2\right)^{5/2}+C.$$

Method 2

Let $x=\sin\theta$. In this case, $dx=\cos\theta d\theta$. Using this substitution, we have

$$\int x^3\sqrt{1-x^2}dx = \int \sin^3\theta\cos^2\theta d\theta$$
$$= \int \left(1-\cos^2\theta\right)\cos^2\theta\sin\theta d\theta \quad \text{Let } u=\cos\theta. \text{ Thus, } du=-\sin\theta d\theta.$$
$$= \int \left(u^4-u^2\right)du$$
$$= \frac{1}{5}u^5-\frac{1}{3}u^3+C \quad \text{Substitute } \cos\theta=u.$$
$$= \frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta+C \quad \text{Use a reference triangle to see that } \cos\theta=\sqrt{1-x^2}.$$
$$= \frac{1}{5}\left(1-x^2\right)^{5/2}-\frac{1}{3}\left(1-x^2\right)^{3/2}+C.$$

3.14 Rewrite the integral $\int \frac{x^3}{\sqrt{25-x^2}}dx$ using the appropriate trigonometric substitution (do not evaluate the integral).

Integrating Expressions Involving $\sqrt{a^2+x^2}$

For integrals containing $\sqrt{a^2+x^2}$, let's first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of x, we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x=a\tan\theta$ or $x=a\cot\theta$. Either of these substitutions would actually work, but the standard substitution is $x=a\tan\theta$ or, equivalently, $\tan\theta=x/a$. With this substitution, we make the assumption that $-(\pi/2)<\theta<\pi/2$, so that we also have $\theta=\tan^{-1}(x/a)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2+x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x=a\tan\theta$ and $dx=a\sec^2\theta d\theta$. This substitution yields $\sqrt{a^2+x^2}=\sqrt{a^2+(a\tan\theta)^2}=\sqrt{a^2(1+\tan^2\theta)}=\sqrt{a^2\sec^2\theta}=|a\sec\theta|=a\sec\theta$. (Since $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ and $\sec\theta>0$ over this interval, $|a\sec\theta|=a\sec\theta$.)

3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from **Figure 3.7** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\frac{x}{a}\right)$. (*Note*: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)

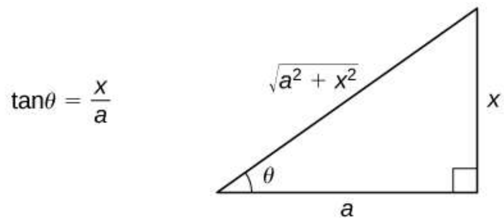


Figure 3.7 A reference triangle can be constructed to express the trigonometric functions evaluated at θ in terms of x.

Example 3.24

Integrating an Expression Involving $\sqrt{a^2+x^2}$

Evaluate $\int \frac{dx}{\sqrt{1+x^2}}$ and check the solution by differentiating.

Solution

Begin with the substitution $x = \tan\theta$ and $dx = \sec^2\theta d\theta$. Since $\tan\theta = x$, draw the reference triangle in the following figure.

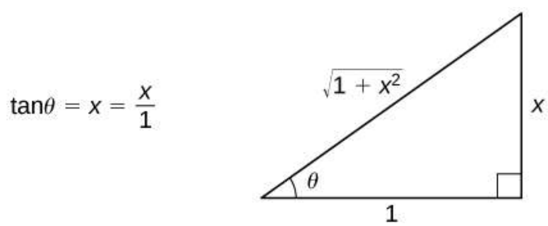


Figure 3.8 The reference triangle for **Example 3.24**.

Thus,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec\theta} d\theta$$

Substitute $x = \tan\theta$ and $dx = \sec^2\theta d\theta$. This substitution makes $\sqrt{1+x^2} = \sec\theta$. Simplify.

$$= \int \sec\theta d\theta$$

Evaluate the integral.

$$= \ln|\sec\theta + \tan\theta| + C$$

Use the reference triangle to express the result in terms of x.

$$= \ln\left|\sqrt{1+x^2} + x\right| + C.$$

To check the solution, differentiate:

$$\begin{aligned}\frac{d}{dx}\left(\ln\left|\sqrt{1+x^2}+x\right|\right) &= \frac{1}{\sqrt{1+x^2}+x}\cdot\left(\frac{x}{\sqrt{1+x^2}}+1\right)\\ &= \frac{1}{\sqrt{1+x^2}+x}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\\ &= \frac{1}{\sqrt{1+x^2}}.\end{aligned}$$

Since $\sqrt{1+x^2}+x>0$ for all values of x, we could rewrite $\ln\left|\sqrt{1+x^2}+x\right|+C=\ln\left(\sqrt{1+x^2}+x\right)+C$, if desired.

Example 3.25

Evaluating $\int\frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Use the substitution $x=\sinh\theta$ to evaluate $\int\frac{dx}{\sqrt{1+x^2}}$.

Solution

Because $\sinh\theta$ has a range of all real numbers, and $1+\sinh^2\theta=\cosh^2\theta$, we may also use the substitution $x=\sinh\theta$ to evaluate this integral. In this case, $dx=\cosh\theta d\theta$. Consequently,

$$\begin{aligned}\int\frac{dx}{\sqrt{1+x^2}} &= \int\frac{\cosh\theta}{\sqrt{1+\sinh^2\theta}}d\theta && \text{Substitute } x=\sinh\theta \text{ and } dx=\cosh\theta d\theta. \text{ Substitute } 1+\sinh^2\theta=\cosh^2\theta.\\ &= \int\frac{\cosh\theta}{\sqrt{\cosh^2\theta}}d\theta && \sqrt{\cosh^2\theta}=|\cosh\theta|\\ &= \int\frac{\cosh\theta}{|\cosh\theta|}d\theta && |\cosh\theta|=\cosh\theta \text{ since } \cosh\theta>0 \text{ for all } \theta.\\ &= \int\frac{\cosh\theta}{\cosh\theta}d\theta && \text{Simplify.}\\ &= \int 1d\theta && \text{Evaluate the integral.}\\ &= \theta+C && \text{Since } x=\sinh\theta, \text{ we know } \theta=\sinh^{-1}x.\\ &= \sinh^{-1}x+C.\end{aligned}$$

Analysis

This answer looks quite different from the answer obtained using the substitution $x=\tan\theta$. To see that the solutions are the same, set $y=\sinh^{-1}x$. Thus, $\sinh y=x$. From this equation we obtain:

$$\frac{e^y-e^{-y}}{2}=x.$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:

$$e^{2y} - 2xe^y - 1 = 0.$$

Use the quadratic equation to solve for e^y:

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}.$$

Simplifying, we have:

$$e^y = x \pm \sqrt{x^2 + 1}.$$

Since $x - \sqrt{x^2 + 1} < 0,$ it must be the case that $e^y = x + \sqrt{x^2 + 1}.$ Thus,

$$y = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Last, we obtain

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

After we make the final observation that, since $x + \sqrt{x^2 + 1} > 0,$

$$\ln\left(x + \sqrt{x^2 + 1}\right) = \ln\left|\sqrt{1 + x^2} + x\right|,$$

we see that the two different methods produced equivalent solutions.

Example 3.26

Finding an Arc Length

Find the length of the curve $y = x^2$ over the interval $[0, \frac{1}{2}]$.

Solution

Because $\frac{dy}{dx} = 2x,$ the arc length is given by

$$\int_0^{1/2} \sqrt{1 + (2x)^2}\, dx = \int_0^{1/2} \sqrt{1 + 4x^2}\, dx.$$

To evaluate this integral, use the substitution $x = \frac{1}{2}\tan\theta$ and $dx = \frac{1}{2}\sec^2\theta d\theta.$ We also need to change the limits of integration. If $x = 0,$ then $\theta = 0$ and if $x = \frac{1}{2},$ then $\theta = \frac{\pi}{4}.$ Thus,

$$\int_0^{1/2} \sqrt{1+4x^2}\,dx = \int_0^{\pi/4} \sqrt{1+\tan^2\theta}\frac{1}{2}\sec^2\theta d\theta$$ After substitution, $\sqrt{1+4x^2} = \tan\theta$. Substitute $1+\tan^2\theta = \sec^2\theta$ and simplify.

$$= \frac{1}{2}\int_0^{\pi/4} \sec^3\theta d\theta$$ We derived this integral in the previous section.

$$= \frac{1}{2}\left(\frac{1}{2}\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\pi/4}$$ Evaluate and simplify.

$$= \frac{1}{4}(\sqrt{2} + \ln(\sqrt{2}+1)).$$

3.15 Rewrite $\int x^3\sqrt{x^2+4}\,dx$ by using a substitution involving $\tan\theta$.

Integrating Expressions Involving $\sqrt{x^2-a^2}$

The domain of the expression $\sqrt{x^2-a^2}$ is $(-\infty, -a] \cup [a, +\infty)$. Thus, either $x < -a$ or $x > a$. Hence, $\frac{x}{a} \le -1$ or $\frac{x}{a} \ge 1$. Since these intervals correspond to the range of $\sec\theta$ on the set $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, it makes sense to use the substitution $\sec\theta = \frac{x}{a}$ or, equivalently, $x = a\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$. The corresponding substitution for dx is $dx = a\sec\theta\tan\theta d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2-a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$. This substitution yields

$$\sqrt{x^2-a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta+1)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|.$$

   For $x \ge a$, $|a\tan\theta| = a\tan\theta$ and for $x \le -a$, $|a\tan\theta| = -a\tan\theta$.

3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from **Figure 3.9** to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (*Note*: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

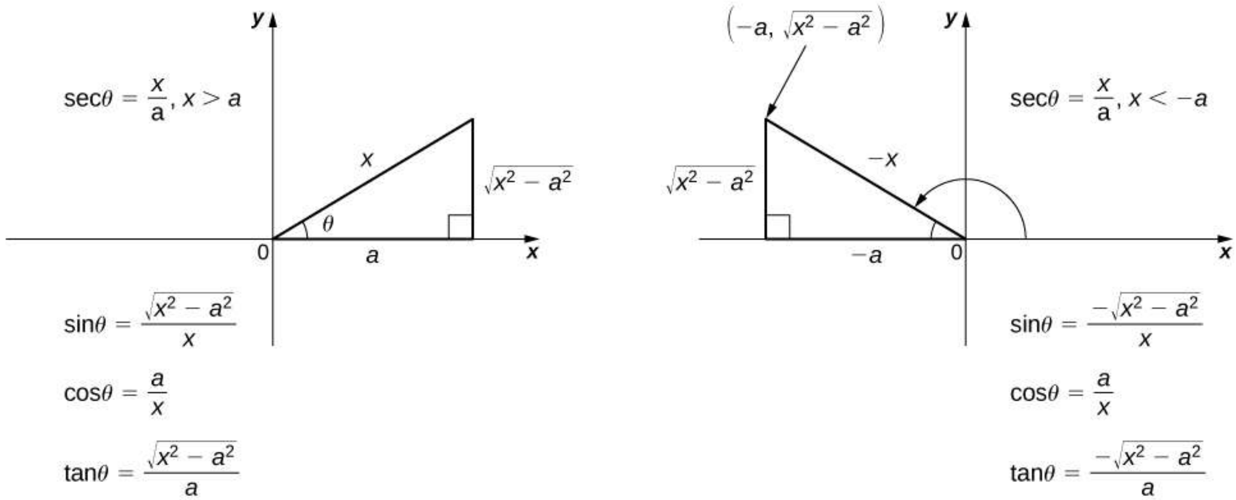


Figure 3.9 Use the appropriate reference triangle to express the trigonometric functions evaluated at θ in terms of x.

Example 3.27

Finding the Area of a Region

Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$.

Solution

First, sketch a rough graph of the region described in the problem, as shown in the following figure.

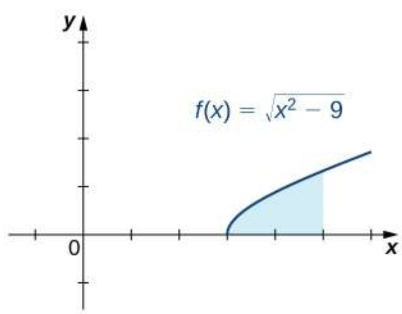


Figure 3.10 Calculating the area of the shaded region requires evaluating an integral with a trigonometric substitution.

We can see that the area is $A = \int_3^5 \sqrt{x^2 - 9}\,dx$. To evaluate this definite integral, substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec\theta$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}\left(\frac{5}{3}\right)$. After making these substitutions and simplifying, we have

$$
\begin{aligned}
\text{Area} &= \int_3^5 \sqrt{x^2-9}\,dx \\
&= \int_0^{\sec^{-1}(5/3)} 9\tan^2\theta\sec\theta\,d\theta && \text{Use } \tan^2\theta = 1-\sec^2\theta. \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^2\theta-1)\sec\theta\,d\theta && \text{Expand.} \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^3\theta-\sec\theta)\,d\theta && \text{Evaluate the integral.} \\
&= \left(\frac{9}{2}\ln|\sec\theta+\tan\theta| + \frac{9}{2}\sec\theta\tan\theta\right) - 9\ln|\sec\theta+\tan\theta|\Big|_0^{\sec^{-1}(5/3)} && \text{Simplify.} \\
&= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta+\tan\theta|\Big|_0^{\sec^{-1}(5/3)} && \text{Evaluate. Use } \sec\left(\sec^{-1}\frac{5}{3}\right)=\frac{5}{3} \text{ and } \tan\left(\sec^{-1}\frac{5}{3}\right)=\frac{4}{3}. \\
&= \frac{9}{2}\cdot\frac{5}{3}\cdot\frac{4}{3} - \frac{9}{2}\ln\left|\frac{5}{3}+\frac{4}{3}\right| - \left(\frac{9}{2}\cdot 1\cdot 0 - \frac{9}{2}\ln|1+0|\right) \\
&= 10 - \frac{9}{2}\ln 3.
\end{aligned}
$$

3.16 Evaluate $\int \frac{dx}{\sqrt{x^2-4}}$. Assume that $x > 2$.

3.3 EXERCISES

Simplify the following expressions by writing each one using a single trigonometric function.

126. $4-4\sin^2\theta$

127. $9\sec^2\theta-9$

128. $a^2+a^2\tan^2\theta$

129. $a^2+a^2\sinh^2\theta$

130. $16\cosh^2\theta-16$

Use the technique of completing the square to express each trinomial as the square of a binomial.

131. $4x^2-4x+1$

132. $2x^2-8x+3$

133. $-x^2-2x+4$

Integrate using the method of trigonometric substitution. Express the final answer in terms of the variable.

134. $\int\frac{dx}{\sqrt{4-x^2}}$

135. $\int\frac{dx}{\sqrt{x^2-a^2}}$

136. $\int\sqrt{4-x^2}dx$

137. $\int\frac{dx}{\sqrt{1+9x^2}}$

138. $\int\frac{x^2\,dx}{\sqrt{1-x^2}}$

139. $\int\frac{dx}{x^2\sqrt{1-x^2}}$

140. $\int\frac{dx}{(1+x^2)^2}$

141. $\int\sqrt{x^2+9}dx$

142. $\int\frac{\sqrt{x^2-25}}{x}dx$

143. $\int\frac{\theta^3\,d\theta}{\sqrt{9-\theta^2}}d\theta$

144. $\int\frac{dx}{\sqrt{x^6-x^2}}$

145. $\int\sqrt{x^6-x^8}dx$

146. $\int\frac{dx}{\left(1+x^2\right)^{3/2}}$

147. $\int\frac{dx}{\left(x^2-9\right)^{3/2}}$

148. $\int\frac{\sqrt{1+x^2}dx}{x}$

149. $\int\frac{x^2\,dx}{\sqrt{x^2-1}}$

150. $\int\frac{x^2\,dx}{x^2+4}$

151. $\int\frac{dx}{x^2\sqrt{x^2+1}}$

152. $\int\frac{x^2\,dx}{\sqrt{1+x^2}}$

153. $\int_{-1}^{1}(1-x^2)^{3/2}\,dx$

In the following exercises, use the substitutions $x=\sinh\theta,\ \cosh\theta,$ or $\tanh\theta.$ Express the final answers in terms of the variable *x*.

154. $\int\frac{dx}{\sqrt{x^2-1}}$

155. $\int\frac{dx}{x\sqrt{1-x^2}}$

156. $\int\sqrt{x^2-1}dx$

157. $\int\frac{\sqrt{x^2-1}}{x^2}dx$

158. $\int \frac{dx}{1 - x^2}$

159. $\int \frac{\sqrt{1 + x^2}}{x^2} dx$

Use the technique of completing the square to evaluate the following integrals.

160. $\int \frac{1}{x^2 - 6x} dx$

161. $\int \frac{1}{x^2 + 2x + 1} dx$

162. $\int \frac{1}{\sqrt{-x^2 + 2x + 8}} dx$

163. $\int \frac{1}{\sqrt{-x^2 + 10x}} dx$

164. $\int \frac{1}{\sqrt{x^2 + 4x - 12}} dx$

165. Evaluate the integral without using calculus: $\int_{-3}^{3} \sqrt{9 - x^2} dx.$

166. Find the area enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1.$

167. Evaluate the integral $\int \frac{dx}{\sqrt{1 - x^2}}$ using two different substitutions. First, let $x = \cos\theta$ and evaluate using trigonometric substitution. Second, let $x = \sin\theta$ and use trigonometric substitution. Are the answers the same?

168. Evaluate the integral $\int \frac{dx}{x\sqrt{x^2 - 1}}$ using the substitution $x = \sec\theta.$ Next, evaluate the same integral using the substitution $x = \csc\theta.$ Show that the results are equivalent.

169. Evaluate the integral $\int \frac{x}{x^2 + 1} dx$ using the form $\int \frac{1}{u} du.$ Next, evaluate the same integral using $x = \tan\theta.$ Are the results the same?

170. State the method of integration you would use to evaluate the integral $\int x\sqrt{x^2 + 1}\, dx.$ Why did you choose this method?

171. State the method of integration you would use to evaluate the integral $\int x^2 \sqrt{x^2 - 1}\, dx.$ Why did you choose this method?

172. Evaluate $\int_{-1}^{1} \frac{x\,dx}{x^2 + 1}$

173. Find the length of the arc of the curve over the specified interval: $y = \ln x,\ [1, 5].$ Round the answer to three decimal places.

174. Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2,\ y = 0,\ x = 0,$ and $x = \sqrt{2}$ about the x-axis. (Round the answer to three decimal places).

175. The region bounded by the graph of $f(x) = \frac{1}{1 + x^2}$ and the x-axis between $x = 0$ and $x = 1$ is revolved about the x-axis. Find the volume of the solid that is generated.

Solve the initial-value problem for y as a function of x.

176. $\left(x^2 + 36\right)\frac{dy}{dx} = 1,\ y(6) = 0$

177. $\left(64 - x^2\right)\frac{dy}{dx} = 1,\ y(0) = 3$

178. Find the area bounded by $y = \frac{2}{\sqrt{64 - 4x^2}},\ x = 0,\ y = 0,$ and $x = 2.$

179. An oil storage tank can be described as the volume generated by revolving the area bounded by $y = \frac{16}{\sqrt{64 + x^2}},\ x = 0,\ y = 0,\ x = 2$ about the x-axis. Find the volume of the tank (in cubic meters).

180. During each cycle, the velocity v (in feet per second) of a robotic welding device is given by $v = 2t - \frac{14}{4 + t^2},$ where t is time in seconds. Find the expression for the displacement s (in feet) as a function of t if $s = 0$ when $t = 0.$

181. Find the length of the curve $y = \sqrt{16 - x^2}$ between $x = 0$ and $x = 2.$

3.4 | Partial Fractions

Learning Objectives

3.4.1 Integrate a rational function using the method of partial fractions.
3.4.2 Recognize simple linear factors in a rational function.
3.4.3 Recognize repeated linear factors in a rational function.
3.4.4 Recognize quadratic factors in a rational function.

We have seen some techniques that allow us to integrate specific rational functions. For example, we know that

$$\int \frac{du}{u} = \ln|u| + C \text{ and } \int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C.$$

However, we do not yet have a technique that allows us to tackle arbitrary quotients of this type. Thus, it is not immediately obvious how to go about evaluating $\int \frac{3x}{x^2 - x - 2}dx$. However, we know from material previously developed that

$$\int \left(\frac{1}{x+1} + \frac{2}{x-2}\right)dx = \ln|x+1| + 2\ln|x-2| + C.$$

In fact, by getting a common denominator, we see that

$$\frac{1}{x+1} + \frac{2}{x-2} = \frac{3x}{x^2 - x - 2}.$$

Consequently,

$$\int \frac{3x}{x^2 - x - 2}dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2}\right)dx.$$

In this section, we examine the method of **partial fraction decomposition**, which allows us to decompose rational functions into sums of simpler, more easily integrated rational functions. Using this method, we can rewrite an expression such as: $\frac{3x}{x^2 - x - 2}$ as an expression such as $\frac{1}{x+1} + \frac{2}{x-2}$.

The key to the method of partial fraction decomposition is being able to anticipate the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also extremely important to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $\deg(P(x)) < \deg(Q(x))$. In the case when $\deg(P(x)) \geq \deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x) + \frac{R(x)}{Q(x)}$, where $\deg(R(x)) < \deg(Q(x))$. We then do a partial fraction decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int \frac{P(x)}{Q(x)}dx$, where $\deg(P(x)) \geq \deg(Q(x))$.

Example 3.28

Integrating $\int \frac{P(x)}{Q(x)}dx$, where $\deg(P(x)) \geq \deg(Q(x))$

Evaluate $\int \frac{x^2 + 3x + 5}{x + 1}dx$.

Solution

Since $\deg\left(x^2 + 3x + 5\right) \geq \deg(x + 1),$ we perform long division to obtain

$$\frac{x^2 + 3x + 5}{x + 1} = x + 2 + \frac{3}{x + 1}.$$

Thus,

$$\begin{aligned} \int \frac{x^2 + 3x + 5}{x + 1} dx &= \int \left(x + 2 + \frac{3}{x + 1}\right) dx \\ &= \frac{1}{2}x^2 + 2x + 3\ln|x + 1| + C. \end{aligned}$$

Visit this **website (http://www.openstaxcollege.org/l/20_polylongdiv)** for a review of long division of polynomials.

3.17 Evaluate $\int \frac{x - 3}{x + 2} dx.$

To integrate $\int \frac{P(x)}{Q(x)} dx,$ where $\deg(P(x)) < \deg(Q(x)),$ we must begin by factoring $Q(x).$

Nonrepeated Linear Factors

If $Q(x)$ can be factored as $(a_1 x + b_1)(a_2 x + b_2)\ldots(a_n x + b_n),$ where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_n}{a_n x + b_n}.$$

The proof that such constants exist is beyond the scope of this course.

In this next example, we see how to use partial fractions to integrate a rational function of this type.

Example 3.29

Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x + 2}{x^3 - x^2 - 2x} dx.$

Solution

Since $\deg(3x + 2) < \deg\left(x^3 - x^2 - 2x\right),$ we begin by factoring the denominator of $\frac{3x + 2}{x^3 - x^2 - 2x}.$ We can see that $x^3 - x^2 - 2x = x(x - 2)(x + 1).$ Thus, there are constants $A,$ $B,$ and C satisfying

$$\frac{3x + 2}{x(x - 2)(x + 1)} = \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{x + 1}.$$

We must now find these constants. To do so, we begin by getting a common denominator on the right. Thus,

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}{x(x-2)(x+1)}.$$

Now, we set the numerators equal to each other, obtaining

$$3x + 2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2). \quad \textbf{(3.8)}$$

There are two different strategies for finding the coefficients $A,\quad B,\quad$ and C. We refer to these as the *method of equating coefficients* and the *method of strategic substitution*.

Rule: Method of Equating Coefficients

Rewrite **Equation 3.8** in the form

$$3x + 2 = (A + B + C)x^2 + (-A + B - 2C)x + (-2A).$$

Equating coefficients produces the system of equations

$$\begin{aligned} A + B + C &= 0 \\ -A + B - 2C &= 3 \\ -2A &= 2. \end{aligned}$$

To solve this system, we first observe that $-2A = 2 \Rightarrow A = -1$. Substituting this value into the first two equations gives us the system

$$\begin{aligned} B + C &= 1 \\ B - 2C &= 2. \end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces

$$-3C = 1,$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B + C = 1$ yields $B = \frac{4}{3}$. Thus, solving these equations yields $A = -1,\quad B = \frac{4}{3},\quad$ and $C = -\frac{1}{3}$.

It is important to note that the system produced by this method is consistent if and only if we have set up the decomposition correctly. If the system is inconsistent, there is an error in our decomposition.

Rule: Method of Strategic Substitution

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of $A,\quad B,\quad$ and C that satisfy **Equation 3.8** for *all* values of x. That is, this equation must be true for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find $A,\quad B,\quad$ and C easily. For example, if we substitute $x = 0,$ the equation reduces to $2 = A(-2)(1).$ Solving for A yields $A = -1.$ Next, by substituting $x = 2,$ the equation reduces to $8 = B(2)(3),$ or equivalently $B = 4/3.$ Last, we substitute $x = -1$ into the equation and obtain $-1 = C(-1)(-3).$ Solving, we have $C = -\frac{1}{3}.$

It is important to keep in mind that if we attempt to use this method with a decomposition that has not been set up correctly, we are still able to find values for the constants, but these constants are meaningless. If we do opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Now that we have the values of $A, \quad B, \quad$ and $C, \quad$ we rewrite the original integral:

$$\int \frac{3x+2}{x^3-x^2-2x}dx = \int \left(-\frac{1}{x} + \frac{4}{3} \cdot \frac{1}{(x-2)} - \frac{1}{3} \cdot \frac{1}{(x+1)}\right)dx.$$

Evaluating the integral gives us

$$\int \frac{3x+2}{x^3-x^2-2x}dx = -\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C.$$

In the next example, we integrate a rational function in which the degree of the numerator is not less than the degree of the denominator.

Example 3.30

Dividing before Applying Partial Fractions

Evaluate $\int \frac{x^2+3x+1}{x^2-4}dx.$

Solution

Since $\text{degree}(x^2+3x+1) \geq \text{degree}(x^2-4), \quad$ we must perform long division of polynomials. This results in

$$\frac{x^2+3x+1}{x^2-4} = 1 + \frac{3x+5}{x^2-4}.$$

Next, we perform partial fraction decomposition on $\frac{3x+5}{x^2-4} = \frac{3x+5}{(x+2)(x-2)}.$ We have

$$\frac{3x+5}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Thus,

$$3x+5 = A(x+2) + B(x-2).$$

Solving for A and B using either method, we obtain $A = 11/4$ and $B = 1/4.$

Rewriting the original integral, we have

$$\int \frac{x^2+3x+1}{x^2-4}dx = \int \left(1 + \frac{11}{4} \cdot \frac{1}{x-2} + \frac{1}{4} \cdot \frac{1}{x+2}\right)dx.$$

Evaluating the integral produces

$$\int \frac{x^2+3x+1}{x^2-4}dx = x + \frac{11}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C.$$

As we see in the next example, it may be possible to apply the technique of partial fraction decomposition to a nonrational function. The trick is to convert the nonrational function to a rational function through a substitution.

Example 3.31

Applying Partial Fractions after a Substitution

Evaluate $\int \frac{\cos x}{\sin^2 x - \sin x} dx.$

Solution

Let's begin by letting $u = \sin x.$ Consequently, $du = \cos x \, dx.$ After making these substitutions, we have

$$\int \frac{\cos x}{\sin^2 x - \sin x} dx = \int \frac{du}{u^2 - u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $1/u(u-1)$ gives $\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1}.$

Thus,

$$\begin{aligned} \int \frac{\cos x}{\sin^2 x - \sin x} dx &= -\ln|u| + \ln|u-1| + C \\ &= -\ln|\sin x| + \ln|\sin x - 1| + C. \end{aligned}$$

3.18 Evaluate $\int \frac{x+1}{(x+3)(x-2)} dx.$

Repeated Linear Factors

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, rational functions with at least one factor of the form $(ax+b)^n,$ where n is a positive integer greater than or equal to $2.$ If the denominator contains the repeated linear factor $(ax+b)^n,$ then the decomposition must contain

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

As we see in our next example, the basic technique used for solving for the coefficients is the same, but it requires more algebra to determine the numerators of the partial fractions.

Example 3.32

Partial Fractions with Repeated Linear Factors

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)} dx.$

Solution

We have $\text{degree}(x-2) < \text{degree}\left((2x-1)^2(x-1)\right),$ so we can proceed with the decomposition. Since

$(2x-1)^2$ is a repeated linear factor, include $\frac{A}{2x-1}+\frac{B}{(2x-1)^2}$ in the decomposition. Thus,

$$\frac{x-2}{(2x-1)^2(x-1)}=\frac{A}{2x-1}+\frac{B}{(2x-1)^2}+\frac{C}{x-1}.$$

After getting a common denominator and equating the numerators, we have

$$x-2=A(2x-1)(x-1)+B(x-1)+C(2x-1)^2. \tag{3.9}$$

We then use the method of equating coefficients to find the values of $A,\ B,$ and $C.$

$$x-2=(2A+4C)x^2+(-3A+B-4C)x+(A-B+C).$$

Equating coefficients yields $2A+4C=0,\ -3A+B-4C=1,$ and $A-B+C=-2.$ Solving this system yields $A=2,\ B=3,$ and $C=-1.$

Alternatively, we can use the method of strategic substitution. In this case, substituting $x=1$ and $x=1/2$ into **Equation 3.9** easily produces the values $B=3$ and $C=-1.$ At this point, it may seem that we have run out of good choices for $x,$ however, since we already have values for B and $C,$ we can substitute in these values and choose any value for x not previously used. The value $x=0$ is a good option. In this case, we obtain the equation $-2=A(-1)(-1)+3(-1)+(-1)(-1)^2$ or, equivalently, $A=2.$

Now that we have the values for $A,\ B,$ and $C,$ we rewrite the original integral and evaluate it:

$$\begin{aligned}\int\frac{x-2}{(2x-1)^2(x-1)}dx &= \int\left(\frac{2}{2x-1}+\frac{3}{(2x-1)^2}-\frac{1}{x-1}\right)dx \\ &= \ln|2x-1|-\frac{3}{2(2x-1)}-\ln|x-1|+C.\end{aligned}$$

3.19 Set up the partial fraction decomposition for $\int\frac{x+2}{(x+3)^3(x-4)^2}dx.$ (Do not solve for the coefficients or complete the integration.)

The General Method

Now that we are beginning to get the idea of how the technique of partial fraction decomposition works, let's outline the basic method in the following problem-solving strategy.

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $P(x)/Q(x),$ use the following steps:

1. Make sure that $\text{degree}(P(x))<\text{degree}(Q(x)).$ If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $\deg(P(x))<\deg(Q(x)),$ the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x).$

   a. If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n),$ where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_n}{a_n x + b_n}.$$

b. If $Q(x)$ contains the repeated linear factor $(ax + b)^n$, then the decomposition must contain

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n}.$$

c. For each irreducible quadratic factor $ax^2 + bx + c$ that $Q(x)$ contains, the decomposition must include

$$\frac{Ax + B}{ax^2 + bx + c}.$$

d. For each repeated irreducible quadratic factor $\left(ax^2 + bx + c\right)^n$, the decomposition must include

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}.$$

e. After the appropriate decomposition is determined, solve for the constants.

f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Simple Quadratic Factors

Now let's look at integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic $ax^2 + bx + c$ is irreducible if $ax^2 + bx + c = 0$ has no real zeros—that is, if $b^2 - 4ac < 0$.

Example 3.33

Rational Expressions with an Irreducible Quadratic Factor

Evaluate $\int \frac{2x - 3}{x^3 + x} dx$.

Solution

Since $\deg(2x - 3) < \deg(x^3 + x)$, factor the denominator and proceed with partial fraction decomposition. Since $x^3 + x = x(x^2 + 1)$ contains the irreducible quadratic factor $x^2 + 1$, include $\frac{Ax + B}{x^2 + 1}$ as part of the decomposition, along with $\frac{C}{x}$ for the linear term x. Thus, the decomposition has the form

$$\frac{2x - 3}{x(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x}.$$

After getting a common denominator and equating the numerators, we obtain the equation

$$2x - 3 = (Ax + B)x + C\left(x^2 + 1\right).$$

Solving for $A, B,$ and C, we get $A = 3$, $B = 2$, and $C = -3$.

Thus,

$$\frac{2x-3}{x^3+x} = \frac{3x+2}{x^2+1} - \frac{3}{x}.$$

Substituting back into the integral, we obtain

$$\begin{aligned}\int \frac{2x-3}{x^3+x}dx &= \int \left(\frac{3x+2}{x^2+1} - \frac{3}{x}\right)dx \\ &= 3\int \frac{x}{x^2+1}dx + 2\int \frac{1}{x^2+1}dx - 3\int \frac{1}{x}dx && \text{Split up the integral.} \\ &= \frac{3}{2}\ln\left|x^2+1\right| + 2\tan^{-1}x - 3\ln|x| + C. && \text{Evaluate each integral.}\end{aligned}$$

Note: We may rewrite $\ln\left|x^2+1\right| = \ln(x^2+1)$, if we wish to do so, since $x^2+1 > 0$.

Example 3.34

Partial Fractions with an Irreducible Quadratic Factor

Evaluate $\int \frac{dx}{x^3-8}$.

Solution

We can start by factoring $x^3 - 8 = (x-2)(x^2+2x+4)$. We see that the quadratic factor x^2+2x+4 is irreducible since $2^2 - 4(1)(4) = -12 < 0$. Using the decomposition described in the problem-solving strategy, we get

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}.$$

After obtaining a common denominator and equating the numerators, this becomes

$$1 = A\left(x^2+2x+4\right) + (Bx+C)(x-2).$$

Applying either method, we get $A = \frac{1}{12}$, $B = -\frac{1}{12}$, and $C = -\frac{1}{3}$.

Rewriting $\int \frac{dx}{x^3-8}$, we have

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\int \frac{1}{x-2}dx - \frac{1}{12}\int \frac{x+4}{x^2+2x+4}dx.$$

We can see that

$\int \frac{1}{x-2}dx = \ln|x-2| + C$, but $\int \frac{x+4}{x^2+2x+4}dx$ requires a bit more effort. Let's begin by completing the square on x^2+2x+4 to obtain

$$x^2+2x+4 = (x+1)^2 + 3.$$

By letting $u = x + 1$ and consequently $du = dx$, we see that

$$\begin{aligned}
\int \frac{x+4}{x^2+2x+4}dx &= \int \frac{x+4}{(x+1)^2+3}dx && \text{Complete the square on the denominator.} \\
&= \int \frac{u+3}{u^2+3}du && \text{Substitute } u = x+1,\ x = u-1, \text{ and } du = dx. \\
&= \int \frac{u}{u^2+3}du + \int \frac{3}{u^2+3}du && \text{Split the numerator apart.} \\
&= \frac{1}{2}\ln\left|u^2+3\right| + \frac{3}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + C && \text{Evaluate each integral.} \\
&= \frac{1}{2}\ln\left|x^2+2x+4\right| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C. && \text{Rewrite in terms of } x \text{ and simplify.}
\end{aligned}$$

Substituting back into the original integral and simplifying gives

$$\int \frac{dx}{x^3-8} = \frac{1}{12}\ln|x-2| - \frac{1}{24}\ln\left|x^2+2x+4\right| - \frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

Here again, we can drop the absolute value if we wish to do so, since $x^2 + 2x + 4 > 0$ for all x.

Example 3.35

Finding a Volume

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x) = \frac{x^2}{\left(x^2+1\right)^2}$ and the x-axis over the interval $[0, 1]$ about the y-axis.

Solution

Let’s begin by sketching the region to be revolved (see **Figure 3.11**). From the sketch, we see that the shell method is a good choice for solving this problem.

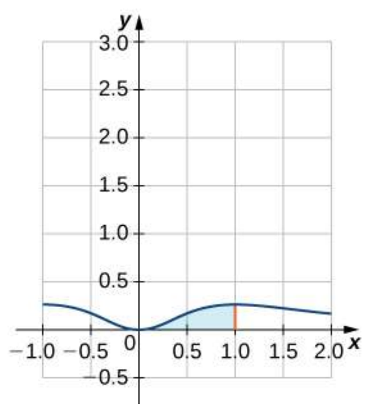


Figure 3.11 We can use the shell method to find the volume of revolution obtained by revolving the region shown about the y-axis.

