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Calc-ulus

Volume 2

5 | SEQUENCES AND SERIES

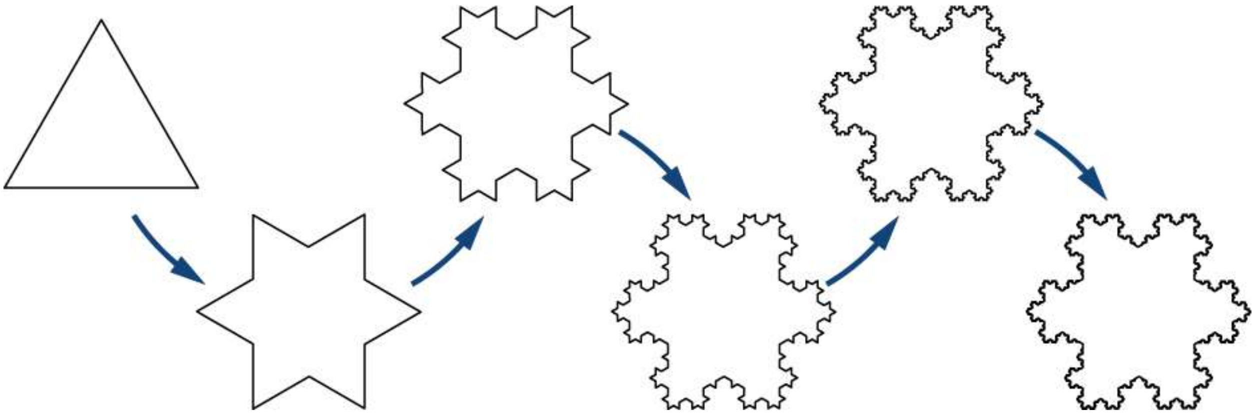


Figure 5.1 The Koch snowflake is constructed by using an iterative process. Starting with an equilateral triangle, at each step of the process the middle third of each line segment is removed and replaced with an equilateral triangle pointing outward.

Chapter Outline

Introduction

The Koch snowflake is constructed from an infinite number of nonoverlapping equilateral triangles. Consequently, we can express its area as a sum of infinitely many terms. How do we add an infinite number of terms? Can a sum of an infinite number of terms be finite? To answer these questions, we need to introduce the concept of an infinite series, a sum with infinitely many terms. Having defined the necessary tools, we will be able to calculate the area of the Koch snowflake (see **Example 5.8**).

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as "infinite polynomials." We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

5.1 | Sequences

Learning Objectives

5.1.1 Find the formula for the general term of a sequence.
5.1.2 Calculate the limit of a sequence if it exists.
5.1.3 Determine the convergence or divergence of a given sequence.

In this section, we introduce sequences and define what it means for a sequence to converge or diverge. We show how to find limits of sequences that converge, often by using the properties of limits for functions discussed earlier. We close this section with the Monotone Convergence Theorem, a tool we can use to prove that certain types of sequences converge.

Terminology of Sequences

To work with this new topic, we need some new terms and definitions. First, an infinite sequence is an ordered list of numbers of the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots .$$

Each of the numbers in the sequence is called a term. The symbol n is called the index variable for the sequence. We use the notation

$$\{a_n\}_{n=1}^{\infty}, \text{ or simply } \{a_n\},$$

to denote this sequence. A similar notation is used for sets, but a sequence is an ordered list, whereas a set is not ordered. Because a particular number a_n exists for each positive integer $n,$ we can also define a sequence as a function whose domain is the set of positive integers.

Let's consider the infinite, ordered list

$$2, 4, 8, 16, 32, \ldots .$$

This is a sequence in which the first, second, and third terms are given by $a_1 = 2,$ $a_2 = 4,$ and $a_3 = 8.$ You can probably see that the terms in this sequence have the following pattern:

$$a_1 = 2^1,\ a_2 = 2^2,\ a_3 = 2^3,\ a_4 = 2^4, \text{ and } a_5 = 2^5.$$

Assuming this pattern continues, we can write the nth term in the sequence by the explicit formula $a_n = 2^n.$ Using this notation, we can write this sequence as

$$\{2^n\}_{n=1}^{\infty} \text{ or } \{2^n\}.$$

Alternatively, we can describe this sequence in a different way. Since each term is twice the previous term, this sequence can be defined recursively by expressing the nth term a_n in terms of the previous term $a_{n-1}.$ In particular, we can define this sequence as the sequence $\{a_n\}$ where $a_1 = 2$ and for all $n \geq 2,$ each term a_n is defined by the **recurrence relation** $a_n = 2a_{n-1}.$

Definition

An **infinite sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$a_1, a_2, \ldots, a_n, \ldots .$$

The subscript n is called the **index variable** of the sequence. Each number a_n is a **term** of the sequence. Sometimes sequences are defined by **explicit formulas**, in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Note that the index does not have to start at $n = 1$ but could start with other integers. For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0,$ in which case the sequence would be

$$a_0, a_1, a_2, \ldots .$$

Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geq 1$ may be defined in terms of $a_{n-1}.$ Since a sequence $\{a_n\}$ has exactly one value for each positive integer $n,$ it can be described as a function whose domain is the set of positive integers. As a result, it makes sense to discuss the graph of a sequence. The graph of a sequence $\{a_n\}$ consists of all points (n, a_n) for all positive integers n. Figure 5.2 shows the graph of $\{2^n\}$.

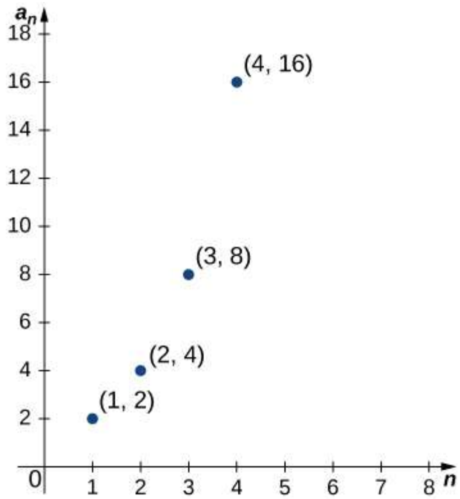


Figure 5.2 The plotted points are a graph of the sequence $\{2^n\}$.

Two types of sequences occur often and are given special names: arithmetic sequences and geometric sequences. In an **arithmetic sequence**, the *difference* between every pair of consecutive terms is the same. For example, consider the sequence

$$3,\ 7,\ 11,\ 15,\ 19, \ldots .$$

You can see that the difference between every consecutive pair of terms is 4. Assuming that this pattern continues, this sequence is an arithmetic sequence. It can be described by using the recurrence relation

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4 \text{ for } n \geq 2. \end{cases}$$

Note that

$$\begin{aligned} a_2 &= 3 + 4 \\ a_3 &= 3 + 4 + 4 = 3 + 2 \cdot 4 \\ a_4 &= 3 + 4 + 4 + 4 = 3 + 3 \cdot 4. \end{aligned}$$

Thus the sequence can also be described using the explicit formula

$$\begin{aligned} a_n &= 3 + 4(n - 1) \\ &= 4n - 1. \end{aligned}$$

In general, an arithmetic sequence is any sequence of the form $a_n = cn + b$.

In a **geometric sequence**, the *ratio* of every pair of consecutive terms is the same. For example, consider the sequence

$$2,\ -\frac{2}{3}, \frac{2}{9},\ -\frac{2}{27}, \frac{2}{81}, \ldots .$$

We see that the ratio of any term to the preceding term is $-\frac{1}{3}$. Assuming this pattern continues, this sequence is a geometric sequence. It can be defined recursively as

$$\begin{aligned} a_1 &= 2 \\ a_n &= -\frac{1}{3} \cdot a_{n-1} \text{ for } n \geq 2. \end{aligned}$$

Alternatively, since

$$a_2 = -\frac{1}{3} \cdot 2$$
$$a_3 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^2 \cdot 2$$
$$a_4 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^3 \cdot 2,$$

we see that the sequence can be described by using the explicit formula

$$a_n = 2\left(-\frac{1}{3}\right)^{n-1}.$$

The sequence $\{2^n\}$ that we discussed earlier is a geometric sequence, where the ratio of any term to the previous term is 2. In general, a geometric sequence is any sequence of the form $a_n = cr^n$.

Example 5.1

Finding Explicit Formulas

For each of the following sequences, find an explicit formula for the nth term of the sequence.

a. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6},\ldots$

b. $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16},\ldots$

Solution

a. First, note that the sequence is alternating from negative to positive. The odd terms in the sequence are negative, and the even terms are positive. Therefore, the nth term includes a factor of $(-1)^n$. Next, consider the sequence of numerators $\{1, 2, 3,\ldots\}$ and the sequence of denominators $\{2, 3, 4,\ldots\}$. We can see that both of these sequences are arithmetic sequences. The nth term in the sequence of numerators is n, and the nth term in the sequence of denominators is $n + 1$. Therefore, the sequence can be described by the explicit formula

$$a_n = \frac{(-1)^n n}{n + 1}.$$

b. The sequence of numerators $3, 9, 27, 81, 243,\ldots$ is a geometric sequence. The numerator of the nth term is 3^n The sequence of denominators $4, 7, 10, 13, 16,\ldots$ is an arithmetic sequence. The denominator of the nth term is $4 + 3(n - 1) = 3n + 1$. Therefore, we can describe the sequence by the explicit formula $a_n = \frac{3^n}{3n + 1}$.

5.1 Find an explicit formula for the nth term of the sequence $\left\{\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11},\ldots\right\}$.

Example 5.2

Defined by Recurrence Relations

For each of the following recursively defined sequences, find an explicit formula for the sequence.

a. $a_1 = 2, \quad a_n = -3a_{n-1}$ for $n \geq 2$

b. $a_1 = \frac{1}{2}, \quad a_n = a_{n-1} + \left(\frac{1}{2}\right)^n$ for $n \geq 2$

Solution

a. Writing out the first few terms, we have

$$\begin{aligned} a_1 &= 2 \\ a_2 &= -3a_1 = -3(2) \\ a_3 &= -3a_2 = (-3)^2 2 \\ a_4 &= -3a_3 = (-3)^3 2. \end{aligned}$$

In general,

$$a_n = 2(-3)^{n-1}.$$

b. Write out the first few terms:

$$\begin{aligned} a_1 &= \frac{1}{2} \\ a_2 &= a_1 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ a_3 &= a_2 + \left(\frac{1}{2}\right)^3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8} \\ a_4 &= a_3 + \left(\frac{1}{2}\right)^4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}. \end{aligned}$$

From this pattern, we derive the explicit formula

$$a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

5.2 Find an explicit formula for the sequence defined recursively such that $a_1 = -4$ and $a_n = a_{n-1} + 6$.

Limit of a Sequence

A fundamental question that arises regarding infinite sequences is the behavior of the terms as n gets larger. Since a sequence is a function defined on the positive integers, it makes sense to discuss the limit of the terms as $n \to \infty$. For example, consider the following four sequences and their different behaviors as $n \to \infty$ (see **Figure 5.3**):

a. $\{1 + 3n\} = \{4, 7, 10, 13, \ldots\}$. The terms $1 + 3n$ become arbitrarily large as $n \to \infty$. In this case, we say that $1 + 3n \to \infty$ as $n \to \infty$.

b. $\left\{1 - \left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\right\}$. The terms $1 - \left(\frac{1}{2}\right)^n \to 1$ as $n \to \infty$.

c. $\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$. The terms alternate but do not approach one single value as $n \to \infty$.

d. $\left\{\frac{(-1)^n}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right\}$. The terms alternate for this sequence as well, but $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

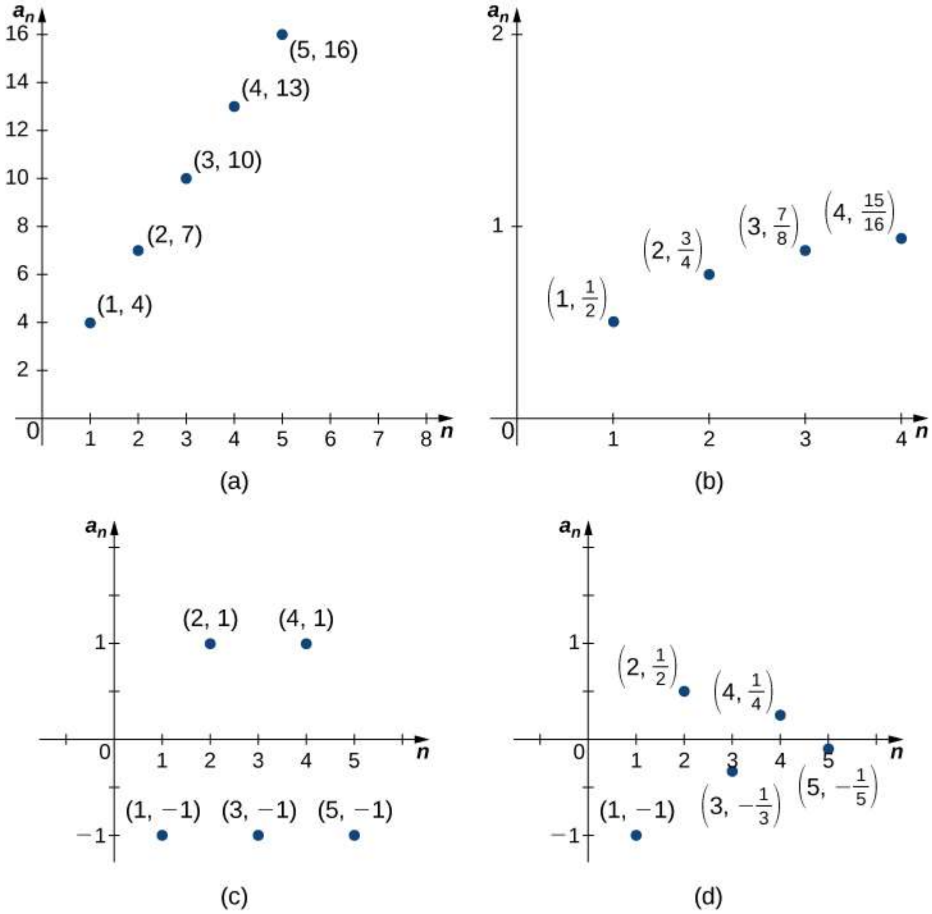


Figure 5.3 (a) The terms in the sequence become arbitrarily large as $n \to \infty$. (b) The terms in the sequence approach 1 as $n \to \infty$. (c) The terms in the sequence alternate between 1 and -1 as $n \to \infty$. (d) The terms in the sequence alternate between positive and negative values but approach 0 as $n \to \infty$.

From these examples, we see several possibilities for the behavior of the terms of a sequence as $n \to \infty$. In two of the sequences, the terms approach a finite number as $n \to \infty$. In the other two sequences, the terms do not. If the terms of a sequence approach a finite number L as $n \to \infty$, we say that the sequence is a convergent sequence and the real number L is the limit of the sequence. We can give an informal definition here.

Definition

Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence**. In this case, we write

$$\lim_{n \to \infty} a_n = L.$$

If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence**.

From **Figure 5.3**, we see that the terms in the sequence $\left\{1 - \left(\frac{1}{2}\right)^n\right\}$ are becoming arbitrarily close to 1 as n becomes very large. We conclude that $\left\{1 - \left(\frac{1}{2}\right)^n\right\}$ is a convergent sequence and its limit is 1. In contrast, from **Figure 5.3**, we see that the terms in the sequence $1 + 3n$ are not approaching a finite number as n becomes larger. We say that $\{1 + 3n\}$ is a divergent sequence.

In the informal definition for the limit of a sequence, we used the terms "arbitrarily close" and "sufficiently large." Although these phrases help illustrate the meaning of a converging sequence, they are somewhat vague. To be more precise, we now present the more formal definition of limit for a sequence and show these ideas graphically in **Figure 5.4**.

Definition

A sequence $\{a_n\}$ converges to a real number L if for all $\varepsilon > 0,$ there exists an integer N such that $|a_n - L| < \varepsilon$ if $n \geq N$. The number L is the limit of the sequence and we write

$$\lim_{n \to \infty} a_n = L \text{ or } a_n \to L.$$

In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

We remark that the convergence or divergence of a sequence $\{a_n\}$ depends only on what happens to the terms a_n as $n \to \infty$. Therefore, if a finite number of terms $b_1, b_2, \ldots, b_N$ are placed before a_1 to create a new sequence

$$b_1, b_2, \ldots, b_N, a_1, a_2, \ldots,$$

this new sequence will converge if $\{a_n\}$ converges and diverge if $\{a_n\}$ diverges. Further, if the sequence $\{a_n\}$ converges to $L,$ this new sequence will also converge to L.

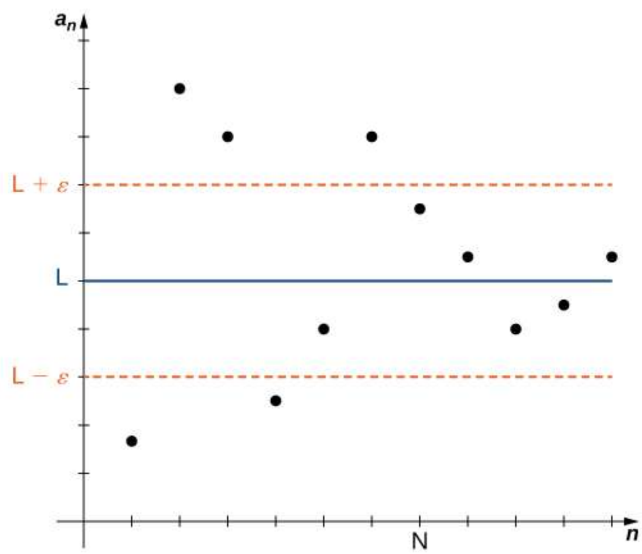


Figure 5.4 As n increases, the terms a_n become closer to L. For values of $n \geq N,$ the distance between each point (n, a_n) and the line $y = L$ is less than ε.

As defined above, if a sequence does not converge, it is said to be a divergent sequence. For example, the sequences $\{1 + 3n\}$ and $\left\{(-1)^n\right\}$ shown in **Figure 5.4** diverge. However, different sequences can diverge in different ways. The sequence $\left\{(-1)^n\right\}$ diverges because the terms alternate between 1 and $-1,$ but do not approach one value as $n \to \infty$. On the other hand, the sequence $\{1 + 3n\}$ diverges because the terms $1 + 3n \to \infty$ as $n \to \infty$. We say the sequence $\{1 + 3n\}$ diverges to infinity and write $\lim_{n \to \infty}(1 + 3n) = \infty$. It is important to recognize that this notation does not imply the limit of the sequence $\{1 + 3n\}$ exists. The sequence is, in fact, divergent. Writing that the limit is infinity is intended only to provide more information about why the sequence is divergent. A sequence can also diverge to negative infinity. For example, the sequence $\{-5n+2\}$ diverges to negative infinity because $-5n+2 \to -\infty$ as $n \to -\infty$. We write this as $\lim_{n \to \infty}(-5n+2) = \to -\infty$.

Because a sequence is a function whose domain is the set of positive integers, we can use properties of limits of functions to determine whether a sequence converges. For example, consider a sequence $\{a_n\}$ and a related function f defined on all positive real numbers such that $f(n) = a_n$ for all integers $n \geq 1$. Since the domain of the sequence is a subset of the domain of f, if $\lim_{x \to \infty} f(x)$ exists, then the sequence converges and has the same limit. For example, consider the sequence $\left\{\frac{1}{n}\right\}$ and the related function $f(x) = \frac{1}{x}$. Since the function f defined on all real numbers $x > 0$ satisfies $f(x) = \frac{1}{x} \to 0$ as $x \to \infty$, the sequence $\left\{\frac{1}{n}\right\}$ must satisfy $\frac{1}{n} \to 0$ as $n \to \infty$.

Theorem 5.1: Limit of a Sequence Defined by a Function

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that

$$\lim_{x \to \infty} f(x) = L,$$

then $\{a_n\}$ converges and

$$\lim_{n \to \infty} a_n = L.$$

We can use this theorem to evaluate $\lim_{n \to \infty} r^n$ for $0 \leq r \leq 1$. For example, consider the sequence $\{(1/2)^n\}$ and the related exponential function $f(x) = (1/2)^x$. Since $\lim_{x \to \infty}(1/2)^x = 0$, we conclude that the sequence $\{(1/2)^n\}$ converges and its limit is 0. Similarly, for any real number r such that $0 \leq r < 1$, $\lim_{x \to \infty} r^x = 0$, and therefore the sequence $\{r^n\}$ converges. On the other hand, if $r = 1$, then $\lim_{x \to \infty} r^x = 1$, and therefore the limit of the sequence $\{1^n\}$ is 1. If $r > 1$, $\lim_{x \to \infty} r^x = \infty$, and therefore we cannot apply this theorem. However, in this case, just as the function r^x grows without bound as $n \to \infty$, the terms r^n in the sequence become arbitrarily large as $n \to \infty$, and we conclude that the sequence $\{r^n\}$ diverges to infinity if $r > 1$.

We summarize these results regarding the geometric sequence $\{r^n\}$:

$$\begin{aligned} &r^n \to 0 \; if \; 0 < r < 1 \\ &r^n \to 1 \; if \; r = 1 \\ &r^n \to \infty \; if \; r > 1. \end{aligned}$$

Later in this section we consider the case when $r < 0$.

We now consider slightly more complicated sequences. For example, consider the sequence $\{(2/3)^n + (1/4)^n\}$. The terms in this sequence are more complicated than other sequences we have discussed, but luckily the limit of this sequence is determined by the limits of the two sequences $\{(2/3)^n\}$ and $\{(1/4)^n\}$. As we describe in the following algebraic limit laws, since $\{(2/3)^n\}$ and $\{1/4)^n\}$ both converge to 0, the sequence $\{(2/3)^n + (1/4)^n\}$ converges to $0 + 0 = 0$. Just as we were able to evaluate a limit involving an algebraic combination of functions f and g by looking at the limits of f and g (see **Introduction to Limits (http://cnx.org/content/m53483/latest/)**), we are able to evaluate the limit of a sequence whose terms are algebraic combinations of a_n and b_n by evaluating the limits of $\{a_n\}$ and $\{b_n\}$.

Theorem 5.2: Algebraic Limit Laws

Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$, then

i. $\lim_{n \to \infty} c = c$

ii. $\lim_{n \to \infty} ca_n = c \lim_{n \to \infty} a_n = cA$

iii. $\lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n = A \pm B$

iv. $\lim_{n \to \infty} (a_n \cdot b_n) = \left(\lim_{n \to \infty} a_n\right) \cdot \left(\lim_{n \to \infty} b_n\right) = A \cdot B$

v. $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

Proof

We prove part iii.

Let $\epsilon > 0$. Since $\lim_{n \to \infty} a_n = A$, there exists a constant positive integer N_1 such that for all $n \geq N_1$. Since $\lim_{n \to \infty} b_n = B$, there exists a constant N_2 such that $|b_n - B| < \varepsilon/2$ for all $n \geq N_2$. Let N be the largest of N_1 and N_2. Therefore, for all $n \geq N$,

$$|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

The algebraic limit laws allow us to evaluate limits for many sequences. For example, consider the sequence $\left\{\frac{1}{n^2}\right\}$. As shown earlier, $\lim_{n \to \infty} 1/n = 0$. Similarly, for any positive integer k, we can conclude that

$$\lim_{n \to \infty} \frac{1}{n^k} = 0.$$

In the next example, we make use of this fact along with the limit laws to evaluate limits for other sequences.

Example 5.3

Determining Convergence and Finding Limits

For each of the following sequences, determine whether or not the sequence converges. If it converges, find its limit.

a. $\left\{5 - \frac{3}{n^2}\right\}$

b. $\left\{\frac{3n^4 - 7n^2 + 5}{6 - 4n^4}\right\}$

c. $\left\{\frac{2^n}{n^2}\right\}$

d. $\left\{\left(1+\frac{4}{n}\right)^n\right\}$

Solution

a. We know that $1/n \to 0$. Using this fact, we conclude that

$$\lim_{n\to\infty}\frac{1}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n}\right) \cdot \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0.$$

Therefore,

$$\lim_{n\to\infty}\left(5-\frac{3}{n^2}\right) = \lim_{n\to\infty}5 - 3\lim_{n\to\infty}\frac{1}{n^2} = 5 - 3.0 = 5.$$

The sequence converges and its limit is 5.

b. By factoring n^4 out of the numerator and denominator and using the limit laws above, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{3n^4-7n^2+5}{6-4n^4} &= \lim_{n\to\infty}\frac{3-\frac{7}{n^2}+\frac{5}{n^4}}{\frac{6}{n^4}-4} \\
&= \frac{\lim_{n\to\infty}\left(3-\frac{7}{n^2}+\frac{5}{n^4}\right)}{\lim_{n\to\infty}\left(\frac{6}{n^4}-4\right)} \\
&= \frac{\left(\lim_{n\to\infty}(3)-\lim_{n\to\infty}\frac{7}{n^2}+\lim_{n\to\infty}\frac{5}{n^4}\right)}{\left(\lim_{n\to\infty}\frac{6}{n^4}-\lim_{n\to\infty}(4)\right)} \\
&= \frac{\left(\lim_{n\to\infty}(3)-7\cdot\lim_{n\to\infty}\frac{1}{n^2}+5\cdot\lim_{n\to\infty}\frac{1}{n^4}\right)}{\left(6\cdot\lim_{n\to\infty}\frac{1}{n^4}-\lim_{n\to\infty}(4)\right)} \\
&= \frac{3-7\cdot 0+5\cdot 0}{6\cdot 0-4} = -\frac{3}{4}.
\end{aligned}$$

The sequence converges and its limit is $-3/4$.

c. Consider the related function $f(x) = 2^x/x^2$ defined on all real numbers $x > 0$. Since $2^x \to \infty$ and $x^2 \to \infty$ as $x \to \infty$, apply L'Hôpital's rule and write

$$\begin{aligned}
\lim_{x\to\infty}\frac{2^x}{x^2} &= \lim_{x\to\infty}\frac{2^x \ln 2}{2x} && \text{Take the derivatives of the numerator and denominator.} \\
&= \lim_{x\to\infty}\frac{2^x(\ln 2)^2}{2} && \text{Take the derivatives again.} \\
&= \infty.
\end{aligned}$$

We conclude that the sequence diverges.

d. Consider the function $f(x) = \left(1+\frac{4}{x}\right)^x$ defined on all real numbers $x > 0$. This function has the indeterminate form 1^∞ as $x \to \infty$. Let

$$y = \lim_{x \to \infty} \left(1 + \frac{4}{x}\right)^x.$$

Now taking the natural logarithm of both sides of the equation, we obtain

$$\ln(y) = \ln\left[\lim_{x \to \infty} \left(1 + \frac{4}{x}\right)^x\right].$$

Since the function $f(x) = \ln x$ is continuous on its domain, we can interchange the limit and the natural logarithm. Therefore,

$$\ln(y) = \lim_{x \to \infty} \left[\ln\left(1 + \frac{4}{x}\right)^x\right].$$

Using properties of logarithms, we write

$$\lim_{x \to \infty} \left[\ln\left(1 + \frac{4}{x}\right)^x\right] = \lim_{x \to \infty} x \ln\left(1 + \frac{4}{x}\right).$$

Since the right-hand side of this equation has the indeterminate form $\infty \cdot 0,$ rewrite it as a fraction to apply L'Hôpital's rule. Write

$$\lim_{x \to \infty} x \ln\left(1 + \frac{4}{x}\right) = \lim_{x \to \infty} \frac{\ln(1 + 4/x)}{1/x}.$$

Since the right-hand side is now in the indeterminate form $0/0,$ we are able to apply L'Hôpital's rule. We conclude that

$$\lim_{x \to \infty} \frac{\ln(1 + 4/x)}{1/x} = \lim_{x \to \infty} \frac{4}{1 + 4/x} = 4.$$

Therefore, $\ln(y) = 4$ and $y = e^4$. Therefore, since $\lim_{x \to \infty} \left(1 + \frac{4}{x}\right)^x = e^4,$ we can conclude that the sequence $\left\{\left(1 + \frac{4}{n}\right)^n\right\}$ converges to e^4.

5.3 Consider the sequence $\left\{\left(5n^2 + 1\right)/e^n\right\}$. Determine whether or not the sequence converges. If it converges, find its limit.

Recall that if f is a continuous function at a value $L,$ then $f(x) \to f(L)$ as $x \to L.$ This idea applies to sequences as well. Suppose a sequence $a_n \to L,$ and a function f is continuous at $L.$ Then $f(a_n) \to f(L).$ This property often enables us to find limits for complicated sequences. For example, consider the sequence $\sqrt{5 - \frac{3}{n^2}}.$ From **Example 5.3**a. we know the sequence $5 - \frac{3}{n^2} \to 5.$ Since $\sqrt{x}$ is a continuous function at $x = 5,$

$$\lim_{n \to \infty} \sqrt{5 - \frac{3}{n^2}} = \sqrt{\lim_{n \to \infty} \left(5 - \frac{3}{n^2}\right)} = \sqrt{5}.$$

Theorem 5.3: Continuous Functions Defined on Convergent Sequences

Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values a_n for $n \ge N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$ (**Figure 5.5**).

Proof

Let $\epsilon > 0$. Since f is continuous at L, there exists $\delta > 0$ such that $|f(x) - f(L)| < \varepsilon$ if $|x - L| < \delta$. Since the sequence $\{a_n\}$ converges to L, there exists N such that $|a_n - L| < \delta$ for all $n \ge N$. Therefore, for all $n \ge N$, $|a_n - L| < \delta$, which implies $|f(a_n) - f(L)| < \varepsilon$. We conclude that the sequence $\{f(a_n)\}$ converges to $f(L)$.

□

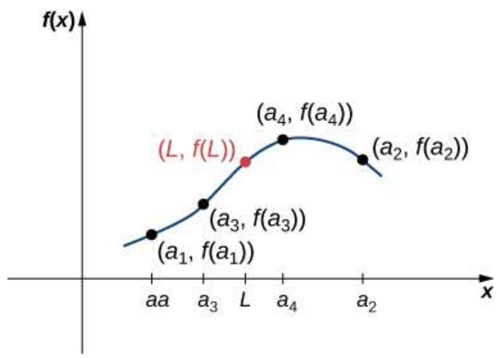


Figure 5.5 Because f is a continuous function as the inputs $a_1, a_2, a_3, \ldots$ approach L, the outputs $f(a_1), f(a_2), f(a_3), \ldots$ approach $f(L)$.

Example 5.4

Limits Involving Continuous Functions Defined on Convergent Sequences

Determine whether the sequence $\{\cos(3/n^2)\}$ converges. If it converges, find its limit.

Solution

Since the sequence $\{3/n^2\}$ converges to 0 and $\cos x$ is continuous at $x = 0$, we can conclude that the sequence $\{\cos(3/n^2)\}$ converges and

$$\lim_{n \to \infty} \cos\left(\frac{3}{n^2}\right) = \cos(0) = 1.$$

5.4 Determine if the sequence $\left\{\sqrt{\frac{2n+1}{3n+5}}\right\}$ converges. If it converges, find its limit.

Another theorem involving limits of sequences is an extension of the Squeeze Theorem for limits discussed in **Introduction to Limits (http://cnx.org/content/m53483/latest/)** .

Theorem 5.4: Squeeze Theorem for Sequences

Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that

$$a_n \le b_n \le c_n \text{ for all } n \ge N.$$

If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$ (**Figure 5.6**).

Proof

Let $\varepsilon > 0.$ Since the sequence $\{a_n\}$ converges to $L,$ there exists an integer N_1 such that $|a_n - L| < \varepsilon$ for all $n \ge N_1.$ Similarly, since $\{c_n\}$ converges to $L,$ there exists an integer N_2 such that $|c_n - L| < \varepsilon$ for all $n \ge N_2.$ By assumption, there exists an integer N such that $a_n \le b_n \le c_n$ for all $n \ge N.$ Let M be the largest of $N_1, N_2,$ and $N.$ We must show that $|b_n - L| < \varepsilon$ for all $n \ge M.$ For all $n \ge M,$

$$-\varepsilon < -|a_n - L| \le a_n - L \le b_n - L \le c_n - L \le |c_n - L| < \varepsilon.$$

Therefore, $-\varepsilon < b_n - L < \varepsilon,$ and we conclude that $|b_n - L| < \varepsilon$ for all $n \ge M,$ and we conclude that the sequence $\{b_n\}$ converges to $L.$

□

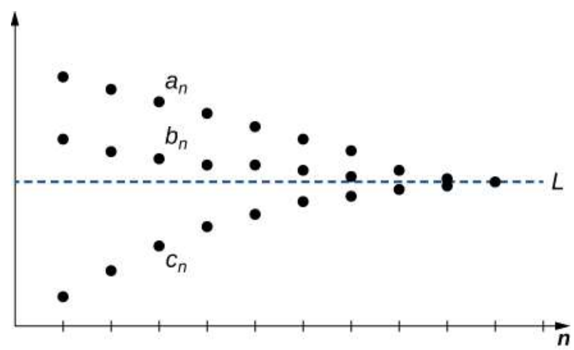


Figure 5.6 Each term b_n satisfies $a_n \le b_n \le c_n$ and the sequences $\{a_n\}$ and $\{c_n\}$ converge to the same limit, so the sequence $\{b_n\}$ must converge to the same limit as well.

Example 5.5

Using the Squeeze Theorem

Use the Squeeze Theorem to find the limit of each of the following sequences.

a. $\left\{\dfrac{\cos n}{n^2}\right\}$

b. $\left\{\left(-\dfrac{1}{2}\right)^n\right\}$

Solution

a. Since $-1 \leq \cos n \leq 1$ for all integers n, we have

$$-\frac{1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2}.$$

Since $-1/n^2 \to 0$ and $1/n^2 \to 0$, we conclude that $\cos n/n^2 \to 0$ as well.

b. Since

$$-\frac{1}{2^n} \leq \left(-\frac{1}{2}\right)^n \leq \frac{1}{2^n}$$

for all positive integers n, $-1/2^n \to 0$ and $1/2^n \to 0$, we can conclude that $(-1/2)^n \to 0$.

5.5 Find $\lim\limits_{n \to \infty} \dfrac{2n - \sin n}{n}$.

Using the idea from **Example 5.5**b. we conclude that $r^n \to 0$ for any real number r such that $-1 < r < 0$. If $r < -1$, the sequence $\{r^n\}$ diverges because the terms oscillate and become arbitrarily large in magnitude. If $r = -1$, the sequence $\{r^n\} = \{(-1)^n\}$ diverges, as discussed earlier. Here is a summary of the properties for geometric sequences.

$$r^n \to 0 \text{ if } |r| < 1 \tag{5.1}$$

$$r^n \to 1 \text{ if } r = 1 \tag{5.2}$$

$$r^n \to \infty \text{ if } r > 1 \tag{5.3}$$

$$\{r^n\} \text{ diverges if } r \leq -1 \tag{5.4}$$

Bounded Sequences

We now turn our attention to one of the most important theorems involving sequences: the Monotone Convergence Theorem. Before stating the theorem, we need to introduce some terminology and motivation. We begin by defining what it means for a sequence to be bounded.

Definition

A sequence $\{a_n\}$ is **bounded above** if there exists a real number M such that

$$a_n \leq M$$

for all positive integers n.

A sequence $\{a_n\}$ is **bounded below** if there exists a real number M such that

$$M \leq a_n$$

for all positive integers n.

A sequence $\{a_n\}$ is a **bounded sequence** if it is bounded above and bounded below.

If a sequence is not bounded, it is an **unbounded sequence**.

For example, the sequence $\{1/n\}$ is bounded above because $1/n \leq 1$ for all positive integers n. It is also bounded below because $1/n \geq 0$ for all positive integers n. Therefore, $\{1/n\}$ is a bounded sequence. On the other hand, consider the sequence $\{2^n\}$. Because $2^n \geq 2$ for all $n \geq 1$, the sequence is bounded below. However, the sequence is not bounded above. Therefore, $\{2^n\}$ is an unbounded sequence.

We now discuss the relationship between boundedness and convergence. Suppose a sequence $\{a_n\}$ is unbounded. Then it is not bounded above, or not bounded below, or both. In either case, there are terms a_n that are arbitrarily large in magnitude as n gets larger. As a result, the sequence $\{a_n\}$ cannot converge. Therefore, being bounded is a necessary condition for a sequence to converge.

Theorem 5.5: Convergent Sequences Are Bounded

If a sequence $\{a_n\}$ converges, then it is bounded.

Note that a sequence being bounded is not a sufficient condition for a sequence to converge. For example, the sequence $\left\{(-1)^n\right\}$ is bounded, but the sequence diverges because the sequence oscillates between 1 and -1 and never approaches a finite number. We now discuss a sufficient (but not necessary) condition for a bounded sequence to converge.

Consider a bounded sequence $\{a_n\}$. Suppose the sequence $\{a_n\}$ is increasing. That is, $a_1 \leq a_2 \leq a_3 \ldots$. Since the sequence is increasing, the terms are not oscillating. Therefore, there are two possibilities. The sequence could diverge to infinity, or it could converge. However, since the sequence is bounded, it is bounded above and the sequence cannot diverge to infinity. We conclude that $\{a_n\}$ converges. For example, consider the sequence

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}.$$

Since this sequence is increasing and bounded above, it converges. Next, consider the sequence

$$\left\{2, 0, 3, 0, 4, 0, 1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}.$$

Even though the sequence is not increasing for all values of n, we see that $-1/2 < -1/3 < -1/4 < \cdots$. Therefore, starting with the eighth term, $a_8 = -1/2$, the sequence is increasing. In this case, we say the sequence is *eventually* increasing. Since the sequence is bounded above, it converges. It is also true that if a sequence is decreasing (or eventually decreasing) and bounded below, it also converges.

Definition

A sequence $\{a_n\}$ is increasing for all $n \geq n_0$ if

$$a_n \leq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is decreasing for all $n \geq n_0$ if

$$a_n \geq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is a **monotone sequence** for all $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

We now have the necessary definitions to state the Monotone Convergence Theorem, which gives a sufficient condition for convergence of a sequence.

Theorem 5.6: Monotone Convergence Theorem

If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geq n_0$, then $\{a_n\}$ converges.

The proof of this theorem is beyond the scope of this text. Instead, we provide a graph to show intuitively why this theorem makes sense (**Figure 5.7**).

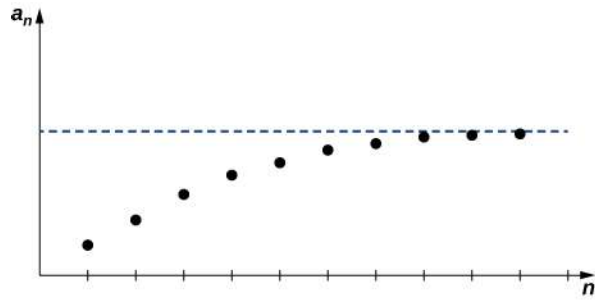


Figure 5.7 Since the sequence $\{a_n\}$ is increasing and bounded above, it must converge.

In the following example, we show how the Monotone Convergence Theorem can be used to prove convergence of a sequence.

Example 5.6

Using the Monotone Convergence Theorem

For each of the following sequences, use the Monotone Convergence Theorem to show the sequence converges and find its limit.

a. $\left\{\frac{4^n}{n!}\right\}$

b. $\{a_n\}$ defined recursively such that

$$a_1 = 2 \text{ and } a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \text{ for all } n \geq 2.$$

Solution

a. Writing out the first few terms, we see that

$$\left\{\frac{4^n}{n!}\right\} = \left\{4,\ 8,\ \frac{32}{3},\ \frac{32}{3},\ \frac{128}{15},\ldots\right\}.$$

At first, the terms increase. However, after the third term, the terms decrease. In fact, the terms decrease for all $n \geq 3$. We can show this as follows.

$$a_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4}{n+1} \cdot \frac{4^n}{n!} = \frac{4}{n+1} \cdot a_n \leq a_n \ if\ n \geq 3.$$

Therefore, the sequence is decreasing for all $n \geq 3$. Further, the sequence is bounded below by 0 because $4^n/n! \geq 0$ for all positive integers n. Therefore, by the Monotone Convergence Theorem, the sequence converges.

To find the limit, we use the fact that the sequence converges and let $L = \lim_{n \to \infty} a_n$. Now note this important observation. Consider $\lim_{n \to \infty} a_{n+1}$. Since

$$\{a_{n+1}\} = \{a_{2,}\ a_3,\ a_4, \ldots\},$$

the only difference between the sequences $\{a_{n+1}\}$ and $\{a_n\}$ is that $\{a_{n+1}\}$ omits the first term. Since a finite number of terms does not affect the convergence of a sequence,

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n = L.$$

Combining this fact with the equation

$$a_{n+1} = \frac{4}{n+1} a_n$$

and taking the limit of both sides of the equation

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{4}{n+1} a_n,$$

we can conclude that

$$L = 0 \cdot L = 0.$$

b. Writing out the first several terms,

$$\left\{2,\ \frac{5}{4},\ \frac{41}{40},\ \frac{3281}{3280}, \ldots\right\}.$$

we can conjecture that the sequence is decreasing and bounded below by 1. To show that the sequence is bounded below by 1, we can show that

$$\frac{a_n}{2} + \frac{1}{2a_n} \geq 1.$$

To show this, first rewrite

$$\frac{a_n}{2} + \frac{1}{2a_n} = \frac{a_n^2 + 1}{2a_n}.$$

Since $a_1 > 0$ and a_2 is defined as a sum of positive terms, $a_2 > 0$. Similarly, all terms $a_n > 0$. Therefore,

$$\frac{a_n^2 + 1}{2a_n} \geq 1$$

if and only if

$$a_n^2 + 1 \geq 2a_n.$$

Rewriting the inequality $a_n^2 + 1 \geq 2a_n$ as $a_n^2 - 2a_n + 1 \geq 0$, and using the fact that

$$a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geq 0$$

because the square of any real number is nonnegative, we can conclude that

$$\frac{a_n}{2} + \frac{1}{2a_n} \geq 1.$$

To show that the sequence is decreasing, we must show that $a_{n+1} \le a_n$ for all $n \ge 1$. Since $1 \le a_n^2$, it follows that

$$a_n^2 + 1 \le 2a_n^2.$$

Dividing both sides by $2a_n$, we obtain

$$\frac{a_n}{2} + \frac{1}{2a_n} \le a_n.$$

Using the definition of a_{n+1}, we conclude that

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \le a_n.$$

Since $\{a_n\}$ is bounded below and decreasing, by the Monotone Convergence Theorem, it converges.

To find the limit, let $L = \lim_{n \to \infty} a_n$. Then using the recurrence relation and the fact that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1}$, we have

$$\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \left(\frac{a_n}{2} + \frac{1}{2a_n} \right),$$

and therefore

$$L = \frac{L}{2} + \frac{1}{2L}.$$

Multiplying both sides of this equation by $2L$, we arrive at the equation

$$2L^2 = L^2 + 1.$$

Solving this equation for L, we conclude that $L^2 = 1$, which implies $L = \pm 1$. Since all the terms are positive, the limit $L = 1$.

5.6 Consider the sequence $\{a_n\}$ defined recursively such that $a_1 = 1$, $a_n = a_{n-1}/2$. Use the Monotone Convergence Theorem to show that this sequence converges and find its limit.

Student PROJECT

Fibonacci Numbers

The Fibonacci numbers are defined recursively by the sequence $\{F_n\}$ where $F_0 = 0, \quad F_1 = 1$ and for $n \geq 2,$

$$F_n = F_{n-1} + F_{n-2}.$$

Here we look at properties of the Fibonacci numbers.

1. Write out the first twenty Fibonacci numbers.
2. Find a closed formula for the Fibonacci sequence by using the following steps.
    a. Consider the recursively defined sequence $\{x_n\}$ where $x_o = c$ and $x_{n+1} = ax_n$. Show that this sequence can be described by the closed formula $x_n = ca^n$ for all $n \geq 0$.
    b. Using the result from part a. as motivation, look for a solution of the equation
    $$F_n = F_{n-1} + F_{n-2}$$
    of the form $F_n = c\lambda^n$. Determine what two values for λ will allow F_n to satisfy this equation.
    c. Consider the two solutions from part b.: λ_1 and λ_2. Let $F_n = c_1 \lambda_1{}^n + c_2 \lambda_2{}^n$. Use the initial conditions F_0 and F_1 to determine the values for the constants c_1 and c_2 and write the closed formula F_n.
3. Use the answer in 2 c. to show that
$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2}.$$

The number $\phi = \left(1 + \sqrt{5}\right)/2$ is known as the golden ratio (**Figure 5.8** and **Figure 5.9**).

Figure 5.8 The seeds in a sunflower exhibit spiral patterns curving to the left and to the right. The number of spirals in each direction is always a Fibonacci number—always. (credit: modification of work by Esdras Calderan, Wikimedia Commons)

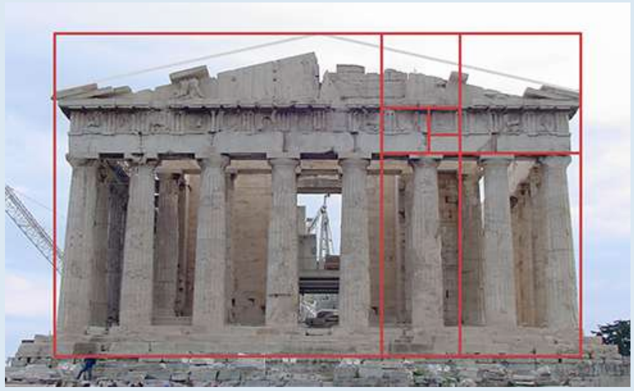


Figure 5.9 The proportion of the golden ratio appears in many famous examples of art and architecture. The ancient Greek temple known as the Parthenon was designed with these proportions, and the ratio appears again in many of the smaller details. (credit: modification of work by TravelingOtter, Flickr)

5.1 EXERCISES

Find the first six terms of each of the following sequences, starting with $n = 1$.

1. $a_n = 1 + (-1)^n$ for $n \geq 1$

2. $a_n = n^2 - 1$ for $n \geq 1$

3. $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$

4. $a_1 = 1$, $a_2 = 1$ and $a_{n+2} = a_n + a_{n+1}$ for $n \geq 1$

5. Find an explicit formula for a_n where $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$.

6. Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n-1} - a_n = 17$ for $n \geq 1$.

7. Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n-1} - a_n = 4$ for $n \geq 1$.

8. Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geq 1$.

9. Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = 1/10$ for $n \geq 1$.

10. Find an explicit formula for the nth term of the sequence whose first several terms are $\{0, 3, 8, 15, 24, 35, 48, 63, 80, 99,\ldots\}$. (*Hint:* First add one to each term.)

11. Find an explicit formula for the nth term of the sequence satisfying $a_1 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

Find a formula for the general term a_n of each of the following sequences.

12. $\{1, 0, -1, 0, 1, 0, -1, 0,\ldots\}$ (*Hint:* Find where $\sin x$ takes these values)

13. $\{1, -1/3, 1/5, -1/7,\ldots\}$

Find a function $f(n)$ that identifies the nth term a_n of the following recursively defined sequences, as $a_n = f(n)$.

14. $a_1 = 1$ and $a_{n+1} = -a_n$ for $n \geq 1$

15. $a_1 = 2$ and $a_{n+1} = 2a_n$ for $n \geq 1$

16. $a_1 = 1$ and $a_{n+1} = (n+1)a_n$ for $n \geq 1$

17. $a_1 = 2$ and $a_{n+1} = (n+1)a_n/2$ for $n \geq 1$

18. $a_1 = 1$ and $a_{n+1} = a_n/2^n$ for $n \geq 1$

Plot the first N terms of each sequence. State whether the graphical evidence suggests that the sequence converges or diverges.

19. **[T]** $a_1 = 1$, $a_2 = 2$, and for $n \geq 2$, $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$; $N = 30$

20. **[T]** $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and for $n \geq 4$, $a_n = \frac{1}{3}(a_{n-1} + a_{n-2} + a_{n-3})$, $N = 30$

21. **[T]** $a_1 = 1$, $a_2 = 2$, and for $n \geq 3$, $a_n = \sqrt{a_{n-1}a_{n-2}}$; $N = 30$

22. **[T]** $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and for $n \geq 4$, $a_n = \sqrt{a_{n-1}a_{n-2}a_{n-3}}$; $N = 30$

Suppose that $\lim_{n \to \infty} a_n = 1$, $\lim_{n \to \infty} b_n = -1$, and $0 < -b_n < a_n$ for all n. Evaluate each of the following limits, or state that the limit does not exist, or state that there is not enough information to determine whether the limit exists.

23. $\lim_{n \to \infty} 3a_n - 4b_n$

24. $\lim_{n \to \infty} \frac{1}{2}b_n - \frac{1}{2}a_n$

25. $\lim_{n \to \infty} \frac{a_n + b_n}{a_n - b_n}$

26. $\lim_{n \to \infty} \frac{a_n - b_n}{a_n + b_n}$

Find the limit of each of the following sequences, using L'Hôpital's rule when appropriate.

27. $\frac{n^2}{2^n}$

28. $\frac{(n-1)^2}{(n+1)^2}$

29. $\frac{\sqrt{n}}{\sqrt{n+1}}$

30. $n^{1/n}$ (*Hint:* $n^{1/n} = e^{\frac{1}{n}\ln n}$)

For each of the following sequences, whose nth terms are indicated, state whether the sequence is bounded and whether it is eventually monotone, increasing, or decreasing.

31. $n/2^n, \quad n \geq 2$

32. $\ln\left(1 + \frac{1}{n}\right)$

33. $\sin n$

34. $\cos\left(n^2\right)$

35. $n^{1/n}, \quad n \geq 3$

36. $n^{-1/n}, \quad n \geq 3$

37. $\tan n$

38. Determine whether the sequence defined as follows has a limit. If it does, find the limit. $a_1 = \sqrt{2}$, $a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$ etc.

39. Determine whether the sequence defined as follows has a limit. If it does, find the limit. $a_1 = 3$, $a_n = \sqrt{2a_{n-1}}, \quad n = 2, 3, \ldots.$

Use the Squeeze Theorem to find the limit of each of the following sequences.

40. $n\sin(1/n)$

41. $\frac{\cos(1/n) - 1}{1/n}$

42. $a_n = \frac{n!}{n^n}$

43. $a_n = \sin n \sin(1/n)$

For the following sequences, plot the first 25 terms of the sequence and state whether the graphical evidence suggests that the sequence converges or diverges.

44. **[T]** $a_n = \sin n$

45. **[T]** $a_n = \cos n$

Determine the limit of the sequence or show that the sequence diverges. If it converges, find its limit.

46. $a_n = \tan^{-1}(n^2)$

47. $a_n = (2n)^{1/n} - n^{1/n}$

48. $a_n = \frac{\ln(n^2)}{\ln(2n)}$

49. $a_n = \left(1 - \frac{2}{n}\right)^n$

50. $a_n = \ln\left(\frac{n+2}{n^2 - 3}\right)$

51. $a_n = \frac{2^n + 3^n}{4^n}$

52. $a_n = \frac{(1000)^n}{n!}$

53. $a_n = \frac{(n!)^2}{(2n)!}$

Newton's method seeks to approximate a solution $f(x) = 0$ that starts with an initial approximation x_0 and successively defines a sequence $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. For the given choice of f and x_0, write out the formula for x_{n+1}. If the sequence appears to converge, give an exact formula for the solution x, then identify the limit x accurate to four decimal places and the smallest n such that x_n agrees with x up to four decimal places.

54. **[T]** $f(x) = x^2 - 2, \quad x_0 = 1$

55. **[T]** $f(x) = (x-1)^2 - 2, \quad x_0 = 2$

56. **[T]** $f(x) = e^x - 2, \quad x_0 = 1$

57. **[T]** $f(x) = \ln x - 1, \quad x_0 = 2$

58. **[T]** Suppose you start with one liter of vinegar and repeatedly remove 0.1 L, replace with water, mix, and repeat.

a. Find a formula for the concentration after n steps.
b. After how many steps does the mixture contain less than 10% vinegar?

59. **[T]** A lake initially contains 2000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 6% each month. However, factoring in all causes, 150 fish are lost each month.

a. Explain why the fish population after n months is modeled by $P_n = 1.06P_{n-1} - 150$ with $P_0 = 2000.$
b. How many fish will be in the pond after one year?

60. **[T]** A bank account earns 5% interest compounded monthly. Suppose that $\$1000$ is initially deposited into the account, but that $\$10$ is withdrawn each month.

a. Show that the amount in the account after n months is $A_n = (1 + .05/12)A_{n-1} - 10;$ $A_0 = 1000.$
b. How much money will be in the account after 1 year?
c. Is the amount increasing or decreasing?
d. Suppose that instead of $\$10,$ a fixed amount d dollars is withdrawn each month. Find a value of d such that the amount in the account after each month remains $\$1000.$
e. What happens if d is greater than this amount?

61. **[T]** A student takes out a college loan of $\$10{,}000$ at an annual percentage rate of $6\%,$ compounded monthly.

a. If the student makes payments of $\$100$ per month, how much does the student owe after 12 months?
b. After how many months will the loan be paid off?

62. **[T]** Consider a series combining geometric growth and arithmetic decrease. Let $a_1 = 1.$ Fix $a > 1$ and $0 < b < a.$ Set $a_{n+1} = a.a_n - b.$ Find a formula for a_{n+1} in terms of $a^n,$ $a,$ and b and a relationship between a and b such that a_n converges.

63. **[T]** The binary representation $x = 0.b_1 b_2 b_3 \ldots$ of a number x between 0 and 1 can be defined as follows. Let $b_1 = 0$ if $x < 1/2$ and $b_1 = 1$ if $1/2 \le x < 1.$ Let $x_1 = 2x - b_1.$ Let $b_2 = 0$ if $x_1 < 1/2$ and $b_2 = 1$ if $1/2 \le x < 1.$ Let $x_2 = 2x_1 - b_2$ and in general, $x_n = 2x_{n-1} - b_n$ and $b_{n-1} = 0$ if $x_n < 1/2$ and $b_{n-1} = 1$ if $1/2 \le x_n < 1.$ Find the binary expansion of $1/3.$

64. **[T]** To find an approximation for $\pi,$ set $a_0 = \sqrt{2+1},$ $a_1 = \sqrt{2+a_0},$ and, in general, $a_{n+1} = \sqrt{2+a_n}.$ Finally, set $p_n = 3.2^n \sqrt{2-a_n}.$ Find the first ten terms of p_n and compare the values to $\pi.$

For the following two exercises, assume that you have access to a computer program or Internet source that can generate a list of zeros and ones of any desired length. Pseudorandom number generators (PRNGs) play an important role in simulating random noise in physical systems by creating sequences of zeros and ones that appear like the result of flipping a coin repeatedly. One of the simplest types of PRNGs recursively defines a random-looking sequence of N integers $a_1, a_2, \ldots, a_N$ by fixing two special integers K and M and letting a_{n+1} be the remainder after dividing $K.a_n$ into $M,$ then creates a bit sequence of zeros and ones whose nth term b_n is equal to one if a_n is odd and equal to zero if a_n is even. If the bits b_n are pseudorandom, then the behavior of their average $(b_1 + b_2 + \cdots + b_N)/N$ should be similar to behavior of averages of truly randomly generated bits.

65. **[T]** Starting with $K = 16{,}807$ and $M = 2{,}147{,}483{,}647,$ using ten different starting values of $a_1,$ compute sequences of bits b_n up to $n = 1000,$ and compare their averages to ten such sequences generated by a random bit generator.

66. **[T]** Find the first 1000 digits of π using either a computer program or Internet resource. Create a bit sequence b_n by letting $b_n = 1$ if the nth digit of π is odd and $b_n = 0$ if the nth digit of π is even. Compute the average value of b_n and the average value of $d_n = |b_{n+1} - b_n|,$ $n = 1,\ldots, 999.$ Does the sequence b_n appear random? Do the differences between successive elements of b_n appear random?

5.2 | Infinite Series

Learning Objectives

5.2.1 Explain the meaning of the sum of an infinite series.
5.2.2 Calculate the sum of a geometric series.
5.2.3 Evaluate a telescoping series.

We have seen that a sequence is an ordered set of terms. If you add these terms together, you get a series. In this section we define an infinite series and show how series are related to sequences. We also define what it means for a series to converge or diverge. We introduce one of the most important types of series: the geometric series. We will use geometric series in the next chapter to write certain functions as polynomials with an infinite number of terms. This process is important because it allows us to evaluate, differentiate, and integrate complicated functions by using polynomials that are easier to handle. We also discuss the harmonic series, arguably the most interesting divergent series because it just fails to converge.

Sums and Series

An infinite series is a sum of infinitely many terms and is written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

But what does this mean? We cannot add an infinite number of terms in the same way we can add a finite number of terms. Instead, the value of an infinite series is defined in terms of the *limit* of partial sums. A partial sum of an infinite series is a finite sum of the form

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

To see how we use partial sums to evaluate infinite series, consider the following example. Suppose oil is seeping into a lake such that 1000 gallons enters the lake the first week. During the second week, an additional 500 gallons of oil enters the lake. The third week, 250 more gallons enters the lake. Assume this pattern continues such that each week half as much oil enters the lake as did the previous week. If this continues forever, what can we say about the amount of oil in the lake? Will the amount of oil continue to get arbitrarily large, or is it possible that it approaches some finite amount? To answer this question, we look at the amount of oil in the lake after k weeks. Letting S_k denote the amount of oil in the lake (measured in thousands of gallons) after k weeks, we see that

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + 0.5 = 1 + \frac{1}{2} \\
S_3 &= 1 + 0.5 + 0.25 = 1 + \frac{1}{2} + \frac{1}{4} \\
S_4 &= 1 + 0.5 + 0.25 + 0.125 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\
S_5 &= 1 + 0.5 + 0.25 + 0.125 + 0.0625 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}.
\end{aligned}$$

Looking at this pattern, we see that the amount of oil in the lake (in thousands of gallons) after k weeks is

$$S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{k-1}} = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1}.$$

We are interested in what happens as $k \to \infty$. Symbolically, the amount of oil in the lake as $k \to \infty$ is given by the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

At the same time, as $k \to \infty$, the amount of oil in the lake can be calculated by evaluating $\lim_{k \to \infty} S_k$. Therefore, the behavior of the infinite series can be determined by looking at the behavior of the sequence of partial sums $\{S_k\}$. If the sequence of partial sums $\{S_k\}$ converges, we say that the infinite series converges, and its sum is given by $\lim_{k \to \infty} S_k$. If the sequence $\{S_k\}$ diverges, we say the infinite series diverges. We now turn our attention to determining the limit of this sequence $\{S_k\}$.

First, simplifying some of these partial sums, we see that

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} \\ S_4 &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8} \\ S_5 &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{31}{16}. \end{aligned}$$

Plotting some of these values in **Figure 5.10**, it appears that the sequence $\{S_k\}$ could be approaching 2.

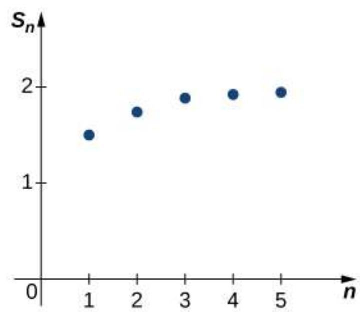


Figure 5.10 The graph shows the sequence of partial sums $\{S_k\}$. It appears that the sequence is approaching the value 2.

Let's look for more convincing evidence. In the following table, we list the values of S_k for several values of k.

k	5	10	15	20
S_k	1.9375	1.998	1.999939	1.999998

These data supply more evidence suggesting that the sequence $\{S_k\}$ converges to 2. Later we will provide an analytic argument that can be used to prove that $\lim_{k \to \infty} S_k = 2$. For now, we rely on the numerical and graphical data to convince ourselves that the sequence of partial sums does actually converge to 2. Since this sequence of partial sums converges to 2, we say the infinite series converges to 2 and write

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2.$$

Returning to the question about the oil in the lake, since this infinite series converges to 2, we conclude that the amount of oil in the lake will get arbitrarily close to 2000 gallons as the amount of time gets sufficiently large.

This series is an example of a geometric series. We discuss geometric series in more detail later in this section. First, we summarize what it means for an infinite series to converge.

Definition

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the kth **partial sum** of the infinite series. The partial sums form a sequence $\{S_k\}$. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the **convergence of a series** to S, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums diverges, we have the **divergence of a series**.

This **website (http://www.openstaxcollege.org/l/20_series)** shows a more whimsical approach to series.

Note that the index for a series need not begin with $n = 1$ but can begin with any value. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

can also be written as

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \text{ or } \sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^{n-5}.$$

Often it is convenient for the index to begin at 1, so if for some reason it begins at a different value, we can reindex by making a change of variables. For example, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

By introducing the variable $m = n - 1$, so that $n = m + 1$, we can rewrite the series as

$$\sum_{m=1}^{\infty} \frac{1}{(m+1)^2}.$$

Example 5.7

Evaluating Limits of Sequences of Partial Sums

For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution

a. The sequence of partial sums $\{S_k\}$ satisfies

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{2}{3} \\ S_3 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} \\ S_4 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}. \end{aligned}$$

Notice that each term added is greater than $1/2$. As a result, we see that

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{2}{3} > \frac{1}{2} + \frac{1}{2} = 2\left(\frac{1}{2}\right) \\ S_3 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3\left(\frac{1}{2}\right) \\ S_4 &= \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4\left(\frac{1}{2}\right). \end{aligned}$$

From this pattern we can see that $S_k > k\left(\frac{1}{2}\right)$ for every integer k. Therefore, $\{S_k\}$ is unbounded and consequently, diverges. Therefore, the infinite series $\displaystyle\sum_{n=1}^{\infty} n/(n+1)$ diverges.

b. The sequence of partial sums $\{S_k\}$ satisfies

$$\begin{aligned} S_1 &= -1 \\ S_2 &= -1 + 1 = 0 \\ S_3 &= -1 + 1 - 1 = -1 \\ S_4 &= -1 + 1 - 1 + 1 = 0. \end{aligned}$$

From this pattern we can see the sequence of partial sums is

$$\{S_k\} = \{-1, 0, -1, 0, \ldots\}.$$

Since this sequence diverges, the infinite series $\displaystyle\sum_{n=1}^{\infty} (-1)^n$ diverges.

c. The sequence of partial sums $\{S_k\}$ satisfies

$$\begin{aligned}
S_1 &= \frac{1}{1 \cdot 2} = \frac{1}{2} \\
S_2 &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\
S_3 &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4} \\
S_4 &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{4}{5} \\
S_5 &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} = \frac{5}{6}.
\end{aligned}$$

From this pattern, we can see that the kth partial sum is given by the explicit formula

$$S_k = \frac{k}{k+1}.$$

Since $k/(k+1) \to 1,$ we conclude that the sequence of partial sums converges, and therefore the infinite series converges to $1.$ We have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

5.7 Determine whether the series $\sum_{n=1}^{\infty} (n+1)/n$ converges or diverges.

The Harmonic Series

A useful series to know about is the **harmonic series**. The harmonic series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots. \tag{5.5}$$

This series is interesting because it diverges, but it diverges very slowly. By this we mean that the terms in the sequence of partial sums $\{S_k\}$ approach infinity, but do so very slowly. We will show that the series diverges, but first we illustrate the slow growth of the terms in the sequence $\{S_k\}$ in the following table.

k	10	100	1000	10,000	100,000	1,000,000
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after $1,000,000$ terms, the partial sum is still relatively small. From this table, it is not clear that this series actually diverges. However, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

To show that the sequence of partial sums diverges, we show that the sequence of partial sums is unbounded. We begin by writing the first several partial sums:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\ S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}. \end{aligned}$$

Notice that for the last two terms in S_4,

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}.$$

Therefore, we conclude that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right).$$

Using the same idea for S_8, we see that

$$\begin{aligned} S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right). \end{aligned}$$

From this pattern, we see that $S_1 = 1$, $S_2 = 1 + 1/2$, $S_4 > 1 + 2(1/2)$, and $S_8 > 1 + 3(1/2)$. More generally, it can be shown that $S_{2^j} > 1 + j(1/2)$ for all $j > 1$. Since $1 + j(1/2) \to \infty$, we conclude that the sequence $\{S_k\}$ is unbounded and therefore diverges. In the previous section, we stated that convergent sequences are bounded. Consequently, since $\{S_k\}$ is unbounded, it diverges. Thus, the harmonic series diverges.

Algebraic Properties of Convergent Series

Since the sum of a convergent infinite series is defined as a limit of a sequence, the algebraic properties for series listed below follow directly from the algebraic properties for sequences.

Theorem 5.7: Algebraic Properties of Convergent Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold.

i. The series $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. (Sum Rule)

ii. The series $\sum_{n=1}^{\infty} (a_n - b_n)$ converges and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$. (Difference Rule)

iii. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$. (Constant Multiple Rule)

Example 5.8

Using Algebraic Properties of Convergent Series

Evaluate

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right].$$

Solution

We showed earlier that

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=2.$$

Since both of those series converge, we can apply the properties of **Algebraic Properties of Convergent Series** to evaluate

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right].$$

Using the sum rule, write

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right]=\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2}.$$

Then, using the constant multiple rule and the sums above, we can conclude that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2} &= 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\left(\frac{1}{2}\right)^{-1}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}\\ &= 3(1)+\left(\frac{1}{2}\right)^{-1}(2)=3+2(2)=7.\end{aligned}$$

5.8 Evaluate $\sum_{n=1}^{\infty}\frac{5}{2^{n-1}}.$

Geometric Series

A **geometric series** is any series that we can write in the form

$$a+ar+ar^2+ar^3+\cdots=\sum_{n=1}^{\infty}ar^{n-1}. \tag{5.6}$$

Because the ratio of each term in this series to the previous term is r, the number r is called the ratio. We refer to a as the initial term because it is the first term in the series. For example, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

is a geometric series with initial term $a=1$ and ratio $r=1/2$.

In general, when does a geometric series converge? Consider the geometric series

$$\sum_{n=1}^{\infty}ar^{n-1}$$

when $a>0$. Its sequence of partial sums $\{S_k\}$ is given by

$$S_k = \sum_{n=1}^{k} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{k-1}.$$

Consider the case when $r = 1$. In that case,

$$S_k = a + a(1) + a(1)^2 + \cdots + a(1)^{k-1} = ak.$$

Since $a > 0$, we know $ak \to \infty$ as $k \to \infty$. Therefore, the sequence of partial sums is unbounded and thus diverges. Consequently, the infinite series diverges for $r = 1$. For $r \neq 1$, to find the limit of $\{S_k\}$, multiply **Equation 5.6** by $1 - r$. Doing so, we see that

$$\begin{aligned}(1-r)S_k &= a(1-r)\left(1 + r + r^2 + r^3 + \cdots + r^{k-1}\right) \\ &= a[(1 + r + r^2 + r^3 + \cdots + r^{k-1}) - (r + r^2 + r^3 + \cdots + r^k)] \\ &= a\left(1 - r^k\right).\end{aligned}$$

All the other terms cancel out.

Therefore,

$$S_k = \frac{a\left(1 - r^k\right)}{1 - r} \text{ for } r \neq 1.$$

From our discussion in the previous section, we know that the geometric sequence $r^k \to 0$ if $|r| < 1$ and that r^k diverges if $|r| > 1$ or $r = \pm 1$. Therefore, for $|r| < 1$, $S_k \to a/(1 - r)$ and we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \text{ if } |r| < 1.$$

If $|r| \geq 1$, S_k diverges, and therefore

$$\sum_{n=1}^{\infty} ar^{n-1} \text{ diverges if } |r| \geq 1.$$

Definition

A geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots.$$

If $|r| < 1$, the series converges, and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \text{ for } |r| < 1. \tag{5.7}$$

If $|r| \geq 1$, the series diverges.

Geometric series sometimes appear in slightly different forms. For example, sometimes the index begins at a value other than $n = 1$ or the exponent involves a linear expression for n other than $n - 1$. As long as we can rewrite the series in the form given by **Equation 5.5**, it is a geometric series. For example, consider the series

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}.$$

To see that this is a geometric series, we write out the first several terms:

$$\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n+2} = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots$$
$$= \frac{4}{9} + \frac{4}{9}\cdot\left(\frac{2}{3}\right) + \frac{4}{9}\cdot\left(\frac{2}{3}\right)^2 + \cdots.$$

We see that the initial term is $a = 4/9$ and the ratio is $r = 2/3$. Therefore, the series can be written as

$$\sum_{n=1}^{\infty}\frac{4}{9}\cdot\left(\frac{2}{3}\right)^{n-1}.$$

Since $r = 2/3 < 1,$ this series converges, and its sum is given by

$$\sum_{n=1}^{\infty}\frac{4}{9}\cdot\left(\frac{2}{3}\right)^{n-1} = \frac{4/9}{1-2/3} = \frac{4}{3}.$$

Example 5.9

Determining Convergence or Divergence of a Geometric Series

Determine whether each of the following geometric series converges or diverges, and if it converges, find its sum.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{4^{n-1}}$

b. $\displaystyle\sum_{n=1}^{\infty} e^{2n}$

Solution

a. Writing out the first several terms in the series, we have

$$\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{4^{n-1}} = \frac{(-3)^2}{4^0} + \frac{(-3)^3}{4} + \frac{(-3)^4}{4^2} + \cdots$$
$$= (-3)^2 + (-3)^2\cdot\left(\frac{-3}{4}\right) + (-3)^2\cdot\left(\frac{-3}{4}\right)^2 + \cdots$$
$$= 9 + 9\cdot\left(\frac{-3}{4}\right) + 9\cdot\left(\frac{-3}{4}\right)^2 + \cdots.$$

The initial term $a = -3$ and the ratio $r = -3/4$. Since $|r| = 3/4 < 1,$ the series converges to

$$\frac{9}{1-(-3/4)} = \frac{9}{7/4} = \frac{36}{7}.$$

b. Writing this series as

$$e^2\sum_{n=1}^{\infty}\left(e^2\right)^{n-1}$$

we can see that this is a geometric series where $r = e^2 > 1$. Therefore, the series diverges.

5.9 Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-2}{5}\right)^{n-1}$ converges or diverges. If it converges, find its sum.

We now turn our attention to a nice application of geometric series. We show how they can be used to write repeating decimals as fractions of integers.

Example 5.10

Writing Repeating Decimals as Fractions of Integers

Use a geometric series to write $3.\overline{26}$ as a fraction of integers.

Solution

Since $3.\overline{26} = 3.262626\ldots,$ first we write

$$\begin{aligned} 3.262626\ldots &= 3 + \frac{26}{100} + \frac{26}{1000} + \frac{26}{100{,}000} + \cdots \\ &= 3 + \frac{26}{10^2} + \frac{26}{10^4} + \frac{26}{10^6} + \cdots. \end{aligned}$$

Ignoring the term 3, the rest of this expression is a geometric series with initial term $a = 26/10^2$ and ratio $r = 1/10^2$. Therefore, the sum of this series is

$$\frac{26/10^2}{1 - (1/10^2)} = \frac{26/10^2}{99/10^2} = \frac{26}{99}.$$

Thus,

$$3.262626\ldots = 3 + \frac{26}{99} = \frac{323}{99}.$$

5.10 Write $5.2\overline{7}$ as a fraction of integers.

Example 5.11

Chapter Opener: Finding the Area of the Koch Snowflake

Define a sequence of figures $\{F_n\}$ recursively as follows (**Figure 5.11**). Let F_0 be an equilateral triangle with sides of length $1.$ For $n \geq 1,$ let F_n be the curve created by removing the middle third of each side of F_{n-1} and replacing it with an equilateral triangle pointing outward. The limiting figure as $n \to \infty$ is known as Koch's snowflake.

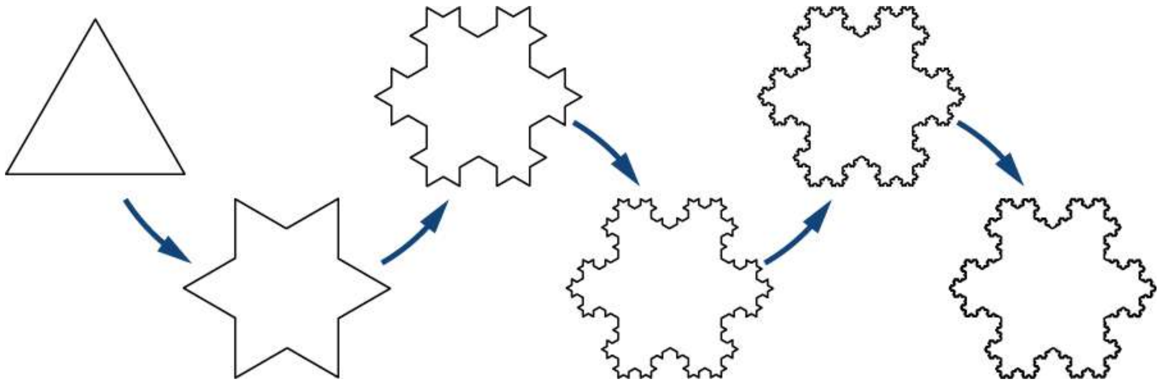


Figure 5.11 The first four figures, $F_0, F_1, F_2,$ and $F_3,$ in the construction of the Koch snowflake.

a. Find the length L_n of the perimeter of F_n. Evaluate $\lim_{n \to \infty} L_n$ to find the length of the perimeter of Koch's snowflake.

b. Find the area A_n of figure F_n. Evaluate $\lim_{n \to \infty} A_n$ to find the area of Koch's snowflake.

Solution

a. Let N_n denote the number of sides of figure F_n. Since F_0 is a triangle, $N_0 = 3$. Let l_n denote the length of each side of F_n. Since F_0 is an equilateral triangle with sides of length $l_0 = 1,$ we now need to determine N_1 and l_1. Since F_1 is created by removing the middle third of each side and replacing that line segment with two line segments, for each side of $F_0,$ we get four sides in F_1. Therefore, the number of sides for F_1 is

$$N_1 = 4 \cdot 3.$$

Since the length of each of these new line segments is $1/3$ the length of the line segments in $F_0,$ the length of the line segments for F_1 is given by

$$l_1 = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

Similarly, for $F_2,$ since the middle third of each side of F_1 is removed and replaced with two line segments, the number of sides in F_2 is given by

$$N_2 = 4N_1 = 4(4 \cdot 3) = 4^2 \cdot 3.$$

Since the length of each of these sides is $1/3$ the length of the sides of $F_1,$ the length of each side of figure F_2 is given by

$$l_2 = \frac{1}{3} \cdot l_1 = \frac{1}{3} \cdot \frac{1}{3} = \left(\frac{1}{3}\right)^2.$$

More generally, since F_n is created by removing the middle third of each side of F_{n-1} and replacing that line segment with two line segments of length $\frac{1}{3} l_{n-1}$ in the shape of an equilateral triangle, we know that $N_n = 4N_{n-1}$ and $l_n = \frac{l_{n-1}}{3}$. Therefore, the number of sides of figure F_n is

$$N_n = 4^n \cdot 3$$

and the length of each side is

$$l_n = \left(\frac{1}{3}\right)^n.$$

Therefore, to calculate the perimeter of F_n, we multiply the number of sides N_n and the length of each side l_n. We conclude that the perimeter of F_n is given by

$$L_n = N_n \cdot l_n = 3 \cdot \left(\frac{4}{3}\right)^n.$$

Therefore, the length of the perimeter of Koch's snowflake is

$$L = \lim_{n \to \infty} L_n = \infty.$$

b. Let T_n denote the area of each new triangle created when forming F_n. For $n = 0$, T_0 is the area of the original equilateral triangle. Therefore, $T_0 = A_0 = \sqrt{3}/4$. For $n \geq 1$, since the lengths of the sides of the new triangle are $1/3$ the length of the sides of F_{n-1}, we have

$$T_n = \left(\frac{1}{3}\right)^2 T_{n-1} = \frac{1}{9} \cdot T_{n-1}.$$

Therefore, $T_n = \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4}$. Since a new triangle is formed on each side of F_{n-1},

$$\begin{aligned} A_n &= A_{n-1} + N_{n-1} \cdot T_n \\ &= A_{n-1} + \left(3 \cdot 4^{n-1}\right) \cdot \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4} \\ &= A_{n-1} + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot \frac{\sqrt{3}}{4}. \end{aligned}$$

Writing out the first few terms A_0, A_1, A_2, we see that

$$\begin{aligned} &A_0 = \frac{\sqrt{3}}{4} \\ &A_1 = A_0 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right] \\ &A_2 = A_1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right] + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2\right]. \end{aligned}$$

More generally,

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4}\left(\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^n\right)\right].$$

Factoring $4/9$ out of each term inside the inner parentheses, we rewrite our expression as

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right)\right].$$

The expression $1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}$ is a geometric sum. As shown earlier, this sum satisfies

$$1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1} = \frac{1 - (4/9)^n}{1 - (4/9)}.$$

Substituting this expression into the expression above and simplifying, we conclude that

$$\begin{aligned} A_n &= \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(\frac{1 - (4/9)^n}{1 - (4/9)}\right)\right] \\ &= \frac{\sqrt{3}}{4}\left[\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^n\right]. \end{aligned}$$

Therefore, the area of Koch's snowflake is

$$A = \lim_{n \to \infty} A_n = \frac{2\sqrt{3}}{5}.$$

Analysis

The Koch snowflake is interesting because it has finite area, yet infinite perimeter. Although at first this may seem impossible, recall that you have seen similar examples earlier in the text. For example, consider the region bounded by the curve $y = 1/x^2$ and the x-axis on the interval $[1, \infty)$. Since the improper integral

$$\int_1^\infty \frac{1}{x^2} dx$$

converges, the area of this region is finite, even though the perimeter is infinite.

Telescoping Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We discussed this series in **Example 5.7**, showing that the series converges by writing out the first several partial sums $S_1, S_2, \ldots, S_6$ and noticing that they are all of the form $S_k = \frac{k}{k+1}$. Here we use a different technique to show that this series converges. By using partial fractions, we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore, the series can be written as

$$\sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots.$$

Writing out the first several terms in the sequence of partial sums $\{S_k\}$, we see that

$$\begin{aligned} S_1 &= 1 - \frac{1}{2} \\ S_2 &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3} \\ S_3 &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}. \end{aligned}$$

In general,

$$S_k = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{k+1}.$$

We notice that the middle terms cancel each other out, leaving only the first and last terms. In a sense, the series collapses like a spyglass with tubes that disappear into each other to shorten the telescope. For this reason, we call a series that has this property a telescoping series. For this series, since $S_k = 1 - 1/(k+1)$ and $1/(k+1) \to 0$ as $k \to \infty$, the sequence of partial sums converges to 1, and therefore the series converges to 1.

Definition

A **telescoping series** is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example, any series of the form

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

is a telescoping series. We can see this by writing out some of the partial sums. In particular, we see that

$$\begin{aligned} S_1 &= b_1 - b_2 \\ S_2 &= (b_1 - b_2) + (b_2 - b_3) = b_1 - b_3 \\ S_3 &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) = b_1 - b_4. \end{aligned}$$

In general, the *k*th partial sum of this series is

$$S_k = b_1 - b_{k+1}.$$

Since the *k*th partial sum can be simplified to the difference of these two terms, the sequence of partial sums $\{S_k\}$ will converge if and only if the sequence $\{b_{k+1}\}$ converges. Moreover, if the sequence b_{k+1} converges to some finite number B, then the sequence of partial sums converges to $b_1 - B$, and therefore

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = b_1 - B.$$

In the next example, we show how to use these ideas to analyze a telescoping series of this form.

Example 5.12

Evaluating a Telescoping Series

Determine whether the telescoping series

$$\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$$

converges or diverges. If it converges, find its sum.

Solution

By writing out terms in the sequence of partial sums, we can see that

$$\begin{aligned}
S_1 &= \cos(1) - \cos\left(\frac{1}{2}\right) \\
S_2 &= \left(\cos(1) - \cos\left(\frac{1}{2}\right)\right) + \left(\cos\left(\frac{1}{2}\right) - \cos\left(\frac{1}{3}\right)\right) = \cos(1) - \cos\left(\frac{1}{3}\right) \\
S_3 &= \left(\cos(1) - \cos\left(\frac{1}{2}\right)\right) + \left(\cos\left(\frac{1}{2}\right) - \cos\left(\frac{1}{3}\right)\right) + \left(\cos\left(\frac{1}{3}\right) - \cos\left(\frac{1}{4}\right)\right) \\
&= \cos(1) - \cos\left(\frac{1}{4}\right).
\end{aligned}$$

In general,

$$S_k = \cos(1) - \cos\left(\frac{1}{k+1}\right).$$

Since $1/(k+1) \to 0$ as $k \to \infty$ and $\cos x$ is a continuous function, $\cos(1/(k+1)) \to \cos(0) = 1$. Therefore, we conclude that $S_k \to \cos(1) - 1$. The telescoping series converges and the sum is given by

$$\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right] = \cos(1) - 1.$$

5.11 Determine whether $\sum_{n=1}^{\infty} \left[e^{1/n} - e^{1/(n+1)}\right]$ converges or diverges. If it converges, find its sum.

Student PROJECT

Euler's Constant

We have shown that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Here we investigate the behavior of the partial sums S_k as $k \to \infty$. In particular, we show that they behave like the natural logarithm function by showing that there exists a constant γ such that

$$\sum_{n=1}^{k} \frac{1}{n} - \ln k \to \gamma \text{ as } k \to \infty.$$

This constant γ is known as Euler's constant.

1. Let $T_k = \sum_{n=1}^{k} \frac{1}{n} - \ln k$. Evaluate T_k for various values of k.
2. For T_k as defined in part 1. show that the sequence $\{T_k\}$ converges by using the following steps.
   a. Show that the sequence $\{T_k\}$ is monotone decreasing. (*Hint:* Show that $\ln(1 + 1/k > 1/(k+1))$
   b. Show that the sequence $\{T_k\}$ is bounded below by zero. (*Hint:* Express $\ln k$ as a definite integral.)
   c. Use the Monotone Convergence Theorem to conclude that the sequence $\{T_k\}$ converges. The limit γ is Euler's constant.
3. Now estimate how far T_k is from γ for a given integer k. Prove that for $k \geq 1$, $0 < T_k - \gamma \leq 1/k$ by using the following steps.
   a. Show that $\ln(k+1) - \ln k < 1/k$.
   b. Use the result from part a. to show that for any integer k,
   $$T_k - T_{k+1} < \frac{1}{k} - \frac{1}{k+1}.$$
   c. For any integers k and j such that $j > k$, express $T_k - T_j$ as a telescoping sum by writing
   $$T_k - T_j = (T_k - T_{k+1}) + (T_{k+1} - T_{k+2}) + (T_{k+2} - T_{k+3}) + \cdots + (T_{j-1} - T_j).$$
   Use the result from part b. combined with this telescoping sum to conclude that
   $$T_k - T_j < \frac{1}{k} - \frac{1}{j}.$$
   d. Apply the limit to both sides of the inequality in part c. to conclude that
   $$T_k - \gamma \leq \frac{1}{k}.$$
   e. Estimate γ to an accuracy of within 0.001.

5.2 EXERCISES

Using sigma notation, write the following expressions as infinite series.

67. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

68. $1-1+1-1+\cdots$

69. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...$

70. $\sin 1+\sin 1/2+\sin 1/3+\sin 1/4+\cdots$

Compute the first four partial sums $S_1,\ldots,S_4$ for the series having nth term a_n starting with $n=1$ as follows.

71. $a_n=n$

72. $a_n=1/n$

73. $a_n=\sin(n\pi/2)$

74. $a_n=(-1)^n$

In the following exercises, compute the general term a_n of the series with the given partial sum S_n. If the sequence of partial sums converges, find its limit S.

75. $S_n=1-\frac{1}{n},\quad n\geq 2$

76. $S_n=\frac{n(n+1)}{2},\quad n\geq 1$

77. $S_n=\sqrt{n},\ n\geq 2$

78. $S_n=2-(n+2)/2^n,\ n\geq 1$

For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

79. $\sum_{n=1}^{\infty}\frac{n}{n+2}$

80. $\sum_{n=1}^{\infty}(1-(-1)^n))$

81. $\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ (*Hint:* Use a partial fraction decomposition like that for $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.)

82. $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ (*Hint:* Follow the reasoning for $\sum_{n=1}^{\infty}\frac{1}{n}$.)

Suppose that $\sum_{n=1}^{\infty}a_n=1$, that $\sum_{n=1}^{\infty}b_n=-1$, that $a_1=2$, and $b_1=-3$. Find the sum of the indicated series.

83. $\sum_{n=1}^{\infty}(a_n+b_n)$

84. $\sum_{n=1}^{\infty}(a_n-2b_n)$

85. $\sum_{n=2}^{\infty}(a_n-b_n)$

86. $\sum_{n=1}^{\infty}(3a_{n+1}-4b_{n+1})$

State whether the given series converges and explain why.

87. $\sum_{n=1}^{\infty}\frac{1}{n+1000}$ (*Hint:* Rewrite using a change of index.)

88. $\sum_{n=1}^{\infty}\frac{1}{n+10^{80}}$ (*Hint:* Rewrite using a change of index.)

89. $1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$

90. $1+\frac{e}{\pi}+\frac{e^2}{\pi^2}+\frac{e^3}{\pi^3}+\cdots$

91. $1+\frac{\pi}{e}+\frac{\pi^2}{e^4}+\frac{\pi^3}{e^6}+\frac{\pi^4}{e^8}+\cdots$

92. $1-\sqrt{\frac{\pi}{3}}+\sqrt{\frac{\pi^2}{9}}-\sqrt{\frac{\pi^3}{27}}+\cdots$

For a_n as follows, write the sum as a geometric series of the form $\sum_{n=1}^{\infty}ar^n$. State whether the series converges and if it does, find the value of $\sum a_n$.

93. $a_1 = -1$ and $a_n/a_{n+1} = -5$ for $n \geq 1$.

94. $a_1 = 2$ and $a_n/a_{n+1} = 1/2$ for $n \geq 1$.

95. $a_1 = 10$ and $a_n/a_{n+1} = 10$ for $n \geq 1$.

96. $a_1 = 1/10$ and $a_n/a_{n+1} = -10$ for $n \geq 1$.

Use the identity $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ to express the function as a geometric series in the indicated term.

97. $\frac{x}{1+x}$ in x

98. $\frac{\sqrt{x}}{1-x^{3/2}}$ in $\sqrt{x}$

99. $\frac{1}{1+\sin^2 x}$ in $\sin x$

100. $\sec^2 x$ in $\sin x$

Evaluate the following telescoping series or state whether the series diverges.

101. $\sum_{n=1}^{\infty} 2^{1/n} - 2^{1/(n+1)}$

102. $\sum_{n=1}^{\infty} \frac{1}{n^{13}} - \frac{1}{(n+1)^{13}}$

103. $\sum_{n=1}^{\infty} \left(\sqrt{n} - \sqrt{n+1}\right)$

104. $\sum_{n=1}^{\infty} (\sin n - \sin(n+1))$

Express the following series as a telescoping sum and evaluate its *n*th partial sum.

105. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

106. $\sum_{n=1}^{\infty} \frac{2n+1}{\left(n^2+n\right)^2}$ (*Hint:* Factor denominator and use partial fractions.)

107. $\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n \ln(n+1)}$

108. $\sum_{n=1}^{\infty} \frac{(n+2)}{n(n+1)2^{n+1}}$ (*Hint:* Look at $1/(n2^n)$.)

A general telescoping series is one in which all but the first few terms cancel out after summing a given number of successive terms.

109. Let $a_n = f(n) - 2f(n+1) + f(n+2)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

110. $a_n = f(n) - f(n+1) - f(n+2) + f(n+3)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

111. Suppose that $a_n = c_0 f(n) + c_1 f(n+1) + c_2 f(n+2) + c_3 f(n+3) + c_4 f(n+4)$, where $f(n) \to 0$ as $n \to \infty$. Find a condition on the coefficients $c_0, \ldots, c_4$ that make this a general telescoping series.

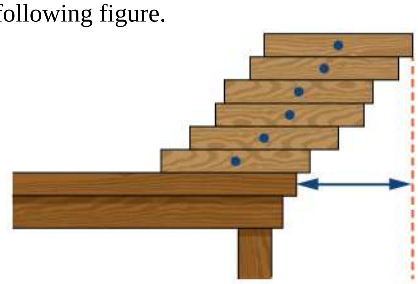
112. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (*Hint:* $\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$)

113. Evaluate $\sum_{n=2}^{\infty} \frac{2}{n^3 - n}$.

114. Find a formula for $\sum_{n=1}^{\infty} \frac{1}{n(n+N)}$ where N is a positive integer.

115. **[T]** Define a sequence $t_k = \sum_{n=1}^{k-1} (1/k) - \ln k$. Use the graph of $1/x$ to verify that t_k is increasing. Plot t_k for $k = 1 \ldots 100$ and state whether it appears that the sequence converges.

116. **[T]** Suppose that N equal uniform rectangular blocks are stacked one on top of the other, allowing for some overhang. Archimedes' law of the lever implies that the stack of N blocks is stable as long as the center of mass of the top $(N-1)$ blocks lies at the edge of the bottom block. Let x denote the position of the edge of the bottom block, and think of its position as relative to the center of the next-to-bottom block. This implies that $(N-1)x = \left(\frac{1}{2} - x\right)$ or $x = 1/(2N)$. Use this expression to compute the maximum overhang (the position of the edge of the top block over the edge of the bottom block.) See the following figure.



Each of the following infinite series converges to the given multiple of π or $1/\pi$.

In each case, find the minimum value of N such that the Nth partial sum of the series accurately approximates the left-hand side to the given number of decimal places, and give the desired approximate value. Up to 15 decimals place, $\pi = 3.141592653589793....$

117. **[T]** $\pi = -3 + \sum_{n=1}^{\infty} \frac{n2^n n!^2}{(2n)!}$, error < 0.0001

118. **[T]** $\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!}$, error $< 10^{-4}$

119. **[T]** $\frac{9801}{2\pi} = \frac{4}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$, error $< 10^{-12}$

120. **[T]** $\frac{1}{12\pi} = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$, error $< 10^{-15}$

121. **[T]** A fair coin is one that has probability $1/2$ of coming up heads when flipped.

a. What is the probability that a fair coin will come up tails n times in a row?

b. Find the probability that a coin comes up heads for the first time after an even number of coin flips.

122. **[T]** Find the probability that a fair coin is flipped a multiple of three times before coming up heads.

123. **[T]** Find the probability that a fair coin will come up heads for the second time after an even number of flips.

124. **[T]** Find a series that expresses the probability that a fair coin will come up heads for the second time on a multiple of three flips.

125. **[T]** The expected number of times that a fair coin will come up heads is defined as the sum over $n = 1, 2,\ldots$ of n times the probability that the coin will come up heads exactly n times in a row, or $n/2^{n+1}$. Compute the expected number of consecutive times that a fair coin will come up heads.

126. **[T]** A person deposits $\$10$ at the beginning of each quarter into a bank account that earns 4% annual interest compounded quarterly (four times a year).

a. Show that the interest accumulated after n quarters is $\$10\left(\frac{1.01^{n+1} - 1}{0.01} - n\right)$.

b. Find the first eight terms of the sequence.

c. How much interest has accumulated after 2 years?

127. **[T]** Suppose that the amount of a drug in a patient's system diminishes by a multiplicative factor $r < 1$ each hour. Suppose that a new dose is administered every N hours. Find an expression that gives the amount $A(n)$ in the patient's system after n hours for each n in terms of the dosage d and the ratio r. (*Hint:* Write $n = mN + k$, where $0 \le k < N$, and sum over values from the different doses administered.)

128. **[T]** A certain drug is effective for an average patient only if there is at least 1 mg per kg in the patient's system, while it is safe only if there is at most 2 mg per kg in an average patient's system. Suppose that the amount in a patient's system diminishes by a multiplicative factor of 0.9 each hour after a dose is administered. Find the maximum interval N of hours between doses, and corresponding dose range d (in mg/kg) for this N that will enable use of the drug to be both safe and effective in the long term.

129. Suppose that $a_n \ge 0$ is a sequence of numbers. Explain why the sequence of partial sums of a_n is increasing.

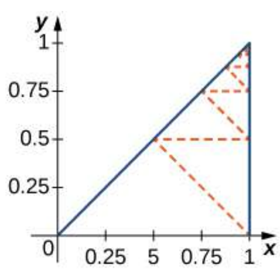
130. **[T]** Suppose that a_n is a sequence of positive numbers and the sequence S_n of partial sums of a_n is bounded above. Explain why $\sum_{n=1}^{\infty} a_n$ converges. Does the conclusion remain true if we remove the hypothesis $a_n \geq 0$?

131. **[T]** Suppose that $a_1 = S_1 = 1$ and that, for given numbers $S > 1$ and $0 < k < 1$, one defines $a_{n+1} = k(S - S_n)$ and $S_{n+1} = a_{n+1} + S_n$. Does S_n converge? If so, to what? (*Hint:* First argue that $S_n < S$ for all n and S_n is increasing.)

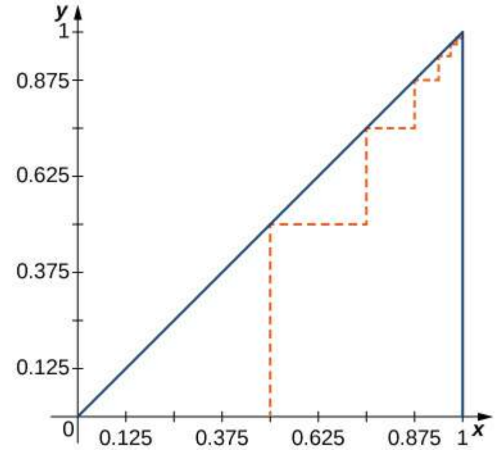
132. **[T]** A version of von Bertalanffy growth can be used to estimate the age of an individual in a homogeneous species from its length if the annual increase in year $n+1$ satisfies $a_{n+1} = k(S - S_n)$, with S_n as the length at year n, S as a limiting length, and k as a relative growth constant. If $S_1 = 3$, $S = 9$, and $k = 1/2$, numerically estimate the smallest value of n such that $S_n \geq 8$. Note that $S_{n+1} = S_n + a_{n+1}$. Find the corresponding n when $k = 1/4$.

133. **[T]** Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Explain why $\lim_{N \to \infty} \sum_{n=N+1}^{\infty} a_n = 0$.

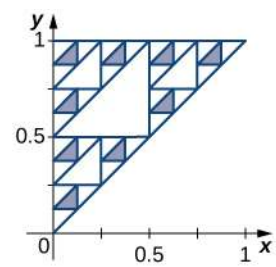
134. **[T]** Find the length of the dashed zig-zag path in the following figure.



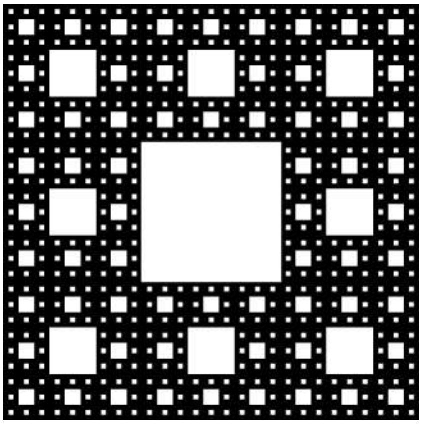
135. **[T]** Find the total length of the dashed path in the following figure.



136. **[T]** The Sierpinski triangle is obtained from a triangle by deleting the middle fourth as indicated in the first step, by deleting the middle fourths of the remaining three congruent triangles in the second step, and in general deleting the middle fourths of the remaining triangles in each successive step. Assuming that the original triangle is shown in the figure, find the areas of the remaining parts of the original triangle after N steps and find the total length of all of the boundary triangles after N steps.



137. **[T]** The Sierpinski gasket is obtained by dividing the unit square into nine equal sub-squares, removing the middle square, then doing the same at each stage to the remaining sub-squares. The figure shows the remaining set after four iterations. Compute the total area removed after N stages, and compute the length the total perimeter of the remaining set after N stages.



5.3 | The Divergence and Integral Tests

Learning Objectives

5.3.1 Use the divergence test to determine whether a series converges or diverges.
5.3.2 Use the integral test to determine the convergence of a series.
5.3.3 Estimate the value of a series by finding bounds on its remainder term.

In the previous section, we determined the convergence or divergence of several series by explicitly calculating the limit of the sequence of partial sums $\{S_k\}$. In practice, explicitly calculating this limit can be difficult or impossible. Luckily, several tests exist that allow us to determine convergence or divergence for many types of series. In this section, we discuss two of these tests: the divergence test and the integral test. We will examine several other tests in the rest of this chapter and then summarize how and when to use them.

Divergence Test

For a series $\sum_{n=1}^{\infty} a_n$ to converge, the nth term a_n must satisfy $a_n \to 0$ as $n \to \infty$.

Therefore, from the algebraic limit properties of sequences,

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} (S_k - S_{k-1}) = \lim_{k \to \infty} S_k - \lim_{k \to \infty} S_{k-1} = S - S = 0.$$

Therefore, if $\sum_{n=1}^{\infty} a_n$ converges, the nth term $a_n \to 0$ as $n \to \infty$. An important consequence of this fact is the following statement:

$$\text{If } a_n \nrightarrow 0 \text{ as } n \to \infty, \sum_{n=1}^{\infty} a_n \text{ diverges.} \tag{5.8}$$

This test is known as the **divergence test** because it provides a way of proving that a series diverges.

Theorem 5.8: Divergence Test

If $\lim_{n \to \infty} a_n = c \neq 0$ or $\lim_{n \to \infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

It is important to note that the converse of this theorem is not true. That is, if $\lim_{n \to \infty} a_n = 0$, we cannot make any conclusion about the convergence of $\sum_{n=1}^{\infty} a_n$. For example, $\lim_{n \to 0} (1/n) = 0$, but the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges. In this section and the remaining sections of this chapter, we show many more examples of such series. Consequently, although we can use the divergence test to show that a series diverges, we cannot use it to prove that a series converges. Specifically, if $a_n \to 0$, the divergence test is inconclusive.

Example 5.13

Using the divergence test

For each of the following series, apply the divergence test. If the divergence test proves that the series diverges, state so. Otherwise, indicate that the divergence test is inconclusive.

a. $\sum_{n=1}^{\infty} \frac{n}{3n-1}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

c. $\sum_{n=1}^{\infty} e^{1/n^2}$

Solution

a. Since $n/(3n-1) \to 1/3 \neq 0,$ by the divergence test, we can conclude that

$$\sum_{n=1}^{\infty} \frac{n}{3n-1}$$

diverges.

b. Since $1/n^3 \to 0,$ the divergence test is inconclusive.

c. Since $e^{1/n^2} \to 1 \neq 0,$ by the divergence test, the series

$$\sum_{n=1}^{\infty} e^{1/n^2}$$

diverges.

5.12 What does the divergence test tell us about the series $\sum_{n=1}^{\infty} \cos(1/n^2)$?

Integral Test

In the previous section, we proved that the harmonic series diverges by looking at the sequence of partial sums $\{S_k\}$ and showing that $S_{2^k} > 1 + k/2$ for all positive integers k. In this section we use a different technique to prove the divergence of the harmonic series. This technique is important because it is used to prove the divergence or convergence of many other series. This test, called the **integral test**, compares an infinite sum to an improper integral. It is important to note that this test can only be applied when we are considering a series whose terms are all positive.

To illustrate how the integral test works, use the harmonic series as an example. In **Figure 5.12**, we depict the harmonic series by sketching a sequence of rectangles with areas $1, 1/2, 1/3, 1/4, \ldots$ along with the function $f(x) = 1/x$. From the graph, we see that

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > \int_1^{k+1} \frac{1}{x} dx.$$

Therefore, for each $k,$ the kth partial sum S_k satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n} > \int_1^{k+1} \frac{1}{x} dx = \ln x \Big|_1^{k+1} = \ln(k+1) - \ln(1) = \ln(k+1).$$

Since $\lim_{k \to \infty} \ln(k+1) = \infty,$ we see that the sequence of partial sums $\{S_k\}$ is unbounded. Therefore, $\{S_k\}$ diverges, and, consequently, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

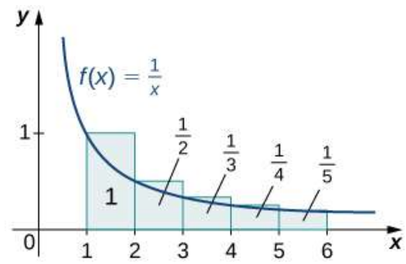


Figure 5.12 The sum of the areas of the rectangles is greater than the area between the curve $f(x) = 1/x$ and the x-axis for $x \geq 1.$ Since the area bounded by the curve is infinite (as calculated by an improper integral), the sum of the areas of the rectangles is also infinite.

Now consider the series $\sum_{n=1}^{\infty} 1/n^2.$ We show how an integral can be used to prove that this series converges. In **Figure 5.13**, we sketch a sequence of rectangles with areas $1, 1/2^2, 1/3^2, \ldots$ along with the function $f(x) = 1/x^2.$ From the graph we see that

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 1 + \int_1^k \frac{1}{x^2} dx.$$

Therefore, for each $k,$ the kth partial sum S_k satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2} < 1 + \int_1^k \frac{1}{x^2} dx = 1 - \frac{1}{x}\Big|_1^k = 1 - \frac{1}{k} + 1 = 2 - \frac{1}{k} < 2.$$

We conclude that the sequence of partial sums $\{S_k\}$ is bounded. We also see that $\{S_k\}$ is an increasing sequence:

$$S_k = S_{k-1} + \frac{1}{k^2} \text{ for } k \geq 2.$$

Since $\{S_k\}$ is increasing and bounded, by the Monotone Convergence Theorem, it converges. Therefore, the series $\sum_{n=1}^{\infty} 1/n^2$ converges.

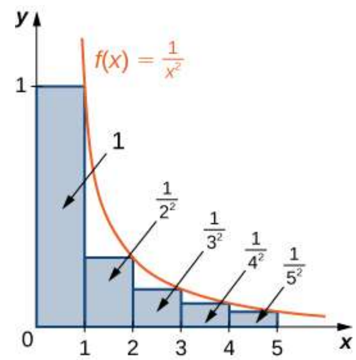


Figure 5.13 The sum of the areas of the rectangles is less than the sum of the area of the first rectangle and the area between the curve $f(x) = 1/x^2$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is finite, the sum of the areas of the rectangles is also finite.

We can extend this idea to prove convergence or divergence for many different series. Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n such that there exists a continuous, positive, decreasing function f where $f(n) = a_n$ for all positive integers. Then, as in **Figure 5.14**(a), for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x)dx < 1 + \int_1^{\infty} f(x)dx.$$

Therefore, if $\int_1^{\infty} f(x)dx$ converges, then the sequence of partial sums $\{S_k\}$ is bounded. Since $\{S_k\}$ is an increasing sequence, if it is also a bounded sequence, then by the Monotone Convergence Theorem, it converges. We conclude that if $\int_1^{\infty} f(x)dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. On the other hand, from **Figure 5.14**(b), for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)dx.$$

If $\lim_{k \to \infty} \int_1^{k+1} f(x)dx = \infty$, then $\{S_k\}$ is an unbounded sequence and therefore diverges. As a result, the series $\sum_{n=1}^{\infty} a_n$ also diverges. Since f is a positive function, if $\int_1^{\infty} f(x)dx$ diverges, then $\lim_{k \to \infty} \int_1^{k+1} f(x)dx = \infty$. We conclude that if $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

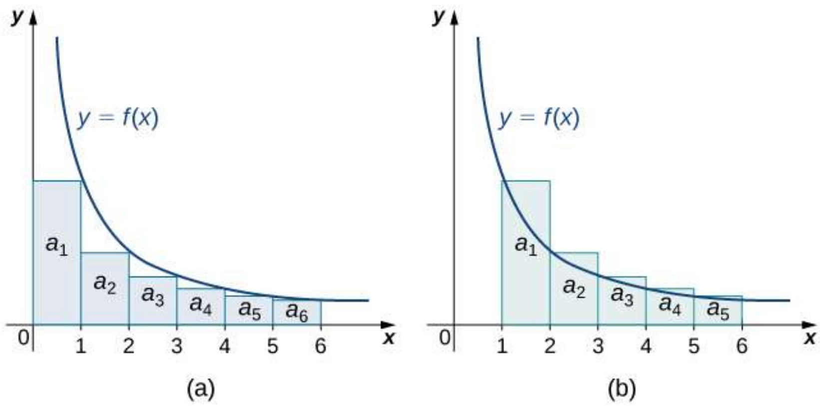


Figure 5.14 (a) If we can inscribe rectangles inside a region bounded by a curve $y = f(x)$ and the x-axis, and the area bounded by those curves for $x \geq 1$ is finite, then the sum of the areas of the rectangles is also finite. (b) If a set of rectangles circumscribes the region bounded by $y = f(x)$ and the x axis for $x \geq 1$ and the region has infinite area, then the sum of the areas of the rectangles is also infinite.

Theorem 5.9: Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

i. f is continuous,

ii. f is decreasing, and

iii. $f(n) = a_n$ for all integers $n \geq N$.

Then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_N^{\infty} f(x)dx$$

both converge or both diverge (see **Figure 5.14**).

Although convergence of $\int_N^{\infty} f(x)dx$ implies convergence of the related series $\sum_{n=1}^{\infty} a_n,$ it does not imply that the value of the integral and the series are the same. They may be different, and often are. For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \left(\frac{1}{e}\right)^3 + \cdots$$

is a geometric series with initial term $a = 1/e$ and ratio $r = 1/e,$ which converges to

$$\frac{1/e}{1-(1/e)} = \frac{1/e}{(e-1)/e} = \frac{1}{e-1}.$$

However, the related integral $\int_1^{\infty} (1/e)^x dx$ satisfies

$$\int_1^{\infty} \left(\frac{1}{e}\right)^x dx = \int_1^{\infty} e^{-x} dx = \lim_{b \to \infty} \int_1^b e^{-x} dx = \lim_{b \to \infty} -e^{-x}\Big|_1^b = \lim_{b \to \infty} \left[-e^{-b} + e^{-1}\right] = \frac{1}{e}.$$

Example 5.14

Using the Integral Test

For each of the following series, use the integral test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} 1/n^3$

b. $\sum_{n=1}^{\infty} 1/\sqrt{2n-1}$

Solution

a. Compare

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ and } \int_1^{\infty} \frac{1}{x^3} dx.$$

We have

$$\int_1^{\infty} \frac{1}{x^3} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^3} dx = \lim_{b \to \infty} \left[-\frac{1}{2x^2} \Big|_1^b \right] = \lim_{b \to \infty} \left[-\frac{1}{2b^2} + \frac{1}{2} \right] = \frac{1}{2}.$$

Thus the integral $\int_1^{\infty} 1/x^3\, dx$ converges, and therefore so does the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

b. Compare

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} \text{ and } \int_1^{\infty} \frac{1}{\sqrt{2x-1}} dx.$$

Since

$$\begin{aligned} \int_1^{\infty} \frac{1}{\sqrt{2x-1}} dx &= \lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{2x-1}} dx = \lim_{b \to \infty} \sqrt{2x-1} \Big|_1^b \\ &= \lim_{b \to \infty} \left[\sqrt{2b-1} - 1 \right] = \infty, \end{aligned}$$

the integral $\int_1^{\infty} 1/\sqrt{2x-1}\, dx$ diverges, and therefore

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$$

diverges.

5.13 Use the integral test to determine whether the series $\sum_{n=1}^{\infty} \frac{n}{3n^2+1}$ converges or diverges.

The *p*-Series

The harmonic series $\sum_{n=1}^{\infty} 1/n$ and the series $\sum_{n=1}^{\infty} 1/n^2$ are both examples of a type of series called a *p*-series.

Definition

For any real number $p,$ the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a ***p*-series**.

We know the *p*-series converges if $p = 2$ and diverges if $p = 1.$ What about other values of p? In general, it is difficult, if not impossible, to compute the exact value of most p-series. However, we can use the tests presented thus far to prove whether a p-series converges or diverges.

If $p < 0,$ then $1/n^p \to \infty,$ and if $p = 0,$ then $1/n^p \to 1.$ Therefore, by the divergence test,

$$\sum_{n=1}^{\infty} 1/n^p \text{ diverges if } p \le 0.$$

If $p > 0,$ then $f(x) = 1/x^p$ is a positive, continuous, decreasing function. Therefore, for $p > 0,$ we use the integral test, comparing

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ and } \int_1^{\infty} \frac{1}{x^p}dx.$$

We have already considered the case when $p = 1.$ Here we consider the case when $p > 0, p \ne 1.$ For this case,

$$\int_1^{\infty} \frac{1}{x^p}dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^p}dx = \lim_{b \to \infty} \frac{1}{1-p}x^{1-p}\Big|_1^b = \lim_{b \to \infty} \frac{1}{1-p}\left[b^{1-p} - 1\right].$$

Because

$$b^{1-p} \to 0 \text{ if } p > 1 \text{ and } b^{1-p} \to \infty \text{ if } p < 1,$$

we conclude that

$$\int_1^{\infty} \frac{1}{x^p}dx = \begin{cases} \frac{1}{p-1} \text{ if } p > 1 \\ \infty \text{ if } p < 1 \end{cases}.$$

Therefore, $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $0 < p < 1.$

In summary,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}. \tag{5.9}$$

Example 5.15

Testing for Convergence of *p*-series

For each of the following series, determine whether it converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

Solution

a. This is a *p*-series with $p = 4 > 1,$ so the series converges.

b. Since $p = 2/3 < 1,$ the series diverges.

5.14 Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converge or diverge?

Estimating the Value of a Series

Suppose we know that a series $\sum_{n=1}^{\infty} a_n$ converges and we want to estimate the sum of that series. Certainly we can approximate that sum using any finite sum $\sum_{n=1}^{N} a_n$ where N is any positive integer. The question we address here is, for a convergent series $\sum_{n=1}^{\infty} a_n,$ how good is the approximation $\sum_{n=1}^{N} a_n$? More specifically, if we let

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$$

be the remainder when the sum of an infinite series is approximated by the Nth partial sum, how large is R_N? For some types of series, we are able to use the ideas from the integral test to estimate R_N.

Theorem 5.10: Remainder Estimate from the Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the following three conditions:

i. f is continuous,

ii. f is decreasing, and

iii. $f(n) = a_n$ for all integers $n \geq 1.$

Let S_N be the *N*th partial sum of $\sum_{n=1}^{\infty} a_n.$ For all positive integers $N,$

$$S_N + \int_{N+1}^{\infty} f(x)dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)dx.$$

In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)dx < R_N < \int_{N}^{\infty} f(x)dx. \tag{5.10}$$

This is known as the **remainder estimate**.

We illustrate **Remainder Estimate from the Integral Test** in **Figure 5.15**. In particular, by representing the remainder $R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots$ as the sum of areas of rectangles, we see that the area of those rectangles is bounded above by $\int_{N}^{\infty} f(x)dx$ and bounded below by $\int_{N+1}^{\infty} f(x)dx$. In other words,

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)dx$$

and

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_{N}^{\infty} f(x)dx.$$

We conclude that

$$\int_{N+1}^{\infty} f(x)dx < R_N < \int_{N}^{\infty} f(x)dx.$$

Since

$$\sum_{n=1}^{\infty} a_n = S_N + R_N,$$

where S_N is the Nth partial sum, we conclude that

$$S_N + \int_{N+1}^{\infty} f(x)dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)dx.$$

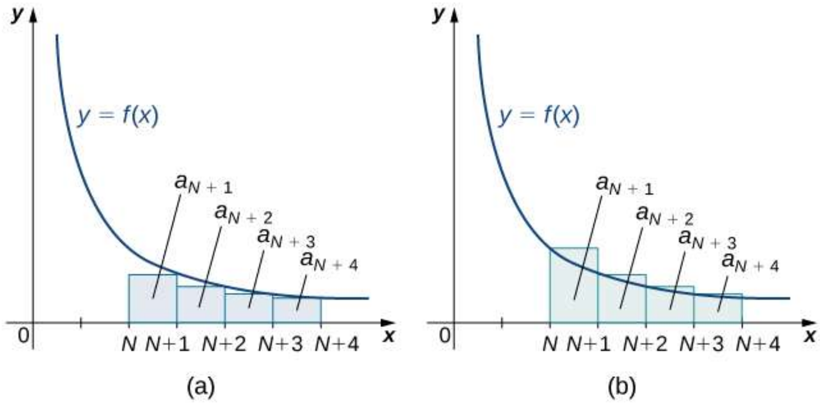


Figure 5.15 Given a continuous, positive, decreasing function f and a sequence of positive terms a_n such that $a_n = f(n)$ for all positive integers n, (a) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_N^{\infty} f(x)dx$, or (b) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)dx$. Therefore, the integral is either an overestimate or an underestimate of the error.

Example 5.16

Estimating the Value of a Series

Consider the series $\sum_{n=1}^{\infty} 1/n^3$.

a. Calculate $S_{10} = \sum_{n=1}^{10} 1/n^3$ and estimate the error.

b. Determine the least value of N necessary such that S_N will estimate $\sum_{n=1}^{\infty} 1/n^3$ to within 0.001.

Solution

a. Using a calculating utility, we have

$$S_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} \approx 1.19753.$$

By the remainder estimate, we know

$$R_N < \int_N^{\infty} \frac{1}{x^3}dx.$$

We have

$$\int_{10}^{\infty} \frac{1}{x^3}dx = \lim_{b \to \infty} \int_{10}^{b} \frac{1}{x^3}dx = \lim_{b \to \infty}\left[-\frac{1}{2x^2}\right]_N^b = \lim_{b \to \infty}\left[-\frac{1}{2b^2} + \frac{1}{2N^2}\right] = \frac{1}{2N^2}.$$

Therefore, the error is $R_{10} < 1/2(10)^2 = 0.005.$

b. Find N such that $R_N < 0.001.$ In part a. we showed that $R_N < 1/2N^2.$ Therefore, the remainder $R_N < 0.001$ as long as $1/2N^2 < 0.001.$ That is, we need $2N^2 > 1000.$ Solving this inequality for $N,$ we see that we need $N > 22.36.$ To ensure that the remainder is within the desired amount, we need to round up to the nearest integer. Therefore, the minimum necessary value is $N = 23.$

5.15 For $\sum_{n=1}^{\infty} \frac{1}{n^4},$ calculate S_5 and estimate the error $R_5.$

5.3 EXERCISES

For each of the following sequences, if the divergence test applies, either state that $\lim\limits_{n \to \infty} a_n$ does not exist or find $\lim\limits_{n \to \infty} a_n$. If the divergence test does not apply, state why.

138. $a_n = \frac{n}{n+2}$

139. $a_n = \frac{n}{5n^2 - 3}$

140. $a_n = \frac{n}{\sqrt{3n^2 + 2n + 1}}$

141. $a_n = \frac{(2n+1)(n-1)}{(n+1)^2}$

142. $a_n = \frac{(2n+1)^{2n}}{\left(3n^2+1\right)^n}$

143. $a_n = \frac{2^n}{3^{n/2}}$

144. $a_n = \frac{2^n + 3^n}{10^{n/2}}$

145. $a_n = e^{-2/n}$

146. $a_n = \cos n$

147. $a_n = \tan n$

148. $a_n = \frac{1 - \cos^2(1/n)}{\sin^2(2/n)}$

149. $a_n = \left(1 - \frac{1}{n}\right)^{2n}$

150. $a_n = \frac{\ln n}{n}$

151. $a_n = \frac{(\ln n)^2}{\sqrt{n}}$

State whether the given p-series converges.

152. $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

153. $\sum\limits_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

154. $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

155. $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

156. $\sum\limits_{n=1}^{\infty} \frac{n^e}{n^\pi}$

157. $\sum\limits_{n=1}^{\infty} \frac{n^\pi}{n^{2e}}$

Use the integral test to determine whether the following sums converge.

158. $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$

159. $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+5}}$

160. $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$

161. $\sum\limits_{n=1}^{\infty} \frac{n}{1+n^2}$

162. $\sum\limits_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

163. $\sum\limits_{n=1}^{\infty} \frac{2n}{1+n^4}$

164. $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

Express the following sums as p-series and determine whether each converges.

165. $\sum\limits_{n=1}^{\infty} 2^{-\ln n}$ (*Hint:* $2^{-\ln n} = 1/n^{\ln 2}$.)

166. $\sum\limits_{n=1}^{\infty} 3^{-\ln n}$ (*Hint:* $3^{-\ln n} = 1/n^{\ln 3}$.)

167. $\sum\limits_{n=1}^{\infty} n2^{-2\ln n}$

168. $\sum_{n=1}^{\infty} n3^{-2\ln n}$

Use the estimate $R_N \le \int_N^{\infty} f(t)dt$ to find a bound for the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ where $a_n = f(n)$.

169. $\sum_{n=1}^{1000} \frac{1}{n^2}$

170. $\sum_{n=1}^{1000} \frac{1}{n^3}$

171. $\sum_{n=1}^{1000} \frac{1}{1+n^2}$

172. $\sum_{n=1}^{100} n/2^n$

[T] Find the minimum value of N such that the remainder estimate $\int_{N+1}^{\infty} f < R_N < \int_N^{\infty} f$ guarantees that $\sum_{n=1}^{N} a_n$ estimates $\sum_{n=1}^{\infty} a_n$, accurate to within the given error.

173. $a_n = \frac{1}{n^2}$, error $< 10^{-4}$

174. $a_n = \frac{1}{n^{1.1}}$, error $< 10^{-4}$

175. $a_n = \frac{1}{n^{1.01}}$, error $< 10^{-4}$

176. $a_n = \frac{1}{n\ln^2 n}$, error $< 10^{-3}$

177. $a_n = \frac{1}{1+n^2}$, error $< 10^{-3}$

In the following exercises, find a value of N such that R_N is smaller than the desired error. Compute the corresponding sum $\sum_{n=1}^{N} a_n$ and compare it to the given estimate of the infinite series.

178. $a_n = \frac{1}{n^{11}}$, error $< 10^{-4}$, $\sum_{n=1}^{\infty} \frac{1}{n^{11}} = 1.000494\ldots$

179. $a_n = \frac{1}{e^n}$, error $< 10^{-5}$, $\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e-1} = 0.581976\ldots$

180. $a_n = \frac{1}{e^{n^2}}$, error $< 10^{-5}$, $\sum_{n=1}^{\infty} n/e^{n2} = 0.40488139857\ldots$

181. $a_n = 1/n^4$, error $< 10^{-4}$, $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90 = 1.08232...$

182. $a_n = 1/n^6$, error $< 10^{-6}$, $\sum_{n=1}^{\infty} 1/n^4 = \pi^6/945 = 1.01734306...,$

183. Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$. (*Hint:* Compare to $\int_n^{2n} \frac{1}{t}dt$.)

184. Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n}$

The next few exercises are intended to give a sense of applications in which partial sums of the harmonic series arise.

185. In certain applications of probability, such as the so-called Watterson estimator for predicting mutation rates in population genetics, it is important to have an accurate estimate of the number $H_k = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right)$. Recall that $T_k = H_k - \ln k$ is decreasing. Compute $T = \lim_{k \to \infty} T_k$ to four decimal places. (*Hint:* $\frac{1}{k+1} < \int_k^{k+1} \frac{1}{x}dx$.)

186. **[T]** Complete sampling with replacement, sometimes called the *coupon collector's problem*, is phrased as follows: Suppose you have N unique items in a bin. At each step, an item is chosen at random, identified, and put back in the bin. The problem asks what is the expected number of steps $E(N)$ that it takes to draw each unique item at least once. It turns out that $E(N) = N.H_N = N\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right).$ Find $E(N)$ for $N = 10,\ 20,$ and $50.$

187. **[T]** The simplest way to shuffle cards is to take the top card and insert it at a random place in the deck, called top random insertion, and then repeat. We will consider a deck to be randomly shuffled once enough top random insertions have been made that the card originally at the bottom has reached the top and then been randomly inserted. If the deck has n cards, then the probability that the insertion will be below the card initially at the bottom (call this card B) is $1/n.$ Thus the expected number of top random insertions before B is no longer at the bottom is n. Once one card is below $B,$ there are two places below B and the probability that a randomly inserted card will fall below B is $2/n.$ The expected number of top random insertions before this happens is $n/2.$ The two cards below B are now in random order. Continuing this way, find a formula for the expected number of top random insertions needed to consider the deck to be randomly shuffled.

188. Suppose a scooter can travel 100 km on a full tank of fuel. Assuming that fuel can be transferred from one scooter to another but can only be carried in the tank, present a procedure that will enable one of the scooters to travel $100H_N$ km, where $H_N = 1 + 1/2 + \cdots + 1/N.$

189. Show that for the remainder estimate to apply on $[N, \infty)$ it is sufficient that $f(x)$ be decreasing on $[N, \infty),$ but f need not be decreasing on $[1, \infty).$

190. **[T]** Use the remainder estimate and integration by parts to approximate $\sum_{n=1}^{\infty} n/e^n$ within an error smaller than $0.0001.$

191. Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge if p is large enough? If so, for which p?

192. **[T]** Suppose a computer can sum one million terms per second of the divergent series $\sum_{n=1}^{N} \frac{1}{n}.$ Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed $100.$

193. **[T]** A fast computer can sum one million terms per second of the divergent series $\sum_{n=2}^{N} \frac{1}{n \ln n}.$ Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed $100.$

5.4 | Comparison Tests

Learning Objectives

5.4.1 Use the comparison test to test a series for convergence.

5.4.2 Use the limit comparison test to determine convergence of a series.

We have seen that the integral test allows us to determine the convergence or divergence of a series by comparing it to a related improper integral. In this section, we show how to use comparison tests to determine the convergence or divergence of a series by comparing it to a series whose convergence or divergence is known. Typically these tests are used to determine convergence of series that are similar to geometric series or *p*-series.

Comparison Test

In the preceding two sections, we discussed two large classes of series: geometric series and *p*-series. We know exactly when these series converge and when they diverge. Here we show how to use the convergence or divergence of these series to prove convergence or divergence for other series, using a method called the **comparison test**.

For example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

This series looks similar to the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the terms in each of the series are positive, the sequence of partial sums for each series is monotone increasing. Furthermore, since

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$$

for all positive integers n, the kth partial sum S_k of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+1} < \sum_{n=1}^{k} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(See **Figure 5.16**(a) and **Table 5.1**.) Since the series on the right converges, the sequence $\{S_k\}$ is bounded above. We conclude that $\{S_k\}$ is a monotone increasing sequence that is bounded above. Therefore, by the Monotone Convergence Theorem, $\{S_k\}$ converges, and thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges.

Similarly, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}.$$

This series looks similar to the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The sequence of partial sums for each series is monotone increasing and

$$\frac{1}{n-1/2} > \frac{1}{n} > 0$$

for every positive integer n. Therefore, the kth partial sum S_k of $\sum_{n=1}^{\infty} \frac{1}{n-1/2}$ satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n-1/2} > \sum_{n=1}^{k} \frac{1}{n}.$$

(See **Figure 5.16**(b) and **Table 5.2**.) Since the series $\sum_{n=1}^{\infty} 1/n$ diverges to infinity, the sequence of partial sums $\sum_{n=1}^{k} 1/n$ is unbounded. Consequently, $\{S_k\}$ is an unbounded sequence, and therefore diverges. We conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}$$

diverges.

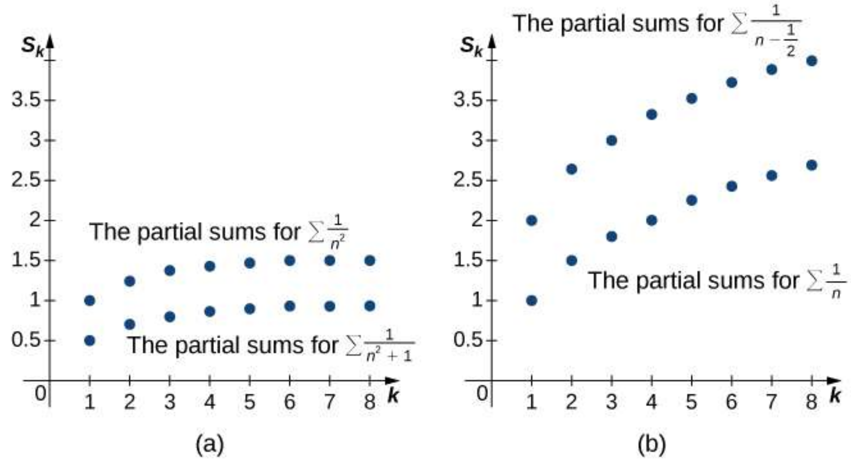


Figure 5.16 (a) Each of the partial sums for the given series is less than the corresponding partial sum for the converging $p-$ series. (b) Each of the partial sums for the given series is greater than the corresponding partial sum for the diverging harmonic series.

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n^2+1}$	0.5	0.7	0.8	0.8588	0.8973	0.9243	0.9443	0.9597
$\sum_{n=1}^{k} \frac{1}{n^2}$	1	1.25	1.3611	1.4236	1.4636	1.4914	1.5118	1.5274

Table 5.1 Comparing a series with a *p*-series ($p = 2$)

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n - 1/2}$	2	2.6667	3.0667	3.3524	3.5746	3.7564	3.9103	4.0436
$\sum_{n=1}^{k} \frac{1}{n}$	1	1.5	1.8333	2.0933	2.2833	2.45	2.5929	2.7179

Table 5.2 Comparing a series with the harmonic series

Theorem 5.11: Comparison Test

i. Suppose there exists an integer N such that $0 \le a_n \le b_n$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. Suppose there exists an integer N such that $a_n \ge b_n \ge 0$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

We prove part i. The proof of part ii. is the contrapositive of part i. Let $\{S_k\}$ be the sequence of partial sums associated with $\sum_{n=1}^{\infty} a_n$, and let $L = \sum_{n=1}^{\infty} b_n$. Since the terms $a_n \ge 0$,

$$S_k = a_1 + a_2 + \cdots + a_k \le a_1 + a_2 + \cdots + a_k + a_{k+1} = S_{k+1}.$$

Therefore, the sequence of partial sums is increasing. Further, since $a_n \le b_n$ for all $n \ge N$, then

$$\sum_{n=N}^{k} a_n \le \sum_{n=N}^{k} b_n \le \sum_{n=1}^{\infty} b_n = L.$$

Therefore, for all $k \ge 1$,

$$S_k = (a_1 + a_2 + \cdots + a_{N-1}) + \sum_{n=N}^{k} a_n \le (a_1 + a_2 + \cdots + a_{N-1}) + L.$$

Since $a_1 + a_2 + \cdots + a_{N-1}$ is a finite number, we conclude that the sequence $\{S_k\}$ is bounded above. Therefore, $\{S_k\}$ is an increasing sequence that is bounded above. By the Monotone Convergence Theorem, we conclude that $\{S_k\}$ converges, and therefore the series $\sum_{n=1}^{\infty} a_n$ converges.

□

To use the comparison test to determine the convergence or divergence of a series $\sum_{n=1}^{\infty} a_n$, it is necessary to find a suitable series with which to compare it. Since we know the convergence properties of geometric series and *p*-series, these series are often used. If there exists an integer N such that for all $n \geq N,$ each term a_n is less than each corresponding term of a known convergent series, then $\sum_{n=1}^{\infty} a_n$ converges. Similarly, if there exists an integer N such that for all $n \geq N,$ each term a_n is greater than each corresponding term of a known divergent series, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 5.17

Using the Comparison Test

For each of the following series, use the comparison test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 3n + 1}$

b. $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

c. $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

Solution

a. Compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$ Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a *p*-series with $p = 3,$ it converges. Further,

$$\frac{1}{n^3 + 3n + 1} < \frac{1}{n^3}$$

for every positive integer $n.$ Therefore, we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^3 + 3n + 1}$ converges.

b. Compare to $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n.$ Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with $r = 1/2$ and $|1/2| < 1,$ it converges. Also,

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

for every positive integer $n.$ Therefore, we see that $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$ converges.

c. Compare to $\sum_{n=2}^{\infty} \frac{1}{n}.$ Since

$$\frac{1}{\ln(n)} > \frac{1}{n}$$

for every integer $n \geq 2$ and $\sum_{n=2}^{\infty} 1/n$ diverges, we have that $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges.

5.16 Use the comparison test to determine if the series $\sum_{n=1}^{\infty} \frac{n}{n^3 + n + 1}$ converges or diverges.

Limit Comparison Test

The comparison test works nicely if we can find a comparable series satisfying the hypothesis of the test. However, sometimes finding an appropriate series can be difficult. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}.$$

It is natural to compare this series with the convergent series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

However, this series does not satisfy the hypothesis necessary to use the comparison test because

$$\frac{1}{n^2 - 1} > \frac{1}{n^2}$$

for all integers $n \geq 2$. Although we could look for a different series with which to compare $\sum_{n=2}^{\infty} 1/(n^2 - 1)$, instead we show how we can use the **limit comparison test** to compare

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \text{ and } \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Let us examine the idea behind the limit comparison test. Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. with positive terms a_n and b_n and evaluate

$$\lim_{n \to \infty} \frac{a_n}{b_n}.$$

If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L \neq 0,$$

then, for n sufficiently large, $a_n \approx Lb_n$. Therefore, either both series converge or both series diverge. For the series $\sum_{n=2}^{\infty} 1/(n^2 - 1)$ and $\sum_{n=2}^{\infty} 1/n^2$, we see that

$$\lim_{n \to \infty} \frac{1/(n^2 - 1)}{1/n^2} = \lim_{n \to \infty} \frac{n^2}{n^2 - 1} = 1.$$

Since $\sum_{n=2}^{\infty} 1/n^2$ converges, we conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

converges.

The limit comparison test can be used in two other cases. Suppose

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0.$$

In this case, $\{a_n/b_n\}$ is a bounded sequence. As a result, there exists a constant M such that $a_n \le Mb_n$. Therefore, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. On the other hand, suppose

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \infty.$$

In this case, $\{a_n/b_n\}$ is an unbounded sequence. Therefore, for every constant M there exists an integer N such that $a_n \ge Mb_n$ for all $n \ge N$. Therefore, if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges as well.

Theorem 5.12: Limit Comparison Test

Let $a_n,\ b_n \ge 0$ for all $n \ge 1$.

i. If $\lim_{n \to \infty} a_n/b_n = L \ne 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

ii. If $\lim_{n \to \infty} a_n/b_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

iii. If $\lim_{n \to \infty} a_n/b_n = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note that if $a_n/b_n \to 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, the limit comparison test gives no information. Similarly, if $a_n/b_n \to \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, the test also provides no information. For example, consider the two series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ and $\sum_{n=1}^{\infty} 1/n^2$. These series are both p-series with $p = 1/2$ and $p = 2$, respectively. Since $p = 1/2 > 1$, the series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges. On the other hand, since $p = 2 < 1$, the series $\sum_{n=1}^{\infty} 1/n^2$ converges. However, suppose we attempted to apply the limit comparison test, using the convergent $p -$ series $\sum_{n=1}^{\infty} 1/n^3$ as our comparison series. First, we see that

$$\frac{1/\sqrt{n}}{1/n^3} = \frac{n^3}{\sqrt{n}} = n^{5/2} \to \infty \text{ as } n \to \infty.$$

Similarly, we see that

$$\frac{1/n^2}{1/n^3} = n \to \infty \text{ as } n \to \infty.$$

Therefore, if $a_n/b_n \to \infty$ when $\sum_{n=1}^{\infty} b_n$ converges, we do not gain any information on the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Example 5.18

Using the Limit Comparison Test

For each of the following series, use the limit comparison test to determine whether the series converges or diverges. If the test does not apply, say so.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

b. $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$

c. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$

Solution

a. Compare this series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Calculate

$$\lim_{n\to\infty} \frac{1/(\sqrt{n}+1)}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/\sqrt{n}} = 1.$$

By the limit comparison test, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges.

b. Compare this series to $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$. We see that

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = \lim_{n\to\infty} \frac{2^n+1}{3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = \lim_{n\to\infty} \left[1+\left(\frac{1}{2}\right)^n\right] = 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{(2^n+1)/3^n}{2^n/3^n} = 1.$$

Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges, we conclude that $\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$ converges.

c. Since $\ln n < n$, compare with $\sum_{n=1}^{\infty} \frac{1}{n}$. We see that

$$\lim_{n\to\infty} \frac{\ln n/n^2}{1/n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\ln n}{n}.$$

In order to evaluate $\lim_{n\to\infty} \ln n/n$, evaluate the limit as $x \to \infty$ of the real-valued function $\ln(x)/x$. These two limits are equal, and making this change allows us to use L'Hôpital's rule. We obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Therefore, $\lim_{n \to \infty} \ln n/n = 0,$ and, consequently,

$$\lim_{n \to \infty} \frac{\ln n/n^2}{1/n} = 0.$$

Since the limit is 0 but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the limit comparison test does not provide any information.

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ instead. In this case,

$$\lim_{n \to \infty} \frac{\ln n/n^2}{1/n^2} = \lim_{n \to \infty} \frac{\ln n}{n^2} \cdot \frac{n^2}{1} = \lim_{n \to \infty} \ln n = \infty.$$

Since the limit is ∞ but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the test still does not provide any information.

So now we try a series between the two we already tried. Choosing the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}},$ we see that

$$\lim_{n \to \infty} \frac{\ln n/n^2}{1/n^{3/2}} = \lim_{n \to \infty} \frac{\ln n}{n^2} \cdot \frac{n^{3/2}}{1} = \lim_{n \to \infty} \frac{\ln n}{\sqrt{n}}.$$

As above, in order to evaluate $\lim_{n \to \infty} \ln n/\sqrt{n},$ evaluate the limit as $x \to \infty$ of the real-valued function $\ln x/\sqrt{x}.$ Using L'Hôpital's rule,

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0.$$

Since the limit is 0 and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we can conclude that $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

5.17 Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{5^n}{3^n + 2}$ converges or diverges.

5.4 EXERCISES

Use the comparison test to determine whether the following series converge.

194. $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{2}{n(n+1)}$

195. $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n(n+1/2)}$

196. $\sum_{n=1}^{\infty} \frac{1}{2(n+1)}$

197. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

198. $\sum_{n=2}^{\infty} \frac{1}{(n\ln n)^2}$

199. $\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$

200. $\sum_{n=1}^{\infty} \frac{1}{n!}$

201. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n}$

202. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

203. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

204. $\sum_{n=1}^{\infty} \frac{n^{1.2}-1}{n^{2.3}+1}$

205. $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

206. $\sum_{n=1}^{\infty} \frac{\sqrt[4]{n}}{\sqrt[3]{n^4+n^2}}$

Use the limit comparison test to determine whether each of the following series converges or diverges.

207. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^2$

208. $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n^{0.6}}\right)^2$

209. $\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$

210. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

211. $\sum_{n=1}^{\infty} \frac{1}{4^n-3^n}$

212. $\sum_{n=1}^{\infty} \frac{1}{n^2-n\sin n}$

213. $\sum_{n=1}^{\infty} \frac{1}{e^{(1.1)n}-3^n}$

214. $\sum_{n=1}^{\infty} \frac{1}{e^{(1.01)n}-3^n}$

215. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

216. $\sum_{n=1}^{\infty} \frac{1}{2^{1+1/n}n^{1+1/n}}$

217. $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\sin\left(\frac{1}{n}\right)\right)$

218. $\sum_{n=1}^{\infty} \left(1-\cos\left(\frac{1}{n}\right)\right)$

219. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\tan^{-1} n-\frac{\pi}{2}\right)$

220. $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n.n}$ (*Hint:* $\left(1-\frac{1}{n}\right)^n \to 1/e$.)

221. $\sum_{n=1}^{\infty} \left(1-e^{-1/n}\right)$ (*Hint:* $1/e \approx (1-1/n)^n$, so $1-e^{-1/n} \approx 1/n$.)

222. Does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ converge if p is large enough? If so, for which p?

223. Does $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^p$ converge if p is large enough? If so, for which p?

224. For which p does the series $\sum_{n=1}^{\infty} 2^{pn}/3^n$ converge?

225. For which $p>0$ does the series $\sum_{n=1}^{\infty}\frac{n^p}{2^n}$ converge?

226. For which $r>0$ does the series $\sum_{n=1}^{\infty}\frac{r^{n^2}}{2^n}$ converge?

227. For which $r>0$ does the series $\sum_{n=1}^{\infty}\frac{2^n}{r^{n^2}}$ converge?

228. Find all values of p and q such that $\sum_{n=1}^{\infty}\frac{n^p}{(n!)^q}$ converges.

229. Does $\sum_{n=1}^{\infty}\frac{\sin^2(nr/2)}{n}$ converge or diverge? Explain.

230. Explain why, for each n, at least one of $\{|\sin n|, |\sin(n+1)|,..., |\sin n+6|\}$ is larger than $1/2$. Use this relation to test convergence of $\sum_{n=1}^{\infty}\frac{|\sin n|}{\sqrt{n}}$.

231. Suppose that $a_n \geq 0$ and $b_n \geq 0$ and that $\sum_{n=1}^{\infty} a^2{}_n$ and $\sum_{n=1}^{\infty} b^2{}_n$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges and $\sum_{n=1}^{\infty} a_n b_n \leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right)$.

232. Does $\sum_{n=1}^{\infty} 2^{-\ln\ln n}$ converge? (*Hint:* Write $2^{\ln\ln n}$ as a power of $\ln n$.)

233. Does $\sum_{n=1}^{\infty}(\ln n)^{-\ln n}$ converge? (*Hint:* Use $t=e^{\ln(t)}$ to compare to a $p-$series.)

234. Does $\sum_{n=2}^{\infty}(\ln n)^{-\ln\ln n}$ converge? (*Hint:* Compare a_n to $1/n$.)

235. Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a^2{}_n$ converges. If $\sum_{n=1}^{\infty} a^2{}_n$ converges, does $\sum_{n=1}^{\infty} a_n$ necessarily converge?

236. Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ converges. Suppose that b_n is an arbitrary sequence of zeros and ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

237. Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ diverges. Suppose that b_n is an arbitrary sequence of zeros and ones with infinitely many terms equal to one. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily diverge?

238. Complete the details of the following argument: If $\sum_{n=1}^{\infty}\frac{1}{n}$ converges to a finite sum s, then $\frac{1}{2}s=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots$ and $s-\frac{1}{2}s=1+\frac{1}{3}+\frac{1}{5}+\cdots$. Why does this lead to a contradiction?

239. Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a^2{}_n$ converges, then $\sum_{n=1}^{\infty}\sin^2(a_n)$ converges.

240. Suppose that $a_n/b_n \to 0$ in the comparison test, where $a_n \geq 0$ and $b_n \geq 0$. Prove that if $\sum b_n$ converges, then $\sum a_n$ converges.

241. Let b_n be an infinite sequence of zeros and ones. What is the largest possible value of $x=\sum_{n=1}^{\infty} b_n/2^n$?

242. Let d_n be an infinite sequence of digits, meaning d_n takes values in $\{0, 1, \ldots, 9\}$. What is the largest possible value of $x = \sum_{n=1}^{\infty} d_n/10^n$ that converges?

243. Explain why, if $x > 1/2$, then x cannot be written $x = \sum_{n=2}^{\infty} \frac{b_n}{2^n} (b_n = 0 \text{ or } 1, \ b_1 = 0)$.

244. **[T]** Evelyn has a perfect balancing scale, an unlimited number of 1-kg weights, and one each of $1/2$-kg, $1/4$-kg, $1/8$-kg, and so on weights. She wishes to weigh a meteorite of unspecified origin to arbitrary precision. Assuming the scale is big enough, can she do it? What does this have to do with infinite series?

245. **[T]** Robert wants to know his body mass to arbitrary precision. He has a big balancing scale that works perfectly, an unlimited collection of 1-kg weights, and nine each of 0.1-kg, 0.01-kg, 0.001-kg, and so on weights. Assuming the scale is big enough, can he do this? What does this have to do with infinite series?

246. The series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is half the harmonic series and hence diverges. It is obtained from the harmonic series by deleting all terms in which n is odd. Let $m > 1$ be fixed. Show, more generally, that deleting all terms $1/n$ where $n = mk$ for some integer k also results in a divergent series.

247. In view of the previous exercise, it may be surprising that a subseries of the harmonic series in which about one in every five terms is deleted might converge. A *depleted harmonic series* is a series obtained from $\sum_{n=1}^{\infty} \frac{1}{n}$ by removing any term $1/n$ if a given digit, say 9, appears in the decimal expansion of n. Argue that this depleted harmonic series converges by answering the following questions.

a. How many whole numbers n have d digits?
b. How many d-digit whole numbers $h(d)$. do not contain 9 as one or more of their digits?
c. What is the smallest d-digit number $m(d)$?
d. Explain why the deleted harmonic series is bounded by $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$.
e. Show that $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$ converges.

248. Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{n+1} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (*Hint:* S_n is monotone.)

249. Suppose that a sequence of numbers $a_n > 0$ has the property that $a_1 = 1$ and $a_{n+1} = \frac{1}{(n+1)^2} S_n$, where $S_n = a_1 + \cdots + a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (*Hint:* $S_2 = a_2 + a_1 = a_2 + S_1 = a_2 + 1 = 1 + 1/4 = (1 + 1/4)S_1$, $S_3 = \frac{1}{3^2} S_2 + S_2 = (1 + 1/9)S_2 = (1 + 1/9)(1 + 1/4)S_1$, etc. Look at $\ln(S_n)$, and use $\ln(1 + t) \le t$, $t > 0$.)

5.5 | Alternating Series

Learning Objectives

5.5.1 Use the alternating series test to test an alternating series for convergence.
5.5.2 Estimate the sum of an alternating series.
5.5.3 Explain the meaning of absolute convergence and conditional convergence.

So far in this chapter, we have primarily discussed series with positive terms. In this section we introduce alternating series—those series whose terms alternate in sign. We will show in a later chapter that these series often arise when studying power series. After defining alternating series, we introduce the alternating series test to determine whether such a series converges.

The Alternating Series Test

A series whose terms alternate between positive and negative values is an **alternating series**. For example, the series

$$\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots \tag{5.11}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{5.12}$$

are both alternating series.

Definition

Any series whose terms alternate between positive and negative values is called an alternating series. An alternating series can be written in the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots \tag{5.13}$$

or

$$\sum_{n-1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \cdots \tag{5.14}$$

Where $b_n \geq 0$ for all positive integers n.

Series (1), shown in **Equation 5.11**, is a geometric series. Since $|r| = |-1/2| < 1,$ the series converges. Series (2), shown in **Equation 5.12**, is called the alternating harmonic series. We will show that whereas the harmonic series diverges, the alternating harmonic series converges.

To prove this, we look at the sequence of partial sums $\{S_k\}$ (**Figure 5.17**).

Proof

Consider the odd terms S_{2k+1} for $k \geq 0.$ Since $1/(2k+1) < 1/2k,$

$$S_{2k+1} = S_{2k-1} - \frac{1}{2k} + \frac{1}{2k+1} < S_{2k-1}.$$

Therefore, $\{S_{2k+1}\}$ is a decreasing sequence. Also,

$$S_{2k+1} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) + \frac{1}{2k+1} > 0.$$

Therefore, $\{S_{2k+1}\}$ is bounded below. Since $\{S_{2k+1}\}$ is a decreasing sequence that is bounded below, by the Monotone Convergence Theorem, $\{S_{2k+1}\}$ converges. Similarly, the even terms $\{S_{2k}\}$ form an increasing sequence that is bounded above because

$$S_{2k} = S_{2k-2} + \frac{1}{2k-1} - \frac{1}{2k} > S_{2k-2}$$

and

$$S_{2k} = 1 + \left(-\frac{1}{2} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{2k-2} + \frac{1}{2k-1}\right) - \frac{1}{2k} < 1.$$

Therefore, by the Monotone Convergence Theorem, the sequence $\{S_{2k}\}$ also converges. Since

$$S_{2k+1} = S_{2k} + \frac{1}{2k+1},$$

we know that

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} \frac{1}{2k+1}.$$

Letting $S = \lim_{k\to\infty} S_{2k+1}$ and using the fact that $1/(2k+1) \to 0,$ we conclude that $\lim_{k\to\infty} S_{2k} = S.$ Since the odd terms and the even terms in the sequence of partial sums converge to the same limit $S,$ it can be shown that the sequence of partial sums converges to $S,$ and therefore the alternating harmonic series converges to $S.$

It can also be shown that $S = \ln 2,$ and we can write

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2).$$

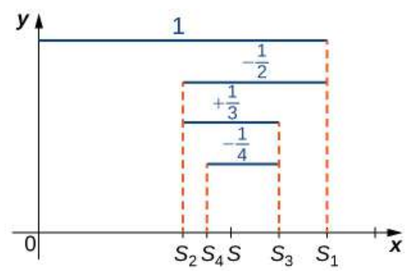


Figure 5.17 For the alternating harmonic series, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

□

More generally, any alternating series of form (3) (**Equation 5.13**) or (4) (**Equation 5.14**) converges as long as $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $b_n \to 0$ (**Figure 5.18**). The proof is similar to the proof for the alternating harmonic series.

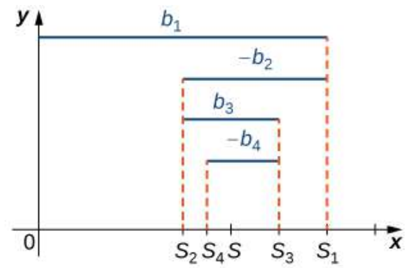


Figure 5.18 For an alternating series $b_1 - b_2 + b_3 - \cdots$ in which $b_1 > b_2 > b_3 > \cdots,$ the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

Theorem 5.13: Alternating Series Test

An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ or } \sum_{n=1}^{\infty} (-1)^n b_n$$

converges if

i. $0 \le b_{n+1} \le b_n$ for all $n \ge 1$ and

ii. $\lim_{n \to \infty} b_n = 0.$

This is known as the **alternating series test**.

We remark that this theorem is true more generally as long as there exists some integer N such that $0 \le b_{n+1} \le b_n$ for all $n \ge N.$

Example 5.19

Convergence of Alternating Series

For each of the following alternating series, determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}/n^2$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n/(n+1)$

Solution

a. Since
$$\frac{1}{(n+1)^2} < \frac{1}{n^2} \quad \text{and} \quad \frac{1}{n^2} \to 0,$$
the series converges.

b. Since $n/(n+1) \nrightarrow 0$ as $n \to \infty,$ we cannot apply the alternating series test. Instead, we use the *n*th term test for divergence. Since

$$\lim_{n \to \infty} \frac{(-1)^{n+1} n}{n+1} \neq 0,$$

the series diverges.

5.18 Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} n/2^n$ converges or diverges.

Remainder of an Alternating Series

It is difficult to explicitly calculate the sum of most alternating series, so typically the sum is approximated by using a partial sum. When doing so, we are interested in the amount of error in our approximation. Consider an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

satisfying the hypotheses of the alternating series test. Let S denote the sum of this series and $\{S_k\}$ be the corresponding sequence of partial sums. From **Figure 5.18**, we see that for any integer $N \geq 1,$ the remainder R_N satisfies

$$|R_N| = |S - S_N| \leq |S_{N+1} - S_N| = b_{n+1}.$$

Theorem 5.14: Remainders in Alternating Series

Consider an alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n \text{ or } \sum_{n=1}^{\infty} (-1)^n b_n$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the Nth partial sum. For any integer $N \geq 1,$ the remainder $R_N = S - S_N$ satisfies

$$|R_N| \leq b_{N+1}.$$

In other words, if the conditions of the alternating series test apply, then the error in approximating the infinite series by the Nth partial sum S_N is in magnitude at most the size of the next term b_{N+1}.

Example 5.20

Estimating the Remainder of an Alternating Series

Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Use the remainder estimate to determine a bound on the error R_{10} if we approximate the sum of the series by the partial sum S_{10}.

Solution

From the theorem stated above,

$$|R_{10}| \leq b_{11} = \frac{1}{11^2} \approx 0.008265.$$

5.19 Find a bound for R_{20} when approximating $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ by S_{20}.

Absolute and Conditional Convergence

Consider a series $\sum_{n=1}^{\infty} a_n$ and the related series $\sum_{n=1}^{\infty} |a_n|$. Here we discuss possibilities for the relationship between the convergence of these two series. For example, consider the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$. The series whose terms are the absolute value of these terms is the harmonic series, since $\sum_{n=1}^{\infty} |(-1)^{n+1}/n| = \sum_{n=1}^{\infty} 1/n$. Since the alternating harmonic series converges, but the harmonic series diverges, we say the alternating harmonic series exhibits conditional convergence.

By comparison, consider the series $\sum_{n=1}^{\infty} (-1)^{n+1}/n^2$. The series whose terms are the absolute values of the terms of this series is the series $\sum_{n=1}^{\infty} 1/n^2$. Since both of these series converge, we say the series $\sum_{n=1}^{\infty} (-1)^{n+1}/n^2$ exhibits absolute convergence.

Definition

A series $\sum_{n=1}^{\infty} a_n$ exhibits **absolute convergence** if $\sum_{n=1}^{\infty} |a_n|$ converges. A series $\sum_{n=1}^{\infty} a_n$ exhibits **conditional convergence** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

As shown by the alternating harmonic series, a series $\sum_{n=1}^{\infty} a_n$ may converge, but $\sum_{n=1}^{\infty} |a_n|$ may diverge. In the following theorem, however, we show that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 5.15: Absolute Convergence Implies Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof

Suppose that $\sum_{n=1}^{\infty} |a_n|$ converges. We show this by using the fact that $a_n = |a_n|$ or $a_n = -|a_n|$ and therefore $|a_n| + a_n = 2|a_n|$ or $|a_n| + a_n = 0$. Therefore, $0 \le |a_n| + a_n \le 2|a_n|$. Consequently, by the comparison test, since $2\sum_{n=1}^{\infty} |a_n|$ converges, the series

$$\sum_{n=1}^{\infty} (|a_n| + a_n)$$

converges. By using the algebraic properties for convergent series, we conclude that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (|a_n| + a_n) - \sum_{n=1}^{\infty} |a_n|$$

converges.

□

Example 5.21

Absolute versus Conditional Convergence

For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} (-1)^{n+1}/(3n+1)$

b. $\sum_{n=1}^{\infty} \cos(n)/n^2$

Solution

a. We can see that

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{3n+1}\right| = \sum_{n=1}^{\infty} \frac{1}{3n+1}$$

diverges by using the limit comparison test with the harmonic series. In fact,

$$\lim_{n \to \infty} \frac{1/(3n+1)}{1/n} = \frac{1}{3}.$$

Therefore, the series does not converge absolutely. However, since

$$\frac{1}{3(n+1)+1} < \frac{1}{3n+1} \text{ and } \frac{1}{3n+1} \to 0,$$

the series converges. We can conclude that $\sum_{n=1}^{\infty} (-1)^{n+1}/(3n+1)$ converges conditionally.

b. Noting that $|\cos n| \le 1$, to determine whether the series converges absolutely, compare

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$$

with the series $\sum_{n=1}^{\infty} 1/n^2$. Since $\sum_{n=1}^{\infty} 1/n^2$ converges, by the comparison test, $\sum_{n=1}^{\infty} |\cos n/n^2|$ converges, and therefore $\sum_{n=1}^{\infty} \cos n/n^2$ converges absolutely.

5.20 Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} n/(2n^3 + 1)$ converges absolutely, converges conditionally, or diverges.

To see the difference between absolute and conditional convergence, look at what happens when we *rearrange* the terms of the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$. We show that we can rearrange the terms so that the new series diverges. Certainly if we rearrange the terms of a finite sum, the sum does not change. When we work with an infinite sum, however, interesting things can happen.

Begin by adding enough of the positive terms to produce a sum that is larger than some real number $M > 0$. For example, let $M = 10$, and find an integer k such that

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} > 10.$$

(We can do this because the series $\sum_{n=1}^{\infty} 1/(2n-1)$ diverges to infinity.) Then subtract $1/2$. Then add more positive terms until the sum reaches 100. That is, find another integer $j > k$ such that

$$1 + \frac{1}{3} + \cdots + \frac{1}{2k-1} - \frac{1}{2} + \frac{1}{2k+1} + \cdots + \frac{1}{2j+1} > 100.$$

Then subtract $1/4$. Continuing in this way, we have found a way of rearranging the terms in the alternating harmonic series so that the sequence of partial sums for the rearranged series is unbounded and therefore diverges.

The terms in the alternating harmonic series can also be rearranged so that the new series converges to a different value. In **Example 5.22**, we show how to rearrange the terms to create a new series that converges to $3\ln(2)/2$. We point out that the alternating harmonic series can be rearranged to create a series that converges to any real number r; however, the proof of that fact is beyond the scope of this text.

In general, any series $\sum_{n=1}^{\infty} a_n$ that converges conditionally can be rearranged so that the new series diverges or converges to a different real number. A series that converges absolutely does not have this property. For any series $\sum_{n=1}^{\infty} a_n$ that converges absolutely, the value of $\sum_{n=1}^{\infty} a_n$ is the same for any rearrangement of the terms. This result is known as the Riemann Rearrangement Theorem, which is beyond the scope of this book.

Example 5.22

Rearranging Series

Use the fact that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\ln 2$$

to rearrange the terms in the alternating harmonic series so the sum of the rearranged series is $3\ln(2)/2$.

Solution

Let

$$\sum_{n=1}^{\infty} a_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots.$$

Since $\sum_{n=1}^{\infty} a_n = \ln(2)$, by the algebraic properties of convergent series,

$$\sum_{n=1}^{\infty} \frac{1}{2}a_n = \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots=\frac{1}{2}\sum_{n=1}^{\infty} a_n = \frac{\ln 2}{2}.$$

Now introduce the series $\sum_{n=1}^{\infty} b_n$ such that for all $n \geq 1$, $b_{2n-1}=0$ and $b_{2n}=a_n/2$. Then

$$\sum_{n=1}^{\infty} b_n = 0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\cdots=\frac{\ln 2}{2}.$$

Then using the algebraic limit properties of convergent series, since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, the series $\sum_{n=1}^{\infty} (a_n+b_n)$ converges and

$$\sum_{n=1}^{\infty} (a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \ln 2+\frac{\ln 2}{2}=\frac{3\ln 2}{2}.$$

Now adding the corresponding terms, a_n and b_n, we see that

$$\begin{aligned}\sum_{n=1}^{\infty} (a_n+b_n) &= (1+0)+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{3}+0\right)+\left(-\frac{1}{4}-\frac{1}{4}\right)+\left(\frac{1}{5}+0\right)+\left(-\frac{1}{6}+\frac{1}{6}\right)\\ &\quad+\left(\frac{1}{7}+0\right)+\left(\frac{1}{8}-\frac{1}{8}\right)+\cdots\\ &= 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots.\end{aligned}$$

We notice that the series on the right side of the equal sign is a rearrangement of the alternating harmonic series. Since $\sum_{n=1}^{\infty} (a_n+b_n) = 3\ln(2)/2$, we conclude that

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3\ln(2)}{2}.$$

Therefore, we have found a rearrangement of the alternating harmonic series having the desired property.

5.5 EXERCISES

State whether each of the following series converges absolutely, conditionally, or not at all.

250. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+3}$

251. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{\sqrt{n}+3}$

252. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n+3}}$

253. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n+3}}{n}$

254. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!}$

255. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n!}$

256. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n-1}{n}\right)^n$

257. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+1}{n}\right)^n$

258. $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 n$

259. $\sum_{n=1}^{\infty} (-1)^{n+1} \cos^2 n$

260. $\sum_{n=1}^{\infty} (-1)^{n+1} \sin^2 (1/n)$

261. $\sum_{n=1}^{\infty} (-1)^{n+1} \cos^2 (1/n)$

262. $\sum_{n=1}^{\infty} (-1)^{n+1} \ln(1/n)$

263. $\sum_{n=1}^{\infty} (-1)^{n+1} \ln\left(1+\frac{1}{n}\right)$

264. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{1+n^4}$

265. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^e}{1+n^{\pi}}$

266. $\sum_{n=1}^{\infty} (-1)^{n+1} 2^{1/n}$

267. $\sum_{n=1}^{\infty} (-1)^{n+1} n^{1/n}$

268. $\sum_{n=1}^{\infty} (-1)^{n} \left(1-n^{1/n}\right)$ (*Hint:* $n^{1/n} \approx 1+\ln(n)/n$ for large n.)

269. $\sum_{n=1}^{\infty} (-1)^{n+1} n\left(1-\cos\left(\frac{1}{n}\right)\right)$ (*Hint:* $\cos(1/n) \approx 1-1/n^2$ for large n.)

270. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\sqrt{n+1}-\sqrt{n}\right)$ (*Hint:* Rationalize the numerator.)

271. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ (*Hint:* Cross-multiply then rationalize numerator.)

272. $\sum_{n=1}^{\infty} (-1)^{n+1} (\ln(n+1)-\ln n)$

273. $\sum_{n=1}^{\infty} (-1)^{n+1} n\left(\tan^{-1}(n+1)-\tan^{-1} n\right)$ (*Hint:* Use Mean Value Theorem.)

274. $\sum_{n=1}^{\infty} (-1)^{n+1} \left((n+1)^2-n^2\right)$

275. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n}-\frac{1}{n+1}\right)$

276. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

277. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{1/n}}$

278. $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$

279. $\sum_{n=1}^{\infty} \sin(n\pi/2)\sin(1/n)$

In each of the following problems, use the estimate $|R_N| \leq b_{N+1}$ to find a value of N that guarantees that the sum of the first N terms of the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ differs from the infinite sum by at most the given error. Calculate the partial sum S_N for this N.

280. **[T]** $b_n = 1/n$, error $< 10^{-5}$

281. **[T]** $b_n = 1/\ln(n)$, $n \geq 2$, error $< 10^{-1}$

282. **[T]** $b_n = 1/\sqrt{n}$, error $< 10^{-3}$

283. **[T]** $b_n = 1/2^n$, error $< 10^{-6}$

284. **[T]** $b_n = \ln\left(1 + \frac{1}{n}\right)$, error $< 10^{-3}$

285. **[T]** $b_n = 1/n^2$, error $< 10^{-6}$

For the following exercises, indicate whether each of the following statements is true or false. If the statement is false, provide an example in which it is false.

286. If $b_n \geq 0$ is decreasing and $\lim_{n \to \infty} b_n = 0$, then $\sum_{n=1}^{\infty} \left(b_{2n-1} - b_{2n}\right)$ converges absolutely.

287. If $b_n \geq 0$ is decreasing, then $\sum_{n=1}^{\infty} \left(b_{2n-1} - b_{2n}\right)$ converges absolutely.

288. If $b_n \geq 0$ and $\lim_{n \to \infty} b_n = 0$ then $\sum_{n=1}^{\infty} \left(\frac{1}{2}(b_{3n-2} + b_{3n-1}) - b_{3n}\right)$ converges.

289. If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty} (b_{3n-2} + b_{3n-1} - b_{3n})$ converges then $\sum_{n=1}^{\infty} b_{3n-2}$ converges.

290. If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges conditionally but not absolutely, then b_n does not tend to zero.

291. Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^- = -a_n$ if $a_n < 0$. (Also, $a_n^+ = 0$ if $a_n < 0$ and $a_n^- = 0$ if $a_n \geq 0$.) If $\sum_{n=1}^{\infty} a_n$ converges conditionally but not absolutely, then neither $\sum_{n=1}^{\infty} a_n^+$ nor $\sum_{n=1}^{\infty} a_n^-$ converge.

292. Suppose that a_n is a sequence of positive real numbers and that $\sum_{n=1}^{\infty} a_n$ converges. Suppose that b_n is an arbitrary sequence of ones and minus ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

293. Suppose that a_n is a sequence such that $\sum_{n=1}^{\infty} a_n b_n$ converges for every possible sequence b_n of zeros and ones. Does $\sum_{n=1}^{\infty} a_n$ converge absolutely?

The following series do not satisfy the hypotheses of the alternating series test as stated.

In each case, state which hypothesis is not satisfied. State whether the series converges absolutely.

294. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin^2 n}{n}$

295. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos^2 n}{n}$

296. $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$

297. $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

298. Show that the alternating series $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \cdots$ does not converge. What hypothesis of the alternating series test is not met?

299. Suppose that $\sum a_n$ converges absolutely. Show that the series consisting of the positive terms a_n also converges.

300. Show that the alternating series $\frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \cdots$ does not converge. What hypothesis of the alternating series test is not met?

301. The formula $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\cos\theta$ by the fifth partial sum $1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \theta^8/8!$ for $\theta = 1$, $\theta = \pi/6$, and $\theta = \pi$.

302. The formula $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ will be derived in the next chapter. Use the remainder $|R_N| \leq b_{N+1}$ to find a bound for the error in estimating $\sin\theta$ by the fifth partial sum $\theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \theta^9/9!$ for $\theta = 1$, $\theta = \pi/6$, and $\theta = \pi$.

303. How many terms in $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$ are needed to approximate $\cos 1$ accurate to an error of at most 0.00001?

304. How many terms in $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$ are needed to approximate $\sin 1$ accurate to an error of at most 0.00001?

305. Sometimes the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges to a certain fraction of an absolutely convergent series $\sum_{n=1}^{\infty} b_n$ at a faster rate. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find $S = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$. Which of the series $6\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $S\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ gives a better estimation of π^2 using 1000 terms?

The following alternating series converge to given multiples of π. Find the value of N predicted by the remainder estimate such that the Nth partial sum of the series accurately approximates the left-hand side to within the given error. Find the minimum N for which the error bound holds, and give the desired approximate value in each case. Up to 15 decimals places, $\pi = 3.141592653589793\ldots.$

306. **[T]** $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, error < 0.0001

307. **[T]** $\frac{\pi}{\sqrt{12}} = \sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k+1}$, error < 0.0001

308. **[T]** The series $\sum_{n=0}^{\infty} \frac{\sin(x + \pi n)}{x + \pi n}$ plays an important role in signal processing. Show that $\sum_{n=0}^{\infty} \frac{\sin(x + \pi n)}{x + \pi n}$ converges whenever $0 < x < \pi$. (*Hint:* Use the formula for the sine of a sum of angles.)

309. **[T]** If $\sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n} \to \ln 2$, what is $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \cdots$?

310. **[T]** Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ for $0 \leq x < 1$. Explain why $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ diverges when $x = 0, 1$. How does the series behave for other x?

311. **[T]** Plot the series $\sum_{n=1}^{100} \frac{\sin(2\pi nx)}{n}$ for $0 \leq x < 1$ and comment on its behavior

312. **[T]** Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n^2}$ for $0 \leq x < 1$ and describe its graph.

313. **[T]** The alternating harmonic series converges because of cancellation among its terms. Its sum is known because the cancellation can be described explicitly. A random harmonic series is one of the form $\sum_{n=1}^{\infty} \frac{S_n}{n}$, where s_n is a randomly generated sequence of ± 1's in which the values ± 1 are equally likely to occur. Use a random number generator to produce 1000 random ± 1s and plot the partial sums $S_N = \sum_{n=1}^{N} \frac{s_n}{n}$ of your random harmonic sequence for $N = 1$ to 1000. Compare to a plot of the first 1000 partial sums of the harmonic series.

314. **[T]** Estimates of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be *accelerated* by writing its partial sums as $\sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{N} \frac{1}{n(n+1)} + \sum_{n=1}^{N} \frac{1}{n^2(n+1)}$ and recalling that $\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$ converges to one as $N \to \infty$. Compare the estimate of $\pi^2/6$ using the sums $\sum_{n=1}^{1000} \frac{1}{n^2}$ with the estimate using $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$.

315. **[T]** The *Euler transform* rewrites $S = \sum_{n=0}^{\infty} (-1)^n b_n$ as $S = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} \sum_{m=0}^{n} \binom{n}{m} b_{n-m}$. For the alternating harmonic series, it takes the form $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n}$. Compute partial sums of $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ until they approximate $\ln(2)$ accurate to within 0.0001. How many terms are needed? Compare this answer to the number of terms of the alternating harmonic series are needed to estimate $\ln(2)$.

316. **[T]** In the text it was stated that a conditionally convergent series can be rearranged to converge to any number. Here is a slightly simpler, but similar, fact. If $a_n \geq 0$ is such that $a_n \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} a_n$ diverges, then, given any number A there is a sequence s_n of ± 1's such that $\sum_{n=1}^{\infty} a_n s_n \to A$. Show this for $A > 0$ as follows.

a. Recursively define s_n by $s_n = 1$ if $S_{n-1} = \sum_{k=1}^{n-1} a_k s_k < A$ and $s_n = -1$ otherwise.
b. Explain why eventually $S_n \geq A$, and for any m larger than this n, $A - a_m \leq S_m \leq A + a_m$.
c. Explain why this implies that $S_n \to A$ as $n \to \infty$.

5.6 | Ratio and Root Tests

Learning Objectives

5.6.1 Use the ratio test to determine absolute convergence of a series.
5.6.2 Use the root test to determine absolute convergence of a series.
5.6.3 Describe a strategy for testing the convergence of a given series.

In this section, we prove the last two series convergence tests: the ratio test and the root test. These tests are particularly nice because they do not require us to find a comparable series. The ratio test will be especially useful in the discussion of power series in the next chapter.

Throughout this chapter, we have seen that no single convergence test works for all series. Therefore, at the end of this section we discuss a strategy for choosing which convergence test to use for a given series.

Ratio Test

Consider a series $\sum_{n=1}^{\infty} a_n$. From our earlier discussion and examples, we know that $\lim_{n \to \infty} a_n = 0$ is not a sufficient condition for the series to converge. Not only do we need $a_n \to 0$, but we need $a_n \to 0$ quickly enough. For example, consider the series $\sum_{n=1}^{\infty} 1/n$ and the series $\sum_{n=1}^{\infty} 1/n^2$. We know that $1/n \to 0$ and $1/n^2 \to 0$. However, only the series $\sum_{n=1}^{\infty} 1/n^2$ converges. The series $\sum_{n=1}^{\infty} 1/n$ diverges because the terms in the sequence $\{1/n\}$ do not approach zero fast enough as $n \to \infty$. Here we introduce the **ratio test**, which provides a way of measuring how fast the terms of a series approach zero.

Theorem 5.16: Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

Proof

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms.