The volume is given by

$$V = 2\pi\int_0^1 x \cdot \frac{x^2}{\left(x^2+1\right)^2}dx = 2\pi\int_0^1 \frac{x^3}{(x^2+1)^2}dx.$$

Since $\deg\left(\left(x^2+1\right)^2\right) = 4 > 3 = \deg(x^3)$, we can proceed with partial fraction decomposition. Note that $(x^2+1)^2$ is a repeated irreducible quadratic. Using the decomposition described in the problem-solving strategy, we get

$$\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Finding a common denominator and equating the numerators gives

$$x^3 = (Ax+B)\left(x^2+1\right) + Cx + D.$$

Solving, we obtain $A = 1,\quad B = 0,\quad C = -1,$ and $D = 0.$ Substituting back into the integral, we have

$$\begin{aligned} V &= 2\pi\int_0^1 \frac{x^3}{(x^2+1)^2}dx \\ &= 2\pi\int_0^1\left(\frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right)dx \\ &= 2\pi\left(\frac{1}{2}\ln(x^2+1) + \frac{1}{2}\cdot\frac{1}{x^2+1}\right)\Bigg|_0^1 \\ &= \pi\left(\ln 2 - \frac{1}{2}\right). \end{aligned}$$

3.20 Set up the partial fraction decomposition for $\int \frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}dx.$

3.4 EXERCISES

Express the rational function as a sum or difference of two simpler rational expressions.

182. $\frac{1}{(x-3)(x-2)}$

183. $\frac{x^2+1}{x(x+1)(x+2)}$

184. $\frac{1}{x^3-x}$

185. $\frac{3x+1}{x^2}$

186. $\frac{3x^2}{x^2+1}$ (*Hint:* Use long division first.)

187. $\frac{2x^4}{x^2-2x}$

188. $\frac{1}{(x-1)(x^2+1)}$

189. $\frac{1}{x^2(x-1)}$

190. $\frac{x}{x^2-4}$

191. $\frac{1}{x(x-1)(x-2)(x-3)}$

192. $\frac{1}{x^4-1} = \frac{1}{(x+1)(x-1)\left(x^2+1\right)}$

193. $\frac{3x^2}{x^3-1} = \frac{3x^2}{(x-1)(x^2+x+1)}$

194. $\frac{2x}{(x+2)^2}$

195. $\frac{3x^4+x^3+20x^2+3x+31}{(x+1)\left(x^2+4\right)^2}$

Use the method of partial fractions to evaluate each of the following integrals.

196. $\int \frac{dx}{(x-3)(x-2)}$

197. $\int \frac{3x}{x^2+2x-8}dx$

198. $\int \frac{dx}{x^3-x}$

199. $\int \frac{x}{x^2-4}dx$

200. $\int \frac{dx}{x(x-1)(x-2)(x-3)}$

201. $\int \frac{2x^2+4x+22}{x^2+2x+10}dx$

202. $\int \frac{dx}{x^2-5x+6}$

203. $\int \frac{2-x}{x^2+x}dx$

204. $\int \frac{2}{x^2-x-6}dx$

205. $\int \frac{dx}{x^3-2x^2-4x+8}$

206. $\int \frac{dx}{x^4-10x^2+9}$

Evaluate the following integrals, which have irreducible quadratic factors.

207. $\int \frac{2}{(x-4)\left(x^2+2x+6\right)}dx$

208. $\int \frac{x^2}{x^3-x^2+4x-4}dx$

209. $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}dx$

210. $\int \frac{x}{(x-1)\left(x^2+2x+2\right)^2}dx$

Use the method of partial fractions to evaluate the following integrals.

211. $\int \frac{3x+4}{\left(x^2+4\right)(3-x)}dx$

212. $\int \frac{2}{(x+2)^2(2-x)}dx$

213. $\int \frac{3x+4}{x^3-2x-4}dx$ (*Hint:* Use the rational root theorem.)

Use substitution to convert the integrals to integrals of rational functions. Then use partial fractions to evaluate the integrals.

214. $\int_0^1 \frac{e^x}{36-e^{2x}}dx$ (Give the exact answer and the decimal equivalent. Round to five decimal places.)

215. $\int \frac{e^x\,dx}{e^{2x}-e^x}dx$

216. $\int \frac{\sin x\,dx}{1-\cos^2 x}$

217. $\int \frac{\sin x}{\cos^2 x+\cos x-6}dx$

218. $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}dx$

219. $\int \frac{dt}{\left(e^t-e^{-t}\right)^2}$

220. $\int \frac{1+e^x}{1-e^x}dx$

221. $\int \frac{dx}{1+\sqrt{x+1}}$

222. $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$

223. $\int \frac{\cos x}{\sin x(1-\sin x)}dx$

224. $\int \frac{e^x}{\left(e^{2x}-4\right)^2}dx$

225. $\int_1^2 \frac{1}{x^2\sqrt{4-x^2}}dx$

226. $\int \frac{1}{2+e^{-x}}dx$

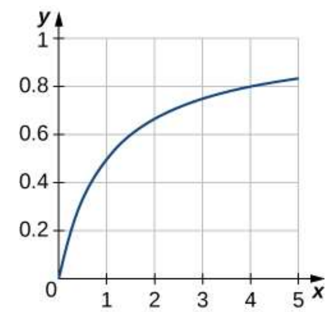
227. $\int \frac{1}{1+e^x}dx$

Use the given substitution to convert the integral to an integral of a rational function, then evaluate.

228. $\int \frac{1}{t-\sqrt[3]{t}}dt\, t=x^3$

229. $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx;\ x=u^6$

230. Graph the curve $y=\frac{x}{1+x}$ over the interval $[0,\ 5]$. Then, find the area of the region bounded by the curve, the x-axis, and the line $x=4$.



231. Find the volume of the solid generated when the region bounded by $y=1/\sqrt{x(3-x)}$, $y=0$, $x=1$, and $x=2$ is revolved about the x-axis.

232. The velocity of a particle moving along a line is a function of time given by $v(t)=\frac{88t^2}{t^2+1}$. Find the distance that the particle has traveled after $t=5$ sec.

Solve the initial-value problem for x as a function of t.

233. $\left(t^2-7t+12\right)\frac{dx}{dt}=1,\ (t>4,\ x(5)=0)$

234. $(t+5)\frac{dx}{dt}=x^2+1,\ t>-5,\ x(1)=\tan 1$

235. $\left(2t^3-2t^2+t-1\right)\frac{dx}{dt}=3,\ x(2)=0$

236. Find the x-coordinate of the centroid of the area bounded by $y\left(x^2-9\right)=1$, $y=0$, $x=4$, and $x=5$. (Round the answer to two decimal places.)

237. Find the volume generated by revolving the area bounded by $y=\frac{1}{x^3+7x^2+6x}x=1$, $x=7$, and $y=0$ about the y-axis.

238. Find the area bounded by $y=\frac{x-12}{x^2-8x-20}$, $y=0$, $x=2$, and $x=4$. (Round the answer to the nearest hundredth.)

239. Evaluate the integral $\int \frac{dx}{x^3 + 1}$.

For the following problems, use the substitutions $\tan\left(\frac{x}{2}\right) = t$, $dx = \frac{2}{1 + t^2}dt$, $\sin x = \frac{2t}{1 + t^2}$, and $\cos x = \frac{1 - t^2}{1 + t^2}$.

240. $\int \frac{dx}{3 - 5\sin x}$

241. Find the area under the curve $y = \frac{1}{1 + \sin x}$ between $x = 0$ and $x = \pi$. (Assume the dimensions are in inches.)

242. Given $\tan\left(\frac{x}{2}\right) = t$, derive the formulas $dx = \frac{2}{1 + t^2}dt$, $\sin x = \frac{2t}{1 + t^2}$, and $\cos x = \frac{1 - t^2}{1 + t^2}$.

243. Evaluate $\int \frac{\sqrt[3]{x - 8}}{x}dx$.

3.5 | Other Strategies for Integration

Learning Objectives

3.5.1 Use a table of integrals to solve integration problems.

3.5.2 Use a computer algebra system (CAS) to solve integration problems.

In addition to the techniques of integration we have already seen, several other tools are widely available to assist with the process of integration. Among these tools are **integration tables**, which are readily available in many books, including the appendices to this one. Also widely available are **computer algebra systems (CAS)**, which are found on calculators and in many campus computer labs, and are free online.

Tables of Integrals

Integration tables, if used in the right manner, can be a handy way either to evaluate or check an integral quickly. Keep in mind that when using a table to check an answer, it is possible for two completely correct solutions to look very different. For example, in **Trigonometric Substitution**, we found that, by using the substitution $x = \tan\theta,$ we can arrive at

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left(x + \sqrt{x^2+1}\right) + C.$$

However, using $x = \sinh\theta,$ we obtained a different solution—namely,

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C.$$

We later showed algebraically that the two solutions are equivalent. That is, we showed that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right).$

In this case, the two antiderivatives that we found were actually equal. This need not be the case. However, as long as the difference in the two antiderivatives is a constant, they are equivalent.

Example 3.36

Using a Formula from a Table to Evaluate an Integral

Use the table formula

$$\int \frac{\sqrt{a^2-u^2}}{u^2}du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$$

to evaluate $\int \frac{\sqrt{16-e^{2x}}}{e^x}dx.$

Solution

If we look at integration tables, we see that several formulas contain expressions of the form $\sqrt{a^2-u^2}.$ This expression is actually similar to $\sqrt{16-e^{2x}},$ where $a = 4$ and $u = e^x.$ Keep in mind that we must also have $du = e^x.$ Multiplying the numerator and the denominator of the given integral by e^x should help to put this integral in a useful form. Thus, we now have

$$\int \frac{\sqrt{16-e^{2x}}}{e^x}dx = \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x\,dx.$$

Substituting $u = e^x$ and $du = e^x$ produces $\int \frac{\sqrt{a^2 - u^2}}{u^2} du$. From the integration table (#88 in **Appendix A**),

$$\int \frac{\sqrt{a^2 - u^2}}{u^2} du = -\frac{\sqrt{a^2 - u^2}}{u} - \sin^{-1}\frac{u}{a} + C.$$

Thus,

$$\begin{aligned}
\int \frac{\sqrt{16 - e^{2x}}}{e^x} dx &= \int \frac{\sqrt{16 - e^{2x}}}{e^{2x}} e^x\, dx && \text{Substitute } u = e^x \text{ and } du = e^x\, dx. \\
&= \int \frac{\sqrt{4^2 - u^2}}{u^2} du && \text{Apply the formula using } a = 4. \\
&= -\frac{\sqrt{4^2 - u^2}}{u} - \sin^{-1}\frac{u}{4} + C && \text{Substitute } u = e^x. \\
&= -\frac{\sqrt{16 - e^{2x}}}{u} - \sin^{-1}\left(\frac{e^x}{4}\right) + C.
\end{aligned}$$

Computer Algebra Systems

If available, a CAS is a faster alternative to a table for solving an integration problem. Many such systems are widely available and are, in general, quite easy to use.

Example 3.37

Using a Computer Algebra System to Evaluate an Integral

Use a computer algebra system to evaluate $\int \frac{dx}{\sqrt{x^2 - 4}}$. Compare this result with $\ln\left|\frac{\sqrt{x^2 - 4}}{2} + \frac{x}{2}\right| + C$, a result we might have obtained if we had used trigonometric substitution.

Solution

Using Wolfram Alpha, we obtain

$$\int \frac{dx}{\sqrt{x^2 - 4}} = \ln\left|\sqrt{x^2 - 4} + x\right| + C.$$

Notice that

$$\ln\left|\frac{\sqrt{x^2 - 4}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{\sqrt{x^2 - 4} + x}{2}\right| + C = \ln\left|\sqrt{x^2 - 4} + x\right| - \ln 2 + C.$$

Since these two antiderivatives differ by only a constant, the solutions are equivalent. We could have also demonstrated that each of these antiderivatives is correct by differentiating them.

You can access an **integral calculator (http://www.openstaxcollege.org/l/20_intcalc)** for more examples.

Example 3.38

Using a CAS to Evaluate an Integral

Evaluate $\int \sin^3 x\,dx$ using a CAS. Compare the result to $\frac{1}{3}\cos^3 x - \cos x + C,$ the result we might have obtained using the technique for integrating odd powers of $\sin x$ discussed earlier in this chapter.

Solution

Using Wolfram Alpha, we obtain

$$\int \sin^3 x\,dx = \frac{1}{12}(\cos(3x) - 9\cos x) + C.$$

This looks quite different from $\frac{1}{3}\cos^3 x - \cos x + C.$ To see that these antiderivatives are equivalent, we can make use of a few trigonometric identities:

$$\begin{aligned}\frac{1}{12}(\cos(3x) - 9\cos x) &= \frac{1}{12}(\cos(x + 2x) - 9\cos x)\\ &= \frac{1}{12}(\cos(x)\cos(2x) - \sin(x)\sin(2x) - 9\cos x)\\ &= \frac{1}{12}(\cos x\left(2\cos^2 x - 1\right) - \sin x(2\sin x\cos x) - 9\cos x)\\ &= \frac{1}{12}(2\cos\ x - \cos x - 2\cos x\left(1 - \cos^2 x\right) - 9\cos x)\\ &= \frac{1}{12}(4\cos\ x - 12\cos x)\\ &= \frac{1}{3}\cos\ x - \cos x.\end{aligned}$$

Thus, the two antiderivatives are identical.

We may also use a CAS to compare the graphs of the two functions, as shown in the following figure.

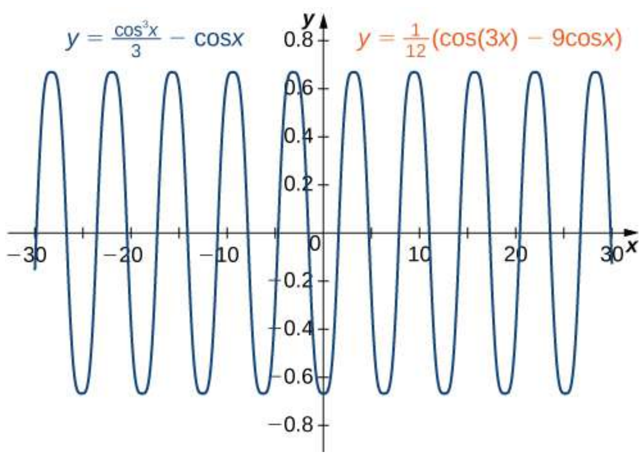


Figure 3.12 The graphs of $y = \frac{1}{3}\cos^3 x - \cos x$ and $y = \frac{1}{12}(\cos(3x) - 9\cos x)$ are identical.

3.21 Use a CAS to evaluate $\int \frac{dx}{\sqrt{x^2 + 4}}.$

3.5 EXERCISES

Use a table of integrals to evaluate the following integrals.

244. $\int_0^4 \frac{x}{\sqrt{1+2x}}dx$

245. $\int \frac{x+3}{x^2+2x+2}dx$

246. $\int x^3\sqrt{1+2x^2}\,dx$

247. $\int \frac{1}{\sqrt{x^2+6x}}dx$

248. $\int \frac{x}{x+1}dx$

249. $\int x\cdot 2^{x^2}dx$

250. $\int \frac{1}{4x^2+25}dx$

251. $\int \frac{dy}{\sqrt{4-y^2}}$

252. $\int \sin^3(2x)\cos(2x)dx$

253. $\int \csc(2w)\cot(2w)dw$

254. $\int 2^y\,dy$

255. $\int_0^1 \frac{3x\,dx}{\sqrt{x^2+8}}$

256. $\int_{-1/4}^{1/4}\sec^2(\pi x)\tan(\pi x)dx$

257. $\int_0^{\pi/2}\tan^2\left(\frac{x}{2}\right)dx$

258. $\int \cos^3 x\,dx$

259. $\int \tan^5(3x)dx$

260. $\int \sin^2 y\cos^3 y\,dy$

Use a CAS to evaluate the following integrals. Tables can also be used to verify the answers.

261. **[T]** $\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)}$

262. **[T]** $\int \frac{dw}{1-\cos(7w)}$

263. **[T]** $\int_0^t \frac{dt}{4\cos t+3\sin t}$

264. **[T]** $\int \frac{\sqrt{x^2-9}}{3x}dx$

265. **[T]** $\int \frac{dx}{x^{1/2}+x^{1/3}}$

266. **[T]** $\int \frac{dx}{x\sqrt{x-1}}$

267. **[T]** $\int x^3\sin x\,dx$

268. **[T]** $\int x\sqrt{x^4-9}\,dx$

269. **[T]** $\int \frac{x}{1+e^{-x^2}}dx$

270. **[T]** $\int \frac{\sqrt{3-5x}}{2x}dx$

271. **[T]** $\int \frac{dx}{x\sqrt{x-1}}$

272. **[T]** $\int e^x\cos^{-1}(e^x)dx$

Use a calculator or CAS to evaluate the following integrals.

273. **[T]** $\int_0^{\pi/4}\cos(2x)dx$

274. **[T]** $\int_0^1 x\cdot e^{-x^2}dx$

275. **[T]** $\int_0^8 \frac{2x}{\sqrt{x^2+36}}dx$

276. **[T]** $\int_0^{2/\sqrt{3}}\frac{1}{4+9x^2}dx$

277. **[T]** $\int \frac{dx}{x^2 + 4x + 13}$

278. **[T]** $\int \frac{dx}{1 + \sin x}$

Use tables to evaluate the integrals. You may need to complete the square or change variables to put the integral into a form given in the table.

279. $\int \frac{dx}{x^2 + 2x + 10}$

280. $\int \frac{dx}{\sqrt{x^2 - 6x}}$

281. $\int \frac{e^x}{\sqrt{e^{2x} - 4}} dx$

282. $\int \frac{\cos x}{\sin^2 x + 2\sin x} dx$

283. $\int \frac{\arctan\left(x^3\right)}{x^4} dx$

284. $\int \frac{\ln|x| \arcsin(\ln|x|)}{x} dx$

Use tables to perform the integration.

285. $\int \frac{dx}{\sqrt{x^2 + 16}}$

286. $\int \frac{3x}{2x + 7} dx$

287. $\int \frac{dx}{1 - \cos(4x)}$

288. $\int \frac{dx}{\sqrt{4x + 1}}$

289. Find the area bounded by $y\left(4 + 25x^2\right) = 5,\ x = 0,\ y = 0,$ and $x = 4.$ Use a table of integrals or a CAS.

290. The region bounded between the curve $y = \frac{1}{\sqrt{1 + \cos x}},\ 0.3 \le x \le 1.1,$ and the x-axis is revolved about the x-axis to generate a solid. Use a table of integrals to find the volume of the solid generated. (Round the answer to two decimal places.)

291. Use substitution and a table of integrals to find the area of the surface generated by revolving the curve $y = e^x,\ 0 \le x \le 3,$ about the x-axis. (Round the answer to two decimal places.)

292. **[T]** Use an integral table and a calculator to find the area of the surface generated by revolving the curve $y = \frac{x^2}{2},\ 0 \le x \le 1,$ about the x-axis. (Round the answer to two decimal places.)

293. **[T]** Use a CAS or tables to find the area of the surface generated by revolving the curve $y = \cos x,\ 0 \le x \le \frac{\pi}{2},$ about the x-axis. (Round the answer to two decimal places.)

294. Find the length of the curve $y = \frac{x^2}{4}$ over $[0,\ 8].$

295. Find the length of the curve $y = e^x$ over $[0,\ \ln(2)].$

296. Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ over the interval $[0,\ 9]$ about the x-axis.

297. Find the average value of the function $f(x) = \frac{1}{x^2 + 1}$ over the interval $[-3,\ 3].$

298. Approximate the arc length of the curve $y = \tan(\pi x)$ over the interval $\left[0,\ \frac{1}{4}\right].$ (Round the answer to three decimal places.)

3.6 | Numerical Integration

Learning Objectives

3.6.1 Approximate the value of a definite integral by using the midpoint and trapezoidal rules.
3.6.2 Determine the absolute and relative error in using a numerical integration technique.
3.6.3 Estimate the absolute and relative error using an error-bound formula.
3.6.4 Recognize when the midpoint and trapezoidal rules over- or underestimate the true value of an integral.
3.6.5 Use Simpson's rule to approximate the value of a definite integral to a given accuracy.

The antiderivatives of many functions either cannot be expressed or cannot be expressed easily in closed form (that is, in terms of known functions). Consequently, rather than evaluate definite integrals of these functions directly, we resort to various techniques of **numerical integration** to approximate their values. In this section we explore several of these techniques. In addition, we examine the process of estimating the error in using these techniques.

The Midpoint Rule

Earlier in this text we defined the definite integral of a function over an interval as the limit of Riemann sums. In general, any Riemann sum of a function $f(x)$ over an interval $[a, b]$ may be viewed as an estimate of $\int_a^b f(x)dx.$ Recall that a Riemann sum of a function $f(x)$ over an interval $[a, b]$ is obtained by selecting a partition

$$P = \{x_0, x_1, x_2, \ldots, x_n\}, \text{ where } a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and a set

$$S = \{x_1^*, x_2^*, \ldots, x_n^*\}, \text{ where } x_{i-1} \le x_i^* \le x_i \text{ for all } i.$$

The Riemann sum corresponding to the partition P and the set S is given by $\sum_{i=1}^{n} f(x_i^*)\Delta x_i,$ where $\Delta x_i = x_i - x_{i-1},$ the length of the ith subinterval.

The **midpoint rule** for estimating a definite integral uses a Riemann sum with subintervals of equal width and the midpoints, $m_i,$ of each subinterval in place of x_i^*. Formally, we state a theorem regarding the convergence of the midpoint rule as follows.

Theorem 3.3: The Midpoint Rule

Assume that $f(x)$ is continuous on $[a, b]$. Let n be a positive integer and $\Delta x = \frac{b-a}{n}$. If $[a, b]$ is divided into n subintervals, each of length $\Delta x,$ and m_i is the midpoint of the ith subinterval, set

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x. \quad (3.10)$$

Then $\lim_{n \to \infty} M_n = \int_a^b f(x)dx.$

As we can see in Figure 3.13, if $f(x) \ge 0$ over $[a, b],$ then $\sum_{i=1}^{n} f(m_i)\Delta x$ corresponds to the sum of the areas of rectangles approximating the area between the graph of $f(x)$ and the x-axis over $[a, b]$. The graph shows the rectangles corresponding to M_4 for a nonnegative function over a closed interval $[a, b]$.

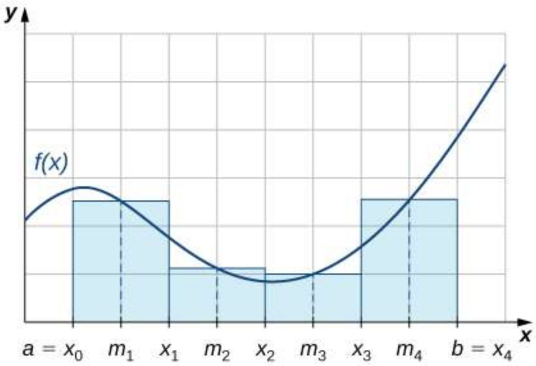


Figure 3.13 The midpoint rule approximates the area between the graph of $f(x)$ and the x-axis by summing the areas of rectangles with midpoints that are points on $f(x)$.

Example 3.39

Using the Midpoint Rule with M_4

Use the midpoint rule to estimate $\int_0^1 x^2\,dx$ using four subintervals. Compare the result with the actual value of this integral.

Solution

Each subinterval has length $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Therefore, the subintervals consist of

$$\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \text{ and } \left[\frac{3}{4}, 1\right].$$

The midpoints of these subintervals are $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,

$$M_4 = \frac{1}{4}f\left(\frac{1}{8}\right) + \frac{1}{4}f\left(\frac{3}{8}\right) + \frac{1}{4}f\left(\frac{5}{8}\right) + \frac{1}{4}f\left(\frac{7}{8}\right) = \frac{1}{4}\cdot\frac{1}{64} + \frac{1}{4}\cdot\frac{9}{64} + \frac{1}{4}\cdot\frac{25}{64} + \frac{1}{4}\cdot\frac{21}{64} = \frac{21}{64}.$$

Since

$$\int_0^1 x^2\,dx = \frac{1}{3} \text{ and } \left|\frac{1}{3} - \frac{21}{64}\right| = \frac{1}{192} \approx 0.0052,$$

we see that the midpoint rule produces an estimate that is somewhat close to the actual value of the definite integral.

Example 3.40

Using the Midpoint Rule with M_6

Use M_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ on $[1, 4]$.

Solution

The length of $y = \frac{1}{2}x^2$ on $[1, 4]$ is

$$\int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Since $\frac{dy}{dx} = x,$ this integral becomes $\int_1^4 \sqrt{1 + x^2}\, dx.$

If $[1, 4]$ is divided into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$ and the midpoints of the subintervals are $\left\{\frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}\right\}.$ If we set $f(x) = \sqrt{1 + x^2},$

$$\begin{aligned} M_6 &= \frac{1}{2}f\left(\frac{5}{4}\right) + \frac{1}{2}f\left(\frac{7}{4}\right) + \frac{1}{2}f\left(\frac{9}{4}\right) + \frac{1}{2}f\left(\frac{11}{4}\right) + \frac{1}{2}f\left(\frac{13}{4}\right) + \frac{1}{2}f\left(\frac{15}{4}\right) \\ &\approx \frac{1}{2}(1.6008 + 2.0156 + 2.4622 + 2.9262 + 3.4004 + 3.8810) = 8.1431. \end{aligned}$$

3.22 Use the midpoint rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x}\, dx.$

The Trapezoidal Rule

We can also approximate the value of a definite integral by using trapezoids rather than rectangles. In **Figure 3.14**, the area beneath the curve is approximated by trapezoids rather than by rectangles.

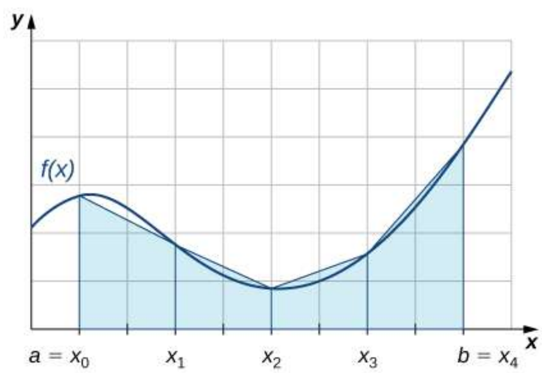


Figure 3.14 Trapezoids may be used to approximate the area under a curve, hence approximating the definite integral.

The **trapezoidal rule** for estimating definite integrals uses trapezoids rather than rectangles to approximate the area under a curve. To gain insight into the final form of the rule, consider the trapezoids shown in **Figure 3.14**. We assume that the length of each subinterval is given by Δx. First, recall that the area of a trapezoid with a height of h and bases of length b_1 and b_2 is given by $\text{Area} = \frac{1}{2}h(b_1 + b_2)$. We see that the first trapezoid has a height Δx and parallel bases of length $f(x_0)$ and $f(x_1)$. Thus, the area of the first trapezoid in **Figure 3.14** is

$$\frac{1}{2}\Delta x(f(x_0) + f(x_1)).$$

The areas of the remaining three trapezoids are

$$\frac{1}{2}\Delta x(f(x_1)+f(x_2)),\ \frac{1}{2}\Delta x(f(x_2)+f(x_3)),\text{ and }\frac{1}{2}\Delta x(f(x_3)+f(x_4)).$$

Consequently,

$$\int_a^b f(x)dx \approx \frac{1}{2}\Delta x(f(x_0)+f(x_1))+\frac{1}{2}\Delta x(f(x_1)+f(x_2))+\frac{1}{2}\Delta x(f(x_2)+f(x_3))+\frac{1}{2}\Delta x(f(x_3)+f(x_4)).$$

After taking out a common factor of $\frac{1}{2}\Delta x$ and combining like terms, we have

$$\int_a^b f(x)dx \approx \frac{1}{2}\Delta x\big(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\big).$$

Generalizing, we formally state the following rule.

Theorem 3.4: The Trapezoidal Rule

Assume that $f(x)$ is continuous over $[a, b]$. Let n be a positive integer and $\Delta x = \frac{b-a}{n}$. Let $[a, b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2 \ldots, x_n\}$. Set

$$T_n = \frac{1}{2}\Delta x\big(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\big). \tag{3.11}$$

Then, $\lim\limits_{n\to+\infty} T_n = \int_a^b f(x)dx.$

Before continuing, let's make a few observations about the trapezoidal rule. First of all, it is useful to note that

$$T_n = \frac{1}{2}(L_n + R_n)\text{ where }L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x\text{ and }R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

That is, L_n and R_n approximate the integral using the left-hand and right-hand endpoints of each subinterval, respectively. In addition, a careful examination of **Figure 3.15** leads us to make the following observations about using the trapezoidal rules and midpoint rules to estimate the definite integral of a nonnegative function. The trapezoidal rule tends to overestimate the value of a definite integral systematically over intervals where the function is concave up and to underestimate the value of a definite integral systematically over intervals where the function is concave down. On the other hand, the midpoint rule tends to average out these errors somewhat by partially overestimating and partially underestimating the value of the definite integral over these same types of intervals. This leads us to hypothesize that, in general, the midpoint rule tends to be more accurate than the trapezoidal rule.

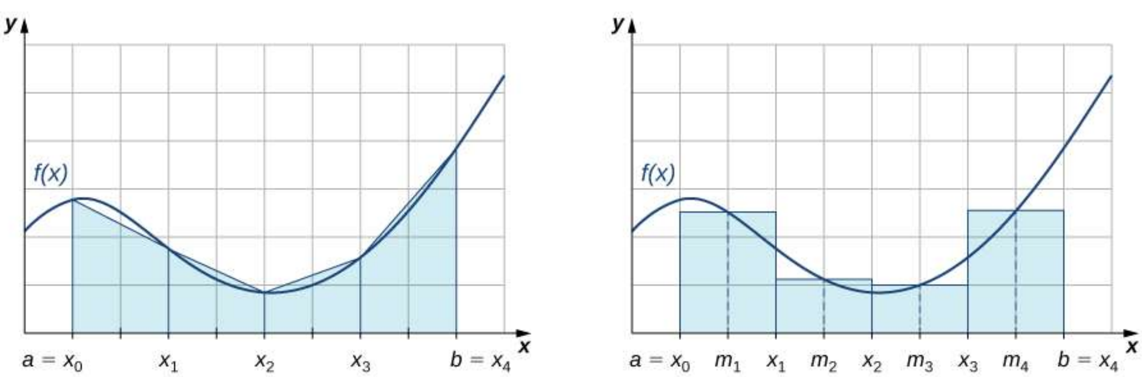


Figure 3.15 The trapezoidal rule tends to be less accurate than the midpoint rule.

Example 3.41

Using the Trapezoidal Rule

Use the trapezoidal rule to estimate $\int_0^1 x^2\,dx$ using four subintervals.

Solution

The endpoints of the subintervals consist of elements of the set $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and $\Delta x = \frac{1-0}{4} = \frac{1}{4}.$

Thus,

$$\begin{aligned}\int_0^1 x^2\,dx &\approx \frac{1}{2}\cdot\frac{1}{4}\left(f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right)\\ &= \frac{1}{8}\left(0 + 2\cdot\frac{1}{16} + 2\cdot\frac{1}{4} + 2\cdot\frac{9}{16} + 1\right)\\ &= \frac{11}{32}.\end{aligned}$$

3.23 Use the trapezoidal rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x}dx.$

Absolute and Relative Error

An important aspect of using these numerical approximation rules consists of calculating the error in using them for estimating the value of a definite integral. We first need to define **absolute error** and **relative error**.

Definition

If B is our estimate of some quantity having an actual value of $A,$ then the absolute error is given by $|A - B|.$ The relative error is the error as a percentage of the absolute value and is given by $\left|\frac{A-B}{A}\right| = \left|\frac{A-B}{A}\right|\cdot 100\%.$

Example 3.42

Calculating Error in the Midpoint Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the midpoint rule, found in **Example 3.39**.

Solution

The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $M_4 = \frac{21}{64}.$ Thus, the absolute error is given by $\left|\left(\frac{1}{3}\right) - \left(\frac{21}{64}\right)\right| = \frac{1}{192} \approx 0.0052.$ The relative error is

$$\frac{1/192}{1/3} = \frac{1}{64} \approx 0.015625 \approx 1.6\%.$$

Example 3.43

Calculating Error in the Trapezoidal Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the trapezoidal rule, found in **Example 3.41**.

Solution

The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $T_4 = \frac{11}{32}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{11}{32}\right| = \frac{1}{96} \approx 0.0104$. The relative error is given by

$$\frac{1/96}{1/3} = 0.03125 \approx 3.1\%.$$

3.24 In an earlier checkpoint, we estimated $\int_1^2 \frac{1}{x}dx$ to be $\frac{24}{35}$ using T_2. The actual value of this integral is $\ln 2$. Using $\frac{24}{35} \approx 0.6857$ and $\ln 2 \approx 0.6931$, calculate the absolute error and the relative error.

In the two previous examples, we were able to compare our estimate of an integral with the actual value of the integral; however, we do not typically have this luxury. In general, if we are approximating an integral, we are doing so because we cannot compute the exact value of the integral itself easily. Therefore, it is often helpful to be able to determine an upper bound for the error in an approximation of an integral. The following theorem provides error bounds for the midpoint and trapezoidal rules. The theorem is stated without proof.

Theorem 3.5: Error Bounds for the Midpoint and Trapezoidal Rules

Let $f(x)$ be a continuous function over $[a, b]$, having a second derivative $f''(x)$ over this interval. If M is the maximum value of $|f''(x)|$ over $[a, b]$, then the upper bounds for the error in using M_n and T_n to estimate $\int_a^b f(x)dx$ are

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \tag{3.12}$$

and

$$\text{Error in } T_n \le \frac{M(b-a)^3}{12n^2}. \tag{3.13}$$

We can use these bounds to determine the value of n necessary to guarantee that the error in an estimate is less than a specified value.

Example 3.44

Determining the Number of Intervals to Use

What value of n should be used to guarantee that an estimate of $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.01 if we use the midpoint rule?

Solution

We begin by determining the value of M, the maximum value of $|f''(x)|$ over $[0, 1]$ for $f(x) = e^{x^2}$. Since $f'(x) = 2xe^{x^2}$, we have

$$f''(x) = 2e^{x^2} + 4x^2 e^{x^2}.$$

Thus,

$$|f''(x)| = 2e^{x^2}\left(1 + 2x^2\right) \le 2 \cdot e \cdot 3 = 6e.$$

From the error-bound **Equation 3.12**, we have

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \le \frac{6e(1-0)^3}{24n^2} = \frac{6e}{24n^2}.$$

Now we solve the following inequality for n:

$$\frac{6e}{24n^2} \le 0.01.$$

Thus, $n \ge \sqrt{\frac{600e}{24}} \approx 8.24$. Since n must be an integer satisfying this inequality, a choice of $n = 9$ would guarantee that $\left|\int_0^1 e^{x^2}\,dx - M_n\right| < 0.01$.

Analysis

We might have been tempted to round 8.24 down and choose $n = 8$, but this would be incorrect because we must have an integer greater than or equal to 8.24. We need to keep in mind that the error estimates provide an upper bound only for the error. The actual estimate may, in fact, be a much better approximation than is indicated by the error bound.

3.25 Use **Equation 3.13** to find an upper bound for the error in using M_4 to estimate $\int_0^1 x^2\,dx$.

Simpson's Rule

With the midpoint rule, we estimated areas of regions under curves by using rectangles. In a sense, we approximated the curve with piecewise constant functions. With the trapezoidal rule, we approximated the curve by using piecewise linear functions. What if we were, instead, to approximate a curve using piecewise quadratic functions? With **Simpson's rule**, we do just this. We partition the interval into an even number of subintervals, each of equal width. Over the first pair of subintervals we approximate $\int_{x_0}^{x_2} f(x)dx$ with $\int_{x_0}^{x_2} p(x)dx,$ where $p(x) = Ax^2 + Bx + C$ is the quadratic function passing through $(x_0, f(x_0)),$ $(x_1, f(x_1)),$ and $(x_2, f(x_2))$ (**Figure 3.16**). Over the next pair of subintervals we approximate $\int_{x_2}^{x_4} f(x)dx$ with the integral of another quadratic function passing through $(x_2, f(x_2)),$ $(x_3, f(x_3)),$ and $(x_4, f(x_4)).$ This process is continued with each successive pair of subintervals.

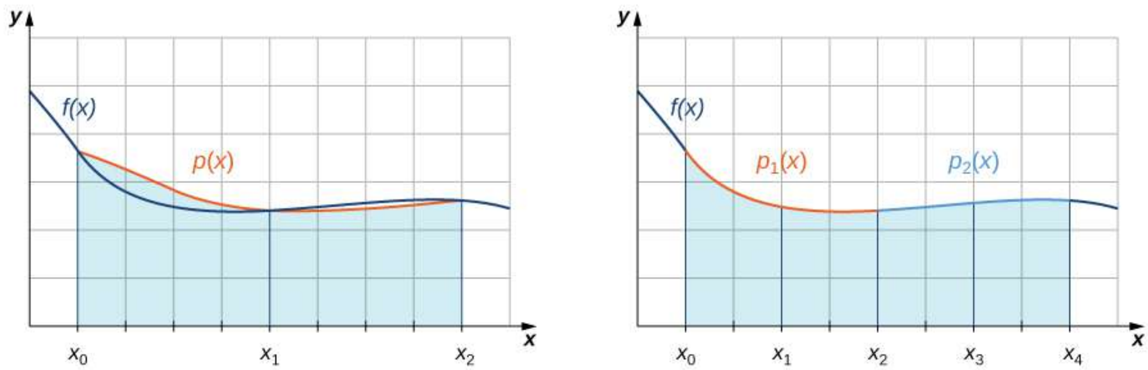


Figure 3.16 With Simpson's rule, we approximate a definite integral by integrating a piecewise quadratic function.

To understand the formula that we obtain for Simpson's rule, we begin by deriving a formula for this approximation over the first two subintervals. As we go through the derivation, we need to keep in mind the following relationships:

$$f(x_0) = p(x_0) = Ax_0{}^2 + Bx_0 + C$$
$$f(x_1) = p(x_1) = Ax_1{}^2 + Bx_1 + C$$
$$f(x_2) = p(x_2) = Ax_2{}^2 + Bx_2 + C$$

$x_2 - x_0 = 2\Delta x,$ where Δx is the length of a subinterval.

$$x_2 + x_0 = 2x_1, \text{ since } x_1 = \frac{(x_2 + x_0)}{2}.$$

Thus,

$$
\begin{aligned}
\int_{x_0}^{x_2} f(x)dx &\approx \int_{x_0}^{x_2} p(x)dx \\
&= \int_{x_0}^{x_2} (Ax^2 + Bx + C)dx \\
&= \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx \Big|_{x_0}^{x_2} && \text{Find the antiderivative.} \\
&= \frac{A}{3}\left(x_2{}^3 - x_0{}^3\right) + \frac{B}{2}\left(x_2{}^2 - x_0{}^2\right) + C(x_2 - x_0) && \text{Evaluate the antiderivative.} \\
&= \frac{A}{3}(x_2 - x_0)\left(x_2{}^2 + x_2 x_0 + x_0{}^2\right) \\
&\quad + \frac{B}{2}(x_2 - x_0)(x_2 + x_0) + C(x_2 - x_0) \\
&= \frac{x_2 - x_0}{6}\left(2A\left(x_2{}^2 + x_2 x_0 + x_0{}^2\right) + 3B(x_2 + x_0) + 6C\right) && \text{Factor out } \frac{x_2 - x_0}{6}. \\
&= \frac{\Delta x}{3}\left(\left(Ax_2{}^2 + Bx_2 + C\right) + (Ax_0{}^2 + Bx_0 + C)\right. \\
&\quad \left. + A\left(x_2{}^2 + 2x_2 x_0 + x_0{}^2\right) + 2B(x_2 + x_0) + 4C\right) && \text{Rearrange the terms.} \\
&= \frac{\Delta x}{3}\left(f(x_2) + f(x_0) + A(x_2 + x_0)^2 + 2B(x_2 + x_0) + 4C\right) && \text{Factor and substitute. } f(x_2) = Ax_0{}^2 + Bx_0 + C \text{ and } f(x_0) = Ax_0{}^2 + Bx_0 + C. \\
&= \frac{\Delta x}{3}\left(f(x_2) + f(x_0) + A(2x_1)^2 + 2B(2x_1) + 4C\right) && \text{Substitute } x_2 + x_0 = 2x_1. \\
&= \frac{\Delta x}{3}(f(x_2) + 4f(x_1) + f(x_0)). && \text{Expand and substitute } f(x_1) = Ax_1{}^2 + Bx_1 + .
\end{aligned}
$$

If we approximate $\int_{x_2}^{x_4} f(x)dx$ using the same method, we see that we have

$$\int_{x_0}^{x_4} f(x)dx \approx \frac{\Delta x}{3}(f(x_4) + 4f(x_3) + f(x_2)).$$

Combining these two approximations, we get

$$\int_{x_0}^{x_4} f(x)dx = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)).$$

The pattern continues as we add pairs of subintervals to our approximation. The general rule may be stated as follows.

Theorem 3.6: Simpson's Rule

Assume that $f(x)$ is continuous over $[a, b]$. Let n be a positive even integer and $\Delta x = \frac{b - a}{n}$. Let $[a, b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2, \ldots, x_n\}$. Set

$$S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)). \qquad (3.14)$$

Then,

$$\lim_{n \to +\infty} S_n = \int_a^b f(x)dx.$$

Just as the trapezoidal rule is the average of the left-hand and right-hand rules for estimating definite integrals, Simpson's rule may be obtained from the midpoint and trapezoidal rules by using a weighted average. It can be shown that $S_{2n} = \left(\frac{2}{3}\right)M_n + \left(\frac{1}{3}\right)T_n.$

It is also possible to put a bound on the error when using Simpson's rule to approximate a definite integral. The bound in the error is given by the following rule:

Rule: Error Bound for Simpson's Rule

Let $f(x)$ be a continuous function over $[a, b]$ having a fourth derivative, $f^{(4)}(x)$, over this interval. If M is the maximum value of $\left|f^{(4)}(x)\right|$ over $[a, b]$, then the upper bound for the error in using S_n to estimate $\int_a^b f(x)dx$ is given by

$$\text{Error in } S_n \le \frac{M(b-a)^5}{180n^4}. \tag{3.15}$$

Example 3.45

Applying Simpson's Rule 1

Use S_2 to approximate $\int_0^1 x^3\,dx.$ Estimate a bound for the error in S_2.

Solution

Since $[0, 1]$ is divided into two intervals, each subinterval has length $\Delta x = \frac{1-0}{2} = \frac{1}{2}.$ The endpoints of these subintervals are $\left\{0, \frac{1}{2}, 1\right\}.$ If we set $f(x) = x^3,$ then

$S_4 = \frac{1}{3} \cdot \frac{1}{2}\left(f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right) = \frac{1}{6}\left(0 + 4 \cdot \frac{1}{8} + 1\right) = \frac{1}{4}.$ Since $f^{(4)}(x) = 0$ and consequently $M = 0,$ we see that

$$\text{Error in } S_2 \le \frac{0(1)^5}{180 \cdot 2^4} = 0.$$

This bound indicates that the value obtained through Simpson's rule is exact. A quick check will verify that, in fact, $\int_0^1 x^3\,dx = \frac{1}{4}.$

Example 3.46

Applying Simpson's Rule 2

Use S_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ over $[1, 4]$.

Solution

The length of $y = \frac{1}{2}x^2$ over $[1, 4]$ is $\int_1^4 \sqrt{1 + x^2}\,dx$. If we divide $[1, 4]$ into six subintervals, then each subinterval has length $\Delta x = \frac{4 - 1}{6} = \frac{1}{2}$, and the endpoints of the subintervals are $\left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$. Setting $f(x) = \sqrt{1 + x^2}$,

$$S_6 = \frac{1}{3} \cdot \frac{1}{2}\left(f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right).$$

After substituting, we have

$$\begin{aligned} S_6 &= \frac{1}{6}(1.4142 + 4 \cdot 1.80278 + 2 \cdot 2.23607 + 4 \cdot 2.69258 + 2 \cdot 3.16228 + 4 \cdot 3.64005 + 4.12311) \\ &\approx 8.14594. \end{aligned}$$

3.26 Use S_2 to estimate $\int_1^2 \frac{1}{x}\,dx$.

3.6 EXERCISES

Approximate the following integrals using either the midpoint rule, trapezoidal rule, or Simpson's rule as indicated. (Round answers to three decimal places.)

299. $\int_1^2 \frac{dx}{x}$; trapezoidal rule; $n = 5$

300. $\int_0^3 \sqrt{4 + x^3}\,dx$; trapezoidal rule; $n = 6$

301. $\int_0^3 \sqrt{4 + x^3}\,dx$; Simpson's rule; $n = 3$

302. $\int_0^{12} x^2\,dx$; midpoint rule; $n = 6$

303. $\int_0^1 \sin^2(\pi x)dx$; midpoint rule; $n = 3$

304. Use the midpoint rule with eight subdivisions to estimate $\int_2^4 x^2\,dx$.

305. Use the trapezoidal rule with four subdivisions to estimate $\int_2^4 x^2\,dx$.

306. Find the exact value of $\int_2^4 x^2\,dx$. Find the error of approximation between the exact value and the value calculated using the trapezoidal rule with four subdivisions. Draw a graph to illustrate.

Approximate the integral to three decimal places using the indicated rule.

307. $\int_0^1 \sin^2(\pi x)dx$; trapezoidal rule; $n = 6$

308. $\int_0^3 \frac{1}{1 + x^3}dx$; trapezoidal rule; $n = 6$

309. $\int_0^3 \frac{1}{1 + x^3}dx$; Simpson's rule; $n = 3$

310. $\int_0^{0.8} e^{-x^2}\,dx$; trapezoidal rule; $n = 4$

311. $\int_0^{0.8} e^{-x^2}\,dx$; Simpson's rule; $n = 4$

312. $\int_0^{0.4} \sin(x^2)dx$; trapezoidal rule; $n = 4$

313. $\int_0^{0.4} \sin(x^2)dx$; Simpson's rule; $n = 4$

314. $\int_{0.1}^{0.5} \frac{\cos x}{x}dx$; trapezoidal rule; $n = 4$

315. $\int_{0.1}^{0.5} \frac{\cos x}{x}dx$; Simpson's rule; $n = 4$

316. Evaluate $\int_0^1 \frac{dx}{1 + x^2}$ exactly and show that the result is $\pi/4$. Then, find the approximate value of the integral using the trapezoidal rule with $n = 4$ subdivisions. Use the result to approximate the value of π.

317. Approximate $\int_2^4 \frac{1}{\ln x}dx$ using the midpoint rule with four subdivisions to four decimal places.

318. Approximate $\int_2^4 \frac{1}{\ln x}dx$ using the trapezoidal rule with eight subdivisions to four decimal places.

319. Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3\,dx$ to four decimal places.

320. Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3\,dx$. Compare this value with the exact value and find the error estimate.

321. Using Simpson's rule with four subdivisions, find $\int_0^{\pi/2} \cos(x)dx$.

322. Show that the exact value of $\int_0^1 xe^{-x}\,dx = 1 - \frac{2}{e}$. Find the absolute error if you approximate the integral using the midpoint rule with 16 subdivisions.

323. Given $\int_0^1 xe^{-x}\,dx = 1 - \frac{2}{e}$, use the trapezoidal rule with 16 subdivisions to approximate the integral and find the absolute error.

324. Find an upper bound for the error in estimating $\int_0^3 (5x+4)dx$ using the trapezoidal rule with six steps.

325. Find an upper bound for the error in estimating $\int_4^5 \frac{1}{(x-1)^2}dx$ using the trapezoidal rule with seven subdivisions.

326. Find an upper bound for the error in estimating $\int_0^3 (6x^2-1)dx$ using Simpson's rule with $n=10$ steps.

327. Find an upper bound for the error in estimating $\int_2^5 \frac{1}{x-1}dx$ using Simpson's rule with $n=10$ steps.

328. Find an upper bound for the error in estimating $\int_0^{\pi} 2x\cos(x)dx$ using Simpson's rule with four steps.

329. Estimate the minimum number of subintervals needed to approximate the integral $\int_1^4 \left(5x^2+8\right)dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

330. Determine a value of *n* such that the trapezoidal rule will approximate $\int_0^1 \sqrt{1+x^2}\,dx$ with an error of no more than 0.01.

331. Estimate the minimum number of subintervals needed to approximate the integral $\int_2^3 \left(2x^3+4x\right)dx$ with an error of magnitude less than 0.0001 using the trapezoidal rule.

332. Estimate the minimum number of subintervals needed to approximate the integral $\int_3^4 \frac{1}{(x-1)^2}dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

333. Use Simpson's rule with four subdivisions to approximate the area under the probability density function $y=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ from $x=0$ to $x=0.4$.

334. Use Simpson's rule with $n=14$ to approximate (to three decimal places) the area of the region bounded by the graphs of $y=0,\quad x=0,$ and $x=\pi/2$.

335. The length of one arch of the curve $y=3\sin(2x)$ is given by $L=\int_0^{\pi/2}\sqrt{1+36\cos^2(2x)}\,dx$. Estimate *L* using the trapezoidal rule with $n=6$.

336. The length of the ellipse $x=a\cos(t),\ y=b\sin(t),\ 0\le t\le 2\pi$ is given by $L=4a\int_0^{\pi/2}\sqrt{1-e^2\cos^2(t)}\,dt$, where e is the eccentricity of the ellipse. Use Simpson's rule with $n=6$ subdivisions to estimate the length of the ellipse when $a=2$ and $e=1/3$.

337. Estimate the area of the surface generated by revolving the curve $y=\cos(2x),\ 0\le x\le\frac{\pi}{4}$ about the *x*-axis. Use the trapezoidal rule with six subdivisions.

338. Estimate the area of the surface generated by revolving the curve $y=2x^2,\quad 0\le x\le 3$ about the *x*-axis. Use Simpson's rule with $n=6$.

339. The growth rate of a certain tree (in feet) is given by $y=\frac{2}{t+1}+e^{-t^2/2}$, where *t* is time in years. Estimate the growth of the tree through the end of the second year by using Simpson's rule, using two subintervals. (Round the answer to the nearest hundredth.)

340. **[T]** Use a calculator to approximate $\int_0^1 \sin(\pi x)dx$ using the midpoint rule with 25 subdivisions. Compute the relative error of approximation.

341. **[T]** Given $\int_1^5 \left(3x^2-2x\right)dx=100$, approximate the value of this integral using the midpoint rule with 16 subdivisions and determine the absolute error.

342. Given that we know the Fundamental Theorem of Calculus, why would we want to develop numerical methods for definite integrals?

343. The table represents the coordinates (x, y) that give the boundary of a lot. The units of measurement are meters. Use the trapezoidal rule to estimate the number of square meters of land that is in this lot.

x	*y*	*x*	*y*
0	125	600	95
100	125	700	88
200	120	800	75
300	112	900	35
400	90	1000	0
500	90		

344. Choose the correct answer. When Simpson's rule is used to approximate the definite integral, it is necessary that the number of partitions be____

a. an even number
b. odd number
c. either an even or an odd number
d. a multiple of 4

345. The "Simpson" sum is based on the area under a ____.

346. The error formula for Simpson's rule depends on___.

a. $f(x)$
b. $f'(x)$
c. $f^{(4)}(x)$
d. the number of steps

3.7 | Improper Integrals

Learning Objectives

- **3.7.1** Evaluate an integral over an infinite interval.
- **3.7.2** Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
- **3.7.3** Use the comparison theorem to determine whether a definite integral is convergent.

Is the area between the graph of $f(x) = \frac{1}{x}$ and the *x*-axis over the interval $[1, +\infty)$ finite or infinite? If this same region is revolved about the *x*-axis, is the volume finite or infinite? Surprisingly, the area of the region described is infinite, but the volume of the solid obtained by revolving this region about the *x*-axis is finite.

In this section, we define integrals over an infinite interval as well as integrals of functions containing a discontinuity on the interval. Integrals of these types are called improper integrals. We examine several techniques for evaluating improper integrals, all of which involve taking limits.

Integrating over an Infinite Interval

How should we go about defining an integral of the type $\int_a^{+\infty} f(x)dx$? We can integrate $\int_a^t f(x)dx$ for any value of t, so it is reasonable to look at the behavior of this integral as we substitute larger values of t. **Figure 3.17** shows that $\int_a^t f(x)dx$ may be interpreted as area for various values of t. In other words, we may define an improper integral as a limit, taken as one of the limits of integration increases or decreases without bound.

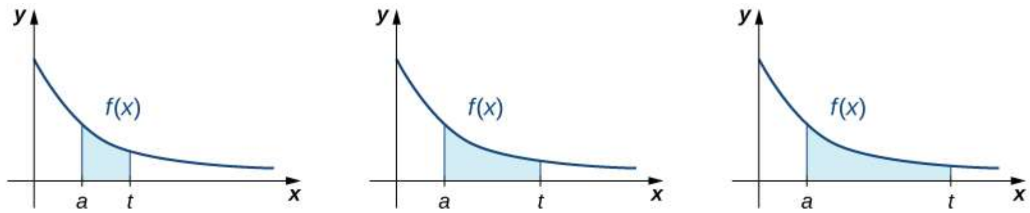


Figure 3.17 To integrate a function over an infinite interval, we consider the limit of the integral as the upper limit increases without bound.

Definition

1. Let $f(x)$ be continuous over an interval of the form $[a, +\infty)$. Then

$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx, \tag{3.16}$$

provided this limit exists.

2. Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then

$$\int_{-\infty}^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx, \tag{3.17}$$

provided this limit exists.
In each case, if the limit exists, then the **improper integral** is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. Let $f(x)$ be continuous over $(-\infty, +\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{+\infty} f(x)dx, \tag{3.18}$$

provided that $\int_{-\infty}^{0} f(x)dx$ and $\int_{0}^{+\infty} f(x)dx$ both converge. If either of these two integrals diverge, then $\int_{-\infty}^{+\infty} f(x)dx$ diverges. (It can be shown that, in fact, $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx$ for any value of a.)

In our first example, we return to the question we posed at the start of this section: Is the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ finite or infinite?

Example 3.47

Finding an Area

Determine whether the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ is finite or infinite.

Solution

We first do a quick sketch of the region in question, as shown in the following graph.

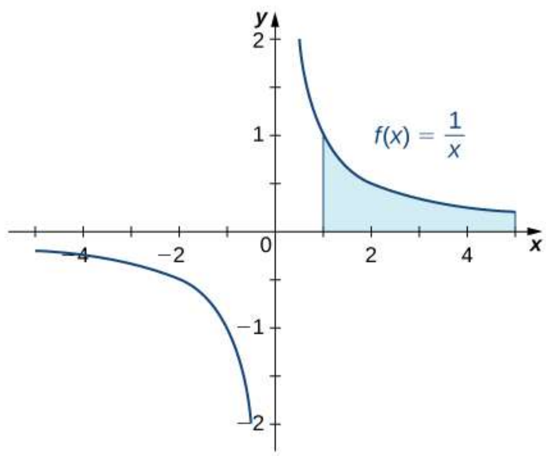


Figure 3.18 We can find the area between the curve $f(x) = 1/x$ and the x-axis on an infinite interval.

We can see that the area of this region is given by $A = \int_{1}^{\infty} \frac{1}{x} dx$. Then we have

$$\begin{aligned}
A &= \int_1^{\infty} \frac{1}{x}dx \\
&= \lim_{t \to +\infty} \int_1^t \frac{1}{x}dx && \text{Rewrite the improper integral as a limit.} \\
&= \lim_{t \to +\infty} \ln|x| \Big|_1^t && \text{Find the antiderivative.} \\
&= \lim_{t \to +\infty} (\ln|t| - \ln 1) && \text{Evaluate the antiderivative.} \\
&= +\infty. && \text{Evaluate the limit.}
\end{aligned}$$

Since the improper integral diverges to $+\infty$, the area of the region is infinite.

Example 3.48

Finding a Volume

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \frac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ about the x-axis.

Solution

The solid is shown in **Figure 3.19**. Using the disk method, we see that the volume V is

$$V = \pi \int_1^{+\infty} \frac{1}{x^2}dx.$$

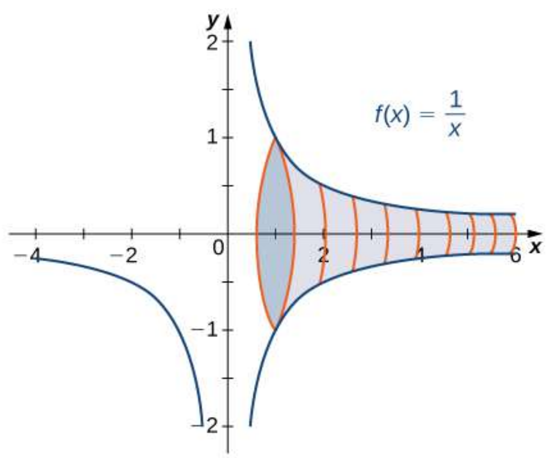


Figure 3.19 The solid of revolution can be generated by rotating an infinite area about the x-axis.

Then we have

$$\begin{aligned} V &= \pi \int_1^{+\infty} \frac{1}{x^2} dx \\ &= \pi \lim_{t \to +\infty} \int_1^t \frac{1}{x^2} dx && \text{Rewrite as a limit.} \\ &= \pi \lim_{t \to +\infty} -\frac{1}{x}\Big|_1^t && \text{Find the antiderivative.} \\ &= \pi \lim_{t \to +\infty} \left(-\frac{1}{t} + 1\right) && \text{Evaluate the antiderivative.} \\ &= \pi. \end{aligned}$$

The improper integral converges to π. Therefore, the volume of the solid of revolution is π.

In conclusion, although the area of the region between the x-axis and the graph of $f(x) = 1/x$ over the interval $[1, +\infty)$ is infinite, the volume of the solid generated by revolving this region about the x-axis is finite. The solid generated is known as *Gabriel's Horn*.

Visit this **website (http://www.openstaxcollege.org/l/20_GabrielsHorn)** to read more about Gabriel's Horn.

Example 3.49

Chapter Opener: Traffic Accidents in a City

Figure 3.20 (credit: modification of work by David McKelvey, Flickr)

In the chapter opener, we stated the following problem: Suppose that at a busy intersection, traffic accidents occur at an average rate of one every three months. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the 8-month interval without an accident a result of chance?

Probability theory tells us that if the average time between events is k, the probability that X, the time between events, is between a and b is given by

$$P(a \le x \le b) = \int_a^b f(x)dx \text{ where } f(x) = \begin{cases} 0 \text{ if } x < 0 \\ ke^{-kx} \text{ if } x \ge 0 \end{cases}.$$

Thus, if accidents are occurring at a rate of one every 3 months, then the probability that X, the time between accidents, is between a and b is given by

$$P(a \le x \le b) = \int_a^b f(x)dx \text{ where } f(x) = \begin{cases} 0 \text{ if } x < 0 \\ 3e^{-3x} \text{ if } x \ge 0 \end{cases}.$$

To answer the question, we must compute $P(X \ge 8) = \int_8^{+\infty} 3e^{-3x}\,dx$ and decide whether it is likely that 8 months could have passed without an accident if there had been no improvement in the traffic situation.

Solution

We need to calculate the probability as an improper integral:

$$\begin{aligned} P(X \ge 8) &= \int_8^{+\infty} 3e^{-3x}\,dx \\ &= \lim_{t \to +\infty} \int_8^t 3e^{-3x}\,dx \\ &= \lim_{t \to +\infty} -e^{-3x}\Big|_8^t \\ &= \lim_{t \to +\infty} (-e^{-3t} + e^{-24}) \\ &\approx 3.8 \times 10^{-11}. \end{aligned}$$

The value 3.8×10^{-11} represents the probability of no accidents in 8 months under the initial conditions. Since this value is very, very small, it is reasonable to conclude the changes were effective.

Example 3.50

Evaluating an Improper Integral over an Infinite Interval

Evaluate $\int_{-\infty}^0 \frac{1}{x^2+4}dx$. State whether the improper integral converges or diverges.

Solution

Begin by rewriting $\int_{-\infty}^0 \frac{1}{x^2+4}dx$ as a limit using **Equation 3.17** from the definition. Thus,

$$\begin{aligned} \int_{-\infty}^0 \frac{1}{x^2+4}dx &= \lim_{t \to -\infty} \int_t^0 \frac{1}{x^2+4}dx && \text{Rewrite as a limit.} \\ &= \lim_{t \to -\infty} \tan^{-1}\frac{x}{2}\Big|_t^0 && \text{Find the antiderivative.} \\ &= \lim_{t \to -\infty} (\tan^{-1} 0 - \tan^{-1}\frac{t}{2}) && \text{Evaluate the antiderivative.} \\ &= \frac{\pi}{2}. && \text{Evaluate the limit and simplify.} \end{aligned}$$

The improper integral converges to $\frac{\pi}{2}$.

Example 3.51

Evaluating an Improper Integral on $(-\infty, +\infty)$

Evaluate $\int_{-\infty}^{+\infty} xe^x\,dx$. State whether the improper integral converges or diverges.

Solution

Start by splitting up the integral:

$$\int_{-\infty}^{+\infty} xe^x\,dx = \int_{-\infty}^{0} xe^x\,dx + \int_{0}^{+\infty} xe^x\,dx.$$

If either $\int_{-\infty}^{0} xe^x\,dx$ or $\int_{0}^{+\infty} xe^x\,dx$ diverges, then $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges. Compute each integral separately.

For the first integral,

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t \to -\infty} \int_{t}^{0} xe^x\,dx$$ Rewrite as a limit.

$$= \lim_{t \to -\infty} (xe^x - e^x)\Big|_t^0$$ Use integration by parts to find he antiderivative. (Here $u = x$ and $dv = e^x$.)

$$= \lim_{t \to -\infty} \left(-1 - te^t + e^t\right)$$ Evaluate the antiderivative.

$$= -1.$$ Evaluate the limit. *Note:* $\lim_{t \to -\infty} te^t$ is indeterminate of the form $0 \cdot \infty$. Thus, $\lim_{t \to -\infty} te^t = \lim_{t \to -\infty} \frac{t}{e^{-t}} = \lim_{t \to -\infty} \frac{-1}{e^{-t}} = \lim_{t \to -\infty} -e^t = 0$ by L'Hôpital's Rule.

The first improper integral converges. For the second integral,

$$\int_{0}^{+\infty} xe^x\,dx = \lim_{t \to +\infty} \int_{0}^{t} xe^x\,dx$$ Rewrite as a limit.

$$= \lim_{t \to +\infty} (xe^x - e^x)\Big|_0^t$$ Find the antiderivative.

$$= \lim_{t \to +\infty} \left(te^t - e^t + 1\right)$$ Evaluate the antiderivative.

$$= \lim_{t \to +\infty} \left((t-1)e^t + 1\right)$$ Rewrite. ($te^t - e^t$ is indeterminate.)

$$= +\infty.$$ Evaluate the limit.

Thus, $\int_{0}^{+\infty} xe^x\,dx$ diverges. Since this integral diverges, $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges as well.

3.27 Evaluate $\int_{-3}^{+\infty} e^{-x}\,dx$. State whether the improper integral converges or diverges.

Integrating a Discontinuous Integrand

Now let's examine integrals of functions containing an infinite discontinuity in the interval over which the integration occurs. Consider an integral of the form $\int_a^b f(x)dx,$ where $f(x)$ is continuous over $[a, b)$ and discontinuous at $b.$ Since the function $f(x)$ is continuous over $[a, t]$ for all values of t satisfying $a < t < b,$ the integral $\int_a^t f(x)dx$ is defined for all such values of $t.$ Thus, it makes sense to consider the values of $\int_a^t f(x)dx$ as t approaches b for $a < t < b.$ That is, we define $\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx,$ provided this limit exists. **Figure 3.21** illustrates $\int_a^t f(x)dx$ as areas of regions for values of t approaching $b.$

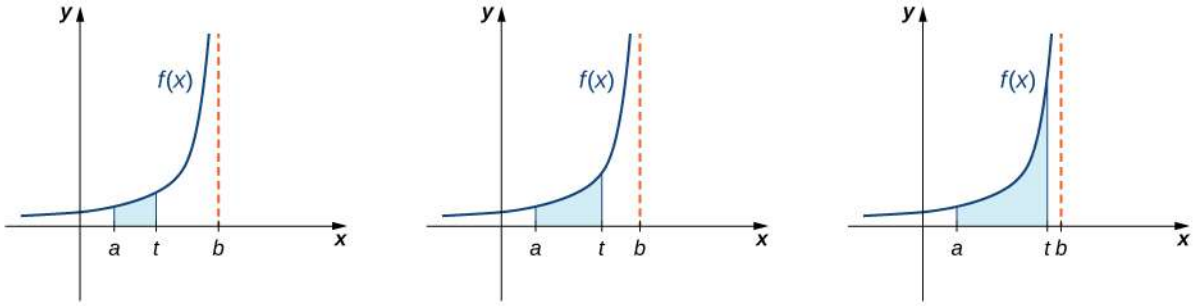


Figure 3.21 As *t* approaches *b* from the left, the value of the area from *a* to *t* approaches the area from *a* to *b*.

We use a similar approach to define $\int_a^b f(x)dx,$ where $f(x)$ is continuous over $(a, b]$ and discontinuous at $a.$ We now proceed with a formal definition.

Definition

1. Let $f(x)$ be continuous over $[a, b).$ Then,

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx. \quad (3.19)$$

2. Let $f(x)$ be continuous over $(a, b].$ Then,

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx. \quad (3.20)$$

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. If $f(x)$ is continuous over $[a, b]$ except at a point c in $(a, b),$ then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad (3.21)$$

provided both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)dx$ diverges.

The following examples demonstrate the application of this definition.

Example 3.52

Integrating a Discontinuous Integrand

Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}dx,$ if possible. State whether the integral converges or diverges.

Solution

The function $f(x) = \frac{1}{\sqrt{4-x}}$ is continuous over $[0, 4)$ and discontinuous at 4. Using **Equation 3.19** from the definition, rewrite $\int_0^4 \frac{1}{\sqrt{4-x}}dx$ as a limit:

$$\begin{aligned} \int_0^4 \frac{1}{\sqrt{4-x}}dx &= \lim_{t \to 4^-} \int_0^t \frac{1}{\sqrt{4-x}}dx && \text{Rewrite as a limit.} \\ &= \lim_{t \to 4^-} \left(-2\sqrt{4-x}\right)\Big|_0^t && \text{Find the antiderivative.} \\ &= \lim_{t \to 4^-} \left(-2\sqrt{4-t}+4\right) && \text{Evaluate the antiderivative.} \\ &= 4. && \text{Evaluate the limit.} \end{aligned}$$

The improper integral converges.

Example 3.53

Integrating a Discontinuous Integrand

Evaluate $\int_0^2 x\ln x\,dx.$ State whether the integral converges or diverges.

Solution

Since $f(x) = x\ln x$ is continuous over $(0, 2]$ and is discontinuous at zero, we can rewrite the integral in limit form using **Equation 3.20**:

$$\begin{aligned} \int_0^2 x\ln x\,dx &= \lim_{t \to 0^+} \int_t^2 x\ln x\,dx && \text{Rewrite as a limit.} \\ &= \lim_{t \to 0^+} \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big|_t^2 && \text{Evaluate } \int x\ln x\,dx \text{ using integration by parts with } u = \ln x \text{ and } dv = x. \\ &= \lim_{t \to 0^+} \left(2\ln 2 - 1 - \frac{1}{2}t^2\ln t + \frac{1}{4}t^2\right). && \text{Evaluate the antiderivative.} \\ &= 2\ln 2 - 1. && \text{Evaluate the limit. } \lim_{t \to 0^+} t^2\ln t \text{ is indeterminate. To evaluate it, rewrite as a quotient and apply L'Hôpital's rule.} \end{aligned}$$

The improper integral converges.

Example 3.54

Integrating a Discontinuous Integrand

Evaluate $\int_{-1}^{1} \frac{1}{x^3} dx.$ State whether the improper integral converges or diverges.

Solution

Since $f(x) = 1/x^3$ is discontinuous at zero, using **Equation 3.21**, we can write

$$\int_{-1}^{1} \frac{1}{x^3} dx = \int_{-1}^{0} \frac{1}{x^3} dx + \int_{0}^{1} \frac{1}{x^3} dx.$$

If either of the two integrals diverges, then the original integral diverges. Begin with $\int_{-1}^{0} \frac{1}{x^3} dx$:

$$\begin{aligned}
\int_{-1}^{0} \frac{1}{x^3} dx &= \lim_{t \to 0^-} \int_{-1}^{t} \frac{1}{x^3} dx && \text{Rewrite as a limit.} \\
&= \lim_{t \to 0^-} \left(-\frac{1}{2x^2} \right) \Big|_{-1}^{t} && \text{Find the antiderivative.} \\
&= \lim_{t \to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{2} \right) && \text{Evaluate the antiderivative.} \\
&= +\infty. && \text{Evaluate the limit.}
\end{aligned}$$

Therefore, $\int_{-1}^{0} \frac{1}{x^3} dx$ diverges. Since $\int_{-1}^{0} \frac{1}{x^3} dx$ diverges, $\int_{-1}^{1} \frac{1}{x^3} dx$ diverges.

3.28 Evaluate $\int_{0}^{2} \frac{1}{x} dx.$ State whether the integral converges or diverges.

A Comparison Theorem

It is not always easy or even possible to evaluate an improper integral directly; however, by comparing it with another carefully chosen integral, it may be possible to determine its convergence or divergence. To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \le f(x) \le g(x)$ for $x \ge a$ (**Figure 3.22**). In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas, so we have the relationship

$$0 \le \int_{a}^{t} f(x)dx \le \int_{a}^{t} g(x)dx \text{ for } t \ge a.$$

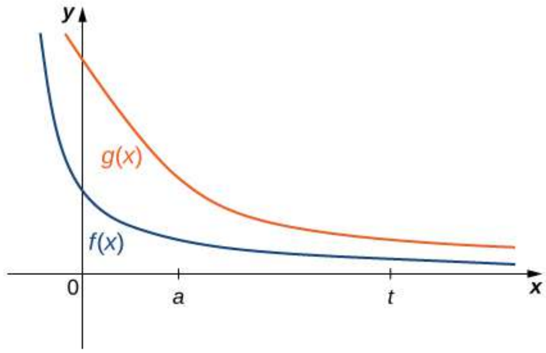


Figure 3.22 If $0 \le f(x) \le g(x)$ for $x \ge a$, then for $t \ge a$, $\int_a^t f(x)dx \le \int_a^t g(x)dx$.

Thus, if

$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx = +\infty,$$

then

$\int_a^{+\infty} g(x)dx = \lim_{t \to +\infty} \int_a^t g(x)dx = +\infty$ as well. That is, if the area of the region between the graph of $f(x)$ and the x-axis over $[a, +\infty)$ is infinite, then the area of the region between the graph of $g(x)$ and the x-axis over $[a, +\infty)$ is infinite too.

On the other hand, if

$\int_a^{+\infty} g(x)dx = \lim_{t \to +\infty} \int_a^t g(x)dx = L$ for some real number L, then

$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$ must converge to some value less than or equal to L, since $\int_a^t f(x)dx$ increases as t increases and $\int_a^t f(x)dx \le L$ for all $t \ge a$.

If the area of the region between the graph of $g(x)$ and the x-axis over $[a, +\infty)$ is finite, then the area of the region between the graph of $f(x)$ and the x-axis over $[a, +\infty)$ is also finite.

These conclusions are summarized in the following theorem.

Theorem 3.7: A Comparison Theorem

Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \le f(x) \le g(x)$ for $x \ge a$.

i. If $\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx = +\infty$, then $\int_a^{+\infty} g(x)dx = \lim_{t \to +\infty} \int_a^t g(x)dx = +\infty$.

ii. If $\int_a^{+\infty} g(x)dx = \lim_{t \to +\infty} \int_a^t g(x)dx = L$, where L is a real number, then $\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx = M$ for some real number $M \le L$.

Example 3.55

Applying the Comparison Theorem

Use a comparison to show that $\int_1^{+\infty} \frac{1}{xe^x} dx$ converges.

Solution

We can see that

$$0 \le \frac{1}{xe^x} \le \frac{1}{e^x} = e^{-x},$$

so if $\int_1^{+\infty} e^{-x}\, dx$ converges, then so does $\int_1^{+\infty} \frac{1}{xe^x} dx$. To evaluate $\int_1^{+\infty} e^{-x}\, dx$, first rewrite it as a limit:

$$\begin{aligned}\int_1^{+\infty} e^{-x} dx &= \lim_{t \to +\infty} \int_1^t e^{-x}\, dx \\ &= \lim_{t \to +\infty} (-e^{-x})\Big|_1^t \\ &= \lim_{t \to +\infty} \left(-e^{-t} + e^1\right) \\ &= e^1.\end{aligned}$$

Since $\int_1^{+\infty} e^{-x}\, dx$ converges, so does $\int_1^{+\infty} \frac{1}{xe^x} dx$.