We begin with the proof of part i. In this case, $\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$. Since $0 \le \rho < 1$, there exists R such that $0 \le \rho < R < 1$. Let $\varepsilon = R - \rho > 0$. By the definition of limit of a sequence, there exists some integer N such that

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - \rho \right| < \varepsilon \text{ for all } n \ge N.$$

Therefore,

$$\left|\frac{a_{n+1}}{a_n}\right| < \rho + \varepsilon = R \text{ for all } n \geq N$$

and, thus,

$$\begin{aligned}
&|a_{N+1}| < R|a_N| \\
&|a_{N+2}| < R|a_{N+1}| < R^2|a_N| \\
&|a_{N+3}| < R|a_{N+2}| < R^2|a_{N+1}| < R^3|a_N| \\
&|a_{N+4}| < R|a_{N+3}| < R^2|a_{N+2}| < R^3|a_{N+1}| < R^4|a_N| \\
&\vdots .
\end{aligned}$$

Since $R < 1,$ the geometric series

$$R|a_N| + R^2|a_N| + R^3|a_N| + \cdots$$

converges. Given the inequalities above, we can apply the comparison test and conclude that the series

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + |a_{N+4}| + \cdots$$

converges. Therefore, since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

where $\sum_{n=1}^{N} |a_n|$ is a finite sum and $\sum_{n=N+1}^{\infty} |a_n|$ converges, we conclude that $\sum_{n=1}^{\infty} |a_n|$ converges.

For part ii.

$$\rho = \lim_{n \to \infty}\left|\frac{a_{n+1}}{a_n}\right| > 1.$$

Since $\rho > 1,$ there exists R such that $\rho > R > 1.$ Let $\varepsilon = \rho - R > 0.$ By the definition of the limit of a sequence, there exists an integer N such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon \text{ for all } n \geq N.$$

Therefore,

$$R = \rho - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| \text{ for all } n \geq N,$$

and, thus,

$$\begin{aligned}
&|a_{N+1}| > R|a_N| \\
&|a_{N+2}| > R|a_{N+1}| > R^2|a_N| \\
&|a_{N+3}| > R|a_{N+2}| > R^2|a_{N+1}| > R^3|a_N| \\
&|a_{N+4}| > R|a_{N+3}| > R^2|a_{N+2}| > R^3|a_{N+1}| > R^4|a_N|.
\end{aligned}$$

Since $R > 1,$ the geometric series

$$R|a_N| + R^2|a_N| + R^3|a_N| + \cdots$$

diverges. Applying the comparison test, we conclude that the series

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

diverges, and therefore the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

For part iii. we show that the test does not provide any information if $\rho = 1$ by considering the $p-$series $\sum_{n=1}^{\infty} 1/n^p$.

For any real number p,

$$\rho = \lim_{n \to \infty} \frac{1/(n+1)^p}{1/n^p} = \lim_{n \to \infty} \frac{n^p}{(n+1)^p} = 1.$$

However, we know that if $p \leq 1$, the $p-$series $\sum_{n=1}^{\infty} 1/n^p$ diverges, whereas $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$.

□

The ratio test is particularly useful for series whose terms contain factorials or exponentials, where the ratio of terms simplifies the expression. The ratio test is convenient because it does not require us to find a comparative series. The drawback is that the test sometimes does not provide any information regarding convergence.

Example 5.23

Using the Ratio Test

For each of the following series, use the ratio test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

Solution

a. From the ratio test, we can see that

$$\rho = \lim_{n \to \infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}.$$

Since $(n+1)! = (n+1) \cdot n!$,

$$\rho = \lim_{n \to \infty} \frac{2}{n+1} = 0.$$

Since $\rho < 1$, the series converges.

b. We can see that

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} \\ &= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} \\ &= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.\end{aligned}$$

Since $\rho > 1,$ the series diverges.

c. Since

$$\begin{aligned}\left|\frac{(-1)^{n+1}((n+1)!)^2/(2(n+1))!}{(-1)^n(n!)^2/(2n)!}\right| &= \frac{(n+1)!(n+1)!}{(2n+2)!}\cdot\frac{(2n)!}{n!n!} \\ &= \frac{(n+1)(n+1)}{(2n+2)(2n+1)}\end{aligned}$$

we see that

$$\rho = \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4}.$$

Since $\rho < 1,$ the series converges.

5.21 Use the ratio test to determine whether the series $\sum_{n=1}^{\infty}\frac{n^3}{3^n}$ converges or diverges.

Root Test

The approach of the **root test** is similar to that of the ratio test. Consider a series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \rho$ for some real number ρ. Then for N sufficiently large, $|a_N| \approx \rho^N$. Therefore, we can approximate $\sum_{n=N}^{\infty}|a_n|$ by writing

$$|a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \approx \rho^N + \rho^{N+1} + \rho^{N+2} + \cdots.$$

The expression on the right-hand side is a geometric series. As in the ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $0 \le \rho < 1$ and the series diverges if $\rho \ge 1$. If $\rho = 1,$ the test does not provide any information. For example, for any *p*-series, $\sum_{n=1}^{\infty} 1/n^p,$ we see that

$$\rho = \lim_{n\to\infty}\sqrt[n]{\left|\frac{1}{n^p}\right|} = \lim_{n\to\infty}\frac{1}{n^{p/n}}.$$

To evaluate this limit, we use the natural logarithm function. Doing so, we see that

$$\ln\rho = \ln\left(\lim_{n\to\infty}\frac{1}{n^{p/n}}\right) = \lim_{n\to\infty}\ln\left(\frac{1}{n}\right)^{p/n} = \lim_{n\to\infty}\frac{p}{n}\cdot\ln\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{p\ln(1/n)}{n}.$$

Using L'Hôpital's rule, it follows that $\ln \rho = 0$, and therefore $\rho = 1$ for all p. However, we know that the p-series only converges if $p > 1$ and diverges if $p < 1$.

Theorem 5.17: Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$. Let

$$\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|}.$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

The root test is useful for series whose terms involve exponentials. In particular, for a series whose terms a_n satisfy $|a_n| = b_n^n$, then $\sqrt[n]{|a_n|} = b_n$ and we need only evaluate $\lim_{n \to \infty} b_n$.

Example 5.24

Using the Root Test

For each of the following series, use the root test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{(n^2 + 3n)^n}{(4n^2 + 5)^n}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{(\ln(n))^n}$

Solution

a. To apply the root test, we compute

$$\rho = \lim_{n \to \infty} \sqrt[n]{(n^2 + 3n)^n / (4n^2 + 5)^n} = \lim_{n \to \infty} \frac{n^2 + 3n}{4n^2 + 5} = \frac{1}{4}.$$

Since $\rho < 1$, the series converges absolutely.

b. We have

$$\rho = \lim_{n \to \infty} \sqrt[n]{n^n / (\ln n)^n} = \lim_{n \to \infty} \frac{n}{\ln n} = \infty \text{ by L'Hôpital's rule}.$$

Since $\rho = \infty$, the series diverges.

5.22 Use the root test to determine whether the series $\sum_{n=1}^{\infty} 1/n^n$ converges or diverges.

Choosing a Convergence Test

At this point, we have a long list of convergence tests. However, not all tests can be used for all series. When given a series, we must determine which test is the best to use. Here is a strategy for finding the best test to apply.

Problem-Solving Strategy: Choosing a Convergence Test for a Series

Consider a series $\sum_{n=1}^{\infty} a_n$. In the steps below, we outline a strategy for determining whether the series converges.

1. Is $\sum_{n=1}^{\infty} a_n$ a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a $p-$series or geometric series? If so, check the power p or the ratio r to determine if the series converges.
2. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the alternating series test. If we are interested in absolute convergence, proceed to step $3,$ considering the series of absolute values $\sum_{n=1}^{\infty} |a_n|.$
3. Is the series similar to a $p-$series or geometric series? If so, try the comparison test or limit comparison test.
4. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = b_n^n,$ try the root test first. Otherwise, try the ratio test first.
5. Use the divergence test. If this test does not provide any information, try the integral test.

Visit this **website (http://www.openstaxcollege.org/l/20_series2)** for more information on testing series for convergence, plus general information on sequences and series.

Example 5.25

Using Convergence Tests

For each of the following series, determine which convergence test is the best to use and explain why. Then determine if the series converges or diverges. If the series is an alternating series, determine whether it converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty} \frac{n^2 + 2n}{n^3 + 3n^2 + 1}$

b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3n+1)}{n!}$

c. $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$

d. $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

Solution

a. Step 1. The series is not a $p-\text{series}$ or geometric series.

Step 2. The series is not alternating.
Step 3. For large values of $n,$ we approximate the series by the expression

$$\frac{n^2+2n}{n^3+3n^2+1} \approx \frac{n^2}{n^3} = \frac{1}{n}.$$

Therefore, it seems reasonable to apply the comparison test or limit comparison test using the series $\sum_{n=1}^{\infty} 1/n.$ Using the limit comparison test, we see that

$$\lim_{n \to \infty} \frac{(n^2+2n)/(n^3+3n^2+1)}{1/n} = \lim_{n \to \infty} \frac{n^3+2n^2}{n^3+3n^2+1} = 1.$$

Since the series $\sum_{n=1}^{\infty} 1/n$ diverges, this series diverges as well.

b. Step 1.The series is not a familiar series.
Step 2. The series is alternating. Since we are interested in absolute convergence, consider the series

$$\sum_{n=1}^{\infty} \frac{3n}{(n+1)!}.$$

Step 3. The series is not similar to a p-series or geometric series.
Step 4. Since each term contains a factorial, apply the ratio test. We see that

$$\lim_{n \to \infty} \frac{(3(n+1))/(n+1)!}{(3n+1)/n!} = \lim_{n \to \infty} \frac{3n+3}{(n+1)!} \cdot \frac{n!}{3n+1} = \lim_{n \to \infty} \frac{3n+3}{(n+1)(3n+1)} = 0.$$

Therefore, this series converges, and we conclude that the original series converges absolutely, and thus converges.

c. Step 1. The series is not a familiar series.
Step 2. It is not an alternating series.
Step 3. There is no obvious series with which to compare this series.
Step 4. There is no factorial. There is a power, but it is not an ideal situation for the root test.
Step 5. To apply the divergence test, we calculate that

$$\lim_{n \to \infty} \frac{e^n}{n^3} = \infty.$$

Therefore, by the divergence test, the series diverges.

d. Step 1. This series is not a familiar series.
Step 2. It is not an alternating series.
Step 3. There is no obvious series with which to compare this series.
Step 4. Since each term is a power of $n,$ we can apply the root test. Since

$$\lim_{n \to \infty} \sqrt[n]{\left(\frac{3}{n+1}\right)^n} = \lim_{n \to \infty} \frac{3}{n+1} = 0,$$

by the root test, we conclude that the series converges.

5.23 For the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n + n}$, determine which convergence test is the best to use and explain why.

In **Table 5.3**, we summarize the convergence tests and when each can be applied. Note that while the comparison test, limit comparison test, and integral test require the series $\sum_{n=1}^{\infty} a_n$ to have nonnegative terms, if $\sum_{n=1}^{\infty} a_n$ has negative terms, these tests can be applied to $\sum_{n=1}^{\infty} |a_n|$ to test for absolute convergence.

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n \to \infty} a_n$.	If $\lim_{n \to \infty} a_n = 0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n \to \infty} a_n \neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\lvert r \rvert < 1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
	If $\lvert r \rvert \geq 1$, the series diverges.	
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
	If $p \leq 1$, the series diverges.	
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n \to \infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.

Table 5.3 Summary of Convergence Tests

Series or Test	Conclusions	Comments
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx.$	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0,$ then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n \to \infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert.$	If $0 \leq \rho < 1,$ the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1 \text{ or } \rho = \infty,$ the series diverges.	
	If $\rho = 1,$ the test is inconclusive.	
Root Test For any series $\sum_{n=1}^{\infty} a_n,$ let $\rho = \lim_{n \to \infty} \sqrt[n]{\lvert a_n \rvert}.$	If $0 \leq \rho < 1,$ the series converges absolutely.	Often used for series where $\lvert a_n \rvert = b_n^n.$
	If $\rho > 1 \text{ or } \rho = \infty,$ the series diverges.	

Table 5.3 Summary of Convergence Tests

Series or Test	Conclusions	Comments
	If $\rho = 1,$ the test is inconclusive.	

Table 5.3 Summary of Convergence Tests

Student PROJECT

Series Converging to π and $1/\pi$

Dozens of series exist that converge to π or an algebraic expression containing π. Here we look at several examples and compare their rates of convergence. By rate of convergence, we mean the number of terms necessary for a partial sum to be within a certain amount of the actual value. The series representations of π in the first two examples can be explained using Maclaurin series, which are discussed in the next chapter. The third example relies on material beyond the scope of this text.

1. The series

$$\pi = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

was discovered by Gregory and Leibniz in the late 1600s. This result follows from the Maclaurin series for $f(x) = \tan^{-1} x$. We will discuss this series in the next chapter.

   a. Prove that this series converges.
   b. Evaluate the partial sums S_n for $n = 10, 20, 50, 100$.
   c. Use the remainder estimate for alternating series to get a bound on the error R_n.
   d. What is the smallest value of N that guarantees $|R_N| < 0.01$? Evaluate S_N.

2. The series

$$\begin{aligned}\pi &= 6\sum_{n=0}^{\infty} \frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)} \\ &= 6\left(\frac{1}{2} + \frac{1}{2\cdot 3}\left(\frac{1}{2}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}\cdot\left(\frac{1}{2}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\left(\frac{1}{2}\right)^7 + \cdots\right)\end{aligned}$$

has been attributed to Newton in the late 1600s. The proof of this result uses the Maclaurin series for $f(x) = \sin^{-1} x$.

   a. Prove that the series converges.
   b. Evaluate the partial sums S_n for $n = 5, 10, 20$.
   c. Compare S_n to π for $n = 5, 10, 20$ and discuss the number of correct decimal places.

3. The series

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

was discovered by Ramanujan in the early 1900s. William Gosper, Jr., used this series to calculate π to an accuracy of more than 17 million digits in the $\text{mid-}1980\text{s}$. At the time, that was a world record. Since that time, this series and others by Ramanujan have led mathematicians to find many other series representations for π and $1/\pi$.

   a. Prove that this series converges.
   b. Evaluate the first term in this series. Compare this number with the value of π from a calculating

utility. To how many decimal places do these two numbers agree? What if we add the first two terms in the series?

c. Investigate the life of Srinivasa Ramanujan $(1887–1920)$ and write a brief summary. Ramanujan is one of the most fascinating stories in the history of mathematics. He was basically self-taught, with no formal training in mathematics, yet he contributed in highly original ways to many advanced areas of mathematics.

5.6 EXERCISES

Use the ratio test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is given in the following problems. State if the ratio test is inconclusive.

317. $a_n = 1/n!$

318. $a_n = 10^n/n!$

319. $a_n = n^2/2^n$

320. $a_n = n^{10}/2^n$

321. $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n!)}$

322. $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n!)}$

323. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$

324. $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n}$

325. $\sum_{n=1}^{\infty} \frac{n!}{(n/e)^n}$

326. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n/e)^{2n}}$

327. $\sum_{n=1}^{\infty} \frac{(2^n n!)^2}{(2n)^{2n}}$

Use the root test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is as follows.

328. $a_k = \left(\frac{k-1}{2k+3}\right)^k$

329. $a_k = \left(\frac{2k^2-1}{k^2+3}\right)^k$

330. $a_n = \frac{(\ln n)^{2n}}{n^n}$

331. $a_n = n/2^n$

332. $a_n = n/e^n$

333. $a_k = \frac{k^e}{e^k}$

334. $a_k = \frac{\pi^k}{k^\pi}$

335. $a_n = \left(\frac{1}{e} + \frac{1}{n}\right)^n$

336. $a_k = \frac{1}{(1+\ln k)^k}$

337. $a_n = \frac{(\ln(1+\ln n))^n}{(\ln n)^n}$

In the following exercises, use either the ratio test or the root test as appropriate to determine whether the series $\sum_{k=1}^{\infty} a_k$ with given terms a_k converges, or state if the test is inconclusive.

338. $a_k = \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$

339. $a_k = \frac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!}$

340. $a_k = \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{3^k k!}$

341. $a_n = \left(1 - \frac{1}{n}\right)^{n^2}$

342. $a_k = \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right)^k$ (*Hint:* Compare $a_k^{1/k}$ to $\int_k^{2k} \frac{dt}{t}$.)

343. $a_k = \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{3k}\right)^k$

344. $a_n = (n^{1/n} - 1)^n$

Use the ratio test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, or state if the ratio test is inconclusive.

345. $\sum_{n=1}^{\infty} \frac{3^{n^2}}{2^{n^3}}$

346. $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n^n n!}$

Use the root and limit comparison tests to determine whether $\sum_{n=1}^{\infty} a_n$ converges.

347. $a_n = 1/x_n^n$ where $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$, $x_1 = 1$ (*Hint:* Find limit of $\{x_n\}$.)

In the following exercises, use an appropriate test to determine whether the series converges.

348. $\sum_{n=1}^{\infty} \frac{(n+1)}{n^3 + n^2 + n + 1}$

349. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n^3 + 3n^2 + 3n + 1}$

350. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3 + (1.1)^n}$

351. $\sum_{n=1}^{\infty} \frac{(n-1)^n}{(n+1)^n}$

352. $a_n = \left(1 + \frac{1}{n^2}\right)^n$ (*Hint:* $\left(1 + \frac{1}{n^2}\right)^{n^2} \approx e$.)

353. $a_k = 1/2^{\sin^2 k}$

354. $a_k = 2^{-\sin(1/k)}$

355. $a_n = 1/\binom{n+2}{n}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

356. $a_k = 1/\binom{2k}{k}$

357. $a_k = 2^k/\binom{3k}{k}$

358. $a_k = \left(\frac{k}{k + \ln k}\right)^k$ (*Hint:* $a_k = \left(1 + \frac{\ln k}{k}\right)^{-(k/\ln k)\ln k} \approx e^{-\ln k}$.)

359. $a_k = \left(\frac{k}{k + \ln k}\right)^{2k}$ (*Hint:* $a_k = \left(1 + \frac{\ln k}{k}\right)^{-(k/\ln k)\ln k^2}$.)

The following series converge by the ratio test. Use summation by parts, $\sum_{k=1}^{n} a_k(b_{k+1} - b_k) = [a_{n+1}b_{n+1} - a_1 b_1] - \sum_{k=1}^{n} b_{k+1}(a_{k+1} - a_k)$, to find the sum of the given series.

360. $\sum_{k=1}^{\infty} \frac{k}{2^k}$ (*Hint:* Take $a_k = k$ and $b_k = 2^{1-k}$.)

361. $\sum_{k=1}^{\infty} \frac{k}{c^k}$, where $c > 1$ (*Hint:* Take $a_k = k$ and $b_k = c^{1-k}/(c-1)$.)

362. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

363. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{2^n}$

The kth term of each of the following series has a factor x^k. Find the range of x for which the ratio test implies that the series converges.

364. $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$

365. $\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$

366. $\sum_{k=1}^{\infty} \frac{x^{2k}}{3^k}$

367. $\sum_{k=1}^{\infty} \frac{x^k}{k!}$

368. Does there exist a number p such that $\sum_{n=1}^{\infty} \frac{2^n}{n^p}$ converges?

369. Let $0 < r < 1$. For which real numbers p does $\sum_{n=1}^{\infty} n^p r^n$ converge?

370. Suppose that $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = p$. For which values of p must $\sum_{n=1}^{\infty} 2^n a_n$ converge?

371. Suppose that $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = p$. For which values of $r > 0$ is $\sum_{n=1}^{\infty} r^n a_n$ guaranteed to converge?

372. Suppose that $\left| \frac{a_{n+1}}{a_n} \right| \le (n+1)^p$ for all $n = 1, 2,\ldots$ where p is a fixed real number. For which values of p is $\sum_{n=1}^{\infty} n!\, a_n$ guaranteed to converge?

373. For which values of $r > 0$, if any, does $\sum_{n=1}^{\infty} r^{\sqrt{n}}$ converge? (*Hint:* $\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2 - 1} a_n$.)

374. Suppose that $\left| \frac{a_{n+2}}{a_n} \right| \le r < 1$ for all n. Can you conclude that $\sum_{n=1}^{\infty} a_n$ converges?

375. Let $a_n = 2^{-[n/2]}$ where $[x]$ is the greatest integer less than or equal to x. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and justify your answer.

The following *advanced* exercises use a generalized ratio test to determine convergence of some series that arise in particular applications when tests in this chapter, including the ratio and root test, are not powerful enough to determine their convergence. The test states that if $\lim_{n \to \infty} \frac{a_{2n}}{a_n} < 1/2$, then $\sum a_n$ converges, while if $\lim_{n \to \infty} \frac{a_{2n+1}}{a_n} > 1/2$, then $\sum a_n$ diverges.

376. Let $a_n = \frac{1}{4} \frac{3}{6} \frac{5}{8} \cdots \frac{2n-1}{2n+2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n (n+1)!}$. Explain why the ratio test cannot determine convergence of $\sum_{n=1}^{\infty} a_n$. Use the fact that $1 - 1/(4k)$ is increasing k to estimate $\lim_{n \to \infty} \frac{a_{2n}}{a_n}$.

377. Let $a_n = \frac{1}{1+x} \frac{2}{2+x} \cdots \frac{n}{n+x} \frac{1}{n} = \frac{(n-1)!}{(1+x)(2+x)\cdots(n+x)}$. Show that $a_{2n}/a_n \le e^{-x/2}/2$. For which $x > 0$ does the generalized ratio test imply convergence of $\sum_{n=1}^{\infty} a_n$? (*Hint:* Write $2a_{2n}/a_n$ as a product of n factors each smaller than $1/(1 + x/(2n))$.)

378. Let $a_n = \frac{n^{\ln n}}{(\ln n)^n}$. Show that $\frac{a_{2n}}{a_n} \to 0$ as $n \to \infty$.

CHAPTER 5 REVIEW

KEY TERMS

absolute convergence if the series $\sum_{n=1}^{\infty} |a_n|$ converges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely

alternating series a series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n \geq 0$, is called an alternating series

alternating series test for an alternating series of either form, if $b_{n+1} \leq b_n$ for all integers $n \geq 1$ and $b_n \to 0$, then an alternating series converges

arithmetic sequence a sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence

bounded above a sequence $\{a_n\}$ is bounded above if there exists a constant M such that $a_n \leq M$ for all positive integers n

bounded below a sequence $\{a_n\}$ is bounded below if there exists a constant M such that $M \leq a_n$ for all positive integers n

bounded sequence a sequence $\{a_n\}$ is bounded if there exists a constant M such that $|a_n| \leq M$ for all positive integers n

comparison test if $0 \leq a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; if $a_n \geq b_n \geq 0$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

conditional convergence if the series $\sum_{n=1}^{\infty} a_n$ converges, but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, the series $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally

convergence of a series a series converges if the sequence of partial sums for that series converges

convergent sequence a convergent sequence is a sequence $\{a_n\}$ for which there exists a real number L such that a_n is arbitrarily close to L as long as n is sufficiently large

divergence of a series a series diverges if the sequence of partial sums for that series diverges

divergence test if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges

divergent sequence a sequence that is not convergent is divergent

explicit formula a sequence may be defined by an explicit formula such that $a_n = f(n)$

geometric sequence a sequence $\{a_n\}$ in which the ratio a_{n+1}/a_n is the same for all positive integers n is called a geometric sequence

geometric series a geometric series is a series that can be written in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

harmonic series the harmonic series takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

index variable the subscript used to define the terms in a sequence is called the index

infinite series an infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

integral test for a series $\sum_{n=1}^{\infty} a_n$ with positive terms $a_n,$ if there exists a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers $n,$ then

$$\sum_{n=1}^{\infty} a_n \text{ and} \int_1^{\infty} f(x)dx$$

either both converge or both diverge

limit comparison test suppose $a_n,\ b_n \geq 0$ for all $n \geq 1.$ If $\lim_{n \to \infty} a_n/b_n \to L \neq 0,$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge; if $\lim_{n \to \infty} a_n/b_n \to 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\lim_{n \to \infty} a_n/b_n \to \infty,$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

limit of a sequence the real number L to which a sequence converges is called the limit of the sequence

monotone sequence an increasing or decreasing sequence

***p*-series** a series of the form $\sum_{n=1}^{\infty} 1/n^p$

partial sum the kth partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

ratio test for a series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n \to \infty} |a_{n+1}/a_n|;$ if $0 \leq \rho < 1,$ the series converges absolutely; if $\rho > 1,$ the series diverges; if $\rho = 1,$ the test is inconclusive

recurrence relation a recurrence relation is a relationship in which a term a_n in a sequence is defined in terms of earlier terms in the sequence

remainder estimate for a series $\sum_{n=1}^{\infty} a_n$ with positive terms a_n and a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers $n,$ the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)dx < R_N < \int_N^{\infty} f(x)dx$$

root test for a series $\sum_{n=1}^{\infty} a_n,$ let $\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|};$ if $0 \le \rho < 1,$ the series converges absolutely; if $\rho > 1,$ the series diverges; if $\rho = 1,$ the test is inconclusive

sequence an ordered list of numbers of the form $a_1, a_2, a_3, \ldots$ is a sequence

telescoping series a telescoping series is one in which most of the terms cancel in each of the partial sums

term the number a_n in the sequence $\{a_n\}$ is called the nth term of the sequence

unbounded sequence a sequence that is not bounded is called unbounded

KEY EQUATIONS

- **Harmonic series**
  $$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$
- **Sum of a geometric series**
  $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1$$
- **Divergence test**
  $$\text{If } a_n \nrightarrow 0 \text{ as } n \to \infty, \sum_{n=1}^{\infty} a_n \text{ diverges}.$$
- ***p*-series**
  $$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}$$
- **Remainder estimate from the integral test**
  $$\int_{N+1}^{\infty} f(x)dx < R_N < \int_{N}^{\infty} f(x)dx$$
- **Alternating series**
  $$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots \text{ or}$$
  $$\sum_{n=1}^{\infty} (-1)^{n} b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

KEY CONCEPTS

5.1 Sequences

- To determine the convergence of a sequence given by an explicit formula $a_n = f(n),$ we use the properties of limits for functions.
- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences that converge to A and $B,$ respectively, and c is any real number, then the sequence $\{ca_n\}$ converges to $c \cdot A,$ the sequences $\{a_n \pm b_n\}$ converge to $A \pm B,$ the sequence $\{a_n \cdot b_n\}$ converges to $A \cdot B,$ and the sequence $\{a_n / b_n\}$ converges to $A/B,$ provided $B \ne 0.$
- If a sequence is bounded and monotone, then it converges, but not all convergent sequences are monotone.
- If a sequence is unbounded, it diverges, but not all divergent sequences are unbounded.
- The geometric sequence $\{r^n\}$ converges if and only if $|r| < 1$ or $r = 1.$

5.2 Infinite Series

- Given the infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

  and the corresponding sequence of partial sums $\{S_k\}$ where

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k,$$

  the series converges if and only if the sequence $\{S_k\}$ converges.

- The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. For $|r| < 1$,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

- The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

  diverges.

- A series of the form $\sum_{n=1}^{\infty} [b_n - b_{n+1}] = [b_1 - b_2] + [b_2 - b_3] + [b_3 - b_4] + \cdots + [b_n - b_{n+1}] + \cdots$ is a telescoping series. The kth partial sum of this series is given by $S_k = b_1 - b_{k+1}$. The series will converge if and only if $\lim_{k \to \infty} b_{k+1}$ exists. In that case,

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = b_1 - \lim_{k \to \infty} (b_{k+1}).$$

5.3 The Divergence and Integral Tests

- If $\lim_{n \to \infty} a_n \neq 0,$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

- If $\lim_{n \to \infty} a_n = 0,$ the series $\sum_{n=1}^{\infty} a_n$ may converge or diverge.

- If $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n and f is a continuous, decreasing function such that $f(n) = a_n$ for all positive integers $n,$ then

$$\sum_{n=1}^{\infty} a_n \text{and} \int_1^{\infty} f(x)dx$$

  either both converge or both diverge. Furthermore, if $\sum_{n=1}^{\infty} a_n$ converges, then the Nth partial sum approximation

S_N is accurate up to an error R_N where $\int_{N+1}^{\infty} f(x)dx < R_N < \int_N^{\infty} f(x)dx.$

- The p-series $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1.$

5.4 Comparison Tests

- The comparison tests are used to determine convergence or divergence of series with positive terms.
- When using the comparison tests, a series $\sum_{n=1}^{\infty} a_n$ is often compared to a geometric or p-series.

5.5 Alternating Series

- For an alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n,$ if $b_{k+1} \leq b_k$ for all k and $b_k \to 0$ as $k \to \infty,$ the alternating series converges.
- If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

5.6 Ratio and Root Tests

- For the ratio test, we consider

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $\rho < 1,$ the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1,$ the series diverges. If $\rho = 1,$ the test does not provide any information. This test is useful for series whose terms involve factorials.

- For the root test, we consider

$$\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|}.$$

If $\rho < 1,$ the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1,$ the series diverges. If $\rho = 1,$ the test does not provide any information. The root test is useful for series whose terms involve powers.

- For a series that is similar to a geometric series or $p-$ series, consider one of the comparison tests.

CHAPTER 5 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample.

379. If $\lim_{n \to \infty} a_n = 0,$ then $\sum_{n=1}^{\infty} a_n$ converges.

380. If $\lim_{n \to \infty} a_n \neq 0,$ then $\sum_{n=1}^{\infty} a_n$ diverges.

381. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

382. If $\sum_{n=1}^{\infty} 2^n a_n$ converges, then $\sum_{n=1}^{\infty} (-2)^n a_n$ converges.

Is the sequence bounded, monotone, and convergent or divergent? If it is convergent, find the limit.

383. $a_n = \frac{3+n^2}{1-n}$

384. $a_n = \ln\left(\frac{1}{n}\right)$

385. $a_n = \frac{\ln(n+1)}{\sqrt{n+1}}$

386. $a_n = \frac{2^{n+1}}{5^n}$

387. $a_n = \frac{\ln(\cos n)}{n}$

Is the series convergent or divergent?

388. $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+4}$

389. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

390. $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$

391. $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

392. $\sum_{n=1}^{\infty} n^{-(n+1/n)}$

Is the series convergent or divergent? If convergent, is it absolutely convergent?

393. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

394. $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$

395. $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

396. $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

397. $\sum_{n=1}^{\infty} \cos(\pi n)e^{-n}$

Evaluate

398. $\sum_{n=1}^{\infty} \frac{2^{n+4}}{7^n}$

399. $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

400. A legend from India tells that a mathematician invented chess for a king. The king enjoyed the game so much he allowed the mathematician to demand any payment. The mathematician asked for one grain of rice for the first square on the chessboard, two grains of rice for the second square on the chessboard, and so on. Find an exact expression for the total payment (in grains of rice) requested by the mathematician. Assuming there are $30{,}000$ grains of rice in 1 pound, and 2000 pounds in 1 ton, how many tons of rice did the mathematician attempt to receive?

The following problems consider a simple population model of the housefly, which can be exhibited by the recursive formula $x_{n+1} = bx_n$, where x_n is the population of houseflies at generation n, and b is the average number of offspring per housefly who survive to the next generation. Assume a starting population x_0.

401. Find $\lim_{n \to \infty} x_n$ if $b > 1$, $b < 1$, and $b = 1$.

402. Find an expression for $S_n = \sum_{i=0}^{n} x_i$ in terms of b and x_0. What does it physically represent?

403. If $b = \frac{3}{4}$ and $x_0 = 100$, find S_{10} and $\lim_{n \to \infty} S_n$

404. For what values of b will the series converge and diverge? What does the series converge to?

6 | POWER SERIES

Figure 6.1 If you win a lottery, do you get more money by taking a lump-sum payment or by accepting fixed payments over time? (credit: modification of work by Robert Huffstutter, Flickr)

Chapter Outline

Introduction

When winning a lottery, sometimes an individual has an option of receiving winnings in one lump-sum payment or receiving smaller payments over fixed time intervals. For example, you might have the option of receiving 20 million dollars today or receiving 1.5 million dollars each year for the next 20 years. Which is the better deal? Certainly 1.5 million dollars over 20 years is equivalent to 30 million dollars. However, receiving the 20 million dollars today would allow you to invest the money.

Alternatively, what if you were guaranteed to receive 1 million dollars every year indefinitely (extending to your heirs) or receive 20 million dollars today. Which would be the better deal? To answer these questions, you need to know how to use infinite series to calculate the value of periodic payments over time in terms of today's dollars (see **Example 6.7**).

An infinite series of the form $\sum_{n=0}^{\infty} c_n x^n$ is known as a power series. Since the terms contain the variable x, power series can be used to define functions. They can be used to represent given functions, but they are also important because they allow us to write functions that cannot be expressed any other way than as “infinite polynomials.” In addition, power series can be easily differentiated and integrated, thus being useful in solving differential equations and integrating complicated functions. An infinite series can also be truncated, resulting in a finite polynomial that we can use to approximate functional values. Power series have applications in a variety of fields, including physics, chemistry, biology, and economics. As we will see in this chapter, representing functions using power series allows us to solve mathematical problems that cannot be solved with other techniques.

6.1 | Power Series and Functions

Learning Objectives

6.1.1 Identify a power series and provide examples of them.
6.1.2 Determine the radius of convergence and interval of convergence of a power series.
6.1.3 Use a power series to represent a function.

A power series is a type of series with terms involving a variable. More specifically, if the variable is x, then all the terms of the series involve powers of x. As a result, a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions. In this section we define power series and show how to determine when a power series converges and when it diverges. We also show how to represent certain functions using power series.

Form of a Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots,$$

where x is a variable and the coefficients c_n are constants, is known as a **power series**. The series

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

is an example of a power series. Since this series is a geometric series with ratio $r = |x|,$ we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1.$

Definition

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \tag{6.1}$$

is a power series centered at $x = 0.$ A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots \tag{6.2}$$

is a power series centered at $x = a.$

To make this definition precise, we stipulate that $x^0 = 1$ and $(x-a)^0 = 1$ even when $x = 0$ and $x = a,$ respectively.

The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and

$$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$$

are both power series centered at $x = 0$. The series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n} = 1 + \frac{x-2}{2 \cdot 3} + \frac{(x-2)^2}{3 \cdot 3^2} + \frac{(x-2)^3}{4 \cdot 3^3} + \cdots$$

is a power series centered at $x = 2$.

Convergence of a Power Series

Since the terms in a power series involve a variable x, the series may converge for certain values of x and diverge for other values of x. For a power series centered at $x = a$, the value of the series at $x = a$ is given by c_0. Therefore, a power series always converges at its center. Some power series converge only at that value of x. Most power series, however, converge for more than one value of x. In that case, the power series either converges for all real numbers x or converges for all x in a finite interval. For example, the geometric series $\sum_{n=0}^{\infty} x^n$ converges for all x in the interval $(-1, 1)$, but diverges for all x outside that interval. We now summarize these three possibilities for a general power series.

Theorem 6.1: Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$. The series satisfies exactly one of the following properties:

i. The series converges at $x = a$ and diverges for all $x \neq a$.

ii. The series converges for all real numbers x.

iii. There exists a real number $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. At the values x where $|x-a| = R$, the series may converge or diverge.

Proof

Suppose that the power series is centered at $a = 0$. (For a series centered at a value of a other than zero, the result follows by letting $y = x - a$ and considering the series $\sum_{n=1}^{\infty} c_n y^n$.) We must first prove the following fact:

If there exists a real number $d \neq 0$ such that $\sum_{n=0}^{\infty} c_n d^n$ converges, then the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for all x such that $|x| < |d|$.

Since $\sum_{n=0}^{\infty} c_n d^n$ converges, the nth term $c_n d^n \to 0$ as $n \to \infty$. Therefore, there exists an integer N such that $|c_n d^n| \leq 1$ for all $n \geq N$. Writing

$$|c_n x^n| = |c_n d^n| \left|\frac{x}{d}\right|^n,$$

we conclude that, for all $n \geq N$,

$$|c_n x^n| \leq \left|\frac{x}{d}\right|^n.$$

The series

$$\sum_{n=N}^{\infty} \left|\frac{x}{d}\right|^n$$

is a geometric series that converges if $\left|\frac{x}{d}\right| < 1.$ Therefore, by the comparison test, we conclude that $\sum_{n=N}^{\infty} c_n x^n$ also converges for $|x| < |d|.$ Since we can add a finite number of terms to a convergent series, we conclude that $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < |d|.$

With this result, we can now prove the theorem. Consider the series

$$\sum_{n=0}^{\infty} a_n x^n$$

and let S be the set of real numbers for which the series converges. Suppose that the set $S = \{0\}.$ Then the series falls under case i. Suppose that the set S is the set of all real numbers. Then the series falls under case ii. Suppose that $S \neq \{0\}$ and S is not the set of real numbers. Then there exists a real number $x^* \neq 0$ such that the series does not converge. Thus, the series cannot converge for any x such that $|x| > |x^*|.$ Therefore, the set S must be a bounded set, which means that it must have a smallest upper bound. (This fact follows from the Least Upper Bound Property for the real numbers, which is beyond the scope of this text and is covered in real analysis courses.) Call that smallest upper bound R. Since $S \neq \{0\},$ the number $R > 0.$ Therefore, the series converges for all x such that $|x| < R,$ and the series falls into case iii.

□

If a series $\sum_{n=0}^{\infty} c_n (x-a)^n$ falls into case iii. of **Convergence of a Power Series**, then the series converges for all x such that $|x-a| < R$ for some $R > 0,$ and diverges for all x such that $|x-a| > R.$ The series may converge or diverge at the values x where $|x-a| = R.$ The set of values x for which the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges is known as the **interval of convergence**. Since the series diverges for all values x where $|x-a| > R,$ the length of the interval is $2R$, and therefore, the radius of the interval is R. The value R is called the **radius of convergence**. For example, since the series $\sum_{n=0}^{\infty} x^n$ converges for all values x in the interval $(-1,\ 1)$ and diverges for all values x such that $|x| \geq 1,$ the interval of convergence of this series is $(-1,\ 1).$ Since the length of the interval is 2, the radius of convergence is 1.

Definition

Consider the power series $\sum_{n=0}^{\infty} c_n (x-a)^n.$ The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R,$ then R is the radius of convergence. If the series converges only at $x = a,$ we say the radius of convergence is $R = 0.$ If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$ (**Figure 6.2**).

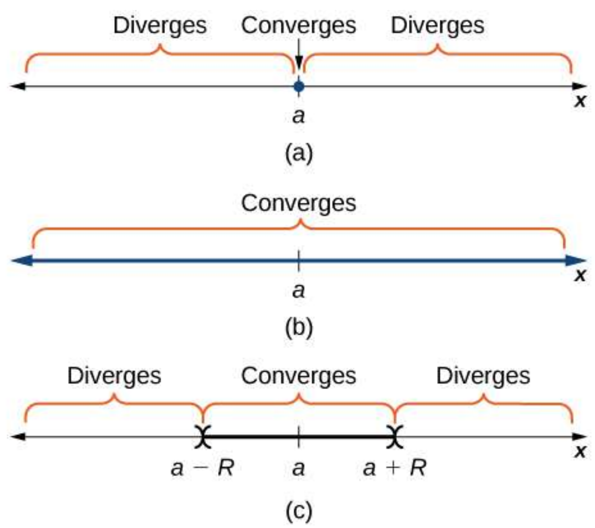


Figure 6.2 For a series $\sum_{n=0}^{\infty} c_n (x-a)^n$ graph (a) shows a radius of convergence at $R = 0$, graph (b) shows a radius of convergence at $R = \infty$, and graph (c) shows a radius of convergence at R. For graph (c) we note that the series may or may not converge at the endpoints $x = a + R$ and $x = a - R$.

To determine the interval of convergence for a power series, we typically apply the ratio test. In **Example 6.1**, we show the three different possibilities illustrated in **Figure 6.2**.

Example 6.1

Finding the Interval and Radius of Convergence

For each of the following series, find the interval and radius of convergence.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\sum_{n=0}^{\infty} n!x^n$

c. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

Solution

a. To check for convergence, apply the ratio test. We have

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)\cdot n!}\cdot\frac{n!}{x^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| \\ &= |x|\lim_{n\to\infty}\frac{1}{n+1} \\ &= 0 < 1\end{aligned}$$

for all values of *x*. Therefore, the series converges for all real numbers *x*. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

b. Apply the ratio test. For $x \neq 0,$ we see that

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}|(n+1)x| \\ &= |x|\lim_{n\to\infty}(n+1) \\ &= \infty.\end{aligned}$$

Therefore, the series diverges for all $x \neq 0.$ Since the series is centered at $x = 0,$ it must converge there, so the series converges only for $x \neq 0.$ The interval of convergence is the single value $x = 0$ and the radius of convergence is $R = 0.$

c. In order to apply the ratio test, consider

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\frac{(x-2)^n}{(n+1)3^n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}\cdot\frac{(n+1)3^n}{(x-2)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-2)(n+1)}{3(n+2)}\right| \\ &= \frac{|x-2|}{3}.\end{aligned}$$

The ratio $\rho < 1$ if $|x-2| < 3.$ Since $|x-2| < 3$ implies that $-3 < x - 2 < 3,$ the series converges absolutely if $-1 < x < 5.$ The ratio $\rho > 1$ if $|x-2| > 3.$ Therefore, the series diverges if $x < -1$ or $x > 5.$ The ratio test is inconclusive if $\rho = 1.$ The ratio $\rho = 1$ if and only if $x = -1$ or $x = 5.$ We need to test these values of *x* separately. For $x = -1,$ the series is given by

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Since this is the alternating harmonic series, it converges. Thus, the series converges at $x = -1$. For $x = 5$, the series is given by

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

This is the harmonic series, which is divergent. Therefore, the power series diverges at $x = 5$. We conclude that the interval of convergence is $[-1, 5)$ and the radius of convergence is $R = 3$.

6.1 Find the interval and radius of convergence for the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

Representing Functions as Power Series

Being able to represent a function by an "infinite polynomial" is a powerful tool. Polynomial functions are the easiest functions to analyze, since they only involve the basic arithmetic operations of addition, subtraction, multiplication, and division. If we can represent a complicated function by an infinite polynomial, we can use the polynomial representation to differentiate or integrate it. In addition, we can use a truncated version of the polynomial expression to approximate values of the function. So, the question is, when can we represent a function by a power series?

Consider again the geometric series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n. \tag{6.3}$$

Recall that the geometric series

$$a + ar + ar^2 + ar^3 + \cdots$$

converges if and only if $|r| < 1$. In that case, it converges to $\frac{a}{1-r}$. Therefore, if $|x| < 1$, the series in **Example 6.3** converges to $\frac{1}{1-x}$ and we write

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \text{ for } |x| < 1.$$

As a result, we are able to represent the function $f(x) = \frac{1}{1-x}$ by the power series

$$1 + x + x^2 + x^3 + \cdots \text{ when } |x| < 1.$$

We now show graphically how this series provides a representation for the function $f(x) = \frac{1}{1-x}$ by comparing the graph of f with the graphs of several of the partial sums of this infinite series.

Example 6.2

Graphing a Function and Partial Sums of its Power Series

Sketch a graph of $f(x) = \frac{1}{1-x}$ and the graphs of the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^n$ for $N = 2, 4, 6$ on the interval $(-1, 1)$. Comment on the approximation S_N as N increases.

Solution

From the graph in **Figure 6.3** you see that as N increases, S_N becomes a better approximation for $f(x) = \frac{1}{1-x}$ for x in the interval $(-1, 1)$.

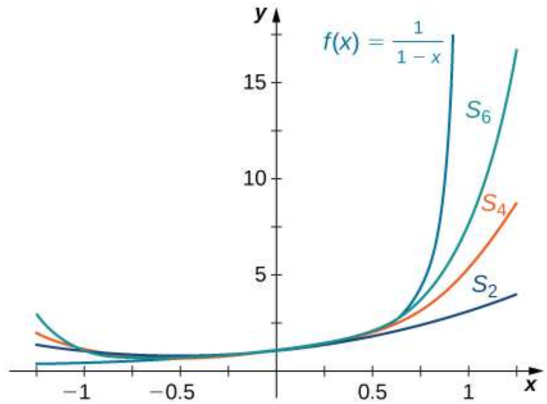


Figure 6.3 The graph shows a function and three approximations of it by partial sums of a power series.

6.2 Sketch a graph of $f(x) = \frac{1}{1-x^2}$ and the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^{2n}$ for $N = 2, 4, 6$ on the interval $(-1, 1)$.

Next we consider functions involving an expression similar to the sum of a geometric series and show how to represent these functions using power series.

Example 6.3

Representing a Function with a Power Series

Use a power series to represent each of the following functions f. Find the interval of convergence.

a. $f(x) = \frac{1}{1+x^3}$

b. $f(x) = \frac{x^2}{4-x^2}$

Solution

a. You should recognize this function f as the sum of a geometric series, because

$$\frac{1}{1+x^3} = \frac{1}{1-\left(-x^3\right)}.$$

Using the fact that, for $|r| < 1$, $\frac{a}{1-r}$ is the sum of the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots,$$

we see that, for $\left|-x^3\right| < 1$,

$$\begin{aligned}\frac{1}{1+x^3} &= \frac{1}{1-\left(-x^3\right)} \\ &= \sum_{n=0}^{\infty} \left(-x^3\right)^n \\ &= 1 - x^3 + x^6 - x^9 + \cdots.\end{aligned}$$

Since this series converges if and only if $\left|-x^3\right| < 1$, the interval of convergence is $(-1, 1)$, and we have

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots \text{ for } |x| < 1.$$

b. This function is not in the exact form of a sum of a geometric series. However, with a little algebraic manipulation, we can relate *f* to a geometric series. By factoring 4 out of the two terms in the denominator, we obtain

$$\begin{aligned}\frac{x^2}{4-x^2} &= \frac{x^2}{4\left(\frac{1-x^2}{4}\right)} \\ &= \frac{x^2}{4\left(1-\left(\frac{x}{2}\right)^2\right)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{x^2}{4-x^2} &= \frac{x^2}{4\left(1-\left(\frac{x}{2}\right)^2\right)} \\ &= \frac{\frac{x^2}{4}}{1-\left(\frac{x}{2}\right)^2} \\ &= \sum_{n=0}^{\infty} \frac{x^2}{4}\left(\frac{x}{2}\right)^{2n}.\end{aligned}$$

The series converges as long as $\left|\left(\frac{x}{2}\right)^2\right| < 1$ (note that when $\left|\left(\frac{x}{2}\right)^2\right| = 1$ the series does not converge). Solving this inequality, we conclude that the interval of convergence is $(-2, 2)$ and

$$\begin{aligned}\frac{x^2}{4-x^2} &= \sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}} \\ &= \frac{x^2}{4} + \frac{x^4}{4^2} + \frac{x^6}{4^3} + \cdots\end{aligned}$$

for $|x| < 2$.

6.3 Represent the function $f(x) = \frac{x^3}{2-x}$ using a power series and find the interval of convergence.

In the remaining sections of this chapter, we will show ways of deriving power series representations for many other functions, and how we can make use of these representations to evaluate, differentiate, and integrate various functions.

6.1 EXERCISES

In the following exercises, state whether each statement is true, or give an example to show that it is false.

1. If $\sum_{n=1}^{\infty} a_n x^n$ converges, then $a_n x^n \to 0$ as $n \to \infty$.

2. $\sum_{n=1}^{\infty} a_n x^n$ converges at $x = 0$ for any real numbers a_n.

3. Given any sequence a_n, there is always some $R > 0$, possibly very small, such that $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$.

4. If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R > 0$ and if $|b_n| \le |a_n|$ for all n, then the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is greater than or equal to R.

5. Suppose that $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 6$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.
   a. $x = 1$
   b. $x = 2$
   c. $x = 3$
   d. $x = 0$
   e. $x = 5.99$
   f. $x = 0.000001$

6. Suppose that $\sum_{n=0}^{\infty} a_n (x+1)^n$ converges at $x = -2$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.
   a. $x = 2$
   b. $x = -1$
   c. $x = -3$
   d. $x = 0$
   e. $x = 0.99$
   f. $x = 0.000001$

In the following exercises, suppose that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$. Find the radius of convergence for each series.

7. $\sum_{n=0}^{\infty} a_n 2^n x^n$

8. $\sum_{n=0}^{\infty} \frac{a_n x^n}{2^n}$

9. $\sum_{n=0}^{\infty} \frac{a_n \pi^n x^n}{e^n}$

10. $\sum_{n=0}^{\infty} \frac{a_n (-1)^n x^n}{10^n}$

11. $\sum_{n=0}^{\infty} a_n (-1)^n x^{2n}$

12. $\sum_{n=0}^{\infty} a_n (-4)^n x^{2n}$

In the following exercises, find the radius of convergence R and interval of convergence for $\sum a_n x^n$ with the given coefficients a_n.

13. $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

14. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$

15. $\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$

16. $\sum_{n=1}^{\infty} \frac{nx^n}{e^n}$

17. $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$

18. $\sum_{k=1}^{\infty} \frac{k^e x^k}{e^k}$

19. $\sum_{k=1}^{\infty} \frac{\pi^k x^k}{k^\pi}$

20. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

21. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$

22. $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\ln(2n)}$

In the following exercises, find the radius of convergence of each series.

23. $\sum_{k=1}^{\infty} \frac{(k!)^2 x^k}{(2k)!}$

24. $\sum_{n=1}^{\infty} \frac{(2n)!x^n}{n^{2n}}$

25. $\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} x^k$

26. $\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!} x^k$

27. $\sum_{n=1}^{\infty} \frac{x^n}{\binom{2n}{n}}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

28. $\sum_{n=1}^{\infty} \sin^2 n x^n$

In the following exercises, use the ratio test to determine the radius of convergence of each series.

29. $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$

30. $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!} x^n$

31. $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

32. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} x^n$

In the following exercises, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1, 1)$, find the power series for each function with the given center a, and identify its interval of convergence.

33. $f(x) = \frac{1}{x}$; $a = 1$ (*Hint:* $\frac{1}{x} = \frac{1}{1-(1-x)}$)

34. $f(x) = \frac{1}{1-x^2}$; $a = 0$

35. $f(x) = \frac{x}{1-x^2}$; $a = 0$

36. $f(x) = \frac{1}{1+x^2}$; $a = 0$

37. $f(x) = \frac{x^2}{1+x^2}$; $a = 0$

38. $f(x) = \frac{1}{2-x}$; $a = 1$

39. $f(x) = \frac{1}{1-2x}$; $a = 0$.

40. $f(x) = \frac{1}{1-4x^2}$; $a = 0$

41. $f(x) = \frac{x^2}{1-4x^2}$; $a = 0$

42. $f(x) = \frac{x^2}{5-4x+x^2}$; $a = 2$

Use the next exercise to find the radius of convergence of the given series in the subsequent exercises.

43. Explain why, if $|a_n|^{1/n} \to r > 0$, then $|a_n x^n|^{1/n} \to |x|r < 1$ whenever $|x| < \frac{1}{r}$ and, therefore, the radius of convergence of $\sum_{n=1}^{\infty} a_n x^n$ is $R = \frac{1}{r}$.

44. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

45. $\sum_{k=1}^{\infty} \left(\frac{k-1}{2k+3}\right)^k x^k$

46. $\sum_{k=1}^{\infty} \left(\frac{2k^2-1}{k^2+3}\right)^k x^k$

47. $\sum_{n=1}^{\infty} a_n = \left(n^{1/n} - 1\right)^n x^n$

48. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is even. Explain why $p(x) = p(-x)$.

49. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $a_n = 0$ if n is odd. Explain why $p(x) = -p(-x)$.

50. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(Ax)$.

51. Suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-1, 1]$. Find the interval of convergence of $p(2x-1)$.

In the following exercises, suppose that $p(x) = \sum_{n=0}^{\infty} a_n x^n$ satisfies $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$ where $a_n \geq 0$ for each n. State whether each series converges on the full interval $(-1, 1)$, or if there is not enough information to draw a conclusion. Use the comparison test when appropriate.

52. $\sum_{n=0}^{\infty} a_n x^{2n}$

53. $\sum_{n=0}^{\infty} a_{2n} x^{2n}$

54. $\sum_{n=0}^{\infty} a_{2n} x^n \left(Hint: x = \pm\sqrt{x^2}\right)$

55. $\sum_{n=0}^{\infty} a_{n^2} x^{n^2}$ (*Hint:* Let $b_k = a_k$ if $k = n^2$ for some n, otherwise $b_k = 0$.)

56. Suppose that $p(x)$ is a polynomial of degree N. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} p(n)x^n$.

57. **[T]** Plot the graphs of $\frac{1}{1-x}$ and of the partial sums $S_N = \sum_{n=0}^{N} x^n$ for $n = 10, 20, 30$ on the interval $[-0.99, 0.99]$. Comment on the approximation of $\frac{1}{1-x}$ by S_N near $x = -1$ and near $x = 1$ as N increases.

58. **[T]** Plot the graphs of $-\ln(1-x)$ and of the partial sums $S_N = \sum_{n=1}^{N} \frac{x^n}{n}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

59. **[T]** Plot the graphs of the partial sums $S_n = \sum_{n=1}^{N} \frac{x^n}{n^2}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

60. **[T]** Plot the graphs of the partial sums $S_N = \sum_{n=1}^{N} \sin nx^n$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$. Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

61. **[T]** Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\sin x$ as N increases.

62. **[T]** Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n}}{(2n)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\cos x$ as N increases.

6.2 | Properties of Power Series

Learning Objectives

6.2.1 Combine power series by addition or subtraction.
6.2.2 Create a new power series by multiplication by a power of the variable or a constant, or by substitution.
6.2.3 Multiply two power series together.
6.2.4 Differentiate and integrate power series term-by-term.

In the preceding section on power series and functions we showed how to represent certain functions using power series. In this section we discuss how power series can be combined, differentiated, or integrated to create new power series. This capability is particularly useful for a couple of reasons. First, it allows us to find power series representations for certain elementary functions, by writing those functions in terms of functions with known power series. For example, given the power series representation for $f(x) = \frac{1}{1-x}$, we can find a power series representation for $f'(x) = \frac{1}{(1-x)^2}$. Second, being able to create power series allows us to define new functions that cannot be written in terms of elementary functions. This capability is particularly useful for solving differential equations for which there is no solution in terms of elementary functions.

Combining Power Series

If we have two power series with the same interval of convergence, we can add or subtract the two series to create a new power series, also with the same interval of convergence. Similarly, we can multiply a power series by a power of x or evaluate a power series at x^m for a positive integer m to create a new power series. Being able to do this allows us to find power series representations for certain functions by using power series representations of other functions. For example, since we know the power series representation for $f(x) = \frac{1}{1-x}$, we can find power series representations for related functions, such as

$$y = \frac{3x}{1-x^2} \text{ and } y = \frac{1}{(x-1)(x-3)}.$$

In **Combining Power Series** we state results regarding addition or subtraction of power series, composition of a power series, and multiplication of a power series by a power of the variable. For simplicity, we state the theorem for power series centered at $x = 0$. Similar results hold for power series centered at $x = a$.

Theorem 6.2: Combining Power Series

Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.

i. The power series $\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.

ii. For any integer $m \geq 0$ and any real number b, the power series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ on I.

iii. For any integer $m \geq 0$ and any real number b, the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ for all x such that bx^m is in I.

Proof

We prove i. in the case of the series $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions *f* and *g*, respectively, on the interval *I*. Let *x* be a point in *I* and let $S_N(x)$ and $T_N(x)$ denote the *N*th partial sums of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$, respectively. Then the sequence $\{S_N(x)\}$ converges to $f(x)$ and the sequence $\{T_N(x)\}$ converges to $g(x)$. Furthermore, the *N*th partial sum of $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$ is

$$\begin{aligned}\sum_{n=0}^{N}(c_n x^n + d_n x^n) &= \sum_{n=0}^{N} c_n x^n + \sum_{n=0}^{N} d_n x^n \\ &= S_N(x) + T_N(x).\end{aligned}$$

Because

$$\begin{aligned}\lim_{N\to\infty}(S_N(x) + T_N(x)) &= \lim_{N\to\infty} S_N(x) + \lim_{N\to\infty} T_N(x) \\ &= f(x) + g(x),\end{aligned}$$

we conclude that the series $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$ converges to $f(x) + g(x)$.

□

We examine products of power series in a later theorem. First, we show several applications of **Combining Power Series** and how to find the interval of convergence of a power series given the interval of convergence of a related power series.

Example 6.4

Combining Power Series

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a power series whose interval of convergence is $(-1,\ 1)$, and suppose that $\sum_{n=0}^{\infty} b_n x^n$ is a power series whose interval of convergence is $(-2,\ 2)$.

a. Find the interval of convergence of the series $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$.

b. Find the interval of convergence of the series $\sum_{n=0}^{\infty} a_n 3^n x^n$.

Solution

a. Since the interval $(-1,\ 1)$ is a common interval of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, the interval of convergence of the series $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$ is $(-1,\ 1)$.

b. Since $\sum_{n=0}^{\infty} a_n x^n$ is a power series centered at zero with radius of convergence 1, it converges for all x in the interval $(-1, 1)$. By **Combining Power Series**, the series

$$\sum_{n=0}^{\infty} a_n 3^n x^n = \sum_{n=0}^{\infty} a_n (3x)^n$$

converges if $3x$ is in the interval $(-1, 1)$. Therefore, the series converges for all x in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

6.4 Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has an interval of convergence of $(-1, 1)$. Find the interval of convergence of $\sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^n$.

In the next example, we show how to use **Combining Power Series** and the power series for a function f to construct power series for functions related to f. Specifically, we consider functions related to the function $f(x) = \frac{1}{1-x}$ and we use the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

for $|x| < 1$.

Example 6.5

Constructing Power Series from Known Power Series

Use the power series representation for $f(x) = \frac{1}{1-x}$ combined with **Combining Power Series** to construct a power series for each of the following functions. Find the interval of convergence of the power series.

a. $f(x) = \frac{3x}{1+x^2}$

b. $f(x) = \frac{1}{(x-1)(x-3)}$

Solution

a. First write $f(x)$ as

$$f(x) = 3x\left(\frac{1}{1-\left(-x^2\right)}\right).$$

Using the power series representation for $f(x) = \frac{1}{1-x}$ and parts ii. and iii. of **Combining Power Series**, we find that a power series representation for f is given by

$$\sum_{n=0}^{\infty} 3x\left(-x^2\right)^n = \sum_{n=0}^{\infty} 3(-1)^n x^{2n+1}.$$

Since the interval of convergence of the series for $\frac{1}{1-x}$ is $(-1,\ 1)$, the interval of convergence for this new series is the set of real numbers x such that $\left|x^2\right| < 1.$ Therefore, the interval of convergence is $(-1,\ 1)$.

b. To find the power series representation, use partial fractions to write $f(x) = \frac{1}{(1-x)(x-3)}$ as the sum of two fractions. We have

$$\begin{aligned}\frac{1}{(x-1)(x-3)} &= \frac{-1/2}{x-1} + \frac{1/2}{x-3} \\ &= \frac{1/2}{1-x} - \frac{1/2}{3-x} \\ &= \frac{1/2}{1-x} - \frac{1/6}{1-\frac{x}{3}}.\end{aligned}$$

First, using part ii. of **Combining Power Series**, we obtain

$$\frac{1/2}{1-x} = \sum_{n=0}^{\infty} \frac{1}{2}x^n \text{ for } |x| < 1.$$

Then, using parts ii. and iii. of **Combining Power Series**, we have

$$\frac{1/6}{1-x/3} = \sum_{n=0}^{\infty} \frac{1}{6}\left(\frac{x}{3}\right)^n \text{ for } |x| < 3.$$

Since we are combining these two power series, the interval of convergence of the difference must be the smaller of these two intervals. Using this fact and part i. of **Combining Power Series**, we have

$$\frac{1}{(x-1)(x-3)} = \sum_{n=0}^{\infty} \left(\frac{1}{2} - \frac{1}{6 \cdot 3^n}\right)x^n$$

where the interval of convergence is $(-1,\ 1)$.

6.5 Use the series for $f(x) = \frac{1}{1-x}$ on $|x| < 1$ to construct a series for $\frac{1}{(1-x)(x-2)}$. Determine the interval of convergence.

In **Example 6.5**, we showed how to find power series for certain functions. In **Example 6.6** we show how to do the opposite: given a power series, determine which function it represents.

Example 6.6

Finding the Function Represented by a Given Power Series

Consider the power series $\sum_{n=0}^{\infty} 2^n x^n$. Find the function f represented by this series. Determine the interval of convergence of the series.

Solution

Writing the given series as

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n,$$

we can recognize this series as the power series for

$$f(x) = \frac{1}{1 - 2x}.$$

Since this is a geometric series, the series converges if and only if $|2x| < 1$. Therefore, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

6.6 Find the function represented by the power series $\sum_{n=0}^{\infty} \frac{1}{3^n} x^n$. Determine its interval of convergence.

Recall the questions posed in the chapter opener about which is the better way of receiving payouts from lottery winnings. We now revisit those questions and show how to use series to compare values of payments over time with a lump sum payment today. We will compute how much future payments are worth in terms of today's dollars, assuming we have the ability to invest winnings and earn interest. The value of future payments in terms of today's dollars is known as the *present value* of those payments.

Example 6.7

Chapter Opener: Present Value of Future Winnings

Figure 6.4 (credit: modification of work by Robert Huffstutter, Flickr)

Suppose you win the lottery and are given the following three options: (1) Receive 20 million dollars today; (2) receive 1.5 million dollars per year over the next 20 years; or (3) receive 1 million dollars per year indefinitely (being passed on to your heirs). Which is the best deal, assuming that the annual interest rate is 5%? We answer this by working through the following sequence of questions.

a. How much is the 1.5 million dollars received annually over the course of 20 years worth in terms of today's dollars, assuming an annual interest rate of 5%?

b. Use the answer to part a. to find a general formula for the present value of payments of C dollars received each year over the next n years, assuming an average annual interest rate r.

c. Find a formula for the present value if annual payments of C dollars continue indefinitely, assuming an average annual interest rate r.

d. Use the answer to part c. to determine the present value of 1 million dollars paid annually indefinitely.

e. Use your answers to parts a. and d. to determine which of the three options is best.

Solution

a. Consider the payment of 1.5 million dollars made at the end of the first year. If you were able to receive that payment today instead of one year from now, you could invest that money and earn 5% interest. Therefore, the present value of that money P_1 satisfies $P_1(1+0.05) = 1.5$ million dollars. We conclude that

$$P_1 = \frac{1.5}{1.05} = \$1.429 \text{ million dollars.}$$

Similarly, consider the payment of 1.5 million dollars made at the end of the second year. If you were able to receive that payment today, you could invest that money for two years, earning 5% interest, compounded annually. Therefore, the present value of that money P_2 satisfies $P_2(1+0.05)^2 = 1.5$ million dollars. We conclude that

$$P_2 = \frac{1.5}{(1.05)^2} = \$1.361 \text{ million dollars.}$$

The value of the future payments today is the sum of the present values $P_1, P_2, \ldots, P_{20}$ of each of those annual payments. The present value P_k satisfies

$$P_k = \frac{1.5}{(1.05)^k}.$$

Therefore,

$$\begin{aligned} P &= \frac{1.5}{1.05} + \frac{1.5}{(1.05)^2} + \cdots + \frac{1.5}{(1.05)^{20}} \\ &= \$18.693 \text{ million dollars.} \end{aligned}$$

b. Using the result from part a. we see that the present value P of C dollars paid annually over the course of n years, assuming an annual interest rate r, is given by

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^n} \text{ dollars.}$$

c. Using the result from part b. we see that the present value of an annuity that continues indefinitely is given by the infinite series

$$P = \sum_{n=0}^{\infty} \frac{C}{(1+r)^{n+1}}.$$

We can view the present value as a power series in r, which converges as long as $\left|\frac{1}{1+r}\right| < 1$. Since $r > 0$, this series converges. Rewriting the series as

$$P = \frac{C}{(1+r)} \sum_{n=0}^{\infty} \left(\frac{1}{1+r}\right)^n,$$

we recognize this series as the power series for

$$f(r) = \frac{1}{1 - \left(\frac{1}{1+r}\right)} = \frac{1}{\left(\frac{r}{1+r}\right)} = \frac{1+r}{r}.$$

We conclude that the present value of this annuity is

$$P = \frac{C}{1+r} \cdot \frac{1+r}{r} = \frac{C}{r}.$$

d. From the result to part c. we conclude that the present value P of $C = 1$ million dollars paid out every year indefinitely, assuming an annual interest rate $r = 0.05$, is given by

$$P = \frac{1}{0.05} = 20 \text{ million dollars.}$$

e. From part a. we see that receiving \$1.5 million dollars over the course of 20 years is worth \$18.693 million dollars in today's dollars. From part d. we see that receiving \$1 million dollars per year indefinitely is worth \$20 million dollars in today's dollars. Therefore, either receiving a lump-sum payment of \$20 million dollars today or receiving \$1 million dollars indefinitely have the same present value.

Multiplication of Power Series

We can also create new power series by multiplying power series. Being able to multiply two power series provides another way of finding power series representations for functions.

The way we multiply them is similar to how we multiply polynomials. For example, suppose we want to multiply

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

and

$$\sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + d_2 x^2 + \cdots.$$

It appears that the product should satisfy

$$\begin{aligned} \left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=-0}^{\infty} d_n x^n\right) &= \left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \cdot \left(d_0 + d_1 x + d_2 x^2 + \cdots\right) \\ &= c_0 d_0 + (c_1 d_0 + c_0 d_1)x + (c_2 d_0 + c_1 d_1 + c_0 d_2)x^2 + \cdots. \end{aligned}$$

In **Multiplying Power Series**, we state the main result regarding multiplying power series, showing that if $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge on a common interval I, then we can multiply the series in this way, and the resulting series also converges on the interval I.

Theorem 6.3: Multiplying Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I. Let

$$\begin{aligned} e_n &= c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \cdots + c_{n-1} d_1 + c_n d_0 \\ &= \sum_{k=0}^{n} c_k d_{n-k}. \end{aligned}$$

Then

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$

and

$$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x) \cdot g(x) \text{ on } I.$$

The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the Cauchy product of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.

We omit the proof of this theorem, as it is beyond the level of this text and is typically covered in a more advanced course. We now provide an example of this theorem by finding the power series representation for

$$f(x) = \frac{1}{(1-x)\left(1-x^2\right)}$$

using the power series representations for

$$y = \frac{1}{1-x} \text{ and } y = \frac{1}{1-x^2}.$$

Example 6.8

Multiplying Power Series

Multiply the power series representation

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

for $|x| < 1$ with the power series representation

$$\begin{aligned} \frac{1}{1-x^2} &= \sum_{n=0}^{\infty} \left(x^2\right)^n \\ &= 1 + x^2 + x^4 + x^6 + \cdots \end{aligned}$$

for $|x| < 1$ to construct a power series for $f(x) = \dfrac{1}{(1-x)\left(1-x^2\right)}$ on the interval $(-1, 1)$.

Solution

We need to multiply

$$\left(1+x+x^2+x^3+\cdots\right)\left(1+x^2+x^4+x^6+\cdots\right).$$

Writing out the first several terms, we see that the product is given by

$$\begin{aligned}&\left(1+x^2+x^4+x^6+\cdots\right)+\left(x+x^3+x^5+x^7+\cdots\right)+\left(x^2+x^4+x^6+x^8+\cdots\right)+\left(x^3+x^5+x^7+x^9+\cdots\right)\\&=1+x+(1+1)x^2+(1+1)x^3+(1+1+1)x^4+(1+1+1)x^5+\cdots\\&=1+x+2x^2+2x^3+3x^4+3x^5+\cdots.\end{aligned}$$

Since the series for $y=\dfrac{1}{1-x}$ and $y=\dfrac{1}{1-x^2}$ both converge on the interval $(-1,\ 1)$, the series for the product also converges on the interval $(-1,\ 1)$.

6.7 Multiply the series $\dfrac{1}{1-x}=\displaystyle\sum_{n=0}^{\infty}x^n$ by itself to construct a series for $\dfrac{1}{(1-x)(1-x)}$.

Differentiating and Integrating Power Series

Consider a power series $\displaystyle\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+\cdots$ that converges on some interval I, and let f be the function defined by this series. Here we address two questions about f.

- Is f differentiable, and if so, how do we determine the derivative f'?
- How do we evaluate the indefinite integral $\int f(x)\,dx$?

We know that, for a polynomial with a finite number of terms, we can evaluate the derivative by differentiating each term separately. Similarly, we can evaluate the indefinite integral by integrating each term separately. Here we show that we can do the same thing for convergent power series. That is, if

$$f(x)=c_nx^n=c_0+c_1x+c_2x^2+\cdots$$

converges on some interval I, then

$$f'(x)=c_1+2c_2x+3c_3x^2+\cdots$$

and

$$\int f(x)\,dx=C+c_0x+c_1\frac{x^2}{2}+c_2\frac{x^3}{3}+\cdots.$$

Evaluating the derivative and indefinite integral in this way is called **term-by-term differentiation of a power series** and **term-by-term integration of a power series**, respectively. The ability to differentiate and integrate power series term-by-term also allows us to use known power series representations to find power series representations for other functions. For example, given the power series for $f(x)=\dfrac{1}{1-x}$, we can differentiate term-by-term to find the power series for $f'(x)=\dfrac{1}{(1-x)^2}$. Similarly, using the power series for $g(x)=\dfrac{1}{1+x}$, we can integrate term-by-term to find the power series for $G(x)=\ln(1+x)$, an antiderivative of g. We show how to do this in **Example 6.9** and **Example 6.10**. First, we state **Term-by-Term Differentiation and Integration for Power Series**, which provides the main result regarding differentiation and integration of power series.

Theorem 6.4: Term-by-Term Differentiation and Integration for Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges on the interval $(a-R,\ a+R)$ for some $R > 0$. Let f be the function defined by the series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \\ &= c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots \end{aligned}$$

for $|x-a| < R$. Then f is differentiable on the interval $(a-R,\ a+R)$ and we can find f' by differentiating the series term-by-term:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} nc_n (x-a)^{n-1} \\ &= c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + \cdots \end{aligned}$$

for $|x-a| < R$. Also, to find $\int f(x)dx,$ we can integrate the series term-by-term. The resulting series converges on $(a-R,\ a+R),$ and we have

$$\begin{aligned} \int f(x)dx &= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \\ &= C + c_0 (x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots \end{aligned}$$

for $|x-a| < R$.

The proof of this result is beyond the scope of the text and is omitted. Note that although **Term-by-Term Differentiation and Integration for Power Series** guarantees the same radius of convergence when a power series is differentiated or integrated term-by-term, it says nothing about what happens at the endpoints. It is possible that the differentiated and integrated power series have different behavior at the endpoints than does the original series. We see this behavior in the next examples.

Example 6.9

Differentiating Power Series

a. Use the power series representation

$$\begin{aligned} f(x) &= \frac{1}{1-x} \\ &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

for $|x| < 1$ to find a power series representation for

$$g(x) = \frac{1}{(1-x)^2}$$

on the interval $(-1,\ 1)$. Determine whether the resulting series converges at the endpoints.

b. Use the result of part a. to evaluate the sum of the series $\sum_{n=0}^{\infty} \frac{n+1}{4^n}$.

Solution

a. Since $g(x) = \frac{1}{(1-x)^2}$ is the derivative of $f(x) = \frac{1}{1-x}$, we can find a power series representation for g by differentiating the power series for f term-by-term. The result is

$$\begin{aligned} g(x) &= \frac{1}{(1-x)^2} \\ &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ &= \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) \\ &= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right) \\ &= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1)x^n \end{aligned}$$

for $|x| < 1$. **Term-by-Term Differentiation and Integration for Power Series** does not guarantee anything about the behavior of this series at the endpoints. Testing the endpoints by using the divergence test, we find that the series diverges at both endpoints $x = \pm 1$. Note that this is the same result found in **Example 6.8**.

b. From part a. we know that

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{n+1}{4^n} &= \sum_{n=0}^{\infty} (n+1)\left(\frac{1}{4}\right)^n \\ &= \frac{1}{\left(1 - \frac{1}{4}\right)^2} \\ &= \frac{1}{\left(\frac{3}{4}\right)^2} \\ &= \frac{16}{9}. \end{aligned}$$

6.8 Differentiate the series $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$ term-by-term to find a power series representation for $\frac{2}{(1-x)^3}$ on the interval $(-1, 1)$.

Example 6.10

Integrating Power Series

For each of the following functions f, find a power series representation for f by integrating the power series for f' and find its interval of convergence.

a. $f(x) = \ln(1 + x)$

b. $f(x) = \tan^{-1} x$

Solution

a. For $f(x) = \ln(1 + x)$, the derivative is $f'(x) = \frac{1}{1 + x}$. We know that

$$\begin{aligned} \frac{1}{1 + x} &= \frac{1}{1 - (-x)} \\ &= \sum_{n=0}^{\infty} (-x)^n \\ &= 1 - x + x^2 - x^3 + \cdots \end{aligned}$$

for $|x| < 1$. To find a power series for $f(x) = \ln(1 + x)$, we integrate the series term-by-term.

$$\begin{aligned} \int f'(x)dx &= \int \left(1 - x + x^2 - x^3 + \cdots\right)dx \\ &= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \end{aligned}$$

Since $f(x) = \ln(1 + x)$ is an antiderivative of $\frac{1}{1 + x}$, it remains to solve for the constant C. Since $\ln(1 + 0) = 0$, we have $C = 0$. Therefore, a power series representation for $f(x) = \ln(1 + x)$ is

$$\begin{aligned} \ln(1 + x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \end{aligned}$$

for $|x| < 1$. **Term-by-Term Differentiation and Integration for Power Series** does not guarantee anything about the behavior of this power series at the endpoints. However, checking the endpoints, we find that at $x = 1$ the series is the alternating harmonic series, which converges. Also, at $x = -1$, the series is the harmonic series, which diverges. It is important to note that, even though this series converges at $x = 1$, **Term-by-Term Differentiation and Integration for Power Series** does not guarantee that the series actually converges to $\ln(2)$. In fact, the series does converge to $\ln(2)$, but showing this fact requires more advanced techniques. (Abel's theorem, covered in more advanced texts, deals with this more technical point.) The interval of convergence is $(-1, 1]$.

b. The derivative of $f(x) = \tan^{-1} x$ is $f'(x) = \frac{1}{1 + x^2}$. We know that

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-\left(-x^2\right)} \\ &= \sum_{n=0}^{\infty}\left(-x^2\right)^n \\ &= 1 - x^2 + x^4 - x^6 + \cdots\end{aligned}$$

for $|x| < 1$. To find a power series for $f(x) = \tan^{-1} x$, we integrate this series term-by-term.

$$\begin{aligned}\int f'(x)dx &= \int\left(1 - x^2 + x^4 - x^6 + \cdots\right)dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\end{aligned}$$

Since $\tan^{-1}(0) = 0$, we have $C = 0$. Therefore, a power series representation for $f(x) = \tan^{-1} x$ is

$$\begin{aligned}\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}\end{aligned}$$

for $|x| < 1$. Again, **Term-by-Term Differentiation and Integration for Power Series** does not guarantee anything about the convergence of this series at the endpoints. However, checking the endpoints and using the alternating series test, we find that the series converges at $x = 1$ and $x = -1$. As discussed in part a., using Abel's theorem, it can be shown that the series actually converges to $\tan^{-1}(1)$ and $\tan^{-1}(-1)$ at $x = 1$ and $x = -1$, respectively. Thus, the interval of convergence is $[-1, 1]$.

6.9 Integrate the power series $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ term-by-term to evaluate $\int \ln(1+x)dx$.

Up to this point, we have shown several techniques for finding power series representations for functions. However, how do we know that these power series are unique? That is, given a function *f* and a power series for *f* at *a*, is it possible that there is a different power series for *f* at *a* that we could have found if we had used a different technique? The answer to this question is no. This fact should not seem surprising if we think of power series as polynomials with an infinite number of terms. Intuitively, if

$$c_0 + c_1 x + c_2 x^2 + \cdots = d_0 + d_1 x + d_2 x^2 + \cdots$$

for all values *x* in some open interval *I* about zero, then the coefficients c_n should equal d_n for $n \geq 0$. We now state this result formally in **Uniqueness of Power Series**.

Theorem 6.5: Uniqueness of Power Series

Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ and $\sum_{n=0}^{\infty} d_n(x-a)^n$ be two convergent power series such that

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} d_n(x-a)^n$$

for all x in an open interval containing a. Then $c_n = d_n$ for all $n \geq 0$.

Proof

Let

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \\ &= d_0 + d_1(x-a) + d_2(x-a)^2 + d_3(x-a)^3 + \cdots. \end{aligned}$$

Then $f(a) = c_0 = d_0$. By **Term-by-Term Differentiation and Integration for Power Series**, we can differentiate both series term-by-term. Therefore,

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \\ &= d_1 + 2d_2(x-a) + 3d_3(x-a)^2 + \cdots, \end{aligned}$$

and thus, $f'(a) = c_1 = d_1$. Similarly,

$$\begin{aligned} f''(x) &= 2c_2 + 3 \cdot 2c_3(x-a) + \cdots \\ &= 2d_2 + 3 \cdot 2d_3(x-a) + \cdots \end{aligned}$$

implies that $f''(a) = 2c_2 = 2d_2$, and therefore, $c_2 = d_2$. More generally, for any integer $n \geq 0$, $f^{(n)}(a) = n!c_n = n!d_n$, and consequently, $c_n = d_n$ for all $n \geq 0$.

□

In this section we have shown how to find power series representations for certain functions using various algebraic operations, differentiation, or integration. At this point, however, we are still limited as to the functions for which we can find power series representations. Next, we show how to find power series representations for many more functions by introducing Taylor series.

6.2 EXERCISES

63. If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$, find the power series of $\frac{1}{2}(f(x) + g(x))$ and of $\frac{1}{2}(f(x) - g(x))$.

64. If $C(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ and $S(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$, find the power series of $C(x) + S(x)$ and of $C(x) - S(x)$.

In the following exercises, use partial fractions to find the power series of each function.

65. $\frac{4}{(x-3)(x+1)}$

66. $\frac{3}{(x+2)(x-1)}$

67. $\frac{5}{(x^2+4)(x^2-1)}$

68. $\frac{30}{(x^2+1)(x^2-9)}$

In the following exercises, express each series as a rational function.

69. $\sum_{n=1}^{\infty} \frac{1}{x^n}$

70. $\sum_{n=1}^{\infty} \frac{1}{x^{2n}}$

71. $\sum_{n=1}^{\infty} \frac{1}{(x-3)^{2n-1}}$

72. $\sum_{n=1}^{\infty} \left(\frac{1}{(x-3)^{2n-1}} - \frac{1}{(x-2)^{2n-1}} \right)$

The following exercises explore applications of annuities.

73. Calculate the present values P of an annuity in which \$10,000 is to be paid out annually for a period of 20 years, assuming interest rates of $r = 0.03$, $r = 0.05$, and $r = 0.07$.

74. Calculate the present values P of annuities in which \$9,000 is to be paid out annually perpetually, assuming interest rates of $r = 0.03$, $r = 0.05$ and $r = 0.07$.

75. Calculate the annual payouts C to be given for 20 years on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03$, $r = 0.05$, and $r = 0.07$.

76. Calculate the annual payouts C to be given perpetually on annuities having present value \$100,000 assuming respective interest rates of $r = 0.03$, $r = 0.05$, and $r = 0.07$.

77. Suppose that an annuity has a present value $P = 1$ million dollars. What interest rate r would allow for perpetual annual payouts of \$50,000?

78. Suppose that an annuity has a present value $P = 10$ million dollars. What interest rate r would allow for perpetual annual payouts of \$100,000?

In the following exercises, express the sum of each power series in terms of geometric series, and then express the sum as a rational function.

79. $x + x^2 - x^3 + x^4 + x^5 - x^6 + \cdots$ (*Hint:* Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

80. $x + x^2 - x^3 - x^4 + x^5 + x^6 - x^7 - x^8 + \cdots$ (*Hint:* Group powers x^{4k}, x^{4k-1}, etc.)

81. $x - x^2 - x^3 + x^4 - x^5 - x^6 + x^7 - \cdots$ (*Hint:* Group powers x^{3k}, x^{3k-1}, and x^{3k-2}.)

82. $\frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16} + \frac{x^5}{32} - \frac{x^6}{64} + \cdots$ (*Hint:* Group powers $\left(\frac{x}{2}\right)^{3k}$, $\left(\frac{x}{2}\right)^{3k-1}$, and $\left(\frac{x}{2}\right)^{3k-2}$.)

In the following exercises, find the power series of $f(x)g(x)$ given f and g as defined.

83. $f(x) = 2\sum_{n=0}^{\infty} x^n$, $g(x) = \sum_{n=0}^{\infty} nx^n$

84. $f(x) = \sum_{n=1}^{\infty} x^n$, $g(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$. Express the coefficients of $f(x)g(x)$ in terms of $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

85. $f(x) = g(x) = \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$

86. $f(x) = g(x) = \sum_{n=1}^{\infty} nx^n$

In the following exercises, differentiate the given series expansion of f term-by-term to obtain the corresponding series expansion for the derivative of f.

87. $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

88. $f(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$

In the following exercises, integrate the given series expansion of f term-by-term from zero to x to obtain the corresponding series expansion for the indefinite integral of f.

89. $f(x) = \frac{2x}{\left(1+x^2\right)^2} = \sum_{n=1}^{\infty} (-1)^n (2n) x^{2n-1}$

90. $f(x) = \frac{2x}{1+x^2} = 2\sum_{n=0}^{\infty} (-1)^n x^{2n+1}$

In the following exercises, evaluate each infinite series by identifying it as the value of a derivative or integral of geometric series.

91. Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$ as $f'\left(\frac{1}{2}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

92. Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ as $f'\left(\frac{1}{3}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

93. Evaluate $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$ as $f''\left(\frac{1}{2}\right)$ where $f(x) = \sum_{n=0}^{\infty} x^n$.

94. Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ as $\int_0^1 f(t)\,dt$ where $f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$.

In the following exercises, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, use term-by-term differentiation or integration to find power series for each function centered at the given point.

95. $f(x) = \ln x$ centered at $x = 1$ (*Hint:* $x = 1 - (1-x)$)

96. $\ln(1-x)$ at $x = 0$

97. $\ln\left(1-x^2\right)$ at $x = 0$

98. $f(x) = \frac{2x}{\left(1-x^2\right)^2}$ at $x = 0$

99. $f(x) = \tan^{-1}\left(x^2\right)$ at $x = 0$

100. $f(x) = \ln\left(1+x^2\right)$ at $x = 0$

101. $f(x) = \int_0^x \ln t\,dt$ where $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$

102. **[T]** Evaluate the power series expansion $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ at $x = 1$ to show that $\ln(2)$ is the sum of the alternating harmonic series. Use the alternating series test to determine how many terms of the sum are needed to estimate $\ln(2)$ accurate to within 0.001, and find such an approximation.

103. **[T]** Subtract the infinite series of $\ln(1-x)$ from $\ln(1+x)$ to get a power series for $\ln\left(\frac{1+x}{1-x}\right)$. Evaluate at $x = \frac{1}{3}$. What is the smallest N such that the Nth partial sum of this series approximates $\ln(2)$ with an error less than 0.001?

In the following exercises, using a substitution if indicated, express each series in terms of elementary functions and find the radius of convergence of the sum.