Example 3.56

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^{+\infty} \frac{1}{x^p} dx$ diverges for all $p < 1$.

Solution

For $p < 1$, $1/x \le 1/(x^p)$ over $[1, +\infty)$. In **Example 3.47**, we showed that $\int_1^{+\infty} \frac{1}{x} dx = +\infty$. Therefore, $\int_1^{+\infty} \frac{1}{x^p} dx$ diverges for all $p < 1$.

3.29 Use a comparison to show that $\int_e^{+\infty} \frac{\ln x}{x} dx$ diverges.

Student PROJECT

Laplace Transforms

In the last few chapters, we have looked at several ways to use integration for solving real-world problems. For this next project, we are going to explore a more advanced application of integration: integral transforms. Specifically, we describe the Laplace transform and some of its properties. The Laplace transform is used in engineering and physics to simplify the computations needed to solve some problems. It takes functions expressed in terms of time and *transforms* them to functions expressed in terms of frequency. It turns out that, in many cases, the computations needed to solve problems in the frequency domain are much simpler than those required in the time domain.

The Laplace transform is defined in terms of an integral as

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt.$$

Note that the input to a Laplace transform is a function of time, $f(t)$, and the output is a function of frequency, $F(s)$. Although many real-world examples require the use of complex numbers (involving the imaginary number $i = \sqrt{-1}$), in this project we limit ourselves to functions of real numbers.

Let's start with a simple example. Here we calculate the Laplace transform of $f(t) = t$. We have

$$L\{t\} = \int_0^{\infty} te^{-st} dt.$$

This is an improper integral, so we express it in terms of a limit, which gives

$$L\{t\} = \int_0^{\infty} te^{-st} dt = \lim_{z \to \infty} \int_0^z te^{-st} dt.$$

Now we use integration by parts to evaluate the integral. Note that we are integrating with respect to *t*, so we treat the variable *s* as a constant. We have

$$\begin{aligned} u &= t & dv &= e^{-st} dt \\ du &= dt & v &= -\tfrac{1}{s}e^{-st}. \end{aligned}$$

Then we obtain

$$\begin{aligned}
\lim_{z \to \infty} \int_0^z te^{-st} dt &= \lim_{z \to \infty} \left[\left[-\tfrac{t}{s}e^{-st} \right]\Big|_0^z + \tfrac{1}{s} \int_0^z e^{-st} dt \right] \\
&= \lim_{z \to \infty} \left[\left[-\tfrac{z}{s}e^{-sz} + \tfrac{0}{s}e^{-0s} \right] + \tfrac{1}{s} \int_0^z e^{-st} dt \right] \\
&= \lim_{z \to \infty} \left[\left[-\tfrac{z}{s}e^{-sz} + 0 \right] - \tfrac{1}{s} \left[\tfrac{e^{-st}}{s} \right]\Big|_0^z \right] \\
&= \lim_{z \to \infty} \left[\left[-\tfrac{z}{s}e^{-sz} \right] - \tfrac{1}{s^2} \left[e^{-sz} - 1 \right] \right] \\
&= \lim_{z \to \infty} \left[-\frac{z}{se^{sz}} \right] - \lim_{z \to \infty} \left[\frac{1}{s^2 e^{sz}} \right] + \lim_{z \to \infty} \frac{1}{s^2} \\
&= 0 - 0 + \frac{1}{s^2} \\
&= \frac{1}{s^2}.
\end{aligned}$$

1. Calculate the Laplace transform of $f(t) = 1$.

2. Calculate the Laplace transform of $f(t) = e^{-3t}$.

3. Calculate the Laplace transform of $f(t) = t^2$. (Note, you will have to integrate by parts twice.)

Laplace transforms are often used to solve differential equations. Differential equations are not covered in detail until later in this book; but, for now, let's look at the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

Let's start with the definition of the Laplace transform. We have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \lim_{z \to \infty} \int_0^z e^{-st} f(t)dt.$$

4. Use integration by parts to evaluate $\lim_{z \to \infty} \int_0^z e^{-st} f(t)dt$. (Let $u = f(t)$ and $dv = e^{-st}dt$.)

After integrating by parts and evaluating the limit, you should see that

$$L\{f(t)\} = \frac{f(0)}{s} + \frac{1}{s}[L\{f'(t)\}].$$

Then,

$$L\{f'(t)\} = sL\{f(t)\} - f(0).$$

Thus, differentiation in the time domain simplifies to multiplication by s in the frequency domain.

The final thing we look at in this project is how the Laplace transforms of $f(t)$ and its antiderivative are related. Let $g(t) = \int_0^t f(u)du$. Then,

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t)dt = \lim_{z \to \infty} \int_0^z e^{-st} g(t)dt.$$

5. Use integration by parts to evaluate $\lim_{z \to \infty} \int_0^z e^{-st} g(t)dt$. (Let $u = g(t)$ and $dv = e^{-st}dt$. Note, by the way, that we have defined $g(t)$, $du = f(t)dt$.)

As you might expect, you should see that

$$L\{g(t)\} = \frac{1}{s} \cdot L\{f(t)\}.$$

Integration in the time domain simplifies to division by s in the frequency domain.

3.7 EXERCISES

Evaluate the following integrals. If the integral is not convergent, answer "divergent."

347. $\int_2^4 \frac{dx}{(x-3)^2}$

348. $\int_0^{\infty} \frac{1}{4+x^2}dx$

349. $\int_0^2 \frac{1}{\sqrt{4-x^2}}dx$

350. $\int_1^{\infty} \frac{1}{x\ln x}dx$

351. $\int_1^{\infty} xe^{-x}\,dx$

352. $\int_{-\infty}^{\infty} \frac{x}{x^2+1}dx$

353. Without integrating, determine whether the integral $\int_1^{\infty} \frac{1}{\sqrt{x^3+1}}dx$ converges or diverges by comparing the function $f(x) = \frac{1}{\sqrt{x^3+1}}$ with $g(x) = \frac{1}{\sqrt{x^3}}$.

354. Without integrating, determine whether the integral $\int_1^{\infty} \frac{1}{\sqrt{x+1}}dx$ converges or diverges.

Determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.

355. $\int_0^{\infty} e^{-x}\cos x\,dx$

356. $\int_1^{\infty} \frac{\ln x}{x}dx$

357. $\int_0^1 \frac{\ln x}{\sqrt{x}}dx$

358. $\int_0^1 \ln x\,dx$

359. $\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx$

360. $\int_1^5 \frac{dx}{\sqrt{x-1}}$

361. $\int_{-2}^2 \frac{dx}{(1+x)^2}$

362. $\int_0^{\infty} e^{-x}\,dx$

363. $\int_0^{\infty} \sin x\,dx$

364. $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}dx$

365. $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

366. $\int_0^2 \frac{dx}{x^3}$

367. $\int_{-1}^2 \frac{dx}{x^3}$

368. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

369. $\int_0^3 \frac{1}{x-1}dx$

370. $\int_1^{\infty} \frac{5}{x^3}dx$

371. $\int_3^5 \frac{5}{(x-4)^2}dx$

Determine the convergence of each of the following integrals by comparison with the given integral. If the integral converges, find the number to which it converges.

372. $\int_1^{\infty} \frac{dx}{x^2+4x}$; compare with $\int_1^{\infty} \frac{dx}{x^2}$.

373. $\int_1^{\infty} \frac{dx}{\sqrt{x}+1}$; compare with $\int_1^{\infty} \frac{dx}{2\sqrt{x}}$.

Evaluate the integrals. If the integral diverges, answer "diverges."

374. $\int_1^{\infty} \frac{dx}{x^e}$

375. $\int_0^1 \frac{dx}{x^{\pi}}$

376. $\int_0^1 \frac{dx}{\sqrt{1-x}}$

377. $\int_0^1 \frac{dx}{1-x}$

378. $\int_{-\infty}^0 \frac{dx}{x^2+1}$

379. $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

380. $\int_0^1 \frac{\ln x}{x} dx$

381. $\int_0^e \ln(x) dx$

382. $\int_0^{\infty} xe^{-x} dx$

383. $\int_{-\infty}^{\infty} \frac{x}{\left(x^2+1\right)^2} dx$

384. $\int_0^{\infty} e^{-x} dx$

Evaluate the improper integrals. Each of these integrals has an infinite discontinuity either at an endpoint or at an interior point of the interval.

385. $\int_0^9 \frac{dx}{\sqrt{9-x}}$

386. $\int_{-27}^1 \frac{dx}{x^{2/3}}$

387. $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

388. $\int_6^{24} \frac{dt}{t\sqrt{t^2-36}}$

389. $\int_0^4 x\ln(4x) dx$

390. $\int_0^3 \frac{x}{\sqrt{9-x^2}} dx$

391. Evaluate $\int_{.5}^t \frac{dx}{\sqrt{1-x^2}}$. (Be careful!) (Express your answer using three decimal places.)

392. Evaluate $\int_1^4 \frac{dx}{\sqrt{x^2-1}}$. (Express the answer in exact form.)

393. Evaluate $\int_2^{\infty} \frac{dx}{(x^2-1)^{3/2}}$.

394. Find the area of the region in the first quadrant between the curve $y = e^{-6x}$ and the x-axis.

395. Find the area of the region bounded by the curve $y = \frac{7}{x^2}$, the x-axis, and on the left by $x = 1$.

396. Find the area under the curve $y = \frac{1}{(x+1)^{3/2}}$, bounded on the left by $x = 3$.

397. Find the area under $y = \frac{5}{1+x^2}$ in the first quadrant.

398. Find the volume of the solid generated by revolving about the x-axis the region under the curve $y = \frac{3}{x}$ from $x = 1$ to $x = \infty$.

399. Find the volume of the solid generated by revolving about the y-axis the region under the curve $y = 6e^{-2x}$ in the first quadrant.

400. Find the volume of the solid generated by revolving about the x-axis the area under the curve $y = 3e^{-x}$ in the first quadrant.

The Laplace transform of a continuous function over the interval $[0, \infty)$ is defined by $F(s) = \int_0^{\infty} e^{-sx} f(x) dx$ (see the Student Project). This definition is used to solve some important initial-value problems in differential equations, as discussed later. The domain of F is the set of all real numbers s such that the improper integral converges. Find the Laplace transform F of each of the following functions and give the domain of F.

401. $f(x) = 1$

402. $f(x) = x$

403. $f(x) = \cos(2x)$

404. $f(x) = e^{ax}$

405. Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is 2π.

A function is a probability density function if it satisfies the following definition: $\int_{-\infty}^{\infty} f(t)dt = 1$. The probability that a random variable x lies between a and b is given by $P(a \leq x \leq b) = \int_a^b f(t)dt$.

406. Show that $f(x) = \begin{cases} 0 \text{ if } x < 0 \\ 7e^{-7x} \text{ if } x \geq 0 \end{cases}$ is a probability density function.

407. Find the probability that x is between 0 and 0.3. (Use the function defined in the preceding problem.) Use four-place decimal accuracy.

CHAPTER 3 REVIEW

KEY TERMS

absolute error if B is an estimate of some quantity having an actual value of A, then the absolute error is given by $|A-B|$

computer algebra system (CAS) technology used to perform many mathematical tasks, including integration

improper integral an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

integration by parts a technique of integration that allows the exchange of one integral for another using the formula $\int u\,dv = uv - \int v\,du$

integration table a table that lists integration formulas

midpoint rule a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where m_i is the midpoint of the ith subinterval to approximate $\int_a^b f(x)dx$

numerical integration the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

partial fraction decomposition a technique used to break down a rational function into the sum of simple rational functions

power reduction formula a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

relative error error as a percentage of the absolute value, given by $\left|\frac{A-B}{A}\right| = \left|\frac{A-B}{A}\right| \cdot 100\%$

Simpson's rule a rule that approximates $\int_a^b f(x)dx$ using the integrals of a piecewise quadratic function. The approximation S_n to $\int_a^b f(x)dx$ is given by $S_n = \frac{\Delta x}{3}\begin{pmatrix} f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) \\ + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \end{pmatrix}$

trapezoidal rule a rule that approximates $\int_a^b f(x)dx$ using trapezoids

trigonometric integral an integral involving powers and products of trigonometric functions

trigonometric substitution an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$ into a trigonometric integral

KEY EQUATIONS

- **Integration by parts formula**
  $\int u\,dv = uv - \int v\,du$
- **Integration by parts for definite integrals**
  $\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions.

- **Sine Products**
  $\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$
- **Sine and Cosine Products**
  $\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$
- **Cosine Products**
  $\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$
- **Power Reduction Formula**
  $\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-1} x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$
- **Power Reduction Formula**
  $\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$
- **Midpoint rule**
  $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$
- **Trapezoidal rule**
  $T_n = \frac{1}{2}\Delta x\big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\big)$
- **Simpson's rule**
  $S_n = \frac{\Delta x}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big)$
- **Error bound for midpoint rule**
  Error in $M_n \le \frac{M(b-a)^3}{24n^2}$
- **Error bound for trapezoidal rule**
  Error in $T_n \le \frac{M(b-a)^3}{12n^2}$
- **Error bound for Simpson's rule**
  Error in $S_n \le \frac{M(b-a)^5}{180n^4}$
- **Improper integrals**
  $\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$
  $\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx$
  $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{+\infty} f(x)dx$

KEY CONCEPTS

3.1 Integration by Parts

- The integration-by-parts formula allows the exchange of one integral for another, possibly easier, integral.
- Integration by parts applies to both definite and indefinite integrals.

3.2 Trigonometric Integrals

- Integrals of trigonometric functions can be evaluated by the use of various strategies. These strategies include
  1. Applying trigonometric identities to rewrite the integral so that it may be evaluated by *u*-substitution
  2. Using integration by parts
  3. Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions
  4. Applying reduction formulas

3.3 Trigonometric Substitution

- For integrals involving $\sqrt{a^2 - x^2}$, use the substitution $x = a\sin\theta$ and $dx = a\cos\theta d\theta$.
- For integrals involving $\sqrt{a^2 + x^2}$, use the substitution $x = a\tan\theta$ and $dx = a\sec^2\theta d\theta$.
- For integrals involving $\sqrt{x^2 - a^2}$, substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$.

3.4 Partial Fractions

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must make sure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.
- The form the decomposition takes depends on the type of factors in the denominator. The types of factors include nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

3.5 Other Strategies for Integration

- An integration table may be used to evaluate indefinite integrals.
- A CAS (or computer algebra system) may be used to evaluate indefinite integrals.
- It may require some effort to reconcile equivalent solutions obtained using different methods.

3.6 Numerical Integration

- We can use numerical integration to estimate the values of definite integrals when a closed form of the integral is difficult to find or when an approximate value only of the definite integral is needed.
- The most commonly used techniques for numerical integration are the midpoint rule, trapezoidal rule, and Simpson’s rule.
- The midpoint rule approximates the definite integral using rectangular regions whereas the trapezoidal rule approximates the definite integral using trapezoidal approximations.
- Simpson’s rule approximates the definite integral by first approximating the original function using piecewise quadratic functions.

3.7 Improper Integrals

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.

- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

CHAPTER 3 REVIEW EXERCISES

For the following exercises, determine whether the statement is true or false. Justify your answer with a proof or a counterexample.

408. $\int e^x \sin(x)dx$ cannot be integrated by parts.

409. $\int \frac{1}{x^4 + 1}dx$ cannot be integrated using partial fractions.

410. In numerical integration, increasing the number of points decreases the error.

411. Integration by parts can always yield the integral.

For the following exercises, evaluate the integral using the specified method.

412. $\int x^2 \sin(4x)dx$ using integration by parts

413. $\int \frac{1}{x^2\sqrt{x^2 + 16}}dx$ using trigonometric substitution

414. $\int \sqrt{x}\ln(x)dx$ using integration by parts

415. $\int \frac{3x}{x^3 + 2x^2 - 5x - 6}dx$ using partial fractions

416. $\int \frac{x^5}{\left(4x^2 + 4\right)^{5/2}}dx$ using trigonometric substitution

417. $\int \frac{\sqrt{4 - \sin^2(x)}}{\sin^2(x)}\cos(x)dx$ using a table of integrals or a CAS

For the following exercises, integrate using whatever method you choose.

418. $\int \sin^2(x)\cos^2(x)dx$

419. $\int x^3\sqrt{x^2 + 2}\,dx$

420. $\int \frac{3x^2 + 1}{x^4 - 2x^3 - x^2 + 2x}dx$

421. $\int \frac{1}{x^4 + 4}dx$

422. $\int \frac{\sqrt{3 + 16x^4}}{x^4}dx$

For the following exercises, approximate the integrals using the midpoint rule, trapezoidal rule, and Simpson's rule using four subintervals, rounding to three decimals.

423. **[T]** $\int_1^2 \sqrt{x^5 + 2}\,dx$

424. **[T]** $\int_0^{\sqrt{\pi}} e^{-\sin(x^2)}dx$

425. **[T]** $\int_1^4 \frac{\ln(1/x)}{x}dx$

For the following exercises, evaluate the integrals, if possible.

426. $\int_1^\infty \frac{1}{x^n}dx,$ for what values of n does this integral converge or diverge?

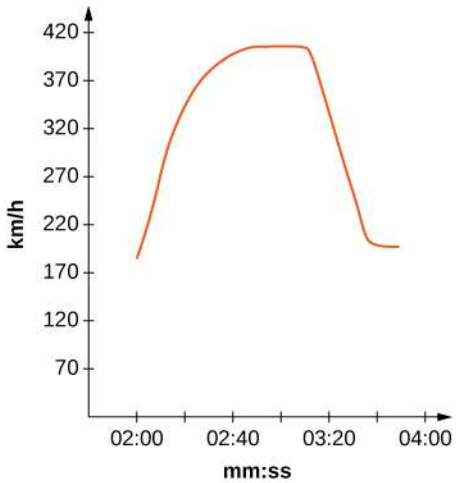
427. $\int_1^\infty \frac{e^{-x}}{x}dx$

For the following exercises, consider the gamma function given by $\Gamma(a) = \int_0^\infty e^{-y}y^{a-1}dy.$

428. Show that $\Gamma(a) = (a - 1)\Gamma(a - 1).$

429. Extend to show that $\Gamma(a) = (a-1)!,$ assuming a is a positive integer.

The fastest car in the world, the Bugati Veyron, can reach a top speed of 408 km/h. The graph represents its velocity.



430. **[T]** Use the graph to estimate the velocity every 20 sec and fit to a graph of the form $v(t) = a\exp^{bx}\sin(cx) + d.$ (*Hint:* Consider the time units.)

431. **[T]** Using your function from the previous problem, find exactly how far the Bugati Veyron traveled in the 1 min 40 sec included in the graph.

4 | INTRODUCTION TO DIFFERENTIAL EQUATIONS

Figure 4.1 The white-tailed deer (*Odocoileus virginianus*) of the eastern United States. Differential equations can be used to study animal populations. (credit: modification of work by Rachel Kramer, Flickr)

Chapter Outline

Introduction

Many real-world phenomena can be modeled mathematically by using differential equations. Population growth, radioactive decay, predator-prey models, and spring-mass systems are four examples of such phenomena. In this chapter we study some of these applications.

Suppose we wish to study a population of deer over time and determine the total number of animals in a given area. We can first observe the population over a period of time, estimate the total number of deer, and then use various assumptions to derive a mathematical model for different scenarios. Some factors that are often considered are environmental impact, threshold population values, and predators. In this chapter we see how differential equations can be used to predict populations over time (see **Example 4.14**).

Another goal of this chapter is to develop solution techniques for different types of differential equations. As the equations become more complicated, the solution techniques also become more complicated, and in fact an entire course could be dedicated to the study of these equations. In this chapter we study several types of differential equations and their corresponding methods of solution.

4.1 | Basics of Differential Equations

Learning Objectives

- **4.1.1** Identify the order of a differential equation.
- **4.1.2** Explain what is meant by a solution to a differential equation.
- **4.1.3** Distinguish between the general solution and a particular solution of a differential equation.
- **4.1.4** Identify an initial-value problem.
- **4.1.5** Identify whether a given function is a solution to a differential equation or an initial-value problem.

Calculus is the mathematics of change, and rates of change are expressed by derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y = f(x)$ and its derivative, known as a *differential equation*. Solving such equations often provides information about how quantities change and frequently provides insight into how and why the changes occur.

Techniques for solving differential equations can take many different forms, including direct solution, use of graphs, or computer calculations. We introduce the main ideas in this chapter and describe them in a little more detail later in the course. In this section we study what differential equations are, how to verify their solutions, some methods that are used for solving them, and some examples of common and useful equations.

General Differential Equations

Consider the equation $y' = 3x^2,$ which is an example of a differential equation because it includes a derivative. There is a relationship between the variables x and y: y is an unknown function of x. Furthermore, the left-hand side of the equation is the derivative of y. Therefore we can interpret this equation as follows: Start with some function $y = f(x)$ and take its derivative. The answer must be equal to $3x^2$. What function has a derivative that is equal to $3x^2$? One such function is $y = x^3,$ so this function is considered a **solution to a differential equation**.

Definition

A **differential equation** is an equation involving an unknown function $y = f(x)$ and one or more of its derivatives. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Go to this **website (http://www.openstaxcollege.org/l/20_Differential)** to explore more on this topic.

Some examples of differential equations and their solutions appear in **Table 4.1**.

Equation	Solution
$y' = 2x$	$y = x^2$
$y' + 3y = 6x + 11$	$y = e^{-3x} + 2x + 3$
$y'' - 3y' + 2y = 24e^{-2x}$	$y = 3e^x - 4e^{2x} + 2e^{-2x}$

Table 4.1 Examples of Differential Equations and Their Solutions

Note that a solution to a differential equation is not necessarily unique, primarily because the derivative of a constant is zero. For example, $y = x^2 + 4$ is also a solution to the first differential equation in **Table 4.1**. We will return to this idea a little bit later in this section. For now, let's focus on what it means for a function to be a solution to a differential equation.

Example 4.1

Verifying Solutions of Differential Equations

Verify that the function $y = e^{-3x} + 2x + 3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

Solution

To verify the solution, we first calculate y' using the chain rule for derivatives. This gives $y' = -3e^{-3x} + 2$. Next we substitute y and y' into the left-hand side of the differential equation:

$$(-3e^{-2x} + 2) + 3(e^{-2x} + 2x + 3).$$

The resulting expression can be simplified by first distributing to eliminate the parentheses, giving

$$-3e^{-2x} + 2 + 3e^{-2x} + 6x + 9.$$

Combining like terms leads to the expression $6x + 11,$ which is equal to the right-hand side of the differential equation. This result verifies that $y = e^{-3x} + 2x + 3$ is a solution of the differential equation.

4.1 Verify that $y = 2e^{3x} - 2x - 2$ is a solution to the differential equation $y' - 3y = 6x + 4$.

It is convenient to define characteristics of differential equations that make it easier to talk about them and categorize them. The most basic characteristic of a differential equation is its order.

Definition

The **order of a differential equation** is the highest order of any derivative of the unknown function that appears in the equation.

Example 4.2

Identifying the Order of a Differential Equation

What is the order of each of the following differential equations?

a. $y' - 4y = x^2 - 3x + 4$

b. $x^2 y''' - 3xy'' + xy' - 3y = \sin x$

c. $\frac{4}{x}y^{(4)} - \frac{6}{x^2}y'' + \frac{12}{x^4}y = x^3 - 3x^2 + 4x - 12$

Solution

a. The highest derivative in the equation is y', so the order is 1.

b. The highest derivative in the equation is y''', so the order is 3.

c. The highest derivative in the equation is $y^{(4)}$, so the order is 4.

4.2 What is the order of the following differential equation?

$$\left(x^4 - 3x\right)y^{(5)} - \left(3x^2 + 1\right)y' + 3y = \sin x \cos x$$

General and Particular Solutions

We already noted that the differential equation $y' = 2x$ has at least two solutions: $y = x^2$ and $y = x^2 + 4$. The only difference between these two solutions is the last term, which is a constant. What if the last term is a different constant? Will this expression still be a solution to the differential equation? In fact, any function of the form $y = x^2 + C$, where C represents any constant, is a solution as well. The reason is that the derivative of $x^2 + C$ is $2x$, regardless of the value of C. It can be shown that any solution of this differential equation must be of the form $y = x^2 + C$. This is an example of a **general solution** to a differential equation. A graph of some of these solutions is given in Figure 4.2. (*Note*: in this graph we used even integer values for C ranging between -4 and 4. In fact, there is no restriction on the value of C; it can be an integer or not.)

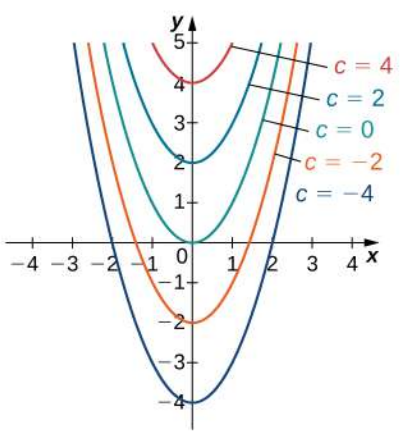


Figure 4.2 Family of solutions to the differential equation $y' = 2x$.

In this example, we are free to choose any solution we wish; for example, $y = x^2 - 3$ is a member of the family of solutions to this differential equation. This is called a **particular solution** to the differential equation. A particular solution can often be uniquely identified if we are given additional information about the problem.

Example 4.3

Finding a Particular Solution

Find the particular solution to the differential equation $y' = 2x$ passing through the point $(2, 7)$.

Solution

Any function of the form $y = x^2 + C$ is a solution to this differential equation. To determine the value of C, we substitute the values $x = 2$ and $y = 7$ into this equation and solve for C:

$$y = x^2 + C$$
$$7 = 2^2 + C = 4 + C$$
$$C = 3.$$

Therefore the particular solution passing through the point $(2, 7)$ is $y = x^2 + 3$.

4.3 Find the particular solution to the differential equation

$$y' = 4x + 3$$

passing through the point $(1, 7)$, given that $y = 2x^2 + 3x + C$ is a general solution to the differential equation.

Initial-Value Problems

Usually a given differential equation has an infinite number of solutions, so it is natural to ask which one we want to use. To choose one solution, more information is needed. Some specific information that can be useful is an **initial value**, which is an ordered pair that is used to find a particular solution.

A differential equation together with one or more initial values is called an **initial-value problem**. The general rule is that the number of initial values needed for an initial-value problem is equal to the order of the differential equation. For example, if we have the differential equation $y' = 2x$, then $y(3) = 7$ is an initial value, and when taken together, these equations form an initial-value problem. The differential equation $y'' - 3y' + 2y = 4e^x$ is second order, so we need two initial values. With initial-value problems of order greater than one, the same value should be used for the independent variable. An example of initial values for this second-order equation would be $y(0) = 2$ and $y'(0) = -1$. These two initial values together with the differential equation form an initial-value problem. These problems are so named because often the independent variable in the unknown function is t, which represents time. Thus, a value of $t = 0$ represents the beginning of the problem.

Example 4.4

Verifying a Solution to an Initial-Value Problem

Verify that the function $y = 2e^{-2t} + e^t$ is a solution to the initial-value problem

$$y' + 2y = 3e^t, \quad y(0) = 3.$$

Solution

For a function to satisfy an initial-value problem, it must satisfy both the differential equation and the initial condition. To show that y satisfies the differential equation, we start by calculating y'. This gives $y' = -4e^{-2t} + e^t$. Next we substitute both y and y' into the left-hand side of the differential equation and simplify:

$$\begin{aligned} y' + 2y &= \left(-4e^{-2t} + e^t\right) + 2\left(2e^{-2t} + e^t\right) \\ &= -4e^{-2t} + e^t + 4e^{-2t} + 2e^t \\ &= 3e^t. \end{aligned}$$

This is equal to the right-hand side of the differential equation, so $y = 2e^{-2t} + e^t$ solves the differential equation. Next we calculate $y(0)$:

$$\begin{aligned} y(0) &= 2e^{-2(0)} + e^0 \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

This result verifies the initial value. Therefore the given function satisfies the initial-value problem.

4.4 Verify that $y = 3e^{2t} + 4\sin t$ is a solution to the initial-value problem

$$y' - 2y = 4\cos t - 8\sin t, \quad y(0) = 3.$$

In **Example 4.4**, the initial-value problem consisted of two parts. The first part was the differential equation $y' + 2y = 3e^x$, and the second part was the initial value $y(0) = 3$. These two equations together formed the initial-value problem.

The same is true in general. An initial-value problem will consists of two parts: the differential equation and the initial condition. The differential equation has a family of solutions, and the initial condition determines the value of C. The family of solutions to the differential equation in **Example 4.4** is given by $y = 2e^{-2t} + Ce^t$. This family of solutions is shown in **Figure 4.3**, with the particular solution $y = 2e^{-2t} + e^t$ labeled.

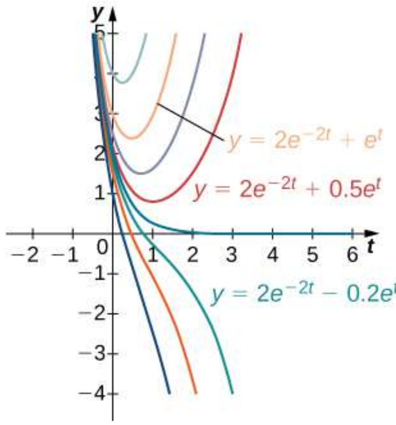


Figure 4.3 A family of solutions to the differential equation $y' + 2y = 3e^t$. The particular solution $y = 2e^{-2t} + e^t$ is labeled.

Example 4.5

Solving an Initial-value Problem

Solve the following initial-value problem:

$$y' = 3e^x + x^2 - 4, \quad y(0) = 5.$$

Solution

The first step in solving this initial-value problem is to find a general family of solutions. To do this, we find an antiderivative of both sides of the differential equation

$$\int y'\,dx = \int \left(3e^x + x^2 - 4\right)dx,$$

namely,

$$y + C_1 = 3e^x + \frac{1}{3}x^3 - 4x + C_2. \quad (4.1)$$

We are able to integrate both sides because the y term appears by itself. Notice that there are two integration constants: C_1 and C_2. Solving **Equation 4.1** for y gives

$$y = 3e^x + \frac{1}{3}x^3 - 4x + C_2 - C_1.$$

Because C_1 and C_2 are both constants, $C_2 - C_1$ is also a constant. We can therefore define $C = C_2 - C_1$, which leads to the equation

$$y = 3e^x + \frac{1}{3}x^3 - 4x + C.$$

Next we determine the value of C. To do this, we substitute $x = 0$ and $y = 5$ into **Equation 4.1** and solve for C:

$$\begin{aligned} 5 &= 3e^0 + \frac{1}{3}0^3 - 4(0) + C \\ 5 &= 3 + C \\ C &= 2. \end{aligned}$$

Now we substitute the value $C = 2$ into **Equation 4.1**. The solution to the initial-value problem is $y = 3e^x + \frac{1}{3}x^3 - 4x + 2.$

Analysis

The difference between a general solution and a particular solution is that a general solution involves a family of functions, either explicitly or implicitly defined, of the independent variable. The initial value or values determine which particular solution in the family of solutions satisfies the desired conditions.

4.5 Solve the initial-value problem

$$y' = x^2 - 4x + 3 - 6e^x, \quad y(0) = 8.$$

In physics and engineering applications, we often consider the forces acting upon an object, and use this information to understand the resulting motion that may occur. For example, if we start with an object at Earth's surface, the primary force acting upon that object is gravity. Physicists and engineers can use this information, along with Newton's second law of motion (in equation form $F = ma$, where F represents force, m represents mass, and a represents acceleration), to derive an equation that can be solved.

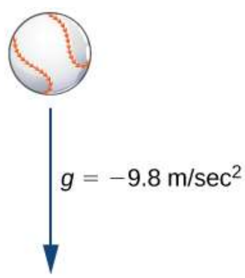


Figure 4.4 For a baseball falling in air, the only force acting on it is gravity (neglecting air resistance).

In **Figure 4.4** we assume that the only force acting on a baseball is the force of gravity. This assumption ignores air resistance. (The force due to air resistance is considered in a later discussion.) The acceleration due to gravity at Earth's surface, $g,$ is approximately $9.8 \text{ m/s}^2.$ We introduce a frame of reference, where Earth's surface is at a height of 0 meters. Let $v(t)$ represent the velocity of the object in meters per second. If $v(t) > 0,$ the ball is rising, and if $v(t) < 0,$ the ball is falling (**Figure 4.5**).

Figure 4.5 Possible velocities for the rising/falling baseball.

Our goal is to solve for the velocity $v(t)$ at any time t. To do this, we set up an initial-value problem. Suppose the mass of the ball is m, where m is measured in kilograms. We use Newton's second law, which states that the force acting on an object is equal to its mass times its acceleration $(F = ma)$. Acceleration is the derivative of velocity, so $a(t) = v'(t)$. Therefore the force acting on the baseball is given by $F = m\,v'(t)$. However, this force must be equal to the force of gravity acting on the object, which (again using Newton's second law) is given by $F_g = -mg$, since this force acts in a downward direction. Therefore we obtain the equation $F = F_g$, which becomes $m\,v'(t) = -mg$. Dividing both sides of the equation by m gives the equation

$$v'(t) = -g.$$

Notice that this differential equation remains the same regardless of the mass of the object.

We now need an initial value. Because we are solving for velocity, it makes sense in the context of the problem to assume that we know the **initial velocity**, or the velocity at time $t = 0$. This is denoted by $v(0) = v_0$.

Example 4.6

Velocity of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kg at Earth's surface.

a. Find the velocity $v(t)$ of the baseball at time t.

b. What is its velocity after 2 seconds?

Solution

a. From the preceding discussion, the differential equation that applies in this situation is

$$v'(t) = -g,$$

where $g = 9.8\text{ m/s}^2$. The initial condition is $v(0) = v_0$, where $v_0 = 10\text{ m/s}$. Therefore the initial-value problem is $v'(t) = -9.8\text{ m/s}^2,\ v(0) = 10\text{ m/s}$.

The first step in solving this initial-value problem is to take the antiderivative of both sides of the differential equation. This gives

$$\begin{aligned} \int v'(t)\,dt &= \int -9.8\,dt \\ v(t) &= -9.8t + C. \end{aligned}$$

The next step is to solve for C. To do this, substitute $t = 0$ and $v(0) = 10$:

$$\begin{aligned} v(t) &= -9.8t + C \\ v(0) &= -9.8(0) + C \\ 10 &= C. \end{aligned}$$

Therefore $C = 10$ and the velocity function is given by $v(t) = -9.8t + 10$.

b. To find the velocity after 2 seconds, substitute $t = 2$ into $v(t)$.

$$\begin{aligned} v(t) &= -9.8t + 10 \\ v(2) &= -9.8(2) + 10 \\ v(2) &= -9.6. \end{aligned}$$

The units of velocity are meters per second. Since the answer is negative, the object is falling at a speed of 9.6 m/s.

4.6 Suppose a rock falls from rest from a height of 100 meters and the only force acting on it is gravity. Find an equation for the velocity $v(t)$ as a function of time, measured in meters per second.

A natural question to ask after solving this type of problem is how high the object will be above Earth's surface at a given point in time. Let $s(t)$ denote the height above Earth's surface of the object, measured in meters. Because velocity is the derivative of position (in this case height), this assumption gives the equation $s'(t) = v(t)$. An initial value is necessary; in this case the initial height of the object works well. Let the initial height be given by the equation $s(0) = s_0$. Together these assumptions give the initial-value problem

$$s'(t) = v(t), \quad s(0) = s_0.$$

If the velocity function is known, then it is possible to solve for the position function as well.

Example 4.7

Height of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kilogram at Earth's surface.

a. Find the position $s(t)$ of the baseball at time t.

b. What is its height after 2 seconds?

Solution

a. We already know the velocity function for this problem is $v(t) = -9.8t + 10$. The initial height of the baseball is 3 meters, so $s_0 = 3$. Therefore the initial-value problem for this example is

To solve the initial-value problem, we first find the antiderivatives:

$$\begin{aligned} \int s'(t)\,dt &= \int -9.8t + 10 dt \\ s(t) &= -4.9t^2 + 10t + C. \end{aligned}$$

Next we substitute $t = 0$ and solve for C:

$$\begin{aligned} s(t) &= -4.9t^2 + 10t + C \\ s(0) &= -4.9(0)^2 + 10(0) + C \\ 3 &= C. \end{aligned}$$

Therefore the position function is $s(t) = -4.9t^2 + 10t + 3$.

b. The height of the baseball after 2 s is given by $s(2)$:

$$\begin{aligned} s(2) &= -4.9(2)^2 + 10(2) + 3 \\ &= -4.9(4) + 23 \\ &= 3.4. \end{aligned}$$

Therefore the baseball is 3.4 meters above Earth's surface after 2 seconds. It is worth noting that the mass of the ball cancelled out completely in the process of solving the problem.