104. $\sum_{k=0}^{\infty} \left(x^k - x^{2k+1}\right)$

105. $\sum_{k=1}^{\infty} \frac{x^{3k}}{6k}$

106. $\sum_{k=1}^{\infty} \left(1+x^2\right)^{-k}$ using $y = \frac{1}{1+x^2}$

107. $\sum_{k=1}^{\infty} 2^{-kx}$ using $y = 2^{-x}$

108. Show that, up to powers x^3 and y^3, $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ satisfies $E(x+y) = E(x)E(y)$.

109. Differentiate the series $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ term-by-term to show that $E(x)$ is equal to its derivative.

110. Show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a sum of even powers, that is, $a_n = 0$ if n is odd, then $F = \int_0^x f(t)\,dt$ is a sum of odd powers, while if f is a sum of odd powers, then F is a sum of even powers.

111. **[T]** Suppose that the coefficients a_n of the series $\sum_{n=0}^{\infty} a_n x^n$ are defined by the recurrence relation $a_n = \frac{a_{n-1}}{n} + \frac{a_{n-2}}{n(n-1)}$. For $a_0 = 0$ and $a_1 = 1$, compute and plot the sums $S_N = \sum_{n=0}^{N} a_n x^n$ for $N = 2, 3, 4, 5$ on $[-1, 1]$.

112. **[T]** Suppose that the coefficients a_n of the series $\sum_{n=0}^{\infty} a_n x^n$ are defined by the recurrence relation $a_n = \frac{a_{n-1}}{\sqrt{n}} - \frac{a_{n-2}}{\sqrt{n(n-1)}}$. For $a_0 = 1$ and $a_1 = 0$, compute and plot the sums $S_N = \sum_{n=0}^{N} a_n x^n$ for $N = 2, 3, 4, 5$ on $[-1, 1]$.

113. **[T]** Given the power series expansion $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$, determine how many terms N of the sum evaluated at $x = -1/2$ are needed to approximate $\ln(2)$ accurate to within 1/1000. Evaluate the corresponding partial sum $\sum_{n=1}^{N} (-1)^{n-1} \frac{x^n}{n}$.

114. **[T]** Given the power series expansion $\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$, use the alternating series test to determine how many terms N of the sum evaluated at $x = 1$ are needed to approximate $\tan^{-1}(1) = \frac{\pi}{4}$ accurate to within 1/1000. Evaluate the corresponding partial sum $\sum_{k=0}^{N} (-1)^k \frac{x^{2k+1}}{2k+1}$.

115. **[T]** Recall that $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. Assuming an exact value of $\left(\frac{1}{\sqrt{3}}\right)$, estimate $\frac{\pi}{6}$ by evaluating partial sums $S_N\left(\frac{1}{\sqrt{3}}\right)$ of the power series expansion $\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ at $x = \frac{1}{\sqrt{3}}$. What is the smallest number N such that $6S_N\left(\frac{1}{\sqrt{3}}\right)$ approximates π accurately to within 0.001? How many terms are needed for accuracy to within 0.00001?

6.3 | Taylor and Maclaurin Series

Learning Objectives

6.3.1 Describe the procedure for finding a Taylor polynomial of a given order for a function.
6.3.2 Explain the meaning and significance of Taylor's theorem with remainder.
6.3.3 Estimate the remainder for a Taylor series approximation of a given function.

In the previous two sections we discussed how to find power series representations for certain types of functions—specifically, functions related to geometric series. Here we discuss power series representations for other types of functions. In particular, we address the following questions: Which functions can be represented by power series and how do we find such representations? If we can find a power series representation for a particular function f and the series converges on some interval, how do we prove that the series actually converges to f?

Overview of Taylor/Maclaurin Series

Consider a function f that has a power series representation at $x = a$. Then the series has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots. \tag{6.4}$$

What should the coefficients be? For now, we ignore issues of convergence, but instead focus on what the series should be, if one exists. We return to discuss convergence later in this section. If the series **Equation 6.4** is a representation for f at $x = a$, we certainly want the series to equal $f(a)$ at $x = a$. Evaluating the series at $x = a$, we see that

$$\begin{aligned}\sum_{n=0}^{\infty} c_n (x-a)^n &= c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots \\ &= c_0.\end{aligned}$$

Thus, the series equals $f(a)$ if the coefficient $c_0 = f(a)$. In addition, we would like the first derivative of the power series to equal $f'(a)$ at $x = a$. Differentiating **Equation 6.4** term-by-term, we see that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots.$$

Therefore, at $x = a$, the derivative is

$$\begin{aligned}\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) &= c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots \\ &= c_1.\end{aligned}$$

Therefore, the derivative of the series equals $f'(a)$ if the coefficient $c_1 = f'(a)$. Continuing in this way, we look for coefficients c_n such that all the derivatives of the power series **Equation 6.4** will agree with all the corresponding derivatives of f at $x = a$. The second and third derivatives of **Equation 6.4** are given by

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a)^2 + \cdots$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + 5 \cdot 4 \cdot 3c_5(x-a)^2 + \cdots.$$

Therefore, at $x = a$, the second and third derivatives

$$\begin{aligned}\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) &= 2c_2 + 3\cdot 2c_3 (a-a) + 4\cdot 3c_4 (a-a)^2 + \cdots \\ &= 2c_2\end{aligned}$$

and

$$\begin{aligned}\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) &= 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4 (a-a) + 5\cdot 4\cdot 3c_5 (a-a)^2 + \cdots \\ &= 3\cdot 2c_3\end{aligned}$$

equal $f''(a)$ and $f'''(a)$, respectively, if $c_2 = \frac{f''(a)}{2}$ and $c_3 = \frac{f'''(a)}{3}\cdot 2.$ More generally, we see that if f has a power series representation at $x = a$, then the coefficients should be given by $c_n = \frac{f^{(n)}(a)}{n!}.$ That is, the series should be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

This power series for f is known as the Taylor series for f at a. If $x = 0$, then this series is known as the Maclaurin series for f.

Definition

If f has derivatives of all orders at $x = a$, then the **Taylor series** for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \tag{6.5}$$

The Taylor series for f at 0 is known as the **Maclaurin series** for f.

Later in this section, we will show examples of finding Taylor series and discuss conditions under which the Taylor series for a function will converge to that function. Here, we state an important result. Recall from **Uniqueness of Power Series** that power series representations are unique. Therefore, if a function f has a power series at a, then it must be the Taylor series for f at a.

Theorem 6.6: Uniqueness of Taylor Series

If a function f has a power series at *a* that converges to f on some open interval containing *a*, then that power series is the Taylor series for f at *a*.

The proof follows directly from **Uniqueness of Power Series**.

To determine if a Taylor series converges, we need to look at its sequence of partial sums. These partial sums are finite polynomials, known as **Taylor polynomials**.

Visit the MacTutor History of Mathematics archive to read brief biographies of **Brook Taylor (http://www.openstaxcollege.org/l/20_BTaylor)** and **Colin Maclaurin (http://www.openstaxcollege.org/l/20_CMaclaurin)** and how they developed the concepts named after them.

Taylor Polynomials

The *n*th partial sum of the Taylor series for a function f at a is known as the *n*th Taylor polynomial. For example, the 0th, 1st, 2nd, and 3rd partial sums of the Taylor series are given by

$$\begin{aligned}
p_0(x) &= f(a), \\
p_1(x) &= f(a) + f'(a)(x-a), \\
p_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2, \\
p_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3,
\end{aligned}$$

respectively. These partial sums are known as the 0th, 1st, 2nd, and 3rd Taylor polynomials of f at a, respectively. If $x = a$, then these polynomials are known as **Maclaurin polynomials** for f. We now provide a formal definition of Taylor and Maclaurin polynomials for a function f.

Definition

If f has n derivatives at $x = a$, then the nth Taylor polynomial for f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth Taylor polynomial for f at 0 is known as the nth Maclaurin polynomial for f.

We now show how to use this definition to find several Taylor polynomials for $f(x) = \ln x$ at $x = 1$.

Example 6.11

Finding Taylor Polynomials

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \ln x$ at $x = 1$. Use a graphing utility to compare the graph of f with the graphs of p_0, p_1, p_2 and p_3.

Solution

To find these Taylor polynomials, we need to evaluate f and its first three derivatives at $x = 1$.

$$\begin{array}{rclrcl}
f(x) &=& \ln x & f(1) &=& 0 \\
f'(x) &=& \frac{1}{x} & f'(1) &=& 1 \\
f''(x) &=& -\frac{1}{x^2} & f''(1) &=& -1 \\
f'''(x) &=& \frac{2}{x^3} & f'''(1) &=& 2
\end{array}$$

Therefore,

$$\begin{aligned}
p_0(x) &= f(1) = 0, \\
p_1(x) &= f(1) + f'(1)(x-1) = x - 1, \\
p_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2, \\
p_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3.
\end{aligned}$$

The graphs of $y = f(x)$ and the first three Taylor polynomials are shown in **Figure 6.5**.

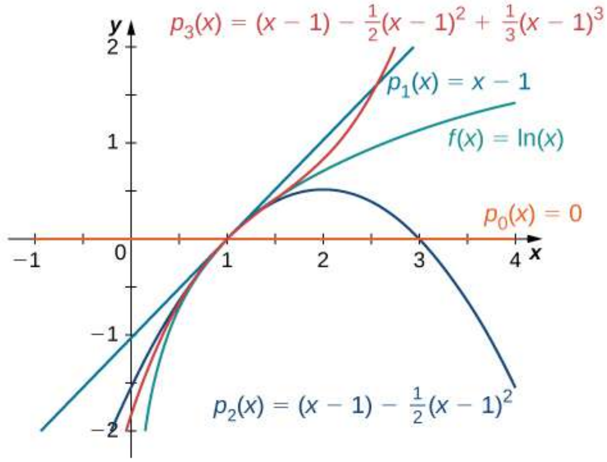


Figure 6.5 The function $y = \ln x$ and the Taylor polynomials $p_0,\ p_1,\ p_2$ and p_3 at $x = 1$ are plotted on this graph.

6.10 Find the Taylor polynomials $p_0,\ p_1,\ p_2$ and p_3 for $f(x) = \dfrac{1}{x^2}$ at $x = 1.$

We now show how to find Maclaurin polynomials for $e^x,\ \sin x,$ and $\cos x.$ As stated above, Maclaurin polynomials are Taylor polynomials centered at zero.

Example 6.12

Finding Maclaurin Polynomials

For each of the following functions, find formulas for the Maclaurin polynomials $p_0,\ p_1,\ p_2$ and $p_3.$ Find a formula for the *n*th Maclaurin polynomial and write it using sigma notation. Use a graphing utilty to compare the graphs of $p_0,\ p_1,\ p_2$ and p_3 with $f.$

a. $f(x) = e^x$

b. $f(x) = \sin x$

c. $f(x) = \cos x$

Solution

a. Since $f(x) = e^x,$ we know that $f(x) = f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$ for all positive integers *n*. Therefore,

$$f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$$

for all positive integers n. Therefore, we have

$$\begin{aligned}
p_0(x) &= f(0) = 1, \\
p_1(x) &= f(0) + f'(0)x = 1 + x, \\
p_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2, \\
p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3, \\
p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \\
&= \sum_{k=0}^{n} \frac{x^k}{k!}.
\end{aligned}$$

The function and the first three Maclaurin polynomials are shown in **Figure 6.6**.

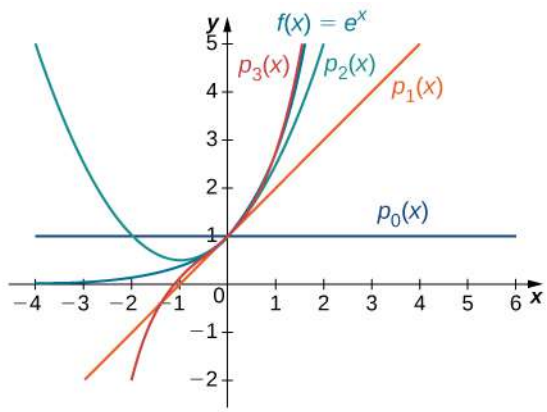


Figure 6.6 The graph shows the function $y = e^x$ and the Maclaurin polynomials p_0, p_1, p_2 and p_3.

b. For $f(x) = \sin x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0.
\end{aligned}$$

Since the fourth derivative is $\sin x$, the pattern repeats. That is, $f^{(2m)}(0) = 0$ and $f^{(2m+1)}(0) = (-1)^m$ for $m \geq 0$. Thus, we have

$$\begin{aligned}
&p_0(x) = 0,\\
&p_1(x) = 0 + x = x,\\
&p_2(x) = 0 + x + 0 = x,\\
&p_3(x) = 0 + x + 0 - \frac{1}{3!}x^3 = x - \frac{x^3}{3!},\\
&p_4(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 = x - \frac{x^3}{3!},\\
&p_5(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!},
\end{aligned}$$

and for $m \geq 0$,

$$\begin{aligned}
p_{2m+1}(x) &= p_{2m+2}(x)\\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}\\
&= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.
\end{aligned}$$

Graphs of the function and its Maclaurin polynomials are shown in **Figure 6.7**.

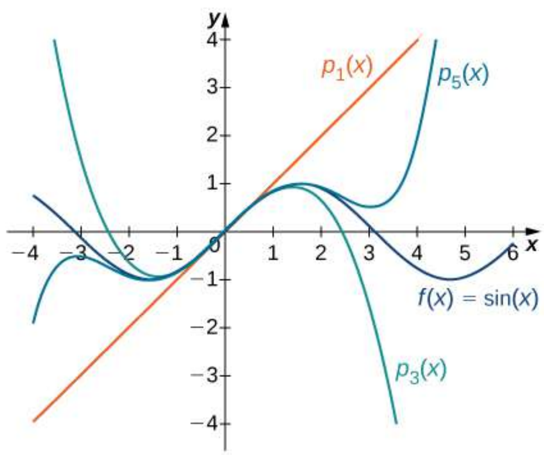


Figure 6.7 The graph shows the function $y = \sin x$ and the Maclaurin polynomials p_1, p_3 and p_5.

c. For $f(x) = \cos x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$\begin{array}{rclrcl}
f(x) &=& \cos x & f(0) &=& 1\\
f'(x) &=& -\sin x & f'(0) &=& 0\\
f''(x) &=& -\cos x & f''(0) &=& -1\\
f'''(x) &=& \sin x & f'''(0) &=& 0\\
f^{(4)}(x) &=& \cos x & f^{(4)}(0) &=& 1.
\end{array}$$

Since the fourth derivative is $\sin x$, the pattern repeats. In other words, $f^{(2m)}(0) = (-1)^m$ and

$f^{(2m+1)} = 0$ for $m \geq 0$. Therefore,

$$\begin{aligned}
p_0(x) &= 1, \\
p_1(x) &= 1 + 0 = 1, \\
p_2(x) &= 1 + 0 - \frac{1}{2!}x^2 = 1 - \frac{x^2}{2!}, \\
p_3(x) &= 1 + 0 - \frac{1}{2!}x^2 + 0 = 1 - \frac{x^2}{2!}, \\
p_4(x) &= 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}, \\
p_5(x) &= 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},
\end{aligned}$$

and for $n \geq 0$,

$$\begin{aligned}
p_{2m}(x) &= p_{2m+1}(x) \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} \\
&= \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!}.
\end{aligned}$$

Graphs of the function and the Maclaurin polynomials appear in **Figure 6.8**.

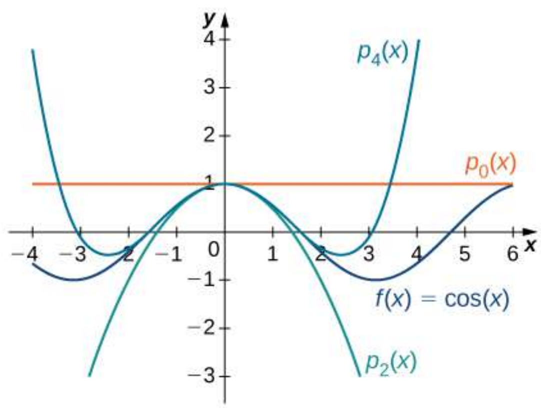


Figure 6.8 The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2 and p_4 are plotted on this graph.

6.11 Find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3 for $f(x) = \frac{1}{1+x}$. Find a formula for the *n*th Maclaurin polynomial. Write your anwer using sigma notation.

Taylor's Theorem with Remainder

Recall that the *n*th Taylor polynomial for a function f at *a* is the *n*th partial sum of the Taylor series for f at *a*. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials $\{p_n\}$ converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if it converges to f. To answer this question, we define the remainder $R_n(x)$ as

$$R_n(x) = f(x) - p_n(x).$$

For the sequence of Taylor polynomials to converge to f, we need the remainder R_n to converge to zero. To determine if R_n converges to zero, we introduce **Taylor's theorem with remainder**. Not only is this theorem useful in proving that a Taylor series converges to its related function, but it will also allow us to quantify how well the nth Taylor polynomial approximates the function.

Here we look for a bound on $|R_n|$. Consider the simplest case: $n = 0$. Let p_0 be the 0th Taylor polynomial at a for a function f. The remainder R_0 satisfies

$$\begin{aligned} R_0(x) &= f(x) - p_0(x) \\ &= f(x) - f(a). \end{aligned}$$

If f is differentiable on an interval I containing a and x, then by the Mean Value Theorem there exists a real number c between a and x such that $f(x) - f(a) = f'(c)(x - a)$. Therefore,

$$R_0(x) = f'(c)(x - a).$$

Using the Mean Value Theorem in a similar argument, we can show that if f is n times differentiable on an interval I containing a and x, then the nth remainder R_n satisfies

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$$

for some real number c between a and x. It is important to note that the value c in the numerator above is not the center a, but rather an unknown value c between a and x. This formula allows us to get a bound on the remainder R_n. If we happen to know that $\left|f^{(n+1)}(x)\right|$ is bounded by some real number M on this interval I, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x - a|^{n+1}$$

for all x in the interval I.

We now state Taylor's theorem, which provides the formal relationship between a function f and its nth degree Taylor polynomial $p_n(x)$. This theorem allows us to bound the error when using a Taylor polynomial to approximate a function value, and will be important in proving that a Taylor series for f converges to f.

Theorem 6.7: Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n + 1$ times on an interval I containing the real number a. Let p_n be the nth Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x)$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}.$$

If there exists a real number M such that $\left|f^{(n+1)}(x)\right| \le M$ for all $x \in I$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x - a|^{n+1}$$

for all x in I.

Proof

Fix a point $x \in I$ and introduce the function g such that

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}.$$

We claim that g satisfies the criteria of Rolle's theorem. Since g is a polynomial function (in t), it is a differentiable function. Also, g is zero at $t = a$ and $t = x$ because

$$\begin{aligned} g(a) &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n - R_n(x) \\ &= f(x) - p_n(x) - R_n(x) \\ &= 0, \\ g(x) &= f(x) - f(x) - 0 - \cdots - 0 \\ &= 0. \end{aligned}$$

Therefore, g satisfies Rolle's theorem, and consequently, there exists c between a and x such that $g'(c) = 0$. We now calculate g'. Using the product rule, we note that

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = \frac{-f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Consequently,

$$\begin{aligned} g'(t) &= -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \cdots \\ &\quad + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}. \end{aligned}$$

Notice that there is a telescoping effect. Therefore,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}.$$

By Rolle's theorem, we conclude that there exists a number c between a and x such that $g'(c) = 0$. Since

$$g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}}$$

we conclude that

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} = 0.$$

Adding the first term on the left-hand side to both sides of the equation and dividing both sides of the equation by $n+1$, we conclude that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

as desired. From this fact, it follows that if there exists M such that $\left|f^{(n+1)}(x)\right| \le M$ for all x in I, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

□

Not only does Taylor's theorem allow us to prove that a Taylor series converges to a function, but it also allows us to estimate the accuracy of Taylor polynomials in approximating function values. We begin by looking at linear and quadratic approximations of $f(x) = \sqrt[3]{x}$ at $x = 8$ and determine how accurate these approximations are at estimating $\sqrt[3]{11}$.

Example 6.13

Using Linear and Quadratic Approximations to Estimate Function Values

Consider the function $f(x) = \sqrt[3]{x}$.

a. Find the first and second Taylor polynomials for f at $x = 8$. Use a graphing utility to compare these polynomials with f near $x = 8$.

b. Use these two polynomials to estimate $\sqrt[3]{11}$.

c. Use Taylor's theorem to bound the error.

Solution

a. For $f(x) = \sqrt[3]{x}$, the values of the function and its first two derivatives at $x = 8$ are as follows:

$$\begin{aligned} f(x) &= \sqrt[3]{x} & f(8) &= 2 \\ f'(x) &= \frac{1}{3x^{2/3}} & f'(8) &= \frac{1}{12} \\ f''(x) &= \frac{-2}{9x^{5/3}} & f''(8) &= -\frac{1}{144}. \end{aligned}$$

Thus, the first and second Taylor polynomials at $x = 8$ are given by

$$\begin{aligned} p_1(x) &= f(8) + f'(8)(x-8) \\ &= 2 + \frac{1}{12}(x-8) \\ p_2(x) &= f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2. \end{aligned}$$

The function and the Taylor polynomials are shown in **Figure 6.9**.

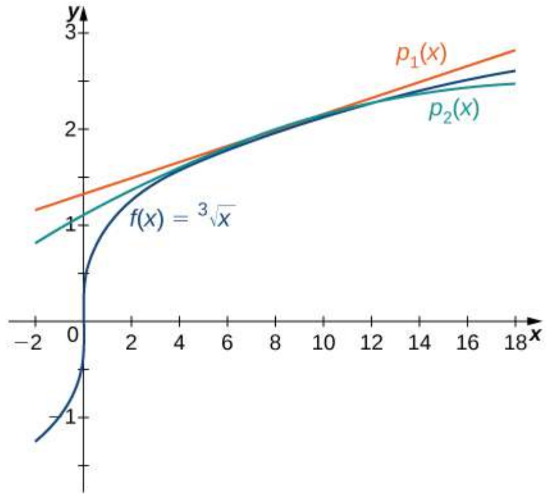


Figure 6.9 The graphs of $f(x) = \sqrt[3]{x}$ and the linear and quadratic approximations $p_1(x)$ and $p_2(x)$.

b. Using the first Taylor polynomial at $x = 8,$ we can estimate

$$\sqrt[3]{11} \approx p_1(11) = 2 + \frac{1}{12}(11 - 8) = 2.25.$$

Using the second Taylor polynomial at $x = 8,$ we obtain

$$\sqrt[3]{11} \approx p_2(11) = 2 + \frac{1}{12}(11 - 8) - \frac{1}{288}(11 - 8)^2 = 2.21875.$$

c. By **Uniqueness of Taylor Series**, there exists a c in the interval $(8, 11)$ such that the remainder when approximating $\sqrt[3]{11}$ by the first Taylor polynomial satisfies

$$R_1(11) = \frac{f''(c)}{2!}(11 - 8)^2.$$

We do not know the exact value of c, so we find an upper bound on $R_1(11)$ by determining the maximum value of f'' on the interval $(8, 11)$. Since $f''(x) = -\frac{2}{9x^{5/3}},$ the largest value for $|f''(x)|$ on that interval occurs at $x = 8$. Using the fact that $f''(8) = -\frac{1}{144},$ we obtain

$$|R_1(11)| \leq \frac{1}{144 \cdot 2!}(11 - 8)^2 = 0.03125.$$

Similarly, to estimate $R_2(11),$ we use the fact that

$$R_2(11) = \frac{f'''(c)}{3!}(11 - 8)^3.$$

Since $f'''(x) = \frac{10}{27x^{8/3}},$ the maximum value of f''' on the interval $(8, 11)$ is $f'''(8) \approx 0.0014468$. Therefore, we have

$$|R_2(11)| \leq \frac{0.0011468}{3!}(11-8)^3 \approx 0.0065104.$$

6.12 Find the first and second Taylor polynomials for $f(x) = \sqrt{x}$ at $x = 4$. Use these polynomials to estimate $\sqrt{6}$. Use Taylor's theorem to bound the error.

Example 6.14

Approximating sin *x* Using Maclaurin Polynomials

From **Example 6.12**b., the Maclaurin polynomials for $\sin x$ are given by

$$\begin{aligned} p_{2m+1}(x) &= p_{2m+2}(x) \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} \end{aligned}$$

for $m = 0, 1, 2, \ldots$.

a. Use the fifth Maclaurin polynomial for $\sin x$ to approximate $\sin\left(\frac{\pi}{18}\right)$ and bound the error.

b. For what values of *x* does the fifth Maclaurin polynomial approximate $\sin x$ to within 0.0001?

Solution

a. The fifth Maclaurin polynomial is

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Using this polynomial, we can estimate as follows:

$$\begin{aligned} \sin\left(\frac{\pi}{18}\right) &\approx p_5\left(\frac{\pi}{18}\right) \\ &= \frac{\pi}{18} - \frac{1}{3!}\left(\frac{\pi}{18}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{18}\right)^5 \\ &\approx 0.173648. \end{aligned}$$

To estimate the error, use the fact that the sixth Maclaurin polynomial is $p_6(x) = p_5(x)$ and calculate a bound on $R_6\left(\frac{\pi}{18}\right)$. By **Uniqueness of Taylor Series**, the remainder is

$$R_6\left(\frac{\pi}{18}\right) = \frac{f^{(7)}(c)}{7!}\left(\frac{\pi}{18}\right)^7$$

for some *c* between 0 and $\frac{\pi}{18}$. Using the fact that $\left|f^{(7)}(x)\right| \leq 1$ for all *x*, we find that the magnitude of the error is at most

$$\frac{1}{7!} \cdot \left(\frac{\pi}{18}\right)^7 \leq 9.8 \times 10^{-10}.$$

b. We need to find the values of x such that

$$\frac{1}{7!}|x|^7 \leq 0.0001.$$

Solving this inequality for x, we have that the fifth Maclaurin polynomial gives an estimate to within 0.0001 as long as $|x| < 0.907$.

6.13 Use the fourth Maclaurin polynomial for $\cos x$ to approximate $\cos\left(\frac{\pi}{12}\right)$.

Now that we are able to bound the remainder $R_n(x)$, we can use this bound to prove that a Taylor series for f at a converges to f.

Representing Functions with Taylor and Maclaurin Series

We now discuss issues of convergence for Taylor series. We begin by showing how to find a Taylor series for a function, and how to find its interval of convergence.

Example 6.15

Finding a Taylor Series

Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 1$. Determine the interval of convergence.

Solution

For $f(x) = \frac{1}{x}$, the values of the function and its first four derivatives at $x = 1$ are

$$\begin{aligned}
f(x) &= \frac{1}{x} & f(1) &= 1 \\
f'(x) &= -\frac{1}{x^2} & f'(1) &= -1 \\
f''(x) &= \frac{2}{x^3} & f''(1) &= 2! \\
f'''(x) &= -\frac{3 \cdot 2}{x^4} & f'''(1) &= -3! \\
f^{(4)}(x) &= \frac{4 \cdot 3 \cdot 2}{x^5} & f^{(4)}(1) &= 4!.
\end{aligned}$$

That is, we have $f^{(n)}(1) = (-1)^n n!$ for all $n \geq 0$. Therefore, the Taylor series for f at $x = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

To find the interval of convergence, we use the ratio test. We find that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left|(-1)^{n+1}(x-1)^{n+1}\right|}{\left|(-1)^n (x-1)^n\right|} = |x-1|.$$

Thus, the series converges if $|x-1| < 1$. That is, the series converges for $0 < x < 2$. Next, we need to check the endpoints. At $x = 2$, we see that

$$\sum_{n=0}^{\infty} (-1)^n (2-1)^n = \sum_{n=0}^{\infty} (-1)^n$$

diverges by the divergence test. Similarly, at $x = 0$,

$$\sum_{n=0}^{\infty} (-1)^n (0-1)^n = \sum_{n=0}^{\infty} (-1)^{2n} = \sum_{n=0}^{\infty} 1$$

diverges. Therefore, the interval of convergence is $(0, 2)$.

6.14 Find the Taylor series for $f(x) = \frac{1}{2}$ at $x = 2$ and determine its interval of convergence.

We know that the Taylor series found in this example converges on the interval $(0, 2)$, but how do we know it actually converges to f? We consider this question in more generality in a moment, but for this example, we can answer this question by writing

$$f(x) = \frac{1}{x} = \frac{1}{1-(1-x)}.$$

That is, f can be represented by the geometric series $\sum_{n=0}^{\infty} (1-x)^n$. Since this is a geometric series, it converges to $\frac{1}{x}$ as long as $|1-x| < 1$. Therefore, the Taylor series found in **Example 6.15** does converge to $f(x) = \frac{1}{x}$ on $(0, 2)$.

We now consider the more general question: if a Taylor series for a function f converges on some interval, how can we determine if it actually converges to f? To answer this question, recall that a series converges to a particular value if and only if its sequence of partial sums converges to that value. Given a Taylor series for f at a, the nth partial sum is given by the nth Taylor polynomial p_n. Therefore, to determine if the Taylor series converges to f, we need to determine whether

$$\lim_{n \to \infty} p_n(x) = f(x).$$

Since the remainder $R_n(x) = f(x) - p_n(x)$, the Taylor series converges to f if and only if

$$\lim_{n \to \infty} R_n(x) = 0.$$

We now state this theorem formally.

Theorem 6.8: Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n \to \infty} R_n(x) = 0$$

for all x in I.

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x) \to 0$. To prove that $R_n(x) \to 0$, we typically use the bound

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

from Taylor's theorem with remainder.

In the next example, we find the Maclaurin series for e^x and $\sin x$ and show that these series converge to the corresponding functions for all real numbers by proving that the remainders $R_n(x) \to 0$ for all real numbers x.

Example 6.16

Finding Maclaurin Series

For each of the following functions, find the Maclaurin series and its interval of convergence. Use **Taylor's Theorem with Remainder** to prove that the Maclaurin series for f converges to f on that interval.

a. e^x

b. $\sin x$

Solution

a. Using the nth Maclaurin polynomial for e^x found in **Example 6.12**a., we find that the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To determine the interval of convergence, we use the ratio test. Since

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1},$$

we have

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|x|}{n+1} = 0$$

for all x. Therefore, the series converges absolutely for all x, and thus, the interval of convergence is $(-\infty, \infty)$. To show that the series converges to e^x for all x, we use the fact that $f^{(n)}(x) = e^x$ for all $n \geq 0$ and e^x is an increasing function on $(-\infty, \infty)$. Therefore, for any real number b, the maximum value of e^x for all $|x| \leq b$ is e^b. Thus,

$$|R_n(x)| \leq \frac{e^b}{(n+1)!}|x|^{n+1}.$$

Since we just showed that

$$\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

converges for all x, by the divergence test, we know that

$$\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for any real number x. By combining this fact with the squeeze theorem, the result is $\lim_{n \to \infty} R_n(x) = 0.$

b. Using the nth Maclaurin polynomial for $\sin x$ found in **Example 6.12**b., we find that the Maclaurin series for $\sin x$ is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

In order to apply the ratio test, consider

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \frac{|x|^2}{(2n+3)(2n+2)}.$$

Since

$$\lim_{n \to \infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0$$

for all x, we obtain the interval of convergence as $(-\infty, \infty)$. To show that the Maclaurin series converges to $\sin x$, look at $R_n(x)$. For each x there exists a real number c between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

Since $\left| f^{(n+1)}(c) \right| \le 1$ for all integers n and all real numbers c, we have

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$$

for all real numbers x. Using the same idea as in part a., the result is $\lim_{n \to \infty} R_n(x) = 0$ for all x, and therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all real x.

6.15 Find the Maclaurin series for $f(x) = \cos x$. Use the ratio test to show that the interval of convergence is $(-\infty, \infty)$. Show that the Maclaurin series converges to $\cos x$ for all real numbers x.

Student PROJECT

Proving that *e* is Irrational

In this project, we use the Maclaurin polynomials for e^x to prove that e is irrational. The proof relies on supposing that e is rational and arriving at a contradiction. Therefore, in the following steps, we suppose $e = r/s$ for some integers r and s where $s \neq 0$.

1. Write the Maclaurin polynomials $p_0(x),\ p_1(x),\ p_2(x),\ p_3(x),\ p_4(x)$ for e^x. Evaluate $p_0(1),\ p_1(1),\ p_2(1),\ p_3(1),\ p_4(1)$ to estimate e.
2. Let $R_n(x)$ denote the remainder when using $p_n(x)$ to estimate e^x. Therefore, $R_n(x) = e^x - p_n(x)$, and $R_n(1) = e - p_n(1)$. Assuming that $e = \frac{r}{s}$ for integers r and s, evaluate $R_0(1),\ R_1(1),\ R_2(1),\ R_3(1),\ R_4(1)$.
3. Using the results from part 2, show that for each remainder $R_0(1),\ R_1(1),\ R_2(1),\ R_3(1),\ R_4(1),$ we can find an integer k such that $kR_n(1)$ is an integer for $n = 0,\ 1,\ 2,\ 3,\ 4$.
4. Write down the formula for the nth Maclaurin polynomial $p_n(x)$ for e^x and the corresponding remainder $R_n(x)$. Show that $sn!R_n(1)$ is an integer.
5. Use Taylor's theorem to write down an explicit formula for $R_n(1)$. Conclude that $R_n(1) \neq 0,$ and therefore, $sn!R_n(1) \neq 0$.
6. Use Taylor's theorem to find an estimate on $R_n(1)$. Use this estimate combined with the result from part 5 to show that $|sn!R_n(1)| < \frac{se}{n+1}$. Conclude that if n is large enough, then $|sn!R_n(1)| < 1$. Therefore, $sn!R_n(1)$ is an integer with magnitude less than 1. Thus, $sn!R_n(1) = 0$. But from part 5, we know that $sn!R_n(1) \neq 0$. We have arrived at a contradiction, and consequently, the original supposition that e is rational must be false.

6.3 EXERCISES

In the following exercises, find the Taylor polynomials of degree two approximating the given function centered at the given point.

116. $f(x) = 1 + x + x^2$ at $a = 1$

117. $f(x) = 1 + x + x^2$ at $a = -1$

118. $f(x) = \cos(2x)$ at $a = \pi$

119. $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

120. $f(x) = \sqrt{x}$ at $a = 4$

121. $f(x) = \ln x$ at $a = 1$

122. $f(x) = \frac{1}{x}$ at $a = 1$

123. $f(x) = e^x$ at $a = 1$

In the following exercises, verify that the given choice of n in the remainder estimate $|R_n| \le \frac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $\left|f^{(n+1)}(z)\right|$ on the interval between a and the indicated point, yields $|R_n| \le \frac{1}{1000}$. Find the value of the Taylor polynomial p_n of f at the indicated point.

124. **[T]** $\sqrt{10}$; $a = 9$, $n = 3$

125. **[T]** $(28)^{1/3}$; $a = 27$, $n = 1$

126. **[T]** $\sin(6)$; $a = 2\pi$, $n = 5$

127. **[T]** e^2; $a = 0$, $n = 9$

128. **[T]** $\cos\left(\frac{\pi}{5}\right)$; $a = 0$, $n = 4$

129. **[T]** $\ln(2)$; $a = 1$, $n = 1000$

130. Integrate the approximation $\sin t \approx t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}$ evaluated at πt to approximate $\int_0^1 \frac{\sin \pi t}{\pi t} dt$.

131. Integrate the approximation $e^x \approx 1 + x + \frac{x^2}{2} + \cdots + \frac{x^6}{720}$ evaluated at $-x^2$ to approximate $\int_0^1 e^{-x^2} dx$.

In the following exercises, find the smallest value of n such that the remainder estimate $|R_n| \le \frac{M}{(n+1)!}(x-a)^{n+1}$, where M is the maximum value of $\left|f^{(n+1)}(z)\right|$ on the interval between a and the indicated point, yields $|R_n| \le \frac{1}{1000}$ on the indicated interval.

132. $f(x) = \sin x$ on $[-\pi, \pi]$, $a = 0$

133. $f(x) = \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $a = 0$

134. $f(x) = e^{-2x}$ on $[-1, 1]$, $a = 0$

135. $f(x) = e^{-x}$ on $[-3, 3]$, $a = 0$

In the following exercises, the maximum of the right-hand side of the remainder estimate $|R_1| \le \frac{\max|f''(z)|}{2}R^2$ on $[a - R, a + R]$ occurs at a or $a \pm R$. Estimate the maximum value of R such that $\frac{\max|f''(z)|}{2}R^2 \le 0.1$ on $[a - R, a + R]$ by plotting this maximum as a function of R.

136. **[T]** e^x approximated by $1 + x$, $a = 0$

137. **[T]** $\sin x$ approximated by x, $a = 0$

138. **[T]** $\ln x$ approximated by $x - 1$, $a = 1$

139. **[T]** $\cos x$ approximated by 1, $a = 0$

In the following exercises, find the Taylor series of the given function centered at the indicated point.

140. x^4 at $a = -1$

141. $1 + x + x^2 + x^3$ at $a = -1$

142. $\sin x$ at $a = \pi$

143. $\cos x$ at $a = 2\pi$

144. $\sin x$ at $x = \frac{\pi}{2}$

145. $\cos x$ at $x = \frac{\pi}{2}$

146. e^x at $a = -1$

147. e^x at $a = 1$

148. $\dfrac{1}{(x-1)^2}$ at $a = 0$ (*Hint:* Differentiate $\frac{1}{1-x}$.)

149. $\dfrac{1}{(x-1)^3}$ at $a = 0$

150. $F(x) = \int_0^x \cos(\sqrt{t})dt;\ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(2n)!}$ at $a = 0$ (*Note*: f is the Taylor series of $\cos(\sqrt{t})$.)

In the following exercises, compute the Taylor series of each function around $x = 1$.

151. $f(x) = 2 - x$

152. $f(x) = x^3$

153. $f(x) = (x-2)^2$

154. $f(x) = \ln x$

155. $f(x) = \frac{1}{x}$

156. $f(x) = \dfrac{1}{2x - x^2}$

157. $f(x) = \dfrac{x}{4x - 2x^2 - 1}$

158. $f(x) = e^{-x}$

159. $f(x) = e^{2x}$

[T] In the following exercises, identify the value of *x* such that the given series $\sum_{n=0}^{\infty} a_n$ is the value of the Maclaurin series of $f(x)$ at x. Approximate the value of $f(x)$ using $S_{10} = \sum_{n=0}^{10} a_n$.

160. $\sum_{n=0}^{\infty} \frac{1}{n!}$

161. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

162. $\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!}$

163. $\sum_{n=0}^{\infty} \frac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!}$

The following exercises make use of the functions $S_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$ and $C_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ on $[-\pi, \pi]$.

164. **[T]** Plot $\sin^2 x - (S_5(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\sin x$.

165. **[T]** Plot $\cos^2 x - (C_4(x))^2$ on $[-\pi, \pi]$. Compare the maximum difference with the square of the Taylor remainder estimate for $\cos x$.

166. **[T]** Plot $|2S_5(x)C_4(x) - \sin(2x)|$ on $[-\pi, \pi]$.

167. **[T]** Compare $\dfrac{S_5(x)}{C_4(x)}$ on $[-1, 1]$ to $\tan x$. Compare this with the Taylor remainder estimate for the approximation of $\tan x$ by $x + \frac{x^3}{3} + \frac{2x^5}{15}$.

168. **[T]** Plot $e^x - e_4(x)$ where $e_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ on $[0, 2]$. Compare the maximum error with the Taylor remainder estimate.

169. (Taylor approximations and root finding.) Recall that Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ approximates solutions of $f(x) = 0$ near the input x_0.

a. If f and g are inverse functions, explain why a solution of $g(x) = a$ is the value $f(a)$ of f.
b. Let $p_N(x)$ be the Nth degree Maclaurin polynomial of e^x. Use Newton's method to approximate solutions of $p_N(x) - 2 = 0$ for $N = 4, 5, 6$.
c. Explain why the approximate roots of $p_N(x) - 2 = 0$ are approximate values of $\ln(2)$.

In the following exercises, use the fact that if

$q(x) = \sum_{n=1}^{\infty} a_n (x-c)^n$ converges in an interval containing c, then $\lim_{x \to c} q(x) = a_0$ to evaluate each limit using Taylor series.

170. $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$

171. $\lim_{x \to 0} \frac{\ln\left(1 - x^2\right)}{x^2}$

172. $\lim_{x \to 0} \frac{e^{x^2} - x^2 - 1}{x^4}$

173. $\lim_{x \to 0^+} \frac{\cos(\sqrt{x}) - 1}{2x}$

6.4 | Working with Taylor Series

Learning Objectives

6.4.1 Write the terms of the binomial series.
6.4.2 Recognize the Taylor series expansions of common functions.
6.4.3 Recognize and apply techniques to find the Taylor series for a function.
6.4.4 Use Taylor series to solve differential equations.
6.4.5 Use Taylor series to evaluate nonelementary integrals.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. In this section we show how to use those Taylor series to derive Taylor series for other functions. We then present two common applications of power series. First, we show how power series can be used to solve differential equations. Second, we show how power series can be used to evaluate integrals when the antiderivative of the integrand cannot be expressed in terms of elementary functions. In one example, we consider $\int e^{-x^2}dx,$ an integral that arises frequently in probability theory.

The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x) = (1+x)^r$ for all real numbers $r.$ The Maclaurin series for this function is known as the **binomial series**. We begin by considering the simplest case: r is a nonnegative integer. We recall that, for $r = 0, 1, 2, 3, 4,$ $f(x) = (1+x)^r$ can be written as

$$\begin{aligned} f(x) &= (1+x)^0 = 1, \\ f(x) &= (1+x)^1 = 1+x, \\ f(x) &= (1+x)^2 = 1+2x+x^2, \\ f(x) &= (1+x)^3 = 1+3x+3x^2+x^3, \\ f(x) &= (1+x)^4 = 1+4x+6x^2+4x^3+x^4. \end{aligned}$$

The expressions on the right-hand side are known as binomial expansions and the coefficients are known as binomial coefficients. More generally, for any nonnegative integer $r,$ the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by

$$\binom{r}{n} = \frac{r!}{n!(r-n)!} \tag{6.6}$$

and

$$\begin{aligned} f(x) &= (1+x)^r \\ &= \binom{r}{0}1 + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots + \binom{r}{r-1}x^{r-1} + \binom{r}{r}x^r \\ &= \sum_{n=0}^{r} \binom{r}{n}x^n. \end{aligned} \tag{6.7}$$

For example, using this formula for $r = 5,$ we see that

$$\begin{aligned} f(x) &= (1+x)^5 \\ &= \binom{5}{0}1 + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5 \\ &= \frac{5!}{0!5!}1 + \frac{5!}{1!4!}x + \frac{5!}{2!3!}x^2 + \frac{5!}{3!2!}x^3 + \frac{5!}{4!1!}x^4 + \frac{5!}{5!0!}x^5 \\ &= 1+5x+10x^2+10x^3+5x^4+x^5. \end{aligned}$$

We now consider the case when the exponent r is any real number, not necessarily a nonnegative integer. If r is not a nonnegative integer, then $f(x) = (1 + x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x = 0$.

$$\begin{array}{rclrcl} f(x) & = & (1+x)^r & f(0) & = & 1 \\ f'(x) & = & r(1+x)^{r-1} & f'(0) & = & r \\ f''(x) & = & r(r-1)(1+x)^{r-2} & f''(0) & = & r(r-1) \\ f'''(x) & = & r(r-1)(r-2)(1+x)^{r-3} & f'''(0) & = & r(r-1)(r-2) \\ f^{(n)}(x) & = & r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} & f^{(n)}(0) & = & r(r-1)(r-2)\cdots(r-n+1) \end{array}$$

We conclude that the coefficients in the binomial series are given by

$$\frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}. \quad (6.8)$$

We note that if r is a nonnegative integer, then the $(r+1)$st derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a nonnegative integer, then **Equation 6.8** for the coefficients agrees with **Equation 6.6** for the coefficients, and the formula for the binomial series agrees with **Equation 6.7** for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

$$\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

With this notation, we can write the binomial series for $(1 + x)^r$ as

$$\sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots. \quad (6.9)$$

We now need to determine the interval of convergence for the binomial series **Equation 6.9**. We apply the ratio test. Consequently, we consider

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|r(r-1)(r-2)\cdots(r-n)||x|^{n+1}}{(n+1)!} \cdot \frac{n}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n} \\ &= \frac{|r-n||x|}{|n+1|}. \end{aligned}$$

Since

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = |x| < 1$$

if and only if $|x| < 1$, we conclude that the interval of convergence for the binomial series is $(-1, 1)$. The behavior at the endpoints depends on r. It can be shown that for $r \geq 0$ the series converges at both endpoints; for $-1 < r < 0$, the series converges at $x = 1$ and diverges at $x = -1$; and for $r < -1$, the series diverges at both endpoints. The binomial series does converge to $(1 + x)^r$ in $(-1, 1)$ for all real numbers r, but proving this fact by showing that the remainder $R_n(x) \to 0$ is difficult.

Definition

For any real number r, the Maclaurin series for $f(x) = (1 + x)^r$ is the binomial series. It converges to f for $|x| < 1$, and we write

$$\begin{aligned}(1+x)^r &= \sum_{n=0}^{\infty} \binom{r}{n} x^n \\ &= 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots\end{aligned}$$

for $|x| < 1$.

We can use this definition to find the binomial series for $f(x) = \sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example 6.17

Finding Binomial Series

a. Find the binomial series for $f(x) = \sqrt{1+x}$.

b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's theorem to bound the error. Use a graphing utility to compare the graphs of f and p_3.

Solution

a. Here $r = \frac{1}{2}$. Using the definition for the binomial series, we obtain

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{2!}\frac{1}{2^2}x^2 + \frac{1}{3!}\frac{1\cdot 3}{2^3}x^3 - \cdots + \frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n.\end{aligned}$$

b. From the result in part a. the third-order Maclaurin polynomial is

$$p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

Therefore,

$$\begin{aligned}\sqrt{1.5} &= \sqrt{1+0.5} \\ &\approx 1 + \frac{1}{2}(0.5) - \frac{1}{8}(0.5)^2 + \frac{1}{16}(0.5)^3 \\ &\approx 1.2266.\end{aligned}$$

From Taylor's theorem, the error satisfies

$$R_3(0.5) = \frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x) = -\frac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $\left|f^{(4)}(x)\right|$ on the interval $(0, 0.5)$ occurs at $x = 0$, we have

$$|R_3(0.5)| \le \frac{15}{4!2^4}(0.5)^4 \approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in **Figure 6.10**.

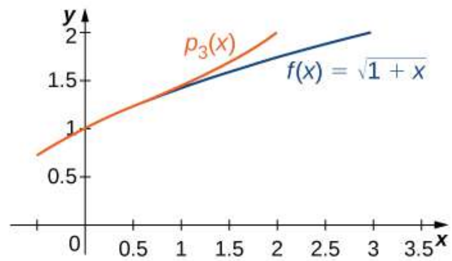


Figure 6.10 The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x) = \sqrt{1+x}$ for x near zero.

6.16 Find the binomial series for $f(x) = \dfrac{1}{(1+x)^2}$.

Common Functions Expressed as Taylor Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions, as well as functions of the form $f(x) = (1+x)^r$. In **Table 6.1**, we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f(x) = \ln(1+x)$ at the endpoint $x = 1$ and the Maclaurin series for $f(x) = \tan^{-1} x$ at the endpoints $x = 1$ and $x = -1$ relies on a more advanced theorem than we present here. (Refer to Abel's theorem for a discussion of this more technical point.)

Function	Maclaurin Series	Interval of Convergence
$f(x) = \frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$f(x) = e^x$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$f(x) = \ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$-1 < x \leq 1$
$f(x) = \tan^{-1} x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$-1 < x \leq 1$
$f(x) = (1+x)^r$	$\sum_{n=0}^{\infty} \binom{r}{n} x^n$	$-1 < x < 1$

Table 6.1 Maclaurin Series for Common Functions

Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties, combined with the Maclaurin series in **Table 6.1**, to create Maclaurin series for other functions.

Example 6.18

Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the following functions by using one of the series listed in **Table 6.1**.

a. $f(x) = \cos\sqrt{x}$

b. $f(x) = \sinh x$

Solution

a. Using the Maclaurin series for $\cos x$ we find that the Maclaurin series for $\cos\sqrt{x}$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$
$$= 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges to $\cos\sqrt{x}$ for all x in the domain of $\cos\sqrt{x}$; that is, for all $x \geq 0$.

b. To find the Maclaurin series for $\sinh x$, we use the fact that

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the nth term in the Maclaurin series for $\sinh x$ is given by

$$\frac{x^n}{n!} - \frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\frac{2x^n}{n!}$. Therefore, the Maclaurin series for $\sinh x$ has only odd-order terms and is given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

6.17 Find the Maclaurin series for $\sin\left(x^2\right)$.

We also showed previously in this chapter how power series can be differentiated term by term to create a new power series. In **Example 6.19**, we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

Example 6.19

Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution

The two functions are related by

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\frac{1}{\sqrt{1+x}}$ is given by

$$\begin{aligned}\frac{1}{\sqrt{1+x}} &= 2\frac{d}{dx}\sqrt{1+x} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n.\end{aligned}$$

6.18 Find the binomial series for $f(x) = \frac{1}{(1+x)^{3/2}}$

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term by term makes them a powerful tool for solving differential equations. We now show how this is accomplished.

Solving Differential Equations with Power Series

Consider the differential equation

$$y'(x) = y.$$

Recall that this is a first-order separable equation and its solution is $y = Ce^x$. This equation is easily solved using techniques discussed earlier in the text. For most differential equations, however, we do not yet have analytical tools to solve them. Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$ and determine what the coefficients would need to be. In the next example, we consider an initial-value problem involving $y' = y$ to illustrate the technique.

Example 6.20

Power Series Solution of a Differential Equation

Use power series to solve the initial-value problem

$$y' = y, \quad y(0) = 3.$$

Solution

Suppose that there exists a power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this series term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots.$$

If y satisfies the differential equation, then

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots.$$

Using **Uniqueness of Power Series** on the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$\begin{aligned} c_0 &= c_1, \\ c_1 &= 2c_2, \\ c_2 &= 3c_3, \\ c_3 &= 4c_4, \\ &\vdots. \end{aligned}$$

Using the initial condition $y(0) = 3$ combined with the power series representation

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

we find that $c_0 = 3.$ We are now ready to solve for the rest of the coefficients. Using the fact that $c_0 = 3,$ we have

$$\begin{aligned} c_1 &= c_0 = 3 = \frac{3}{1!}, \\ c_2 &= \frac{c_1}{2} = \frac{3}{2} = \frac{3}{2!}, \\ c_3 &= \frac{c_2}{3} = \frac{3}{3 \cdot 2} = \frac{3}{3!}, \\ c_4 &= \frac{c_3}{4} = \frac{3}{4 \cdot 3 \cdot 2} = \frac{3}{4!}. \end{aligned}$$

Therefore,

$$\begin{aligned} y &= 3\left[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \frac{1}{4!}x^4 + \cdots\right] \\ &= 3 \sum_{n=0}^{\infty} \frac{x^n}{n!}. \end{aligned}$$

You might recognize

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as the Taylor series for $e^x.$ Therefore, the solution is $y = 3e^x.$

6.19 Use power series to solve $y' = 2y, \quad y(0) = 5.$

We now consider an example involving a differential equation that we cannot solve using previously discussed methods. This differential equation

$$y' - xy = 0$$

is known as Airy's equation. It has many applications in mathematical physics, such as modeling the diffraction of light. Here we show how to solve it using power series.

Example 6.21

Power Series Solution of Airy's Equation

Use power series to solve

$$y'' - xy = 0$$

with the initial conditions $y(0) = a$ and $y'(0) = b$.

Solution

We look for a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this function term by term, we obtain

$$\begin{aligned} y' &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots, \\ y'' &= 2 \cdot 1 c_2 + 3 \cdot 2 c_3 x + 4 \cdot 3 c_4 x^2 + \cdots. \end{aligned}$$

If y satisfies the equation $y'' = xy$, then

$$2 \cdot 1 c_2 + 3 \cdot 2 c_3 x + 4 \cdot 3 c_4 x^2 + \cdots = x\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right).$$

Using **Uniqueness of Power Series** on the uniqueness of power series representations, we know that coefficients of the same degree must be equal. Therefore,

$$\begin{aligned} 2 \cdot 1 c_2 &= 0, \\ 3 \cdot 2 c_3 &= c_0, \\ 4 \cdot 3 c_4 &= c_1, \\ 5 \cdot 4 c_5 &= c_2, \\ &\vdots. \end{aligned}$$

More generally, for $n \geq 3$, we have $n \cdot (n-1) c_n = c_{n-3}$. In fact, all coefficients can be written in terms of c_0 and c_1. To see this, first note that $c_2 = 0$. Then

$$\begin{aligned} c_3 &= \frac{c_0}{3 \cdot 2}, \\ c_4 &= \frac{c_1}{4 \cdot 3}. \end{aligned}$$

For c_5, c_6, c_7, we see that

$$\begin{aligned} c_5 &= \frac{c_2}{5 \cdot 4} = 0, \\ c_6 &= \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}, \\ c_7 &= \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}. \end{aligned}$$

Therefore, the series solution of the differential equation is given by

$$y = c_0 + c_1 x + 0 \cdot x^2 + \frac{c_0}{3 \cdot 2} x^3 + \frac{c_1}{4 \cdot 3} x^4 + 0 \cdot x^5 + \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2} x^6 + \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3} x^7 + \cdots.$$

The initial condition $y(0) = a$ implies $c_0 = a$. Differentiating this series term by term and using the fact that

$y'(0) = b$, we conclude that $c_1 = b$. Therefore, the solution of this initial-value problem is

$$y = a\left(1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \cdots\right) + b\left(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \cdots\right).$$

6.20 Use power series to solve $y'' + x^2 y = 0$ with the initial condition $y(0) = a$ and $y'(0) = b$.

Evaluating Nonelementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

One integral that arises often in applications in probability theory is $\int e^{-x^2} dx$. Unfortunately, the antiderivative of the integrand e^{-x^2} is not an elementary function. By elementary function, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with noncomplicated function. For example, the function $f(x) = \sqrt{x^2 - 3x} + e^{x^3} - \sin(5x + 4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x)\,dx$ where the antiderivative of f cannot be written as an elementary function is considered a **nonelementary integral**.

Nonelementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int e^{-x^2} dx$.

Example 6.22

Using Taylor Series to Evaluate a Definite Integral

a. Express $\int e^{-x^2} dx$ as an infinite series.

b. Evaluate $\int_0^1 e^{-x^2} dx$ to within an error of 0.01.

Solution

a. The Maclaurin series for e^{-x^2} is given by

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int e^{-x^2} dx &= \int\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots. \end{aligned}$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2} dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots.$$

The sum of the first four terms is approximately $0.74.$ By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216} \approx 0.0046296 < 0.01.$

6.21 Express $\int \cos\sqrt{x} dx$ as an infinite series. Evaluate $\int_0^1 \cos\sqrt{x} dx$ to within an error of $0.01.$

As mentioned above, the integral $\int e^{-x^2} dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation $\sigma,$ then the probability that a randomly chosen value lies between $x = a$ and $x = b$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/\left(2\sigma^2\right)} dx. \tag{6.10}$$

(See **Figure 6.11**.)

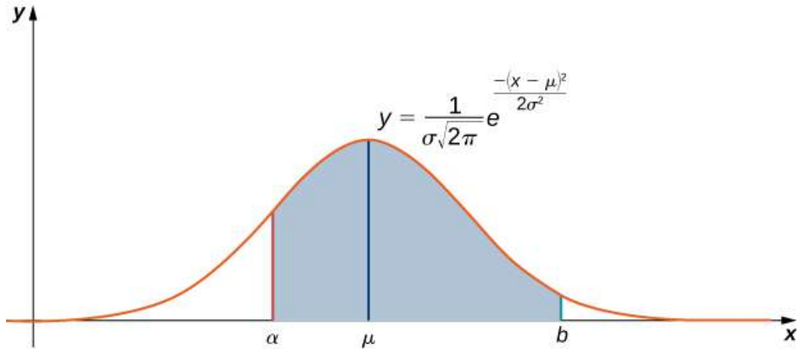


Figure 6.11 If data values are normally distributed with mean μ and standard deviation $\sigma,$ the probability that a randomly selected data value is between a and b is the area under the curve $y = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/\left(2\sigma^2\right)}$ between $x = a$ and $x = b.$

To simplify this integral, we typically let $z = \frac{x - \mu}{\sigma}.$ This quantity z is known as the z score of a data value. With this simplification, integral **Equation 6.10** becomes

$$\frac{1}{\sqrt{2\pi}}\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2}dz. \tag{6.11}$$

In **Example 6.23**, we show how we can use this integral in calculating probabilities.

Example 6.23

Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50.$ Use **Equation 6.11** and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x = 100$ and $x = 200.$ Use the alternating series test to determine how accurate your approximation is.

Solution

Since $\mu = 100, \sigma = 50,$ and we are trying to determine the area under the curve from $a = 100$ to $b = 200,$ integral **Equation 6.11** becomes

$$\frac{1}{\sqrt{2\pi}}\int_0^2 e^{-z^2/2}dz.$$

The Maclaurin series for $e^{-x^2/2}$ is given by

$$
\begin{aligned}
e^{-x^2/2} &= \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!} \\
&= 1 - \frac{x^2}{2^1 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n!} + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} dz &= \frac{1}{\sqrt{2\pi}} \int \left(1 - \frac{z^2}{2^1 \cdot 1!} + \frac{z^4}{2^2 \cdot 2!} - \frac{z^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n}}{2^n \cdot n!} + \cdots \right) dz \\
&= \frac{1}{\sqrt{2\pi}} \left(C + z - \frac{z^3}{3 \cdot 2^1 \cdot 1!} + \frac{z^5}{5 \cdot 2^2 \cdot 2!} - \frac{z^7}{7 \cdot 2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)2^n \cdot n!} + \cdots \right) \\
\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2} dz &= \frac{1}{\sqrt{2\pi}} \left(2 - \frac{8}{6} + \frac{32}{40} - \frac{128}{336} + \frac{512}{3456} - \frac{2^{11}}{11 \cdot 2^5 \cdot 5!} + \cdots \right).
\end{aligned}
$$

Using the first five terms, we estimate that the probability is approximately 0.4922. By the alternating series test, we see that this estimate is accurate to within

$$\frac{1}{\sqrt{2\pi}} \frac{2^{13}}{13 \cdot 2^6 \cdot 6!} \approx 0.00546.$$

Analysis

If you are familiar with probability theory, you may know that the probability that a data value is within two standard deviations of the mean is approximately 95%. Here we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5%. The estimate, combined with the bound on the accuracy, falls within this range.

6.22 Use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the alternating series test to determine the accuracy of this estimate.

Another application in which a nonelementary integral arises involves the period of a pendulum. The integral is

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

An integral of this form is known as an elliptic integral of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

Example 6.24

Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle $\theta_{\max}$ with the vertical, its period T is given by

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$$

where g is the acceleration due to gravity and $k = \sin\left(\frac{\theta_{\max}}{2}\right)$ (see **Figure 6.12**). (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model is used and $\sin\theta$ is approximated by θ.) Use the binomial series

$$\frac{1}{\sqrt{1 + x}} = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n}x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and

b. you use the first two terms in the binomial series.

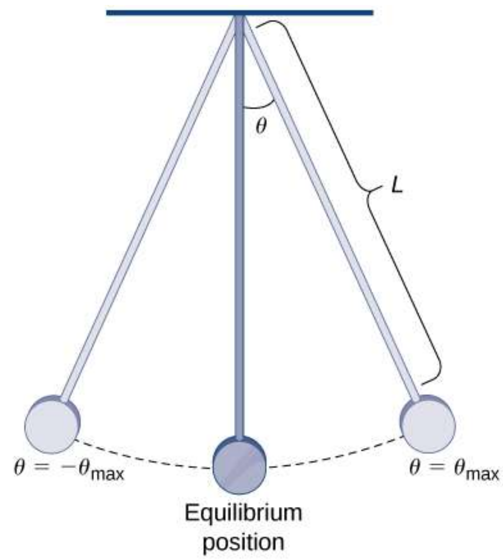


Figure 6.12 This pendulum has length L and makes a maximum angle $\theta_{\max}$ with the vertical.

Solution

We use the binomial series, replacing x with $-k^2\sin^2\theta$. Then we can write the period as

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1 + \frac{1}{2}k^2\sin^2\theta + \frac{1 \cdot 3}{2!2^2}k^4\sin^4\theta + \cdots\right)d\theta.$$

a. Using just the first term in the integrand, the first-order estimate is

$$T \approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}d\theta = 2\pi\sqrt{\frac{L}{g}}.$$

If $\theta_{\max}$ is small, then $k = \sin\left(\frac{\theta_{\max}}{2}\right)$ is small. We claim that when k is small, this is a good estimate. To justify this claim, consider

$$\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta+\frac{1\cdot 3}{2!2^2}k^4\sin^4\theta+\cdots\right)d\theta.$$

Since $|\sin x| \leq 1,$ this integral is bounded by

$$\int_0^{\pi/2}\left(\frac{1}{2}k^2+\frac{1.3}{2!2^2}k^4+\cdots\right)d\theta < \frac{\pi}{2}\left(\frac{1}{2}k^2+\frac{1\cdot 3}{2!2^2}k^4+\cdots\right).$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

$$\frac{\pi k^2}{2}\left(1+k^2+k^4+\cdots\right)=\frac{\pi k^2}{2}\cdot\frac{1}{1-k^2},$$

which is small for k small.

b. For larger values of $\theta_{\text{max}},$ we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

$$\begin{aligned} T &\approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta\right)d\theta \\ &= 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right). \end{aligned}$$

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are useful because they allow us to represent known functions using polynomials, thus providing us a tool for approximating function values and estimating complicated integrals. In addition, they allow us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.

6.4 EXERCISES

In the following exercises, use appropriate substitutions to write down the Maclaurin series for the given binomial.

174. $(1-x)^{1/3}$

175. $\left(1+x^2\right)^{-1/3}$

176. $(1-x)^{1.01}$

177. $(1-2x)^{2/3}$

In the following exercises, use the substitution $(b+x)^r=(b+a)^r\left(1+\frac{x-a}{b+a}\right)^r$ in the binomial expansion to find the Taylor series of each function with the given center.

178. $\sqrt{x+2}$ at $a=0$

179. $\sqrt{x^2+2}$ at $a=0$

180. $\sqrt{x+2}$ at $a=1$

181. $\sqrt{2x-x^2}$ at $a=1$ (*Hint:* $2x-x^2=1-(x-1)^2$)

182. $(x-8)^{1/3}$ at $a=9$

183. $\sqrt{x}$ at $a=4$

184. $x^{1/3}$ at $a=27$

185. $\sqrt{x}$ at $x=9$

In the following exercises, use the binomial theorem to estimate each number, computing enough terms to obtain an estimate accurate to an error of at most $1/1000$.

186. **[T]** $(15)^{1/4}$ using $(16-x)^{1/4}$

187. **[T]** $(1001)^{1/3}$ using $(1000+x)^{1/3}$

In the following exercises, use the binomial approximation $\sqrt{1-x}\approx 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\frac{7x^5}{256}$ for $|x|<1$ to approximate each number. Compare this value to the value given by a scientific calculator.

188. **[T]** $\frac{1}{\sqrt{2}}$ using $x=\frac{1}{2}$ in $(1-x)^{1/2}$

189. **[T]** $\sqrt{5}=5\times\frac{1}{\sqrt{5}}$ using $x=\frac{4}{5}$ in $(1-x)^{1/2}$

190. **[T]** $\sqrt{3}=\frac{3}{\sqrt{3}}$ using $x=\frac{2}{3}$ in $(1-x)^{1/2}$

191. **[T]** $\sqrt{6}$ using $x=\frac{5}{6}$ in $(1-x)^{1/2}$

192. Integrate the binomial approximation of $\sqrt{1-x}$ to find an approximation of $\int_0^x \sqrt{1-t}\,dt$.

193. **[T]** Recall that the graph of $\sqrt{1-x^2}$ is an upper semicircle of radius 1. Integrate the binomial approximation of $\sqrt{1-x^2}$ up to order 8 from $x=-1$ to $x=1$ to estimate $\frac{\pi}{2}$.

In the following exercises, use the expansion $(1+x)^{1/3}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ to write the first five terms (not necessarily a quartic polynomial) of each expression.

194. $(1+4x)^{1/3}; a=0$

195. $(1+4x)^{4/3}; a=0$

196. $(3+2x)^{1/3}; a=-1$

197. $\left(x^2+6x+10\right)^{1/3}; a=-3$

198. Use $(1+x)^{1/3}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots$ with $x=1$ to approximate $2^{1/3}$.

199. Use the approximation $(1-x)^{2/3}=1-\frac{2x}{3}-\frac{x^2}{9}-\frac{4x^3}{81}-\frac{7x^4}{243}-\frac{14x^5}{729}+\cdots$ for $|x|<1$ to approximate $2^{1/3}=2.2^{-2/3}$.

200. Find the 25th derivative of $f(x)=\left(1+x^2\right)^{13}$ at $x=0$.

201. Find the 99 th derivative of $f(x)=\left(1+x^4\right)^{25}$.

In the following exercises, find the Maclaurin series of each function.

202. $f(x) = xe^{2x}$

203. $f(x) = 2^x$

204. $f(x) = \frac{\sin x}{x}$

205. $f(x) = \frac{\sin(\sqrt{x})}{\sqrt{x}}, \quad (x > 0),$

206. $f(x) = \sin\left(x^2\right)$

207. $f(x) = e^{x^3}$

208. $f(x) = \cos^2 x$ using the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

209. $f(x) = \sin^2 x$ using the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

In the following exercises, find the Maclaurin series of $F(x) = \int_0^x f(t)\,dt$ by integrating the Maclaurin series of f term by term. If f is not strictly defined at zero, you may substitute the value of the Maclaurin series at zero.

210. $F(x) = \int_0^x e^{-t^2}dt;\ f(t) = e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$

211. $F(x) = \tan^{-1} x; \quad f(t) = \frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$

212. $F(x) = \tanh^{-1} x; \quad f(t) = \frac{1}{1-t^2} = \sum_{n=0}^{\infty} t^{2n}$

213. $F(x) = \sin^{-1} x; \quad f(t) = \frac{1}{\sqrt{1-t^2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \frac{t^{2k}}{k!}$

214. $F(x) = \int_0^x \frac{\sin t}{t}dt; \quad f(t) = \frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$

215. $F(x) = \int_0^x \cos(\sqrt{t})\,dt; \quad f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$

216. $F(x) = \int_0^x \frac{1-\cos t}{t^2}dt; \quad f(t) = \frac{1-\cos t}{t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+2)!}$

217. $F(x) = \int_0^x \frac{\ln(1+t)}{t}dt; \quad f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n+1}$

In the following exercises, compute at least the first three nonzero terms (not necessarily a quadratic polynomial) of the Maclaurin series of f.

218. $f(x) = \sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)$

219. $f(x) = \tan x$

220. $f(x) = \ln(\cos x)$

221. $f(x) = e^x \cos x$

222. $f(x) = e^{\sin x}$

223. $f(x) = \sec^2 x$

224. $f(x) = \tanh x$

225. $f(x) = \frac{\tan\sqrt{x}}{\sqrt{x}}$ (see expansion for $\tan x$)

In the following exercises, find the radius of convergence of the Maclaurin series of each function.

226. $\ln(1+x)$

227. $\frac{1}{1+x^2}$

228. $\tan^{-1} x$

229. $\ln\left(1+x^2\right)$

230. Find the Maclaurin series of $\sinh x = \frac{e^x - e^{-x}}{2}$.

231. Find the Maclaurin series of $\cosh x = \frac{e^x + e^{-x}}{2}$.

232. Differentiate term by term the Maclaurin series of $\sinh x$ and compare the result with the Maclaurin series of $\cosh x$.

233. **[T]** Let $S_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ and $C_n(x) = \sum_{n=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$ denote the respective Maclaurin polynomials of degree $2n+1$ of $\sin x$ and degree $2n$ of $\cos x$. Plot the errors $\frac{S_n(x)}{C_n(x)} - \tan x$ for $n = 1, .., 5$ and compare them to $x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} - \tan x$ on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

234. Use the identity $2\sin x \cos x = \sin(2x)$ to find the power series expansion of $\sin^2 x$ at $x = 0$. (*Hint:* Integrate the Maclaurin series of $\sin(2x)$ term by term.)

235. If $y = \sum_{n=0}^{\infty} a_n x^n$, find the power series expansions of xy' and $x^2 y''$.

236. **[T]** Suppose that $y = \sum_{k=0}^{\infty} a_k x^k$ satisfies $y' = -2xy$ and $y(0) = 0$. Show that $a_{2k+1} = 0$ for all k and that $a_{2k+2} = \frac{-a_{2k}}{k+1}$. Plot the partial sum S_{20} of y on the interval $[-4, 4]$.

237. **[T]** Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 10$. Set up an integral that represents the probability that a test score will be between 90 and 110 and use the integral of the degree 10 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to estimate this probability.

238. **[T]** Suppose that a set of standardized test scores is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 10$. Set up an integral that represents the probability that a test score will be between 70 and 130 and use the integral of the degree 50 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ to estimate this probability.

239. **[T]** Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 1$, $f'(0) = 0$, and $f''(x) = -f(x)$. Find a formula for a_n and plot the partial sum S_N for $N = 20$ on $[-5, 5]$.

240. **[T]** Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 0$, $f'(0) = 1$, and $f''(x) = -f(x)$. Find a formula for a_n and plot the partial sum S_N for $N = 10$ on $[-5, 5]$.

241. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 1$ and $y'(0) = 0$. Find a formula that relates a_{n+2}, a_{n+1}, and a_n and compute $a_0, ..., a_5$.

242. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 0$ and $y'(0) = 1$. Find a formula that relates a_{n+2}, a_{n+1}, and a_n and compute $a_1, ..., a_5$.

The error in approximating the integral $\int_a^b f(t)\,dt$ by that of a Taylor approximation $\int_a^b P_n(t)\,dt$ is at most $\int_a^b R_n(t)\,dt$. In the following exercises, the Taylor remainder estimate $R_n \le \frac{M}{(n+1)!}|x-a|^{n+1}$ guarantees that the integral of the Taylor polynomial of the given order approximates the integral of f with an error less than $\frac{1}{10}$.

a. Evaluate the integral of the appropriate Taylor polynomial and verify that it approximates the CAS value with an error less than $\frac{1}{100}$.
b. Compare the accuracy of the polynomial integral estimate with the remainder estimate.

243. **[T]** $\int_0^{\pi} \frac{\sin t}{t} dt$; $P_s = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$ (You may assume that the absolute value of the ninth derivative of $\frac{\sin t}{t}$ is bounded by 0.1.)

244. **[T]** $\int_0^2 e^{-x^2} dx$; $p_{11} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots - \frac{x^{22}}{11!}$ (You may assume that the absolute value of the 23rd derivative of e^{-x^2} is less than 2×10^{14}.)

The following exercises deal with Fresnel integrals.

245. The Fresnel integrals are defined by $C(x) = \int_0^x \cos\left(t^2\right) dt$ and $S(x) = \int_0^x \sin\left(t^2\right) dt$. Compute the power series of $C(x)$ and $S(x)$ and plot the sums $C_N(x)$ and $S_N(x)$ of the first $N = 50$ nonzero terms on $[0, 2\pi]$.

246. **[T]** The Fresnel integrals are used in design applications for roadways and railways and other applications because of the curvature properties of the curve with coordinates $(C(t), S(t))$. Plot the curve $\left(C_{50}, S_{50}\right)$ for $0 \le t \le 2\pi$, the coordinates of which were computed in the previous exercise.

247. Estimate $\int_0^{1/4} \sqrt{x - x^2} dx$ by approximating $\sqrt{1 - x}$ using the binomial approximation $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{2128} - \frac{7x^5}{256}$.

248. **[T]** Use Newton's approximation of the binomial $\sqrt{1 - x^2}$ to approximate π as follows. The circle centered at $\left(\frac{1}{2}, 0\right)$ with radius $\frac{1}{2}$ has upper semicircle $y = \sqrt{x}\sqrt{1 - x}$. The sector of this circle bounded by the x -axis between $x = 0$ and $x = \frac{1}{2}$ and by the line joining $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$ corresponds to $\frac{1}{6}$ of the circle and has area $\frac{\pi}{24}$. This sector is the union of a right triangle with height $\frac{\sqrt{3}}{4}$ and base $\frac{1}{4}$ and the region below the graph between $x = 0$ and $x = \frac{1}{4}$. To find the area of this region you can write $y = \sqrt{x}\sqrt{1 - x} = \sqrt{x} \times \left(\text{binomial expansion of} \sqrt{1 - x}\right)$ and integrate term by term. Use this approach with the binomial approximation from the previous exercise to estimate π.

249. Use the approximation $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4}\right)$ to approximate the period of a pendulum having length 10 meters and maximum angle $\theta_{\max} = \frac{\pi}{6}$ where $k = \sin\left(\frac{\theta_{\max}}{2}\right)$. Compare this with the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$.

250. Suppose that a pendulum is to have a period of 2 seconds and a maximum angle of $\theta_{\max} = \frac{\pi}{6}$. Use $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4}\right)$ to approximate the desired length of the pendulum. What length is predicted by the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$?

251. Evaluate $\int_0^{\pi/2} \sin^4 \theta d\theta$ in the approximation $T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1 + \frac{1}{2}k^2 \sin^2 \theta + \frac{3}{8}k^4 \sin^4 \theta + \cdots\right) d\theta$ to obtain an improved estimate for T.

252. **[T]** An equivalent formula for the period of a pendulum with amplitude $\theta_{\max}$ is $T(\theta_{\max}) = 2\sqrt{2}\sqrt{\frac{L}{g}}\int_0^{\theta_{\max}} \frac{d\theta}{\sqrt{\cos\theta - \cos(\theta_{\max})}}$ where L is the pendulum length and g is the gravitational acceleration constant. When $\theta_{\max} = \frac{\pi}{3}$ we get $\frac{1}{\sqrt{\cos t - 1/2}} \approx \sqrt{2}\left(1 + \frac{t^2}{2} + \frac{t^4}{3} + \frac{181t^6}{720}\right)$. Integrate this approximation to estimate $T\left(\frac{\pi}{3}\right)$ in terms of L and g. Assuming $g = 9.806$ meters per second squared, find an approximate length L such that $T\left(\frac{\pi}{3}\right) = 2$ seconds.

CHAPTER 6 REVIEW

KEY TERMS

binomial series the Maclaurin series for $f(x) = (1 + x)^r$; it is given by

$$(1 + x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots \text{ for } |x| < 1$$

interval of convergence the set of real numbers x for which a power series converges

Maclaurin polynomial a Taylor polynomial centered at 0; the nth Taylor polynomial for f at 0 is the nth Maclaurin polynomial for f

Maclaurin series a Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f

nonelementary integral an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

power series a series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x = a$

radius of convergence if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for $|x - a| < R$ and diverges for $|x - a| > R$, then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$

Taylor polynomials the nth Taylor polynomial for f at $x = a$ is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor series a power series at a that converges to a function f on some open interval containing a

Taylor's theorem with remainder for a function f and the nth Taylor polynomial for f at $x = a$, the remainder

$$R_n(x) = f(x) - p_n(x) \text{ satisfies } R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between x and a; if there exists an interval I containing a and a real number M such that $\left|f^{(n+1)}(x)\right| \le M$ for all x in I, then $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$

term-by-term differentiation of a power series a technique for evaluating the derivative of a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ by evaluating the derivative of each term separately to create the new power series $\sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$

term-by-term integration of a power series a technique for integrating a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ by integrating each term separately to create the new power series $C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

KEY EQUATIONS

- **Power series centered at $\mathbf{x = 0}$**

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

- **Power series centered at $\mathbf{x = a}$**

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

- **Taylor series for the function f at the point $\mathbf{x = a}$**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

KEY CONCEPTS

6.1 Power Series and Functions

- For a power series centered at $x = a$, one of the following three properties hold:
  i. The power series converges only at $x = a$. In this case, we say that the radius of convergence is $R = 0$.
  ii. The power series converges for all real numbers x. In this case, we say that the radius of convergence is $R = \infty$.
  iii. There is a real number R such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$. In this case, the radius of convergence is R.
- If a power series converges on a finite interval, the series may or may not converge at the endpoints.
- The ratio test may often be used to determine the radius of convergence.
- The geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$ allows us to represent certain functions using geometric series.

6.2 Properties of Power Series

- Given two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ that converge to functions f and g on a common interval I, the sum and difference of the two series converge to $f \pm g$, respectively, on I. In addition, for any real number b and integer $m \geq 0$, the series $\sum_{n=0}^{\infty} bx^m c_n x^n$ converges to $bx^m f(x)$ and the series $\sum_{n=0}^{\infty} c_n (bx^m)^n$ converges to $f(bx^m)$ whenever bx^m is in the interval I.
- Given two power series that converge on an interval $(-R, R)$, the Cauchy product of the two power series converges on the interval $(-R, R)$.
- Given a power series that converges to a function f on an interval $(-R, R)$, the series can be differentiated term-by-term and the resulting series converges to f' on $(-R, R)$. The series can also be integrated term-by-term and the resulting series converges to $\int f(x)dx$ on $(-R, R)$.

6.3 Taylor and Maclaurin Series

- Taylor polynomials are used to approximate functions near a value $x = a$. Maclaurin polynomials are Taylor polynomials at $x = 0$.

- The nth degree Taylor polynomials for a function f are the partial sums of the Taylor series for f.
- If a function f has a power series representation at $x = a$, then it is given by its Taylor series at $x = a$.
- A Taylor series for f converges to f if and only if $\lim_{n \to \infty} R_n(x) = 0$ where $R_n(x) = f(x) - p_n(x)$.
- The Taylor series for e^x, $\sin x$, and $\cos x$ converge to the respective functions for all real x.

6.4 Working with Taylor Series

- The binomial series is the Maclaurin series for $f(x) = (1 + x)^r$. It converges for $|x| < 1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.
- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

CHAPTER 6 REVIEW EXERCISES

True or False? In the following exercises, justify your answer with a proof or a counterexample.

253. If the radius of convergence for a power series $\sum_{n=0}^{\infty} a_n x^n$ is 5, then the radius of convergence for the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also 5.

254. Power series can be used to show that the derivative of e^x is e^x. (*Hint:* Recall that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.)

255. For small values of x, $\sin x \approx x$.

256. The radius of convergence for the Maclaurin series of $f(x) = 3^x$ is 3.

In the following exercises, find the radius of convergence and the interval of convergence for the given series.

257. $\sum_{n=0}^{\infty} n^2 (x-1)^n$

258. $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$

259. $\sum_{n=0}^{\infty} \frac{3nx^n}{12^n}$

260. $\sum_{n=0}^{\infty} \frac{2^n}{e^n} (x-e)^n$

In the following exercises, find the power series representation for the given function. Determine the radius of convergence and the interval of convergence for that series.

261. $f(x) = \frac{x^2}{x+3}$

262. $f(x) = \frac{8x+2}{2x^2 - 3x + 1}$

In the following exercises, find the power series for the given function using term-by-term differentiation or integration.

263. $f(x) = \tan^{-1}(2x)$

264. $f(x) = \frac{x}{\left(2 + x^2\right)^2}$

In the following exercises, evaluate the Taylor series expansion of degree four for the given function at the specified point. What is the error in the approximation?

265. $f(x) = x^3 - 2x^2 + 4,\ a = -3$

266. $f(x) = e^{1/(4x)},\ a = 4$

In the following exercises, find the Maclaurin series for the given function.

267. $f(x) = \cos(3x)$

268. $f(x) = \ln(x + 1)$

In the following exercises, find the Taylor series at the given value.

269. $f(x) = \sin x,\ a = \frac{\pi}{2}$

270. $f(x) = \frac{3}{x},\ a = 1$

In the following exercises, find the Maclaurin series for the given function.

271. $f(x) = e^{-x^2} - 1$

272. $f(x) = \cos x - x\sin x$

In the following exercises, find the Maclaurin series for $F(x) = \int_0^x f(t)dt$ by integrating the Maclaurin series of $f(x)$ term by term.

273. $f(x) = \frac{\sin x}{x}$

274. $f(x) = 1 - e^x$

275. Use power series to prove Euler's formula: $e^{ix} = \cos x + i\sin x$

The following exercises consider problems of annuity payments.

276. For annuities with a present value of $\$1$ million, calculate the annual payouts given over 25 years assuming interest rates of $1\%,\ 5\%,$ and $10\%.$

277. A lottery winner has an annuity that has a present value of $\$10$ million. What interest rate would they need to live on perpetual annual payments of $\$250{,}000$?

278. Calculate the necessary present value of an annuity in order to support annual payouts of $\$15{,}000$ given over 25 years assuming interest rates of $1\%,\ 5\%,$ and $10\%.$

7 | PARAMETRIC EQUATIONS AND POLAR COORDINATES

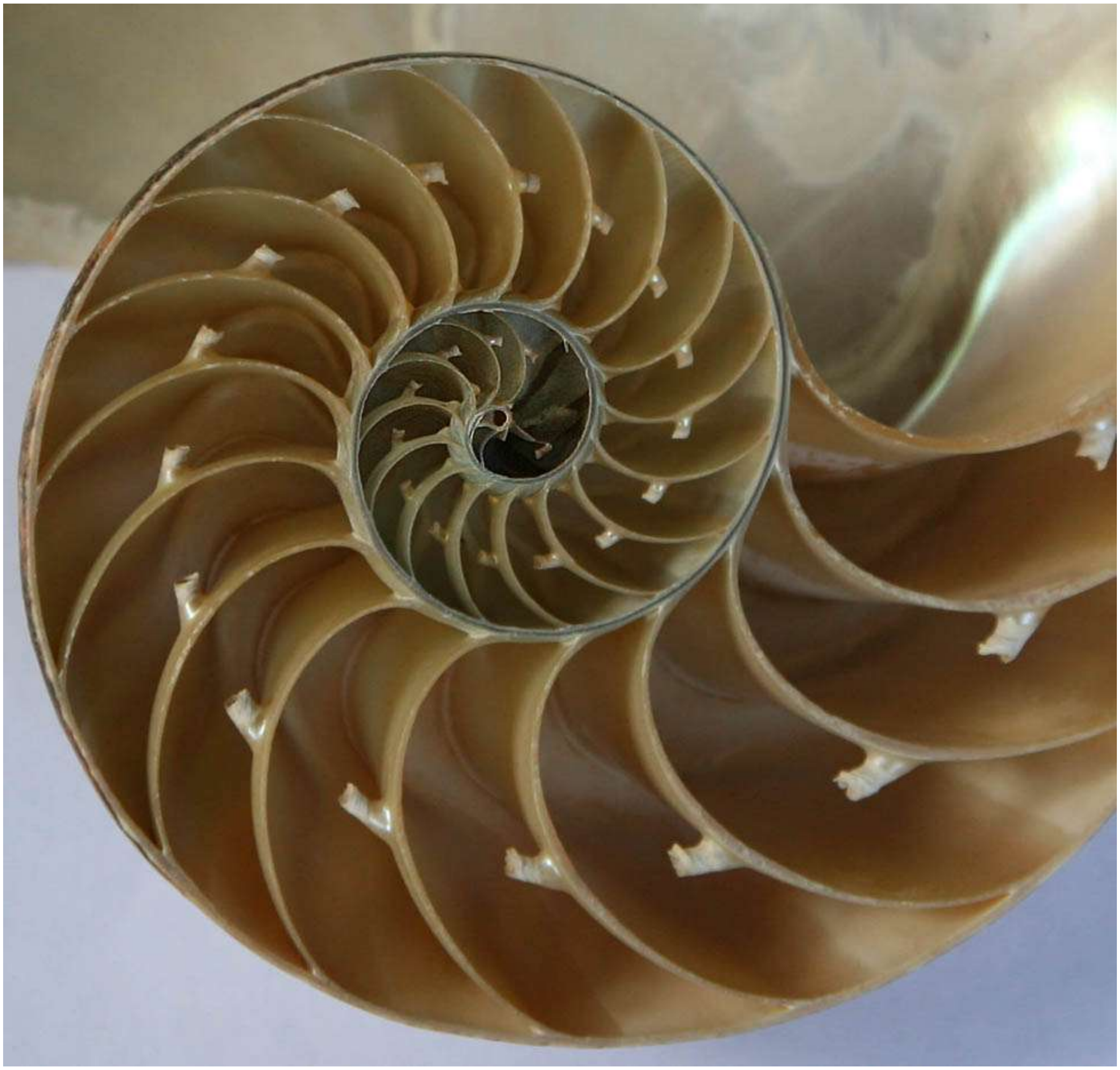


Figure 7.1 The chambered nautilus is a marine animal that lives in the tropical Pacific Ocean. Scientists think they have existed mostly unchanged for about 500 million years.(credit: modification of work by Jitze Couperus, Flickr)

Chapter Outline

Introduction

The chambered nautilus is a fascinating creature. This animal feeds on hermit crabs, fish, and other crustaceans. It has a hard outer shell with many chambers connected in a spiral fashion, and it can retract into its shell to avoid predators. When part of the shell is cut away, a perfect spiral is revealed, with chambers inside that are somewhat similar to growth rings in a tree.