4.1 EXERCISES

Determine the order of the following differential equations.

1. $y' + y = 3y^2$

2. $(y')^2 = y' + 2y$

3. $y''' + y''y' = 3x^2$

4. $y' = y'' + 3t^2$

5. $\frac{dy}{dt} = t$

6. $\frac{dy}{dx} + \frac{d^2y}{dx^2} = 3x^4$

7. $\left(\frac{dy}{dt}\right)^2 + 8\frac{dy}{dt} + 3y = 4t$

Verify that the following functions are solutions to the given differential equation.

8. $y = \frac{x^3}{3}$ solves $y' = x^2$

9. $y = 2e^{-x} + x - 1$ solves $y' = x - y$

10. $y = e^{3x} - \frac{e^x}{2}$ solves $y' = 3y + e^x$

11. $y = \frac{1}{1 - x}$ solves $y' = y^2$

12. $y = e^{x^2/2}$ solves $y' = xy$

13. $y = 4 + \ln x$ solves $xy' = 1$

14. $y = 3 - x + x\ln x$ solves $y' = \ln x$

15. $y = 2e^x - x - 1$ solves $y' = y + x$

16. $y = e^x + \frac{\sin x}{2} - \frac{\cos x}{2}$ solves $y' = \cos x + y$

17. $y = \pi e^{-\cos x}$ solves $y' = y\sin x$

Verify the following general solutions and find the particular solution.

18. Find the particular solution to the differential equation $y' = 4x^2$ that passes through $(-3, -30)$, given that $y = C + \frac{4x^3}{3}$ is a general solution.

19. Find the particular solution to the differential equation $y' = 3x^3$ that passes through $(1, 4.75)$, given that $y = C + \frac{3x^4}{4}$ is a general solution.

20. Find the particular solution to the differential equation $y' = 3x^2 y$ that passes through $(0, 12)$, given that $y = Ce^{x^3}$ is a general solution.

21. Find the particular solution to the differential equation $y' = 2xy$ that passes through $\left(0, \frac{1}{2}\right)$, given that $y = Ce^{x^2}$ is a general solution.

22. Find the particular solution to the differential equation $y' = (2xy)^2$ that passes through $\left(1, -\frac{1}{2}\right)$, given that $y = -\frac{3}{C + 4x^3}$ is a general solution.

23. Find the particular solution to the differential equation $y'x^2 = y$ that passes through $\left(1, \frac{2}{e}\right)$, given that $y = Ce^{-1/x}$ is a general solution.

24. Find the particular solution to the differential equation $8\frac{dx}{dt} = -2\cos(2t) - \cos(4t)$ that passes through (π, π), given that $x = C - \frac{1}{8}\sin(2t) - \frac{1}{32}\sin(4t)$ is a general solution.

25. Find the particular solution to the differential equation $\frac{du}{dt} = \tan u$ that passes through $\left(1, \frac{\pi}{2}\right)$, given that $u = \sin^{-1}\left(e^{C+t}\right)$ is a general solution.

26. Find the particular solution to the differential equation $\frac{dy}{dt} = e^{(t+y)}$ that passes through $(1, 0)$, given that $y = -\ln(C - e^t)$ is a general solution.

27. Find the particular solution to the differential equation $y'(1 - x^2) = 1 + y$ that passes through $(0, -2)$, given that $y = C\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$ is a general solution.

For the following problems, find the general solution to the differential equation.

28. $y' = 3x + e^x$

29. $y' = \ln x + \tan x$

30. $y' = \sin x e^{\cos x}$

31. $y' = 4^x$

32. $y' = \sin^{-1}(2x)$

33. $y' = 2t\sqrt{t^2 + 16}$

34. $x' = \coth t + \ln t + 3t^2$

35. $x' = t\sqrt{4 + t}$

36. $y' = y$

37. $y' = \frac{y}{x}$

Solve the following initial-value problems starting from $y(t = 0) = 1$ and $y(t = 0) = -1$. Draw both solutions on the same graph.

38. $\frac{dy}{dt} = 2t$

39. $\frac{dy}{dt} = -t$

40. $\frac{dy}{dt} = 2y$

41. $\frac{dy}{dt} = -y$

42. $\frac{dy}{dt} = 2$

Solve the following initial-value problems starting from $y_0 = 10$. At what time does y increase to 100 or drop to 1?

43. $\frac{dy}{dt} = 4t$

44. $\frac{dy}{dt} = 4y$

45. $\frac{dy}{dt} = -2y$

46. $\frac{dy}{dt} = e^{4t}$

47. $\frac{dy}{dt} = e^{-4t}$

Recall that a family of solutions includes solutions to a differential equation that differ by a constant. For the following problems, use your calculator to graph a family of solutions to the given differential equation. Use initial conditions from $y(t = 0) = -10$ to $y(t = 0) = 10$ increasing by 2. Is there some critical point where the behavior of the solution begins to change?

48. **[T]** $y' = y(x)$

49. **[T]** $xy' = y$

50. **[T]** $y' = t^3$

51. **[T]** $y' = x + y$ (*Hint:* $y = Ce^x - x - 1$ is the general solution)

52. **[T]** $y' = x \ln x + \sin x$

53. Find the general solution to describe the velocity of a ball of mass 1 lb that is thrown upward at a rate a ft/sec.

54. In the preceding problem, if the initial velocity of the ball thrown into the air is $a = 25$ ft/s, write the particular solution to the velocity of the ball. Solve to find the time when the ball hits the ground.

55. You throw two objects with differing masses m_1 and m_2 upward into the air with the same initial velocity a ft/s. What is the difference in their velocity after 1 second?

56. **[T]** You throw a ball of mass 1 kilogram upward with a velocity of $a = 25$ m/s on Mars, where the force of gravity is $g = -3.711$ m/s^2. Use your calculator to approximate how much longer the ball is in the air on Mars.

57. **[T]** For the previous problem, use your calculator to approximate how much higher the ball went on Mars.

58. **[T]** A car on the freeway accelerates according to $a = 15\cos(\pi t)$, where t is measured in hours. Set up and solve the differential equation to determine the velocity of the car if it has an initial speed of 51 mph. After 40 minutes of driving, what is the driver's velocity?

59. **[T]** For the car in the preceding problem, find the expression for the distance the car has traveled in time t, assuming an initial distance of 0. How long does it take the car to travel 100 miles? Round your answer to hours and minutes.

60. **[T]** For the previous problem, find the total distance traveled in the first hour.

61. Substitute $y = Be^{3t}$ into $y' - y = 8e^{3t}$ to find a particular solution.

62. Substitute $y = a\cos(2t) + b\sin(2t)$ into $y' + y = 4\sin(2t)$ to find a particular solution.

63. Substitute $y = a + bt + ct^2$ into $y' + y = 1 + t^2$ to find a particular solution.

64. Substitute $y = ae^t \cos t + be^t \sin t$ into $y' = 2e^t \cos t$ to find a particular solution.

65. Solve $y' = e^{kt}$ with the initial condition $y(0) = 0$ and solve $y' = 1$ with the same initial condition. As k approaches 0, what do you notice?

4.2 | Direction Fields and Numerical Methods

Learning Objectives

4.2.1 Draw the direction field for a given first-order differential equation.
4.2.2 Use a direction field to draw a solution curve of a first-order differential equation.
4.2.3 Use Euler's Method to approximate the solution to a first-order differential equation.

For the rest of this chapter we will focus on various methods for solving differential equations and analyzing the behavior of the solutions. In some cases it is possible to predict properties of a solution to a differential equation without knowing the actual solution. We will also study numerical methods for solving differential equations, which can be programmed by using various computer languages or even by using a spreadsheet program, such as Microsoft Excel.

Creating Direction Fields

Direction fields (also called slope fields) are useful for investigating first-order differential equations. In particular, we consider a first-order differential equation of the form

$$y' = f(x, y).$$

An applied example of this type of differential equation appears in Newton's law of cooling, which we will solve explicitly later in this chapter. First, though, let us create a direction field for the differential equation

$$T'(t) = -0.4(T - 72).$$

Here $T(t)$ represents the temperature (in degrees Fahrenheit) of an object at time t, and the ambient temperature is $72°\text{F}$. **Figure 4.6** shows the direction field for this equation.

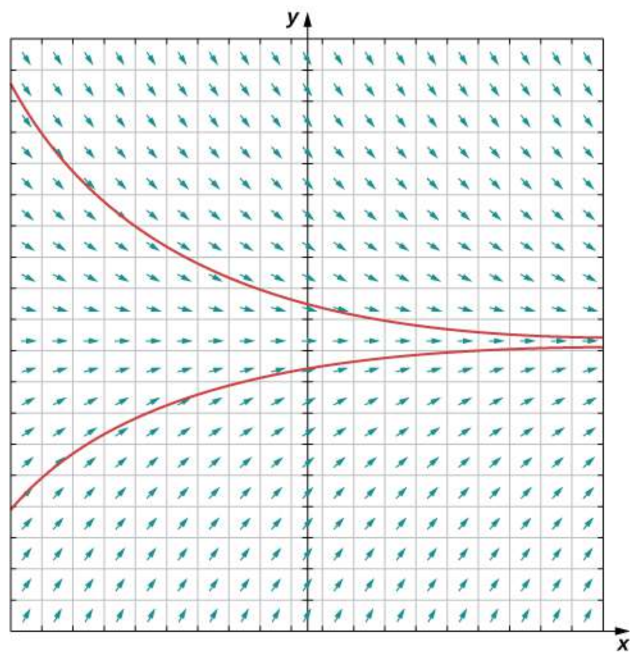


Figure 4.6 Direction field for the differential equation $T'(t) = -0.4(T - 72)$. Two solutions are plotted: one with initial temperature less than $72°\text{F}$ and the other with initial temperature greater than $72°\text{F}$.

The idea behind a direction field is the fact that the derivative of a function evaluated at a given point is the slope of the tangent line to the graph of that function at the same point. Other examples of differential equations for which we can create a direction field include

$$y' = 3x + 2y - 4$$
$$y' = x^2 - y^2$$
$$y' = \frac{2x + 4}{y - 2}.$$

To create a direction field, we start with the first equation: $y' = 3x + 2y - 4.$ We let (x_0, y_0) be any ordered pair, and we substitute these numbers into the right-hand side of the differential equation. For example, if we choose $x = 1$ and $y = 2,$ substituting into the right-hand side of the differential equation yields

$$\begin{aligned} y' &= 3x + 2y - 4 \\ &= 3(1) + 2(2) - 4 = 3. \end{aligned}$$

This tells us that if a solution to the differential equation $y' = 3x + 2y - 4$ passes through the point $(1, 2),$ then the slope of the solution at that point must equal $3.$ To start creating the direction field, we put a short line segment at the point $(1, 2)$ having slope $3.$ We can do this for any point in the domain of the function $f(x, y) = 3x + 2y - 4,$ which consists of all ordered pairs (x, y) in $\mathbb{R}^2.$ Therefore any point in the Cartesian plane has a slope associated with it, assuming that a solution to the differential equation passes through that point. The direction field for the differential equation $y' = 3x + 2y - 4$ is shown in **Figure 4.7**.

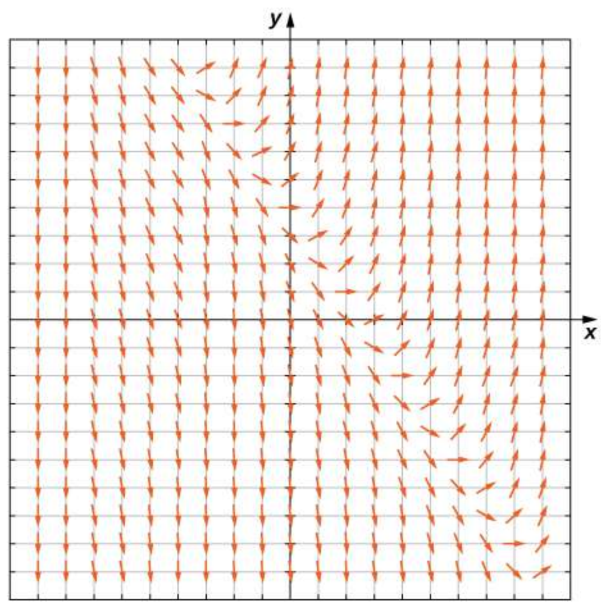


Figure 4.7 Direction field for the differential equation $y' = 3x + 2y - 4.$

We can generate a direction field of this type for any differential equation of the form $y' = f(x, y).$

Definition

A **direction field (slope field)** is a mathematical object used to graphically represent solutions to a first-order differential equation. At each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point.

Using Direction Fields

We can use a direction field to predict the behavior of solutions to a differential equation without knowing the actual solution. For example, the direction field in **Figure 4.7** serves as a guide to the behavior of solutions to the differential equation $y' = 3x + 2y - 4.$

To use a direction field, we start by choosing any point in the field. The line segment at that point serves as a signpost telling us what direction to go from there. For example, if a solution to the differential equation passes through the point $(0,\ 1),$ then the slope of the solution passing through that point is given by $y' = 3(0) + 2(1) - 4 = -2.$ Now let x increase slightly, say to $x = 0.1.$ Using the method of linear approximations gives a formula for the approximate value of y for $x = 0.1.$ In particular,

$$\begin{aligned} L(x) &= y_0 + f'(x_0)(x - x_0) \\ &= 1 - 2(x_0 - 0) \\ &= 1 - 2x_0. \end{aligned}$$

Substituting $x_0 = 0.1$ into $L(x)$ gives an approximate y value of $0.8.$

At this point the slope of the solution changes (again according to the differential equation). We can keep progressing, recalculating the slope of the solution as we take small steps to the right, and watching the behavior of the solution. **Figure 4.8** shows a graph of the solution passing through the point $(0,\ 1).$

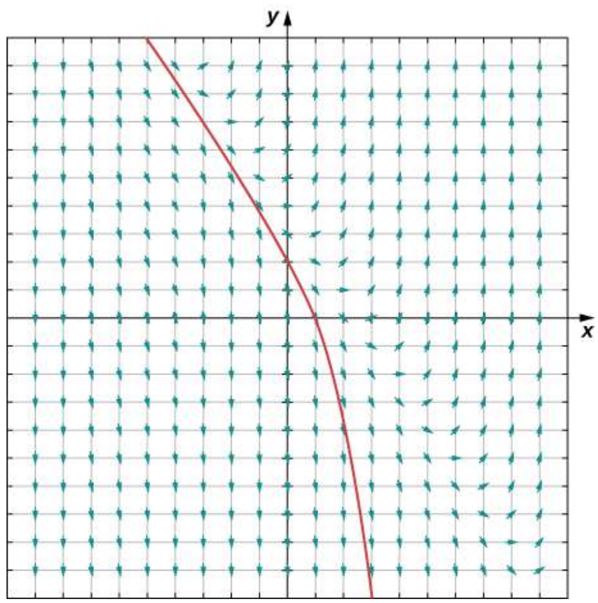


Figure 4.8 Direction field for the differential equation $y' = 3x + 2y - 4$ with the solution passing through the point $(0,\ 1).$

The curve is the graph of the solution to the initial-value problem

$$y' = 3x + 2y - 4, \quad y(0) = 1.$$

This curve is called a **solution curve** passing through the point $(0,\ 1).$ The exact solution to this initial-value problem is

$$y = -\frac{3}{2}x + \frac{5}{4} - \frac{1}{4}e^{2x},$$

and the graph of this solution is identical to the curve in **Figure 4.8**.

4.7 Create a direction field for the differential equation $y' = x^2 - y^2$ and sketch a solution curve passing through the point $(-1,\ 2).$

Go to this **Java applet (http://www.openstaxcollege.org/l/20_DifferEq)** and this **website (http://www.openstaxcollege.org/l/20_SlopeFields)** to see more about slope fields.

Now consider the direction field for the differential equation $y' = (x-3)(y^2-4)$, shown in **Figure 4.9**. This direction field has several interesting properties. First of all, at $y = -2$ and $y = 2$, horizontal dashes appear all the way across the graph. This means that if $y = -2$, then $y' = 0$. Substituting this expression into the right-hand side of the differential equation gives

$$\begin{aligned}(x-3)(y^2-4) &= (x-3)((^{-2}-4)\\ &= (x-3)(0)\\ &= 0\\ &= y'.\end{aligned}$$

Therefore $y = -2$ is a solution to the differential equation. Similarly, $y = 2$ is a solution to the differential equation. These are the only constant-valued solutions to the differential equation, as we can see from the following argument. Suppose $y = k$ is a constant solution to the differential equation. Then $y' = 0$. Substituting this expression into the differential equation yields $0 = (x-3)\left(k^2-4\right)$. This equation must be true for all values of x, so the second factor must equal zero. This result yields the equation $k^2 - 4 = 0$. The solutions to this equation are $k = -2$ and $k = 2$, which are the constant solutions already mentioned. These are called the equilibrium solutions to the differential equation.

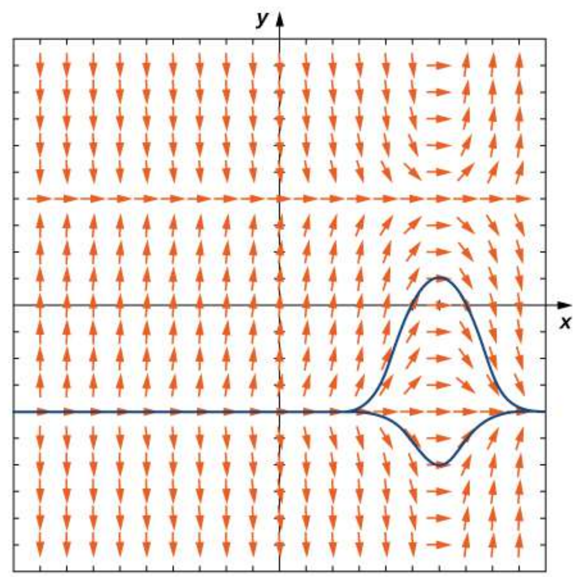


Figure 4.9 Direction field for the differential equation $y' = (x-3)(y^2-4)$ showing two solutions. These solutions are very close together, but one is barely above the equilibrium solution $x = -2$ and the other is barely below the same equilibrium solution.

Definition

Consider the differential equation $y' = f(x, y)$. An **equilibrium solution** is any solution to the differential equation of the form $y = c$, where c is a constant.

To determine the equilibrium solutions to the differential equation $y' = f(x, y)$, set the right-hand side equal to zero. An equilibrium solution of the differential equation is any function of the form $y = k$ such that $f(x, k) = 0$ for all values of x in the domain of f.

An important characteristic of equilibrium solutions concerns whether or not they approach the line $y = k$ as an asymptote for large values of x.

Definition

Consider the differential equation $y' = f(x, y)$, and assume that all solutions to this differential equation are defined for $x \geq x_0$. Let $y = k$ be an equilibrium solution to the differential equation.

1. $y = k$ is an **asymptotically stable solution** to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem

$$y' = f(x, y), \quad y(x_0) = c$$

approaches k as x approaches infinity.

2. $y = k$ is an **asymptotically unstable solution** to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem

$$y' = f(x, y), \quad y(x_0) = c$$

never approaches k as x approaches infinity.

3. $y = k$ is an **asymptotically semi-stable solution** to the differential equation if it is neither asymptotically stable nor asymptotically unstable.

Now we return to the differential equation $y' = (x - 3)(y^2 - 4)$, with the initial condition $y(0) = 0.5$. The direction field for this initial-value problem, along with the corresponding solution, is shown in **Figure 4.10**.

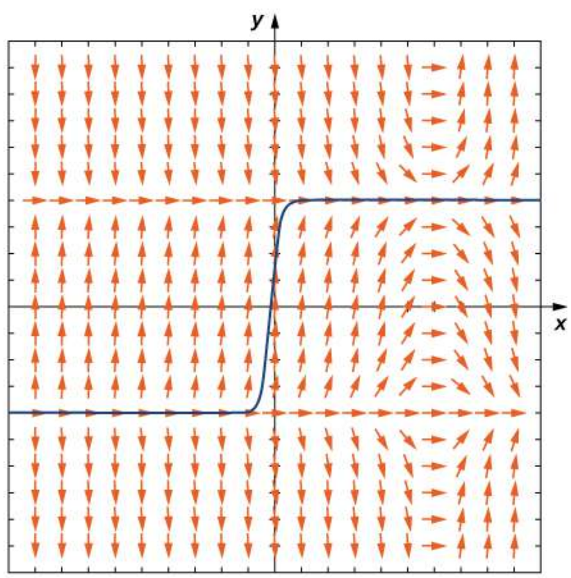


Figure 4.10 Direction field for the initial-value problem $y' = (x - 3)(y^2 - 4),\ y(0) = 0.5$.

The values of the solution to this initial-value problem stay between $y = -2$ and $y = 2$, which are the equilibrium solutions to the differential equation. Furthermore, as x approaches infinity, y approaches 2. The behavior of solutions is similar if the initial value is higher than 2, for example, $y(0) = 2.3$. In this case, the solutions decrease and approach $y = 2$ as x approaches infinity. Therefore $y = 2$ is an asymptotically stable solution to the differential equation.

What happens when the initial value is below $y = -2$? This scenario is illustrated in **Figure 4.11**, with the initial value $y(0) = -3$.

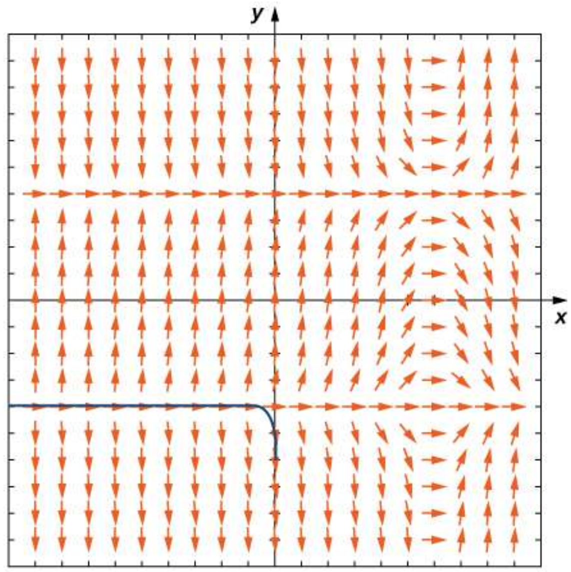


Figure 4.11 Direction field for the initial-value problem $y' = (x - 3)(y^2 - 4), \quad y(0) = -3$.

The solution decreases rapidly toward negative infinity as x approaches infinity. Furthermore, if the initial value is slightly higher than -2, then the solution approaches 2, which is the other equilibrium solution. Therefore in neither case does the solution approach $y = -2$, so $y = -2$ is called an asymptotically unstable, or unstable, equilibrium solution.

Example 4.8

Stability of an Equilibrium Solution

Create a direction field for the differential equation $y' = (y - 3)^2(y^2 + y - 2)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

Solution

The direction field is shown in **Figure 4.12**.

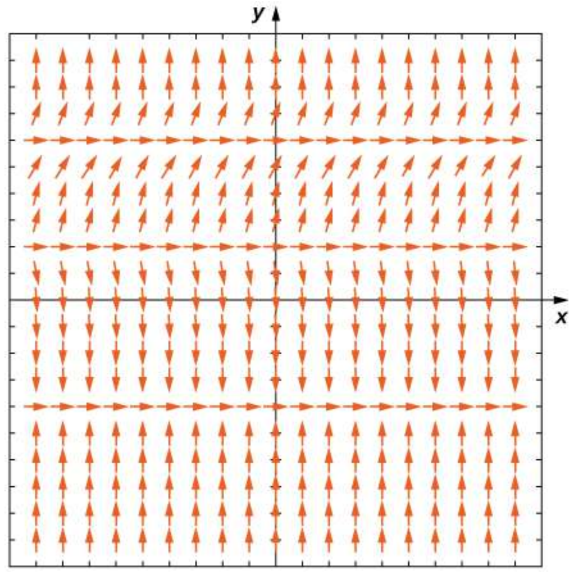


Figure 4.12 Direction field for the differential equation $y' = (y-3)^2(y^2+y-2)$.

The equilibrium solutions are $y = -2$, $y = 1$, and $y = 3$. To classify each of the solutions, look at an arrow directly above or below each of these values. For example, at $y = -2$ the arrows directly below this solution point up, and the arrows directly above the solution point down. Therefore all initial conditions close to $y = -2$ approach $y = -2$, and the solution is stable. For the solution $y = 1$, all initial conditions above and below $y = 1$ are repelled (pushed away) from $y = 1$, so this solution is unstable. The solution $y = 3$ is semi-stable, because for initial conditions slightly greater than 3, the solution approaches infinity, and for initial conditions slightly less than 3, the solution approaches $y = 1$.

Analysis

It is possible to find the equilibrium solutions to the differential equation by setting the right-hand side equal to zero and solving for y. This approach gives the same equilibrium solutions as those we saw in the direction field.

4.8 Create a direction field for the differential equation $y' = (x+5)(y+2)(y^2-4y+4)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

Euler's Method

Consider the initial-value problem

$$y' = 2x - 3, \quad y(0) = 3.$$

Integrating both sides of the differential equation gives $y = x^2 - 3x + C$, and solving for C yields the particular solution $y = x^2 - 3x + 3$. The solution for this initial-value problem appears as the parabola in **Figure 4.13**.

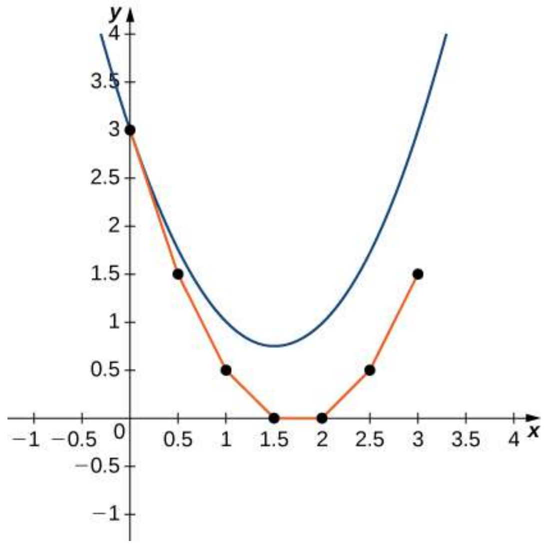


Figure 4.13 Euler's Method for the initial-value problem $y' = 2x - 3, \quad y(0) = 3.$

The red graph consists of line segments that approximate the solution to the initial-value problem. The graph starts at the same initial value of $(0, 3)$. Then the slope of the solution at any point is determined by the right-hand side of the differential equation, and the length of the line segment is determined by increasing the x value by 0.5 each time (the *step size*). This approach is the basis of Euler's Method.

Before we state Euler's Method as a theorem, let's consider another initial-value problem:

$$y' = x^2 - y^2, \quad y(-1) = 2.$$

The idea behind direction fields can also be applied to this problem to study the behavior of its solution. For example, at the point $(-1, 2),$ the slope of the solution is given by $y' = (-1)^2 - 2^2 = -3,$ so the slope of the tangent line to the solution at that point is also equal to $-3.$ Now we define $x_0 = -1$ and $y_0 = 2.$ Since the slope of the solution at this point is equal to $-3,$ we can use the method of linear approximation to approximate y near $(-1, 2).$

$$L(x) = y_0 + f'(x_0)(x - x_0).$$

Here $x_0 = -1,\ y_0 = 2,$ and $f'(x_0) = -3,$ so the linear approximation becomes

$$\begin{aligned} L(x) &= 2 - 3(x - (-1)) \\ &= 2 - 3x - 3 \\ &= -3x - 1. \end{aligned}$$

Now we choose a **step size**. The step size is a small value, typically 0.1 or less, that serves as an increment for x; it is represented by the variable h. In our example, let $h = 0.1.$ Incrementing x_0 by h gives our next x value:

$$x_1 = x_0 + h = -1 + 0.1 = -0.9.$$

We can substitute $x_1 = -0.9$ into the linear approximation to calculate y_1.

$$\begin{aligned} y_1 &= L(x_1) \\ &= -3(-0.9) - 1 \\ &= 1.7. \end{aligned}$$

Therefore the approximate y value for the solution when $x = -0.9$ is $y = 1.7.$ We can then repeat the process, using $x_1 = -0.9$ and $y_1 = 1.7$ to calculate x_2 and y_2. The new slope is given by $y' = (-0.9)^2 - (1.7)^2 = -2.08.$ First, $x_2 = x_1 + h = -0.9 + 0.1 = -0.8.$ Using linear approximation gives

$$\begin{aligned} L(x) &= y_1 + f'(x_1)(x - x_1) \\ &= 1.7 - 2.08(x - (-0.9)) \\ &= 1.7 - 2.08x - 1.872 \\ &= -2.08x - 0.172. \end{aligned}$$

Finally, we substitute $x_2 = -0.8$ into the linear approximation to calculate y_2.

$$\begin{aligned} y_2 &= L(x_2) \\ &= -2.08x_2 - 0.172 \\ &= -2.08(-0.8) - 0.172 \\ &= 1.492. \end{aligned}$$

Therefore the approximate value of the solution to the differential equation is $y = 1.492$ when $x = -0.8$.

What we have just shown is the idea behind **Euler's Method**. Repeating these steps gives a list of values for the solution. These values are shown in **Table 4.2**, rounded off to four decimal places.

n	0	1	2	3	4	5
x_n	−1	−0.9	−0.8	−0.7	−0.6	−0.5
y_n	2	1.7	1.492	1.3334	1.2046	1.0955
n	6	7	8	9	10	
x_n	−0.4	−0.3	−0.2	−0.1	0	
y_n	1.0004	1.9164	1.8414	1.7746	1.7156	

Table 4.2 Using Euler's Method to Approximate Solutions to a Differential Equation

Theorem 4.1: Euler's Method

Consider the initial-value problem

$y' = f(x, y), \quad y(x_0) = y_0.$

To approximate a solution to this problem using Euler's method, define

$$\begin{aligned} x_n &= x_0 + nh \\ y_n &= y_{n-1} + hf(x_{n-1}, y_{n-1}). \end{aligned} \tag{4.2}$$

Here $h > 0$ represents the step size and n is an integer, starting with 1. The number of steps taken is counted by the variable n.

Typically h is a small value, say 0.1 or 0.05. The smaller the value of h, the more calculations are needed. The higher the value of h, the fewer calculations are needed. However, the tradeoff results in a lower degree of accuracy for larger step size, as illustrated in **Figure 4.14**.

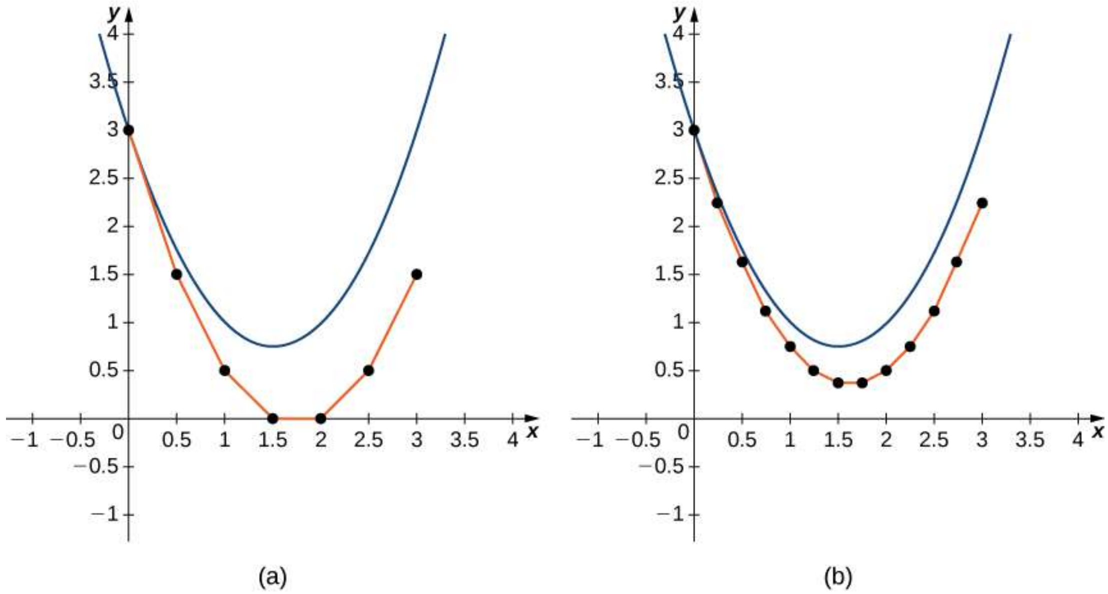


Figure 4.14 Euler's method for the initial-value problem $y' = 2x - 3,\ y(0) = 3$ with (a) a step size of $h = 0.5;$ and (b) a step size of $h = 0.25.$

Example 4.9

Using Euler's Method

Consider the initial-value problem

$$y' = 3x^2 - y^2 + 1, \quad y(0) = 2.$$

Use Euler's method with a step size of 0.1 to generate a table of values for the solution for values of x between 0 and $1.$

Solution

We are given $h = 0.1$ and $f(x, y) = 3x^2 - y^2 + 1.$ Furthermore, the initial condition $y(0) = 2$ gives $x_0 = 0$ and $y_0 = 2.$ Using **Equation 4.2** with $n = 0,$ we can generate **Table 4.3**.

n	x_n	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0	0	2
1	0.1	$y_1 = y_0 + hf(x_0, y_0) = 1.7$
2	0.2	$y_2 = y_1 + hf(x_1, y_1) = 1.514$
3	0.3	$y_3 = y_2 + hf(x_2, y_2) = 1.3968$
4	0.4	$y_4 = y_3 + hf(x_3, y_3) = 1.3287$
5	0.5	$y_5 = y_4 + hf(x_4, y_4) = 1.3001$
6	0.6	$y_6 = y_5 + hf(x_5, y_5) = 1.3061$
7	0.7	$y_7 = y_6 + hf(x_6, y_6) = 1.3435$
8	0.8	$y_8 = y_7 + hf(x_7, y_7) = 1.4100$
9	0.9	$y_9 = y_8 + hf(x_8, y_8) = 1.5032$
10	1.0	$y_{10} = y_9 + hf(x_9, y_9) = 1.6202$

Table 4.3
Using Euler's Method to Approximate Solutions to a Differential Equation

With ten calculations, we are able to approximate the values of the solution to the initial-value problem for values of x between 0 and 1.

Go to this **website (http://www.openstaxcollege.org/l/20_EulersMethod)** for more information on Euler's method.

4.9 Consider the initial-value problem

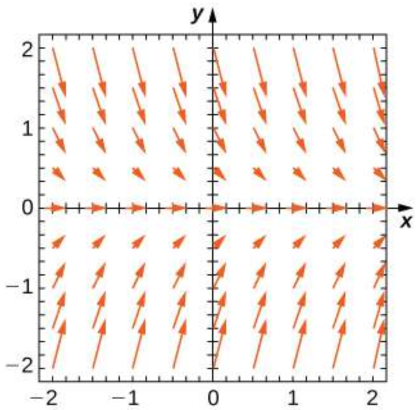
$$y' = x^3 + y^2, \quad y(1) = -2.$$

Using a step size of 0.1, generate a table with approximate values for the solution to the initial-value problem for values of x between 1 and $2.$

Visit this **website (http://www.openstaxcollege.org/l/20_EulerMethod2)** for a practical application of the material in this section.

4.2 EXERCISES

For the following problems, use the direction field below from the differential equation $y' = -2y$. Sketch the graph of the solution for the given initial conditions.



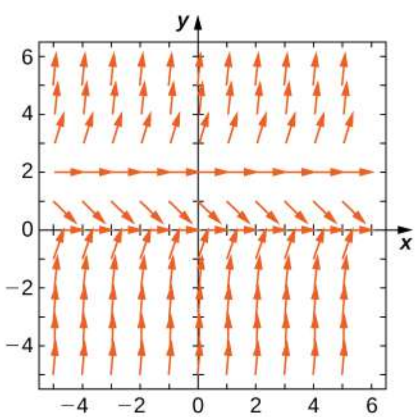
66. $y(0) = 1$

67. $y(0) = 0$

68. $y(0) = -1$

69. Are there any equilibria? What are their stabilities?

For the following problems, use the direction field below from the differential equation $y' = y^2 - 2y$. Sketch the graph of the solution for the given initial conditions.