The mathematical function that describes a spiral can be expressed using rectangular (or Cartesian) coordinates. However, if we change our coordinate system to something that works a bit better with circular patterns, the function becomes much simpler to describe. The polar coordinate system is well suited for describing curves of this type. How can we use this coordinate system to describe spirals and other radial figures? (See **Example 7.14**.)

In this chapter we also study parametric equations, which give us a convenient way to describe curves, or to study the position of a particle or object in two dimensions as a function of time. We will use parametric equations and polar coordinates for describing many topics later in this text.

7.1 | Parametric Equations

Learning Objectives

7.1.1 Plot a curve described by parametric equations.
7.1.2 Convert the parametric equations of a curve into the form $y = f(x)$.
7.1.3 Recognize the parametric equations of basic curves, such as a line and a circle.
7.1.4 Recognize the parametric equations of a cycloid.

In this section we examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Parametric Equations and Their Graphs

Consider the orbit of Earth around the Sun. Our year lasts approximately 365.25 days, but for this discussion we will use 365 days. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar. We call January 1 "day 1" of the year. Then, for example, day 31 is January 31, day 59 is February 28, and so on.

The number of the day in a year can be considered a variable that determines Earth's position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According to Kepler's laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse. We study this idea in more detail in **Conic Sections**.

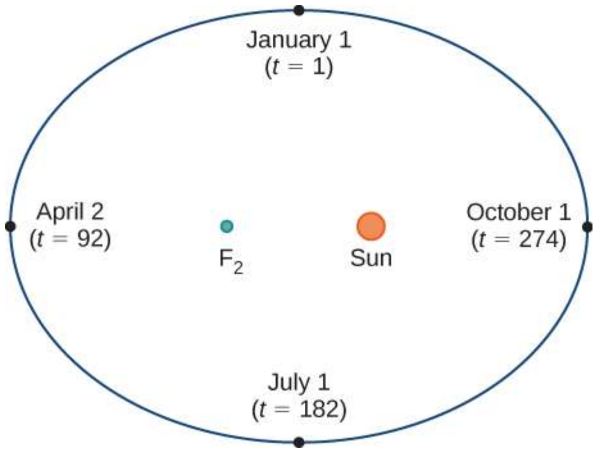


Figure 7.2 Earth's orbit around the Sun in one year.

Figure 7.2 depicts Earth's orbit around the Sun during one year. The point labeled F_2 is one of the foci of the ellipse; the other focus is occupied by the Sun. If we superimpose coordinate axes over this graph, then we can assign ordered pairs to each point on the ellipse (**Figure 7.3**). Then each x value on the graph is a value of position as a function of time, and each y value is also a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth's position as a function of time.

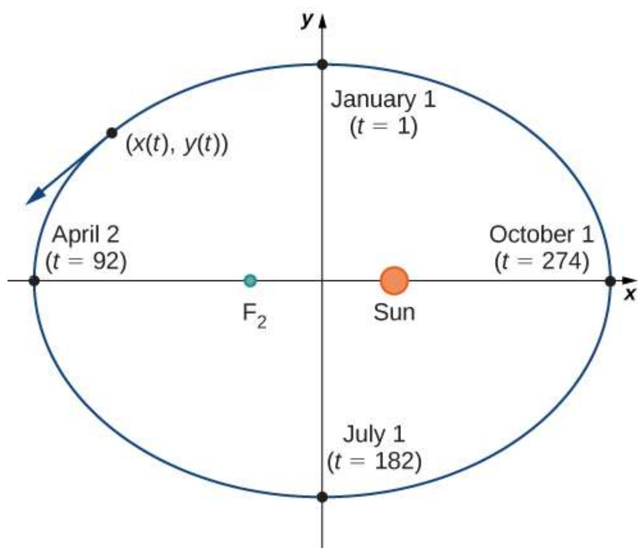


Figure 7.3 Coordinate axes superimposed on the orbit of Earth.

We can determine the functions for $x(t)$ and $y(t)$, thereby parameterizing the orbit of Earth around the Sun. The variable t is called an independent parameter and, in this context, represents time relative to the beginning of each year.

A curve in the (x, y) plane can be represented parametrically. The equations that are used to define the curve are called **parametric equations**.

Definition

If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t) \text{ and } y = y(t)$$

are called parametric equations and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a **parametric curve** or *plane curve*, and is denoted by C.

Notice in this definition that x and y are used in two ways. The first is as functions of the independent variable t. As t varies over the interval I, the functions $x(t)$ and $y(t)$ generate a set of ordered pairs (x, y). This set of ordered pairs generates the graph of the parametric equations. In this second usage, to designate the ordered pairs, x and y are variables. It is important to distinguish the variables x and y from the functions $x(t)$ and $y(t)$.

Example 7.1

Graphing a Parametrically Defined Curve

Sketch the curves described by the following parametric equations:

a. $x(t) = t - 1, \quad y(t) = 2t + 4, \quad -3 \le t \le 2$

b. $x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad -2 \le t \le 3$

c. $x(t) = 4\cos t, \quad y(t) = 4\sin t, \quad 0 \le t \le 2\pi$

Solution

a. To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is t, let t appear in the first column. Then $x(t)$ and $y(t)$ will appear in the second and third columns of the table.

t	$x(t)$	$y(t)$
−3	−4	−2
−2	−3	0
−1	−2	2
0	−1	4
1	0	6
2	1	8

The second and third columns in this table provide a set of points to be plotted. The graph of these points appears in Figure 7.4. The arrows on the graph indicate the **orientation** of the graph, that is, the direction that a point moves on the graph as t varies from −3 to 2.

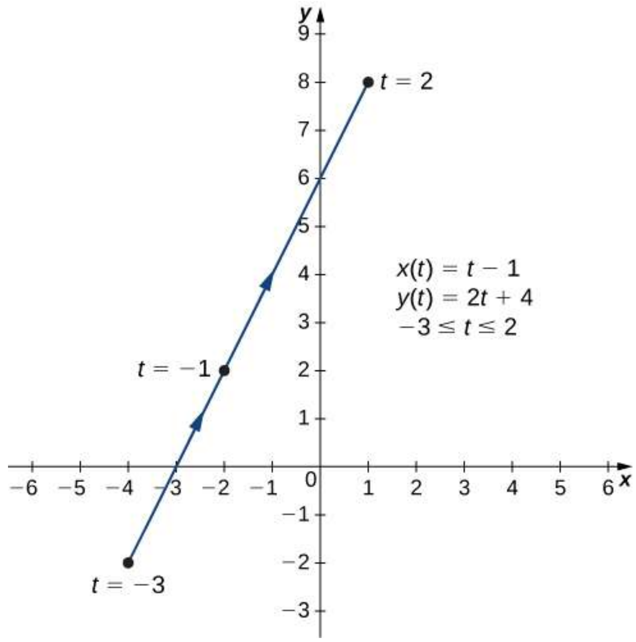


Figure 7.4 Graph of the plane curve described by the parametric equations in part a.

b. To create a graph of this curve, again set up a table of values.

t	$x(t)$	$y(t)$
−2	1	−3
−1	−2	−1
0	−3	1
1	−2	3
2	1	5
3	6	7

The second and third columns in this table give a set of points to be plotted (**Figure 7.5**). The first point on the graph (corresponding to $t = -2$) has coordinates $(1, -3)$, and the last point (corresponding to $t = 3$) has coordinates $(6, 7)$. As t progresses from −2 to 3, the point on the curve travels along a parabola. The direction the point moves is again called the orientation and is indicated on the graph.

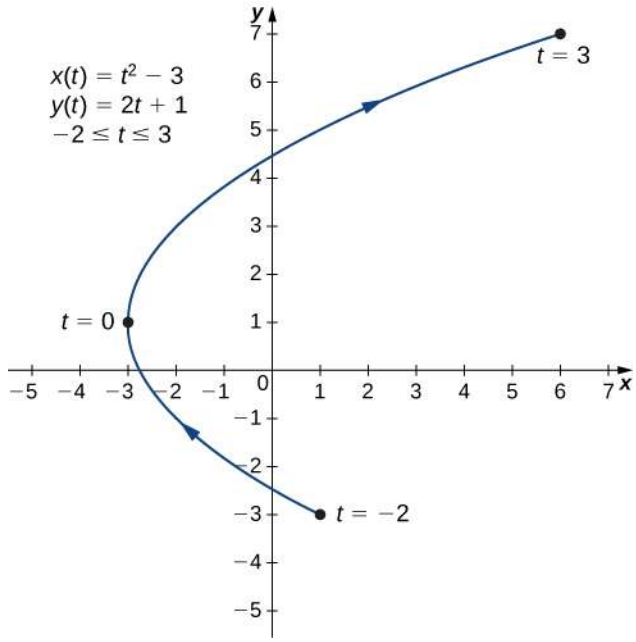


Figure 7.5 Graph of the plane curve described by the parametric equations in part b.

c. In this case, use multiples of $\pi/6$ for t and create another table of values:

t	$x(t)$	$y(t)$		t	$x(t)$	$y(t)$
0	4	0		$\frac{7\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2		$\frac{4\pi}{3}$	-2	$-2\sqrt{3} \approx -3.5$
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$		$\frac{3\pi}{2}$	0	-4
$\frac{\pi}{2}$	0	4		$\frac{5\pi}{3}$	2	$-2\sqrt{3} \approx -3.5$
$\frac{2\pi}{3}$	-2	$2\sqrt{3} \approx 3.5$		$\frac{11\pi}{6}$	$2\sqrt{3} \approx 3.5$	2
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2		2π	4	0
π	-4	0				

The graph of this plane curve appears in the following graph.

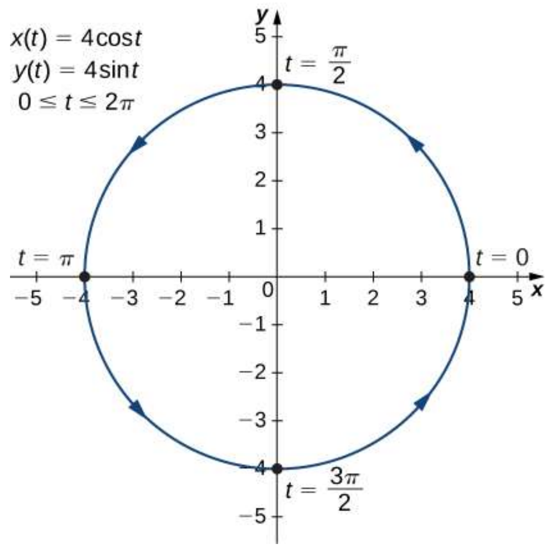


Figure 7.6 Graph of the plane curve described by the parametric equations in part c.

This is the graph of a circle with radius 4 centered at the origin, with a counterclockwise orientation. The starting point and ending points of the curve both have coordinates $(4, 0)$.

7.1 Sketch the curve described by the parametric equations

$$x(t) = 3t + 2, \quad y(t) = t^2 - 1, \quad -3 \leq t \leq 2.$$

Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y. Then we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve in **Example 7.1**b. are

$$x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad -2 \leq t \leq 3.$$

Solving the second equation for t gives

$$t = \frac{y-1}{2}.$$

This can be substituted into the first equation:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 = \frac{y^2 - 2y + 1}{4} - 3 = \frac{y^2 - 2y - 11}{4}.$$

This equation describes x as a function of y. These steps give an example of *eliminating the parameter*. The graph of this function is a parabola opening to the right. Recall that the plane curve started at $(1, -3)$ and ended at $(6, 7)$. These terminations were due to the restriction on the parameter t.

Example 7.2

Eliminating the Parameter

Eliminate the parameter for each of the plane curves described by the following parametric equations and describe the resulting graph.

a. $x(t) = \sqrt{2t+4}, \quad y(t) = 2t+1, \quad -2 \le t \le 6$

b. $x(t) = 4\cos t, \quad y(t) = 3\sin t, \quad 0 \le t \le 2\pi$

Solution

a. To eliminate the parameter, we can solve either of the equations for t. For example, solving the first equation for t gives

$$\begin{aligned} x &= \sqrt{2t+4} \\ x^2 &= 2t+4 \\ x^2 - 4 &= 2t \\ t &= \frac{x^2-4}{2}. \end{aligned}$$

Note that when we square both sides it is important to observe that $x \ge 0$. Substituting $t = \frac{x^2-4}{2}$ this into $y(t)$ yields

$$\begin{aligned} y(t) &= 2t+1 \\ y &= 2\left(\frac{x^2-4}{2}\right)+1 \\ y &= x^2-4+1 \\ y &= x^2-3. \end{aligned}$$

This is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter t. When $t = -2$, $x = \sqrt{2(-2)+4} = 0$, and when $t = 6$, $x = \sqrt{2(6)+4} = 4$. The graph of this plane curve follows.

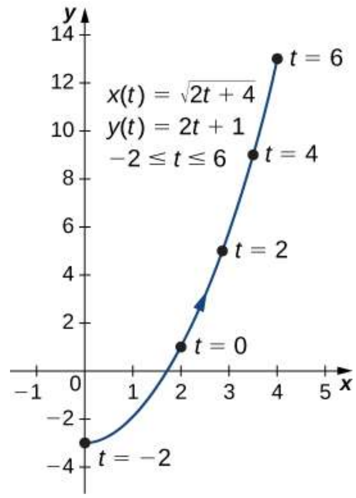


Figure 7.7 Graph of the plane curve described by the parametric equations in part a.

b. Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

$$x(t) = 4\cos t \text{ and } y(t) = 3\sin t.$$

Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

$$\cos t = \frac{x}{4} \text{ and } \sin t = \frac{y}{3}.$$

Now use the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y. This gives

$$\begin{aligned}\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 &= 1\\ \frac{x^2}{16} + \frac{y^2}{9} &= 1.\end{aligned}$$

This is the equation of a horizontal ellipse centered at the origin, with semimajor axis 4 and semiminor axis 3 as shown in the following graph.

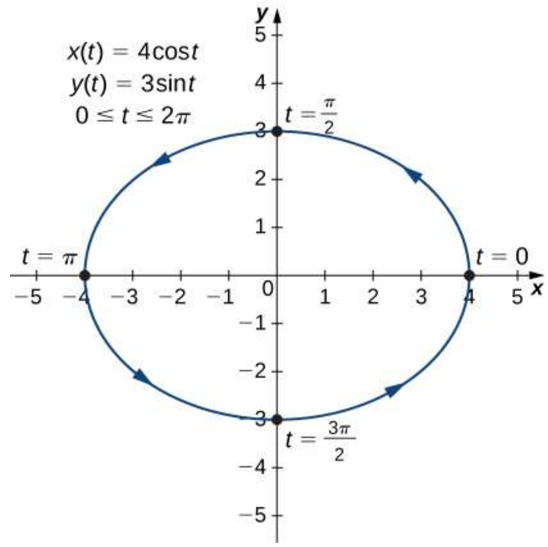


Figure 7.8 Graph of the plane curve described by the parametric equations in part b.

As t progresses from 0 to $2\pi,$ a point on the curve traverses the ellipse once, in a counterclockwise direction. Recall from the section opener that the orbit of Earth around the Sun is also elliptical. This is a perfect example of using parameterized curves to model a real-world phenomenon.

7.2 Eliminate the parameter for the plane curve defined by the following parametric equations and describe the resulting graph.

$$x(t) = 2 + \frac{3}{t}, \quad y(t) = t - 1, \quad 2 \le t \le 6$$

So far we have seen the method of eliminating the parameter, assuming we know a set of parametric equations that describe a plane curve. What if we would like to start with the equation of a curve and determine a pair of parametric equations for that curve? This is certainly possible, and in fact it is possible to do so in many different ways for a given curve. The process is known as **parameterization of a curve**.

Example 7.3

Parameterizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3.$

Solution

First, it is always possible to parameterize a curve by defining $x(t) = t,$ then replacing x with t in the equation for $y(t).$ This gives the parameterization

$$x(t) = t, \quad y(t) = 2t^2 - 3.$$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of t.

We have complete freedom in the choice for the second parameterization. For example, we can choose $x(t) = 3t - 2.$ The only thing we need to check is that there are no restrictions imposed on x; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t - 2.$ Now since $y = 2x^2 - 3,$ we can substitute $x(t) = 3t - 2$ for x. This gives

$$\begin{aligned} y(t) &= 2(3t-2)^2 - 2 \\ &= 2\left(9t^2 - 12t + 4\right) - 2 \\ &= 18t^2 - 24t + 8 - 2 \\ &= 18t^2 - 24t + 6. \end{aligned}$$

Therefore, a second parameterization of the curve can be written as

$$x(t) = 3t - 2 \text{ and } y(t) = 18t^2 - 24t + 6.$$

7.3 Find two different sets of parametric equations to represent the graph of $y = x^2 + 2x$.

Cycloids and Other Parametric Curves

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a **cycloid** (**Figure 7.9**). A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the x-axis at a constant height equal to the radius of the wheel. If the radius is a, then the coordinates of the center can be given by the equations

$$x(t) = at, \quad y(t) = a$$

for any value of t. Next, consider the ant, which rotates around the center along a circular path. If the bicycle is moving from left to right then the wheels are rotating in a clockwise direction. A possible parameterization of the circular motion of the ant (relative to the center of the wheel) is given by

$$x(t) = -a \sin t, \quad y(t) = -a \cos t.$$

(The negative sign is needed to reverse the orientation of the curve. If the negative sign were not there, we would have to imagine the wheel rotating counterclockwise.) Adding these equations together gives the equations for the cycloid.

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

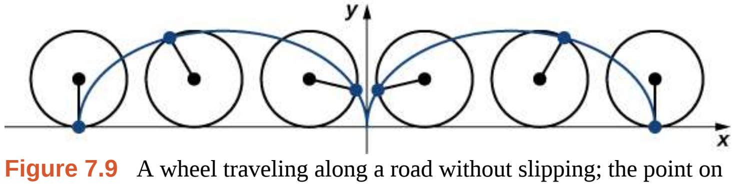


Figure 7.9 A wheel traveling along a road without slipping; the point on the edge of the wheel traces out a cycloid.

Now suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in **Figure 7.10**. In this graph, the green circle is traveling around the blue circle in a counterclockwise direction. A point on the edge of the green circle traces out the red graph, which is called a hypocycloid.

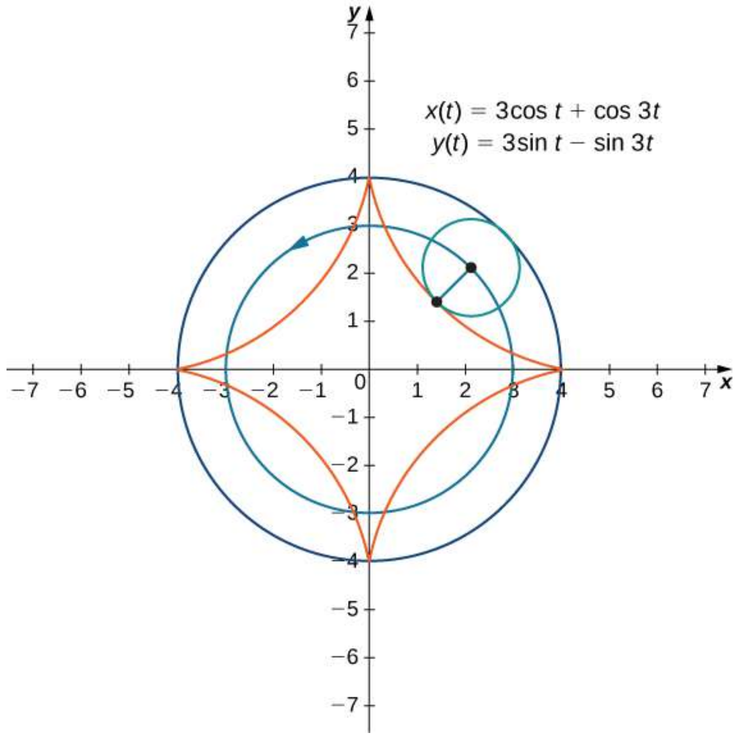


Figure 7.10 Graph of the hypocycloid described by the parametric equations shown.

The general parametric equations for a hypocycloid are

$$x(t) = (a - b)\cos t + b\cos\left(\frac{a-b}{b}\right)t$$
$$y(t) = (a - b)\sin t - b\sin\left(\frac{a-b}{b}\right)t.$$

These equations are a bit more complicated, but the derivation is somewhat similar to the equations for the cycloid. In this case we assume the radius of the larger circle is *a* and the radius of the smaller circle is *b*. Then the center of the wheel travels along a circle of radius $a - b.$ This fact explains the first term in each equation above. The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\frac{2\pi b}{a-b}.$

The ratio $\frac{a}{b}$ is related to the number of **cusps** on the graph (cusps are the corners or pointed ends of the graph), as illustrated in **Figure 7.11**. This ratio can lead to some very interesting graphs, depending on whether or not the ratio is rational. **Figure 7.10** corresponds to $a = 4$ and $b = 1.$ The result is a hypocycloid with four cusps. **Figure 7.11** shows some other possibilities. The last two hypocycloids have irrational values for $\frac{a}{b}.$ In these cases the hypocycloids have an infinite number of cusps, so they never return to their starting point. These are examples of what are known as *space-filling curves*.

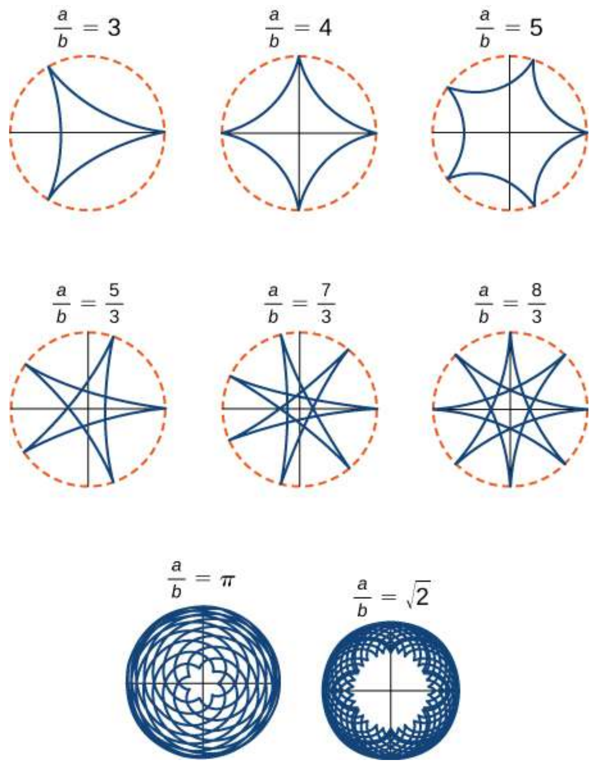


Figure 7.11 Graph of various hypocycloids corresponding to different values of a/b.

Student PROJECT

The Witch of Agnesi

Many plane curves in mathematics are named after the people who first investigated them, like the folium of Descartes or the spiral of Archimedes. However, perhaps the strangest name for a curve is the witch of Agnesi. Why a witch?

Maria Gaetana Agnesi (1718–1799) was one of the few recognized women mathematicians of eighteenth-century Italy. She wrote a popular book on analytic geometry, published in 1748, which included an interesting curve that had been studied by Fermat in 1630. The mathematician Guido Grandi showed in 1703 how to construct this curve, which he later called the “versoria,” a Latin term for a rope used in sailing. Agnesi used the Italian term for this rope, “versiera,” but in Latin, this same word means a “female goblin.” When Agnesi’s book was translated into English in 1801, the translator used the term “witch” for the curve, instead of rope. The name “witch of Agnesi” has stuck ever since.

The witch of Agnesi is a curve defined as follows: Start with a circle of radius *a* so that the points $(0, 0)$ and $(0, 2a)$ are points on the circle (Figure 7.12). Let *O* denote the origin. Choose any other point *A* on the circle, and draw the secant line *OA*. Let *B* denote the point at which the line *OA* intersects the horizontal line through $(0, 2a)$. The vertical line through *B* intersects the horizontal line through *A* at the point *P*. As the point *A* varies, the path that the point *P* travels is the witch of Agnesi curve for the given circle.

Witch of Agnesi curves have applications in physics, including modeling water waves and distributions of spectral lines. In probability theory, the curve describes the probability density function of the Cauchy distribution. In this project you will parameterize these curves.

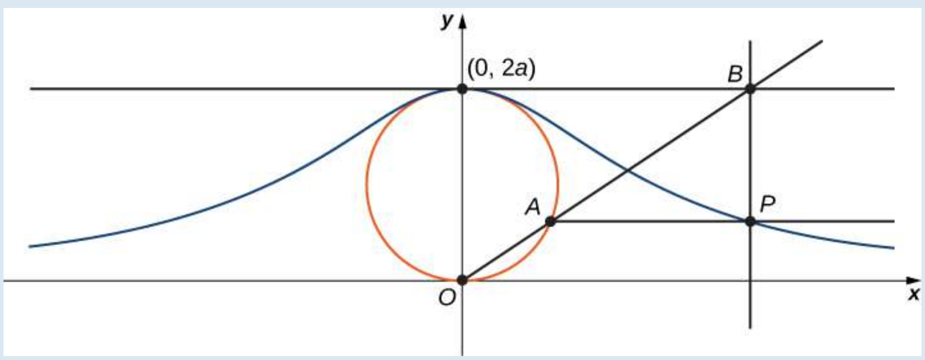


Figure 7.12 As the point *A* moves around the circle, the point *P* traces out the witch of Agnesi curve for the given circle.

1. On the figure, label the following points, lengths, and angle:
   a. *C* is the point on the *x*-axis with the same *x*-coordinate as *A*.
   b. *x* is the *x*-coordinate of *P*, and *y* is the *y*-coordinate of *P*.
   c. *E* is the point $(0, a)$.
   d. *F* is the point on the line segment *OA* such that the line segment *EF* is perpendicular to the line segment *OA*.
   e. *b* is the distance from *O* to *F*.
   f. *c* is the distance from *F* to *A*.
   g. *d* is the distance from *O* to *B*.
   h. θ is the measure of angle $\angle COA$.

   The goal of this project is to parameterize the witch using θ as a parameter. To do this, write equations for *x* and *y* in terms of only θ.

2. Show that $d = \frac{2a}{\sin\theta}$.
3. Note that $x = d\cos\theta$. Show that $x = 2a\cot\theta$. When you do this, you will have parameterized the x-coordinate of the curve with respect to θ. If you can get a similar equation for y, you will have parameterized the curve.
4. In terms of θ, what is the angle $\angle EOA$?
5. Show that $b + c = 2a\cos\left(\frac{\pi}{2} - \theta\right)$.
6. Show that $y = 2a\cos\left(\frac{\pi}{2} - \theta\right)\sin\theta$.
7. Show that $y = 2a\sin^2\theta$. You have now parameterized the y-coordinate of the curve with respect to θ.
8. Conclude that a parameterization of the given witch curve is
$$x = 2a\cot\theta,\ y = 2a\sin^2\theta,\ -\infty < \theta < \infty.$$
9. Use your parameterization to show that the given witch curve is the graph of the function $f(x) = \frac{8a^3}{x^2 + 4a^2}$.

Student PROJECT

Travels with My Ant: The Curtate and Prolate Cycloids

Earlier in this section, we looked at the parametric equations for a cycloid, which is the path a point on the edge of a wheel traces as the wheel rolls along a straight path. In this project we look at two different variations of the cycloid, called the curtate and prolate cycloids.

First, let's revisit the derivation of the parametric equations for a cycloid. Recall that we considered a tenacious ant trying to get home by hanging onto the edge of a bicycle tire. We have assumed the ant climbed onto the tire at the very edge, where the tire touches the ground. As the wheel rolls, the ant moves with the edge of the tire (**Figure 7.13**).

As we have discussed, we have a lot of flexibility when parameterizing a curve. In this case we let our parameter t represent the angle the tire has rotated through. Looking at **Figure 7.13**, we see that after the tire has rotated through an angle of t, the position of the center of the wheel, $C = (x_C, y_C)$, is given by

$$x_C = at \text{ and } y_C = a.$$

Furthermore, letting $A = (x_A, y_A)$ denote the position of the ant, we note that

$$x_C - x_A = a \sin t \text{ and } y_C - y_A = a \cos t.$$

Then

$$\begin{aligned} x_A &= x_C - a \sin t = at - a \sin t = a(t - \sin t) \\ y_A &= y_C - a \cos t = a - a \cos t = a(1 - \cos t). \end{aligned}$$

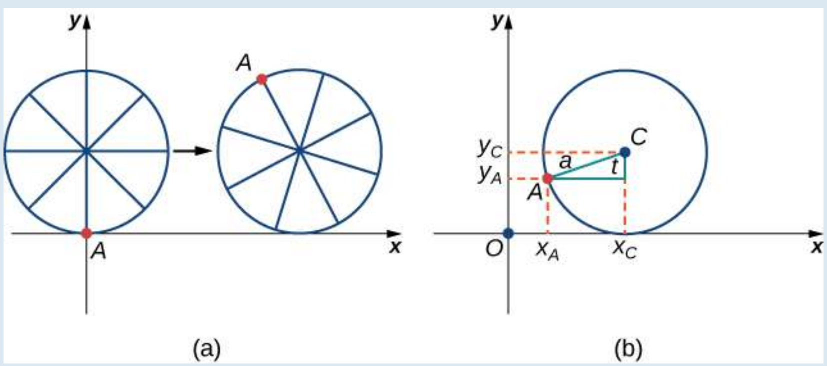


Figure 7.13 (a) The ant clings to the edge of the bicycle tire as the tire rolls along the ground. (b) Using geometry to determine the position of the ant after the tire has rotated through an angle of t.

Note that these are the same parametric representations we had before, but we have now assigned a physical meaning to the parametric variable t.

After a while the ant is getting dizzy from going round and round on the edge of the tire. So he climbs up one of the spokes toward the center of the wheel. By climbing toward the center of the wheel, the ant has changed his path of motion. The new path has less up-and-down motion and is called a curtate cycloid (**Figure 7.14**). As shown in the figure, we let b denote the distance along the spoke from the center of the wheel to the ant. As before, we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant.

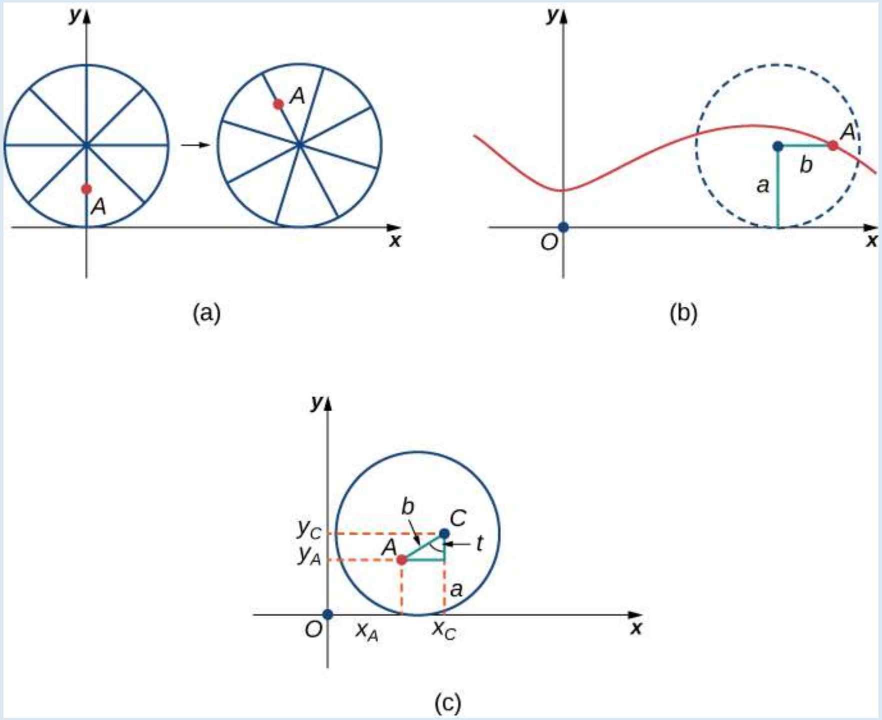


Figure 7.14 (a) The ant climbs up one of the spokes toward the center of the wheel. (b) The ant's path of motion after he climbs closer to the center of the wheel. This is called a curtate cycloid. (c) The new setup, now that the ant has moved closer to the center of the wheel.

1. What is the position of the center of the wheel after the tire has rotated through an angle of t?
2. Use geometry to find expressions for $x_C - x_A$ and for $y_C - y_A$.
3. On the basis of your answers to parts 1 and 2, what are the parametric equations representing the curtate cycloid?

   Once the ant's head clears, he realizes that the bicyclist has made a turn, and is now traveling away from his home. So he drops off the bicycle tire and looks around. Fortunately, there is a set of train tracks nearby, headed back in the right direction. So the ant heads over to the train tracks to wait. After a while, a train goes by, heading in the right direction, and he manages to jump up and just catch the edge of the train wheel (without getting squished!).

   The ant is still worried about getting dizzy, but the train wheel is slippery and has no spokes to climb, so he decides to just hang on to the edge of the wheel and hope for the best. Now, train wheels have a flange to keep the wheel running on the tracks. So, in this case, since the ant is hanging on to the very edge of the flange, the distance from the center of the wheel to the ant is actually greater than the radius of the wheel (**Figure 7.15**).

   The setup here is essentially the same as when the ant climbed up the spoke on the bicycle wheel. We let b denote the distance from the center of the wheel to the ant, and we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant (**Figure 7.15**).

   When the distance from the center of the wheel to the ant is greater than the radius of the wheel, his path of motion is called a prolate cycloid. A graph of a prolate cycloid is shown in the figure.

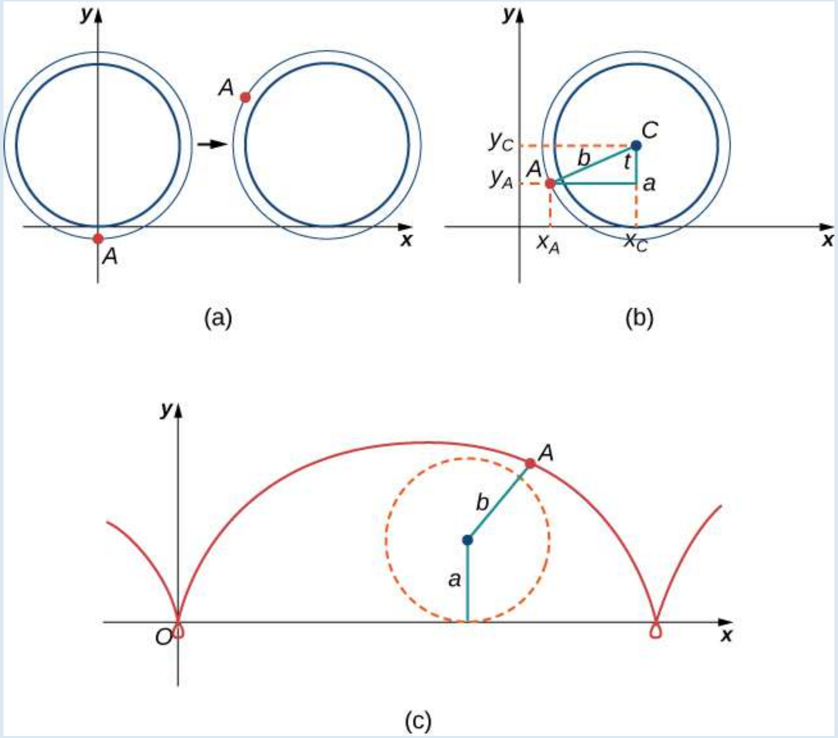


Figure 7.15 (a) The ant is hanging onto the flange of the train wheel. (b) The new setup, now that the ant has jumped onto the train wheel. (c) The ant travels along a prolate cycloid.

4. Using the same approach you used in parts 1– 3, find the parametric equations for the path of motion of the ant.
5. What do you notice about your answer to part 3 and your answer to part 4?
   Notice that the ant is actually traveling backward at times (the “loops” in the graph), even though the train continues to move forward. He is probably going to be *really* dizzy by the time he gets home!

7.1 EXERCISES

For the following exercises, sketch the curves below by eliminating the parameter t. Give the orientation of the curve.

1. $x = t^2 + 2t, \quad y = t + 1$

2. $x = \cos(t),\ y = \sin(t),\ (0,\ 2\pi]$

3. $x = 2t + 4,\ y = t - 1$

4. $x = 3 - t,\ y = 2t - 3,\ 1.5 \le t \le 3$

For the following exercises, eliminate the parameter and sketch the graphs.

5. $x = 2t^2, \quad y = t^4 + 1$

For the following exercises, use technology (CAS or calculator) to sketch the parametric equations.

6. **[T]** $x = t^2 + t, \quad y = t^2 - 1$

7. **[T]** $x = e^{-t}, \quad y = e^{2t} - 1$

8. **[T]** $x = 3\cos t, \quad y = 4\sin t$

9. **[T]** $x = \sec t, \quad y = \cos t$

For the following exercises, sketch the parametric equations by eliminating the parameter. Indicate any asymptotes of the graph.

10. $x = e^t, \quad y = e^{2t} + 1$

11. $x = 6\sin(2\theta),\ y = 4\cos(2\theta)$

12. $x = \cos\theta, \quad y = 2\sin(2\theta)$

13. $x = 3 - 2\cos\theta, \quad y = -5 + 3\sin\theta$

14. $x = 4 + 2\cos\theta, \quad y = -1 + \sin\theta$

15. $x = \sec t, \quad y = \tan t$

16. $x = \ln(2t), \quad y = t^2$

17. $x = e^t, \quad y = e^{2t}$

18. $x = e^{-2t}, \quad y = e^{3t}$

19. $x = t^3, \quad y = 3\ln t$

20. $x = 4\sec\theta, \quad y = 3\tan\theta$

For the following exercises, convert the parametric equations of a curve into rectangular form. No sketch is necessary. State the domain of the rectangular form.

21. $x = t^2 - 1, \quad y = \frac{t}{2}$

22. $x = \frac{1}{\sqrt{t+1}}, \quad y = \frac{t}{1+t},\ t > -1$

23. $x = 4\cos\theta,\ y = 3\sin\theta,\ t \in (0,\ 2\pi]$

24. $x = \cosh t, \quad y = \sinh t$

25. $x = 2t - 3, \quad y = 6t - 7$

26. $x = t^2, \quad y = t^3$

27. $x = 1 + \cos t, \quad y = 3 - \sin t$

28. $x = \sqrt{t}, \quad y = 2t + 4$

29. $x = \sec t, \quad y = \tan t,\ \pi \le t < \frac{3\pi}{2}$

30. $x = 2\cosh t, \quad y = 4\sinh t$

31. $x = \cos(2t), \quad y = \sin t$

32. $x = 4t + 3,\ y = 16t^2 - 9$

33. $x = t^2, \quad y = 2\ln t,\ t \ge 1$

34. $x = t^3, \quad y = 3\ln t,\ t \ge 1$

35. $x = t^n, \quad y = n\ln t,\ t \ge 1,$ where n is a natural number

36. $\begin{array}{l} x = \ln(5t) \\ y = \ln(t^2) \end{array}$ where $1 \le t \le e$

37. $\begin{array}{l} x = 2\sin(8t) \\ y = 2\cos(8t) \end{array}$

38. $\begin{array}{l} x = \tan t \\ y = \sec^2 t - 1 \end{array}$

For the following exercises, the pairs of parametric equations represent lines, parabolas, circles, ellipses, or hyperbolas. Name the type of basic curve that each pair of equations represents.

39. $x = 3t + 4$
$y = 5t - 2$

40. $x - 4 = 5t$
$y + 2 = t$

41. $x = 2t + 1$
$y = t^2 - 3$

42. $x = 3\cos t$
$y = 3\sin t$

43. $x = 2\cos(3t)$
$y = 2\sin(3t)$

44. $x = \cosh t$
$y = \sinh t$

45. $x = 3\cos t$
$y = 4\sin t$

46. $x = 2\cos(3t)$
$y = 5\sin(3t)$

47. $x = 3\cosh(4t)$
$y = 4\sinh(4t)$

48. $x = 2\cosh t$
$y = 2\sinh t$

49. Show that $\begin{matrix} x = h + r\cos\theta \\ y = k + r\sin\theta \end{matrix}$ represents the equation of a circle.

50. Use the equations in the preceding problem to find a set of parametric equations for a circle whose radius is 5 and whose center is $(-2,\ 3)$.

For the following exercises, use a graphing utility to graph the curve represented by the parametric equations and identify the curve from its equation.

51. **[T]** $x = \theta + \sin\theta$
$y = 1 - \cos\theta$

52. **[T]** $x = 2t - 2\sin t$
$y = 2 - 2\cos t$

53. **[T]** $x = t - 0.5\sin t$
$y = 1 - 1.5\cos t$

54. An airplane traveling horizontally at 100 m/s over flat ground at an elevation of 4000 meters must drop an emergency package on a target on the ground. The trajectory of the package is given by $x = 100t,\ y = -4.9t^2 + 4000,\ t \geq 0$ where the origin is the point on the ground directly beneath the plane at the moment of release. How many horizontal meters before the target should the package be released in order to hit the target?

55. The trajectory of a bullet is given by $x = v_0(\cos\alpha)\,t y = v_0(\sin\alpha)\,t - \frac{1}{2}gt^2$ where $v_0 = 500\text{ m/s},$ $g = 9.8 = 9.8\text{ m/s}^2,$ and $\alpha = 30\text{ degrees}.$ When will the bullet hit the ground? How far from the gun will the bullet hit the ground?

56. **[T]** Use technology to sketch the curve represented by $x = \sin(4t),\ y = \sin(3t),\ 0 \leq t \leq 2\pi.$

57. **[T]** Use technology to sketch $x = 2\tan(t),\ y = 3\sec(t),\ -\pi < t < \pi.$

58. Sketch the curve known as an *epitrochoid*, which gives the path of a point on a circle of radius b as it rolls on the outside of a circle of radius a. The equations are

$$x = (a + b)\cos t - c \cdot \cos\left[\frac{(a + b)t}{b}\right]$$
$$y = (a + b)\sin t - c \cdot \sin\left[\frac{(a + b)t}{b}\right].$$

Let $a = 1,\ b = 2,\ c = 1.$

59. **[T]** Use technology to sketch the spiral curve given by $x = t\cos(t),\ y = t\sin(t)$ from $-2\pi \leq t \leq 2\pi.$

60. **[T]** Use technology to graph the curve given by the parametric equations $x = 2\cot(t),\ y = 1 - \cos(2t),\ -\pi/2 \leq t \leq \pi/2.$ This curve is known as the witch of Agnesi.

61. **[T]** Sketch the curve given by parametric equations $\begin{matrix} x = \cosh(t) \\ y = \sinh(t), \end{matrix}$ where $-2 \leq t \leq 2.$

7.2 | Calculus of Parametric Curves

Learning Objectives

7.2.1 Determine derivatives and equations of tangents for parametric curves.
7.2.2 Find the area under a parametric curve.
7.2.3 Use the equation for arc length of a parametric curve.
7.2.4 Apply the formula for surface area to a volume generated by a parametric curve.

Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Another scenario: Suppose we would like to represent the location of a baseball after the ball leaves a pitcher's hand. If the position of the baseball is represented by the plane curve $(x(t), y(t)),$ then we should be able to use calculus to find the speed of the ball at any given time. Furthermore, we should be able to calculate just how far that ball has traveled as a function of time.

Derivatives of Parametric Equations

We start by asking how to calculate the slope of a line tangent to a parametric curve at a point. Consider the plane curve defined by the parametric equations

$$x(t) = 2t + 3, \quad y(t) = 3t - 4, \quad -2 \le t \le 3.$$

The graph of this curve appears in **Figure 7.16**. It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$.

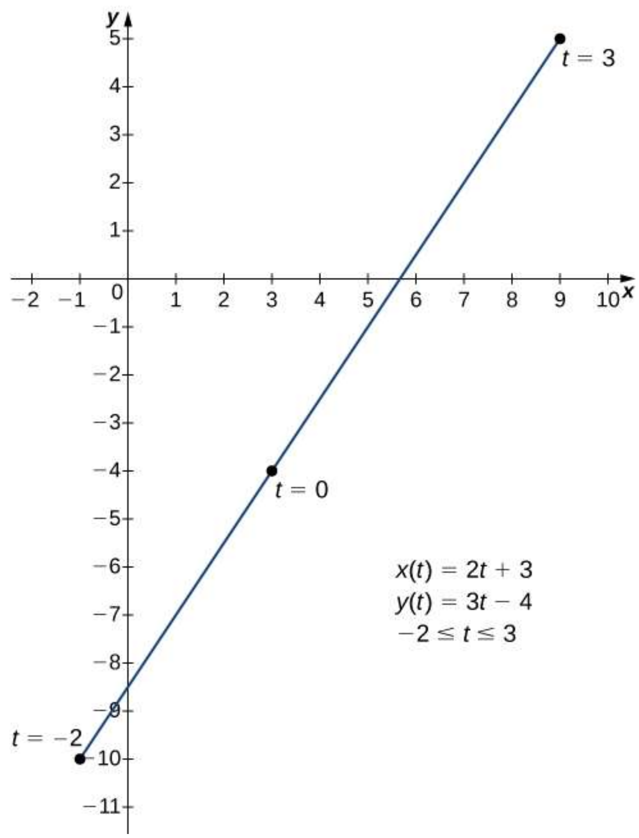


Figure 7.16 Graph of the line segment described by the given parametric equations.

We can eliminate the parameter by first solving the equation $x(t) = 2t + 3$ for t:

$$\begin{aligned} x(t) &= 2t + 3 \\ x - 3 &= 2t \\ t &= \frac{x-3}{2}. \end{aligned}$$

Substituting this into $y(t)$, we obtain

$$\begin{aligned} y(t) &= 3t - 4 \\ y &= 3\left(\frac{x-3}{2}\right) - 4 \\ y &= \frac{3x}{2} - \frac{9}{2} - 4 \\ y &= \frac{3x}{2} - \frac{17}{2}. \end{aligned}$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$. Next we calculate $x'(t)$ and $y'(t)$. This gives $x'(t) = 2$ and $y'(t) = 3$. Notice that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}$. This is no coincidence, as outlined in the following theorem.

Theorem 7.1: Derivative of Parametric Equations

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}. \tag{7.1}$$

Proof

This theorem can be proven using the Chain Rule. In particular, assume that the parameter t can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields

$$y'(t) = F'(x(t))x'(t),$$

so

$$F'(x(t)) = \frac{y'(t)}{x'(t)}.$$

But $F'(x(t)) = \frac{dy}{dx}$, which proves the theorem.

□

Equation 7.1 can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist. **Equation 7.1** gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

Example 7.4

Finding the Derivative of a Parametric Curve

Calculate the derivative $\frac{dy}{dx}$ for each of the following parametrically defined plane curves, and locate any critical points on their respective graphs.

a. $x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad -3 \le t \le 4$

b. $x(t) = 2t + 1, \quad y(t) = t^3 - 3t + 4, \quad -2 \le t \le 5$

c. $x(t) = 5\cos t, \quad y(t) = 5\sin t, \quad 0 \le t \le 2\pi$

Solution

a. To apply **Equation 7.1**, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$
$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{2}{2t}$$
$$\frac{dy}{dx} = \frac{1}{t}.$$

This derivative is undefined when $t = 0$. Calculating $x(0)$ and $y(0)$ gives $x(0) = (0)^2 - 3 = -3$ and $y(0) = 2(0) - 1 = -1$, which corresponds to the point $(-3, -1)$ on the graph. The graph of this curve is a parabola opening to the right, and the point $(-3, -1)$ is its vertex as shown.

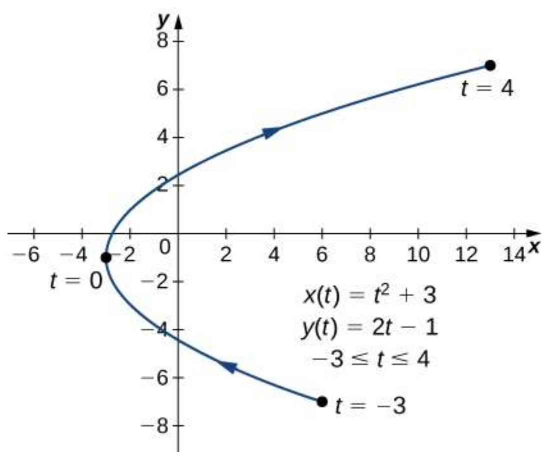


Figure 7.17 Graph of the parabola described by parametric equations in part a.

b. To apply **Equation 7.1**, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2$$
$$y'(t) = 3t^2 - 3.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{3t^2 - 3}{2}.$$

This derivative is zero when $t = \pm 1$. When $t = -1$ we have

$$x(-1) = 2(-1) + 1 = -1 \text{ and } y(-1) = (-1)^3 - 3(-1) + 4 = -1 + 3 + 4 = 6,$$

which corresponds to the point $(-1, 6)$ on the graph. When $t = 1$ we have

$$x(1) = 2(1) + 1 = 3 \text{ and } y(1) = (1)^3 - 3(1) + 4 = 1 - 3 + 4 = 2,$$

which corresponds to the point $(3, 2)$ on the graph. The point $(3, 2)$ is a relative minimum and the point $(-1, 6)$ is a relative maximum, as seen in the following graph.

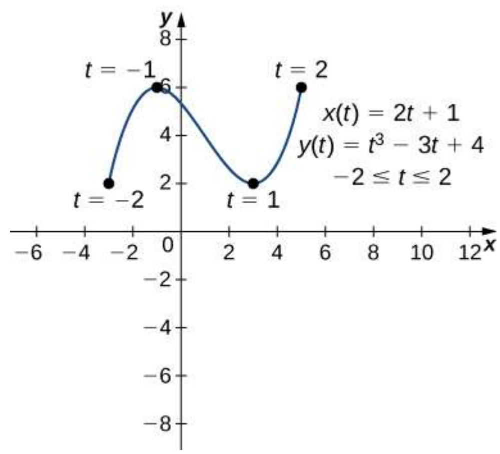


Figure 7.18 Graph of the curve described by parametric equations in part b.

c. To apply **Equation 7.1**, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = -5 \sin t$$
$$y'(t) = 5 \cos t.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{5 \cos t}{-5 \sin t}$$
$$\frac{dy}{dx} = -\cot t.$$

This derivative is zero when $\cos t = 0$ and is undefined when $\sin t = 0$. This gives $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ and } 2\pi$ as critical points for *t*. Substituting each of these into $x(t)$ and $y(t)$, we obtain

t	$x(t)$	$y(t)$
0	5	0
$\frac{\pi}{2}$	0	5
π	−5	0
$\frac{3\pi}{2}$	0	−5
2π	5	0

These points correspond to the sides, top, and bottom of the circle that is represented by the parametric equations (**Figure 7.19**). On the left and right edges of the circle, the derivative is undefined, and on the top and bottom, the derivative equals zero.

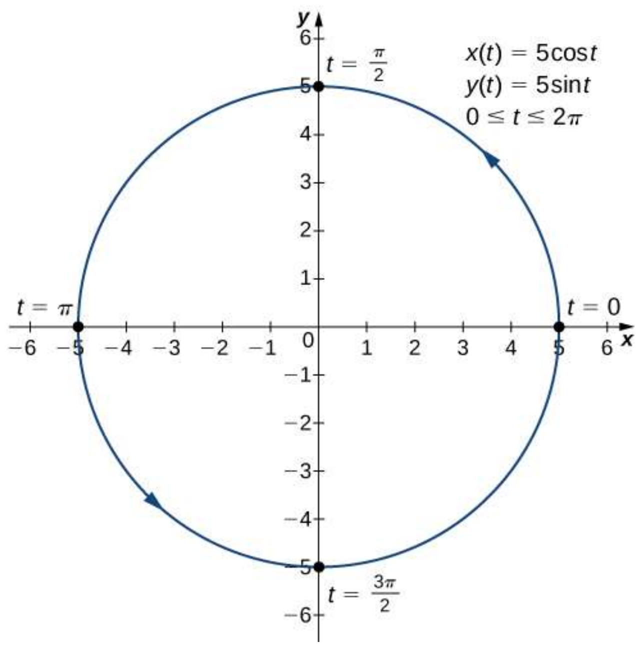


Figure 7.19 Graph of the curve described by parametric equations in part c.

7.4 Calculate the derivative dy/dx for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \le t \le 3$$

and locate any critical points on its graph.

Example 7.5

Finding a Tangent Line

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad -3 \le t \le 4 \text{ when } t = 2.$$

Solution

First find the slope of the tangent line using **Equation 7.1**, which means calculating $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$
$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{2}{2t}$$
$$\frac{dy}{dx} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

which corresponds to the point $(1, 3)$ on the graph (**Figure 7.20**). Now use the point-slope form of the equation of a line to find the equation of the tangent line:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 3 &= \frac{1}{2}(x - 1) \\ y - 3 &= \frac{1}{2}x - \frac{1}{2} \\ y &= \frac{1}{2}x + \frac{5}{2}. \end{aligned}$$

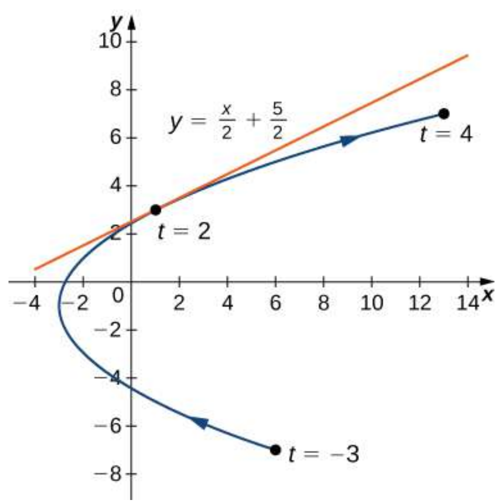


Figure 7.20 Tangent line to the parabola described by the given parametric equations when $t = 2$.

7.5 Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3 \text{ when } t = 5.$$

Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right].$$

Since $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, we can replace the y on both sides of this equation with $\frac{dy}{dx}$. This gives us

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}. \quad (7.2)$$

If we know dy/dx as a function of t, then this formula is straightforward to apply.

Example 7.6

Finding a Second Derivative

Calculate the second derivative $d^2 y/dx^2$ for the plane curve defined by the parametric equations $x(t) = t^2 - 3, \; y(t) = 2t - 1, \; -3 \leq t \leq 4.$

Solution

From **Example 7.4** we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using **Equation 7.2**, we obtain

$$\frac{d^2 y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{(d/dt)(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

7.6 Calculate the second derivative $d^2 y/dx^2$ for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3$$

and locate any critical points on its graph.

Integrals Involving Parametric Equations

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically? Recall the cycloid defined by the equations $x(t) = t - \sin t, \quad y(t) = 1 - \cos t.$ Suppose we want to find the area of the shaded region in the following graph.

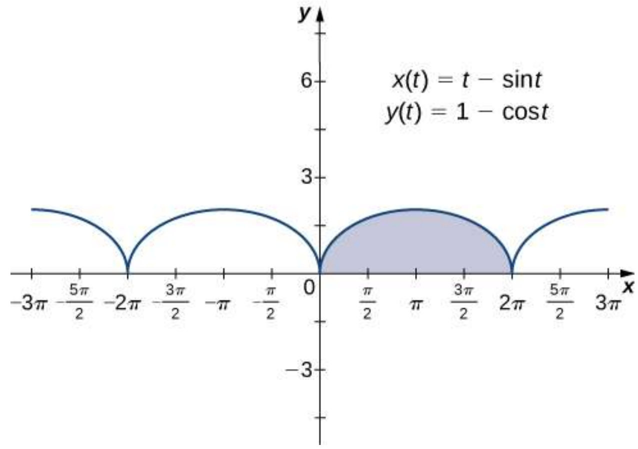


Figure 7.21 Graph of a cycloid with the arch over $[0, 2\pi]$ highlighted.

To derive a formula for the area under the curve defined by the functions

$$x = x(t), \quad y = y(t), \quad a \le t \le b,$$

we assume that $x(t)$ is differentiable and start with an equal partition of the interval $a \le t \le b.$ Suppose $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ and consider the following graph.

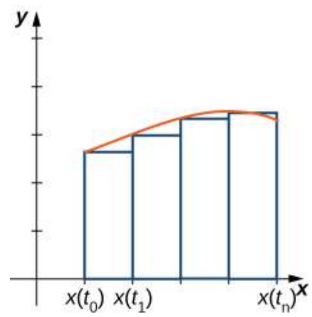


Figure 7.22 Approximating the area under a parametrically defined curve.

We use rectangles to approximate the area under the curve. The height of a typical rectangle in this parametrization is $y(x(\bar{t}_i))$ for some value $\bar{t}_i$ in the ith subinterval, and the width can be calculated as $x(t_i) - x(t_{i-1}).$ Thus the area of the ith rectangle is given by

$$A_i = y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Then a Riemann sum for the area is

$$A_n = \sum_{i=1}^{n} y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Multiplying and dividing each area by $t_i - t_{i-1}$ gives

$$A_n = \sum_{i=1}^{n} y(x(\bar{t}_i))\left(\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}\right)(t_i - t_{i-1}) = \sum_{i=1}^{n} y(x(\bar{t}_i))\left(\frac{x(t_i) - x(t_{i-1})}{\Delta t}\right)\Delta t.$$

Taking the limit as n approaches infinity gives

$$A = \lim_{n \to \infty} A_n = \int_a^b y(t)x'(t)\,dt.$$

This leads to the following theorem.

Theorem 7.2: Area under a Parametric Curve

Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t)\,dt. \tag{7.3}$$

Example 7.7

Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad 0 \le t \le 2\pi.$$

Solution

Using **Equation 7.3**, we have

$$\begin{aligned} A &= \int_a^b y(t)x'(t)\,dt \\ &= \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\,dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)dt \\ &= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2}\right)dt \\ &= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos 2t}{2}\right)dt \\ &= \frac{3t}{2} - 2\sin t + \frac{\sin 2t}{4}\Big|_0^{2\pi} \\ &= 3\pi. \end{aligned}$$

7.7 Find the area under the curve of the hypocycloid defined by the equations

$$x(t) = 3\cos t + \cos 3t, \quad y(t) = 3\sin t - \sin 3t, \quad 0 \le t \le \pi.$$

Arc Length of a Parametric Curve

In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point A to point B along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments as shown in the following graph.

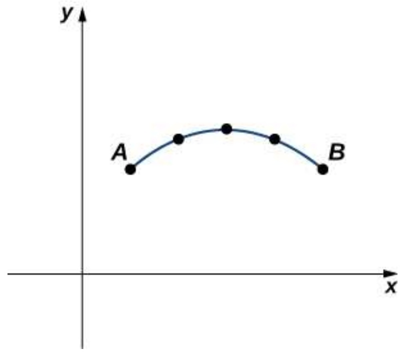


Figure 7.23 Approximation of a curve by line segments.

Given a plane curve defined by the functions $x = x(t),\ y = y(t),\ a \le t \le b,$ we start by partitioning the interval $[a, b]$ into n equal subintervals: $t_0 = a < t_1 < t_2 < \cdots < t_n = b.$ The width of each subinterval is given by $\Delta t = (b - a)/n.$ We can calculate the length of each line segment:

$$\begin{aligned} d_1 &= \sqrt{(x(t_1) - x(t_0))^2 + (y(t_1) - y(t_0))^2} \\ d_2 &= \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2} \text{ etc.} \end{aligned}$$

Then add these up. We let s denote the exact arc length and s_n denote the approximation by n line segments:

$$s \approx \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}. \tag{7.4}$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of t, then the Mean Value Theorem (**Introduction to the Applications of Derivatives (http://cnx.org/content/m53602/latest/)**) applies, so in each subinterval $[t_{k-1}, t_k]$ there exist $\hat{t}_k$ and $\tilde{t}_k$ such that

$$\begin{aligned} x(t_k) - x(t_{k-1}) &= x'\left(\hat{t}_k\right)(t_k - t_{k-1}) = x'\left(\hat{t}_k\right)\Delta t \\ y(t_k) - y(t_{k-1}) &= y'\left(\tilde{t}_k\right)(t_k - t_{k-1}) = y'\left(\tilde{t}_k\right)\Delta t. \end{aligned}$$

Therefore **Equation 7.4** becomes

$$\begin{aligned} s &\approx \sum_{k=1}^{n} s_k \\ &= \sum_{k=1}^{n} \sqrt{\left(x'\left(\hat{t}_k\right)\Delta t\right)^2 + \left(y'\left(\tilde{t}_k\right)\Delta t\right)^2} \\ &= \sum_{k=1}^{n} \sqrt{\left(x'\left(\hat{t}_k\right)\right)^2 (\Delta t)^2 + \left(y'\left(\tilde{t}_k\right)\right)^2 (\Delta t)^2} \\ &= \left(\sum_{k=1}^{n} \sqrt{\left(x'\left(\hat{t}_k\right)\right)^2 + \left(y'\left(\tilde{t}_k\right)\right)^2}\right)\Delta t. \end{aligned}$$

This is a Riemann sum that approximates the arc length over a partition of the interval $[a, b]$. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

$$\begin{aligned} s &= \lim_{n \to \infty} \sum_{k=1}^{n} s_k \\ &= \lim_{n \to \infty} \left(\sum_{k=1}^{n} \sqrt{\left(x'\left(\hat{t}_k\right)\right)^2 + \left(y'\left(\tilde{t}_k\right)\right)^2} \right) \Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt. \end{aligned}$$

When taking the limit, the values of $\hat{t}_k$ and $\tilde{t}_k$ are both contained within the same ever-shrinking interval of width Δt, so they must converge to the same value.

We can summarize this method in the following theorem.

Theorem 7.3: Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad t_1 \le t \le t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \tag{7.5}$$

At this point a side derivation leads to a previous formula for arc length. In particular, suppose the parameter can be eliminated, leading to a function $y = F(x)$. Then $y(t) = F(x(t))$ and the Chain Rule gives $y'(t) = F'(x(t))x'(t)$. Substituting this into **Equation 7.5** gives

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2} dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + (F'(x))^2\right)} dt \\ &= \int_{t_1}^{t_2} x'(t) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dt. \end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t)\,dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

which is the formula for arc length obtained in the **Introduction to the Applications of Integration**.

Example 7.8

Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad 0 \le t \le \pi.$$

Solution

The values $t = 0$ to $t = \pi$ trace out the red curve in **Figure 7.23**. To determine its length, use **Equation 7.5**:

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt \\ &= \int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt \\ &= \int_0^{\pi} \sqrt{9\left(\sin^2 t + \cos^2 t\right)}\, dt \\ &= \int_0^{\pi} 3dt = 3t|_0^{\pi} = 3\pi. \end{aligned}$$

Note that the formula for the arc length of a semicircle is πr and the radius of this circle is 3. This is a great example of using calculus to derive a known formula of a geometric quantity.

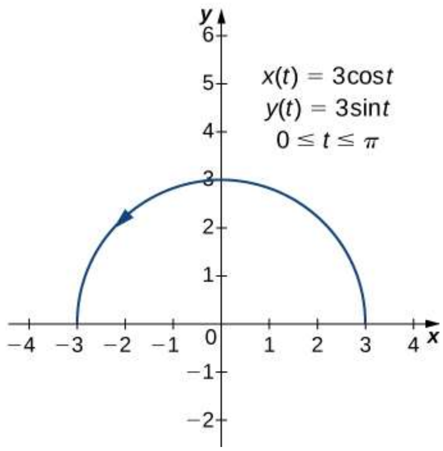


Figure 7.24 The arc length of the semicircle is equal to its radius times π.

7.8 Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad 1 \le t \le 3.$$

We now return to the problem posed at the beginning of the section about a baseball leaving a pitcher's hand. Ignoring the effect of air resistance (unless it is a curve ball!), the ball travels a parabolic path. Assuming the pitcher's hand is at the origin and the ball travels left to right in the direction of the positive *x*-axis, the parametric equations for this curve can be written as

$$x(t) = 140t, \quad y(t) = -16t^2 + 2t$$

where *t* represents time. We first calculate the distance the ball travels as a function of time. This distance is represented by the arc length. We can modify the arc length formula slightly. First rewrite the functions $x(t)$ and $y(t)$ using *v* as an independent variable, so as to eliminate any confusion with the parameter *t*:

$$x(v) = 140v, \quad y(v) = -16v^2 + 2v.$$

Then we write the arc length formula as follows:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{\left(\frac{dx}{dv}\right)^2 + \left(\frac{dy}{dv}\right)^2}\,dv \\ &= \int_0^t \sqrt{140^2 + (-32v+2)^2}\,dv. \end{aligned}$$

The variable v acts as a dummy variable that disappears after integration, leaving the arc length as a function of time t. To integrate this expression we can use a formula from **Appendix A**,

$$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u + \sqrt{a^2+u^2}\right| + C.$$

We set $a = 140$ and $u = -32v + 2$. This gives $du = -32dv$, so $dv = -\frac{1}{32}du$. Therefore

$$\begin{aligned} \int \sqrt{140^2 + (-32v+2)^2}\,dv &= -\frac{1}{32}\int \sqrt{a^2+u^2}\,du \\ &= -\frac{1}{32}\left[\begin{array}{l} \frac{(-32v+2)}{2}\sqrt{140^2+(-32v+2)^2} \\ +\frac{140^2}{2}\ln\left|(-32v+2) + \sqrt{140^2+(-32v+2)^2}\right| \end{array}\right] + C \end{aligned}$$

and

$$\begin{aligned} s(t) &= -\frac{1}{32}\left[\frac{(-32t+2)}{2}\sqrt{140^2+(-32t+2)^2} + \frac{140^2}{2}\ln\left|(-32t+2)+\sqrt{140^2+(-32t+2)^2}\right|\right] \\ &\quad +\frac{1}{32}\left[\sqrt{140^2+2^2} + \frac{140^2}{2}\ln\left|2+\sqrt{140^2+2^2}\right|\right] \\ &= \left(\frac{t}{2} - \frac{1}{32}\right)\sqrt{1024t^2 - 128t + 19604} - \frac{1225}{4}\ln\left|(-32t+2) + \sqrt{1024t^2 - 128t + 19604}\right| \\ &\quad + \frac{\sqrt{19604}}{32} + \frac{1225}{4}\ln\left(2+\sqrt{19604}\right). \end{aligned}$$

This function represents the distance traveled by the ball as a function of time. To calculate the speed, take the derivative of this function with respect to t. While this may seem like a daunting task, it is possible to obtain the answer directly from the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(u)\,du = f(x).$$

Therefore

$$\begin{aligned} s'(t) &= \frac{d}{dt}[s(t)] \\ &= \frac{d}{dt}\left[\int_0^t \sqrt{140^2+(-32v+2)^2}\,dv\right] \\ &= \sqrt{140^2+(-32t+2)^2} \\ &= \sqrt{1024t^2 - 128t + 19604} \\ &= 2\sqrt{256t^2 - 32t + 4901}. \end{aligned}$$

One third of a second after the ball leaves the pitcher's hand, the distance it travels is equal to

$$\begin{aligned} s\left(\frac{1}{3}\right) &= \left(\frac{1/3}{2} - \frac{1}{32}\right)\sqrt{1024\left(\frac{1}{3}\right)^2 - 128\left(\frac{1}{3}\right) + 19604} \\ &\quad -\frac{1225}{4}\ln\left|\left(-32\left(\frac{1}{3}\right) + 2\right) + \sqrt{1024\left(\frac{1}{3}\right)^2 - 128\left(\frac{1}{3}\right) + 19604}\right| \\ &\quad +\frac{\sqrt{19604}}{32} + \frac{1225}{4}\ln\left(2 + \sqrt{19604}\right) \\ &\approx 46.69 \text{ feet.} \end{aligned}$$

This value is just over three quarters of the way to home plate. The speed of the ball is

$$s'\left(\frac{1}{3}\right) = 2\sqrt{256\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 4901} \approx 140.34 \text{ ft/s.}$$

This speed translates to approximately 95 mph—a major-league fastball.

Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. In **Curve Length and Surface Area**, we derived a formula for finding the surface area of a volume generated by a function $y = f(x)$ from $x = a$ to $x = b$, revolved around the x-axis:

$$S = 2\pi\int_a^b f(x)\sqrt{1 + (f'(x))^2}dx.$$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x = x(t),\ y = y(t),\ a \le t \le b$ around the x-axis as shown in the following figure.

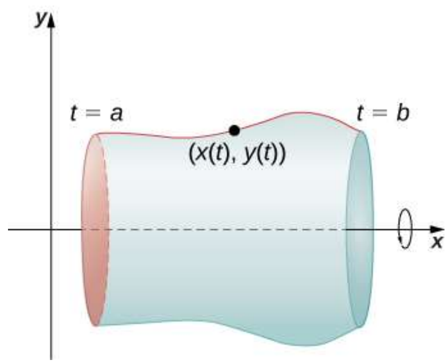


Figure 7.25 A surface of revolution generated by a parametrically defined curve.

The analogous formula for a parametrically defined curve is

$$S = 2\pi\int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}dt \tag{7.6}$$

provided that $y(t)$ is not negative on $[a, b]$.

Example 7.9

Finding Surface Area

Find the surface area of a sphere of radius r centered at the origin.

Solution

We start with the curve defined by the equations

$$x(t) = r\cos t, \quad y(t) = r\sin t, \quad 0 \le t \le \pi.$$

This generates an upper semicircle of radius r centered at the origin as shown in the following graph.

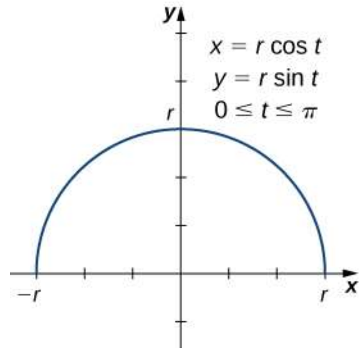


Figure 7.26 A semicircle generated by parametric equations.

When this curve is revolved around the x-axis, it generates a sphere of radius r. To calculate the surface area of the sphere, we use **Equation 7.6**:

$$\begin{aligned} S &= 2\pi\int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \\ &= 2\pi\int_0^\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\,dt \\ &= 2\pi\int_0^\pi r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt \\ &= 2\pi\int_0^\pi r\sin t\sqrt{r^2\left(\sin^2 t + \cos^2 t\right)}\,dt \\ &= 2\pi\int_0^\pi r^2\sin t\,dt \\ &= 2\pi r^2(-\cos t|_0^\pi) \\ &= 2\pi r^2(-\cos\pi + \cos 0) \\ &= 4\pi r^2. \end{aligned}$$

This is, in fact, the formula for the surface area of a sphere.

7.9 Find the surface area generated when the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad 0 \le t \le 1$$

is revolved around the x-axis.

7.2 EXERCISES

For the following exercises, each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.

62. $x = 3 + t, \quad y = 1 - t$

63. $x = 8 + 2t, \quad y = 1$

64. $x = 4 - 3t, \quad y = -2 + 6t$

65. $x = -5t + 7, \quad y = 3t - 1$

For the following exercises, determine the slope of the tangent line, then find the equation of the tangent line at the given value of the parameter.

66. $x = 3\sin t, \quad y = 3\cos t, \quad t = \frac{\pi}{4}$

67. $x = \cos t, \quad y = 8\sin t, \ t = \frac{\pi}{2}$

68. $x = 2t, \quad y = t^3, \quad t = -1$

69. $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t = 1$

70. $x = \sqrt{t}, \quad y = 2t, \quad t = 4$

For the following exercises, find all points on the curve that have the given slope.

71. $x = 4\cos t, \quad y = 4\sin t, \quad \text{slope} = 0.5$

72. $x = 2\cos t, \quad y = 8\sin t, \ \text{slope} = -1$

73. $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \ \text{slope} = 1$

74. $x = 2 + \sqrt{t}, \quad y = 2 - 4t, \ \text{slope} = 0$

For the following exercises, write the equation of the tangent line in Cartesian coordinates for the given parameter t.

75. $x = e^{\sqrt{t}}, \quad y = 1 - \ln t^2, \quad t = 1$

76. $x = t\ln t, \quad y = \sin^2 t, \ t = \frac{\pi}{4}$

77. $x = e^t, \quad y = (t-1)^2, \quad \text{at}(1, 1)$

78. For $x = \sin(2t), \ y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of t at which a horizontal tangent line exists.

79. For $x = \sin(2t), \ y = 2\sin t$ where $0 \le t < 2\pi$. Find all values of t at which a vertical tangent line exists.

80. Find all points on the curve $x = 4\cos(t), \ y = 4\sin(t)$ that have the slope of $\frac{1}{2}$.

81. Find $\frac{dy}{dx}$ for $x = \sin(t), \ y = \cos(t)$.

82. Find the equation of the tangent line to $x = \sin(t), \ y = \cos(t)$ at $t = \frac{\pi}{4}$.

83. For the curve $x = 4t, \ y = 3t - 2,$ find the slope and concavity of the curve at $t = 3$.

84. For the parametric curve whose equation is $x = 4\cos\theta, \ y = 4\sin\theta,$ find the slope and concavity of the curve at $\theta = \frac{\pi}{4}$.

85. Find the slope and concavity for the curve whose equation is $x = 2 + \sec\theta, \ y = 1 + 2\tan\theta$ at $\theta = \frac{\pi}{6}$.

86. Find all points on the curve $x = t + 4, \ y = t^3 - 3t$ at which there are vertical and horizontal tangents.

87. Find all points on the curve $x = \sec\theta, \ y = \tan\theta$ at which horizontal and vertical tangents exist.

For the following exercises, find d^2y/dx^2.

88. $x = t^4 - 1, \quad y = t - t^2$

89. $x = \sin(\pi t), \quad y = \cos(\pi t)$

90. $x = e^{-t}, \quad y = te^{2t}$

For the following exercises, find points on the curve at which tangent line is horizontal or vertical.

91. $x = t(t^2 - 3), \quad y = 3(t^2 - 3)$

92. $x = \dfrac{3t}{1 + t^3}, \quad y = \dfrac{3t^2}{1 + t^3}$

For the following exercises, find dy/dx at the value of the parameter.

93. $x = \cos t, \quad y = \sin t, \quad t = \frac{3\pi}{4}$

94. $x = \sqrt{t}, \quad y = 2t + 4, \quad t = 9$

95. $x = 4\cos(2\pi s), \quad y = 3\sin(2\pi s), \quad s = -\frac{1}{4}$

For the following exercises, find d^2y/dx^2 at the given point without eliminating the parameter.

96. $x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad t = 2$

97. $x = \sqrt{t}, \quad y = 2t + 4, \quad t = 1$

98. Find t intervals on which the curve $x = 3t^2,\ y = t^3 - t$ is concave up as well as concave down.

99. Determine the concavity of the curve $x = 2t + \ln t,\ y = 2t - \ln t.$

100. Sketch and find the area under one arch of the cycloid $x = r(\theta - \sin\theta),\ y = r(1 - \cos\theta).$

101. Find the area bounded by the curve $x = \cos t,\ y = e^t,\ 0 \le t \le \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0.$

102. Find the area enclosed by the ellipse $x = a\cos\theta,\ y = b\sin\theta,\ 0 \le \theta < 2\pi.$

103. Find the area of the region bounded by $x = 2\sin^2\theta,\ y = 2\sin^2\theta\tan\theta,$ for $0 \le \theta \le \frac{\pi}{2}.$

For the following exercises, find the area of the regions bounded by the parametric curves and the indicated values of the parameter.

104. $x = 2\cot\theta,\ y = 2\sin^2\theta,\ 0 \le \theta \le \pi$

105. **[T]** $x = 2a\cos t - a\cos(2t),\ y = 2a\sin t - a\sin(2t),\ 0 \le t < 2\pi$

106. **[T]** $x = a\sin(2t),\ y = b\sin(t),\ 0 \le t < 2\pi$ (the "hourglass")

107. **[T]** $x = 2a\cos t - a\sin(2t),\ y = b\sin t,\ 0 \le t < 2\pi$ (the "teardrop")

For the following exercises, find the arc length of the curve on the indicated interval of the parameter.

108. $x = 4t + 3, \quad y = 3t - 2, \quad 0 \le t \le 2$

109. $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad 0 \le t \le 1$

110. $x = \cos(2t), \quad y = \sin(2t), \quad 0 \le t \le \frac{\pi}{2}$

111. $x = 1 + t^2, \quad y = (1 + t)^3, \quad 0 \le t \le 1$

112. $x = e^t\cos t, \quad y = e^t\sin t, \quad 0 \le t \le \frac{\pi}{2}$ (express answer as a decimal rounded to three places)

113. $x = a\cos^3\theta,\ y = a\sin^3\theta$ on the interval $[0, 2\pi)$ (the hypocycloid)

114. Find the length of one arch of the cycloid $x = 4(t - \sin t),\ y = 4(1 - \cos t).$

115. Find the distance traveled by a particle with position (x, y) as t varies in the given time interval: $x = \sin^2 t, \quad y = \cos^2 t, \quad 0 \le t \le 3\pi.$

116. Find the length of one arch of the cycloid $x = \theta - \sin\theta,\ y = 1 - \cos\theta.$

117. Show that the total length of the ellipse $x = 4\sin\theta,\ y = 3\cos\theta$ is $L = 16\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta,$ where $e = \frac{c}{a}$ and $c = \sqrt{a^2 - b^2}.$

118. Find the length of the curve $x = e^t - t,\ y = 4e^{t/2},\ -8 \le t \le 3.$

For the following exercises, find the area of the surface obtained by rotating the given curve about the x-axis.

119. $x = t^3, \quad y = t^2, \quad 0 \le t \le 1$

120. $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \le \theta \le \frac{\pi}{2}$

121. **[T]** Use a CAS to find the area of the surface generated by rotating $x = t + t^3,\ y = t - \frac{1}{t^2},\ 1 \le t \le 2$ about the x-axis. (Answer to three decimal places.)

122. Find the surface area obtained by rotating $x = 3t^2,\ y = 2t^3,\ 0 \le t \le 5$ about the y-axis.

123. Find the area of the surface generated by revolving $x = t^2,\ y = 2t,\ 0 \le t \le 4$ about the x-axis.

124. Find the surface area generated by revolving $x = t^2,\ y = 2t^2,\ 0 \le t \le 1$ about the y-axis.

7.3 | Polar Coordinates

Learning Objectives

7.3.1 Locate points in a plane by using polar coordinates.
7.3.2 Convert points between rectangular and polar coordinates.
7.3.3 Sketch polar curves from given equations.
7.3.4 Convert equations between rectangular and polar coordinates.
7.3.5 Identify symmetry in polar curves and equations.

The rectangular coordinate system (or Cartesian plane) provides a means of mapping points to ordered pairs and ordered pairs to points. This is called a *one-to-one mapping* from points in the plane to ordered pairs. The polar coordinate system provides an alternative method of mapping points to ordered pairs. In this section we see that in some circumstances, polar coordinates can be more useful than rectangular coordinates.

Defining Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider **Figure 7.27**. The point P has Cartesian coordinates (x, y). The line segment connecting the origin to the point P measures the distance from the origin to P and has length r. The angle between the positive x-axis and the line segment has measure θ. This observation suggests a natural correspondence between the coordinate pair (x, y) and the values r and θ. This correspondence is the basis of the **polar coordinate system**. Note that every point in the Cartesian plane has two values (hence the term *ordered pair*) associated with it. In the polar coordinate system, each point also two values associated with it: r and θ.

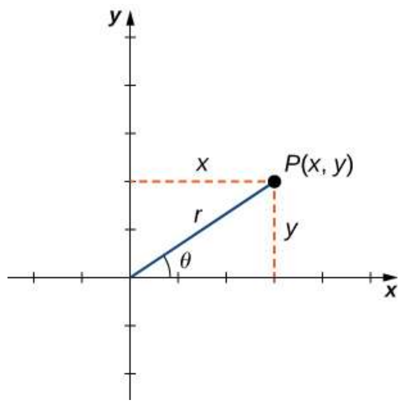


Figure 7.27 An arbitrary point in the Cartesian plane.

Using right-triangle trigonometry, the following equations are true for the point P:

$$\cos\theta = \frac{x}{r} \text{ so } x = r\cos\theta$$
$$\sin\theta = \frac{y}{r} \text{ so } y = r\sin\theta.$$

Furthermore,

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}.$$

Each point (x, y) in the Cartesian coordinate system can therefore be represented as an ordered pair (r, θ) in the polar coordinate system. The first coordinate is called the **radial coordinate** and the second coordinate is called the **angular coordinate**. Every point in the plane can be represented in this form.

Note that the equation $\tan\theta = y/x$ has an infinite number of solutions for any ordered pair (x, y). However, if we restrict the solutions to values between 0 and 2π then we can assign a unique solution to the quadrant in which the original point (x, y) is located. Then the corresponding value of r is positive, so $r^2 = x^2 + y^2$.

Theorem 7.4: Converting Points between Coordinate Systems

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \text{ and } y = r\sin\theta, \tag{7.7}$$

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}. \tag{7.8}$$

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates.