70. $y(0) = 3$

71. $y(0) = 1$

72. $y(0) = -1$

73. Are there any equilibria? What are their stabilities?

Draw the direction field for the following differential equations, then solve the differential equation. Draw your solution on top of the direction field. Does your solution follow along the arrows on your direction field?

74. $y' = t^3$

75. $y' = e^t$

76. $\dfrac{dy}{dx} = x^2 \cos x$

77. $\dfrac{dy}{dt} = te^t$

78. $\dfrac{dx}{dt} = \cosh(t)$

Draw the directional field for the following differential equations. What can you say about the behavior of the solution? Are there equilibria? What stability do these equilibria have?

79. $y' = y^2 - 1$

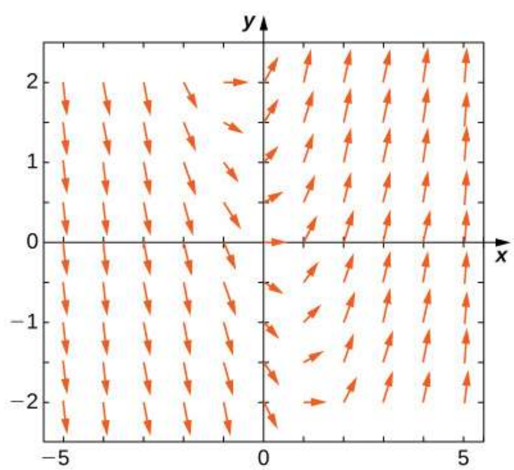
80. $y' = y - x$

81. $y' = 1 - y^2 - x^2$

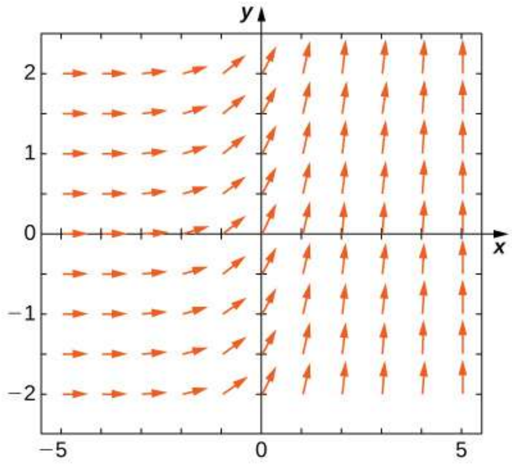
82. $y' = t^2 \sin y$

83. $y' = 3y + xy$

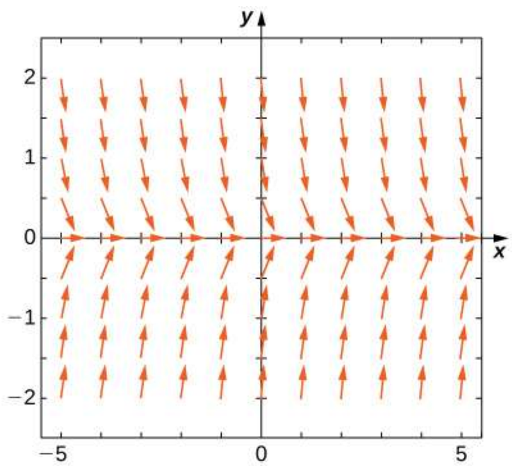
Match the direction field with the given differential equations. Explain your selections.



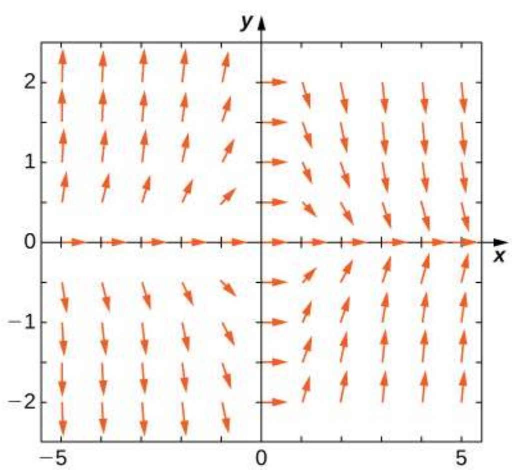
(a)



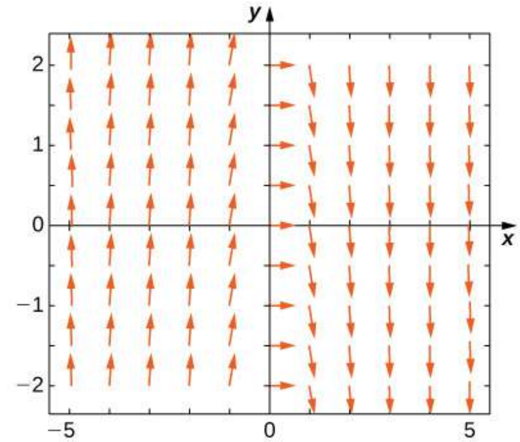
(b)



(c)



(d)



(e)

84. $y' = -3y$

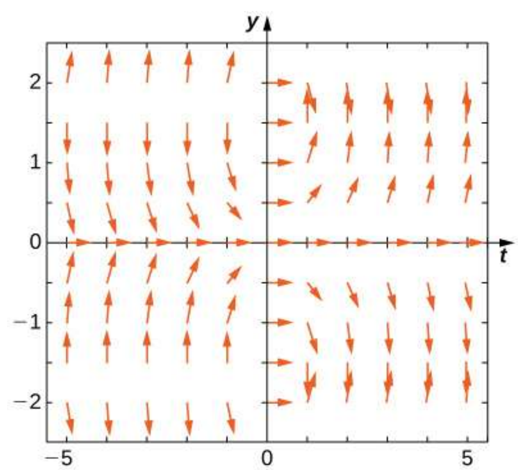
85. $y' = -3t$

86. $y' = e^t$

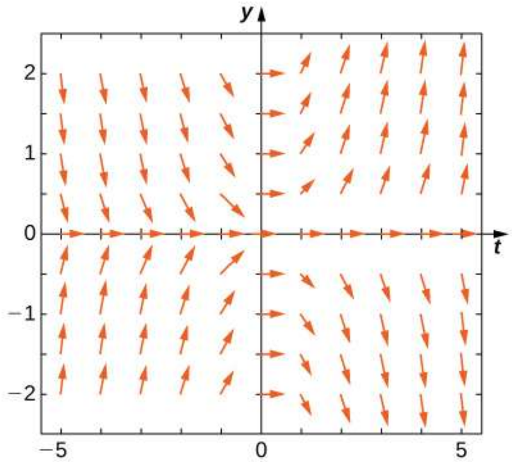
87. $y' = \frac{1}{2}y + t$

88. $y' = -ty$

Match the direction field with the given differential equations. Explain your selections.



(a)



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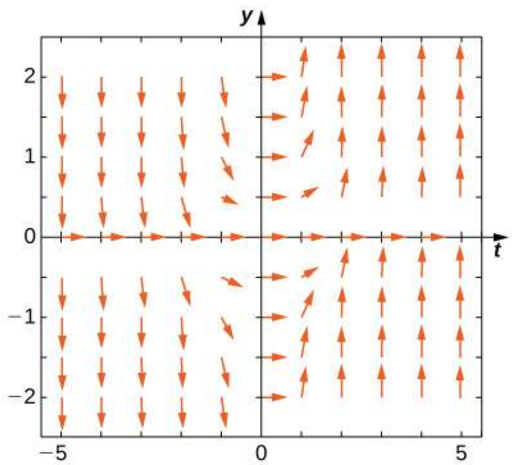
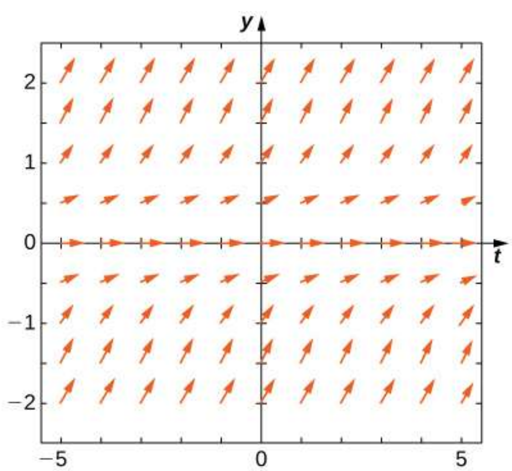
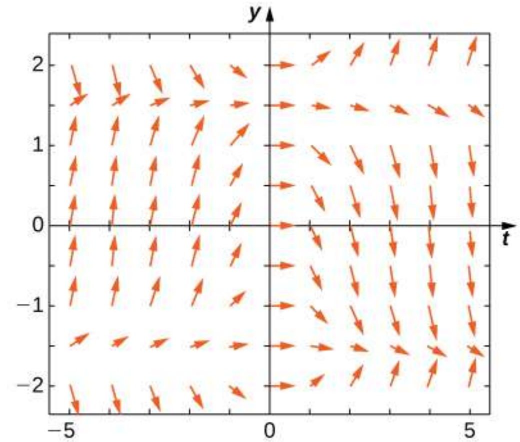


(c)



(d)



(e)

89. $y' = t \sin y$

90. $y' = -t \cos y$

91. $y' = t \tan y$

92. $y' = \sin^2 y$

93. $y' = y^2 t^3$

Estimate the following solutions using Euler's method with $n = 5$ steps over the interval $t = [0, 1]$. If you are able to solve the initial-value problem exactly, compare your solution with the exact solution. If you are unable to solve the initial-value problem, the exact solution will be provided for you to compare with Euler's method. How accurate is Euler's method?

94. $y' = -3y, \quad y(0) = 1$

95. $y' = t^2$

96. $y' = 3t - y,\ y(0) = 1.$ Exact solution is $y = 3t + 4e^{-t} - 3$

97. $y' = y + t^2,\ y(0) = 3.$ Exact solution is $y = 5e^t - 2 - t^2 - 2t$

98. $y' = 2t,\ y(0) = 0$

99. **[T]** $y' = e^{(x + y)},\ y(0) = -1.$ Exact solution is $y = -\ln(e + 1 - e^x)$

100. $y' = y^2 \ln(x+1),\ y(0) = 1.$ Exact solution is $y = -\frac{1}{(x+1)(\ln(x+1)-1)}$

101. $y' = 2^x,\ y(0) = 0,$ Exact solution is $y = \frac{2^x - 1}{\ln(2)}$

102. $y' = y,\ y(0) = -1.$ Exact solution is $y = -e^x.$

103. $y' = -5t,\ y(0) = -2.$ Exact solution is $y = -\frac{5}{2}t^2 - 2$

Differential equations can be used to model disease epidemics. In the next set of problems, we examine the change of size of two sub-populations of people living in a city: individuals who are infected and individuals who are susceptible to infection. S represents the size of the susceptible population, and I represents the size of the infected population. We assume that if a susceptible person interacts with an infected person, there is a probability c that the susceptible person will become infected. Each infected person recovers from the infection at a rate r and becomes susceptible again. We consider the case of influenza, where we assume that no one dies from the disease, so we assume that the total population size of the two sub-populations is a constant number, N. The differential equations that model these population sizes are

$$S' = rI - cSI \quad \text{and}$$
$$I' = cSI - rI.$$

Here c represents the contact rate and r is the recovery rate.

104. Show that, by our assumption that the total population size is constant $(S + I = N),$ you can reduce the system to a single differential equation in $I\colon I' = c(N - I)I - rI.$

105. Assuming the parameters are $c = 0.5,\ N = 5,$ and $r = 0.5,$ draw the resulting directional field.

106. **[T]** Use computational software or a calculator to compute the solution to the initial-value problem $y' = ty,\quad y(0) = 2$ using Euler's Method with the given step size $h.$ Find the solution at $t = 1.$ For a hint, here is "pseudo-code" for how to write a computer program to perform Euler's Method for $y' = f(t, y),\ y(0) = 2:$ Create function $f(t, y)$ Define parameters $y(1) = y_0,\quad t(0) = 0,$ step size $h,$ and total number of steps, N Write a for loop: for $\text{k} = 1 \text{ to N}$ $\text{fn} = \text{f(t(k), y(k))}$ $\text{y(k+1)} = \text{y(k)} + \text{h*fn}$ $\text{t(k+1)} = \text{t(k)} + \text{h}$

107. Solve the initial-value problem for the exact solution.

108. Draw the directional field

109. $h = 1$

110. **[T]** $h = 10$

111. **[T]** $h = 100$

112. **[T]** $h = 1000$

113. **[T]** Evaluate the exact solution at $t = 1.$ Make a table of errors for the relative error between the Euler's method solution and the exact solution. How much does the error change? Can you explain?

Consider the initial-value problem $y' = -2y,\quad y(0) = 2.$

114. Show that $y = 2e^{-2x}$ solves this initial-value problem.

115. Draw the directional field of this differential equation.

116. **[T]** By hand or by calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 5.$

117. **[T]** By calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 100.$

118. **[T]** Plot exact answer and each Euler approximation (for $h = 5$ and $h = 100$) at each h on the directional field. What do you notice?

4.3 | Separable Equations

Learning Objectives

4.3.1 Use separation of variables to solve a differential equation.
4.3.2 Solve applications using separation of variables.

We now examine a solution technique for finding exact solutions to a class of differential equations known as separable differential equations. These equations are common in a wide variety of disciplines, including physics, chemistry, and engineering. We illustrate a few applications at the end of the section.

Separation of Variables

We start with a definition and some examples.

Definition

A **separable differential equation** is any equation that can be written in the form

$$y' = f(x)g(y). \tag{4.3}$$

The term 'separable' refers to the fact that the right-hand side of the equation can be separated into a function of x times a function of y. Examples of separable differential equations include

$$\begin{aligned} y' &= \left(x^2 - 4\right)(3y + 2) \\ y' &= 6x^2 + 4x \\ y' &= \sec y + \tan y \\ y' &= xy + 3x - 2y - 6. \end{aligned}$$

The second equation is separable with $f(x) = 6x^2 + 4x$ and $g(y) = 1,$ the third equation is separable with $f(x) = 1$ and $g(y) = \sec y + \tan y,$ and the right-hand side of the fourth equation can be factored as $(x + 3)(y - 2),$ so it is separable as well. The third equation is also called an **autonomous differential equation** because the right-hand side of the equation is a function of y alone. If a differential equation is separable, then it is possible to solve the equation using the method of **separation of variables**.

Problem-Solving Strategy: Separation of Variables

1. Check for any values of y that make $g(y) = 0.$ These correspond to constant solutions.
2. Rewrite the differential equation in the form $\dfrac{dy}{g(y)} = f(x)dx.$
3. Integrate both sides of the equation.
4. Solve the resulting equation for y if possible.
5. If an initial condition exists, substitute the appropriate values for x and y into the equation and solve for the constant.

Note that Step 4. states "Solve the resulting equation for y if possible." It is not always possible to obtain y as an explicit function of $x.$ Quite often we have to be satisfied with finding y as an implicit function of $x.$

Example 4.10

Using Separation of Variables

Find a general solution to the differential equation $y' = (x^2 - 4)(3y + 2)$ using the method of separation of variables.

Solution

Follow the five-step method of separation of variables.

1. In this example, $f(x) = x^2 - 4$ and $g(y) = 3y + 2$. Setting $g(y) = 0$ gives $y = -\frac{2}{3}$ as a constant solution.
2. Rewrite the differential equation in the form
$$\frac{dy}{3y+2} = (x^2 - 4)dx.$$
3. Integrate both sides of the equation:
$$\int \frac{dy}{3y+2} = \int (x^2 - 4)dx.$$
Let $u = 3y + 2$. Then $du = 3\frac{dy}{dx}dx$, so the equation becomes
$$\begin{aligned} \frac{1}{3}\int \frac{1}{u}du &= \frac{1}{3}x^3 - 4x + C \\ \frac{1}{3}\ln|u| &= \frac{1}{3}x^3 - 4x + C \\ \frac{1}{3}\ln|3y+2| &= \frac{1}{3}x^3 - 4x + C. \end{aligned}$$
4. To solve this equation for y, first multiply both sides of the equation by 3.
$$\ln|3y+2| = x^3 - 12x + 3C$$
Now we use some logic in dealing with the constant C. Since C represents an arbitrary constant, $3C$ also represents an arbitrary constant. If we call the second arbitrary constant C_1, the equation becomes
$$\ln|3y+2| = x^3 - 12x + C_1.$$
Now exponentiate both sides of the equation (i.e., make each side of the equation the exponent for the base e).
$$\begin{aligned} e^{\ln|3y+2|} &= e^{x^3 - 12x + C_1} \\ |3y+2| &= e^{C_1}e^{x^3 - 12x} \end{aligned}$$
Again define a new constant $C_2 = e^{c_1}$ (note that $C_2 > 0$):
$$|3y+2| = C_2 e^{x^3 - 12x}.$$

This corresponds to two separate equations: $3y + 2 = C_2 e^{x^3 - 12x}$ and $3y + 2 = -C_2 e^{x^3 - 12x}$.

The solution to either equation can be written in the form $y = \frac{-2 \pm C_2 e^{x^3 - 12x}}{3}$.

Since $C_2 > 0$, it does not matter whether we use plus or minus, so the constant can actually have either sign. Furthermore, the subscript on the constant C is entirely arbitrary, and can be dropped. Therefore the solution can be written as

$$y = \frac{-2 + Ce^{x^3 - 12x}}{3}.$$

5. No initial condition is imposed, so we are finished.

4.10 Use the method of separation of variables to find a general solution to the differential equation $y' = 2xy + 3y - 4x - 6$.

Example 4.11

Solving an Initial-Value Problem

Using the method of separation of variables, solve the initial-value problem

$$y' = (2x + 3)(y^2 - 4), \quad y(0) = -3.$$

Solution

Follow the five-step method of separation of variables.

1. In this example, $f(x) = 2x + 3$ and $g(y) = y^2 - 4$. Setting $g(y) = 0$ gives $y = \pm 2$ as constant solutions.
2. Divide both sides of the equation by $y^2 - 4$ and multiply by dx. This gives the equation

$$\frac{dy}{y^2 - 4} = (2x + 3)\,dx.$$

3. Next integrate both sides:

$$\int \frac{1}{y^2 - 4}\,dy = \int (2x + 3)\,dx. \tag{4.4}$$

To evaluate the left-hand side, use the method of partial fraction decomposition. This leads to the identity

$$\frac{1}{y^2 - 4} = \frac{1}{4}\left(\frac{1}{y - 2} - \frac{1}{y + 2}\right).$$

Then **Equation 4.4** becomes

$$\begin{aligned}\frac{1}{4}\int\left(\frac{1}{y-2}-\frac{1}{y+2}\right)dy &= \int(2x+3)\,dx\\ \frac{1}{4}(\ln|y-2|-\ln|y+2|) &= x^2+3x+C.\end{aligned}$$

Multiplying both sides of this equation by 4 and replacing $4C$ with C_1 gives

$$\begin{aligned}\ln|y-2|-\ln|y+2| &= 4x^2+12x+C_1\\ \ln\left|\frac{y-2}{y+2}\right| &= 4x^2+12x+C_1.\end{aligned}$$

4. It is possible to solve this equation for *y*. First exponentiate both sides of the equation and define $C_2 = e^{C_1}$:

$$\left|\frac{y-2}{y+2}\right| = C_2 e^{4x^2+12x}.$$

Next we can remove the absolute value and let C_2 be either positive or negative. Then multiply both sides by $y+2$.

$$\begin{aligned}y-2 &= C_2(y+2)e^{4x^2+12x}\\ y-2 &= C_2 y e^{4x^2+12x} + 2C_2 e^{4x^2+12x}.\end{aligned}$$

Now collect all terms involving *y* on one side of the equation, and solve for y:

$$\begin{aligned}y - C_2 y e^{4x^2+12x} &= 2+2C_2 e^{4x^2+12x}\\ y(1-C_2 e^{4x^2+12x}) &= 2+2C_2 e^{4x^2+12x}\\ y &= \frac{2+2C_2 e^{4x^2+12x}}{1-C_2 e^{4x^2+12x}}.\end{aligned}$$

5. To determine the value of $C_2,$ substitute $x=0$ and $y=-1$ into the general solution. Alternatively, we can put the same values into an earlier equation, namely the equation $\frac{y-2}{y+2} = C_2 e^{4x^2+12}$. This is much easier to solve for C_2:

$$\begin{aligned}\frac{y-2}{y+2} &= C_2 e^{4x^2+12x}\\ \frac{-1-2}{-1+2} &= C_2 e^{4(0)^2+12(0)}\\ C_2 &= -3.\end{aligned}$$

Therefore the solution to the initial-value problem is

$$y = \frac{2-6e^{4x^2+12x}}{1+3e^{4x^2+12x}}.$$

A graph of this solution appears in **Figure 4.15**.

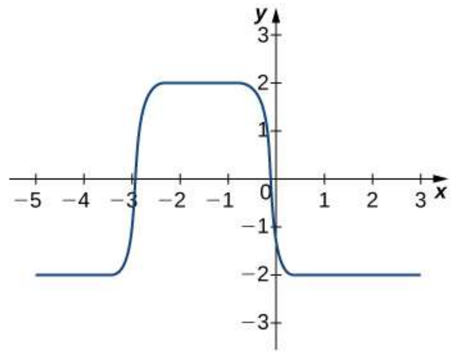


Figure 4.15 Graph of the solution to the initial-value problem $y' = (2x+3)\left(y^2-4\right), \quad y(0) = -3.$

4.11 Find the solution to the initial-value problem

$$6y' = (2x+1)\left(y^2 - 2y - 8\right), \quad y(0) = -3$$

using the method of separation of variables.

Applications of Separation of Variables

Many interesting problems can be described by separable equations. We illustrate two types of problems: solution concentrations and Newton's law of cooling.

Solution concentrations

Consider a tank being filled with a salt solution. We would like to determine the amount of salt present in the tank as a function of time. We can apply the process of separation of variables to solve this problem and similar problems involving solution concentrations.

Example 4.12

Determining Salt Concentration over Time

A tank containing $100\,\text{L}$ of a brine solution initially has $4\,\text{kg}$ of salt dissolved in the solution. At time $t = 0,$ another brine solution flows into the tank at a rate of $2\,\text{L/min}.$ This brine solution contains a concentration of $0.5\,\text{kg/L}$ of salt. At the same time, a stopcock is opened at the bottom of the tank, allowing the combined solution to flow out at a rate of $2\,\text{L/min},$ so that the level of liquid in the tank remains constant (**Figure 4.16**). Find the amount of salt in the tank as a function of time (measured in minutes), and find the limiting amount of salt in the tank, assuming that the solution in the tank is well mixed at all times.

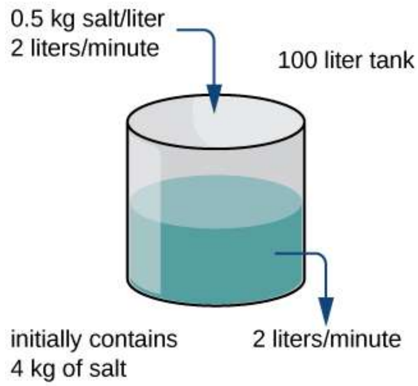


Figure 4.16 A brine tank with an initial amount of salt solution accepts an input flow and delivers an output flow. How does the amount of salt change with time?

Solution

First we define a function $u(t)$ that represents the amount of salt in kilograms in the tank as a function of time. Then $\frac{du}{dt}$ represents the rate at which the amount of salt in the tank changes as a function of time. Also, $u(0)$ represents the amount of salt in the tank at time $t = 0,$ which is 4 kilograms.

The general setup for the differential equation we will solve is of the form

$$\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}. \tag{4.5}$$

INFLOW RATE represents the rate at which salt enters the tank, and OUTFLOW RATE represents the rate at which salt leaves the tank. Because solution enters the tank at a rate of 2 L/min, and each liter of solution contains 0.5 kilogram of salt, every minute $2(0.5) = 1\,\text{kilogram}$ of salt enters the tank. Therefore INFLOW RATE = 1.

To calculate the rate at which salt leaves the tank, we need the concentration of salt in the tank at any point in time. Since the actual amount of salt varies over time, so does the concentration of salt. However, the volume of the solution remains fixed at 100 liters. The number of kilograms of salt in the tank at time t is equal to $u(t)$. Thus, the concentration of salt is $\frac{u(t)}{100}$ kg/L, and the solution leaves the tank at a rate of 2 L/min. Therefore salt leaves the tank at a rate of $\frac{u(t)}{100} \cdot 2 = \frac{u(t)}{50}$ kg/min, and OUTFLOW RATE is equal to $\frac{u(t)}{50}$. Therefore the differential equation becomes $\frac{du}{dt} = 1 - \frac{u}{50},$ and the initial condition is $u(0) = 4.$ The initial-value problem to be solved is

$$\frac{du}{dt} = 1 - \frac{u}{50}, \quad u(0) = 4.$$

The differential equation is a separable equation, so we can apply the five-step strategy for solution.

Step 1. Setting $1 - \frac{u}{50} = 0$ gives $u = 50$ as a constant solution. Since the initial amount of salt in the tank is 4 kilograms, this solution does not apply.

Step 2. Rewrite the equation as

$$\frac{du}{dt} = \frac{50 - u}{50}.$$

Then multiply both sides by dt and divide both sides by $50 - u$:

$$\frac{du}{50 - u} = \frac{dt}{50}.$$

Step 3. Integrate both sides:

$$\begin{aligned} \int \frac{du}{50 - u} &= \int \frac{dt}{50} \\ -\ln|50 - u| &= \frac{t}{50} + C. \end{aligned}$$

Step 4. Solve for $u(t)$:

$$\begin{aligned} \ln|50 - u| &= -\frac{t}{50} - C \\ e^{\ln|50 - u|} &= e^{-(t/50) - C} \\ |50 - u| &= C_1 e^{-t/50}. \end{aligned}$$

Eliminate the absolute value by allowing the constant to be either positive or negative:

$$50 - u = C_1 e^{-t/50}.$$

Finally, solve for $u(t)$:

$$u(t) = 50 - C_1 e^{-t/50}.$$

Step 5. Solve for C_1:

$$\begin{aligned} u(0) &= 50 - C_1 e^{-0/50} \\ 4 &= 50 - C_1 \\ C_1 &= 46. \end{aligned}$$

The solution to the initial value problem is $u(t) = 50 - 46e^{-t/50}$. To find the limiting amount of salt in the tank, take the limit as t approaches infinity:

$$\begin{aligned} \lim_{t \to \infty} u(t) &= 50 - 46e^{-t/50} \\ &= 50 - 46(0) \\ &= 50. \end{aligned}$$

Note that this was the constant solution to the differential equation. If the initial amount of salt in the tank is 50 kilograms, then it remains constant. If it starts at less than 50 kilograms, then it approaches 50 kilograms over time.

4.12 A tank contains 3 kilograms of salt dissolved in 75 liters of water. A salt solution of 0.4 kg salt/L is pumped into the tank at a rate of 6 L/min and is drained at the same rate. Solve for the salt concentration at time t. Assume the tank is well mixed at all times.

Newton’s law of cooling

Newton’s law of cooling states that the rate of change of an object’s temperature is proportional to the difference between its own temperature and the ambient temperature (i.e., the temperature of its surroundings). If we let $T(t)$ represent the temperature of an object as a function of time, then $\frac{dT}{dt}$ represents the rate at which that temperature changes. The temperature of the object's surroundings can be represented by T_s. Then Newton's law of cooling can be written in the form

$$\frac{dT}{dt} = k(T(t) - T_s)$$

or simply

$$\frac{dT}{dt} = k(T - T_s). \tag{4.6}$$

The temperature of the object at the beginning of any experiment is the initial value for the initial-value problem. We call this temperature T_0. Therefore the initial-value problem that needs to be solved takes the form

$$\frac{dT}{dt} = k(T - T_s), \quad T(0) = T_0, \tag{4.7}$$

where k is a constant that needs to be either given or determined in the context of the problem. We use these equations in **Example 4.13**.

Example 4.13

Waiting for a Pizza to Cool

A pizza is removed from the oven after baking thoroughly, and the temperature of the oven is $350°\text{F}$. The temperature of the kitchen is $75°\text{F}$, and after 5 minutes the temperature of the pizza is $340°\text{F}$. We would like to wait until the temperature of the pizza reaches $300°\text{F}$ before cutting and serving it (**Figure 4.17**). How much longer will we have to wait?

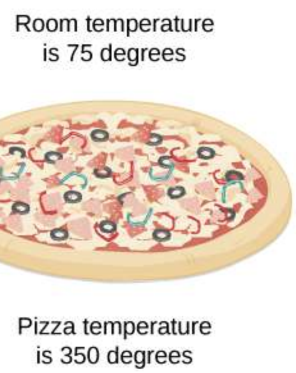


Figure 4.17 From Newton's law of cooling, if the pizza cools $10°\text{F}$ in 5 minutes, how long before it cools to $300°\text{F}$?

Solution

The ambient temperature (surrounding temperature) is $75°\text{F}$, so $T_s = 75$. The temperature of the pizza when it comes out of the oven is $350°\text{F}$, which is the initial temperature (i.e., initial value), so $T_0 = 350$. Therefore **Equation 4.4** becomes

$$\frac{dT}{dt} = k(T - 75), \quad T(0) = 350.$$

To solve the differential equation, we use the five-step technique for solving separable equations.

1. Setting the right-hand side equal to zero gives $T = 75$ as a constant solution. Since the pizza starts at $350°\text{F}$, this is not the solution we are seeking.

2. Rewrite the differential equation by multiplying both sides by dt and dividing both sides by $T-75$:

$$\frac{dT}{T-75}=kdt.$$

3. Integrate both sides:

$$\begin{aligned}\int\frac{dT}{T-75} &= \int kdt\\ \ln|T-75| &= kt+C.\end{aligned}$$

4. Solve for T by first exponentiating both sides:

$$\begin{aligned}e^{\ln|T-75|} &= e^{kt+C}\\ |T-75| &= C_1e^{kt}\\ T-75 &= C_1e^{kt}\\ T(t) &= 75+C_1e^{kt}.\end{aligned}$$

5. Solve for C_1 by using the initial condition $T(0)=350$:

$$\begin{aligned}T(t) &= 75+C_1e^{kt}\\ T(0) &= 75+C_1e^{k(0)}\\ 350 &= 75+C_1\\ C_1 &= 275.\end{aligned}$$

Therefore the solution to the initial-value problem is

$$T(t)=75+275e^{kt}.$$

To determine the value of $k,$ we need to use the fact that after 5 minutes the temperature of the pizza is $340°\text{F}.$ Therefore $T(5)=340.$ Substituting this information into the solution to the initial-value problem, we have

$$\begin{aligned}T(t) &= 75+275e^{kt}\\ T(5) &= 340=75+275e^{5k}\\ 265 &= 275e^{5k}\\ e^{5k} &= \frac{53}{55}\\ \ln e^{5k} &= \ln\left(\frac{53}{55}\right)\\ 5k &= \ln\left(\frac{53}{55}\right)\\ k &= \frac{1}{5}\ln\left(\frac{53}{55}\right)\approx -0.007408.\end{aligned}$$

So now we have $T(t)=75+275e^{-0.007048t}.$ When is the temperature $300°\text{F}?$ Solving for $t,$ we find

$$\begin{aligned} T(t) &= 75 + 275e^{-0.007048t} \\ 300 &= 75 + 275e^{-0.007048t} \\ 225 &= 275e^{-0.007048t} \\ e^{-0.007048t} &= \frac{9}{11} \\ \ln e^{-0.007048t} &= \ln \frac{9}{11} \\ -0.007048t &= \ln \frac{9}{11} \\ t &= -\frac{1}{0.007048} \ln \frac{9}{11} \approx 28.5. \end{aligned}$$

Therefore we need to wait an additional 23.5 minutes (after the temperature of the pizza reached $340°\text{F}$). That should be just enough time to finish this calculation.

4.13 A cake is removed from the oven after baking thoroughly, and the temperature of the oven is $450°\text{F}.$ The temperature of the kitchen is $70°\text{F},$ and after 10 minutes the temperature of the cake is $430°\text{F}.$

a. Write the appropriate initial-value problem to describe this situation.
b. Solve the initial-value problem for $T(t).$
c. How long will it take until the temperature of the cake is within $5°\text{F}$ of room temperature?

4.3 EXERCISES

Solve the following initial-value problems with the initial condition $y_0 = 0$ and graph the solution.

119. $\frac{dy}{dt} = y + 1$

120. $\frac{dy}{dt} = y - 1$

121. $\frac{dy}{dt} = y + 1$

122. $\frac{dy}{dt} = -y - 1$

Find the general solution to the differential equation.

123. $x^2 y' = (x + 1)y$

124. $y' = \tan(y)x$

125. $y' = 2xy^2$

126. $\frac{dy}{dt} = y\cos(3t + 2)$

127. $2x\frac{dy}{dx} = y^2$

128. $y' = e^y x^2$

129. $(1 + x)y' = (x + 2)(y - 1)$

130. $\frac{dx}{dt} = 3t^2\left(x^2 + 4\right)$

131. $t\frac{dy}{dt} = \sqrt{1 - y^2}$

132. $y' = e^x e^y$

Find the solution to the initial-value problem.

133. $y' = e^{y - x}$, $y(0) = 0$

134. $y' = y^2(x + 1)$, $y(0) = 2$

135. $\frac{dy}{dx} = y^3 xe^{x^2}$, $y(0) = 1$

136. $\frac{dy}{dt} = y^2 e^x \sin(3x)$, $y(0) = 1$

137. $y' = \frac{x}{\operatorname{sech}^2 y}$, $y(0) = 0$

138. $y' = 2xy(1 + 2y)$, $y(0) = -1$

139. $\frac{dx}{dt} = \ln(t)\sqrt{1 - x^2}$, $x(0) = 0$

140. $y' = 3x^2(y^2 + 4)$, $y(0) = 0$

141. $y' = e^y 5^x$, $y(0) = \ln(\ln(5))$

142. $y' = -2x\tan(y)$, $y(0) = \frac{\pi}{2}$

For the following problems, use a software program or your calculator to generate the directional fields. Solve explicitly and draw solution curves for several initial conditions. Are there some critical initial conditions that change the behavior of the solution?

143. **[T]** $y' = 1 - 2y$

144. **[T]** $y' = y^2 x^3$

145. **[T]** $y' = y^3 e^x$

146. **[T]** $y' = e^y$

147. **[T]** $y' = y\ln(x)$

148. Most drugs in the bloodstream decay according to the equation $y' = cy$, where y is the concentration of the drug in the bloodstream. If the half-life of a drug is 2 hours, what fraction of the initial dose remains after 6 hours?

149. A drug is administered intravenously to a patient at a rate r mg/h and is cleared from the body at a rate proportional to the amount of drug still present in the body, d Set up and solve the differential equation, assuming there is no drug initially present in the body.

150. **[T]** How often should a drug be taken if its dose is 3 mg, it is cleared at a rate $c = 0.1$ mg/h, and 1 mg is required to be in the bloodstream at all times?

151. A tank contains 1 kilogram of salt dissolved in 100 liters of water. A salt solution of 0.1 kg salt/L is pumped into the tank at a rate of 2 L/min and is drained at the same rate. Solve for the salt concentration at time t. Assume the tank is well mixed.

152. A tank containing 10 kilograms of salt dissolved in 1000 liters of water has two salt solutions pumped in. The first solution of 0.2 kg salt/L is pumped in at a rate of 20 L/min and the second solution of 0.05 kg salt/L is pumped in at a rate of 5 L/min. The tank drains at 25 L/min. Assume the tank is well mixed. Solve for the salt concentration at time t.

153. **[T]** For the preceding problem, find how much salt is in the tank 1 hour after the process begins.

154. Torricelli's law states that for a water tank with a hole in the bottom that has a cross-section of A and with a height of water h above the bottom of the tank, the rate of change of volume of water flowing from the tank is proportional to the square root of the height of water, according to $\frac{dV}{dt} = -A\sqrt{2gh},$ where g is the acceleration due to gravity. Note that $\frac{dV}{dt} = A\frac{dh}{dt}.$ Solve the resulting initial-value problem for the height of the water, assuming a tank with a hole of radius 2 ft. The initial height of water is 100 ft.

155. For the preceding problem, determine how long it takes the tank to drain.

For the following problems, use Newton's law of cooling.

156. The liquid base of an ice cream has an initial temperature of $200°\text{F}$ before it is placed in a freezer with a constant temperature of $0°\text{F}$. After 1 hour, the temperature of the ice-cream base has decreased to $140°\text{F}$. Formulate and solve the initial-value problem to determine the temperature of the ice cream.