Example 7.10

Converting between Rectangular and Polar Coordinates

Convert each of the following points into polar coordinates.

a. $(1, 1)$

b. $(-3, 4)$

c. $(0, 3)$

d. $(5\sqrt{3}, -5)$

Convert each of the following points into rectangular coordinates.

e. $(3, \pi/3)$

f. $(2, 3\pi/2)$

g. $(6, -5\pi/6)$

Solution

a. Use $x = 1$ and $y = 1$ in **Equation 7.8**:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1^2 + 1^2 \\ r &= \sqrt{2} \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{1}{1} = 1 \\ \theta &= \frac{\pi}{4}. \end{aligned}$$

Therefore this point can be represented as $\left(\sqrt{2}, \frac{\pi}{4}\right)$ in polar coordinates.

b. Use $x = -3$ and $y = 4$ in **Equation 7.8**:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (-3)^2 + (4)^2 \\ r &= 5 \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= -\frac{4}{3} \\ \theta &= -\arctan\left(\frac{4}{3}\right) \\ &\approx 2.21. \end{aligned}$$

Therefore this point can be represented as $(5, 2.21)$ in polar coordinates.

c. Use $x = 0$ and $y = 3$ in **Equation 7.8**:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (3)^2 + (0)^2 \\ &= 9 + 0 \\ r &= 3 \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{3}{0}. \end{aligned}$$

Direct application of the second equation leads to division by zero. Graphing the point $(0, 3)$ on the rectangular coordinate system reveals that the point is located on the positive y-axis. The angle between the positive x-axis and the positive y-axis is $\frac{\pi}{2}$. Therefore this point can be represented as $\left(3, \frac{\pi}{2}\right)$ in polar coordinates.

d. Use $x = 5\sqrt{3}$ and $y = -5$ in **Equation 7.8**:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= \left(5\sqrt{3}\right)^2 + (-5)^2 \\ &= 75 + 25 \\ r &= 10 \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{-5}{5\sqrt{3}} = -\frac{\sqrt{3}}{3} \\ \theta &= -\frac{\pi}{6}. \end{aligned}$$

Therefore this point can be represented as $\left(10, -\frac{\pi}{6}\right)$ in polar coordinates.

e. Use $r = 3$ and $\theta = \frac{\pi}{3}$ in **Equation 7.7**:

$$\begin{aligned} x &= r\cos\theta \\ &= 3\cos\left(\frac{\pi}{3}\right) \\ &= 3\left(\frac{1}{2}\right) = \frac{3}{2} \end{aligned} \quad \text{and} \quad \begin{aligned} y &= r\sin\theta \\ &= 3\sin\left(\frac{\pi}{3}\right) \\ &= 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}. \end{aligned}$$

Therefore this point can be represented as $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ in rectangular coordinates.

f. Use $r = 2$ and $\theta = \frac{3\pi}{2}$ in **Equation 7.7**:

$$\begin{aligned} x &= r\cos\theta \\ &= 2\cos\left(\frac{3\pi}{2}\right) \\ &= 2(0) = 0 \end{aligned} \quad \text{and} \quad \begin{aligned} y &= r\sin\theta \\ &= 2\sin\left(\frac{3\pi}{2}\right) \\ &= 2(-1) = -2. \end{aligned}$$

Therefore this point can be represented as $(0, -2)$ in rectangular coordinates.

g. Use $r = 6$ and $\theta = -\frac{5\pi}{6}$ in **Equation 7.7**:

$$\begin{aligned} x &= r\cos\theta \\ &= 6\cos\left(-\frac{5\pi}{6}\right) \\ &= 6\left(-\frac{\sqrt{3}}{2}\right) \\ &= -3\sqrt{3} \end{aligned} \quad \text{and} \quad \begin{aligned} y &= r\sin\theta \\ &= 6\sin\left(-\frac{5\pi}{6}\right) \\ &= 6\left(-\frac{1}{2}\right) \\ &= -3. \end{aligned}$$

Therefore this point can be represented as $\left(-3\sqrt{3},\ -3\right)$ in rectangular coordinates.

7.10 Convert $(-8,\ -8)$ into polar coordinates and $\left(4,\ \frac{2\pi}{3}\right)$ into rectangular coordinates.

The polar representation of a point is not unique. For example, the polar coordinates $\left(2,\ \frac{\pi}{3}\right)$ and $\left(2,\ \frac{7\pi}{3}\right)$ both represent the point $\left(1,\ \sqrt{3}\right)$ in the rectangular system. Also, the value of r can be negative. Therefore, the point with polar coordinates $\left(-2,\ \frac{4\pi}{3}\right)$ also represents the point $\left(1,\ \sqrt{3}\right)$ in the rectangular system, as we can see by using Equation 7.8:

$$\begin{aligned} x &= r\cos\theta \\ &= -2\cos\left(\frac{4\pi}{3}\right) \\ &= -2\left(-\frac{1}{2}\right) = 1 \end{aligned} \quad \text{and} \quad \begin{aligned} y &= r\sin\theta \\ &= -2\sin\left(\frac{4\pi}{3}\right) \\ &= -2\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}. \end{aligned}$$

Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

Note that the polar representation of a point in the plane also has a visual interpretation. In particular, r is the directed distance that the point lies from the origin, and θ measures the angle that the line segment from the origin to the point makes with the positive x-axis. Positive angles are measured in a counterclockwise direction and negative angles are measured in a clockwise direction. The polar coordinate system appears in the following figure.

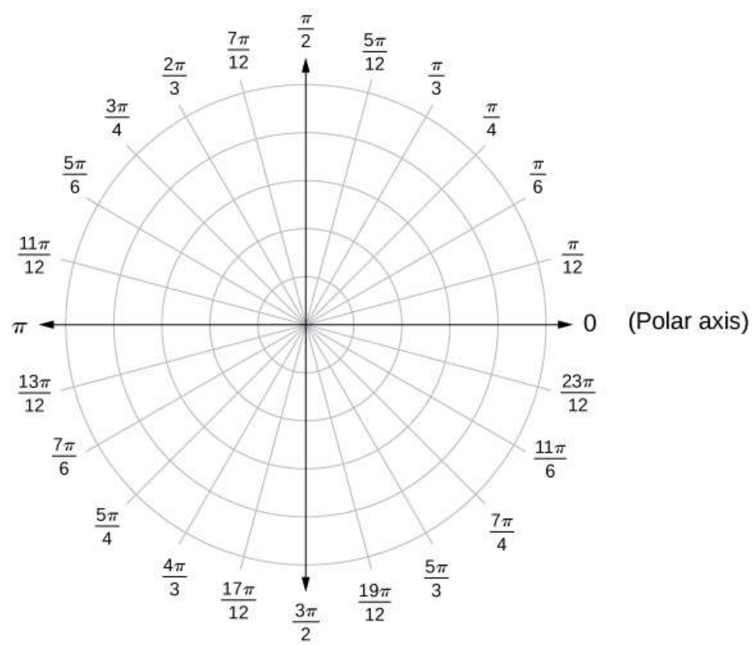


Figure 7.28 The polar coordinate system.

The line segment starting from the center of the graph going to the right (called the positive x-axis in the Cartesian system) is the **polar axis**. The center point is the **pole**, or origin, of the coordinate system, and corresponds to $r = 0$. The innermost circle shown in Figure 7.28 contains all points a distance of 1 unit from the pole, and is represented by the equation $r = 1$.

Then $r = 2$ is the set of points 2 units from the pole, and so on. The line segments emanating from the pole correspond to fixed angles. To plot a point in the polar coordinate system, start with the angle. If the angle is positive, then measure the angle from the polar axis in a counterclockwise direction. If it is negative, then measure it clockwise. If the value of r is positive, move that distance along the terminal ray of the angle. If it is negative, move along the ray that is opposite the terminal ray of the given angle.

Example 7.11

Plotting Points in the Polar Plane

Plot each of the following points on the polar plane.

a. $\left(2, \frac{\pi}{4}\right)$

b. $\left(-3, \frac{2\pi}{3}\right)$

c. $\left(4, \frac{5\pi}{4}\right)$

Solution

The three points are plotted in the following figure.

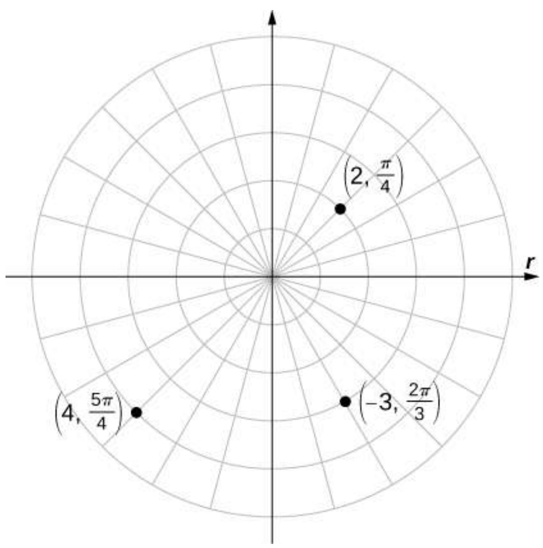


Figure 7.29 Three points plotted in the polar coordinate system.

7.11 Plot $\left(4, \frac{5\pi}{3}\right)$ and $\left(-3, -\frac{7\pi}{2}\right)$ on the polar plane.

Polar Curves

Now that we know how to plot points in the polar coordinate system, we can discuss how to plot curves. In the rectangular coordinate system, we can graph a function $y = f(x)$ and create a curve in the Cartesian plane. In a similar fashion, we can graph a curve that is generated by a function $r = f(\theta)$.

The general idea behind graphing a function in polar coordinates is the same as graphing a function in rectangular coordinates. Start with a list of values for the independent variable $(\theta$ in this case) and calculate the corresponding values of the dependent variable r. This process generates a list of ordered pairs, which can be plotted in the polar coordinate system. Finally, connect the points, and take advantage of any patterns that may appear. The function may be periodic, for example, which indicates that only a limited number of values for the independent variable are needed.

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates

1. Create a table with two columns. The first column is for θ, and the second column is for r.
2. Create a list of values for θ.
3. Calculate the corresponding r values for each θ.
4. Plot each ordered pair (r, θ) on the coordinate axes.
5. Connect the points and look for a pattern.

Watch this **video (http://www.openstaxcollege.org/l/20_polarcurves)** for more information on sketching polar curves.

Example 7.12

Graphing a Function in Polar Coordinates

Graph the curve defined by the function $r = 4 \sin \theta$. Identify the curve and rewrite the equation in rectangular coordinates.

Solution

Because the function is a multiple of a sine function, it is periodic with period 2π, so use values for θ between 0 and 2π. The result of steps 1–3 appear in the following table. **Figure 7.30** shows the graph based on this table.

θ	$r = 4\sin\theta$		θ	$r = 4\sin\theta$
0	0		π	0
$\frac{\pi}{6}$	2		$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$		$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$		$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4		$\frac{3\pi}{2}$	4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$		$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$		$\frac{7\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2		$\frac{11\pi}{6}$	-2
			2π	0

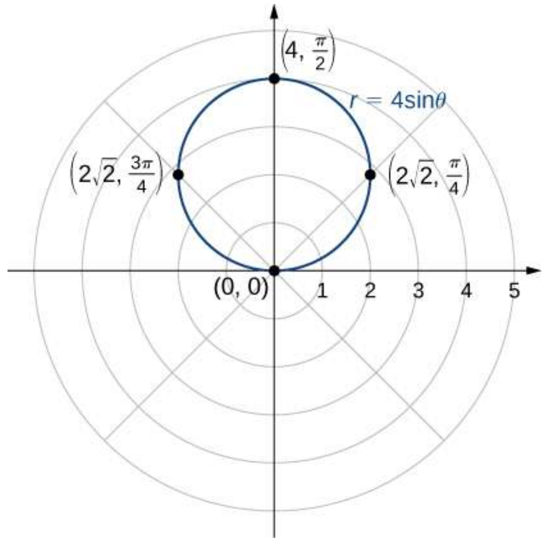


Figure 7.30 The graph of the function $r = 4\sin\theta$ is a circle.

This is the graph of a circle. The equation $r = 4\sin\theta$ can be converted into rectangular coordinates by first multiplying both sides by r. This gives the equation $r^2 = 4r\sin\theta$. Next use the facts that $r^2 = x^2 + y^2$ and $y = r\sin\theta$. This gives $x^2 + y^2 = 4y$. To put this equation into standard form, subtract $4y$ from both sides of the equation and complete the square:

$$\begin{aligned} x^2 + y^2 - 4y &= 0 \\ x^2 + \left(y^2 - 4y\right) &= 0 \\ x^2 + \left(y^2 - 4y + 4\right) &= 0 + 4 \\ x^2 + (y-2)^2 &= 4. \end{aligned}$$

This is the equation of a circle with radius 2 and center $(0, 2)$ in the rectangular coordinate system.

7.12 Create a graph of the curve defined by the function $r = 4 + 4\cos\theta$.

The graph in **Example 7.12** was that of a circle. The equation of the circle can be transformed into rectangular coordinates using the coordinate transformation formulas in **Equation 7.8**. **Example 7.14** gives some more examples of functions for transforming from polar to rectangular coordinates.

Example 7.13

Transforming Polar Equations to Rectangular Coordinates

Rewrite each of the following equations in rectangular coordinates and identify the graph.

a. $\theta = \frac{\pi}{3}$

b. $r = 3$

c. $r = 6\cos\theta - 8\sin\theta$

Solution

a. Take the tangent of both sides. This gives $\tan\theta = \tan(\pi/3) = \sqrt{3}$. Since $\tan\theta = y/x$ we can replace the left-hand side of this equation by y/x. This gives $y/x = \sqrt{3}$, which can be rewritten as $y = x\sqrt{3}$. This is the equation of a straight line passing through the origin with slope $\sqrt{3}$. In general, any polar equation of the form $\theta = K$ represents a straight line through the pole with slope equal to $\tan K$.

b. First, square both sides of the equation. This gives $r^2 = 9$. Next replace r^2 with $x^2 + y^2$. This gives the equation $x^2 + y^2 = 9$, which is the equation of a circle centered at the origin with radius 3. In general, any polar equation of the form $r = k$ where k is a positive constant represents a circle of radius k centered at the origin. (*Note*: when squaring both sides of an equation it is possible to introduce new points unintentionally. This should always be taken into consideration. However, in this case we do not introduce new points. For example, $\left(-3, \frac{\pi}{3}\right)$ is the same point as $\left(3, \frac{4\pi}{3}\right)$.)

c. Multiply both sides of the equation by r. This leads to $r^2 = 6r\cos\theta - 8r\sin\theta$. Next use the formulas

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

This gives

$$\begin{aligned} r^2 &= 6(r\cos\theta) - 8(r\sin\theta) \\ x^2 + y^2 &= 6x - 8y. \end{aligned}$$

To put this equation into standard form, first move the variables from the right-hand side of the equation to the left-hand side, then complete the square.

$$\begin{aligned} x^2 + y^2 &= 6x - 8y \\ x^2 - 6x + y^2 + 8y &= 0 \\ \left(x^2 - 6x\right) + \left(y^2 + 8y\right) &= 0 \\ \left(x^2 - 6x + 9\right) + \left(y^2 + 8y + 16\right) &= 9 + 16 \\ (x-3)^2 + (y+4)^2 &= 25. \end{aligned}$$

This is the equation of a circle with center at $(3, -4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.

7.13 Rewrite the equation $r = \sec\theta\tan\theta$ in rectangular coordinates and identify its graph.

We have now seen several examples of drawing graphs of curves defined by **polar equations**. A summary of some common curves is given in the tables below. In each equation, a and b are arbitrary constants.

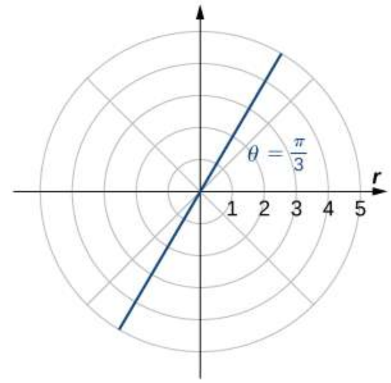
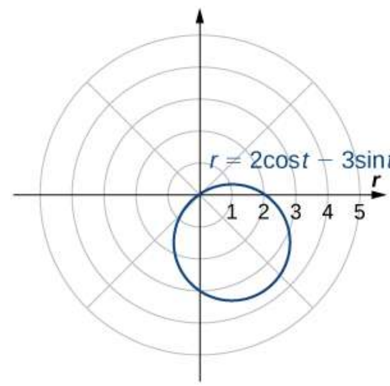
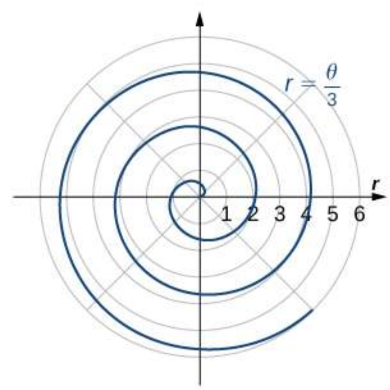
Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Figure 7.31

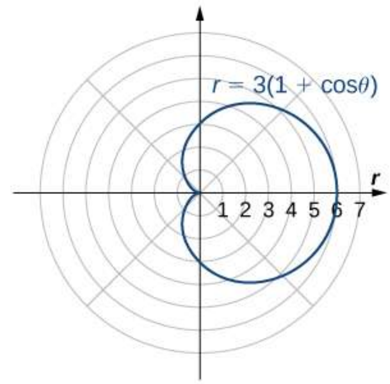
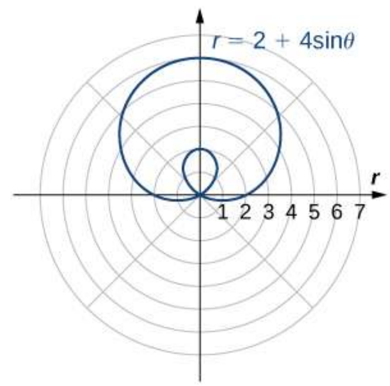
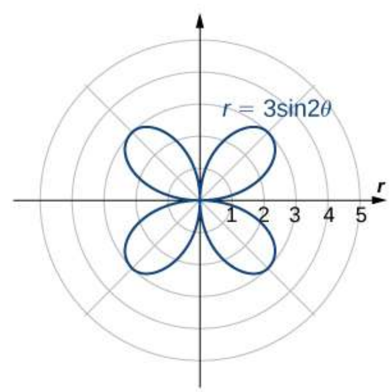
Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

Figure 7.32

A **cardioid** is a special case of a **limaçon** (pronounced "lee-mah-son"), in which $a = b$ or $a = -b$. The **rose** is a very interesting curve. Notice that the graph of $r = 3\sin 2\theta$ has four petals. However, the graph of $r = 3\sin 3\theta$ has three petals as shown.

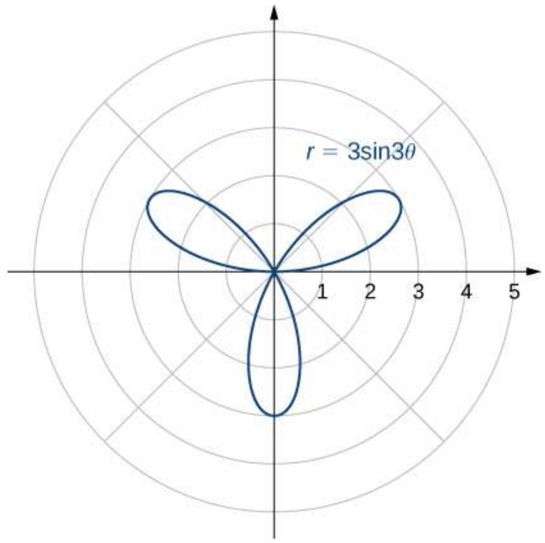


Figure 7.33 Graph of $r = 3 \sin 3\theta$.

If the coefficient of θ is even, the graph has twice as many petals as the coefficient. If the coefficient of θ is odd, then the number of petals equals the coefficient. You are encouraged to explore why this happens. Even more interesting graphs emerge when the coefficient of θ is not an integer. For example, if it is rational, then the curve is closed; that is, it eventually ends where it started (**Figure 7.34**(a)). However, if the coefficient is irrational, then the curve never closes (**Figure 7.34**(b)). Although it may appear that the curve is closed, a closer examination reveals that the petals just above the positive *x* axis are slightly thicker. This is because the petal does not quite match up with the starting point.

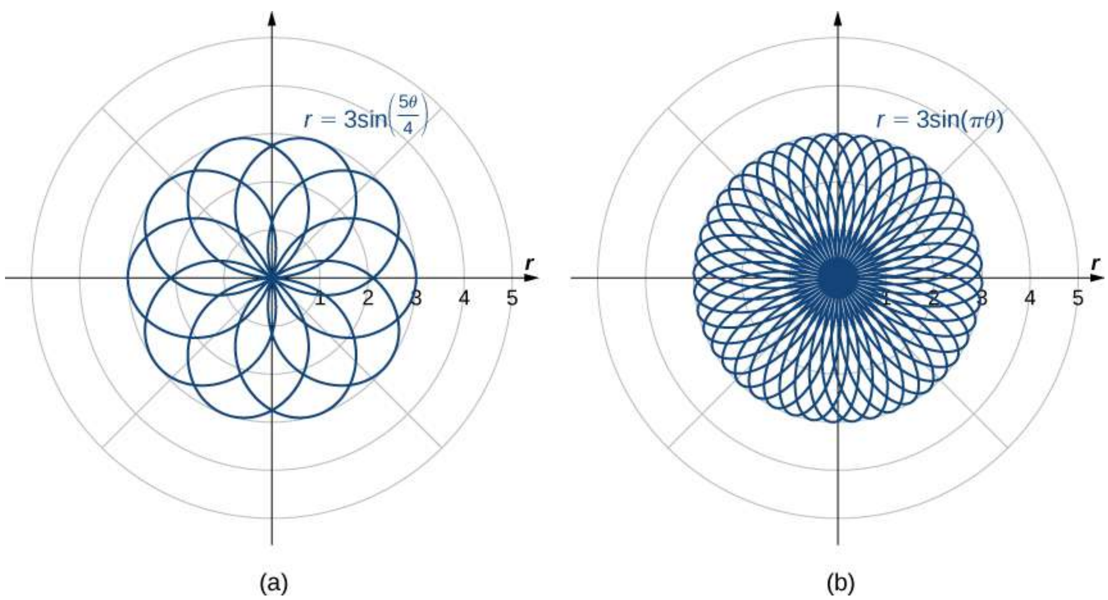


Figure 7.34 Polar rose graphs of functions with (a) rational coefficient and (b) irrational coefficient. Note that the rose in part (b) would actually fill the entire circle if plotted in full.

Since the curve defined by the graph of $r = 3 \sin(\pi\theta)$ never closes, the curve depicted in **Figure 7.34**(b) is only a partial depiction. In fact, this is an example of a **space-filling curve**. A space-filling curve is one that in fact occupies a two-dimensional subset of the real plane. In this case the curve occupies the circle of radius 3 centered at the origin.

Example 7.14

Chapter Opener: Describing a Spiral

Recall the chambered nautilus introduced in the chapter opener. This creature displays a spiral when half the outer shell is cut away. It is possible to describe a spiral using rectangular coordinates. **Figure 7.35** shows a spiral in rectangular coordinates. How can we describe this curve mathematically?

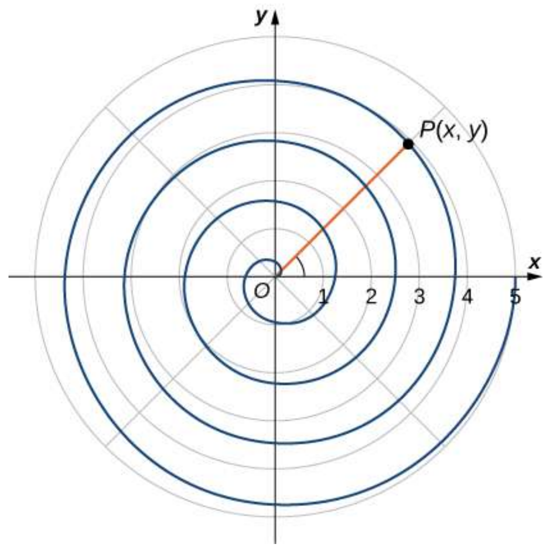


Figure 7.35 How can we describe a spiral graph mathematically?

Solution

As the point P travels around the spiral in a counterclockwise direction, its distance d from the origin increases. Assume that the distance d is a constant multiple k of the angle θ that the line segment OP makes with the positive x-axis. Therefore $d(P, O) = k\theta,$ where O is the origin. Now use the distance formula and some trigonometry:

$$\begin{aligned} d(P, O) &= k\theta \\ \sqrt{(x-0)^2 + (y-0)^2} &= k \arctan\left(\frac{y}{x}\right) \\ \sqrt{x^2 + y^2} &= k \arctan\left(\frac{y}{x}\right) \\ \arctan\left(\frac{y}{x}\right) &= \frac{\sqrt{x^2+y^2}}{k} \\ y &= x \tan\left(\frac{\sqrt{x^2+y^2}}{k}\right). \end{aligned}$$

Although this equation describes the spiral, it is not possible to solve it directly for either x or y. However, if we use polar coordinates, the equation becomes much simpler. In particular, $d(P, O) = r,$ and θ is the second coordinate. Therefore the equation for the spiral becomes $r = k\theta.$ Note that when $\theta = 0$ we also have $r = 0,$ so the spiral emanates from the origin. We can remove this restriction by adding a constant to the equation. Then the equation for the spiral becomes $r = a + k\theta$ for arbitrary constants a and $k.$ This is referred to as an Archimedean spiral, after the Greek mathematician Archimedes.

Another type of spiral is the logarithmic spiral, described by the function $r = a \cdot b^{\theta}.$ A graph of the function $r = 1.2\left(1.25^{\theta}\right)$ is given in **Figure 7.36**. This spiral describes the shell shape of the chambered nautilus.

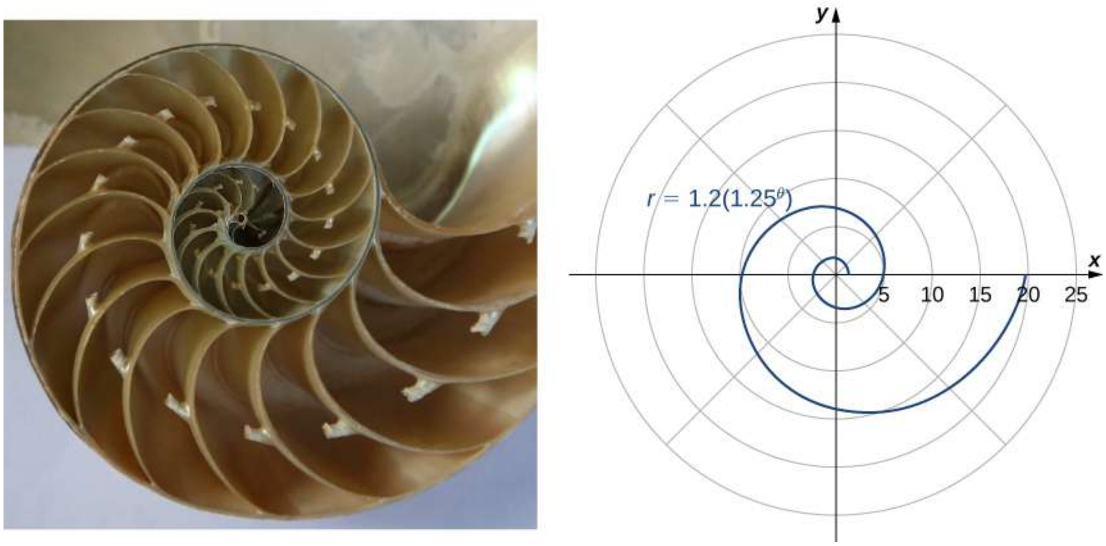


Figure 7.36 A logarithmic spiral is similar to the shape of the chambered nautilus shell. (credit: modification of work by Jitze Couperus, Flickr)

Suppose a curve is described in the polar coordinate system via the function $r = f(\theta)$. Since we have conversion formulas from polar to rectangular coordinates given by

$$x = r\cos\theta$$
$$y = r\sin\theta,$$

it is possible to rewrite these formulas using the function

$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta.$$

This step gives a parameterization of the curve in rectangular coordinates using θ as the parameter. For example, the spiral formula $r = a + b\theta$ from **Figure 7.31** becomes

$$x = (a + b\theta)\cos\theta$$
$$y = (a + b\theta)\sin\theta.$$

Letting θ range from $-\infty$ to ∞ generates the entire spiral.

Symmetry in Polar Coordinates

When studying symmetry of functions in rectangular coordinates (i.e., in the form $y = f(x)$), we talk about symmetry with respect to the *y*-axis and symmetry with respect to the origin. In particular, if $f(-x) = f(x)$ for all x in the domain of f, then f is an even function and its graph is symmetric with respect to the *y*-axis. If $f(-x) = -f(x)$ for all x in the domain of f, then f is an odd function and its graph is symmetric with respect to the origin. By determining which types of symmetry a graph exhibits, we can learn more about the shape and appearance of the graph. Symmetry can also reveal other properties of the function that generates the graph. Symmetry in polar curves works in a similar fashion.

Theorem 7.5: Symmetry in Polar Curves and Equations

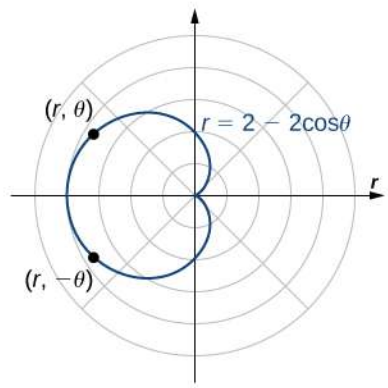
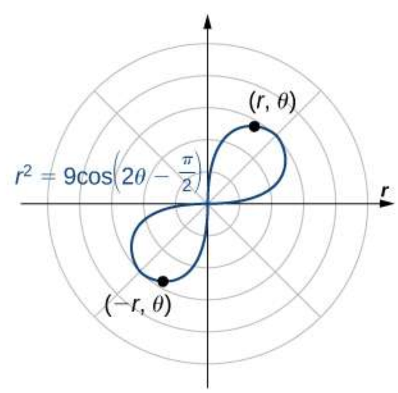
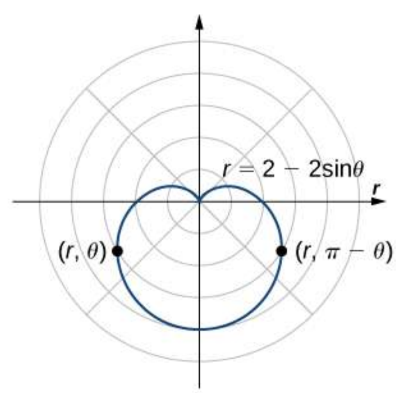
Consider a curve generated by the function $r = f(\theta)$ in polar coordinates.

i. The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point $(r, -\theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged by replacing θ with $-\theta$.

ii. The curve is symmetric about the pole if for every point (r, θ) on the graph, the point $(r, \pi + \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when replacing r with $-r$, or θ with $\pi + \theta$.

iii. The curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$ if for every point (r, θ) on the graph, the point $(r, \pi - \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when θ is replaced by $\pi - \theta$.

The following table shows examples of each type of symmetry.

Symmetry with respect to the polar axis: For every point (r, θ) on the graph, there is also a point reflected directly across the horizontal (polar) axis.	
Symmetry with respect to the pole: For every point (r, θ) on the graph, there is also a point on the graph that is reflected through the pole as well.	
Symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$: For every point (r, θ) on the graph, there is also a point reflected directly across the vertical axis.	

Example 7.15

Using Symmetry to Graph a Polar Equation

Find the symmetry of the rose defined by the equation $r = 3\sin(2\theta)$ and create a graph.

Solution

Suppose the point (r, θ) is on the graph of $r = 3\sin(2\theta)$.

i. To test for symmetry about the polar axis, first try replacing θ with $-\theta$. This gives $r = 3\sin(2(-\theta)) = -3\sin(2\theta)$. Since this changes the original equation, this test is not satisfied. However, returning to the original equation and replacing r with $-r$ and θ with $\pi - \theta$ yields

$$\begin{aligned} -r &= 3\sin(2(\pi - \theta)) \\ -r &= 3\sin(2\pi - 2\theta) \\ -r &= 3\sin(-2\theta) \\ -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta,$ which is the original equation. This demonstrates that the graph is symmetric with respect to the polar axis.

ii. To test for symmetry with respect to the pole, first replace r with $-r,$ which yields $-r = 3\sin(2\theta).$ Multiplying both sides by -1 gives $r = -3\sin(2\theta),$ which does not agree with the original equation. Therefore the equation does not pass the test for this symmetry. However, returning to the original equation and replacing θ with $\theta + \pi$ gives

$$\begin{aligned} r &= 3\sin(2(\theta + \pi)) \\ &= 3\sin(2\theta + 2\pi) \\ &= 3(\sin 2\theta \cos 2\pi + \cos 2\theta \sin 2\pi) \\ &= 3\sin 2\theta. \end{aligned}$$

Since this agrees with the original equation, the graph is symmetric about the pole.

iii. To test for symmetry with respect to the vertical line $\theta = \frac{\pi}{2},$ first replace both r with $-r$ and θ with $-\theta.$

$$\begin{aligned} -r &= 3\sin(2(-\theta)) \\ -r &= 3\sin(-2\theta) \\ -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta,$ which is the original equation. Therefore the graph is symmetric about the vertical line $\theta = \frac{\pi}{2}.$

This graph has symmetry with respect to the polar axis, the origin, and the vertical line going through the pole. To graph the function, tabulate values of θ between 0 and $\pi/2$ and then reflect the resulting graph.

θ	r
0	0
$\frac{\pi}{6}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{4}$	3
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{2}$	0

This gives one petal of the rose, as shown in the following graph.

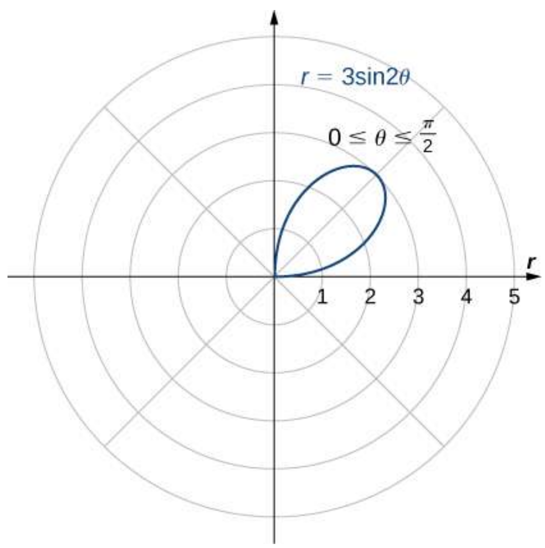


Figure 7.37 The graph of the equation between $\theta = 0$ and $\theta = \pi/2$.

Reflecting this image into the other three quadrants gives the entire graph as shown.

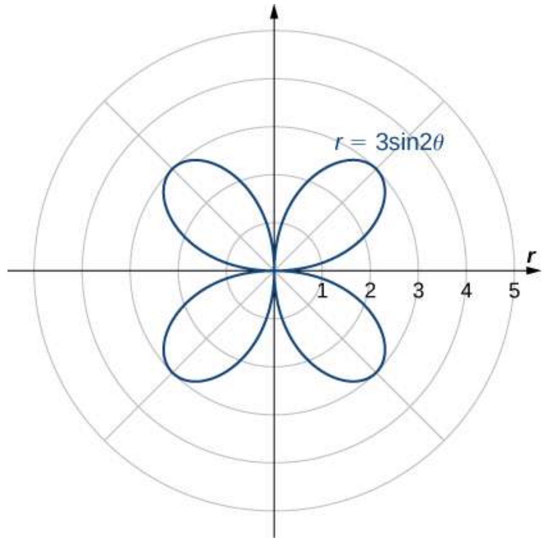


Figure 7.38 The entire graph of the equation is called a four-petaled rose.

7.14 Determine the symmetry of the graph determined by the equation $r = 2\cos(3\theta)$ and create a graph.

7.3 EXERCISES

In the following exercises, plot the point whose polar coordinates are given by first constructing the angle θ and then marking off the distance *r* along the ray.

125. $\left(3, \frac{\pi}{6}\right)$

126. $\left(-2, \frac{5\pi}{3}\right)$

127. $\left(0, \frac{7\pi}{6}\right)$

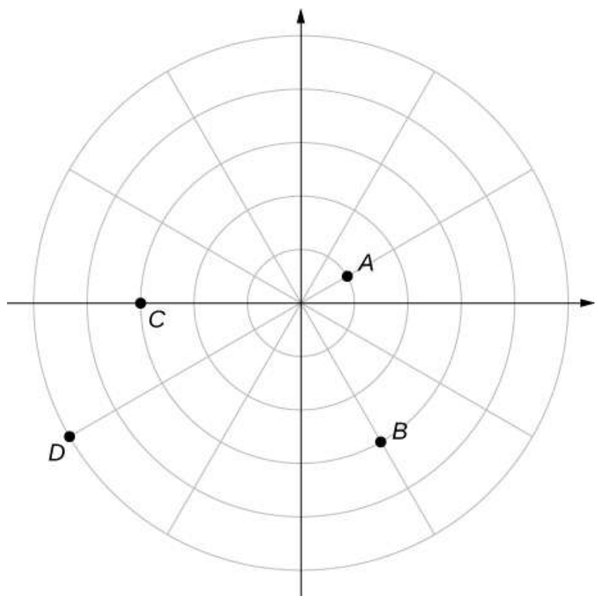
128. $\left(-4, \frac{3\pi}{4}\right)$

129. $\left(1, \frac{\pi}{4}\right)$

130. $\left(2, \frac{5\pi}{6}\right)$

131. $\left(1, \frac{\pi}{2}\right)$

For the following exercises, consider the polar graph below. Give two sets of polar coordinates for each point.



132. Coordinates of point *A*.

133. Coordinates of point *B*.

134. Coordinates of point *C*.

135. Coordinates of point *D*.

For the following exercises, the rectangular coordinates of a point are given. Find two sets of polar coordinates for the point in $(0, 2\pi]$. Round to three decimal places.

136. $(2, 2)$

137. $(3, -4)$ $(3, -4)$

138. $(8, 15)$

139. $(-6, 8)$

140. $(4, 3)$

141. $\left(3, -\sqrt{3}\right)$

For the following exercises, find rectangular coordinates for the given point in polar coordinates.

142. $\left(2, \frac{5\pi}{4}\right)$

143. $\left(-2, \frac{\pi}{6}\right)$

144. $\left(5, \frac{\pi}{3}\right)$

145. $\left(1, \frac{7\pi}{6}\right)$

146. $\left(-3, \frac{3\pi}{4}\right)$

147. $\left(0, \frac{\pi}{2}\right)$

148. $(-4.5, 6.5)$

For the following exercises, determine whether the graphs of the polar equation are symmetric with respect to the x-axis, the y-axis, or the origin.

149. $r = 3\sin(2\theta)$

150. $r^2 = 9\cos\theta$

151. $r = \cos\left(\frac{\theta}{5}\right)$

152. $r = 2\sec\theta$

153. $r = 1 + \cos\theta$

For the following exercises, describe the graph of each polar equation. Confirm each description by converting into a rectangular equation.

154. $r = 3$

155. $\theta = \frac{\pi}{4}$

156. $r = \sec\theta$

157. $r = \csc\theta$

For the following exercises, convert the rectangular equation to polar form and sketch its graph.

158. $x^2 + y^2 = 16$

159. $x^2 - y^2 = 16$

160. $x = 8$

For the following exercises, convert the rectangular equation to polar form and sketch its graph.

161. $3x - y = 2$

162. $y^2 = 4x$

For the following exercises, convert the polar equation to rectangular form and sketch its graph.

163. $r = 4\sin\theta$

164. $r = 6\cos\theta$

165. $r = \theta$

166. $r = \cot\theta\csc\theta$

For the following exercises, sketch a graph of the polar equation and identify any symmetry.

167. $r = 1 + \sin\theta$

168. $r = 3 - 2\cos\theta$

169. $r = 2 - 2\sin\theta$

170. $r = 5 - 4\sin\theta$

171. $r = 3\cos(2\theta)$

172. $r = 3\sin(2\theta)$

173. $r = 2\cos(3\theta)$

174. $r = 3\cos\left(\frac{\theta}{2}\right)$

175. $r^2 = 4\cos(2\theta)$

176. $r^2 = 4\sin\theta$

177. $r = 2\theta$

178. **[T]** The graph of $r = 2\cos(2\theta)\sec(\theta)$. is called a *strophoid*. Use a graphing utility to sketch the graph, and, from the graph, determine the asymptote.

179. **[T]** Use a graphing utility and sketch the graph of $r = \frac{6}{2\sin\theta - 3\cos\theta}$.

180. **[T]** Use a graphing utility to graph $r = \frac{1}{1 - \cos\theta}$.

181. **[T]** Use technology to graph $r = e^{\sin(\theta)} - 2\cos(4\theta)$.

182. **[T]** Use technology to plot $r = \sin\left(\frac{3\theta}{7}\right)$ (use the interval $0 \le \theta \le 14\pi$).

183. Without using technology, sketch the polar curve $\theta = \frac{2\pi}{3}$.

184. **[T]** Use a graphing utility to plot $r = \theta\sin\theta$ for $-\pi \le \theta \le \pi$.

185. **[T]** Use technology to plot $r = e^{-0.1\theta}$ for $-10 \le \theta \le 10$.

186. **[T]** There is a curve known as the "*Black Hole*." Use technology to plot $r = e^{-0.01\theta}$ for $-100 \le \theta \le 100$.

187. **[T]** Use the results of the preceding two problems to explore the graphs of $r = e^{-0.001\theta}$ and $r = e^{-0.0001\theta}$ for $|\theta| > 100$.

7.4 | Area and Arc Length in Polar Coordinates

Learning Objectives

- **7.4.1** Apply the formula for area of a region in polar coordinates.
- **7.4.2** Determine the arc length of a polar curve.

In the rectangular coordinate system, the definite integral provides a way to calculate the area under a curve. In particular, if we have a function $y = f(x)$ defined from $x = a$ to $x = b$ where $f(x) > 0$ on this interval, the area between the curve and the *x*-axis is given by $A = \int_a^b f(x)\,dx$. This fact, along with the formula for evaluating this integral, is summarized in the Fundamental Theorem of Calculus. Similarly, the arc length of this curve is given by $L = \int_a^b \sqrt{1 + (f'(x))^2}dx$. In this section, we study analogous formulas for area and arc length in the polar coordinate system.

Areas of Regions Bounded by Polar Curves

We have studied the formulas for area under a curve defined in rectangular coordinates and parametrically defined curves. Now we turn our attention to deriving a formula for the area of a region bounded by a polar curve. Recall that the proof of the Fundamental Theorem of Calculus used the concept of a Riemann sum to approximate the area under a curve by using rectangles. For polar curves we use the Riemann sum again, but the rectangles are replaced by sectors of a circle.

Consider a curve defined by the function $r = f(\theta)$, where $\alpha \le \theta \le \beta$. Our first step is to partition the interval $[\alpha, \beta]$ into n equal-width subintervals. The width of each subinterval is given by the formula $\Delta\theta = (\beta - \alpha)/n$, and the ith partition point θ_i is given by the formula $\theta_i = \alpha + i\Delta\theta$. Each partition point $\theta = \theta_i$ defines a line with slope $\tan\theta_i$ passing through the pole as shown in the following graph.

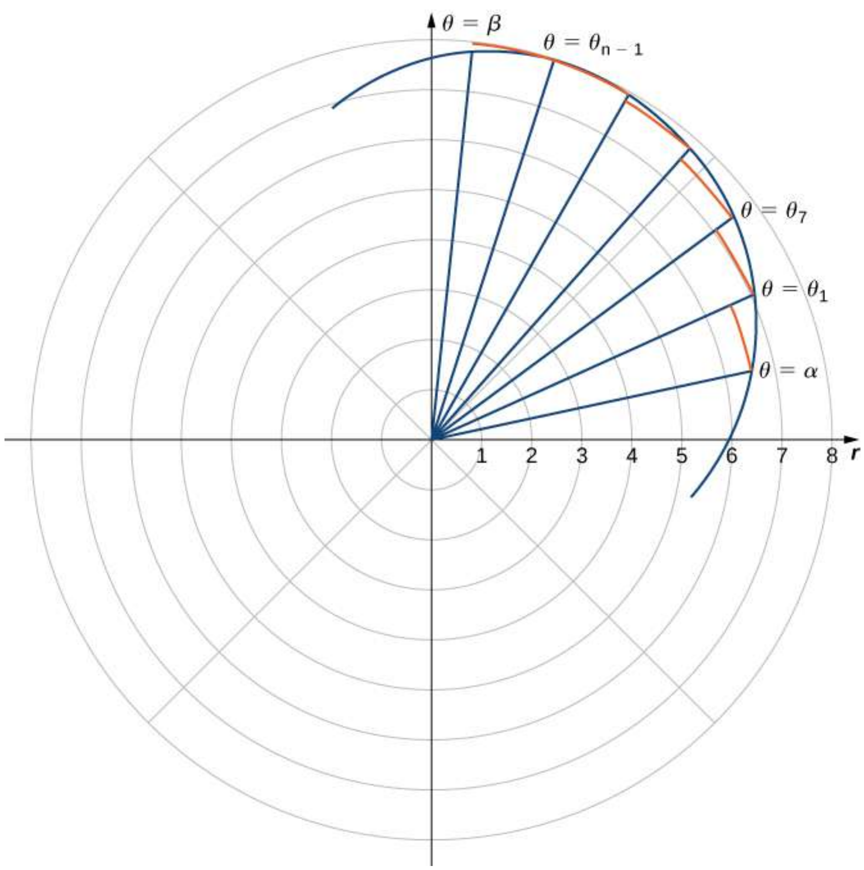


Figure 7.39 A partition of a typical curve in polar coordinates.

The line segments are connected by arcs of constant radius. This defines sectors whose areas can be calculated by using a geometric formula. The area of each sector is then used to approximate the area between successive line segments. We then sum the areas of the sectors to approximate the total area. This approach gives a Riemann sum approximation for the total area. The formula for the area of a sector of a circle is illustrated in the following figure.

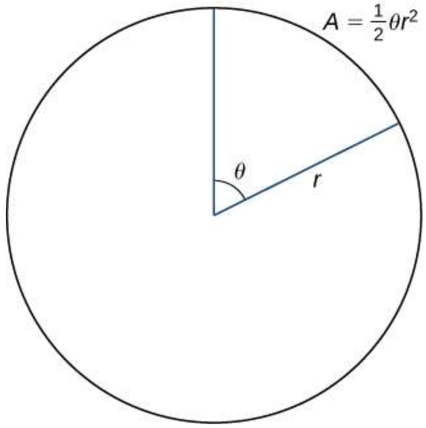


Figure 7.40 The area of a sector of a circle is given by $A = \frac{1}{2}\theta r^2$.

Recall that the area of a circle is $A = \pi r^2$. When measuring angles in radians, 360 degrees is equal to 2π radians. Therefore a fraction of a circle can be measured by the central angle θ. The fraction of the circle is given by $\frac{\theta}{2\pi}$, so the area of the sector is this fraction multiplied by the total area:

$$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}\theta r^2.$$

Since the radius of a typical sector in **Figure 7.39** is given by $r_i = f(\theta_i)$, the area of the ith sector is given by

$$A_i = \frac{1}{2}(\Delta\theta)\left(f(\theta_i)\right)^2.$$

Therefore a Riemann sum that approximates the area is given by

$$A_n = \sum_{i=1}^{n} A_i \approx \sum_{i=1}^{n} \frac{1}{2}(\Delta\theta)\left(f(\theta_i)\right)^2.$$

We take the limit as $n \to \infty$ to get the exact area:

$$A = \lim_{n\to\infty} A_n = \frac{1}{2}\int_{\alpha}^{\beta}(f(\theta))^2\,d\theta.$$

This gives the following theorem.

Theorem 7.6: Area of a Region Bounded by a Polar Curve

Suppose f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$ with $0 < \beta - \alpha \le 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta. \tag{7.9}$$

Example 7.16

Finding an Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.

Solution

The graph of $r = 3\sin(2\theta)$ follows.

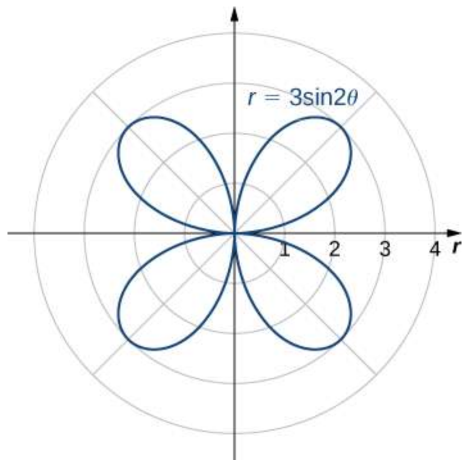


Figure 7.41 The graph of $r = 3\sin(2\theta)$.

When $\theta = 0$ we have $r = 3\sin(2(0)) = 0$. The next value for which $r = 0$ is $\theta = \pi/2$. This can be seen by solving the equation $3\sin(2\theta) = 0$ for θ. Therefore the values $\theta = 0$ to $\theta = \pi/2$ trace out the first petal of the rose. To find the area inside this petal, use **Equation 7.9** with $f(\theta) = 3\sin(2\theta)$, $\alpha = 0$, and $\beta = \pi/2$:

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2}[3\sin(2\theta)]^2\,d\theta \\ &= \frac{1}{2}\int_{0}^{\pi/2}9\sin^2(2\theta)\,d\theta. \end{aligned}$$

To evaluate this integral, use the formula $\sin^2\alpha = (1 - \cos(2\alpha))/2$ with $\alpha = 2\theta$:

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi/2} 9\sin^2(2\theta)\,d\theta \\
&= \frac{9}{2}\int_0^{\pi/2} \frac{(1-\cos(4\theta))}{2}d\theta \\
&= \frac{9}{4}\left(\int_0^{\pi/2} 1-\cos(4\theta)\,d\theta\right) \\
&= \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\Big|_0^{\pi/2}\right. \\
&= \frac{9}{4}\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - \frac{9}{4}\left(0 - \frac{\sin 4(0)}{4}\right) \\
&= \frac{9\pi}{8}.
\end{aligned}$$

7.15 Find the area inside the cardioid defined by the equation $r = 1 - \cos\theta$.

Example 7.16 involved finding the area inside one curve. We can also use **Area of a Region Bounded by a Polar Curve** to find the area between two polar curves. However, we often need to find the points of intersection of the curves and determine which function defines the outer curve or the inner curve between these two points.

Example 7.17

Finding the Area between Two Polar Curves

Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.

Solution

First draw a graph containing both curves as shown.

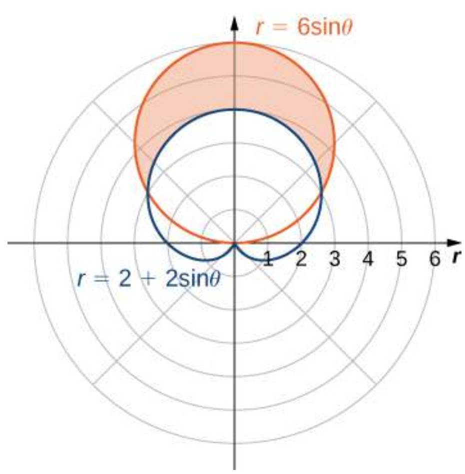


Figure 7.42 The region between the curves $r = 2 + 2\sin\theta$ and $r = 6\sin\theta$.

To determine the limits of integration, first find the points of intersection by setting the two functions equal to each other and solving for θ:

$$\begin{aligned} 6\sin\theta &= 2 + 2\sin\theta \\ 4\sin\theta &= 2 \\ \sin\theta &= \frac{1}{2}. \end{aligned}$$

This gives the solutions $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, which are the limits of integration. The circle $r = 3\sin\theta$ is the red graph, which is the outer function, and the cardioid $r = 2 + 2\sin\theta$ is the blue graph, which is the inner function. To calculate the area between the curves, start with the area inside the circle between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, then subtract the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$:

$$\begin{aligned} A &= \text{circle} - \text{cardioid} \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} [6\sin\theta]^2\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} [2 + 2\sin\theta]^2\, d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 36\sin^2\theta\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4 + 8\sin\theta + 4\sin^2\theta\, d\theta \\ &= 18\int_{\pi/6}^{5\pi/6} \frac{1-\cos(2\theta)}{2} d\theta - 2\int_{\pi/6}^{5\pi/6} 1 + 2\sin\theta + \frac{1-\cos(2\theta)}{2} d\theta \\ &= 9\left[\theta - \frac{\sin(2\theta)}{2}\right]_{\pi/6}^{5\pi/6} - 2\left[\frac{3\theta}{2} - 2\cos\theta - \frac{\sin(2\theta)}{4}\right]_{\pi/6}^{5\pi/6} \\ &= 9\left(\frac{5\pi}{6} - \frac{\sin 2(5\pi/6)}{2}\right) - 9\left(\frac{\pi}{6} - \frac{\sin 2(\pi/6)}{2}\right) \\ &\quad -\left(3\left(\frac{5\pi}{6}\right) - 4\cos\frac{5\pi}{6} - \frac{\sin 2(5\pi/6)}{2}\right) + \left(3\left(\frac{\pi}{6}\right) - 4\cos\frac{\pi}{6} - \frac{\sin 2(\pi/6)}{2}\right) \\ &= 4\pi. \end{aligned}$$

7.16 Find the area inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.

In **Example 7.17** we found the area inside the circle and outside the cardioid by first finding their intersection points. Notice that solving the equation directly for θ yielded two solutions: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. However, in the graph there are three intersection points. The third intersection point is the origin. The reason why this point did not show up as a solution is because the origin is on both graphs but for different values of θ. For example, for the cardioid we get

$$\begin{aligned} 2 + 2\sin\theta &= 0 \\ \sin\theta &= -1, \end{aligned}$$

so the values for θ that solve this equation are $\theta = \frac{3\pi}{2} + 2n\pi$, where n is any integer. For the circle we get

$$6\sin\theta = 0.$$

The solutions to this equation are of the form $\theta = n\pi$ for any integer value of n. These two solution sets have no points in common. Regardless of this fact, the curves intersect at the origin. This case must always be taken into consideration.

Arc Length in Polar Curves

Here we derive a formula for the arc length of a curve defined in polar coordinates.

In rectangular coordinates, the arc length of a parameterized curve $(x(t), y(t))$ for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In polar coordinates we define the curve by the equation $r = f(\theta)$, where $\alpha \le \theta \le \beta$. In order to adapt the arc length formula for a polar curve, we use the equations

$$x = r\cos\theta = f(\theta)\cos\theta \text{ and } y = r\sin\theta = f(\theta)\sin\theta,$$

and we replace the parameter t by θ. Then

$$\begin{aligned}\frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta.\end{aligned}$$

We replace dt by $d\theta$, and the lower and upper limits of integration are α and β, respectively. Then the arc length formula becomes

$$\begin{aligned}L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2\left(\cos^2\theta + \sin^2\theta\right) + (f(\theta))^2\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.\end{aligned}$$

This gives us the following theorem.

Theorem 7.7: Arc Length of a Curve Defined by a Polar Function

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \quad (7.10)$$

Example 7.18

Finding the Arc Length of a Polar Curve

Find the arc length of the cardioid $r = 2 + 2\cos\theta$.

Solution

When $\theta = 0,$ $r = 2 + 2\cos 0 = 4.$ Furthermore, as θ goes from 0 to $2\pi,$ the cardioid is traced out exactly once. Therefore these are the limits of integration. Using $f(\theta) = 2 + 2\cos\theta,$ $\alpha = 0,$ and $\beta = 2\pi,$ **Equation 7.10** becomes

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{[2 + 2\cos\theta]^2 + [-2\sin\theta]^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta \\ &= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta \\ &= 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta. \end{aligned}$$

Next, using the identity $\cos(2\alpha) = 2\cos^2\alpha - 1,$ add 1 to both sides and multiply by 2. This gives $2 + 2\cos(2\alpha) = 4\cos^2\alpha.$ Substituting $\alpha = \theta/2$ gives $2 + 2\cos\theta = 4\cos^2(\theta/2),$ so the integral becomes

$$\begin{aligned} L &= 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\ &= 2\int_0^{2\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)} d\theta \\ &= 2\int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta. \end{aligned}$$

The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue, change the limits from 0 to π and double the answer. This strategy works because cosine is positive between 0 and $\frac{\pi}{2}.$ Thus,

$$\begin{aligned} L &= 4\int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta \\ &= 8\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta \\ &= 8\left(2\sin\left(\frac{\theta}{2}\right)\right|_0^{\pi} \\ &= 16. \end{aligned}$$

7.17 Find the total arc length of $r = 3\sin\theta.$

7.4 EXERCISES

For the following exercises, determine a definite integral that represents the area.

188. Region enclosed by $r = 4$

189. Region enclosed by $r = 3\sin\theta$

190. Region in the first quadrant within the cardioid $r = 1 + \sin\theta$

191. Region enclosed by one petal of $r = 8\sin(2\theta)$

192. Region enclosed by one petal of $r = \cos(3\theta)$

193. Region below the polar axis and enclosed by $r = 1 - \sin\theta$

194. Region in the first quadrant enclosed by $r = 2 - \cos\theta$

195. Region enclosed by the inner loop of $r = 2 - 3\sin\theta$

196. Region enclosed by the inner loop of $r = 3 - 4\cos\theta$

197. Region enclosed by $r = 1 - 2\cos\theta$ and outside the inner loop

198. Region common to $r = 3\sin\theta$ and $r = 2 - \sin\theta$

199. Region common to $r = 2$ and $r = 4\cos\theta$

200. Region common to $r = 3\cos\theta$ and $r = 3\sin\theta$

For the following exercises, find the area of the described region.

201. Enclosed by $r = 6\sin\theta$

202. Above the polar axis enclosed by $r = 2 + \sin\theta$

203. Below the polar axis and enclosed by $r = 2 - \cos\theta$

204. Enclosed by one petal of $r = 4\cos(3\theta)$

205. Enclosed by one petal of $r = 3\cos(2\theta)$

206. Enclosed by $r = 1 + \sin\theta$

207. Enclosed by the inner loop of $r = 3 + 6\cos\theta$

208. Enclosed by $r = 2 + 4\cos\theta$ and outside the inner loop

209. Common interior of $r = 4\sin(2\theta)$ and $r = 2$

210. Common interior of $r = 3 - 2\sin\theta$ and $r = -3 + 2\sin\theta$

211. Common interior of $r = 6\sin\theta$ and $r = 3$

212. Inside $r = 1 + \cos\theta$ and outside $r = \cos\theta$

213. Common interior of $r = 2 + 2\cos\theta$ and $r = 2\sin\theta$

For the following exercises, find a definite integral that represents the arc length.

214. $r = 4\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

215. $r = 1 + \sin\theta$ on the interval $0 \le \theta \le 2\pi$

216. $r = 2\sec\theta$ on the interval $0 \le \theta \le \frac{\pi}{3}$

217. $r = e^{\theta}$ on the interval $0 \le \theta \le 1$

For the following exercises, find the length of the curve over the given interval.

218. $r = 6$ on the interval $0 \le \theta \le \frac{\pi}{2}$

219. $r = e^{3\theta}$ on the interval $0 \le \theta \le 2$

220. $r = 6\cos\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

221. $r = 8 + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

222. $r = 1 - \sin\theta$ on the interval $0 \le \theta \le 2\pi$

For the following exercises, use the integration capabilities of a calculator to approximate the length of the curve.

223. **[T]** $r = 3\theta$ on the interval $0 \le \theta \le \frac{\pi}{2}$

224. **[T]** $r = \frac{2}{\theta}$ on the interval $\pi \le \theta \le 2\pi$

225. **[T]** $r = \sin^2\left(\frac{\theta}{2}\right)$ on the interval $0 \le \theta \le \pi$

226. **[T]** $r = 2\theta^2$ on the interval $0 \le \theta \le \pi$

227. **[T]** $r = \sin(3\cos\theta)$ on the interval $0 \le \theta \le \pi$

For the following exercises, use the familiar formula from geometry to find the area of the region described and then confirm by using the definite integral.

228. $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

229. $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

230. $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

For the following exercises, use the familiar formula from geometry to find the length of the curve and then confirm using the definite integral.

231. $r = 3\sin\theta$ on the interval $0 \le \theta \le \pi$

232. $r = \sin\theta + \cos\theta$ on the interval $0 \le \theta \le \pi$

233. $r = 6\sin\theta + 8\cos\theta$ on the interval $0 \le \theta \le \pi$

234. Verify that if $y = r\sin\theta = f(\theta)\sin\theta$ then $\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$.

For the following exercises, find the slope of a tangent line to a polar curve $r = f(\theta)$. Let $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, so the polar equation $r = f(\theta)$ is now written in parametric form.

235. Use the definition of the derivative $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ and the product rule to derive the derivative of a polar equation.

236. $r = 1 - \sin\theta$; $\left(\frac{1}{2}, \frac{\pi}{6}\right)$

237. $r = 4\cos\theta$; $\left(2, \frac{\pi}{3}\right)$

238. $r = 8\sin\theta$; $\left(4, \frac{5\pi}{6}\right)$

239. $r = 4 + \sin\theta$; $\left(3, \frac{3\pi}{2}\right)$

240. $r = 6 + 3\cos\theta$; $(3, \pi)$

241. $r = 4\cos(2\theta)$; tips of the leaves

242. $r = 2\sin(3\theta)$; tips of the leaves

243. $r = 2\theta$; $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

244. Find the points on the interval $-\pi \le \theta \le \pi$ at which the cardioid $r = 1 - \cos\theta$ has a vertical or horizontal tangent line.

245. For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

For the following exercises, find the slope of the tangent line to the given polar curve at the point given by the value of θ.

246. $r = 3\cos\theta$, $\theta = \frac{\pi}{3}$

247. $r = \theta$, $\theta = \frac{\pi}{2}$

248. $r = \ln\theta$, $\theta = e$

249. **[T]** Use technology: $r = 2 + 4\cos\theta$ at $\theta = \frac{\pi}{6}$

For the following exercises, find the points at which the following polar curves have a horizontal or vertical tangent line.

250. $r = 4\cos\theta$

251. $r^2 = 4\cos(2\theta)$

252. $r = 2\sin(2\theta)$

253. The cardioid $r = 1 + \sin\theta$

254. Show that the curve $r = \sin\theta\tan\theta$ (called a *cissoid of Diocles*) has the line $x = 1$ as a vertical asymptote.

7.5 | Conic Sections

Learning Objectives

7.5.1 Identify the equation of a parabola in standard form with given focus and directrix.
7.5.2 Identify the equation of an ellipse in standard form with given foci.
7.5.3 Identify the equation of a hyperbola in standard form with given foci.
7.5.4 Recognize a parabola, ellipse, or hyperbola from its eccentricity value.
7.5.5 Write the polar equation of a conic section with eccentricity e.
7.5.6 Identify when a general equation of degree two is a parabola, ellipse, or hyperbola.

Conic sections have been studied since the time of the ancient Greeks, and were considered to be an important mathematical concept. As early as 320 BCE, such Greek mathematicians as Menaechmus, Appollonius, and Archimedes were fascinated by these curves. Appollonius wrote an entire eight-volume treatise on conic sections in which he was, for example, able to derive a specific method for identifying a conic section through the use of geometry. Since then, important applications of conic sections have arisen (for example, in astronomy), and the properties of conic sections are used in radio telescopes, satellite dish receivers, and even architecture. In this section we discuss the three basic conic sections, some of their properties, and their equations.

Conic sections get their name because they can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called **nappes**. One nappe is what most people mean by “cone,” having the shape of a party hat. A right circular cone can be generated by revolving a line passing through the origin around the *y*-axis as shown.

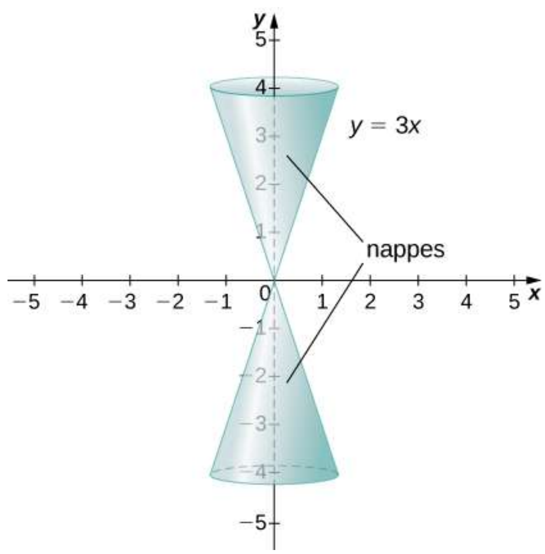


Figure 7.43 A cone generated by revolving the line $y = 3x$ around the y-axis.

Conic sections are generated by the intersection of a plane with a cone (Figure 7.44). If the plane is parallel to the axis of revolution (the *y*-axis), then the **conic section** is a hyperbola. If the plane is parallel to the generating line, the conic section is a parabola. If the plane is perpendicular to the axis of revolution, the conic section is a circle. If the plane intersects one nappe at an angle to the axis (other than $90°$), then the conic section is an ellipse.

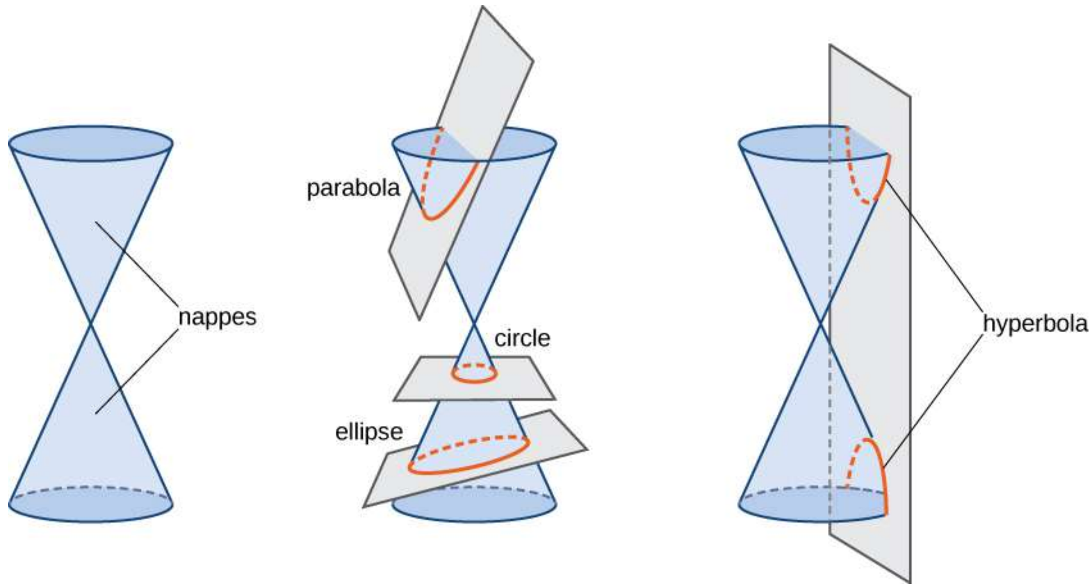


Figure 7.44 The four conic sections. Each conic is determined by the angle the plane makes with the axis of the cone.

Parabolas

A parabola is generated when a plane intersects a cone parallel to the generating line. In this case, the plane intersects only one of the nappes. A parabola can also be defined in terms of distances.

Definition

A parabola is the set of all points whose distance from a fixed point, called the **focus**, is equal to the distance from a fixed line, called the **directrix**. The point halfway between the focus and the directrix is called the **vertex** of the parabola.

A graph of a typical parabola appears in **Figure 7.45**. Using this diagram in conjunction with the distance formula, we can derive an equation for a parabola. Recall the distance formula: Given point P with coordinates (x_1, y_1) and point Q with coordinates $(x_2, \ y_2),$ the distance between them is given by the formula

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Then from the definition of a parabola and **Figure 7.45**, we get

$$\begin{aligned} d(F, P) &= d(P, Q) \\ \sqrt{(0 - x)^2 + (p - y)^2} &= \sqrt{(x - x)^2 + (-p - y)^2}. \end{aligned}$$

Squaring both sides and simplifying yields

$$\begin{aligned} x^2 + (p - y)^2 &= 0^2 + (-p - y)^2 \\ x^2 + p^2 - 2py + y^2 &= p^2 + 2py + y^2 \\ x^2 - 2py &= 2py \\ x^2 &= 4py. \end{aligned}$$

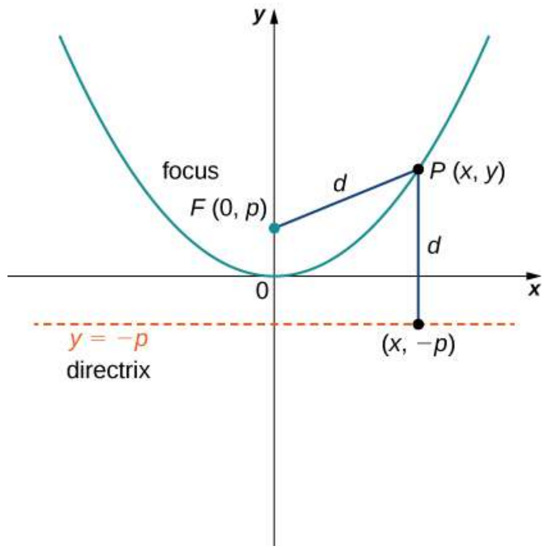


Figure 7.45 A typical parabola in which the distance from the focus to the vertex is represented by the variable p.

Now suppose we want to relocate the vertex. We use the variables (h, k) to denote the coordinates of the vertex. Then if the focus is directly above the vertex, it has coordinates $(h, k + p)$ and the directrix has the equation $y = k - p$. Going through the same derivation yields the formula $(x - h)^2 = 4p(y - k)$. Solving this equation for y leads to the following theorem.

Theorem 7.8: Equations for Parabolas

Given a parabola opening upward with vertex located at (h, k) and focus located at $(h, k + p)$, where p is a constant, the equation for the parabola is given by

$$y = \frac{1}{4p}(x - h)^2 + k. \quad (7.11)$$

This is the **standard form** of a parabola.

We can also study the cases when the parabola opens down or to the left or the right. The equation for each of these cases can also be written in standard form as shown in the following graphs.

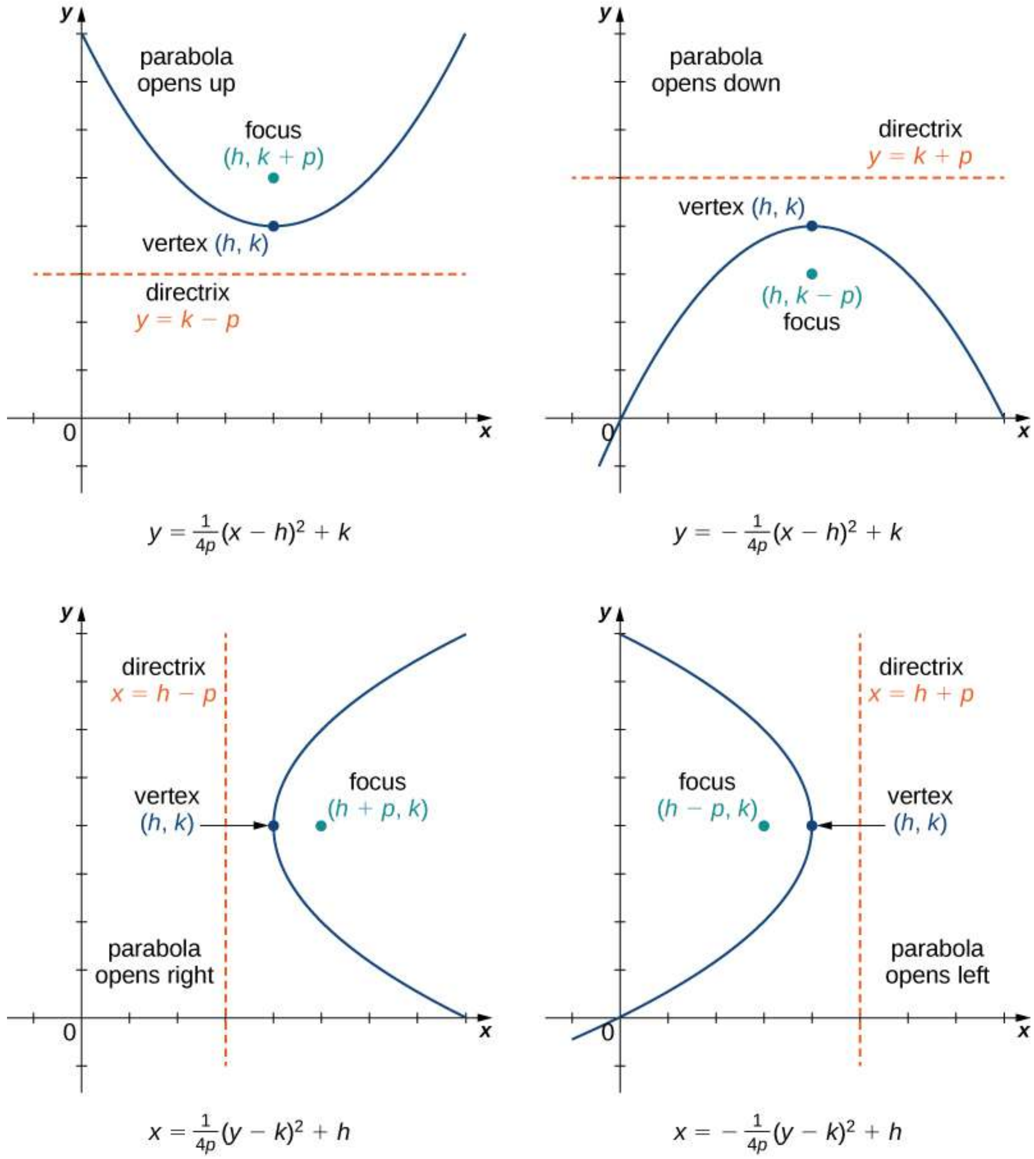


Figure 7.46 Four parabolas, opening in various directions, along with their equations in standard form.

In addition, the equation of a parabola can be written in the **general form**, though in this form the values of h, k, and p are not immediately recognizable. The general form of a parabola is written as

$$ax^2 + bx + cy + d = 0 \quad \text{or} \quad ay^2 + bx + cy + d = 0.$$

The first equation represents a parabola that opens either up or down. The second equation represents a parabola that opens either to the left or to the right. To put the equation into standard form, use the method of completing the square.

Example 7.19

Converting the Equation of a Parabola from General into Standard Form

Put the equation $x^2 - 4x - 8y + 12 = 0$ into standard form and graph the resulting parabola.

Solution

Since y is not squared in this equation, we know that the parabola opens either upward or downward. Therefore we need to solve this equation for y, which will put the equation into standard form. To do that, first add $8y$ to both sides of the equation:

$$8y = x^2 - 4x + 12.$$

The next step is to complete the square on the right-hand side. Start by grouping the first two terms on the right-hand side using parentheses:

$$8y = \left(x^2 - 4x\right) + 12.$$

Next determine the constant that, when added inside the parentheses, makes the quantity inside the parentheses a perfect square trinomial. To do this, take half the coefficient of x and square it. This gives $\left(\frac{-4}{2}\right)^2 = 4.$ Add 4 inside the parentheses and subtract 4 outside the parentheses, so the value of the equation is not changed:

$$8y = \left(x^2 - 4x + 4\right) + 12 - 4.$$

Now combine like terms and factor the quantity inside the parentheses:

$$8y = (x - 2)^2 + 8.$$

Finally, divide by 8:

$$y = \frac{1}{8}(x - 2)^2 + 1.$$

This equation is now in standard form. Comparing this to **Equation 7.11** gives $h = 2,\quad k = 1,\quad \text{and}\quad p = 2.$ The parabola opens up, with vertex at $(2,\ 1),\quad$ focus at $(2,\ 3),\quad$ and directrix $y = -1.$ The graph of this parabola appears as follows.

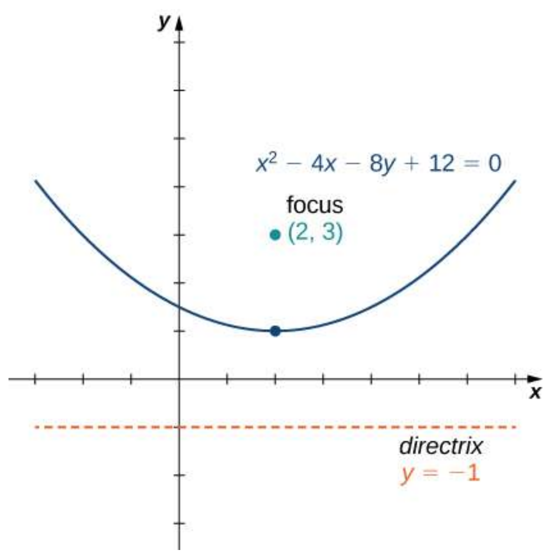
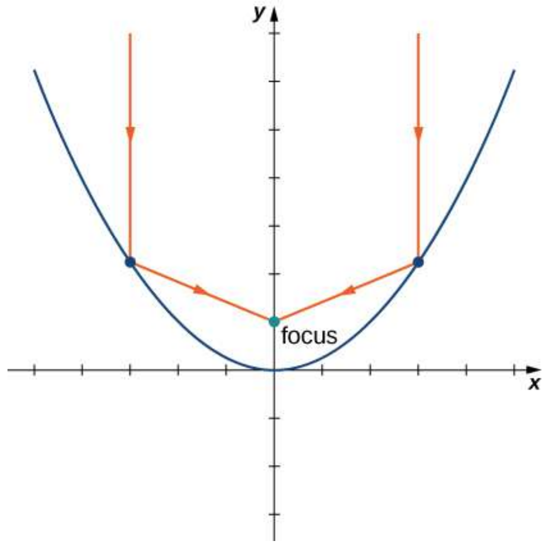


Figure 7.47 The parabola in **Example 7.19**.

7.18 Put the equation $2y^2 - x + 12y + 16 = 0$ into standard form and graph the resulting parabola.

The axis of symmetry of a vertical (opening up or down) parabola is a vertical line passing through the vertex. The parabola has an interesting reflective property. Suppose we have a satellite dish with a parabolic cross section. If a beam of electromagnetic waves, such as light or radio waves, comes into the dish in a straight line from a satellite (parallel to the axis of symmetry), then the waves reflect off the dish and collect at the focus of the parabola as shown.



Consider a parabolic dish designed to collect signals from a satellite in space. The dish is aimed directly at the satellite, and a receiver is located at the focus of the parabola. Radio waves coming in from the satellite are reflected off the surface of the parabola to the receiver, which collects and decodes the digital signals. This allows a small receiver to gather signals from a wide angle of sky. Flashlights and headlights in a car work on the same principle, but in reverse: the source of the light (that is, the light bulb) is located at the focus and the reflecting surface on the parabolic mirror focuses the beam straight ahead. This allows a small light bulb to illuminate a wide angle of space in front of the flashlight or car.

Ellipses

An ellipse can also be defined in terms of distances. In the case of an ellipse, there are two foci (plural of focus), and two directrices (plural of directrix). We look at the directrices in more detail later in this section.

Definition

An *ellipse* is the set of all points for which the sum of their distances from two fixed points (the foci) is constant.

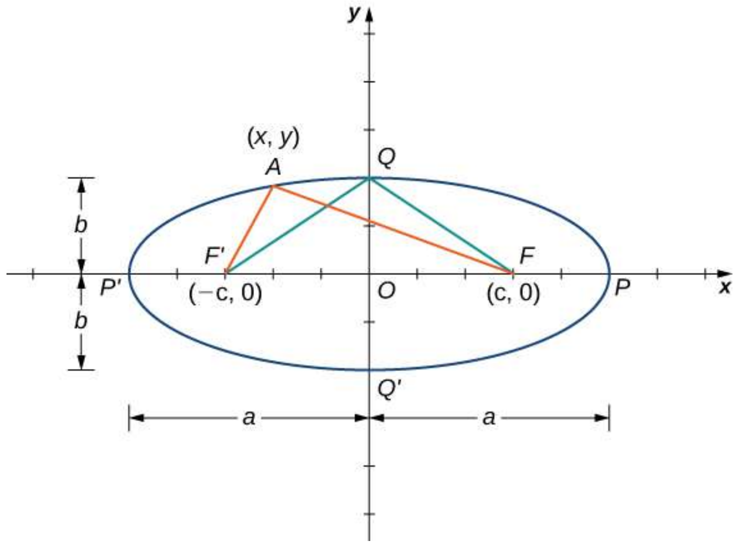


Figure 7.48 A typical ellipse in which the sum of the distances from any point on the ellipse to the foci is constant.

A graph of a typical ellipse is shown in **Figure 7.48**. In this figure the foci are labeled as F and F'. Both are the same fixed distance from the origin, and this distance is represented by the variable c. Therefore the coordinates of F are $(c, 0)$ and the coordinates of F' are $(-c, 0)$. The points P and P' are located at the ends of the **major axis** of the ellipse, and have coordinates $(a, 0)$ and $(-a, 0)$, respectively. The major axis is always the longest distance across the ellipse, and can be horizontal or vertical. Thus, the length of the major axis in this ellipse is $2a$. Furthermore, P and P' are called the vertices of the ellipse. The points Q and Q' are located at the ends of the **minor axis** of the ellipse, and have coordinates $(0, b)$ and $(0, -b)$, respectively. The minor axis is the shortest distance across the ellipse. The minor axis is perpendicular to the major axis.

According to the definition of the ellipse, we can choose any point on the ellipse and the sum of the distances from this point to the two foci is constant. Suppose we choose the point P. Since the coordinates of point P are $(a, 0)$, the sum of the distances is

$$d(P, F) + d(P, F') = (a - c) + (a + c) = 2a.$$

Therefore the sum of the distances from an arbitrary point A with coordinates (x, y) is also equal to $2a$. Using the distance formula, we get

$$\begin{aligned} d(A, F) + d(A, F') &= 2a \\ \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} &= 2a. \end{aligned}$$

Subtract the second radical from both sides and square both sides:

$$\begin{aligned} \sqrt{(x-c)^2 + y^2} &= 2a - \sqrt{(x+c)^2 + y^2} \\ (x-c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\ x^2 - 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\ -2cx &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx. \end{aligned}$$

Now isolate the radical on the right-hand side and square again:

$$\begin{aligned}
-2cx &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx \\
4a\sqrt{(x+c)^2 + y^2} &= 4a^2 + 4cx \\
\sqrt{(x+c)^2 + y^2} &= a + \frac{cx}{a} \\
(x+c)^2 + y^2 &= a^2 + 2cx + \frac{c^2x^2}{a^2} \\
x^2 + 2cx + c^2 + y^2 &= a^2 + 2cx + \frac{c^2x^2}{a^2} \\
x^2 + c^2 + y^2 &= a^2 + \frac{c^2x^2}{a^2}.
\end{aligned}$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

$$\begin{aligned}
x^2 - \frac{c^2x^2}{a^2} + y^2 &= a^2 - c^2 \\
\frac{\left(a^2 - c^2\right)x^2}{a^2} + y^2 &= a^2 - c^2.
\end{aligned}$$

Divide both sides by $a^2 - c^2$. This gives the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we refer back to **Figure 7.48**, then the length of each of the two green line segments is equal to a. This is true because the sum of the distances from the point Q to the foci F and F' is equal to $2a$, and the lengths of these two line segments are equal. This line segment forms a right triangle with hypotenuse length a and leg lengths b and c. From the Pythagorean theorem, $a^2 + b^2 = c^2$ and $b^2 = a^2 - c^2$. Therefore the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Finally, if the center of the ellipse is moved from the origin to a point (h, k), we have the following standard form of an ellipse.

Theorem 7.9: Equation of an Ellipse in Standard Form

Consider the ellipse with center (h, k), a horizontal major axis with length $2a$, and a vertical minor axis with length $2b$. Then the equation of this ellipse in standard form is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{7.12}$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 - b^2$. The equations of the directrices are $x = h \pm \frac{a^2}{c}$.

If the major axis is vertical, then the equation of the ellipse becomes

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \tag{7.13}$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 - b^2$. The equations of the directrices in this case are $y = k \pm \frac{a^2}{c}$.

If the major axis is horizontal, then the ellipse is called horizontal, and if the major axis is vertical, then the ellipse is called vertical. The equation of an ellipse is in general form if it is in the form $Ax^2 + By^2 + Cx + Dy + E = 0,$ where A and B are either both positive or both negative. To convert the equation from general to standard form, use the method of completing the square.

Example 7.20

Finding the Standard Form of an Ellipse

Put the equation $9x^2 + 4y^2 - 36x + 24y + 36 = 0$ into standard form and graph the resulting ellipse.

Solution

First subtract 36 from both sides of the equation:

$$9x^2 + 4y^2 - 36x + 24y = -36.$$

Next group the x terms together and the y terms together, and factor out the common factor:

$$\begin{aligned}\left(9x^2 - 36x\right) + \left(4y^2 + 24y\right) &= -36 \\ 9\left(x^2 - 4x\right) + 4\left(y^2 + 6y\right) &= -36.\end{aligned}$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $\left(\frac{-4}{2}\right)^2 = 4.$ In the second set of parentheses, take half the coefficient of y and square it. This gives $\left(\frac{6}{2}\right)^2 = 9.$ Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are adding 36 to the second set as well. Therefore the equation becomes

$$\begin{aligned}9\left(x^2 - 4x + 4\right) + 4\left(y^2 + 6y + 9\right) &= -36 + 36 + 36 \\ 9\left(x^2 - 4x + 4\right) + 4\left(y^2 + 6y + 9\right) &= 36.\end{aligned}$$

Now factor both sets of parentheses and divide by 36:

$$\begin{aligned}9(x-2)^2 + 4(y+3)^2 &= 36 \\ \frac{9(x-2)^2}{36} + \frac{4(y+3)^2}{36} &= 1 \\ \frac{(x-2)^2}{4} + \frac{(y+3)^2}{9} &= 1.\end{aligned}$$

The equation is now in standard form. Comparing this to **Equation 7.14** gives $h = 2,$ $k = -3,$ $a = 3,$ and $b = 2.$ This is a vertical ellipse with center at $(2, -3),$ major axis 6, and minor axis 4. The graph of this ellipse appears as follows.

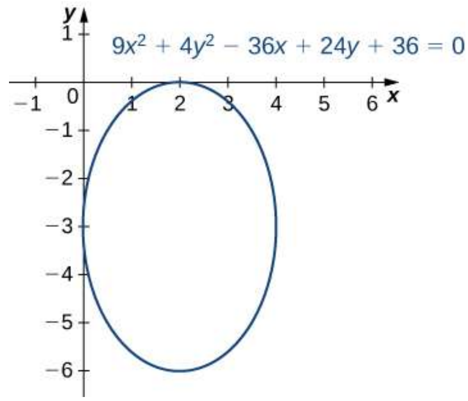


Figure 7.49 The ellipse in **Example 7.20**.

7.19 Put the equation $9x^2 + 16y^2 + 18x - 64y - 71 = 0$ into standard form and graph the resulting ellipse.

According to Kepler's first law of planetary motion, the orbit of a planet around the Sun is an ellipse with the Sun at one of the foci as shown in **Figure 7.50**(a). Because Earth's orbit is an ellipse, the distance from the Sun varies throughout the year. A commonly held misconception is that Earth is closer to the Sun in the summer. In fact, in summer for the northern hemisphere, Earth is farther from the Sun than during winter. The difference in season is caused by the tilt of Earth's axis in the orbital plane. Comets that orbit the Sun, such as Halley's Comet, also have elliptical orbits, as do moons orbiting the planets and satellites orbiting Earth.

Ellipses also have interesting reflective properties: A light ray emanating from one focus passes through the other focus after mirror reflection in the ellipse. The same thing occurs with a sound wave as well. The National Statuary Hall in the U.S. Capitol in Washington, DC, is a famous room in an elliptical shape as shown in **Figure 7.50**(b). This hall served as the meeting place for the U.S. House of Representatives for almost fifty years. The location of the two foci of this semi-elliptical room are clearly identified by marks on the floor, and even if the room is full of visitors, when two people stand on these spots and speak to each other, they can hear each other much more clearly than they can hear someone standing close by. Legend has it that John Quincy Adams had his desk located on one of the foci and was able to eavesdrop on everyone else in the House without ever needing to stand. Although this makes a good story, it is unlikely to be true, because the original ceiling produced so many echoes that the entire room had to be hung with carpets to dampen the noise. The ceiling was rebuilt in 1902 and only then did the now-famous whispering effect emerge. Another famous whispering gallery—the site of many marriage proposals—is in Grand Central Station in New York City.

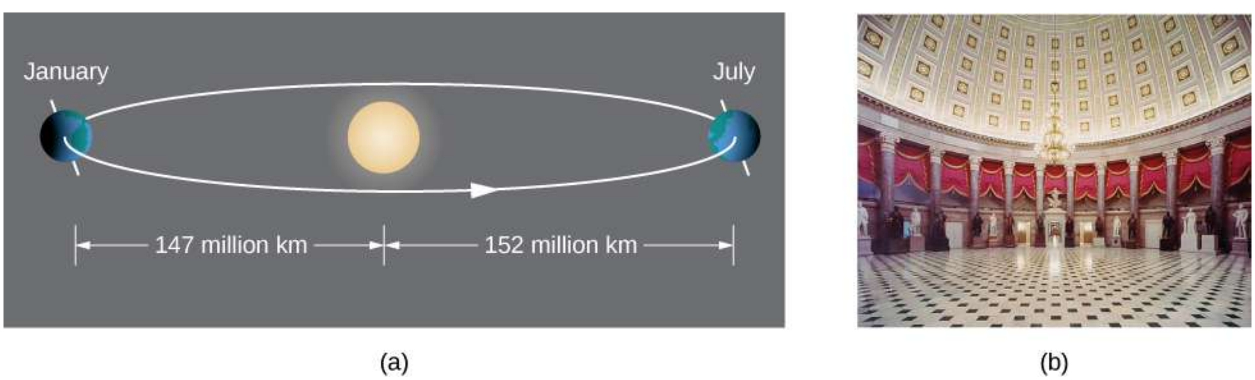


Figure 7.50 (a) Earth's orbit around the Sun is an ellipse with the Sun at one focus. (b) Statuary Hall in the U.S. Capitol is a whispering gallery with an elliptical cross section.

Hyperbolas

A hyperbola can also be defined in terms of distances. In the case of a hyperbola, there are two foci and two directrices. Hyperbolas also have two asymptotes.

Definition

A hyperbola is the set of all points where the difference between their distances from two fixed points (the foci) is constant.

A graph of a typical hyperbola appears as follows.

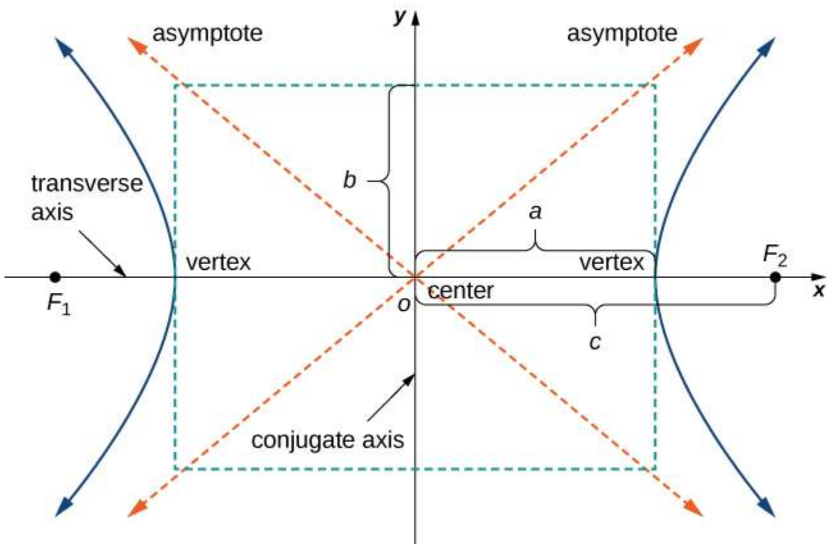


Figure 7.51 A typical hyperbola in which the difference of the distances from any point on the ellipse to the foci is constant. The transverse axis is also called the major axis, and the conjugate axis is also called the minor axis.

The derivation of the equation of a hyperbola in standard form is virtually identical to that of an ellipse. One slight hitch lies in the definition: The difference between two numbers is always positive. Let P be a point on the hyperbola with coordinates (x, y). Then the definition of the hyperbola gives $|d(P, F_1) - d(P, F_2)| = \text{constant}$. To simplify the derivation, assume that P is on the right branch of the hyperbola, so the absolute value bars drop. If it is on the left branch, then the subtraction is reversed. The vertex of the right branch has coordinates $(a, 0)$, so

$$d(P, F_1) - d(P, F_2) = (c + a) - (c - a) = 2a.$$

This equation is therefore true for any point on the hyperbola. Returning to the coordinates (x, y) for P:

$$\begin{aligned} d(P, F_1) - d(P, F_2) &= 2a \\ \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a. \end{aligned}$$

Add the second radical from both sides and square both sides:

$$\begin{aligned} \sqrt{(x-c)^2 + y^2} &= 2a + \sqrt{(x+c)^2 + y^2} \\ (x-c)^2 + y^2 &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\ x^2 - 2cx + c^2 + y^2 &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\ -2cx &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + 2cx. \end{aligned}$$

Now isolate the radical on the right-hand side and square again:

$$\begin{aligned} -2cx &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + 2cx \\ 4a\sqrt{(x+c)^2 + y^2} &= -4a^2 - 4cx \\ \sqrt{(x+c)^2 + y^2} &= -a - \frac{cx}{a} \\ (x+c)^2 + y^2 &= a^2 + 2cx + \frac{c^2x^2}{a^2} \\ x^2 + 2cx + c^2 + y^2 &= a^2 + 2cx + \frac{c^2x^2}{a^2} \\ x^2 + c^2 + y^2 &= a^2 + \frac{c^2x^2}{a^2}. \end{aligned}$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

$$\begin{aligned} x^2 - \frac{c^2x^2}{a^2} + y^2 &= a^2 - c^2 \\ \frac{\left(a^2 - c^2\right)x^2}{a^2} + y^2 &= a^2 - c^2. \end{aligned}$$

Finally, divide both sides by $a^2 - c^2$. This gives the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

We now define b so that $b^2 = c^2 - a^2$. This is possible because $c > a$. Therefore the equation of the ellipse becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Finally, if the center of the hyperbola is moved from the origin to the point (h, k), we have the following standard form of a hyperbola.

Theorem 7.10: Equation of a Hyperbola in Standard Form

Consider the hyperbola with center (h, k), a horizontal major axis, and a vertical minor axis. Then the equation of this ellipse is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{7.14}$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{b}{a}(x - h)$. The equations of the directrices are

$$x = k \pm \frac{a^2}{\sqrt{a^2 + b^2}} = h \pm \frac{a^2}{c}.$$

If the major axis is vertical, then the equation of the hyperbola becomes

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \tag{7.15}$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{a}{b}(x - h)$. The equations of the directrices are

$$y = k \pm \frac{a^2}{\sqrt{a^2 + b^2}} = k \pm \frac{a^2}{c}.$$

If the major axis (transverse axis) is horizontal, then the hyperbola is called horizontal, and if the major axis is vertical then the hyperbola is called vertical. The equation of a hyperbola is in general form if it is in the form $Ax^2 + By^2 + Cx + Dy + E = 0,$ where A and B have opposite signs. In order to convert the equation from general to standard form, use the method of completing the square.

Example 7.21

Finding the Standard Form of a Hyperbola

Put the equation $9x^2 - 16y^2 + 36x + 32y - 124 = 0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Solution

First add 124 to both sides of the equation:

$$9x^2 - 16y^2 + 36x + 32y = 124.$$

Next group the x terms together and the y terms together, then factor out the common factors:

$$\begin{aligned} \left(9x^2 + 36x\right) - \left(16y^2 - 32y\right) &= 124 \\ 9\left(x^2 + 4x\right) - 16\left(y^2 - 2y\right) &= 124. \end{aligned}$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $\left(\frac{4}{2}\right)^2 = 4.$ In the second set of parentheses, take half the coefficient of y and square it. This gives $\left(\frac{-2}{2}\right)^2 = 1.$ Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are subtracting 16 from the second set of parentheses. Therefore the equation becomes

$$\begin{aligned} 9\left(x^2 + 4x + 4\right) - 16\left(y^2 - 2y + 1\right) &= 124 + 36 - 16 \\ 9\left(x^2 + 4x + 4\right) - 16\left(y^2 - 2y + 1\right) &= 144. \end{aligned}$$

Next factor both sets of parentheses and divide by 144:

$$\begin{aligned} 9(x + 2)^2 - 16(y - 1)^2 &= 144 \\ \frac{9(x + 2)^2}{144} - \frac{16(y - 1)^2}{144} &= 1 \\ \frac{(x + 2)^2}{16} - \frac{(y - 1)^2}{9} &= 1. \end{aligned}$$

The equation is now in standard form. Comparing this to **Equation 7.15** gives $h = -2,\quad k = 1,\quad a = 4,$ and $b = 3.$ This is a horizontal hyperbola with center at $(-2, 1)$ and asymptotes given by the equations $y = 1 \pm \frac{3}{4}(x + 2).$ The graph of this hyperbola appears in the following figure.

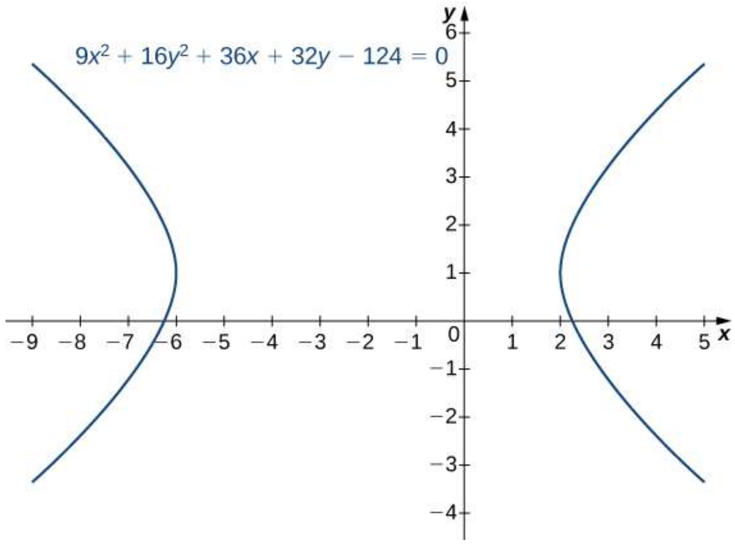


Figure 7.52 Graph of the hyperbola in **Example 7.21**.

7.20 Put the equation $4y^2 - 9x^2 + 16y + 18x - 29 = 0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Hyperbolas also have interesting reflective properties. A ray directed toward one focus of a hyperbola is reflected by a hyperbolic mirror toward the other focus. This concept is illustrated in the following figure.

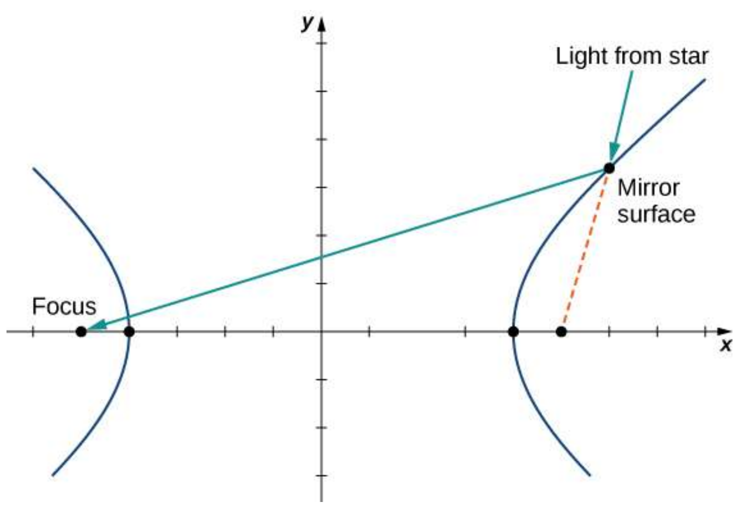


Figure 7.53 A hyperbolic mirror used to collect light from distant stars.

This property of the hyperbola has important applications. It is used in radio direction finding (since the difference in signals from two towers is constant along hyperbolas), and in the construction of mirrors inside telescopes (to reflect light coming from the parabolic mirror to the eyepiece). Another interesting fact about hyperbolas is that for a comet entering the solar system, if the speed is great enough to escape the Sun's gravitational pull, then the path that the comet takes as it passes through the solar system is hyperbolic.

Eccentricity and Directrix

An alternative way to describe a conic section involves the directrices, the foci, and a new property called eccentricity. We will see that the value of the eccentricity of a conic section can uniquely define that conic.

Definition

The **eccentricity** e of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for any conic section, and can define the conic section as well:

1. If $e = 1,$ the conic is a parabola.
2. If $e < 1,$ it is an ellipse.
3. If $e > 1,$ it is a hyperbola.

The eccentricity of a circle is zero. The directrix of a conic section is the line that, together with the point known as the focus, serves to define a conic section. Hyperbolas and noncircular ellipses have two foci and two associated directrices. Parabolas have one focus and one directrix.

The three conic sections with their directrices appear in the following figure.

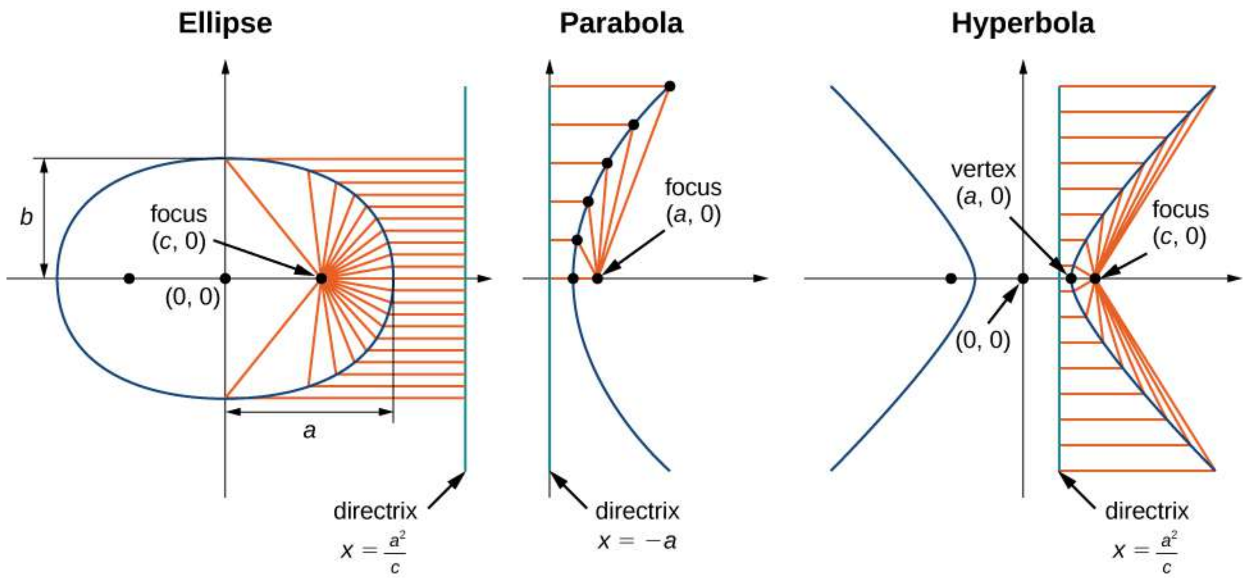


Figure 7.54 The three conic sections with their foci and directrices.

Recall from the definition of a parabola that the distance from any point on the parabola to the focus is equal to the distance from that same point to the directrix. Therefore, by definition, the eccentricity of a parabola must be 1. The equations of the directrices of a horizontal ellipse are $x = \pm\frac{a^2}{c}.$ The right vertex of the ellipse is located at $(a, 0)$ and the right focus is $(c, 0).$ Therefore the distance from the vertex to the focus is $a - c$ and the distance from the vertex to the right directrix is $\frac{a^2}{c} - c.$ This gives the eccentricity as

$$e = \frac{a - c}{\frac{a^2}{c} - a} = \frac{c(a - c)}{a^2 - ac} = \frac{c(a - c)}{a(a - c)} = \frac{c}{a}.$$

Since $c < a,$ this step proves that the eccentricity of an ellipse is less than 1. The directrices of a horizontal hyperbola are also located at $x = \pm\frac{a^2}{c},$ and a similar calculation shows that the eccentricity of a hyperbola is also $e = \frac{c}{a}.$ However in this case we have $c > a,$ so the eccentricity of a hyperbola is greater than 1.

Example 7.22

Determining Eccentricity of a Conic Section

Determine the eccentricity of the ellipse described by the equation

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1.$$

Solution

From the equation we see that $a = 5$ and $b = 4$. The value of c can be calculated using the equation $a^2 = b^2 + c^2$ for an ellipse. Substituting the values of a and b and solving for c gives $c = 3$. Therefore the eccentricity of the ellipse is $e = \frac{c}{a} = \frac{3}{5} = 0.6$.

7.21 Determine the eccentricity of the hyperbola described by the equation

$$\frac{(y-3)^2}{49} - \frac{(x+2)^2}{25} = 1.$$

Polar Equations of Conic Sections

Sometimes it is useful to write or identify the equation of a conic section in polar form. To do this, we need the concept of the focal parameter. The **focal parameter** of a conic section p is defined as the distance from a focus to the nearest directrix. The following table gives the focal parameters for the different types of conics, where a is the length of the semi-major axis (i.e., half the length of the major axis), c is the distance from the origin to the focus, and e is the eccentricity. In the case of a parabola, a represents the distance from the vertex to the focus.

Conic	*e*	*p*
Ellipse	$0 < e < 1$	$\frac{a^2 - c^2}{c} = \frac{a(1-e^2)}{c}$
Parabola	$e = 1$	$2a$
Hyperbola	$e > 1$	$\frac{c^2 - a^2}{c} = \frac{a(e^2-1)}{e}$

Table 7.7 Eccentricities and Focal Parameters of the Conic Sections

Using the definitions of the focal parameter and eccentricity of the conic section, we can derive an equation for any conic section in polar coordinates. In particular, we assume that one of the foci of a given conic section lies at the pole. Then using the definition of the various conic sections in terms of distances, it is possible to prove the following theorem.

Theorem 7.11: Polar Equation of Conic Sections

The polar equation of a conic section with focal parameter p is given by

$$r = \frac{ep}{1 \pm e\cos\theta} \text{ or } r = \frac{ep}{1 \pm e\sin\theta}.$$

In the equation on the left, the major axis of the conic section is horizontal, and in the equation on the right, the major axis is vertical. To work with a conic section written in polar form, first make the constant term in the denominator equal to 1. This can be done by dividing both the numerator and the denominator of the fraction by the constant that appears in front of the plus or minus in the denominator. Then the coefficient of the sine or cosine in the denominator is the eccentricity. This value identifies the conic. If cosine appears in the denominator, then the conic is horizontal. If sine appears, then the conic is vertical. If both appear then the axes are rotated. The center of the conic is not necessarily at the origin. The center is at the origin only if the conic is a circle (i.e., $e = 0$).

Example 7.23

Graphing a Conic Section in Polar Coordinates

Identify and create a graph of the conic section described by the equation

$$r = \frac{3}{1 + 2\cos\theta}.$$

Solution

The constant term in the denominator is 1, so the eccentricity of the conic is 2. This is a hyperbola. The focal parameter p can be calculated by using the equation $ep = 3$. Since $e = 2$, this gives $p = \frac{3}{2}$. The cosine function appears in the denominator, so the hyperbola is horizontal. Pick a few values for θ and create a table of values. Then we can graph the hyperbola (**Figure 7.55**).

θ	r	θ	r
0	1	π	-3
$\frac{\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$	$\frac{5\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$
$\frac{\pi}{2}$	3	$\frac{3\pi}{2}$	3
$\frac{3\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$	$\frac{7\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$

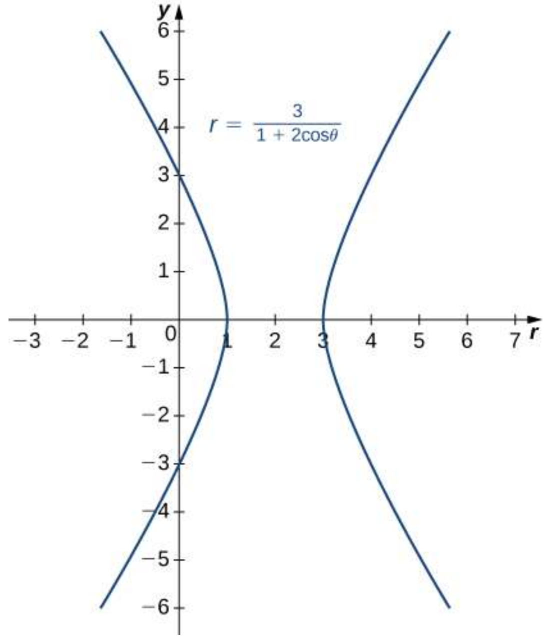


Figure 7.55 Graph of the hyperbola described in **Example 7.23**.

7.22 Identify and create a graph of the conic section described by the equation

$$r = \frac{4}{1 - 0.8 \sin \theta}.$$

General Equations of Degree Two

A general equation of degree two can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The graph of an equation of this form is a conic section. If $B \neq 0$ then the coordinate axes are rotated. To identify the conic section, we use the **discriminant** of the conic section $4AC - B^2$. One of the following cases must be true:

1. $4AC - B^2 > 0$. If so, the graph is an ellipse.
2. $4AC - B^2 = 0$. If so, the graph is a parabola.
3. $4AC - B^2 < 0$. If so, the graph is a hyperbola.

The simplest example of a second-degree equation involving a cross term is $xy = 1$. This equation can be solved for y to obtain $y = \frac{1}{x}$. The graph of this function is called a *rectangular hyperbola* as shown.

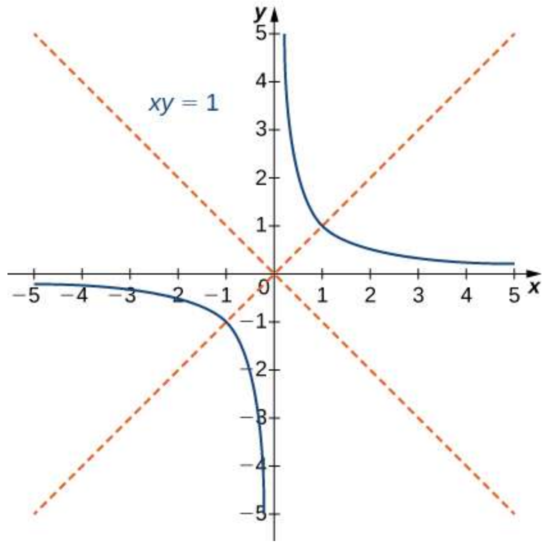


Figure 7.56 Graph of the equation $xy = 1$; The red lines indicate the rotated axes.

The asymptotes of this hyperbola are the *x* and *y* coordinate axes. To determine the angle θ of rotation of the conic section, we use the formula $\cot 2\theta = \frac{A - C}{B}$. In this case $A = C = 0$ and $B = 1$, so $\cot 2\theta = (0 - 0)/1 = 0$ and $\theta = 45°$. The method for graphing a conic section with rotated axes involves determining the coefficients of the conic in the rotated coordinate system. The new coefficients are labeled $A', B', C', D', E',$ and F', and are given by the formulas

$$\begin{aligned}
A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\
B' &= 0 \\
C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\
D' &= D\cos\theta + E\sin\theta \\
E' &= -D\sin\theta + E\cos\theta \\
F' &= F.
\end{aligned}$$

The procedure for graphing a rotated conic is the following:

1. Identify the conic section using the discriminant $4AC - B^2$.
2. Determine θ using the formula $\cot 2\theta = \frac{A - C}{B}$.
3. Calculate $A', B', C', D', E',$ and F'.
4. Rewrite the original equation using $A', B', C', D', E',$ and F'.
5. Draw a graph using the rotated equation.

Example 7.24

Identifying a Rotated Conic

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$$13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0.$$

Solution

In this equation, $A = 13, B = -6\sqrt{3}, C = 7, D = 0, E = 0,$ and $F = -256.$ The discriminant of this equation is $4AC - B^2 = 4(13)(7) - \left(-6\sqrt{3}\right)^2 = 364 - 108 = 256.$ Therefore this conic is an ellipse. To calculate the angle of rotation of the axes, use $\cot 2\theta = \frac{A - C}{B}.$ This gives

$$\begin{aligned} \cot 2\theta &= \frac{A - C}{B} \\ &= \frac{13 - 7}{-6\sqrt{3}} \\ &= -\frac{\sqrt{3}}{3}. \end{aligned}$$

Therefore $2\theta = 120^{\text{o}}$ and $\theta = 60^{\text{o}},$ which is the angle of the rotation of the axes.

To determine the rotated coefficients, use the formulas given above:

$$\begin{aligned} A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\ &= 13\cos^2 60 + \left(-6\sqrt{3}\right)\cos 60 \sin 60 + 7\sin^2 60 \\ &= 13\left(\frac{1}{2}\right)^2 - 6\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 7\left(\frac{\sqrt{3}}{2}\right)^2 \\ &= 4, \\ B' &= 0, \\ C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\ &= 13\sin^2 60 + \left(-6\sqrt{3}\right)\sin 60 \cos 60 = 7\cos^2 60 \\ &= \left(\frac{\sqrt{3}}{2}\right)^2 + 6\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 7\left(\frac{1}{2}\right)^2 \\ &= 16, \\ D' &= D\cos\theta + E\sin\theta \\ &= (0)\cos 60 + (0)\sin 60 \\ &= 0, \\ E' &= -D\sin\theta + E\cos\theta \\ &= -(0)\sin 60 + (0)\cos 60 \\ &= 0, \\ F' &= F \\ &= -256. \end{aligned}$$

The equation of the conic in the rotated coordinate system becomes

$$\begin{aligned} 4(x')^2 + 16(y')^2 &= 256 \\ \frac{(x')^2}{64} + \frac{(y')^2}{16} &= 1. \end{aligned}$$

A graph of this conic section appears as follows.

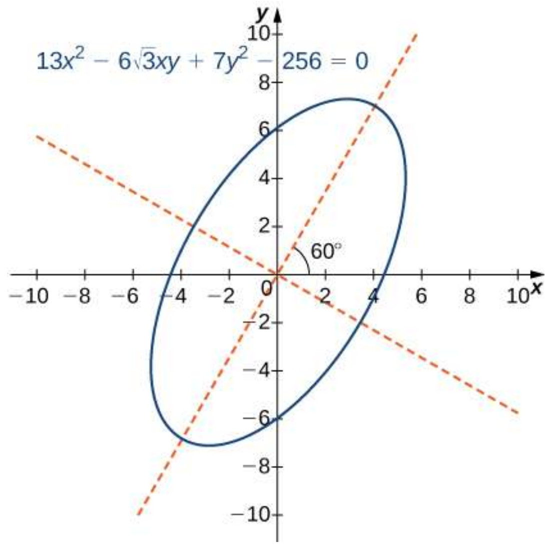


Figure 7.57 Graph of the ellipse described by the equation $13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0$. The axes are rotated $60°$. The red dashed lines indicate the rotated axes.

7.23 Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$$3x^2 + 5xy - 2y^2 - 125 = 0.$$

7.5 EXERCISES

For the following exercises, determine the equation of the parabola using the information given.

255. Focus $(4, 0)$ and directrix $x = -4$

256. Focus $(0, -3)$ and directrix $y = 3$

257. Focus $(0, 0.5)$ and directrix $y = -0.5$

258. Focus $(2, 3)$ and directrix $x = -2$

259. Focus $(0, 2)$ and directrix $y = 4$

260. Focus $(-1, 4)$ and directrix $x = 5$

261. Focus $(-3, 5)$ and directrix $y = 1$

262. Focus $\left(\frac{5}{2}, -4\right)$ and directrix $x = \frac{7}{2}$

For the following exercises, determine the equation of the ellipse using the information given.

263. Endpoints of major axis at $(4, 0), (-4, 0)$ and foci located at $(2, 0), (-2, 0)$

264. Endpoints of major axis at $(0, 5), (0, -5)$ and foci located at $(0, 3), (0, -3)$

265. Endpoints of major axis at $(0, 2), (0, -2)$ and foci located at $(3, 0), (-3, 0)$

266. Endpoints of major axis at $(-3, 3), (7, 3)$ and foci located at $(-2, 3), (6, 3)$

267. Endpoints of major axis at $(-3, 5), (-3, -3)$ and foci located at $(-3, 3), (-3, -1)$

268. Endpoints of major axis at $(0, 0), (0, 4)$ and foci located at $(5, 2), (-5, 2)$

269. Foci located at $(2, 0)$, $(-2, 0)$ and eccentricity of $\frac{1}{2}$

270. Foci located at $(0, -3)$, $(0, 3)$ and eccentricity of $\frac{3}{4}$

For the following exercises, determine the equation of the hyperbola using the information given.

271. Vertices located at $(5, 0), (-5, 0)$ and foci located at $(6, 0), (-6, 0)$

272. Vertices located at $(0, 2), (0, -2)$ and foci located at $(0, 3), (0, -3)$

273. Endpoints of the conjugate axis located at $(0, 3), (0, -3)$ and foci located $(4, 0), (-4, 0)$

274. Vertices located at $(0, 1), (6, 1)$ and focus located at $(8, 1)$

275. Vertices located at $(-2, 0), (-2, -4)$ and focus located at $(-2, -8)$

276. Endpoints of the conjugate axis located at $(3, 2), (3, 4)$ and focus located at $(3, 7)$

277. Foci located at $(6, -0), (6, 0)$ and eccentricity of 3

278. $(0, 10), (0, -10)$ and eccentricity of 2.5

For the following exercises, consider the following polar equations of conics. Determine the eccentricity and identify the conic.

279. $r = \dfrac{-1}{1 + \cos\theta}$

280. $r = \dfrac{8}{2 - \sin\theta}$

281. $r = \dfrac{5}{2 + \sin\theta}$

282. $r = \dfrac{5}{-1 + 2\sin\theta}$

283. $r = \dfrac{3}{2 - 6\sin\theta}$

284. $r = \dfrac{3}{-4 + 3\sin\theta}$

For the following exercises, find a polar equation of the conic with focus at the origin and eccentricity and directrix as given.

285. Directrix: $x = 4;\ e = \frac{1}{5}$

286. Directrix: $x = -4;\ e = 5$

287. Directrix: $\mathrm{y} = 2;\ e = 2$

288. Directrix: y = -2; $e = \frac{1}{2}$

For the following exercises, sketch the graph of each conic.

289. $r = \frac{1}{1 + \sin\theta}$

290. $r = \frac{1}{1 - \cos\theta}$

291. $r = \frac{4}{1 + \cos\theta}$

292. $r = \frac{10}{5 + 4\sin\theta}$

293. $r = \frac{15}{3 - 2\cos\theta}$

294. $r = \frac{32}{3 + 5\sin\theta}$

295. $r(2 + \sin\theta) = 4$

296. $r = \frac{3}{2 + 6\sin\theta}$

297. $r = \frac{3}{-4 + 2\sin\theta}$

298. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

299. $\frac{x^2}{4} + \frac{y^2}{16} = 1$

300. $4x^2 + 9y^2 = 36$

301. $25x^2 - 4y^2 = 100$

302. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

303. $x^2 = 12y$

304. $y^2 = 20x$

305. $12x = 5y^2$

For the following equations, determine which of the conic sections is described.

306. $xy = 4$

307. $x^2 + 4xy - 2y^2 - 6 = 0$

308. $x^2 + 2\sqrt{3}xy + 3y^2 - 6 = 0$

309. $x^2 - xy + y^2 - 2 = 0$

310. $34x^2 - 24xy + 41y^2 - 25 = 0$

311. $52x^2 - 72xy + 73y^2 + 40x + 30y - 75 = 0$

312. The mirror in an automobile headlight has a parabolic cross section, with the lightbulb at the focus. On a schematic, the equation of the parabola is given as $x^2 = 4y$. At what coordinates should you place the lightbulb?

313. A satellite dish is shaped like a paraboloid of revolution. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

314. Consider the satellite dish of the preceding problem. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?

315. A searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

316. Whispering galleries are rooms designed with elliptical ceilings. A person standing at one focus can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet and the foci are located 30 feet from the center, find the height of the ceiling at the center.

317. A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus and the other focus is 80 feet away, what is the length and the height at the center of the gallery?

For the following exercises, determine the polar equation form of the orbit given the length of the major axis and eccentricity for the orbits of the comets or planets. Distance is given in astronomical units (AU).

318. Halley's Comet: length of major axis = 35.88, eccentricity = 0.967

319. Hale-Bopp Comet: length of major axis = 525.91, eccentricity = 0.995

320. Mars: length of major axis = 3.049, eccentricity = 0.0934

321. Jupiter: length of major axis = 10.408, eccentricity = 0.0484

CHAPTER 7 REVIEW

KEY TERMS

angular coordinate θ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial (x) axis, measured counterclockwise

cardioid a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r = a(1 + \sin\theta)$ or $r = a(1 + \cos\theta)$

conic section a conic section is any curve formed by the intersection of a plane with a cone of two nappes

cusp a pointed end or part where two curves meet

cycloid the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

directrix a directrix (plural: directrices) is a line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two

discriminant the value $4AC - B^2$, which is used to identify a conic when the equation contains a term involving xy, is called a discriminant

eccentricity the eccentricity is defined as the distance from any point on the conic section to its focus divided by the perpendicular distance from that point to the nearest directrix

focal parameter the focal parameter is the distance from a focus of a conic section to the nearest directrix

focus a focus (plural: foci) is a point used to construct and define a conic section; a parabola has one focus; an ellipse and a hyperbola have two

general form an equation of a conic section written as a general second-degree equation

limaçon the graph of the equation $r = a + b\sin\theta$ or $r = a + b\cos\theta$. If $a = b$ then the graph is a cardioid

major axis the major axis of a conic section passes through the vertex in the case of a parabola or through the two vertices in the case of an ellipse or hyperbola; it is also an axis of symmetry of the conic; also called the transverse axis

minor axis the minor axis is perpendicular to the major axis and intersects the major axis at the center of the conic, or at the vertex in the case of the parabola; also called the conjugate axis

nappe a nappe is one half of a double cone

orientation the direction that a point moves on a graph as the parameter increases

parameter an independent variable that both x and y depend on in a parametric curve; usually represented by the variable t

parameterization of a curve rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

parametric curve the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \le t \le b$ combined with the equations

parametric equations the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

polar axis the horizontal axis in the polar coordinate system corresponding to $r \ge 0$

polar coordinate system a system for locating points in the plane. The coordinates are r, the radial coordinate, and θ, the angular coordinate

polar equation an equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system

pole the central point of the polar coordinate system, equivalent to the origin of a Cartesian system

radial coordinate r the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole

rose graph of the polar equation $r = a\cos 2\theta$ or $r = a\sin 2\theta$ for a positive constant a

space-filling curve a curve that completely occupies a two-dimensional subset of the real plane

standard form an equation of a conic section showing its properties, such as location of the vertex or lengths of major and minor axes

vertex a vertex is an extreme point on a conic section; a parabola has one vertex at its turning point. An ellipse has two vertices, one at each end of the major axis; a hyperbola has two vertices, one at the turning point of each branch

KEY EQUATIONS

- **Derivative of parametric equations**
  $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$
- **Second-order derivative of parametric equations**
  $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}$$
- **Area under a parametric curve**
  $$A = \int_a^b y(t)x'(t)\,dt$$
- **Arc length of a parametric curve**
  $$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$
- **Surface area generated by a parametric curve**
  $$S = 2\pi\int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$
- **Area of a region bounded by a polar curve**
  $$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\,d\theta = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$
- **Arc length of a polar curve**
  $$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

KEY CONCEPTS

7.1 Parametric Equations

- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parameterized curve to obtain a function or relation describing that curve.
- There is always more than one way to parameterize a curve.
- Parametric equations can describe complicated curves that are difficult or perhaps impossible to describe using rectangular coordinates.

7.2 Calculus of Parametric Curves

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.
- The area between a parametric curve and the *x*-axis can be determined by using the formula $A = \int_{t_1}^{t_2} y(t)x'(t)\,dt$.
- The arc length of a parametric curve can be calculated by using the formula $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.
- The surface area of a volume of revolution revolved around the *x*-axis is given by $S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$. If the curve is revolved around the *y*-axis, then the formula is $S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$.

7.3 Polar Coordinates

- The polar coordinate system provides an alternative way to locate points in the plane.
- Convert points between rectangular and polar coordinates using the formulas

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}.$$

- To sketch a polar curve from a given polar function, make a table of values and take advantage of periodic properties.
- Use the conversion formulas to convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves, which can occur through the pole, the horizontal axis, or the vertical axis.

7.4 Area and Arc Length in Polar Coordinates

- The area of a region in polar coordinates defined by the equation $r = f(\theta)$ with $\alpha \le \theta \le \beta$ is given by the integral $A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\,d\theta$.
- To find the area between two curves in the polar coordinate system, first find the points of intersection, then subtract the corresponding areas.
- The arc length of a polar curve defined by the equation $r = f(\theta)$ with $\alpha \le \theta \le \beta$ is given by the integral $L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$.

7.5 Conic Sections

- The equation of a vertical parabola in standard form with given focus and directrix is $y = \frac{1}{4p}(x-h)^2 + k$ where p is the distance from the vertex to the focus and (h, k) are the coordinates of the vertex.

- The equation of a horizontal ellipse in standard form is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ where the center has coordinates (h, k), the major axis has length $2a$, the minor axis has length $2b$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$.
- The equation of a horizontal hyperbola in standard form is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ where the center has coordinates (h, k), the vertices are located at $(h \pm a, k)$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 + b^2$.
- The eccentricity of an ellipse is less than 1, the eccentricity of a parabola is equal to 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a circle is 0.
- The polar equation of a conic section with eccentricity e is $r = \frac{ep}{1 \pm e\cos\theta}$ or $r = \frac{ep}{1 \pm e\sin\theta}$, where p represents the focal parameter.
- To identify a conic generated by the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, first calculate the discriminant $D = 4AC - B^2$. If $D > 0$ then the conic is an ellipse, if $D = 0$ then the conic is a parabola, and if $D < 0$ then the conic is a hyperbola.

CHAPTER 7 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample.

322. The rectangular coordinates of the point $\left(4, \frac{5\pi}{6}\right)$ are $\left(2\sqrt{3}, -2\right)$.

323. The equations $x = \cosh(3t)$, $y = 2\sinh(3t)$ represent a hyperbola.

324. The arc length of the spiral given by $r = \frac{\theta}{2}$ for $0 \le \theta \le 3\pi$ is $\frac{9}{4}\pi^3$.

325. Given $x = f(t)$ and $y = g(t)$, if $\frac{dx}{dy} = \frac{dy}{dx}$, then $f(t) = g(t) + \mathrm{C}$, where C is a constant.

For the following exercises, sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

326. $x = 1 + t, \quad y = t^2 - 1, \quad -1 \le t \le 1$

327. $x = e^t, \quad y = 1 - e^{3t}, \quad 0 \le t \le 1$

328. $x = \sin\theta, \quad y = 1 - \csc\theta, \quad 0 \le \theta \le 2\pi$

329. $x = 4\cos\phi, \quad y = 1 - \sin\phi, \quad 0 \le \phi \le 2\pi$

For the following exercises, sketch the polar curve and determine what type of symmetry exists, if any.

330. $r = 4\sin\left(\frac{\theta}{3}\right)$

331. $r = 5\cos(5\theta)$

For the following exercises, find the polar equation for the curve given as a Cartesian equation.

332. $x + y = 5$

333. $y^2 = 4 + x^2$

For the following exercises, find the equation of the tangent line to the given curve. Graph both the function and its tangent line.

334. $x = \ln(t), \quad y = t^2 - 1, \quad t = 1$

335. $r = 3 + \cos(2\theta), \quad \theta = \frac{3\pi}{4}$

336. Find $\frac{dy}{dx}$, $\frac{dx}{dy}$, and $\frac{d^2x}{dy^2}$ of $y = \left(2 + e^{-t}\right)$, $x = 1 - \sin(t)$

For the following exercises, find the area of the region.

337. $x = t^2, \quad y = \ln(t), \quad 0 \le t \le e$

338. $r = 1 - \sin\theta$ in the first quadrant

For the following exercises, find the arc length of the curve over the given interval.

339. $x = 3t + 4, \quad y = 9t - 2, \quad 0 \le t \le 3$

340. $r = 6\cos\theta, \quad 0 \le \theta \le 2\pi.$ Check your answer by geometry.

For the following exercises, find the Cartesian equation describing the given shapes.

341. A parabola with focus $(2, -5)$ and directrix $x = 6$

342. An ellipse with a major axis length of 10 and foci at $(-7, 2)$ and $(1, 2)$

343. A hyperbola with vertices at $(3, -2)$ and $(-5, -2)$ and foci at $(-2, -6)$ and $(-2, 4)$

For the following exercises, determine the eccentricity and identify the conic. Sketch the conic.

344. $r = \dfrac{6}{1 + 3\cos(\theta)}$

345. $r = \dfrac{4}{3 - 2\cos\theta}$

346. $r = \dfrac{7}{5 - 5\cos\theta}$

347. Determine the Cartesian equation describing the orbit of Pluto, the most eccentric orbit around the Sun. The length of the major axis is 39.26 AU and minor axis is 38.07 AU. What is the eccentricity?

348. The C/1980 E1 comet was observed in 1980. Given an eccentricity of 1.057 and a perihelion (point of closest approach to the Sun) of 3.364 AU, find the Cartesian equations describing the comet's trajectory. Are we guaranteed to see this comet again? (*Hint*: Consider the Sun at point $(0, 0)$.)

APPENDIX A | TABLE OF INTEGRALS

Basic Integrals

1. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \; n \neq -1$

2. $\int \frac{du}{u} = \ln|u| + C$

3. $\int e^u \, du = e^u + C$

4. $\int a^u \, du = \frac{a^u}{\ln a} + C$

5. $\int \sin u \, du = -\cos u + C$

6. $\int \cos u \, du = \sin u + C$

7. $\int \sec^2 u \, du = \tan u + C$

8. $\int \csc^2 u \, du = -\cot u + C$

9. $\int \sec u \tan u \, du = \sec u + C$

10. $\int \csc u \cot u \, du = -\csc u + C$

11. $\int \tan u \, du = \ln|\sec u| + C$

12. $\int \cot u \, du = \ln|\sin u| + C$

13. $\int \sec u \, du = \ln|\sec u + \tan u| + C$

14. $\int \csc u \, du = \ln|\csc u - \cot u| + C$

15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

Trigonometric Integrals

18. $\int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

19. $\int \cos^2 u\, du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

20. $\int \tan^2 u\, du = \tan u - u + C$

21. $\int \cot^2 u\, du = -\cot u - u + C$

22. $\int \sin^3 u\, du = -\frac{1}{3}\left(2 + \sin^2 u\right)\cos u + C$

23. $\int \cos^3 u\, du = \frac{1}{3}\left(2 + \cos^2 u\right)\sin u + C$

24. $\int \tan^3 u\, du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u\, du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\, du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\, du$

29. $\int \cos^n u\, du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\, du$

30. $\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du$

31. $\int \cot^n u\, du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\, du$

32. $\int \sec^n u\, du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\, du$

33. $\int \csc^n u\, du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\, du$

34. $\int \sin au \sin bu\, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu\, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu\, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u \sin u\, du = \sin u - u \cos u + C$

38. $\int u \cos u\, du = \cos u + u \sin u + C$

39. $\int u^n \sin u\, du = -u^n \cos u + n\int u^{n-1} \cos u\, du$

40. $\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1} \sin u\, du$

41. $$\begin{aligned}\int \sin^n u \cos^m u\, du &= -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u\, du \\ &= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u\, du\end{aligned}$$

Exponential and Logarithmic Integrals

42. $\int ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$

43. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1} e^{au}\,du$

44. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a\sin bu - b\cos bu) + C$

45. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a\cos bu + b\sin bu) + C$

46. $\int \ln u\,du = u\ln u - u + C$

47. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\,du = \cosh u + C$

50. $\int \cosh u\,du = \sinh u + C$

51. $\int \tanh u\,du = \ln\cosh u + C$

52. $\int \coth u\,du = \ln|\sinh u| + C$

53. $\int \operatorname{sech} u\,du = \tan^{-1}|\sinh u| + C$

54. $\int \operatorname{csch} u\,du = \ln\left|\tanh\frac{1}{2}u\right| + C$

55. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

56. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

57. $\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$

58. $\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1 - u^2} + C$

60. $\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1 - u^2} + C$

61. $\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln\left(1 + u^2\right) + C$

62. $\int u\sin^{-1} u\,du = \frac{2u^2 - 1}{4}\sin^{-1} u + \frac{u\sqrt{1 - u^2}}{4} + C$

63. $\int u \cos^{-1} u\, du = \frac{2u^2 - 1}{4}\cos^{-1} u - \frac{u\sqrt{1 - u^2}}{4} + C$

64. $\int u \tan^{-1} u\, du = \frac{u^2 + 1}{2}\tan^{-1} u - \frac{u}{2} + C$

65. $\int u^n \sin^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\, du}{\sqrt{1 - u^2}}\right],\ n \neq -1$

66. $\int u^n \cos^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\, du}{\sqrt{1 - u^2}}\right],\ n \neq -1$

67. $\int u^n \tan^{-1} u\, du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\, du}{1 + u^2}\right],\ n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2 + u^2}\, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

69. $\int u^2 \sqrt{a^2 + u^2}\, du = \frac{u}{8}\left(a^2 + 2u^2\right)\sqrt{a^2 + u^2} - \frac{a^4}{8}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

70. $\int \frac{\sqrt{a^2 + u^2}}{u}\, du = \sqrt{a^2 + u^2} - a\ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2 + u^2}}{u^2}\, du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2 + u^2}} = \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

73. $\int \frac{u^2\, du}{\sqrt{a^2 + u^2}} = \frac{u}{2}\left(\sqrt{a^2 + u^2}\right) - \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2 + u^2} + a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C$

76. $\int \frac{du}{\left(a^2 + u^2\right)^{3/2}} = \frac{u}{a^2\sqrt{a^2 + u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

78. $\int u^2 \sqrt{u^2 - a^2}\, du = \frac{u}{8}\left(2u^2 - a^2\right)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

79. $\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{|u|} + C$

80. $\int \frac{\sqrt{u^2 - a^2}}{u^2}\, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

81. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

82. $\displaystyle\int \frac{u^2\,du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

83. $\displaystyle\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

84. $\displaystyle\int \frac{du}{\left(u^2 - a^2\right)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $\displaystyle\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

86. $\displaystyle\int u^2\sqrt{a^2 - u^2}\,du = \frac{u}{8}\left(2u^2 - a^2\right)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

87. $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

88. $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

89. $\displaystyle\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{u}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

90. $\displaystyle\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

91. $\displaystyle\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

92. $\displaystyle\int \left(a^2 - u^2\right)^{3/2}\,du = -\frac{u}{8}\left(2u^2 - 5a^2\right)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

93. $\displaystyle\int \frac{du}{\left(a^2 - u^2\right)^{3/2}} = -\frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Integrals Involving $2au - u^2$, $a > 0$

94. $\displaystyle\int \sqrt{2au - u^2}\,du = \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

95. $\displaystyle\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\left(\frac{a - u}{a}\right) + C$

96. $\displaystyle\int u\sqrt{2au - u^2}\,du = \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

97. $\displaystyle\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$

Integrals Involving $a + bu$, $a \neq 0$

98. $\displaystyle\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a + bu - a\ln|a+bu|\right) + C$

99. $\displaystyle\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

100. $\displaystyle\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

101. $\displaystyle\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

102. $\displaystyle\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

103. $\displaystyle\int \frac{u\,du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

104. $\displaystyle\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

105. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a+bu)^{3/2} + C$

106. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a+bu} + C$

107. $\displaystyle\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}\left(8a^2 + 3b^2u^2 - 4abu\right)\sqrt{a+bu} + C$

108. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$

$\displaystyle\qquad = \frac{2}{\sqrt{-a}}\tan - 1\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$

109. $\displaystyle\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

110. $\displaystyle\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

111. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

112. $\displaystyle\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

113. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

APPENDIX B | TABLE OF DERIVATIVES

General Formulas

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

4. $\frac{d}{dx}(x^n) = nx^{n-1}$, for real numbers n

5. $\frac{d}{dx}(cf(x)) = cf'(x)$

6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$

10. $\frac{d}{dx}(\tan x) = \sec^2 x$

11. $\frac{d}{dx}(\sec x) = \sec x \tan x$

12. $\frac{d}{dx}(\cos x) = -\sin x$

13. $\frac{d}{dx}(\cot x) = -\csc^2 x$

14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$

16. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2}$

17. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2 - 1}}$

18. $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1 - x^2}}$

19. $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1 + x^2}$

20. $\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2 - 1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$

22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

23. $\frac{d}{dx}(b^x) = b^x \ln b$

24. $\frac{d}{dx}\left(\log_b x\right) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$

26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

28. $\frac{d}{dx}(\cosh x) = \sinh x$

29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2 + 1}}$

32. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1 - x^2}(|x| < 1)$

33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1 - x^2}} \quad (0 < x < 1)$

34. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2 - 1}} \quad (x > 1)$

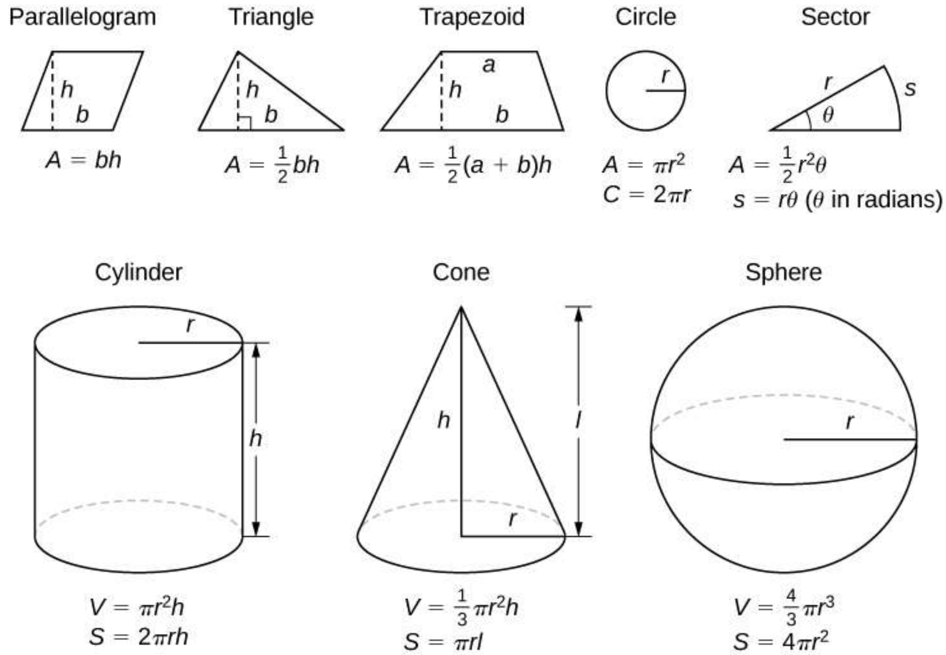
35. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1 - x^2} \quad (|x| > 1)$

36. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{|x|\sqrt{1 + x^2}} (x \neq 0)$

APPENDIX C | REVIEW OF PRE-CALCULUS

Formulas from Geometry

$A = \text{area}, \quad V = \text{Volume}, \quad \text{and} \quad S = \text{lateral surface area}$



Formulas from Algebra

Laws of Exponents

$$x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n} \qquad (x^m)^n = x^{mn}$$

$$x^{-n} = \frac{1}{x^n} \qquad (xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x} \qquad \sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

Special Factorizations

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)\left(x^2 - xy + y^2\right)$$

$$x^3 - y^3 = (x-y)\left(x^2 + xy + y^2\right)$$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ca}}{2a}$.

Binomial Theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n,$$

where $\binom{n}{k} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1} = \dfrac{n!}{k!(n-k)!}$

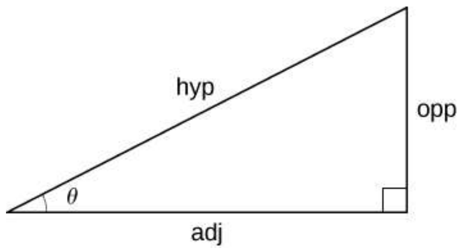
Formulas from Trigonometry

Right-Angle Trigonometry

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ $\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ $\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}}$ $\cot\theta = \dfrac{\text{adj}}{\text{opp}}$



Trigonometric Functions of Important Angles

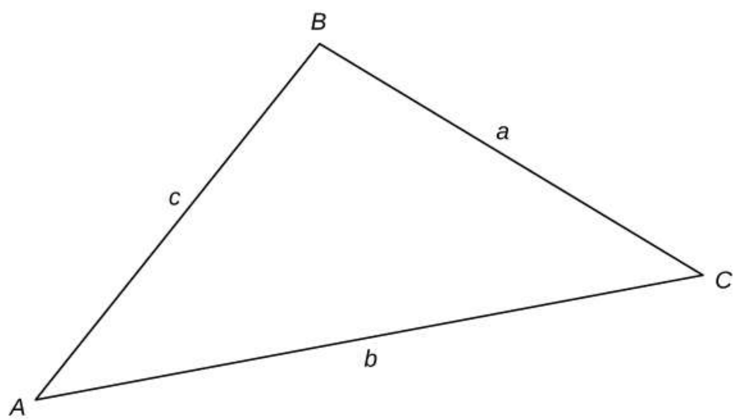
θ	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
90°	$\pi/2$	1	0	—

Fundamental Identities

$$
\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 & \sin(-\theta) &= -\sin\theta \\
1 + \tan^2\theta &= \sec^2\theta & \cos(-\theta) &= \cos\theta \\
1 + \cot^2\theta &= \csc^2\theta & \tan(-\theta) &= -\tan\theta \\
\sin\left(\frac{\pi}{2} - \theta\right) &= \cos\theta & \sin(\theta + 2\pi) &= \sin\theta \\
\cos\left(\frac{\pi}{2} - \theta\right) &= \sin\theta & \cos(\theta + 2\pi) &= \cos\theta \\
\tan\left(\frac{\pi}{2} - \theta\right) &= \cot\theta & \tan(\theta + \pi) &= \tan\theta
\end{aligned}
$$

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Law of Cosines

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \\
b^2 &= a^2 + c^2 - 2ac\cos B \\
c^2 &= a^2 + b^2 - 2ab\cos C
\end{aligned}
$$

Addition and Subtraction Formulas

$$
\begin{aligned}
\sin(x+y) &= \sin x\cos y + \cos x\sin y \\
\sin(x-y) &= \sin x\cos y - \cos x\sin y \\
\cos(x+y) &= \cos x\cos y - \sin x\sin y \\
\cos(x-y) &= \cos x\cos y + \sin x\sin y \\
\tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x\tan y} \\
\tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x\tan y}
\end{aligned}
$$

Double-Angle Formulas

$$
\begin{aligned}
\sin 2x &= 2\sin x\cos x \\
\cos 2x &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \\
\tan 2x &= \frac{2\tan x}{1 - \tan^2 x}
\end{aligned}
$$

Half-Angle Formulas

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2}\\ \cos^2 x &= \frac{1+\cos 2x}{2}\end{aligned}$$

ANSWER KEY

Chapter 1

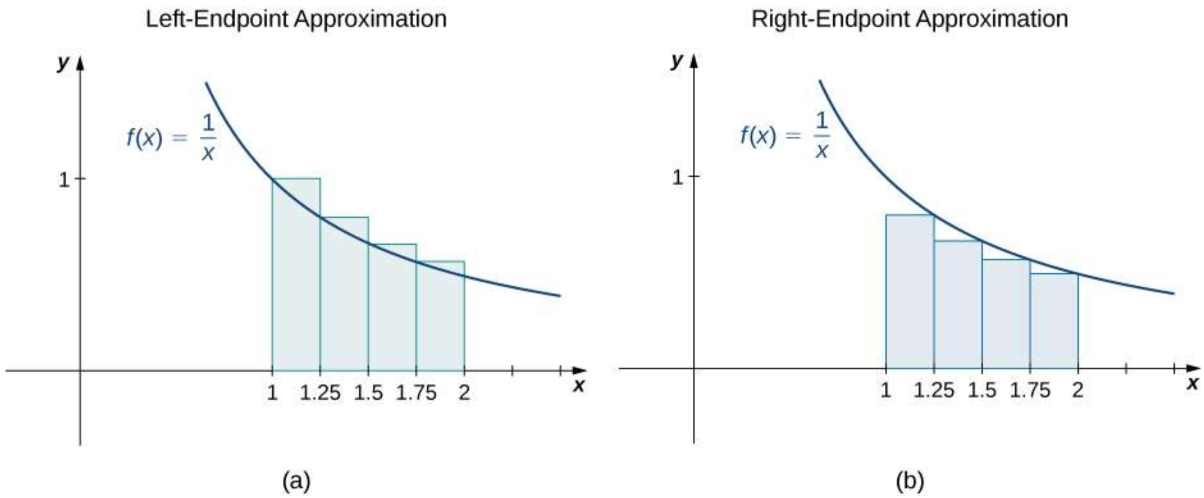
Checkpoint

1.1. $\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$

1.2. 15,550

1.3. 440

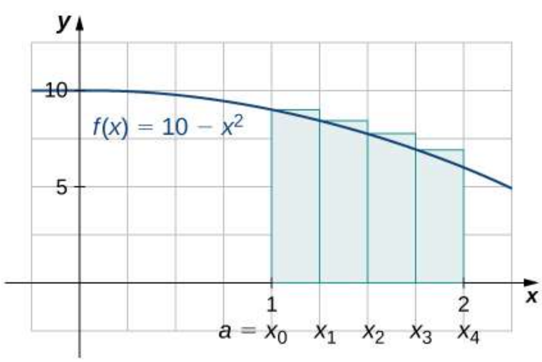
1.4. The left-endpoint approximation is 0.7595. The right-endpoint approximation is 0.6345. See the below image.



1.5.

a. Upper sum $= 8.0313.$

b.



1.6. $A \approx 1.125$

1.7. 6

1.8. 18 square units

1.9. 6

1.10. 18

1.11. $6\int_1^3 x^3\,dx - 4\int_1^3 x^2\,dx + 2\int_1^3 x\,dx - \int_1^3 3\,dx$

1.12. −7

1.13. 3

1.14. Average value $= 1.5$; $c = 3$

1.15. $c = \sqrt{3}$

1.16. $g'(r) = \sqrt{r^2 + 4}$

1.17. $F'(x) = 3x^2 \cos x^3$

1.18. $F'(x) = 2x \cos x^2 - \cos x$

1.19. $\frac{7}{24}$

1.20. Kathy still wins, but by a much larger margin: James skates 24 ft in 3 sec, but Kathy skates 29.3634 ft in 3 sec.

1.21. $-\frac{10}{3}$

1.22. Net displacement: $\frac{e^2 - 9}{2} \approx -0.8055$ m; total distance traveled: $4 \ln 4 - 7.5 + \frac{e^2}{2} \approx 1.740$ m

1.23. 17.5 mi

1.24. $\frac{64}{5}$

1.25. $\int 3x^2 \left(x^3 - 3\right)^2 dx = \frac{1}{3}\left(x^3 - 3\right)^3 + C$

1.26. $\frac{\left(x^3 + 5\right)^{10}}{30} + C$

1.27. $-\frac{1}{\sin t} + C$

1.28. $-\frac{\cos^4 t}{4} + C$

1.29. $\frac{91}{3}$

1.30. $\frac{2}{3\pi} \approx 0.2122$

1.31. $\int x^2 e^{-2x^3} dx = -\frac{1}{6}e^{-2x^3} + C$

1.32. $\int e^x (3e^x - 2)^2 dx = \frac{1}{9}(3e^x - 2)^3$

1.33. $\int 2x^3 e^{x^4} dx = \frac{1}{2}e^{x^4}$

1.34. $\frac{1}{2}\int_0^4 e^u du = \frac{1}{2}\left(e^4 - 1\right)$

1.35. $Q(t) = \frac{2^t}{\ln 2} + 8.557$. There are 20,099 bacteria in the dish after 3 hours.

1.36. There are 116 flies.

1.37. $\int_1^2 \frac{1}{x^3}e^{4x^{-2}} dx = \frac{1}{8}\left[e^4 - e\right]$

1.38. $\ln|x + 2| + C$

1.39. $\frac{x}{\ln 3}(\ln x - 1) + C$

1.40. $\frac{1}{4}\sin^{-1}(4x) + C$

1.41. $\sin^{-1}\left(\frac{x}{3}\right) + C$

1.42. $\frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right) + C$

1.43. $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$

1.44. $\frac{\pi}{8}$

Section Exercises

1. a. They are equal; both represent the sum of the first 10 whole numbers. b. They are equal; both represent the sum of the first 10 whole numbers. c. They are equal by substituting $j = i - 1.$ d. They are equal; the first sum factors the terms of the second.

3. $385 - 30 = 355$

5. $15 - (-12) = 27$

7. $5(15) + 4(-12) = 27$

9. $\sum_{j=1}^{50} j^2 - 2\sum_{j=1}^{50} j = \frac{(50)(51)(101)}{6} - \frac{2(50)(51)}{2} = 40,\ 375$

11. $4\sum_{k=1}^{25} k^2 - 100\sum_{k=1}^{25} k = \frac{4(25)(26)(51)}{9} - 50(25)(26) = -10,\ 400$

13. $R_4 = 0.25$

15. $R_6 = 0.372$

17. $L_4 = 2.20$

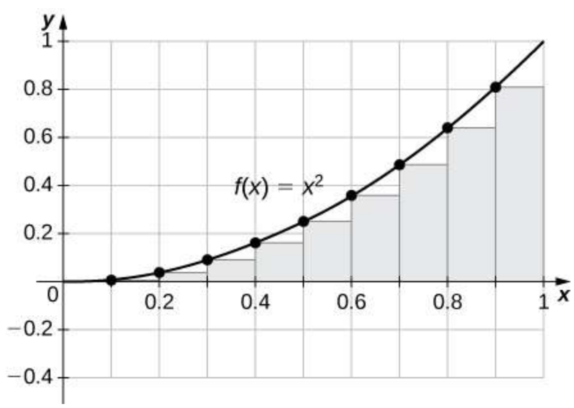
19. $L_8 = 0.6875$

21. $L_6 = 9.000 = R_6.$ The graph of *f* is a triangle with area 9.

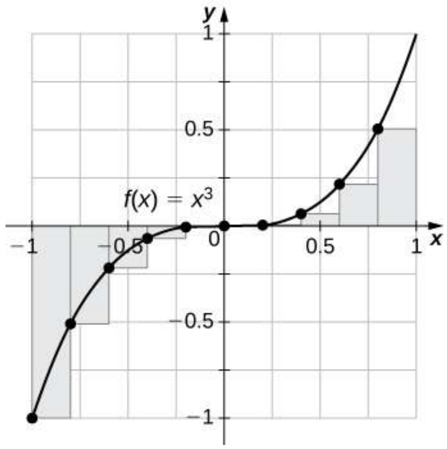
23. $L_6 = 13.12899 = R_6.$ They are equal.

25. $L_{10} = \frac{4}{10}\sum_{i=1}^{10} \sqrt{4 - \left(-2 + 4\frac{(i-1)}{10}\right)}$

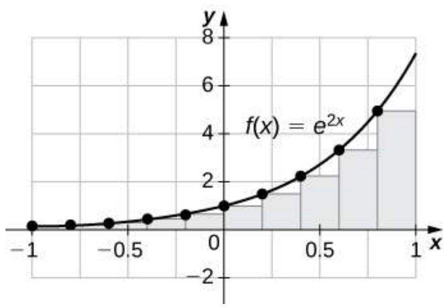
27. $R_{100} = \frac{e-1}{100}\sum_{i=1}^{100} \ln\left(1 + (e-1)\frac{i}{100}\right)$



29.
$R_{100} = 0.33835,\ L_{100} = 0.32835.$ The plot shows that the left Riemann sum is an underestimate because the function is increasing. Similarly, the right Riemann sum is an overestimate. The area lies between the left and right Riemann sums. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.



31.
$L_{100} = -0.02,\ R_{100} = 0.02.$ The left endpoint sum is an underestimate because the function is increasing. Similarly, a right endpoint approximation is an overestimate. The area lies between the left and right endpoint estimates.



33.
$L_{100} = 3.555,\ R_{100} = 3.670.$ The plot shows that the left Riemann sum is an underestimate because the function is increasing. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

35. The sum represents the cumulative rainfall in January 2009.

37. The total mileage is $7 \times \sum_{i=1}^{25} \left(1 + \frac{(i-1)}{10}\right) = 7 \times 25 + \frac{7}{10} \times 12 \times 25 = 385$ mi.

39. Add the numbers to get 8.1-in. net increase.

41. 309,389,957

43. $L_8 = 3 + 2 + 1 + 2 + 3 + 4 + 5 + 4 = 24$

45. $L_8 = 3 + 5 + 7 + 6 + 8 + 6 + 5 + 4 = 44$

47. $L_{10} \approx 1.7604,\ L_{30} \approx 1.7625,\ L_{50} \approx 1.76265$

49. $R_1 = -1,\ L_1 = 1,\ R_{10} = -0.1,\ L_{10} = 0.1,\ L_{100} = 0.01,$ and $R_{100} = -0.1.$ By symmetry of the graph, the exact area is zero.

51. $R_1 = 0,\ L_1 = 0,\ R_{10} = 2.4499,\ L_{10} = 2.4499,\ R_{100} = 2.1365,\ L_{100} = 2.1365$

53. If $[c, d]$ is a subinterval of $[a, b]$ under one of the left-endpoint sum rectangles, then the area of the rectangle contributing to the left-endpoint estimate is $f(c)(d-c)$. But, $f(c) \le f(x)$ for $c \le x \le d$, so the area under the graph of *f* between *c* and *d* is $f(c)(d-c)$ plus the area below the graph of *f* but above the horizontal line segment at height $f(c)$, which is positive. As this is true for each left-endpoint sum interval, it follows that the left Riemann sum is less than or equal to the area below the graph of *f* on $[a, b]$.

55. $L_N = \frac{b-a}{N} \sum_{i=1}^{N} f\left(a + (b-a)\frac{i-1}{N}\right) = \frac{b-a}{N} \sum_{i=0}^{N-1} f\left(a + (b-a)\frac{i}{N}\right)$ and $R_N = \frac{b-a}{N} \sum_{i=1}^{N} f\left(a + (b-a)\frac{i}{N}\right)$. The left sum has a term corresponding to $i = 0$ and the right sum has a term corresponding to $i = N$. In $R_N - L_N$, any term corresponding to $i = 1, 2,\ldots, N-1$ occurs once with a plus sign and once with a minus sign, so each such term cancels and one is left with $R_N - L_N = \frac{b-a}{N}\left(f\left(a + (b-a)\right)\frac{N}{N}\right) - \left(f(a) + (b-a)\frac{0}{N}\right) = \frac{b-a}{N}(f(b) - f(a))$.

57. Graph 1: a. $L(A) = 0,\ B(A) = 20$; b. $U(A) = 20$. Graph 2: a. $L(A) = 9$; b. $B(A) = 11,\ U(A) = 20$. Graph 3: a. $L(A) = 11.0$; b. $B(A) = 4.5,\ U(A) = 15.5$.

59. Let A be the area of the unit circle. The circle encloses n congruent triangles each of area $\frac{\sin\left(\frac{2\pi}{n}\right)}{2}$, so $\frac{n}{2}\sin\left(\frac{2\pi}{n}\right) \le A$. Similarly, the circle is contained inside n congruent triangles each of area $\frac{BH}{2} = \frac{1}{2}\left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)\right)\sin\left(\frac{2\pi}{n}\right)$, so $A \le \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\left(\cos\left(\frac{\pi}{n}\right)\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)$. As $n \to \infty,\ \frac{n}{2}\sin\left(\frac{2\pi}{n}\right) = \frac{\pi\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)} \to \pi$, so we conclude $\pi \le A$. Also, as $n \to \infty,\ \cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right) \to 1$, so we also have $A \le \pi$. By the squeeze theorem for limits, we conclude that $A = \pi$.

61. $\int_0^2 \left(5x^2 - 3x^3\right)dx$

63. $\int_0^1 \cos^2(2\pi x)dx$

65. $\int_0^1 xdx$

67. $\int_3^6 xdx$

69. $\int_1^2 x\log\left(x^2\right)dx$

71. $1 + 2\cdot 2 + 3\cdot 3 = 14$

73. $1 - 4 + 9 = 6$

75. $1 - 2\pi + 9 = 10 - 2\pi$

77. The integral is the area of the triangle, $\frac{1}{2}$

79. The integral is the area of the triangle, 9.

81. The integral is the area $\frac{1}{2}\pi r^2 = 2\pi$.

83. The integral is the area of the "big" triangle less the "missing" triangle, $9 - \frac{1}{2}$.

85. $L = 2 + 0 + 10 + 5 + 4 = 21,\ R = 0 + 10 + 10 + 2 + 0 = 22,\ \frac{L + R}{2} = 21.5$

87. $L = 0 + 4 + 0 + 4 + 2 = 10,\ R = 4 + 0 + 2 + 4 + 0 = 10,\ \frac{L + R}{2} = 10$

89. $\int_2^4 f(x)dx + \int_2^4 g(x)dx = 8 - 3 = 5$

91. $\int_2^4 f(x)dx - \int_2^4 g(x)dx = 8 + 3 = 11$

93. $4\int_2^4 f(x)dx - 3\int_2^4 g(x)dx = 32 + 9 = 41$

95. The integrand is odd; the integral is zero.

97. The integrand is antisymmetric with respect to $x = 3$. The integral is zero.

99. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$

101. $\int_0^1 \left(1 - 2x + 4x^2 - 8x^3\right)dx = 1 - 1 + \frac{4}{3} - 2 = -\frac{2}{3}$

103. $7 - \frac{5}{4} = \frac{23}{4}$

105. The integrand is negative over $[-2, 3]$.

107. $x \leq x^2$ over $[1, 2]$, so $\sqrt{1+x} \leq \sqrt{1+x^2}$ over $[1, 2]$.

109. $\cos(t) \geq \frac{\sqrt{2}}{2}$. Multiply by the length of the interval to get the inequality.

111. $f_{\text{ave}} = 0; c = 0$

113. $\frac{3}{2}$ when $c = \pm\frac{3}{2}$

115. $f_{\text{ave}} = 0; c = \frac{\pi}{2}, \frac{3\pi}{2}$

117. $L_{100} = 1.294, R_{100} = 1.301;$ the exact average is between these values.

119. $L_{100} \times \left(\frac{1}{2}\right) = 0.5178, R_{100} \times \left(\frac{1}{2}\right) = 0.5294$

121. $L_1 = 0, L_{10} \times \left(\frac{1}{2}\right) = 8.743493, L_{100} \times \left(\frac{1}{2}\right) = 12.861728.$ The exact answer $\approx 26.799,$ so L_{100} is not accurate.

123. $L_1 \times \left(\frac{1}{\pi}\right) = 1.352, L_{10} \times \left(\frac{1}{\pi}\right) = -0.1837, L_{100} \times \left(\frac{1}{\pi}\right) = -0.2956.$ The exact answer $\approx -0.303,$ so L_{100} is not accurate to first decimal.

125. Use $\tan^2\theta + 1 = \sec^2\theta.$ Then, $B - A = \int_{-\pi/4}^{\pi/4} 1dx = \frac{\pi}{2}.$

127. $\int_0^{2\pi} \cos^2 tdt = \pi,$ so divide by the length 2π of the interval. $\cos^2 t$ has period π, so yes, it is true.

129. The integral is maximized when one uses the largest interval on which p is nonnegative. Thus, $A = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $B = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$

131. If $f(t_0) > g(t_0)$ for some $t_0 \in [a, b],$ then since $f - g$ is continuous, there is an interval containing t_0 such that $f(t) > g(t)$ over the interval $[c, d],$ and then $\int_d^d f(t)dt > \int_c^d g(t)dt$ over this interval.

133. The integral of f over an interval is the same as the integral of the average of f over that interval. Thus, $\int_a^b f(t)dt = \int_{a_0}^{a_1} f(t)dt + \int_{a_1}^{a_2} f(t)dt + \cdots + \int_{a_{N+1}}^{a_N} f(t)dt = \int_{a_0}^{a_1} 1dt + \int_{a_1}^{a_2} 1dt + \cdots + \int_{a_{N+1}}^{a_N} 1dt$ $= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_N - a_{N-1}) = a_N - a_0 = b - a.$ Dividing through by $b - a$ gives the desired identity.

135. $\int_0^N f(t)dt = \sum_{i=1}^{N} \int_{i-1}^{i} f(t)dt = \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$

137. $L_{10} = 1.815, R_{10} = 1.515, \frac{L_{10} + R_{10}}{2} = 1.665,$ so the estimate is accurate to two decimal places.

139. The average is $1/2,$ which is equal to the integral in this case.

141. a. The graph is antisymmetric with respect to $t = \frac{1}{2}$ over $[0, 1],$ so the average value is zero. b. For any value of a, the graph between $[a, a+1]$ is a shift of the graph over $[0, 1],$ so the net areas above and below the axis do not change and the average remains zero.

143. Yes, the integral over any interval of length 1 is the same.

145. Yes. It is implied by the Mean Value Theorem for Integrals.

147. $F'(2) = -1;$ average value of F' over $[1, 2]$ is $-1/2.$

149. $e^{\cos t}$

151. $\frac{1}{\sqrt{16 - x^2}}$

153. $\sqrt{x}\frac{d}{dx}\sqrt{x} = \frac{1}{2}$

155. $-\sqrt{1-\cos^2 x}\frac{d}{dx}\cos x = |\sin x|\sin x$

157. $2x\frac{|x|}{1+x^2}$

159. $\ln(e^{2x})\frac{d}{dx}e^x = 2xe^x$

161. a. f is positive over $[1, 2]$ and $[5, 6]$, negative over $[0, 1]$ and $[3, 4]$, and zero over $[2, 3]$ and $[4, 5]$. b. The maximum value is 2 and the minimum is −3. c. The average value is 0.

163. a. ℓ is positive over $[0, 1]$ and $[3, 6]$, and negative over $[1, 3]$. b. It is increasing over $[0, 1]$ and $[3, 5]$, and it is constant over $[1, 3]$ and $[5, 6]$. c. Its average value is $\frac{1}{3}$.

165. $T_{10} = 49.08$, $\int_{-2}^{3} x^3 + 6x^2 + x - 5dx = 48$

167. $T_{10} = 260.836$, $\int_1^9 \left(\sqrt{x} + x^2\right)dx = 260$

169. $T_{10} = 3.058$, $\int_1^4 \frac{4}{x^2}dx = 3$

171. $F(x) = \frac{x^3}{3} + \frac{3x^2}{2} - 5x$, $F(3) - F(-2) = -\frac{35}{6}$

173. $F(x) = -\frac{t^5}{5} + \frac{13t^3}{3} - 36t$, $F(3) - F(2) = \frac{62}{15}$

175. $F(x) = \frac{x^{100}}{100}$, $F(1) - F(0) = \frac{1}{100}$

177. $F(x) = \frac{x^3}{3} + \frac{1}{x}$, $F(4) - F\left(\frac{1}{4}\right) = \frac{1125}{64}$

179. $F(x) = \sqrt{x}$, $F(4) - F(1) = 1$

181. $F(x) = \frac{4}{3}t^{3/4}$, $F(16) - F(1) = \frac{28}{3}$

183. $F(x) = -\cos x$, $F\left(\frac{\pi}{2}\right) - F(0) = 1$

185. $F(x) = \sec x$, $F\left(\frac{\pi}{4}\right) - F(0) = \sqrt{2} - 1$

187. $F(x) = -\cot(x)$, $F\left(\frac{\pi}{2}\right) - F\left(\frac{\pi}{4}\right) = 1$

189. $F(x) = -\frac{1}{x} + \frac{1}{2x^2}$, $F(-1) - F(-2) = \frac{7}{8}$

191. $F(x) = e^x - e$

193. $F(x) = 0$

195. $\int_{-2}^{-1}\left(t^2 - 2t - 3\right)dt - \int_{-1}^{3}\left(t^2 - 2t - 3\right)dt + \int_3^4\left(t^2 - 2t - 3\right)dt = \frac{46}{3}$

197. $-\int_{-\pi/2}^{0} \sin t dt + \int_0^{\pi/2} \sin t dt = 2$

199. a. The average is 11.21×10^9 since $\cos\left(\frac{\pi t}{6}\right)$ has period 12 and integral 0 over any period. Consumption is equal to the average when $\cos\left(\frac{\pi t}{6}\right) = 0$, when $t = 3$, and when $t = 9$. b. Total consumption is the average rate times duration: $11.21 \times 12 \times 10^9 = 1.35 \times 10^{11}$ c. $10^9\left(11.21 - \frac{1}{6}\int_3^9 \cos\left(\frac{\pi t}{6}\right)dt\right) = 10^9\left(11.21 + \frac{2}{\pi}\right) = 11.84x10^9$

201. If f is not constant, then its average is strictly smaller than the maximum and larger than the minimum, which are attained over $[a, b]$ by the extreme value theorem.

203. a. $d^2\theta = (a\cos\theta + c)^2 + b^2\sin^2\theta = a^2 + c^2\cos^2\theta + 2ac\cos\theta = (a + c\cos\theta)^2$; b. $\overline{d} = \frac{1}{2\pi}\int_0^{2\pi}(a + 2c\cos\theta)d\theta = a$

205. Mean gravitational force = $\frac{GmM}{2}\displaystyle\int_0^{2\pi}\frac{1}{\left(a + 2\sqrt{a^2 - b^2}\cos\theta\right)^2}d\theta.$

207. $\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx = \int x^{1/2}dx - \int x^{-1/2}dx = \frac{2}{3}x^{3/2} + C_1 - 2x^{1/2} + C_2 = \frac{2}{3}x^{3/2} - 2x^{1/2} + C$

209. $\int\frac{dx}{2x} = \frac{1}{2}\ln|x| + C$

211. $\displaystyle\int_0^{\pi}\sin x dx - \int_0^{\pi}\cos x dx = -\cos x|_0^{\pi} - (\sin x)|_0^{\pi} = (-(-1) + 1) - (0 - 0) = 2$

213. $P(s) = 4s$, so $\frac{dP}{ds} = 4$ and $\int_2^4 4ds = 8.$

215. $\int_1^2 Nds = N$

217. With p as in the previous exercise, each of the 12 pentagons increases in area from $2p$ to $4p$ units so the net increase in the area of the dodecahedron is $36p$ units.