157. **[T]** The liquid base of an ice cream has an initial temperature of $210°\text{F}$ before it is placed in a freezer with a constant temperature of $20°\text{F}$. After 2 hours, the temperature of the ice-cream base has decreased to $170°\text{F}$. At what time will the ice cream be ready to eat? (Assume $30°\text{F}$ is the optimal eating temperature.)

158. **[T]** You are organizing an ice cream social. The outside temperature is $80°\text{F}$ and the ice cream is at $10°\text{F}$. After 10 minutes, the ice cream temperature has risen by $10°\text{F}$. How much longer can you wait before the ice cream melts at $40°\text{F}$?

159. You have a cup of coffee at temperature $70°\text{C}$ and the ambient temperature in the room is $20°\text{C}$. Assuming a cooling rate k of $0.125,$ write and solve the differential equation to describe the temperature of the coffee with respect to time.

160. **[T]** You have a cup of coffee at temperature $70°\text{C}$ that you put outside, where the ambient temperature is $0°\text{C}$. After 5 minutes, how much colder is the coffee?

161. You have a cup of coffee at temperature $70°\text{C}$ and you immediately pour in 1 part milk to 5 parts coffee. The milk is initially at temperature $1°\text{C}$. Write and solve the differential equation that governs the temperature of this coffee.

162. You have a cup of coffee at temperature $70°\text{C},$ which you let cool 10 minutes before you pour in the same amount of milk at $1°\text{C}$ as in the preceding problem. How does the temperature compare to the previous cup after 10 minutes?

163. Solve the generic problem $y' = ay + b$ with initial condition $y(0) = c$.

164. Prove the basic continual compounded interest equation. Assuming an initial deposit of P_0 and an interest rate of $r,$ set up and solve an equation for continually compounded interest.

165. Assume an initial nutrient amount of I kilograms in a tank with L liters. Assume a concentration of c kg/L being pumped in at a rate of r L/min. The tank is well mixed and is drained at a rate of r L/min. Find the equation describing the amount of nutrient in the tank.

166. Leaves accumulate on the forest floor at a rate of 2 g/cm^2/yr and also decompose at a rate of 90% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor, assuming at time 0 there is no leaf litter on the ground. Does this amount approach a steady value? What is that value?

167. Leaves accumulate on the forest floor at a rate of 4 g/cm^2/yr. These leaves decompose at a rate of 10% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor. Does this amount approach a steady value? What is that value?

4.4 | The Logistic Equation

Learning Objectives

4.4.1 Describe the concept of environmental carrying capacity in the logistic model of population growth.
4.4.2 Draw a direction field for a logistic equation and interpret the solution curves.
4.4.3 Solve a logistic equation and interpret the results.

Differential equations can be used to represent the size of a population as it varies over time. We saw this in an earlier chapter in the section on exponential growth and decay, which is the simplest model. A more realistic model includes other factors that affect the growth of the population. In this section, we study the logistic differential equation and see how it applies to the study of population dynamics in the context of biology.

Population Growth and Carrying Capacity

To model population growth using a differential equation, we first need to introduce some variables and relevant terms. The variable $t.$ will represent time. The units of time can be hours, days, weeks, months, or even years. Any given problem must specify the units used in that particular problem. The variable P will represent population. Since the population varies over time, it is understood to be a function of time. Therefore we use the notation $P(t)$ for the population as a function of time. If $P(t)$ is a differentiable function, then the first derivative $\frac{dP}{dt}$ represents the instantaneous rate of change of the population as a function of time.

In **Exponential Growth and Decay**, we studied the exponential growth and decay of populations and radioactive substances. An example of an exponential growth function is $P(t) = P_0 e^{rt}.$ In this function, $P(t)$ represents the population at time t, P_0 represents the **initial population** (population at time $t = 0),$ and the constant $r > 0$ is called the **growth rate**. **Figure 4.18** shows a graph of $P(t) = 100e^{0.03t}.$ Here $P_0 = 100$ and $r = 0.03.$

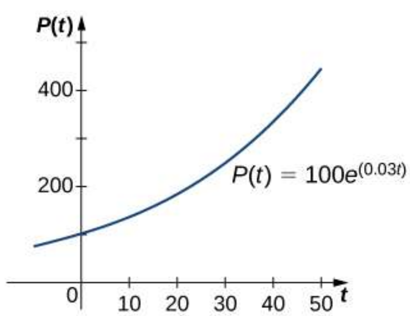


Figure 4.18 An exponential growth model of population.

We can verify that the function $P(t) = P_0 e^{rt}$ satisfies the initial-value problem

$$\frac{dP}{dt} = rP, \quad P(0) = P_0.$$

This differential equation has an interesting interpretation. The left-hand side represents the rate at which the population increases (or decreases). The right-hand side is equal to a positive constant multiplied by the current population. Therefore the differential equation states that the rate at which the population increases is proportional to the population at that point in time. Furthermore, it states that the constant of proportionality never changes.

One problem with this function is its prediction that as time goes on, the population grows without bound. This is unrealistic in a real-world setting. Various factors limit the rate of growth of a particular population, including birth rate, death rate, food supply, predators, and so on. The growth constant r usually takes into consideration the birth and death rates but none of the other factors, and it can be interpreted as a net (birth minus death) percent growth rate per unit time. A natural question to ask is whether the population growth rate stays constant, or whether it changes over time. Biologists have found that in many biological systems, the population grows until a certain steady-state population is reached. This possibility is not taken into account with exponential growth. However, the concept of carrying capacity allows for the possibility that in a given area, only a certain number of a given organism or animal can thrive without running into resource issues.

Definition

The **carrying capacity** of an organism in a given environment is defined to be the maximum population of that organism that the environment can sustain indefinitely.

We use the variable K to denote the carrying capacity. The growth rate is represented by the variable r. Using these variables, we can define the logistic differential equation.

Definition

Let K represent the carrying capacity for a particular organism in a given environment, and let r be a real number that represents the growth rate. The function $P(t)$ represents the population of this organism as a function of time t, and the constant P_0 represents the initial population (population of the organism at time $t = 0$). Then the **logistic differential equation** is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \ = rP. \tag{4.8}$$

See this **website (http://www.openstaxcollege.org/l/20_logisticEq)** for more information on the logistic equation.

The logistic equation was first published by Pierre Verhulst in 1845. This differential equation can be coupled with the initial condition $P(0) = P_0$ to form an initial-value problem for $P(t)$.

Suppose that the initial population is small relative to the carrying capacity. Then $\frac{P}{K}$ is small, possibly close to zero. Thus, the quantity in parentheses on the right-hand side of **Equation 4.8** is close to $1,$ and the right-hand side of this equation is close to rP. If $r > 0,$ then the population grows rapidly, resembling exponential growth.

However, as the population grows, the ratio $\frac{P}{K}$ also grows, because K is constant. If the population remains below the carrying capacity, then $\frac{P}{K}$ is less than $1,$ so $1 - \frac{P}{K} > 0$. Therefore the right-hand side of **Equation 4.8** is still positive, but the quantity in parentheses gets smaller, and the growth rate decreases as a result. If $P = K$ then the right-hand side is equal to zero, and the population does not change.

Now suppose that the population starts at a value higher than the carrying capacity. Then $\frac{P}{K} > 1,$ and $1 - \frac{P}{K} < 0$. Then the right-hand side of **Equation 4.8** is negative, and the population decreases. As long as $P > K,$ the population decreases. It never actually reaches K because $\frac{dP}{dt}$ will get smaller and smaller, but the population approaches the carrying capacity as t approaches infinity. This analysis can be represented visually by way of a phase line. A **phase line** describes the general behavior of a solution to an autonomous differential equation, depending on the initial condition. For the case of a carrying capacity in the logistic equation, the phase line is as shown in **Figure 4.19**.

Figure 4.19 A phase line for the differential equation $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$.

This phase line shows that when P is less than zero or greater than $K,$ the population decreases over time. When P is between 0 and $K,$ the population increases over time.

Example 4.14

Chapter Opener: Examining the Carrying Capacity of a Deer Population

Figure 4.20 (credit: modification of work by Rachel Kramer, Flickr)

Let's consider the population of white-tailed deer (*Odocoileus virginianus*) in the state of Kentucky. The Kentucky Department of Fish and Wildlife Resources (KDFWR) sets guidelines for hunting and fishing in the state. Before the hunting season of $2004,$ it estimated a population of $900{,}000$ deer. Johnson notes: "A deer population that has plenty to eat and is not hunted by humans or other predators will double every three years." (George Johnson, "The Problem of Exploding Deer Populations Has No Attractive Solutions," January $12, 2001,$ accessed April 9, 2015, http://www.txtwriter.com/onscience/Articles/deerpops.html.) This observation corresponds to a rate of increase $r = \frac{\ln(2)}{3} = 0.2311,$ so the approximate growth rate is 23.11% per year. (This assumes that the population grows exponentially, which is reasonable—at least in the short term—with plentiful food supply and no predators.) The KDFWR also reports deer population densities for 32 counties in Kentucky, the average of which is approximately 27 deer per square mile. Suppose this is the deer density for the whole state $(39{,}732$ square miles). The carrying capacity K is $39{,}732$ square miles times 27 deer per square mile, or $1{,}072{,}764$ deer.

a. For this application, we have $P_0 = 900{,}000$, $K = 1{,}072{,}764$, and $r = 0.2311$. Substitute these values into **Equation 4.8** and form the initial-value problem.

b. Solve the initial-value problem from part a.

c. According to this model, what will be the population in 3 years? Recall that the doubling time predicted by Johnson for the deer population was 3 years. How do these values compare?

d. Suppose the population managed to reach $1{,}200{,}000$ deer. What does the logistic equation predict will happen to the population in this scenario?

Solution

a. The initial value problem is

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1{,}072{,}764}\right), \quad P(0) = 900{,}000.$$

b. The logistic equation is an autonomous differential equation, so we can use the method of separation of variables.

Step 1: Setting the right-hand side equal to zero gives $P = 0$ and $P = 1{,}072{,}764$. This means that if the population starts at zero it will never change, and if it starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation and multiply both sides by:

$$\begin{aligned} \frac{dP}{dt} &= 0.2311P\left(\frac{1{,}072{,}764 - P}{1{,}072{,}764}\right) \\ dP &= 0.2311P\left(\frac{1{,}072{,}764 - P}{1{,}072{,}764}\right)dt. \end{aligned}$$

Divide both sides by $P(1{,}072{,}764 - P)$:

$$\frac{dP}{P(1{,}072{,}764 - P)} = \frac{0.2311}{1{,}072{,}764}dt.$$

Step 3: Integrate both sides of the equation using partial fraction decomposition:

$$\begin{aligned} \int \frac{dP}{P(1{,}072{,}764 - P)} &= \int \frac{0.2311}{1{,}072{,}764}dt \\ \frac{1}{1{,}072{,}764}\int\left(\frac{1}{P} + \frac{1}{1{,}072{,}764 - P}\right)dP &= \frac{0.2311t}{1{,}072{,}764} + C \\ \frac{1}{1{,}072{,}764}\left(\ln|P| - \ln|1{,}072{,}764 - P|\right) &= \frac{0.2311t}{1{,}072{,}764} + C. \end{aligned}$$

Step 4: Multiply both sides by $1{,}072{,}764$ and use the quotient rule for logarithms:

$$\ln\left|\frac{P}{1{,}072{,}764 - P}\right| = 0.2311t + C_1.$$

Here $C_1 = 1{,}072{,}764C$. Next exponentiate both sides and eliminate the absolute value:

$$\begin{aligned} e^{\ln\left|\frac{P}{1{,}072{,}764 - P}\right|} &= e^{0.2311t + C_1} \\ \left|\frac{P}{1{,}072{,}764 - P}\right| &= C_2 e^{0.2311t} \\ \frac{P}{1{,}072{,}764 - P} &= C_2 e^{0.2311t}. \end{aligned}$$

Here $C_2 = e^{C_1}$ but after eliminating the absolute value, it can be negative as well. Now solve for:

$$\begin{aligned} P &= C_2 e^{0.2311t}(1{,}072{,}764 - P). \\ P &= 1{,}072{,}764 C_2 e^{0.2311t} - C_2 P e^{0.2311t} \\ P + C_2 P e^{0.2311t} &= 1{,}072{,}764 C_2 e^{0.2311t} \\ P\left(1 + C_2 e^{0.2311t}\right) &= 1{,}072{,}764 C_2 e^{0.2311t} \\ P(t) &= \frac{1{,}072{,}764 C_2 e^{0.2311t}}{1 + C_2 e^{0.2311t}}. \end{aligned}$$

Step 5: To determine the value of C_2, it is actually easier to go back a couple of steps to where C_2 was defined. In particular, use the equation

$$\frac{P}{1{,}072{,}764 - P} = C_2 e^{0.2311t}.$$

The initial condition is $P(0) = 900{,}000$. Replace P with $900{,}000$ and t with zero:

$$\begin{aligned} \frac{P}{1{,}072{,}764 - P} &= C_2 e^{0.2311t} \\ \frac{900{,}000}{1{,}072{,}764 - 900{,}000} &= C_2 e^{0.2311(0)} \\ \frac{900{,}000}{172{,}764} &= C_2 \\ C_2 &= \frac{25{,}000}{4{,}799} \approx 5.209. \end{aligned}$$

Therefore

$$\begin{aligned} P(t) &= \frac{1{,}072{,}764\left(\frac{25000}{4799}\right)e^{0.2311t}}{1 + \left(\frac{25000}{4799}\right)e^{0.2311t}} \\ &= \frac{1{,}072{,}764(25000)e^{0.2311t}}{4799 + 25000e^{0.2311t}}. \end{aligned}$$

Dividing the numerator and denominator by $25{,}000$ gives

$$P(t) = \frac{1{,}072{,}764e^{0.2311t}}{0.19196 + e^{0.2311t}}.$$

Figure 4.21 is a graph of this equation.

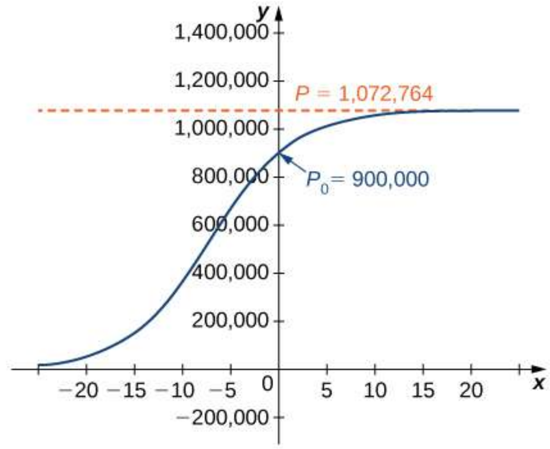


Figure 4.21 Logistic curve for the deer population with an initial population of $900,000$ deer.

c. Using this model we can predict the population in 3 years.

$$P(3) = \frac{1{,}072{,}764e^{0.2311(3)}}{0.19196 + e^{0.2311(3)}} \approx 978{,}830 \text{ deer}$$

This is far short of twice the initial population of $900{,}000.$ Remember that the doubling time is based on the assumption that the growth rate never changes, but the logistic model takes this possibility into account.

d. If the population reached $1{,}200{,}000$ deer, then the new initial-value problem would be

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1{,}072{,}764}\right), \quad P(0) = 1{,}200{,}000.$$

The general solution to the differential equation would remain the same.

$$P(t) = \frac{1{,}072{,}764C_2 e^{0.2311t}}{1 + C_2 e^{0.2311t}}$$

To determine the value of the constant, return to the equation

$$\frac{P}{1{,}072{,}764 - P} = C_2 e^{0.2311t}.$$

Substituting the values $t = 0$ and $P = 1{,}200{,}000,$ you get

$$\begin{aligned} C_2 e^{0.2311(0)} &= \frac{1{,}200{,}000}{1{,}072{,}764 - 1{,}200{,}000} \\ C_2 &= -\frac{100{,}000}{10{,}603} \approx -9.431. \end{aligned}$$

Therefore

$$\begin{aligned} P(t) &= \frac{1{,}072{,}764 C_2 e^{0.2311t}}{1 + C_2 e^{0.2311t}} \\ &= \frac{1{,}072{,}764\left(-\frac{100{,}000}{10{,}603}\right)e^{0.2311t}}{1 + \left(-\frac{100{,}000}{10{,}603}\right)e^{0.2311t}} \\ &= -\frac{107{,}276{,}400{,}000e^{0.2311t}}{100{,}000e^{0.2311t} - 10{,}603} \\ &\approx \frac{10{,}117{,}551e^{0.2311t}}{9.43129e^{0.2311t} - 1}. \end{aligned}$$

This equation is graphed in **Figure 4.22**.

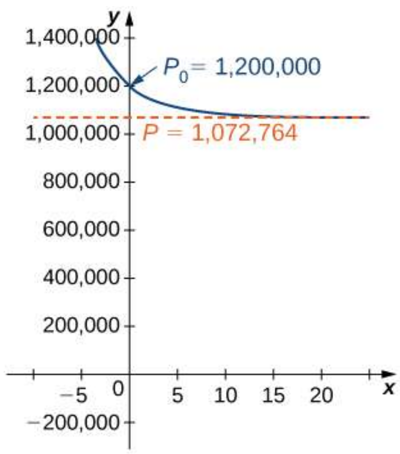


Figure 4.22 Logistic curve for the deer population with an initial population of $1{,}200{,}000$ deer.

Solving the Logistic Differential Equation

The logistic differential equation is an autonomous differential equation, so we can use separation of variables to find the general solution, as we just did in **Example 4.14**.

Step 1: Setting the right-hand side equal to zero leads to $P = 0$ and $P = K$ as constant solutions. The first solution indicates that when there are no organisms present, the population will never grow. The second solution indicates that when the population starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation in the form

$$\frac{dP}{dt} = \frac{rP(K - P)}{K}.$$

Then multiply both sides by dt and divide both sides by $P(K - P)$. This leads to

$$\frac{dP}{P(K - P)} = \frac{r}{K}dt.$$

Multiply both sides of the equation by K and integrate:

$$\int \frac{K}{P(K - P)}dP = \int r\,dt.$$

The left-hand side of this equation can be integrated using partial fraction decomposition. We leave it to you to verify that

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}.$$

Then the equation becomes

$$\begin{aligned} \int \frac{1}{P} + \frac{1}{K-P} dP &= \int r dt \\ \ln|P| - \ln|K-P| &= rt + C \\ \ln\left|\frac{P}{K-P}\right| &= rt + C. \end{aligned}$$

Now exponentiate both sides of the equation to eliminate the natural logarithm:

$$\begin{aligned} e^{\ln\left|\frac{P}{K-P}\right|} &= e^{rt+C} \\ \left|\frac{P}{K-P}\right| &= e^C e^{rt}. \end{aligned}$$

We define $C_1 = e^c$ so that the equation becomes

$$\frac{P}{K-P} = C_1 e^{rt}. \tag{4.9}$$

To solve this equation for $P(t)$, first multiply both sides by $K - P$ and collect the terms containing P on the left-hand side of the equation:

$$\begin{aligned} P &= C_1 e^{rt}(K-P) \\ P &= C_1 Ke^{rt} - C_1 Pe^{rt} \\ P + C_1 Pe^{rt} &= C_1 Ke^{rt}. \end{aligned}$$

Next, factor P from the left-hand side and divide both sides by the other factor:

$$\begin{aligned} P\left(1 + C_1 e^{rt}\right) &= C_1 Ke^{rt} \\ P(t) &= \frac{C_1 Ke^{rt}}{1 + C_1 e^{rt}}. \end{aligned} \tag{4.10}$$

The last step is to determine the value of C_1. The easiest way to do this is to substitute $t = 0$ and P_0 in place of P in **Equation 4.9** and solve for C_1:

$$\begin{aligned} \frac{P}{K-P} &= C_1 e^{rt} \\ \frac{P_0}{K-P_0} &= C_1 e^{r(0)} \\ C_1 &= \frac{P_0}{K-P_0}. \end{aligned}$$

Finally, substitute the expression for C_1 into **Equation 4.10**:

$$P(t) = \frac{C_1 Ke^{rt}}{1 + C_1 e^{rt}} = \frac{\frac{P_0}{K-P_0} Ke^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}}$$

Now multiply the numerator and denominator of the right-hand side by $\left(K - P_0\right)$ and simplify:

$$\begin{aligned} P(t) &= \frac{\frac{P_0}{K - P_0} K e^{rt}}{1 + \frac{P_0}{K - P_0} e^{rt}} \\ &= \frac{\frac{P_0}{K - P_0} K e^{rt}}{1 + \frac{P_0}{K - P_0} e^{rt}} \cdot \frac{K - P_0}{K - P_0} \\ &= \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}. \end{aligned}$$

We state this result as a theorem.

Theorem 4.2: Solution of the Logistic Differential Equation

Consider the logistic differential equation subject to an initial population of P_0 with carrying capacity K and growth rate r. The solution to the corresponding initial-value problem is given by

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}. \quad \text{(4.11)}$$

Now that we have the solution to the initial-value problem, we can choose values for P_0, r, and K and study the solution curve. For example, in **Example 4.14** we used the values $r = 0.2311$, $K = 1{,}072{,}764$, and an initial population of $900{,}000$ deer. This leads to the solution

$$\begin{aligned} P(t) &= \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}} \\ &= \frac{900{,}000(1{,}072{,}764)e^{0.2311t}}{(1{,}072{,}764 - 900{,}000) + 900{,}000e^{0.2311t}} \\ &= \frac{900{,}000(1{,}072{,}764)e^{0.2311t}}{172{,}764 + 900{,}000e^{0.2311t}}. \end{aligned}$$

Dividing top and bottom by $900{,}000$ gives

$$P(t) = \frac{1{,}072{,}764e^{0.2311t}}{0.19196 + e^{0.2311t}}.$$

This is the same as the original solution. The graph of this solution is shown again in blue in **Figure 4.23**, superimposed over the graph of the exponential growth model with initial population $900{,}000$ and growth rate 0.2311 (appearing in green). The red dashed line represents the carrying capacity, and is a horizontal asymptote for the solution to the logistic equation.

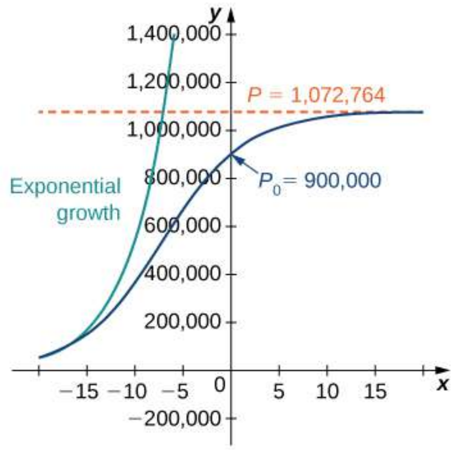


Figure 4.23 A comparison of exponential versus logistic growth for the same initial population of $900,000$ organisms and growth rate of 23.11%.

Working under the assumption that the population grows according to the logistic differential equation, this graph predicts that approximately 20 years earlier $(1984),$ the growth of the population was very close to exponential. The net growth rate at that time would have been around 23.1% per year. As time goes on, the two graphs separate. This happens because the population increases, and the logistic differential equation states that the growth rate decreases as the population increases. At the time the population was measured $(2004),$ it was close to carrying capacity, and the population was starting to level off.

The solution to the logistic differential equation has a point of inflection. To find this point, set the second derivative equal to zero:

$$\begin{aligned} P(t) &= \frac{P_0 K e^{rt}}{(K-P_0)+P_0 e^{rt}} \\ P'(t) &= \frac{rP_0 K(K-P_0)e^{rt}}{((K-P_0)+P_0 e^{rt})^2} \\ P''(t) &= \frac{r^2 P_0 K(K-P_0)^2 e^{rt} - r^2 {P_0}^2 K(K-P_0)e^{2rt}}{((K-P_0)+P_0 e^{rt})^3} \\ &= \frac{r^2 P_0 K(K-P_0)e^{rt}((K-P_0)-P_0 e^{rt})}{((K-P_0)+P_0 e^{rt})^3}. \end{aligned}$$

Setting the numerator equal to zero,

$$r^2 P_0 K(K-P_0)e^{rt}((K-P_0)-P_0 e^{rt}) = 0.$$

As long as $P_0 \neq K,$ the entire quantity before and including e^{rt} is nonzero, so we can divide it out:

$$(K-P_0) - P_0 e^{rt} = 0.$$

Solving for $t,$

$$\begin{aligned} P_0 e^{rt} &= K - P_0 \\ e^{rt} &= \frac{K - P_0}{P_0} \\ \ln e^{rt} &= \ln \frac{K - P_0}{P_0} \\ rt &= \ln \frac{K - P_0}{P_0} \\ t &= \frac{1}{r} \ln \frac{K - P_0}{P_0}. \end{aligned}$$

Notice that if $P_0 > K,$ then this quantity is undefined, and the graph does not have a point of inflection. In the logistic graph, the point of inflection can be seen as the point where the graph changes from concave up to concave down. This is where the "leveling off" starts to occur, because the net growth rate becomes slower as the population starts to approach the carrying capacity.

4.14 A population of rabbits in a meadow is observed to be 200 rabbits at time $t = 0.$ After a month, the rabbit population is observed to have increased by $4\%.$ Using an initial population of 200 and a growth rate of $0.04,$ with a carrying capacity of 750 rabbits,

a. Write the logistic differential equation and initial condition for this model.

b. Draw a slope field for this logistic differential equation, and sketch the solution corresponding to an initial population of 200 rabbits.

c. Solve the initial-value problem for $P(t).$

d. Use the solution to predict the population after 1 year.

Student PROJECT

Student Project: Logistic Equation with a Threshold Population

An improvement to the logistic model includes a **threshold population**. The threshold population is defined to be the minimum population that is necessary for the species to survive. We use the variable T to represent the threshold population. A differential equation that incorporates both the threshold population T and carrying capacity K is

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right) \tag{4.12}$$

where r represents the growth rate, as before.

1. The threshold population is useful to biologists and can be utilized to determine whether a given species should be placed on the endangered list. A group of Australian researchers say they have determined the threshold population for any species to survive: 5000 adults. (Catherine Clabby, "A Magic Number," *American Scientist* 98(1): 24, doi:10.1511/2010.82.24. accessed April 9, 2015, http://www.americanscientist.org/issues/pub/a-magic-number). Therefore we use $T = 5000$ as the threshold population in this project. Suppose that the environmental carrying capacity in Montana for elk is $25{,}000$. Set up **Equation 4.12** using the carrying capacity of $25{,}000$ and threshold population of 5000. Assume an annual net growth rate of 18%.
2. Draw the direction field for the differential equation from step $1,$ along with several solutions for different initial populations. What are the constant solutions of the differential equation? What do these solutions correspond to in the original population model (i.e., in a biological context)?
3. What is the limiting population for each initial population you chose in step 2? (Hint: use the slope field to see what happens for various initial populations, i.e., look for the horizontal asymptotes of your solutions.)
4. This equation can be solved using the method of separation of variables. However, it is very difficult to get the solution as an explicit function of t. Using an initial population of $18{,}000$ elk, solve the initial-value problem and express the solution as an implicit function of $t,$ or solve the general initial-value problem, finding a solution in terms of $r, K, T,$ and P_0.

4.4 EXERCISES

For the following problems, consider the logistic equation in the form $P' = CP - P^2$. Draw the directional field and find the stability of the equilibria.

168. $C = 3$

169. $C = 0$

170. $C = -3$

171. Solve the logistic equation for $C = 10$ and an initial condition of $P(0) = 2$.

172. Solve the logistic equation for $C = -10$ and an initial condition of $P(0) = 2$.

173. A population of deer inside a park has a carrying capacity of 200 and a growth rate of 2%. If the initial population is 50 deer, what is the population of deer at any given time?

174. A population of frogs in a pond has a growth rate of 5%. If the initial population is 1000 frogs and the carrying capacity is 6000, what is the population of frogs at any given time?

175. **[T]** Bacteria grow at a rate of 20% per hour in a petri dish. If there is initially one bacterium and a carrying capacity of 1 million cells, how long does it take to reach $500{,}000$ cells?

176. **[T]** Rabbits in a park have an initial population of 10 and grow at a rate of 4% per year. If the carrying capacity is 500, at what time does the population reach 100 rabbits?

177. **[T]** Two monkeys are placed on an island. After 5 years, there are 8 monkeys, and the estimated carrying capacity is 25 monkeys. When does the population of monkeys reach 16 monkeys?

178. **[T]** A butterfly sanctuary is built that can hold 2000 butterflies, and 400 butterflies are initially moved in. If after 2 months there are now 800 butterflies, when does the population get to 1500 butterflies?

The following problems consider the logistic equation with an added term for depletion, either through death or emigration.

179. **[T]** The population of trout in a pond is given by $P' = 0.4P\left(1 - \frac{P}{10000}\right) - 400$, where 400 trout are caught per year. Use your calculator or computer software to draw a directional field and draw a few sample solutions. What do you expect for the behavior?

180. In the preceding problem, what are the stabilities of the equilibria $0 < P_1 < P_2$?

181. **[T]** For the preceding problem, use software to generate a directional field for the value $f = 400$. What are the stabilities of the equilibria?

182. **[T]** For the preceding problems, use software to generate a directional field for the value $f = 600$. What are the stabilities of the equilibria?

183. **[T]** For the preceding problems, consider the case where a certain number of fish are added to the pond, or $f = -200$. What are the nonnegative equilibria and their stabilities?

It is more likely that the amount of fishing is governed by the current number of fish present, so instead of a constant number of fish being caught, the rate is proportional to the current number of fish present, with proportionality constant k, as

$$P' = 0.4P\left(1 - \frac{P}{10000}\right) - kP.$$

184. **[T]** For the previous fishing problem, draw a directional field assuming $k = 0.1$. Draw some solutions that exhibit this behavior. What are the equilibria and what are their stabilities?

185. **[T]** Use software or a calculator to draw directional fields for $k = 0.4$. What are the nonnegative equilibria and their stabilities?

186. **[T]** Use software or a calculator to draw directional fields for $k = 0.6$. What are the equilibria and their stabilities?

187. Solve this equation, assuming a value of $k = 0.05$ and an initial condition of 2000 fish.

188. Solve this equation, assuming a value of $k = 0.05$ and an initial condition of 5000 fish.

The following problems add in a minimal threshold value for the species to survive, T, which changes the differential equation to $P'(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$.

189. Draw the directional field of the threshold logistic equation, assuming $K = 10, r = 0.1, T = 2$. When does the population survive? When does it go extinct?

190. For the preceding problem, solve the logistic threshold equation, assuming the initial condition $P(0) = P_0$.

191. Bengal tigers in a conservation park have a carrying capacity of 100 and need a minimum of 10 to survive. If they grow in population at a rate of 1% per year, with an initial population of 15 tigers, solve for the number of tigers present.

192. A forest containing ring-tailed lemurs in Madagascar has the potential to support 5000 individuals, and the lemur population grows at a rate of 5% per year. A minimum of 500 individuals is needed for the lemurs to survive. Given an initial population of 600 lemurs, solve for the population of lemurs.

193. The population of mountain lions in Northern Arizona has an estimated carrying capacity of 250 and grows at a rate of 0.25% per year and there must be 25 for the population to survive. With an initial population of 30 mountain lions, how many years will it take to get the mountain lions off the endangered species list (at least 100)?

The following questions consider the Gompertz equation, a modification for logistic growth, which is often used for modeling cancer growth, specifically the number of tumor cells.

194. The Gompertz equation is given by $P(t)' = \alpha \ln\left(\frac{K}{P(t)}\right)P(t)$. Draw the directional fields for this equation assuming all parameters are positive, and given that $K = 1$.

195. Assume that for a population, $K = 1000$ and $\alpha = 0.05$. Draw the directional field associated with this differential equation and draw a few solutions. What is the behavior of the population?

196. Solve the Gompertz equation for generic α and K and $P(0) = P_0$.

197. **[T]** The Gompertz equation has been used to model tumor growth in the human body. Starting from one tumor cell on day 1 and assuming $\alpha = 0.1$ and a carrying capacity of 10 million cells, how long does it take to reach "detection" stage at 5 million cells?

198. **[T]** It is estimated that the world human population reached 3 billion people in 1959 and 6 billion in 1999. Assuming a carrying capacity of 16 billion humans, write and solve the differential equation for logistic growth, and determine what year the population reached 7 billion.

199. **[T]** It is estimated that the world human population reached 3 billion people in 1959 and 6 billion in 1999. Assuming a carrying capacity of 16 billion humans, write and solve the differential equation for Gompertz growth, and determine what year the population reached 7 billion. Was logistic growth or Gompertz growth more accurate, considering world population reached 7 billion on October $31, 2011$?

200. Show that the population grows fastest when it reaches half the carrying capacity for the logistic equation $P' = rP\left(1 - \frac{P}{K}\right)$.

201. When does population increase the fastest in the threshold logistic equation $P'(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$?

202. When does population increase the fastest for the Gompertz equation $P(t)' = \alpha \ln\left(\frac{K}{P(t)}\right)P(t)$?

Below is a table of the populations of whooping cranes in the wild from 1940 to 2000. The population rebounded from near extinction after conservation efforts began. The following problems consider applying population models to fit the data. Assume a carrying capacity of $10,000$ cranes. Fit the data assuming years since 1940 (so your initial population at time 0 would be 22 cranes).

Year (years since conservation began)	**Whooping Crane Population**
1940(0)	22
1950(10)	31
1960(20)	36
1970(30)	57
1980(40)	91
1990(50)	159
2000(60)	256

Source: https://www.savingcranes.org/images/stories/site_images/conservation/whooping_crane/pdfs/historic_wc_numbers.pdf

203. Find the equation and parameter r that best fit the data for the logistic equation.

204. Find the equation and parameters r and T that best fit the data for the threshold logistic equation.

205. Find the equation and parameter α that best fit the data for the Gompertz equation.

206. Graph all three solutions and the data on the same graph. Which model appears to be most accurate?

207. Using the three equations found in the previous problems, estimate the population in 2010 (year 70 after conservation). The real population measured at that time was 437. Which model is most accurate?

4.5 | First-order Linear Equations

Learning Objectives

4.5.1 Write a first-order linear differential equation in standard form.
4.5.2 Find an integrating factor and use it to solve a first-order linear differential equation.
4.5.3 Solve applied problems involving first-order linear differential equations.

Earlier, we studied an application of a first-order differential equation that involved solving for the velocity of an object. In particular, if a ball is thrown upward with an initial velocity of v_0 ft/s, then an initial-value problem that describes the velocity of the ball after t seconds is given by

$$\frac{dv}{dt} = -32, \quad v(0) = v_0.$$

This model assumes that the only force acting on the ball is gravity. Now we add to the problem by allowing for the possibility of air resistance acting on the ball.

Air resistance always acts in the direction opposite to motion. Therefore if an object is rising, air resistance acts in a downward direction. If the object is falling, air resistance acts in an upward direction (**Figure 4.24**). There is no exact relationship between the velocity of an object and the air resistance acting on it. For very small objects, air resistance is proportional to velocity; that is, the force due to air resistance is numerically equal to some constant k times v. For larger (e.g., baseball-sized) objects, depending on the shape, air resistance can be approximately proportional to the square of the velocity. In fact, air resistance may be proportional to $v^{1.5}$, or $v^{0.9}$, or some other power of v.

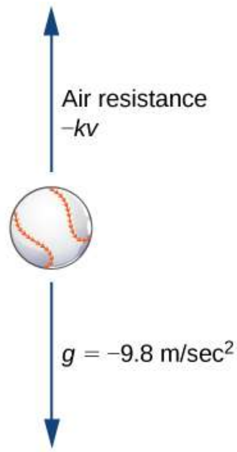


Figure 4.24 Forces acting on a moving baseball: gravity acts in a downward direction and air resistance acts in a direction opposite to the direction of motion.

We will work with the linear approximation for air resistance. If we assume $k > 0$, then the expression for the force F_A due to air resistance is given by $F_A = -kv$. Therefore the sum of the forces acting on the object is equal to the sum of the gravitational force and the force due to air resistance. This, in turn, is equal to the mass of the object multiplied by its acceleration at time t (Newton's second law). This gives us the differential equation

$$m\frac{dv}{dt} = -kv - mg.$$

Finally, we impose an initial condition $v(0) = v_0$, where v_0 is the initial velocity measured in meters per second. This makes $g = 9.8 \text{ m/s}^2$. The initial-value problem becomes

$$m\frac{dv}{dt} = -kv - mg, \quad v(0) = v_0. \tag{4.13}$$

The differential equation in this initial-value problem is an example of a first-order linear differential equation. (Recall that a differential equation is first-order if the highest-order derivative that appears in the equation is $1.$) In this section, we study first-order linear equations and examine a method for finding a general solution to these types of equations, as well as solving initial-value problems involving them.

Definition

A first-order differential equation is **linear** if it can be written in the form

$$a(x)y' + b(x)y = c(x), \tag{4.14}$$

where $a(x)$, $b(x)$, and $c(x)$ are arbitrary functions of x.

Remember that the unknown function y depends on the variable x; that is, x is the independent variable and y is the dependent variable. Some examples of first-order linear differential equations are

$$\begin{aligned}(3x^2 - 4)y' + (x - 3)y &= \sin x \\ (\sin x)y' - (\cos x)y &= \cot x \\ 4xy' + (3\ln x)y &= x^3 - 4x.\end{aligned}$$

Examples of first-order nonlinear differential equations include

$$\begin{aligned}(y')^4 - (y')^3 &= (3x - 2)(y + 4) \\ 4y' + 3y^3 &= 4x - 5 \\ (y')^2 &= \sin y + \cos x.\end{aligned}$$

These equations are nonlinear because of terms like $(y')^4$, y^3, etc. Due to these terms, it is impossible to put these equations into the same form as Equation 4.14.

Standard Form

Consider the differential equation

$$(3x^2 - 4)y' + (x - 3)y = \sin x.$$

Our main goal in this section is to derive a solution method for equations of this form. It is useful to have the coefficient of y' be equal to 1. To make this happen, we divide both sides by $3x^2 - 4.$

$$y' + \left(\frac{x - 3}{3x^2 - 4}\right)y = \frac{\sin x}{3x^2 - 4}$$

This is called the **standard form** of the differential equation. We will use it later when finding the solution to a general first-order linear differential equation. Returning to Equation 4.14, we can divide both sides of the equation by $a(x)$. This leads to the equation

$$y' + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}. \tag{4.15}$$

Now define $p(x) = \frac{b(x)}{a(x)}$ and $q(x) = \frac{c(x)}{a(x)}$. Then Equation 4.14 becomes

$$y' + p(x)y = q(x). \tag{4.16}$$

We can write any first-order linear differential equation in this form, and this is referred to as the standard form for a first-order linear differential equation.

Example 4.15

Writing First-Order Linear Equations in Standard Form

Put each of the following first-order linear differential equations into standard form. Identify $p(x)$ and $q(x)$ for each equation.

a. $y' = 3x - 4y$

b. $\frac{3xy'}{4y-3} = 2$ (here $x > 0$)

c. $y = 3y' - 4x^2 + 5$

Solution

a. Add $4y$ to both sides:

$$y' + 4y = 3x.$$

In this equation, $p(x) = 4$ and $q(x) = 3x$.

b. Multiply both sides by $4y - 3$, then subtract $8y$ from each side:

$$\begin{aligned} \frac{3xy'}{4y-3} &= 2 \\ 3xy' &= 2(4y-3) \\ 3xy' &= 8y - 6 \\ 3xy' - 8y &= -6. \end{aligned}$$

Finally, divide both sides by $3x$ to make the coefficient of y' equal to 1:

$$y' - \frac{8}{3x}y = -\frac{2}{3x}. \qquad (4.17)$$

This is allowable because in the original statement of this problem we assumed that $x > 0$. (If $x = 0$ then the original equation becomes $0 = 2$, which is clearly a false statement.)

In this equation, $p(x) = -\frac{8}{3x}$ and $q(x) = -\frac{2}{3x}$.

c. Subtract y from each side and add $4x^2 - 5$:

$$3y' - y = 4x^2 - 5.$$

Next divide both sides by 3:

$$y' - \frac{1}{3}y = \frac{4}{3}x^2 - \frac{5}{3}.$$

In this equation, $p(x) = -\frac{1}{3}$ and $q(x) = \frac{4}{3}x^2 - \frac{5}{3}$.

4.15 Put the equation $\frac{(x+3)y'}{2x-3y-4} = 5$ into standard form and identify $p(x)$ and $q(x)$.

Integrating Factors

We now develop a solution technique for any first-order linear differential equation. We start with the standard form of a first-order linear differential equation:

$$y' + p(x)y = q(x). \tag{4.18}$$

The first term on the left-hand side of **Equation 4.15** is the derivative of the unknown function, and the second term is the product of a known function with the unknown function. This is somewhat reminiscent of the power rule from the **Differentiation Rules (http://cnx.org/content/m53575/latest/)** section. If we multiply **Equation 4.16** by a yet-to-be-determined function $\mu(x)$, then the equation becomes

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{4.19}$$

The left-hand side **Equation 4.18** can be matched perfectly to the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Matching term by term gives $y = f(x)$, $g(x) = \mu(x)$, and $g'(x) = \mu(x)p(x)$. Taking the derivative of $g(x) = \mu(x)$ and setting it equal to the right-hand side of $g'(x) = \mu(x)p(x)$ leads to

$$\mu'(x) = \mu(x)p(x).$$

This is a first-order, separable differential equation for $\mu(x)$. We know $p(x)$ because it appears in the differential equation we are solving. Separating variables and integrating yields

$$\begin{aligned} \frac{\mu'(x)}{\mu(x)} &= p(x) \\ \int \frac{\mu'(x)}{\mu(x)}dx &= \int p(x)\,dx \\ \ln|\mu(x)| &= \int p(x)\,dx + C \\ e^{\ln|\mu(x)|} &= e^{\int p(x)\,dx + C} \\ |\mu(x)| &= C_1 e^{\int p(x)\,dx} \\ \mu(x) &= C_2 e^{\int p(x)\,dx}. \end{aligned}$$

Here C_2 can be an arbitrary (positive or negative) constant. This leads to a general method for solving a first-order linear differential equation. We first multiply both sides of **Equation 4.16** by the **integrating factor** $\mu(x)$. This gives

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{4.20}$$

The left-hand side of **Equation 4.19** can be rewritten as $\frac{d}{dx}(\mu(x)y)$.

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x). \tag{4.21}$$

Next integrate both sides of **Equation 4.20** with respect to x.

$$\begin{aligned} \int \frac{d}{dx}(\mu(x)y)dx &= \int \mu(x)q(x)\,dx \\ \mu(x)y &= \int \mu(x)q(x)\,dx. \end{aligned} \tag{4.22}$$

Divide both sides of **Equation 4.21** by $\mu(x)$:

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)q(x)\,dx + C\right]. \tag{4.23}$$

Since $\mu(x)$ was previously calculated, we are now finished. An important note about the integrating constant C: It may seem that we are inconsistent in the usage of the integrating constant. However, the integral involving $p(x)$ is necessary in order to find an integrating factor for **Equation 4.15**. Only one integrating factor is needed in order to solve the equation; therefore, it is safe to assign a value for C for this integral. We chose $C = 0.$ When calculating the integral inside the brackets in **Equation 4.21**, it is necessary to keep our options open for the value of the integrating constant, because our goal is to find a general family of solutions to **Equation 4.15**. This integrating factor guarantees just that.

Problem-Solving Strategy: Solving a First-order Linear Differential Equation

1. Put the equation into standard form and identify $p(x)$ and $q(x)$.
2. Calculate the integrating factor $\mu(x) = e^{\int p(x)dx}$.
3. Multiply both sides of the differential equation by $\mu(x)$.
4. Integrate both sides of the equation obtained in step $3,$ and divide both sides by $\mu(x)$.
5. If there is an initial condition, determine the value of C.

Example 4.16

Solving a First-order Linear Equation

Find a general solution for the differential equation $xy' + 3y = 4x^2 - 3x.$ Assume $x > 0.$

Solution

1. To put this differential equation into standard form, divide both sides by x:

$$y' + \frac{3}{x}y = 4x - 3.$$

Therefore $p(x) = \frac{3}{x}$ and $q(x) = 4x - 3.$

2. The integrating factor is $\mu(x) = e^{\int (3/x)dx} = e^{3\ln x} = x^3.$
3. Multiplying both sides of the differential equation by $\mu(x)$ gives us

$$\begin{aligned} x^3 y' + x^3\left(\frac{3}{x}\right)y &= x^3(4x - 3) \\ x^3 y' + 3x^2 y &= 4x^4 - 3x^3 \\ \frac{d}{dx}\left(x^3 y\right) &= 4x^4 - 3x^3. \end{aligned}$$

4. Integrate both sides of the equation.

$$\begin{aligned} \int \frac{d}{dx}\left(x^3 y\right)dx &= \int 4x^4 - 3x^3 dx \\ x^3 y &= \frac{4x^5}{5} - \frac{3x^4}{4} + C \\ y &= \frac{4x^2}{5} - \frac{3x}{4} + Cx^{-3}. \end{aligned}$$

5. There is no initial value, so the problem is complete.

Analysis

You may have noticed the condition that was imposed on the differential equation; namely, $x > 0$. For any nonzero value of C, the general solution is not defined at $x = 0$. Furthermore, when $x < 0$, the integrating factor changes. The integrating factor is given by **Equation 4.19** as $f(x) = e^{\int p(x)dx}$. For this $p(x)$ we get

$$e^{\int p(x)\,dx} = e^{\int (3/x)dx} = e^{3\ln|x|} = |x|^3,$$

since $x < 0$. The behavior of the general solution changes at $x = 0$ largely due to the fact that $p(x)$ is not defined there.

4.16 Find the general solution to the differential equation $(x-2)y' + y = 3x^2 + 2x$. Assume $x > 2$.

Now we use the same strategy to find the solution to an initial-value problem.

Example 4.17

A First-order Linear Initial-Value Problem

Solve the initial-value problem

$$y' + 3y = 2x - 1, \qquad y(0) = 3.$$

Solution

1. This differential equation is already in standard form with $p(x) = 3$ and $q(x) = 2x - 1$.
2. The integrating factor is $\mu(x) = e^{\int 3dx} = e^{3x}$.
3. Multiplying both sides of the differential equation by $\mu(x)$ gives

$$\begin{aligned} e^{3x}y' + 3e^{3x}y &= (2x-1)e^{3x} \\ \frac{d}{dx}\left[ye^{3x}\right] &= (2x-1)e^{3x}. \end{aligned}$$

Integrate both sides of the equation:

$$\begin{aligned} \int \frac{d}{dx}\left[ye^{3x}\right]dx &= \int (2x-1)e^{3x}\,dx \\ ye^{3x} &= \frac{e^{3x}}{3}(2x-1) - \int \frac{2}{3}e^{3x}\,dx \\ ye^{3x} &= \frac{e^{3x}(2x-1)}{3} - \frac{2e^{3x}}{9} + C \\ y &= \frac{2x-1}{3} - \frac{2}{9} + Ce^{-3x} \\ y &= \frac{2x}{3} - \frac{5}{9} + Ce^{-3x}. \end{aligned}$$

4. Now substitute $x = 0$ and $y = 3$ into the general solution and solve for C:

$$\begin{aligned} y &= \frac{2}{3}x - \frac{5}{9} + Ce^{-3x} \\ 3 &= \frac{2}{3}(0) - \frac{5}{9} + Ce^{-3(0)} \\ 3 &= -\frac{5}{9} + C \\ C &= \frac{32}{9}. \end{aligned}$$

Therefore the solution to the initial-value problem is

$$y = \frac{2}{3}x - \frac{5}{9} + \frac{32}{9}e^{-3x}.$$

4.17 Solve the initial-value problem $y' - 2y = 4x + 3 \quad y(0) = -2.$

Applications of First-order Linear Differential Equations

We look at two different applications of first-order linear differential equations. The first involves air resistance as it relates to objects that are rising or falling; the second involves an electrical circuit. Other applications are numerous, but most are solved in a similar fashion.

Free fall with air resistance

We discussed air resistance at the beginning of this section. The next example shows how to apply this concept for a ball in vertical motion. Other factors can affect the force of air resistance, such as the size and shape of the object, but we ignore them here.

Example 4.18

A Ball with Air Resistance

A racquetball is hit straight upward with an initial velocity of 2 m/s. The mass of a racquetball is approximately 0.0427 kg. Air resistance acts on the ball with a force numerically equal to $0.5v,$ where v represents the velocity of the ball at time t.

a. Find the velocity of the ball as a function of time.
b. How long does it take for the ball to reach its maximum height?
c. If the ball is hit from an initial height of 1 meter, how high will it reach?

Solution

a. The mass $m = 0.0427\,\text{kg},$ $k = 0.5,$ and $g = 9.8\,\text{m/s}^2.$ The initial velocity is $v_0 = 2$ m/s. Therefore the initial-value problem is

$$0.0427\frac{dv}{dt} = -0.5v - 0.0427(9.8), \qquad v_0 = 2.$$

Dividing the differential equation by 0.0427 gives

$$\frac{dv}{dt} = -11.7096v - 9.8, \quad v_0 = 2.$$

The differential equation is linear. Using the problem-solving strategy for linear differential equations:

Step 1. Rewrite the differential equation as $\frac{dv}{dt} + 11.7096v = -9.8.$ This gives $p(t) = 11.7096$ and $q(t) = -9.8$

Step 2. The integrating factor is $\mu(t) = e^{\int 11.7096 dt} = e^{11.7096t}.$

Step 3. Multiply the differential equation by $\mu(t)$:

$$\begin{aligned} e^{11.7096t}\frac{dv}{dt} + 11.7096ve^{11.7096t} &= -9.8e^{11.7096t} \\ \frac{d}{dt}\left[ve^{11.7096t}\right] &= -9.8e^{11.7096t}. \end{aligned}$$

Step 4. Integrate both sides:

$$\begin{aligned} \int \frac{d}{dt}\left[ve^{11.7096t}\right]dt &= \int -9.8e^{11.7096t}\,dt \\ ve^{11.7096t} &= \frac{-9.8}{11.7096}e^{11.7096t} + C \\ v(t) &= -0.8369 + Ce^{-11.7096t}. \end{aligned}$$

Step 5. Solve for C using the initial condition $v_0 = v(0) = 2$:

$$\begin{aligned} v(t) &= -0.8369 + Ce^{-11.7096t} \\ v(0) &= -0.8369 + Ce^{-11.7096(0)} \\ 2 &= -0.8369 + C \\ C &= 2.8369. \end{aligned}$$

Therefore the solution to the initial-value problem is $v(t) = 2.8369e^{-11.7096t} - 0.8369.$

b. The ball reaches its maximum height when the velocity is equal to zero. The reason is that when the velocity is positive, it is rising, and when it is negative, it is falling. Therefore when it is zero, it is neither rising nor falling, and is at its maximum height:

$$\begin{aligned} 2.8369e^{-11.7096t} - 0.8369 &= 0 \\ 2.8369e^{-11.7096t} &= 0.8369 \\ e^{-11.7096t} &= \frac{0.8369}{2.8369} \approx 0.295 \\ \ln e^{-11.7096t} &= \ln 0.295 \approx -1.221 \\ -11.7096t &= -1.221 \\ t &\approx 0.104. \end{aligned}$$

Therefore it takes approximately 0.104 second to reach maximum height.

c. To find the height of the ball as a function of time, use the fact that the derivative of position is velocity, i.e., if $h(t)$ represents the height at time $t,$ then $h'(t) = v(t).$ Because we know $v(t)$ and the initial height, we can form an initial-value problem:

$$h'(t) = 2.8369e^{-11.7096t} - 0.8369, \quad h(0) = 1.$$

Integrating both sides of the differential equation with respect to t gives

$$\begin{aligned}\int h'(t)\,dt &= \int 2.8369e^{-11.7096t} - 0.8369dt \\ h(t) &= -\frac{2.8369}{11.7096}e^{-11.7096t} - 0.8369t + C \\ h(t) &= -0.2423e^{-11.7096t} - 0.8369t + C.\end{aligned}$$

Solve for C by using the initial condition:

$$\begin{aligned}h(t) &= -0.2423e^{-11.7096t} - 0.8369t + C \\ h(0) &= -0.2423e^{-11.7096(0)} - 0.8369(0) + C \\ 1 &= -0.2423 + C \\ C &= 1.2423.\end{aligned}$$

Therefore

$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423.$$

After 0.104 second, the height is given by
$h(0.2) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423 \approx 1.0836$ meter.

4.18 The weight of a penny is 2.5 grams (United States Mint, “Coin Specifications,” accessed April 9, 2015, http://www.usmint.gov/about_the_mint/?action=coin_specifications), and the upper observation deck of the Empire State Building is 369 meters above the street. Since the penny is a small and relatively smooth object, air resistance acting on the penny is actually quite small. We assume the air resistance is numerically equal to $0.0025v$. Furthermore, the penny is dropped with no initial velocity imparted to it.

a. Set up an initial-value problem that represents the falling penny.

b. Solve the problem for $v(t)$.

c. What is the terminal velocity of the penny (i.e., calculate the limit of the velocity as t approaches infinity)?

Electrical Circuits

A source of electromotive force (e.g., a battery or generator) produces a flow of current in a closed circuit, and this current produces a voltage drop across each resistor, inductor, and capacitor in the circuit. Kirchhoff’s Loop Rule states that the sum of the voltage drops across resistors, inductors, and capacitors is equal to the total electromotive force in a closed circuit. We have the following three results:

1. The voltage drop across a resistor is given by

$$E_R = Ri,$$

where R is a constant of proportionality called the *resistance*, and i is the current.

2. The voltage drop across an inductor is given by

$$E_L = Li',$$

where L is a constant of proportionality called the *inductance*, and i again denotes the current.

3. The voltage drop across a capacitor is given by

$$E_C = \frac{1}{C}q,$$

where C is a constant of proportionality called the *capacitance*, and q is the instantaneous charge on the capacitor. The relationship between i and q is $i = q'$.

We use units of volts (V) to measure voltage E, amperes (A) to measure current i, coulombs (C) to measure charge q, ohms (Ω) to measure resistance R, henrys (H) to measure inductance L, and farads (F) to measure capacitance C. Consider the circuit in **Figure 4.25**.

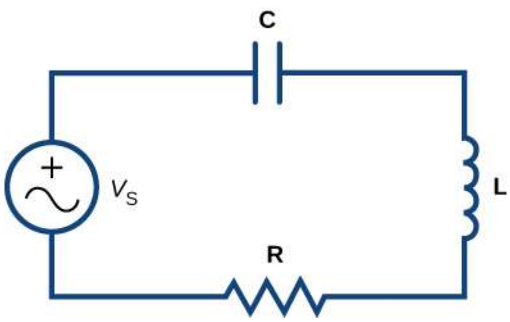


Figure 4.25 A typical electric circuit, containing a voltage generator (V_S), capacitor (C), inductor (L), and resistor (R).

Applying Kirchhoff's Loop Rule to this circuit, we let E denote the electromotive force supplied by the voltage generator. Then

$$E_L + E_R + E_C = E.$$

Substituting the expressions for E_L, E_R, and E_C into this equation, we obtain

$$Li' + Ri + \frac{1}{C}q = E. \tag{4.24}$$

If there is no capacitor in the circuit, then the equation becomes

$$Li' + Ri = E. \tag{4.25}$$

This is a first-order differential equation in i. The circuit is referred to as an LR circuit.

Next, suppose there is no inductor in the circuit, but there is a capacitor and a resistor, so $L = 0$, $R \neq 0$, and $C \neq 0$. Then **Equation 4.23** can be rewritten as

$$Rq' + \frac{1}{C}q = E, \tag{4.26}$$

which is a first-order linear differential equation. This is referred to as an *RC* circuit. In either case, we can set up and solve an initial-value problem.

Example 4.19

Finding Current in an *RL* Electric Circuit

A circuit has in series an electromotive force given by $E = 50\sin 20t\,\text{V}$, a resistor of 5Ω, and an inductor of $0.4\,\text{H}$. If the initial current is 0, find the current at time $t > 0$.

Solution

We have a resistor and an inductor in the circuit, so we use **Equation 4.24**. The voltage drop across the resistor is given by $E_R = Ri = 5i$. The voltage drop across the inductor is given by $E_L = Li' = 0.4i'$. The electromotive force becomes the right-hand side of **Equation 4.24**. Therefore **Equation 4.24** becomes

$$0.4i' + 5i = 50\sin 20t.$$

Dividing both sides by 0.4 gives the equation

$$i' + 12.5i = 125\sin 20t.$$

Since the initial current is 0, this result gives an initial condition of $i(0) = 0$. We can solve this initial-value problem using the five-step strategy for solving first-order differential equations.

Step 1. Rewrite the differential equation as $i' + 12.5i = 125\sin 20t$. This gives $p(t) = 12.5$ and $q(t) = 125\sin 20t$.

Step 2. The integrating factor is $\mu(t) = e^{\int 12.5dt} = e^{12.5t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

$$\begin{aligned} e^{12.5t}i' + 12.5e^{12.5t}i &= 125e^{12.5t}\sin 20t \\ \frac{d}{dt}\left[ie^{12.5t}\right] &= 125e^{12.5t}\sin 20t. \end{aligned}$$

Step 4. Integrate both sides:

$$\begin{aligned} \int \frac{d}{dt}\left[ie^{12.5t}\right]dt &= \int 125e^{12.5t}\sin 20t\,dt \\ ie^{12.5t} &= \left(\frac{250\sin 20t - 400\cos 20t}{89}\right)e^{12.5t} + C \\ i(t) &= \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t}. \end{aligned}$$

Step 5. Solve for C using the initial condition $v(0) = 2$:

$$\begin{aligned} i(t) &= \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t} \\ i(0) &= \frac{250\sin 20(0) - 400\cos 20(0)}{89} + Ce^{-12.5(0)} \\ 0 &= -\frac{400}{89} + C \\ C &= \frac{400}{89}. \end{aligned}$$

Therefore the solution to the initial-value problem is $i(t) = \frac{250\sin 20t - 400\cos 20t + 400e^{-12.5t}}{89} = \frac{250\sin 20t - 400\cos 20t}{89} + \frac{400e^{-12.5t}}{89}$.

The first term can be rewritten as a single cosine function. First, multiply and divide by $\sqrt{250^2 + 400^2} = 50\sqrt{89}$:

$$\begin{aligned} \frac{250\sin 20t - 400\cos 20t}{89} &= \frac{50\sqrt{89}}{89}\left(\frac{250\sin 20t - 400\cos 20t}{50\sqrt{89}}\right) \\ &= -\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}} - \frac{5\sin 20t}{\sqrt{89}}\right). \end{aligned}$$

Next, define φ to be an acute angle such that $\cos\varphi = \frac{8}{\sqrt{89}}$. Then $\sin\varphi = \frac{5}{\sqrt{89}}$ and

$$-\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}} - \frac{5\sin 20t}{\sqrt{89}}\right) = -\frac{50\sqrt{89}}{89}(\cos\varphi\cos 20t - \sin\varphi\sin 20t)$$

$$= -\frac{50\sqrt{89}}{89}\cos(20t + \varphi).$$

Therefore the solution can be written as

$$i(t) = -\frac{50\sqrt{89}}{89}\cos(20t + \varphi) + \frac{400e^{-12.5t}}{89}.$$

The second term is called the *attenuation* term, because it disappears rapidly as t grows larger. The phase shift is given by φ, and the amplitude of the steady-state current is given by $\frac{50\sqrt{89}}{89}$. The graph of this solution appears in **Figure 4.26**:

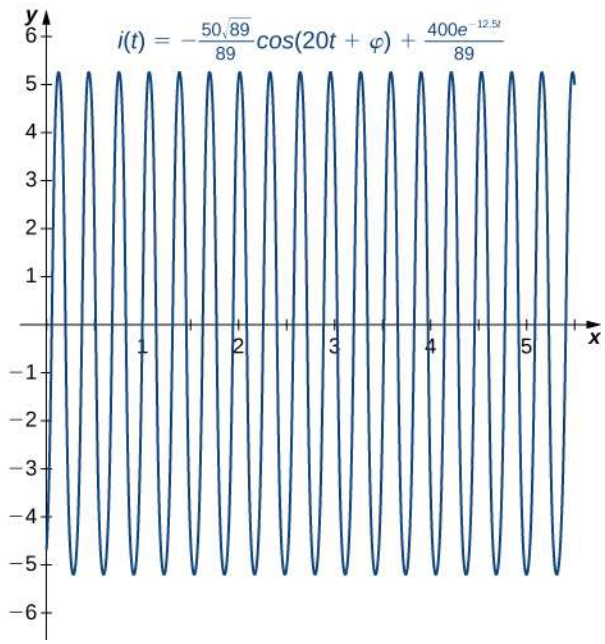


Figure 4.26

4.19 A circuit has in series an electromotive force given by $E = 20\sin 5t$ V, a capacitor with capacitance 0.02 F, and a resistor of $8\ \Omega$. If the initial charge is 4 C, find the charge at time $t > 0$.

4.5 EXERCISES

Are the following differential equations linear? Explain your reasoning.

208. $\frac{dy}{dx} = x^2 y + \sin x$

209. $\frac{dy}{dt} = ty$

210. $\frac{dy}{dt} + y^2 = x$

211. $y' = x^3 + e^x$

212. $y' = y + e^y$

Write the following first-order differential equations in standard form.

213. $y' = x^3 y + \sin x$

214. $y' + 3y - \ln x = 0$

215. $-xy' = (3x + 2)y + xe^x$

216. $\frac{dy}{dt} = 4y + ty + \tan t$

217. $\frac{dy}{dt} = yx(x + 1)$

What are the integrating factors for the following differential equations?

218. $y' = xy + 3$

219. $y' + e^x y = \sin x$

220. $y' = x \ln(x)y + 3x$

221. $\frac{dy}{dx} = \tanh(x)y + 1$

222. $\frac{dy}{dt} + 3ty = e^t y$

Solve the following differential equations by using integrating factors.

223. $y' = 3y + 2$

224. $y' = 2y - x^2$

225. $xy' = 3y - 6x^2$

226. $(x + 2)y' = 3x + y$

227. $y' = 3x + xy$

228. $xy' = x + y$

229. $\sin(x)y' = y + 2x$

230. $y' = y + e^x$

231. $xy' = 3y + x^2$

232. $y' + \ln x = \frac{y}{x}$

Solve the following differential equations. Use your calculator to draw a family of solutions. Are there certain initial conditions that change the behavior of the solution?

233. **[T]** $(x + 2)y' = 2y - 1$

234. **[T]** $y' = 3e^{t/3} - 2y$

235. **[T]** $xy' + \frac{y}{2} = \sin(3t)$

236. **[T]** $xy' = 2\frac{\cos x}{x} - 3y$

237. **[T]** $(x + 1)y' = 3y + x^2 + 2x + 1$

238. **[T]** $\sin(x)y' + \cos(x)y = 2x$

239. **[T]** $\sqrt{x^2 + 1}y' = y + 2$

240. **[T]** $x^3 y' + 2x^2 y = x + 1$

Solve the following initial-value problems by using integrating factors.

241. $y' + y = x,\ y(0) = 3$

242. $y' = y + 2x^2,\ y(0) = 0$

243. $xy' = y - 3x^3,\ y(1) = 0$

244. $x^2 y' = xy - \ln x,\ y(1) = 1$

245. $\left(1 + x^2\right)y' = y - 1,\ y(0) = 0$

246. $xy' = y + 2x \ln x,\ y(1) = 5$

247. $(2+x)y' = y+2+x,\ y(0)=0$

248. $y' = xy+2xe^x,\ y(0)=2$

249. $\sqrt{x}y' = y+2x,\ y(0)=1$

250. $y' = 2y+xe^x,\ y(0)=-1$

251. A falling object of mass m can reach terminal velocity when the drag force is proportional to its velocity, with proportionality constant k. Set up the differential equation and solve for the velocity given an initial velocity of 0.

252. Using your expression from the preceding problem, what is the terminal velocity? (*Hint:* Examine the limiting behavior; does the velocity approach a value?)

253. **[T]** Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is 9.8 m/s^2 and the proportionality constant is 4?

254. A more accurate way to describe terminal velocity is that the drag force is proportional to the square of velocity, with a proportionality constant k. Set up the differential equation and solve for the velocity.

255. Using your expression from the preceding problem, what is the terminal velocity? (*Hint:* Examine the limiting behavior: Does the velocity approach a value?)

256. **[T]** Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is $9.8\,\text{m/s}^2$ and the proportionality constant is 4? Does it take more or less time than your initial estimate?

For the following problems, determine how parameter a affects the solution.

257. Solve the generic equation $y' = ax+y$. How does varying a change the behavior?

258. Solve the generic equation $y' = ax+y$. How does varying a change the behavior?

259. Solve the generic equation $y' = ax+xy$. How does varying a change the behavior?

260. Solve the generic equation $y' = x+axy$. How does varying a change the behavior?

261. Solve $y' - y = e^{kt}$ with the initial condition $y(0)=0$. As k approaches 1, what happens to your formula?

CHAPTER 4 REVIEW

KEY TERMS

asymptotically semi-stable solution $y = k$ if it is neither asymptotically stable nor asymptotically unstable

asymptotically stable solution $y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y), \quad y(x_0) = c$ approaches k as x approaches infinity

asymptotically unstable solution $y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon, k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y), \quad y(x_0) = c$ never approaches k as x approaches infinity

autonomous differential equation an equation in which the right-hand side is a function of y alone

carrying capacity the maximum population of an organism that the environment can sustain indefinitely

differential equation an equation involving a function $y = y(x)$ and one or more of its derivatives

direction field (slope field) a mathematical object used to graphically represent solutions to a first-order differential equation; at each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point

equilibrium solution any solution to the differential equation of the form $y = c,$ where c is a constant

Euler's Method a numerical technique used to approximate solutions to an initial-value problem

general solution (or family of solutions) the entire set of solutions to a given differential equation

growth rate the constant $r > 0$ in the exponential growth function $P(t) = P_0 e^{rt}$

initial population the population at time $t = 0$

initial value(s) a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

initial velocity the velocity at time $t = 0$

initial-value problem a differential equation together with an initial value or values

integrating factor any function $f(x)$ that is multiplied on both sides of a differential equation to make the side involving the unknown function equal to the derivative of a product of two functions

linear description of a first-order differential equation that can be written in the form $a(x)y' + b(x)y = c(x)$

logistic differential equation a differential equation that incorporates the carrying capacity K and growth rate r into a population model

order of a differential equation the highest order of any derivative of the unknown function that appears in the equation

particular solution member of a family of solutions to a differential equation that satisfies a particular initial condition

phase line a visual representation of the behavior of solutions to an autonomous differential equation subject to various initial conditions

separable differential equation any equation that can be written in the form $y' = f(x)g(y)$

separation of variables a method used to solve a separable differential equation

solution curve a curve graphed in a direction field that corresponds to the solution to the initial-value problem passing through a given point in the direction field

solution to a differential equation a function $y = f(x)$ that satisfies a given differential equation

standard form the form of a first-order linear differential equation obtained by writing the differential equation in the form $y' + p(x)y = q(x)$

step size the increment h that is added to the x value at each step in Euler's Method

threshold population the minimum population that is necessary for a species to survive

KEY EQUATIONS

- **Euler's Method**
  $x_n = x_0 + nh$
  $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$, where h is the step size
- **Separable differential equation**
  $y' = f(x)g(y)$
- **Solution concentration**
  $\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}$
- **Newton's law of cooling**
  $\frac{dT}{dt} = k(T - T_s)$
- **Logistic differential equation and initial-value problem**
  $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0$
- **Solution to the logistic differential equation/initial-value problem**
  $P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$
- **Threshold population model**
  $\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$
- **standard form**
  $y' + p(x)y = q(x)$
- **integrating factor**
  $\mu(x) = e^{\int p(x)\,dx}$

KEY CONCEPTS

4.1 Basics of Differential Equations

- A differential equation is an equation involving a function $y = f(x)$ and one or more of its derivatives. A solution is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.
- The order of a differential equation is the highest order of any derivative of the unknown function that appears in the equation.
- A differential equation coupled with an initial value is called an initial-value problem. To solve an initial-value problem, first find the general solution to the differential equation, then determine the value of the constant. Initial-value problems have many applications in science and engineering.

4.2 Direction Fields and Numerical Methods

- A direction field is a mathematical object used to graphically represent solutions to a first-order differential equation.

- Euler's Method is a numerical technique that can be used to approximate solutions to a differential equation.

4.3 Separable Equations

- A separable differential equation is any equation that can be written in the form $y' = f(x)g(y)$.
- The method of separation of variables is used to find the general solution to a separable differential equation.

4.4 The Logistic Equation

- When studying population functions, different assumptions—such as exponential growth, logistic growth, or threshold population—lead to different rates of growth.
- The logistic differential equation incorporates the concept of a carrying capacity. This value is a limiting value on the population for any given environment.
- The logistic differential equation can be solved for any positive growth rate, initial population, and carrying capacity.

4.5 First-order Linear Equations

- Any first-order linear differential equation can be written in the form $y' + p(x)y = q(x)$.
- We can use a five-step problem-solving strategy for solving a first-order linear differential equation that may or may not include an initial value.
- Applications of first-order linear differential equations include determining motion of a rising or falling object with air resistance and finding current in an electrical circuit.

CHAPTER 4 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample.

262. The differential equation $y' = 3x^2 y - \cos(x)y''$ is linear.

263. The differential equation $y' = x - y$ is separable.

264. You can explicitly solve all first-order differential equations by separation or by the method of integrating factors.

265. You can determine the behavior of all first-order differential equations using directional fields or Euler's method.

For the following problems, find the general solution to the differential equations.

266. $y' = x^2 + 3e^x - 2x$

267. $y' = 2^x + \cos^{-1} x$

268. $y' = y\left(x^2 + 1\right)$

269. $y' = e^{-y} \sin x$

270. $y' = 3x - 2y$

271. $y' = y \ln y$

For the following problems, find the solution to the initial value problem.

272. $y' = 8x - \ln x - 3x^4,\ y(1) = 5$

273. $y' = 3x - \cos x + 2,\ y(0) = 4$

274. $xy' = y(x - 2),\ y(1) = 3$

275. $y' = 3y^2(x + \cos x),\ y(0) = -2$

276. $(x - 1)y' = y - 2,\ y(0) = 0$

277. $y' = 3y - x + 6x^2,\ y(0) = -1$

For the following problems, draw the directional field associated with the differential equation, then solve the differential equation. Draw a sample solution on the directional field.

278. $y' = 2y - y^2$

279. $y' = \frac{1}{x} + \ln x - y, \quad \text{for } x > 0$

For the following problems, use Euler's Method with $n = 5$ steps over the interval $t = [0, 1]$. Then solve the initial-value problem exactly. How close is your Euler's Method estimate?

280. $y' = -4yx,\ y(0) = 1$

281. $y' = 3^x - 2y,\ y(0) = 0$

For the following problems, set up and solve the differential equations.

282. A car drives along a freeway, accelerating according to $a = 5\sin(\pi t)$, where t represents time in minutes. Find the velocity at any time t, assuming the car starts with an initial speed of 60 mph.

283. You throw a ball of mass 2 kilograms into the air with an upward velocity of 8 m/s. Find exactly the time the ball will remain in the air, assuming that gravity is given by $g = 9.8\ \text{m/s}^2$.

284. You drop a ball with a mass of 5 kilograms out an airplane window at a height of 5000 m. How long does it take for the ball to reach the ground?

285. You drop the same ball of mass 5 kilograms out of the same airplane window at the same height, except this time you assume a drag force proportional to the ball's velocity, using a proportionality constant of 3 and the ball reaches terminal velocity. Solve for the distance fallen as a function of time. How long does it take the ball to reach the ground?

286. A drug is administered to a patient every 24 hours and is cleared at a rate proportional to the amount of drug left in the body, with proportionality constant 0.2. If the patient needs a baseline level of 5 mg to be in the bloodstream at all times, how large should the dose be?

287. A 1000-liter tank contains pure water and a solution of 0.2 kg salt/L is pumped into the tank at a rate of 1 L/min and is drained at the same rate. Solve for total amount of salt in the tank at time t.

288. You boil water to make tea. When you pour the water into your teapot, the temperature is $100°\text{C}$. After 5 minutes in your $15°\text{C}$ room, the temperature of the tea is $85°\text{C}$. Solve the equation to determine the temperatures of the tea at time t. How long must you wait until the tea is at a drinkable temperature $(72°\text{C})$?

289. The human population (in thousands) of Nevada in 1950 was roughly 160. If the carrying capacity is estimated at 10 million individuals, and assuming a growth rate of 2% per year, develop a logistic growth model and solve for the population in Nevada at any time (use 1950 as time = 0). What population does your model predict for 2000? How close is your prediction to the true value of $1{,}998{,}257$?

290. Repeat the previous problem but use Gompertz growth model. Which is more accurate?