219. $18s^2 = 6\int_s^{2s}2xdx$

221. $12\pi R^2 = 8\pi\int_R^{2R}rdr$

223. $d(t) = \int_0^t v(s)ds = 4t - t^2$. The total distance is $d(2) = 4$ m.

225. $d(t) = \int_0^t v(s)ds.$ For $t < 3,\ d(t) = \int_0^t(6 - 2t)dt = 6t - t^2.$ For $t > 3,\ d(t) = d(3) + \int_3^t(2t - 6)dt = 9 + (t^2 - 6t).$ The total distance is $d(6) = 9$ m.

227. $v(t) = 40 - 9.8t;\ h(t) = 1.5 + 40t - 4.9t^2$ m/s

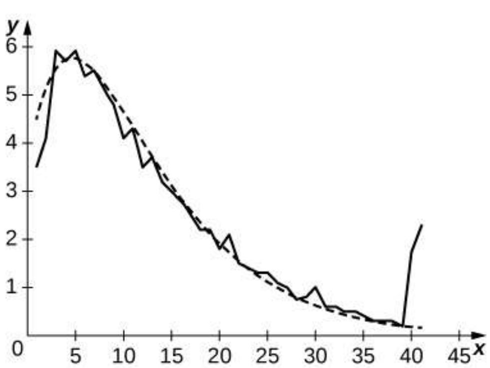
229. The net increase is 1 unit.

231. At $t = 5$, the height of water is $x = \left(\frac{15}{\pi}\right)^{1/3}$ m.. The net change in height from $t = 5$ to $t = 10$ is $\left(\frac{30}{\pi}\right)^{1/3} - \left(\frac{15}{\pi}\right)^{1/3}$ m.

233. The total daily power consumption is estimated as the sum of the hourly power rates, or 911 gW-h.

235. 17 kJ

237. a. 54.3%; b. 27.00%; c. The curve in the following plot is $2.35(t + 3)e^{-0.15(t + 3)}$.

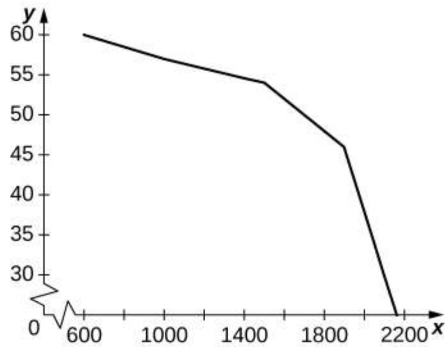


239. In dry conditions, with initial velocity $v_0 = 30$ m/s, $D = 64.3$ and, if $v_0 = 25,\ D = 44.64.$ In wet conditions, if $v_0 = 30,$ and $D = 180$ and if $v_0 = 25,\ D = 125.$

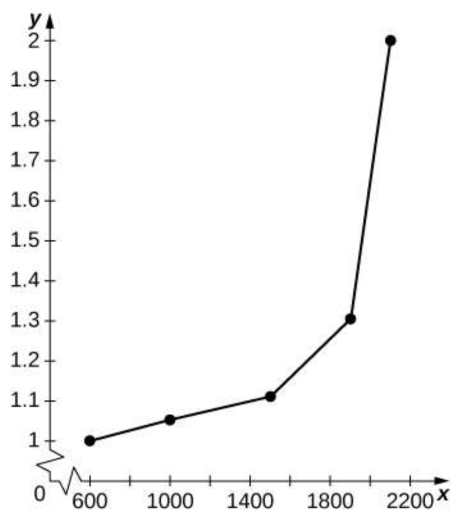
241. 225 cal

243. $E(150) = 28,\ E(300) = 22,\ E(450) = 16$

245. a.



b. Between 600 and 1000 the average decrease in vehicles per hour per lane is −0.0075. Between 1000 and 1500 it is −0.006 per vehicles per hour per lane, and between 1500 and 2100 it is −0.04 vehicles per hour per lane. c.



The graph is nonlinear, with minutes per mile increasing dramatically as vehicles per hour per lane reach 2000.

247. $\frac{1}{37}\int_0^{37} p(t)dt = \frac{0.07(37)^3}{4} + \frac{2.42(37)^2}{3} - \frac{25.63(37)}{2} + 521.23 \approx 2037$

249. Average acceleration is $A = \frac{1}{5}\int_0^5 a(t)dt = -\frac{0.7(5^2)}{3} + \frac{1.44(5)}{2} + 10.44 \approx 8.2$ mph/s

251. $d(t) = \int_0^1 |v(t)|dt = \int_0^t \left(\frac{7}{30}t^3 - 0.72t^2 - 10.44t + 41.033\right)dt = \frac{7}{120}t^4 - 0.24t^3 - 5.22t^3 + 41.033t.$ Then, $d(5) \approx 81.12$ mph $\times$ sec ≈ 119 feet.

253. $\frac{1}{40}\int_0^{40} (-0.068t + 5.14)dt = -\frac{0.068(40)}{2} + 5.14 = 3.78$

255. $u = h(x)$

257. $f(u) = \frac{(u+1)^2}{\sqrt{u}}$

259. $du = 8xdx;\ f(u) = \frac{1}{8\sqrt{u}}$

261. $\frac{1}{5}(x+1)^5 + C$

263. $-\frac{1}{12(3-2x)^6} + C$

265. $\sqrt{x^2+1} + C$

267. $\frac{1}{8}\left(x^2-2x\right)^4 + C$

269. $\sin\theta - \frac{\sin^3\theta}{3} + C$

271. $\frac{(1-x)^{101}}{101} - \frac{(1-x)^{100}}{100} + C$

273. $-\frac{1}{22\left(7-11x^2\right)} + C$

275. $-\frac{\cos^4\theta}{4} + C$

277. $-\frac{\cos^3(\pi t)}{3\pi} + C$

279. $-\frac{1}{4}\cos^2\left(t^2\right) + C$

281. $-\frac{1}{3(x^3-3)} + C$

283. $-\frac{2\left(y^3-2\right)}{3\sqrt{1-y^3}}$

285. $\frac{1}{33}\left(1-\cos^3\theta\right)^{11} + C$

287. $\frac{1}{12}\left(\sin^3\theta - 3\sin^2\theta\right)^4 + C$

289. $L_{50} = -8.5779.$ The exact area is $\frac{-81}{8}$

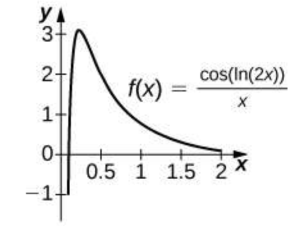
291. $L_{50} = -0.006399$ … The exact area is 0.

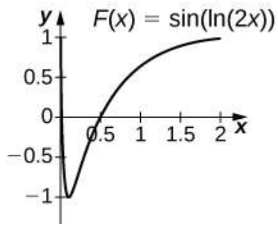
293. $u = 1 + x^2,\ du = 2xdx,\ \frac{1}{2}\int_1^2 u^{-1/2}du = \sqrt{2} - 1$

295. $u = 1 + t^3,\ du = 3t^2,\ \frac{1}{3}\int_1^2 u^{-1/2}du = \frac{2}{3}(\sqrt{2} - 1)$

297. $u = \cos\theta,\ du = -\sin\theta d\theta,\ \int_{1/\sqrt{2}}^1 u^{-4}du = \frac{1}{3}(2\sqrt{2} - 1)$

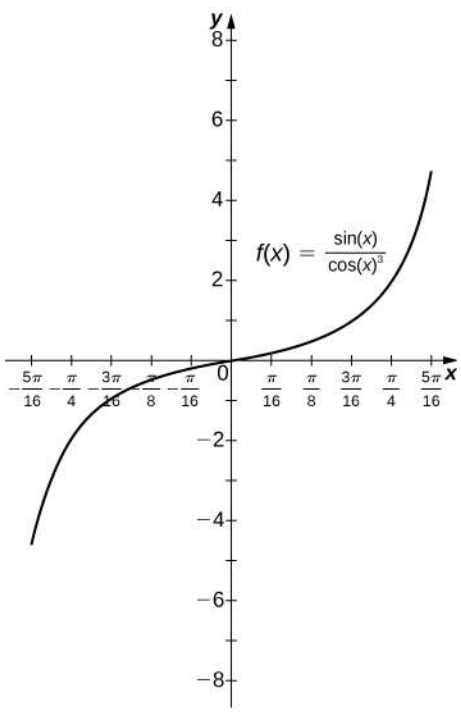
299.

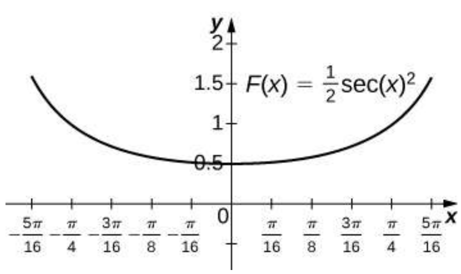




The antiderivative is $y = \sin(\ln(2x))$. Since the antiderivative is not continuous at $x = 0$, one cannot find a value of *C* that would make $y = \sin(\ln(2x)) - C$ work as a definite integral.

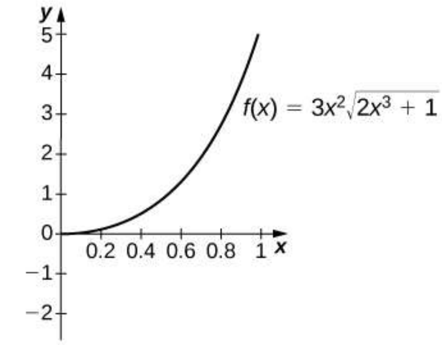
301.

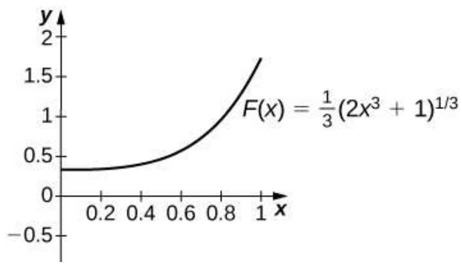




The antiderivative is $y = \frac{1}{2}\sec^2 x$. You should take $C = -2$ so that $F\left(-\frac{\pi}{3}\right) = 0$.

303.





The antiderivative is $y=\frac{1}{3}\left(2x^3+1\right)^{3/2}$. One should take $C=-\frac{1}{3}$.

305. No, because the integrand is discontinuous at $x=1$.

307. $u=\sin\left(t^2\right)$; the integral becomes $\frac{1}{2}\int_0^0 u\,du$.

309. $u=\left(1+\left(t-\frac{1}{2}\right)^2\right)$; the integral becomes $-\int_{5/4}^{5/4}\frac{1}{u}du$.

311. $u=1-t$; the integral becomes

$$\begin{aligned}&\int_1^{-1} u\cos(\pi(1-u))du\\ &=\int_1^{-1} u[\cos\pi\cos u-\sin\pi\sin u]du\\ &=-\int_1^{-1} u\cos u\,du\\ &=\int_{-1}^{1} u\cos u\,du=0\end{aligned}$$

since the integrand is odd.

313. Setting $u=cx$ and $du=cdx$ gets you $\frac{1}{\frac{b}{c}-\frac{a}{c}}\int_{a/c}^{b/c} f(cx)dx=\frac{c}{b-a}\int_{u=a}^{u=b} f(u)\frac{du}{c}=\frac{1}{b-a}\int_a^b f(u)du$.

315. $\int_0^x g(t)dt=\frac{1}{2}\int_{u=1-x^2}^{1}\frac{du}{u^a}=\frac{1}{2(1-a)}u^{1-a}\Big|_{u=1-x^2}^{1}=\frac{1}{2(1-a)}\left(1-\left(1-x^2\right)^{1-a}\right)$. As $x\to 1$ the limit is $\frac{1}{2(1-a)}$ if $a<1$, and the limit diverges to +∞ if $a>1$.

317. $\int_{t=\pi}^{0} b\sqrt{1-\cos^2 t}\times(-a\sin t)dt=\int_{t=0}^{\pi} ab\sin^2 t\,dt$

319. $f(t) = 2\cos(3t) - \cos(2t)$; $\int_0^{\pi/2} (2\cos(3t) - \cos(2t)) = -\frac{2}{3}$

321. $\frac{-1}{3}e^{-3x} + C$

323. $-\frac{3^{-x}}{\ln 3} + C$

325. $\ln\left(x^2\right) + C$

327. $2\sqrt{x} + C$

329. $-\frac{1}{\ln x} + C$

331. $\ln(\ln(\ln x)) + C$

333. $\ln(x\cos x) + C$

335. $-\frac{1}{2}(\ln(\cos(x)))^2 + C$

337. $\frac{-e^{-x^3}}{3} + C$

339. $e^{\tan x} + C$

341. $t + C$

343. $\frac{1}{9}x^3\left(\ln\left(x^3\right) - 1\right) + C$

345. $2\sqrt{x}(\ln x - 2) + C$

347. $\int_0^{\ln x} e^t\,dt = e^t\Big|_0^{\ln x} = e^{\ln x} - e^0 = x - 1$

349. $-\frac{1}{3}\ln(\sin(3x) + \cos(3x))$

351. $-\frac{1}{2}\ln\left|\csc\left(x^2\right) + \cot\left(x^2\right)\right| + C$

353. $-\frac{1}{2}(\ln(\csc x))^2 + C$

355. $\frac{1}{3}\ln\left(\frac{26}{7}\right)$

357. $\ln\left(\sqrt{3} - 1\right)$

359. $\frac{1}{2}\ln\frac{3}{2}$

361. $y - 2\ln|y + 1| + C$

363. $\ln|\sin x - \cos x| + C$

365. $-\frac{1}{3}\left(1 - \left(\ln x^2\right)\right)^{3/2} + C$

367. Exact solution: $\frac{e - 1}{e}$, $R_{50} = 0.6258$. Since f is decreasing, the right endpoint estimate underestimates the area.

369. Exact solution: $\frac{2\ln(3) - \ln(6)}{2}$, $R_{50} = 0.2033$. Since f is increasing, the right endpoint estimate overestimates the area.

371. Exact solution: $-\frac{1}{\ln(4)}$, $R_{50} = -0.7164$. Since f is increasing, the right endpoint estimate overestimates the area (the actual area is a larger negative number).

373. $\frac{11}{2}\ln 2$

375. $\frac{1}{\ln(65,\ 536)}$

377. $\int_N^{N+1} xe^{-x^2}\,dx = \frac{1}{2}\left(e^{-N^2} - e^{-(N+1)^2}\right)$. The quantity is less than 0.01 when $N = 2$.

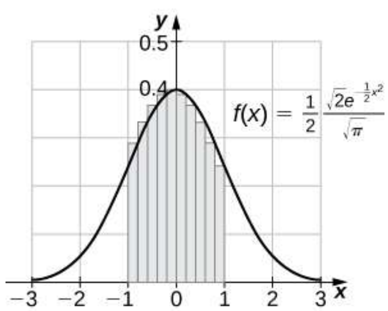
379. $\int_a^b \frac{dx}{x} = \ln(b) - \ln(a) = \ln\left(\frac{1}{a}\right) - \ln\left(\frac{1}{b}\right) = \int_{1/b}^{1/a} \frac{dx}{x}$

381. 23

383. We may assume that $x > 1$, so $\frac{1}{x} < 1$. Then, $\int_1^{1/x} \frac{dt}{t}$. Now make the substitution $u = \frac{1}{t}$, so $du = -\frac{dt}{t^2}$ and $\frac{du}{u} = -\frac{dt}{t}$, and change endpoints: $\int_1^{1/x} \frac{dt}{t} = -\int_1^x \frac{du}{u} = -\ln x$.

387. $x = E(\ln(x))$. Then, $1 = \frac{E'(\ln x)}{x}$ or $x = E'(\ln x)$. Since any number t can be written $t = \ln x$ for some x, and for such t we have $x = E(t)$, it follows that for any t, $E'(t) = E(t)$.

389. $R_{10} = 0.6811$, $R_{100} = 0.6827$



391. $\sin^{-1} x \Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$

393. $\tan^{-1} x \Big|_{\sqrt{3}}^{1} = -\frac{\pi}{12}$

395. $\sec^{-1} x \Big|_1^{\sqrt{2}} = \frac{\pi}{4}$

397. $\sin^{-1}\left(\frac{x}{3}\right) + C$

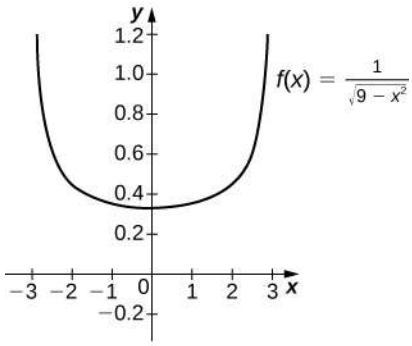
399. $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

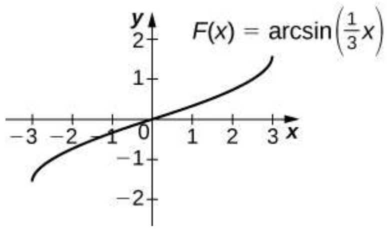
401. $\frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right) + C$

403. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$. So, $\sin^{-1} t = \frac{\pi}{2} - \cos^{-1} t$. They differ by a constant.

405. $\sqrt{1 - t^2}$ is not defined as a real number when $t > 1$.

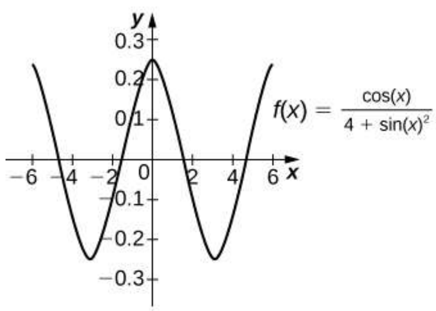
407.

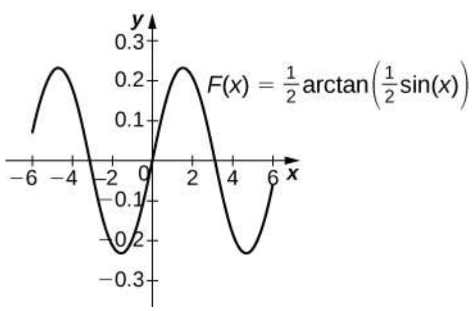




The antiderivative is $\sin^{-1}\left(\frac{x}{3}\right) + C$. Taking $C = \frac{\pi}{2}$ recovers the definite integral.

409.





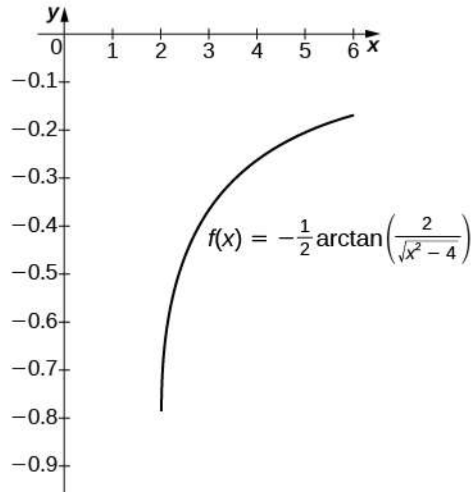
The antiderivative is $\frac{1}{2}\tan^{-1}\left(\frac{\sin x}{2}\right) + C$. Taking $C = \frac{1}{2}\tan^{-1}\left(\frac{\sin(6)}{2}\right)$ recovers the definite integral.

411. $\frac{1}{2}\left(\sin^{-1} t\right)^2 + C$

413. $\frac{1}{4}\left(\tan^{-1}(2t)\right)^2$

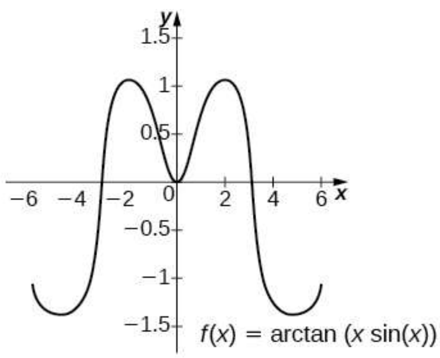
415. $\frac{1}{4}\left(\sec^{-1}\left(\frac{t}{2}\right)^2\right) + C$

417.



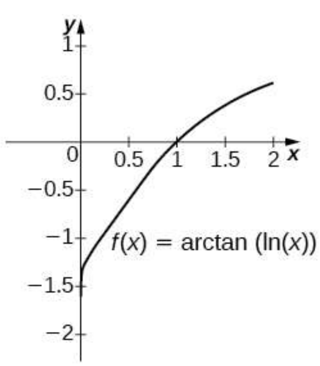
The antiderivative is $\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C$. Taking $C = 0$ recovers the definite integral over $[2, 6]$.

419.



The general antiderivative is $\tan^{-1}(x\sin x) + C$. Taking $C = -\tan^{-1}(6\sin(6))$ recovers the definite integral.

421.



The general antiderivative is $\tan^{-1}(\ln x) + C$. Taking $C = \frac{\pi}{2} = \tan^{-1}\infty$ recovers the definite integral.

423. $\sin^{-1}\left(e^t\right) + C$

425. $\sin^{-1}(\ln t) + C$

427. $-\frac{1}{2}\left(\cos^{-1}(2t)\right)^2 + C$

429. $\frac{1}{2}\ln\left(\frac{4}{3}\right)$

431. $1 - \frac{2}{\sqrt{5}}$

433. $2\tan^{-1}(A) \to \pi$ as $A \to \infty$

435. Using the hint, one has $\int \frac{\csc^2 x}{\csc^2 x + \cot^2 x}dx = \int \frac{\csc^2 x}{1 + 2\cot^2 x}dx.$ Set $u = \sqrt{2}\cot x.$ Then, $du = -\sqrt{2}\csc^2 x$ and the integral is $-\frac{1}{\sqrt{2}}\int \frac{du}{1 + u^2} = -\frac{1}{\sqrt{2}}\tan^{-1} u + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\cot x\right) + C.$ If one uses the identity $\tan^{-1} s + \tan^{-1}\left(\frac{1}{s}\right) = \frac{\pi}{2},$ then this can also be written $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C.$

437. $x \approx \pm 1.13525.$ The left endpoint estimate with $N = 100$ is 2.796 and these decimals persist for $N = 500.$

Review Exercises

439. False

441. True

443. $L_4 = 5.25,\ R_4 = 3.25,$ exact answer: 4

445. $L_4 = 5.364,\ R_4 = 5.364,$ exact answer: 5.870

447. $-\frac{4}{3}$

449. 1

451. $-\frac{1}{2(x+4)^2} + C$

453. $\frac{4}{3}\sin^{-1}\left(x^3\right) + C$

455. $\frac{\sin t}{\sqrt{1 + t^2}}$

457. $4\frac{\ln x}{x} + 1$

459. $6,328,113

461. $73.36

463. $\frac{19117}{12}$ft/sec, or 1593 ft/sec

Chapter 2

Checkpoint

2.1. 12 units^2

2.2. $\frac{3}{10}$ unit^2

2.3. $2 + 2\sqrt{2}$ units^2

2.4. $\frac{5}{3}$ units^2

2.5. $\frac{5}{3}$ units^2

2.7. $\frac{\pi}{2}$

2.8. 8π units^3

2.9. 21π units^3

2.10. $\frac{10\pi}{3}$ units^3

2.11. 60π units^3

2.12. $\frac{15\pi}{2}$ units^3

2.13. 8π units^3

2.14. $12\pi\ \text{units}^3$

2.15. $\frac{11\pi}{6}\ \text{units}^3$

2.16. $\frac{\pi}{6}\ \text{units}^3$

2.17. Use the method of washers; $V = \int_{-1}^{1} \pi\left[\left(2 - x^2\right)^2 - \left(x^2\right)^2\right]dx$

2.18. $\frac{1}{6}\left(5\sqrt{5} - 1\right) \approx 1.697$

2.19. $\text{Arc Length} \approx 3.8202$

2.20. $\text{Arc Length} = 3.15018$

2.21. $\frac{\pi}{6}\left(5\sqrt{5} - 3\sqrt{3}\right) \approx 3.133$

2.22. 12π

2.23. $70/3$

2.24. 24π

2.25. 8 ft-lb

2.26. Approximately $43,255.2$ ft-lb

2.27. $156,800$ N

2.28. Approximately 7,164,520,000 lb or 3,582,260 t

2.29. $M = 24,\ \bar{x} = \frac{2}{5}\text{ m}$

2.30. $(-1, -1)$ m

2.31. The centroid of the region is $(3/2, 6/5)$.

2.32. The centroid of the region is $(1, 13/5)$.

2.33. The centroid of the region is $(0, 2/5)$.

2.34. $6\pi^2\ \text{units}^3$

2.35.

a. $\frac{d}{dx}\ln\left(2x^2 + x\right) = \frac{4x + 1}{2x^2 + x}$

b. $\frac{d}{dx}\left(\ln\left(x^3\right)\right)^2 = \frac{6\ln\left(x^3\right)}{x}$

2.36. $\int \frac{x^2}{x^3 + 6}dx = \frac{1}{3}\ln\left|x^3 + 6\right| + C$

2.37. $4\ln 2$

2.38.

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right) = e^{x^2 - 5x}(2x - 5)$

b. $\frac{d}{dt}\left(e^{2t}\right)^3 = 6e^{6t}$

2.39. $\int \frac{4}{e^{3x}}dx = -\frac{4}{3}e^{-3x} + C$

2.40.

a. $\frac{d}{dt}4^{t^4} = 4^{t^4}(\ln 4)\left(4t^3\right)$

b. $\frac{d}{dx}\log_3\left(\sqrt{x^2 + 1}\right) = \frac{x}{(\ln 3)\left(x^2 + 1\right)}$

2.41. $\int x^2 2^{x^3}dx = \frac{1}{3\ln 2}2^{x^3} + C$

2.42. There are $81,377,396$ bacteria in the population after 4 hours. The population reaches 100 million bacteria after

244.12 minutes.

2.43. At 5% interest, she must invest $\$223{,}130.16$. At 6% interest, she must invest $\$165{,}298.89$.

2.44. 38.90 months

2.45. The coffee is first cool enough to serve about 3.5 minutes after it is poured. The coffee is too cold to serve about 7 minutes after it is poured.

2.46. A total of 94.13 g of carbon remains. The artifact is approximately $13{,}300$ years old.

2.47.

a. $\frac{d}{dx}\left(\tanh\left(x^2+3x\right)\right)=\left(\operatorname{sech}^2\left(x^2+3x\right)\right)(2x+3)$

b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)=\frac{d}{dx}(\sinh x)^{-2}=-2(\sinh x)^{-3}\cosh x$

2.48.

a. $\int \sinh^3 x\cosh x\,dx=\frac{\sinh^4 x}{4}+C$

b. $\int \operatorname{sech}^2(3x)dx=\frac{\tanh(3x)}{3}+C$

2.49.

a. $\frac{d}{dx}\left(\cosh^{-1}(3x)\right)=\frac{3}{\sqrt{9x^2-1}}$

b. $\frac{d}{dx}\left(\coth^{-1}x\right)^3=\frac{3\left(\coth^{-1}x\right)^2}{1-x^2}$

2.50.

a. $\int \frac{1}{\sqrt{x^2-4}}dx=\cosh^{-1}\left(\frac{x}{2}\right)+C$

b. $\int \frac{1}{\sqrt{1-e^{2x}}}dx=-\operatorname{sech}^{-1}\left(e^x\right)+C$

2.51. 52.95 ft

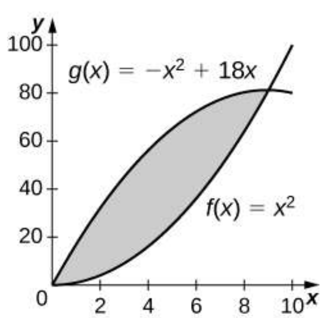
Section Exercises

1. $\frac{32}{3}$

3. $\frac{13}{12}$

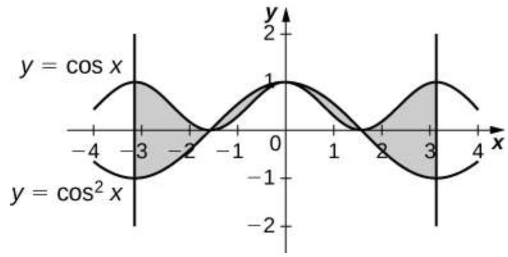
5. 36

7.



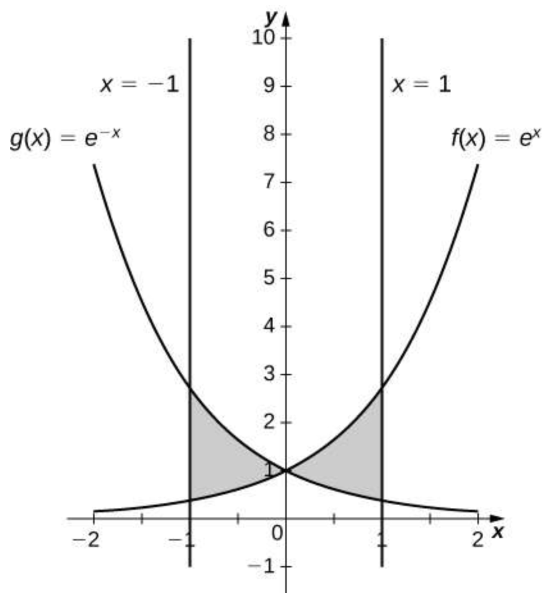
243 square units

9.



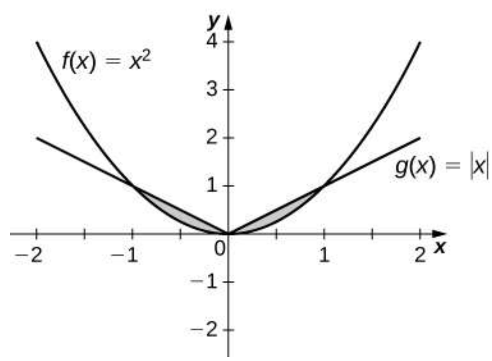
4

11.



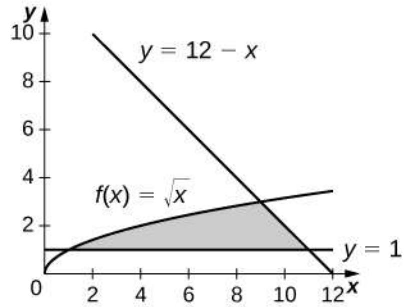
$$\frac{2(e-1)^2}{e}$$

13.



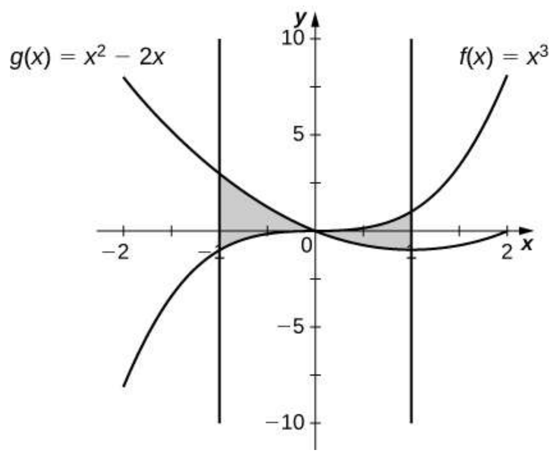
$$\frac{1}{3}$$

15.



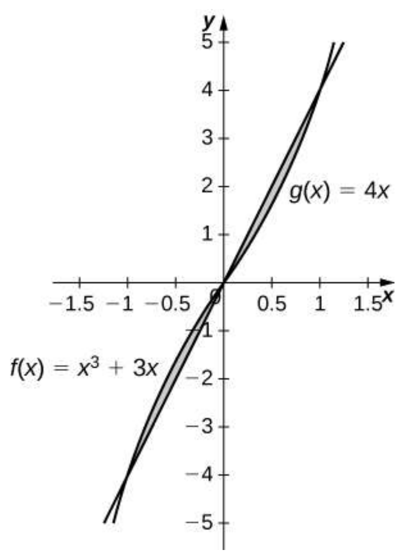
$\frac{34}{3}$

17.



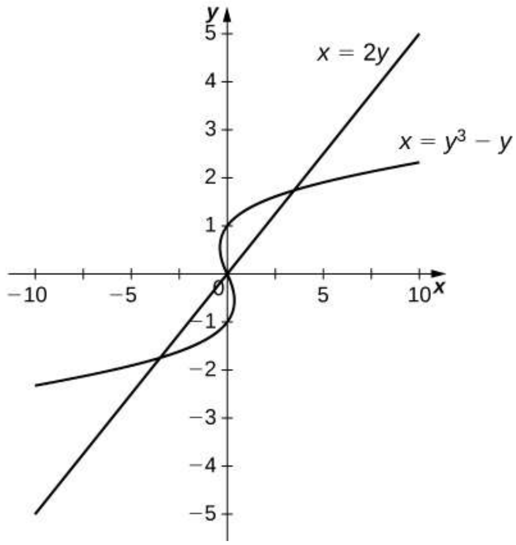
$\frac{5}{2}$

19.



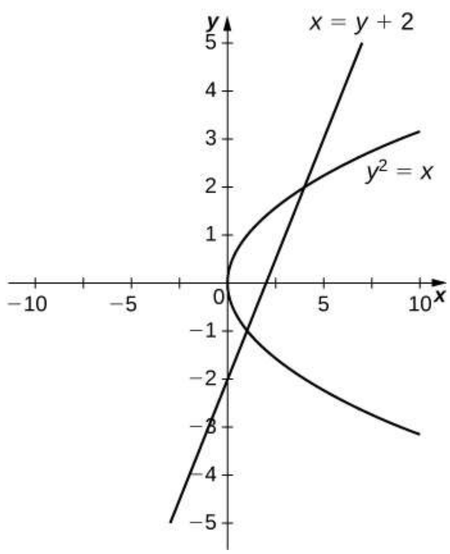
$\frac{1}{2}$

21.



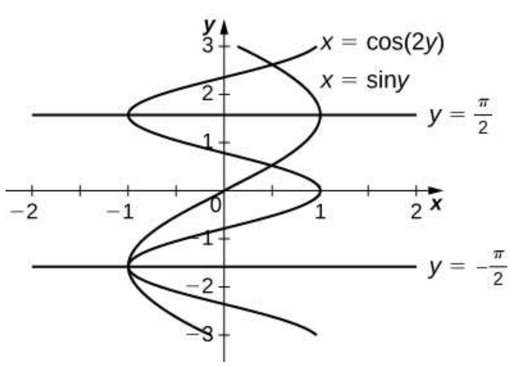
$\frac{9}{2}$

23.



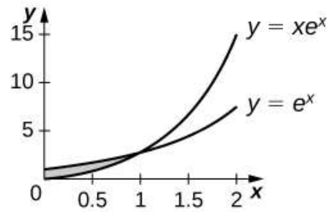
$\frac{9}{2}$

25.



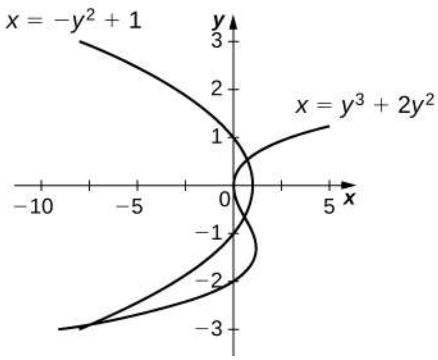
$\frac{3\sqrt{3}}{2}$

27.



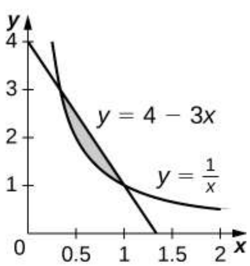
e^{-2}

29.



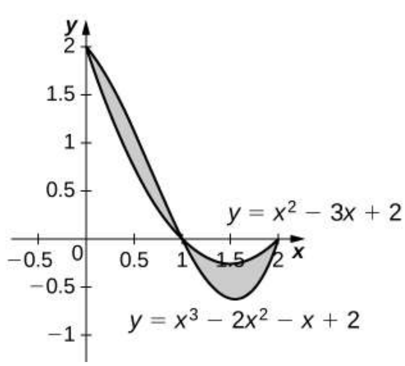
$\frac{27}{4}$

31.



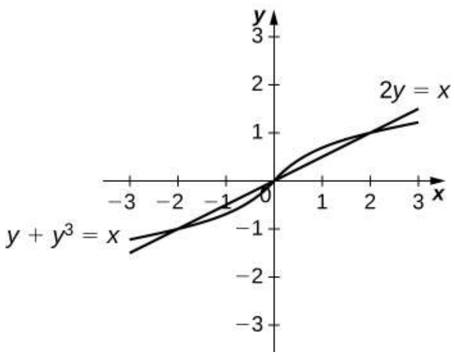
$\frac{4}{3} - \ln(3)$

33.



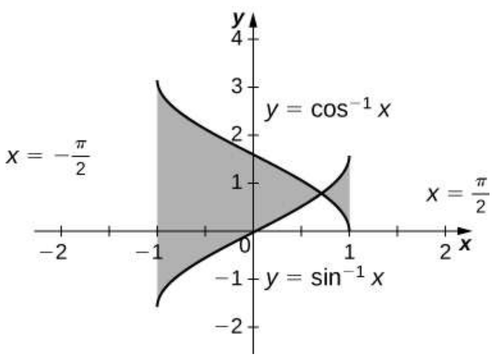
$\frac{1}{2}$

35.



$\frac{1}{2}$

37.



$-2(\sqrt{2}-\pi)$

39. 1.067

41. 0.852

43. 7.523

45. $\frac{3\pi-4}{12}$

47. 1.429

49. $\$33,333.33$ total profit for 200 cell phones sold

51. 3.263 mi represents how far ahead the hare is from the tortoise

53. $\frac{343}{24}$

55. $4\sqrt{3}$

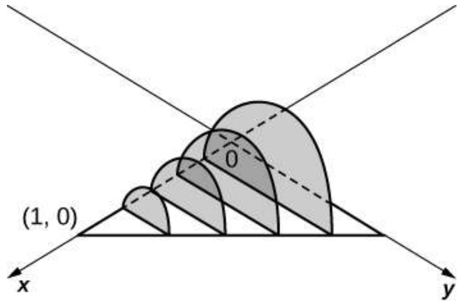
57. $\pi-\frac{32}{25}$

63. 8 units^3

65. $\frac{32}{3\sqrt{2}}$ units^3

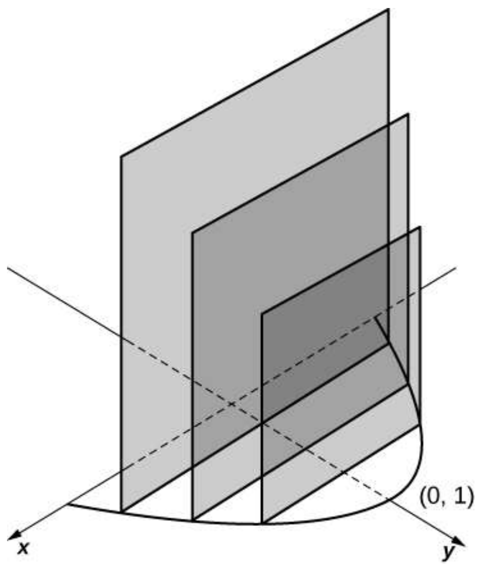
67. $\frac{7\pi}{12}hr^2$ units^3

69.



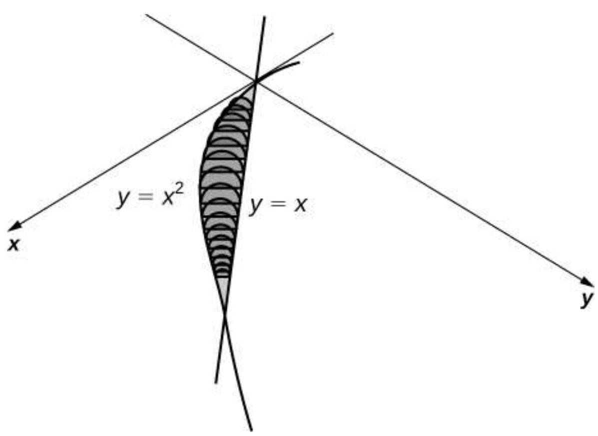
$\frac{\pi}{24}$ units^3

71.



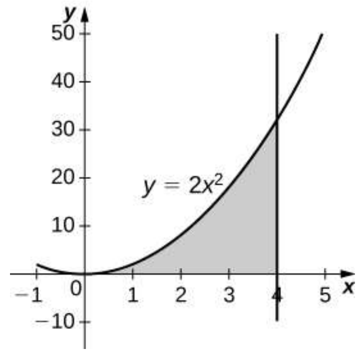
2 units^3

73.



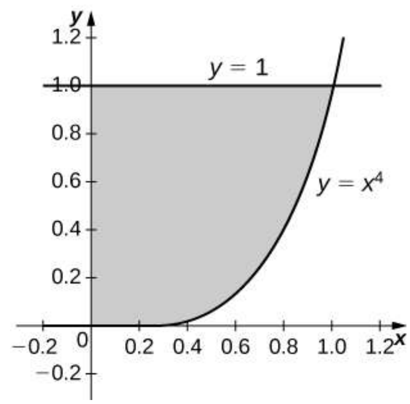
$\frac{\pi}{240}$ units^3

75.



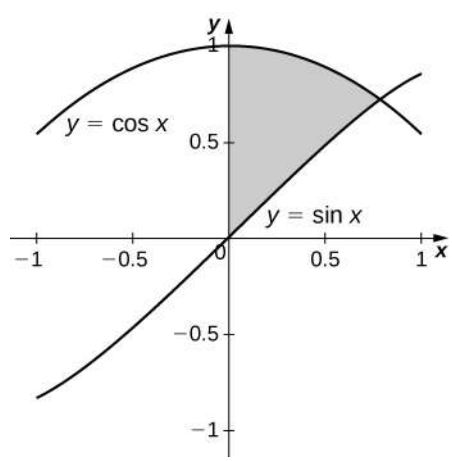
$\frac{4096\pi}{5}$ units^3

77.



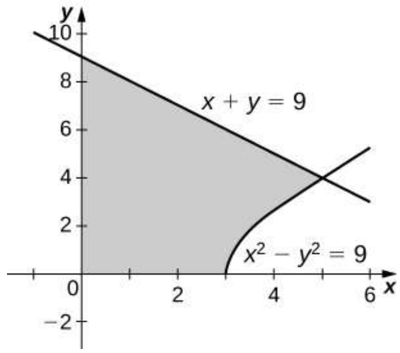
$\frac{8\pi}{9}$ units^3

79.



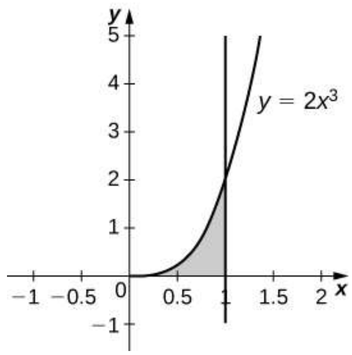
$\frac{\pi}{2}$ units^3

81.



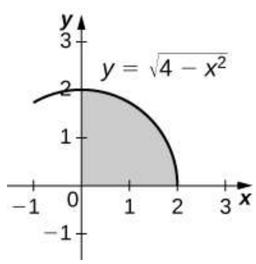
207π units^3

83.



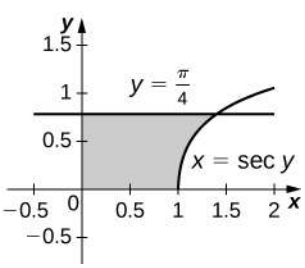
$\frac{4\pi}{5}$ units^3

85.



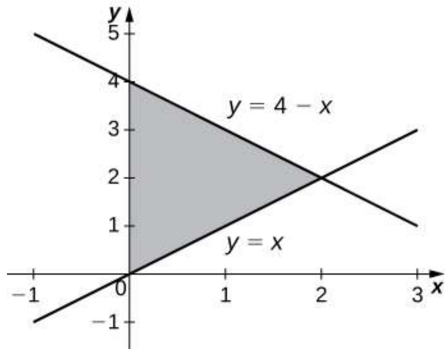
$\frac{16\pi}{3}$ units^3

87.



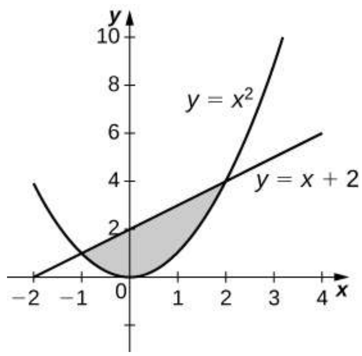
π units^3

89.



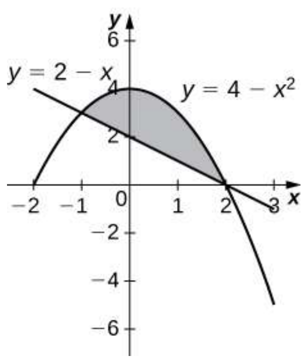
$\frac{16\pi}{3}$ units^3

91.



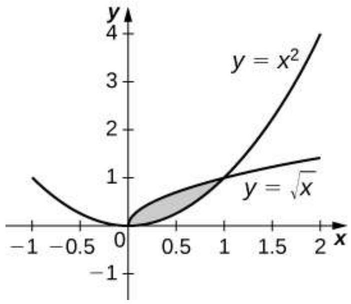
$\frac{72\pi}{5}$ units^3

93.



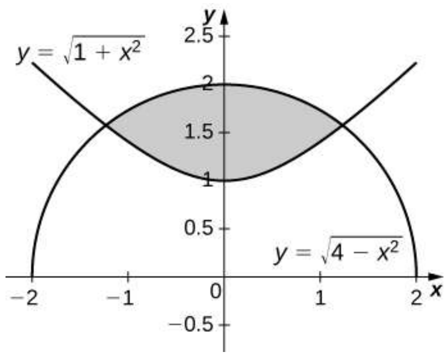
$\frac{108\pi}{5}$ units^3

95.



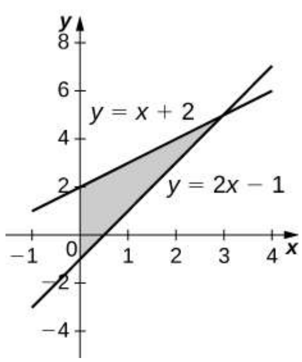
$\frac{3\pi}{10}$ units^3

97.



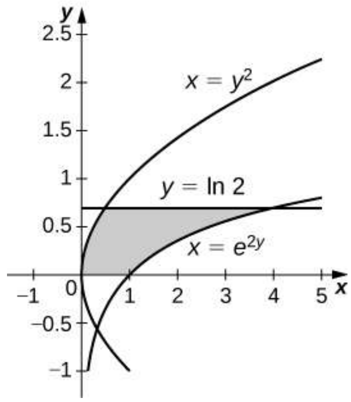
$2\sqrt{6}\pi$ units^3

99.



9π units^3

101.



$\frac{\pi}{20}\left(75 - 4\ln^5(2)\right)$ units^3

103. $\frac{m^2\pi}{3}\left(b^3 - a^3\right)$ units^3

105. $\frac{4a^2 b\pi}{3}$ units^3

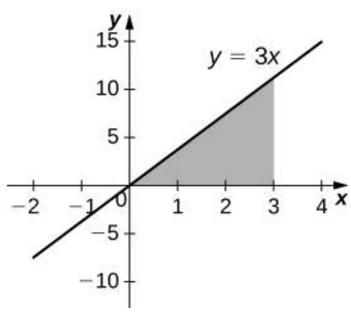
107. $2\pi^2$ units^3

109. $\frac{2ab^2\pi}{3}$ units^3

111. $\frac{\pi}{12}(r+h)^2(6r-h)$ units^3

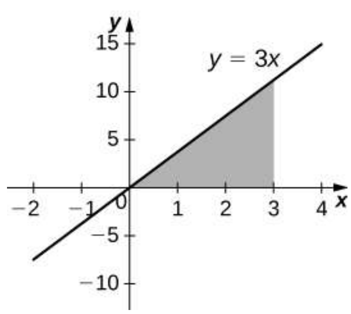
113. $\frac{\pi}{3}(h+R)(h-2R)^2$ units^3

115.



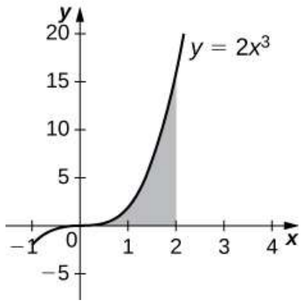
54π units^3

117.



81π units^3

119.



$\frac{512\pi}{7}$ units^3

121. 2π units^3

123. $\frac{2\pi}{3}$ units^3

125. 2π units^3

127. $\frac{4\pi}{5}$ units^3

129. $\frac{64\pi}{3}$ units^3

131. $\frac{32\pi}{5}$ units^3

133. $\pi(e-2)$ units^3

135. $\frac{28\pi}{3}$ units^3

137. $\frac{-84\pi}{5}$ units^3

139. $-e^{\pi}\pi^2$ units^3

141. $\frac{64\pi}{5}$ units^3

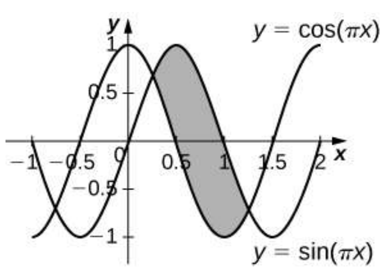
143. $\frac{28\pi}{15}$ units^3

145. $\frac{3\pi}{10}$ units^3

147. $\frac{52\pi}{5}$ units^3

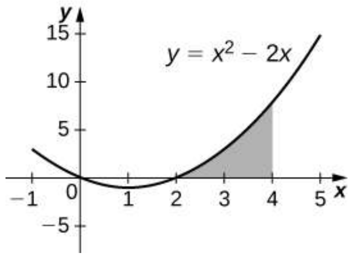
149. 0.9876 units^3

151.



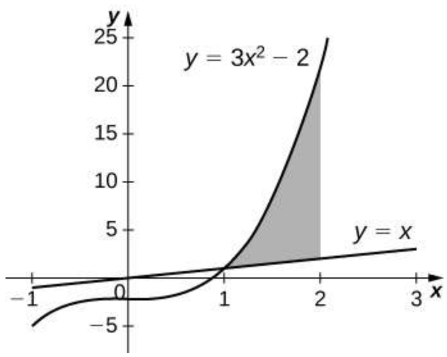
$3\sqrt{2}$ units^3

153.



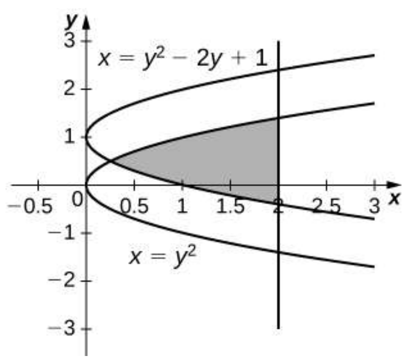
$\frac{496\pi}{15}$ units^3

155.



$\frac{398\pi}{15}$ units^3

157.



15.9074 units^3

159. $\frac{1}{3}\pi r^2 h$ units^3

161. $\pi r^2 h$ units^3

163. πa^2 units^3

165. $2\sqrt{26}$

167. $2\sqrt{17}$

169. $\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$

171. $\frac{13\sqrt{13} - 8}{27}$

173. $\frac{4}{3}$

175. 2.0035

177. $\frac{123}{32}$

179. 10

181. $\frac{20}{3}$

183. $\frac{1}{675}\left(229\sqrt{229} - 8\right)$

185. $\frac{1}{8}\left(4\sqrt{5} + \ln\left(9 + 4\sqrt{5}\right)\right)$

187. 1.201

189. 15.2341

191. $\frac{49\pi}{3}$

193. $70\pi\sqrt{2}$

195. 8π

197. $120\pi\sqrt{26}$

199. $\frac{\pi}{6}(17\sqrt{17} - 1)$

201. $9\sqrt{2}\pi$

203. $\frac{10\sqrt{10}\pi}{27}\left(73\sqrt{73} - 1\right)$

205. 25.645

207. 2π

209. 10.5017

211. 23 ft

213. 2

215. Answers may vary

217. For more information, look up Gabriel's Horn.

219. 150 ft-lb

221. 200 J

223. 1 J

225. $\frac{39}{2}$

227. $\ln(243)$

229. $\frac{332\pi}{15}$

231. 100π

233. $20\pi\sqrt{15}$

235. 6 J

237. 5 cm

239. 36 J

241. $18{,}750$ ft-lb

243. $\frac{32}{3} \times 10^9$ ft-lb

245. 8.65×10^5 J

247. a. $3{,}000{,}000$ lb, b. $749{,}000$ lb

249. 23.25π million ft-lb

251. $\frac{A\rho H^2}{2}$

253. Answers may vary

255. $\frac{5}{4}$

257. $\left(\frac{2}{3}, \frac{2}{3}\right)$

259. $\left(\frac{7}{4}, \frac{3}{2}\right)$

261. $\frac{3L}{4}$

263. $\frac{\pi}{2}$

265. $\frac{e^2+1}{e^2-1}$

267. $\frac{\pi^2-4}{\pi}$

269. $\frac{1}{4}\left(1+e^2\right)$

271. $\left(\frac{a}{3}, \frac{b}{3}\right)$

273. $\left(0, \frac{\pi}{8}\right)$

275. $(0, 3)$

277. $\left(0, \frac{4}{\pi}\right)$

279. $\left(\frac{5}{8}, \frac{1}{3}\right)$

281. $\frac{m\pi}{3}$

283. $\pi a^2 b$

285. $\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$

287. $\left(\frac{1}{2}, \frac{2}{5}\right)$

289. $\left(0, \frac{28}{9\pi}\right)$

291. Center of mass: $\left(\frac{a}{6}, \frac{4a^2}{5}\right)$, volume: $\frac{2\pi a^4}{9}$

293. Volume: $2\pi^2 a^2 (b+a)$

295. $\frac{1}{x}$

297. $-\frac{1}{x(\ln x)^2}$

299. $\ln(x+1)+C$

301. $\ln(x)+1$

303. $\cot(x)$

305. $\frac{7}{x}$

307. $\csc(x)\sec x$

309. $-2\tan x$

311. $\frac{1}{2}\ln\left(\frac{5}{3}\right)$

313. $2-\frac{1}{2}\ln(5)$

315. $\frac{1}{\ln(2)}-1$

317. $\frac{1}{2}\ln(2)$

319. $\frac{1}{3}(\ln x)^3$

321. $\frac{2x^3}{\sqrt{x^2+1}\sqrt{x^2-1}}$

323. $x^{-2-(1/x)}(\ln x - 1)$

325. ex^{e-1}

327. 1

329. $-\frac{1}{x^2}$

331. $\pi - \ln(2)$

333. $\frac{1}{x}$

335. $e^5 - 6 \text{ units}^2$

337. $\ln(4) - 1 \text{ units}^2$

339. 2.8656

341. 3.1502

349. True

351. False; $k = \frac{\ln(2)}{t}$

353. 20 hours

355. No. The relic is approximately 871 years old.

357. 71.92 years

359. 5 days 6 hours 27 minutes

361. 12

363. 8.618%

365. $\$6766.76$

367. 9 hours 13 minutes

369. $239,179$ years

371. $P'(t) = 43e^{0.01604t}$. The population is always increasing.

373. The population reaches 10 billion people in 2027.

375. $P'(t) = 2.259e^{0.06407t}$. The population is always increasing.

377. e^x and e^{-x}

379. Answers may vary

381. Answers may vary

383. Answers may vary

385. $3\sinh(3x+1)$

387. $-\tanh(x)\operatorname{sech}(x)$

389. $4\cosh(x)\sinh(x)$

391. $\frac{x \operatorname{sech}^2\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$

393. $6\sinh^5(x)\cosh(x)$

395. $\frac{1}{2}\sinh(2x+1) + C$

397. $\frac{1}{2}\sinh^2\left(x^2\right) + C$

399. $\frac{1}{3}\cosh^3(x) + C$

401. $\ln(1 + \cosh(x)) + C$

403. $\cosh(x) + \sinh(x) + C$

405. $\frac{4}{1-16x^2}$

407. $\frac{\sinh(x)}{\sqrt{\cosh^2(x)+1}}$

409. $-\csc(x)$

411. $-\frac{1}{(x^2-1)\tanh^{-1}(x)}$

413. $\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C$

415. $\sqrt{x^2+1}+C$

417. $\cosh^{-1}(e^x)+C$

419. Answers may vary

421. 37.30

423. $y=\frac{1}{c}\cosh(cx)$

425. -0.521095

427. 10

Review Exercises

435. False

437. False

439. $32\sqrt{3}$

441. $\frac{162\pi}{5}$

443. a. 4, b. $\frac{128\pi}{7}$, c. $\frac{64\pi}{5}$

445. a. 1.949, b. 21.952, c. 17.099

447. a. $\frac{31}{6}$, b. $\frac{452\pi}{15}$, c. $\frac{31\pi}{6}$

449. 245.282

451. Mass: $\frac{1}{2}$, center of mass: $\left(\frac{18}{35}, \frac{9}{11}\right)$

453. $\sqrt{17}+\frac{1}{8}\ln(33+8\sqrt{17})$

455. Volume: $\frac{3\pi}{4}$, surface area: $\pi\left(\sqrt{2}-\sinh^{-1}(1)+\sinh^{-1}(16)-\frac{\sqrt{257}}{16}\right)$

457. 11:02 a.m.

459. $\pi(1+\sinh(1)\cosh(1))$

Chapter 3

Checkpoint

3.1. $\int xe^{2x}\,dx=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$

3.2. $\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$

3.3. $-x^2\cos x+2x\sin x+2\cos x+C$

3.4. $\frac{\pi}{2}-1$

3.5. $\frac{1}{5}\sin^5 x+C$

3.6. $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$

3.7. $\frac{1}{2}x+\frac{1}{4}\sin(2x)+C$

3.8. $\sin x - \frac{1}{3}\sin^3 x + C$

3.9. $\frac{1}{2}x + \frac{1}{12}\sin(6x) + C$

3.10. $\frac{1}{2}\sin x + \frac{1}{22}\sin(11x) + C$

3.11. $\frac{1}{6}\tan^6 x + C$

3.12. $\frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C$

3.13. $\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x \tan x - \frac{3}{4}\int \sec^3 x$

3.14. $\int 125 \sin^3 \theta d\theta$

3.15. $\int 32 \tan^3 \theta \sec^3 \theta d\theta$

3.16. $\ln\left|\frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2}\right| + C$

3.17. $x - 5\ln|x + 2| + C$

3.18. $\frac{2}{5}\ln|x + 3| + \frac{3}{5}\ln|x - 2| + C$

3.19. $\frac{x + 2}{(x + 3)^3 (x - 4)^2} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2} + \frac{C}{(x + 3)^3} + \frac{D}{(x - 4)} + \frac{E}{(x - 4)^2}$

3.20. $\frac{x^2 + 3x + 1}{(x + 2)(x - 3)^2 (x^2 + 4)^2} = \frac{A}{x + 2} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 4} + \frac{Fx + G}{(x^2 + 4)^2}$

3.21. Possible solutions include $\sinh^{-1}\left(\frac{x}{2}\right) + C$ and $\ln\left|\sqrt{x^2 + 4} + x\right| + C$.

3.22. $\frac{24}{35}$

3.23. $\frac{17}{24}$

3.24. 0.0074, 1.1%

3.25. $\frac{1}{192}$

3.26. $\frac{25}{36}$

3.27. e^3, converges

3.28. $+\infty$, diverges

3.29. Since $\int_e^{+\infty} \frac{1}{x}dx = +\infty$, $\int_e^{+\infty} \frac{\ln x}{x}dx$ diverges.

Section Exercises

1. $u = x^3$

3. $u = y^3$

5. $u = \sin(2x)$

7. $-x + x\ln x + C$

9. $x\tan^{-1} x - \frac{1}{2}\ln(1 + x^2) + C$

11. $-\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C$

13. $e^{-x}(-1 - x) + C$

15. $2x\cos x + \left(-2 + x^2\right)\sin x + C$

17. $\frac{1}{2}(1+2x)(-1+\ln(1+2x))+C$

19. $\frac{1}{2}e^{x}(-\cos x+\sin x)+C$

21. $-\frac{e^{-x^2}}{2}+C$

23. $-\frac{1}{2}x\cos[\ln(2x)]+\frac{1}{2}x\sin[\ln(2x)]+C$

25. $2x-2x\ln x+x(\ln x)^2+C$

27. $\left(-\frac{x^3}{9}+\frac{1}{3}x^3\ln x\right)+C$

29. $-\frac{1}{2}\sqrt{1-4x^2}+x\cos^{-1}(2x)+C$

31. $-\left(-2+x^2\right)\cos x+2x\sin x+C$

33. $-x\left(-6+x^2\right)\cos x+3\left(-2+x^2\right)\sin x+C$

35. $\frac{1}{2}x\left(-\sqrt{1-\frac{1}{x^2}}+x\cdot\sec^{-1}x\right)+C$

37. $-\cosh x+x\sinh x+C$

39. $\frac{1}{4}-\frac{3}{4e^2}$

41. 2

43. 2π

45. $-2+\pi$

47. $-\sin(x)+\ln[\sin(x)]\sin x+C$

49. Answers vary

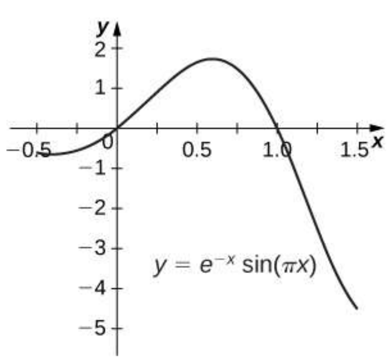
51. a. $\frac{2}{5}(1+x)(-3+2x)^{3/2}+C$ b. $\frac{2}{5}(1+x)(-3+2x)^{3/2}+C$

53. Do not use integration by parts. Choose u to be $\ln x$, and the integral is of the form $\int u^2\,du$.

55. Do not use integration by parts. Let $u=x^2-3$, and the integral can be put into the form $\int e^u\,du$.

57. Do not use integration by parts. Choose u to be $u=3x^3+2$ and the integral can be put into the form $\int \sin(u)du$.

59. The area under graph is 0.39535.



61. $2\pi e$

63. 2.05

65. 12π

67. $8\pi^2$

69. $\cos^2 x$

71. $\frac{1-\cos(2x)}{2}$

73. $\frac{\sin^4 x}{4} + C$

75. $\frac{1}{12}\tan^6(2x) + C$

77. $\sec^2\left(\frac{x}{2}\right) + C$

79. $-\frac{3\cos x}{4} + \frac{1}{12}\cos(3x) + C = -\cos x + \frac{\cos^3 x}{3} + C$

81. $-\frac{1}{2}\cos^2 x + C$

83. $-\frac{5\cos x}{64} - \frac{1}{192}\cos(3x) + \quad \frac{3}{320}\cos(5x) - \frac{1}{448}\cos(7x) + C$

85. $\frac{2}{3}(\sin x)^{2/3} + C$

87. $\sec x + C$

89. $\frac{1}{2}\sec x\tan x - \frac{1}{2}\ln(\sec x + \tan x) + C$

91. $\frac{2\tan x}{3} + \frac{1}{3}\sec(x)^2\tan x \quad = \tan x + \frac{\tan^3 x}{3} + C$

93. $-\ln|\cot x + \csc x| + C$

95. $\frac{\sin^3(ax)}{3a} + C$

97. $\frac{\pi}{2}$

99. $\frac{x}{2} + \frac{1}{12}\sin(6x) + C$

101. $x + C$

103. 0

105. 0

107. 0

109. Approximately 0.239

111. $\sqrt{2}$

113. 1.0

115. 0

117. $\frac{3\theta}{8} - \frac{1}{4\pi}\sin(2\pi\theta) + \frac{1}{32\pi}\sin(4\pi\theta) + C = f(x)$

119. $\ln\left(\sqrt{3}\right)$

121. $\int_{-\pi}^{\pi}\sin(2x)\cos(3x)dx = 0$

123. $\sqrt{\tan(x)}x\left(\frac{8\tan x}{21} + \frac{2}{7}\sec x^2\tan x\right) + C = f(x)$

125. The second integral is more difficult because the first integral is simply a u-substitution type.

127. $9\tan^2\theta$

129. $a^2\cosh^2\theta$

131. $4\left(x - \frac{1}{2}\right)^2$

133. $-(x+1)^2 + 5$

135. $\ln\left|x + \sqrt{-a^2 + x^2}\right| + C$

137. $\frac{1}{3}\ln\left|\sqrt{9x^2 + 1} + 3x\right| + C$

139. $-\frac{\sqrt{1-x^2}}{x}+C$

141. $9\left[\frac{x\sqrt{x^2+9}}{18}+\frac{1}{2}ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right|\right]+C$

143. $-\frac{1}{3}\sqrt{9-\theta^2}\left(18+\theta^2\right)+C$

145. $\frac{\left(-1+x^2\right)\left(2+3x^2\right)\sqrt{x^6-x^8}}{15x^3}+C$

147. $-\frac{x}{9\sqrt{-9+x^2}}+C$

149. $\frac{1}{2}\left(\ln\left|x+\sqrt{x^2-1}\right|+x\sqrt{x^2-1}\right)+C$

151. $-\frac{\sqrt{1+x^2}}{x}+C$

153. $\frac{1}{8}\left(x\left(5-2x^2\right)\sqrt{1-x^2}+3\arcsin x\right)+C$

155. $\ln x-\ln\left|1+\sqrt{1-x^2}\right|+C$

157. $-\frac{\sqrt{-1+x^2}}{x}+\ln\left|x+\sqrt{-1+x^2}\right|+C$

159. $-\frac{\sqrt{1+x^2}}{x}+\operatorname{arcsinh}x+C$

161. $-\frac{1}{1+x}+C$

163. $\frac{2\sqrt{-10+x}\sqrt{x}\ln\left|\sqrt{-10+x}+\sqrt{x}\right|}{\sqrt{(10-x)x}}+C$

165. $\frac{9\pi}{2};$ area of a semicircle with radius 3

167. $\arcsin(x)+C$ is the common answer.

169. $\frac{1}{2}\ln\left(1+x^2\right)+C$ is the result using either method.

171. Use trigonometric substitution. Let $x=\sec(\theta)$.

173. 4.367

175. $\frac{\pi^2}{8}+\frac{\pi}{4}$

177. $y=\frac{1}{16}\ln\left|\frac{x+8}{x-8}\right|+3$

179. 24.6 m^3

181. $\frac{2\pi}{3}$

183. $-\frac{2}{x+1}+\frac{5}{2(x+2)}+\frac{1}{2x}$

185. $\frac{1}{x^2}+\frac{3}{x}$

187. $2x^2+4x+8+\frac{16}{x-2}$

189. $-\frac{1}{x^2}-\frac{1}{x}+\frac{1}{x-1}$

191. $-\frac{1}{2(x-2)}+\frac{1}{2(x-1)}-\frac{1}{6x}+\frac{1}{6(x-3)}$

193. $\frac{1}{x-1}+\frac{2x+1}{x^2+x+1}$

195. $\frac{2}{x+1}+\frac{x}{x^2+4}-\frac{1}{\left(x^2+4\right)^2}$

197. $-\ln|2-x|+2\ln|4+x|+C$

199. $\frac{1}{2}\ln\left|4-x^2\right|+C$

201. $2\left(x+\frac{1}{3}\arctan\left(\frac{1+x}{3}\right)\right)+C$

203. $2\ln|x|-3\ln|1+x|+C$

205. $\frac{1}{16}\left(-\frac{4}{-2+x}-\ln|-2+x|+\ln|2+x|\right)+C$

207. $\frac{1}{30}\left(-2\sqrt{5}\arctan\left[\frac{1+x}{\sqrt{5}}\right]+2\ln|-4+x|-\ln\left|6+2x+x^2\right|\right)+C$

209. $-\frac{3}{x}+4\ln|x+2|+x+C$

211. $-\ln|3-x|+\frac{1}{2}\ln\left|x^2+4\right|+C$

213. $\ln|x-2|-\frac{1}{2}\ln\left|x^2+2x+2\right|+C$

215. $-x+\ln\left|1-e^x\right|+C$

217. $\frac{1}{5}\ln\left|\frac{\cos x+3}{\cos x-2}\right|+C$

219. $\frac{1}{2-2e^{2t}}+C$

221. $2\sqrt{1+x}-2\ln\left|1+\sqrt{1+x}\right|+C$

223. $\ln\left|\frac{\sin x}{1-\sin x}\right|+C$

225. $\frac{\sqrt{3}}{4}$

227. $x-\ln(1+e^x)+C$

229. $6x^{1/6}-3x^{1/3}+2\sqrt{x}-6\ln\left(1+x^{1/6}\right)+C$

231. $\frac{4}{3}\pi\operatorname{arctanh}\left[\frac{1}{3}\right]=\frac{1}{3}\pi\ln 4$

233. $x=-\ln|t-3|+\ln|t-4|+\ln 2$

235. $x=\ln|t-1|-\sqrt{2}\arctan\left(\sqrt{2}t\right)-\frac{1}{2}\ln\left(t^2+\frac{1}{2}\right)+\sqrt{2}\arctan(2\sqrt{2})+\frac{1}{2}\ln 4.5$

237. $\frac{2}{5}\pi\ln\frac{28}{13}$

239. $\frac{\arctan\left[\frac{-1+2x}{\sqrt{3}}\right]}{\sqrt{3}}+\frac{1}{3}\ln|1+x|-\frac{1}{6}\ln\left|1-x+x^2\right|+C$

241. 2.0 in.^{2}

243. $3(-8+x)^{1/3}$

$-2\sqrt{3}\arctan\left[\frac{-1+(-8+x)^{1/3}}{\sqrt{3}}\right]$

$-2\ln\left[2+(-8+x)^{1/3}\right]$

$+\ln\left[4-2(-8+x)^{1/3}+(-8+x)^{2/3}\right]+C$

245. $\frac{1}{2}\ln\left|x^2+2x+2\right|+2\arctan(x+1)+C$

247. $\cosh^{-1}\left(\frac{x+3}{3}\right)+C$

249. $\frac{2^{x^2-1}}{\ln 2} + C$

251. $\arcsin\left(\frac{y}{2}\right) + C$

253. $-\frac{1}{2}\csc(2w) + C$

255. $9 - 6\sqrt{2}$

257. $2 - \frac{\pi}{2}$

259. $\frac{1}{12}\tan^4(3x) - \frac{1}{6}\tan^2(3x) + \frac{1}{3}\ln|\sec(3x)| + C$

261. $2\cot\left(\frac{w}{2}\right) - 2\csc\left(\frac{w}{2}\right) + w + C$

263. $\frac{1}{5}\ln\left|\frac{2(5 + 4\sin t - 3\cos t)}{4\cos t + 3\sin t}\right|$

265. $6x^{1/6} - 3x^{1/3} + 2\sqrt{x} - 6\ln\left[1 + x^{1/6}\right] + C$

267. $-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$

269. $\frac{1}{2}\left(x^2 + \ln\left|1 + e^{-x^2}\right|\right) + C$

271. $2\arctan\left(\sqrt{x - 1}\right) + C$

273. $0.5 = \frac{1}{2}$

275. 8.0

277. $\frac{1}{3}\arctan\left(\frac{1}{3}(x + 2)\right) + C$

279. $\frac{1}{3}\arctan\left(\frac{x + 1}{3}\right) + C$

281. $\ln\left(e^x + \sqrt{4 + e^{2x}}\right) + C$

283. $\ln x - \frac{1}{6}\ln\left(x^6 + 1\right) - \frac{\arctan\left(x^3\right)}{3x^3} + C$

285. $\ln\left|x + \sqrt{16 + x^2}\right| + C$

287. $-\frac{1}{4}\cot(2x) + C$

289. $\frac{1}{2}\arctan 10$

291. 1276.14

293. 7.21

295. $\sqrt{5} - \sqrt{2} + \ln\left|\frac{2 + 2\sqrt{2}}{1 + \sqrt{5}}\right|$

297. $\frac{1}{3}\arctan(3) \approx 0.416$

299. 0.696

301. 9.279

303. 0.5000

305. $T_4 = 18.75$

307. 0.500

309. 1.1614

311. 0.6577

313. 0.0213

315. 1.5629

317. 1.9133

319. $\mathrm{T}(4) = 0.1088$

321. 1.0
323. Approximate error is 0.000325.
325. $\frac{1}{7938}$
327. $\frac{81}{25,000}$
329. 475
331. 174
333. 0.1544
335. 6.2807
337. 4.606
339. 3.41 ft
341. $T_{16} = 100.125;$ absolute error = 0.125
343. about 89,250 m^2
345. parabola
347. divergent
349. $\frac{\pi}{2}$
351. $\frac{2}{e}$
353. Converges
355. Converges to 1/2.
357. −4
359. π
361. diverges
363. diverges
365. 1.5
367. diverges
369. diverges
371. diverges
373. Both integrals diverge.
375. diverges
377. diverges
379. π
381. 0.0
383. 0.0
385. 6.0
387. $\frac{\pi}{2}$
389. $8\ln(16) - 4$
391. 1.047
393. $-1 + \frac{2}{\sqrt{3}}$
395. 7.0
397. $\frac{5\pi}{2}$
399. 3π
401. $\frac{1}{s}, s > 0$
403. $\frac{s}{s^2 + 4}, s > 0$
405. Answers will vary.
407. 0.8775

Review Exercises

409. False
411. False
413. $-\frac{\sqrt{x^2 + 16}}{16x} + C$

415. $\frac{1}{10}(4\ln(2-x)+5\ln(x+1)-9\ln(x+3))+C$

417. $-\frac{\sqrt{4-\sin^2(x)}}{\sin(x)}-\frac{x}{2}+C$

419. $\frac{1}{15}\left(x^2+2\right)^{3/2}\left(3x^2-4\right)+C$

421. $\frac{1}{16}\ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right)-\frac{1}{8}\tan^{-1}(1-x)+\frac{1}{8}\tan^{-1}(x+1)+C$

423. $\mathrm{M}_4=3.312,\ \mathrm{T}_4=3.354,\ S_4=3.326$

425. $\mathrm{M}_4=-0.982,\ \mathrm{T}_4=-0.917,\ S_4=-0.952$

427. approximately 0.2194

431. Answers may vary. Ex: 9.405 km

Chapter 4

Checkpoint

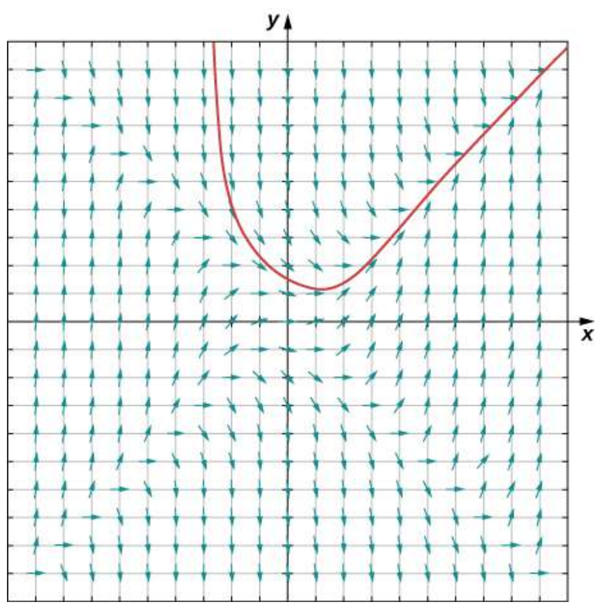
4.2. 5

4.3. $y=2x^2+3x+2$

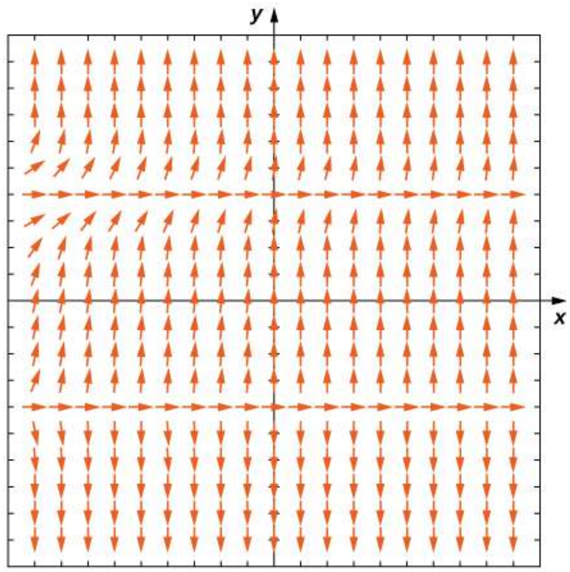
4.5. $y=\frac{1}{3}x^3-2x^2+3x-6e^x+14$

4.6. $v(t)=-9.8t$

4.7.



4.8.



The equilibrium solutions are $y = -2$ and $y = 2.$ For this equation, $y = -2$ is an unstable equilibrium solution, and $y = 2$ is a semi-stable equilibrium solution.

4.9.

n	x_n	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0	1	-2
1	1.1	$y_1 = y_0 + hf(x_0, y_0) = -1.5$
2	1.2	$y_2 = y_1 + hf(x_1, y_1) = -1.1419$
3	1.3	$y_3 = y_2 + hf(x_2, y_2) = -0.8387$
4	1.4	$y_4 = y_3 + hf(x_3, y_3) = -0.5487$
5	1.5	$y_5 = y_4 + hf(x_4, y_4) = -0.2442$
6	1.6	$y_6 = y_5 + hf(x_5, y_5) = 0.0993$
7	1.7	$y_7 = y_6 + hf(x_6, y_6) = 0.5099$
8	1.8	$y_8 = y_7 + hf(x_7, y_7) = 1.0272$
9	1.9	$y_9 = y_8 + hf(x_8, y_8) = 1.7159$
10	2	$y_{10} = y_9 + hf(x_9, y_9) = 2.6962$

4.10. $y = 2 + Ce^{x^2 + 3x}$

4.11. $y = \dfrac{4 + 14e^{x^2 + x}}{1 - 7e^{x^2 + x}}$

4.12. Initial value problem: $\dfrac{du}{dt} = 2.4 - \dfrac{2u}{25}, \quad u(0) = 3$ Solution: $u(t) = 30 - 27e^{-t/50}$

4.13.

a. Initial-value problem
$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 450$$

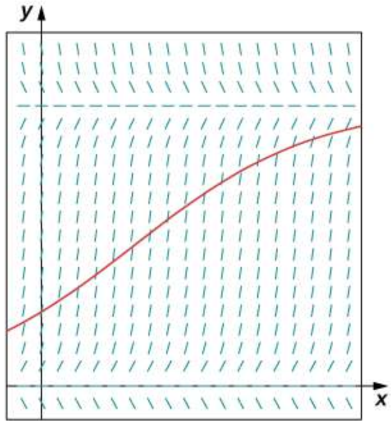
b. $T(t) = 70 + 380e^{kt}$

c. Approximately 114 minutes.

4.14.

a. $\dfrac{dP}{dt} = 0.04\left(1 - \dfrac{P}{750}\right), \quad P(0) = 200$

b.



c. $P(t) = \dfrac{3000e^{.04t}}{11 + 4e^{.04t}}$

d. After 12 months, the population will be $P(12) \approx 278$ rabbits.

4.15. $y' + \dfrac{15}{x+3}y = \dfrac{10x - 20}{x+3}$; $p(x) = \dfrac{15}{x+3}$ and $q(x) = \dfrac{10x - 20}{x+3}$

4.16. $y = \dfrac{x^3 + x^2 + C}{x - 2}$

4.17. $y = -2x - 4 + 2e^{2x}$

4.18.

a. $$\begin{aligned} \frac{dv}{dt} &= -v - 9.8 \\ v(0) &= 0 \end{aligned}$$

b. $v(t) = 9.8\left(e^{-t} - 1\right)$

c. $\lim\limits_{t \to \infty} v(t) = \lim\limits_{t \to \infty}\left(9.8\left(e^{-t} - 1\right)\right) = -9.8 \text{ m/s} \approx -21.922 \text{ mph}$

4.19. Initial-value problem: $8q' + \dfrac{1}{0.02}q = 20\sin 5t, \quad q(0) = 4$ $q(t) = \dfrac{10\sin 5t - 8\cos 5t + 172e^{-6.25t}}{41}$

Section Exercises

1. 1

3. 3

5. 1

7. 1

19. $y = 4 + \dfrac{3x^4}{4}$

21. $y = \dfrac{1}{2}e^{x^2}$

23. $y = 2e^{-1/x}$

25. $u = \sin^{-1}\left(e^{-1+t}\right)$

27. $y = -\dfrac{\sqrt{x+1}}{\sqrt{1-x}} - 1$

29. $y = C - x + x\ln x - \ln(\cos x)$

31. $y = C + \dfrac{4^x}{\ln(4)}$

33. $y = \dfrac{2}{3}\sqrt{t^2 + 16}\left(t^2 + 16\right) + C$

35. $x = \frac{2}{15}\sqrt{4+t}\left(3t^2 + 4t - 32\right) + C$

37. $y = Cx$

39. $y = 1 - \frac{t^2}{2},\ y = -\frac{t^2}{2} - 1$

41. $y = e^{-t},\ y = -e^{-t}$

43. $y = 2\left(t^2 + 5\right),\ t = 3\sqrt{5}$

45. $y = 10e^{-2t},\ t = -\frac{1}{2}\ln\left(\frac{1}{10}\right)$

47. $y = \frac{1}{4}\left(41 - e^{-4t}\right),$ never

49. Solution changes from increasing to decreasing at $y(0) = 0$

51. Solution changes from increasing to decreasing at $y(0) = 0$

53. $v(t) = -32t + a$

55. 0 ft/s

57. 4.86 meters

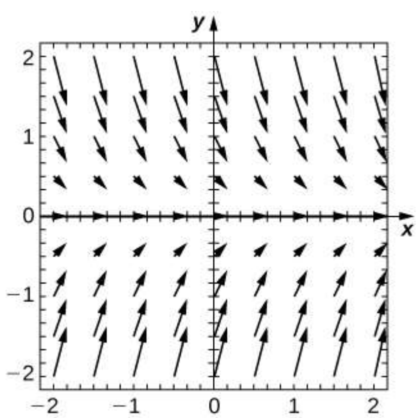
59. $x = 50t - \frac{15}{\pi^2}\cos(\pi t) + \frac{3}{\pi^2},\ 2$ hours 1 minute

61. $y = 4e^{3t}$

63. $y = 1 - 2t + t^2$

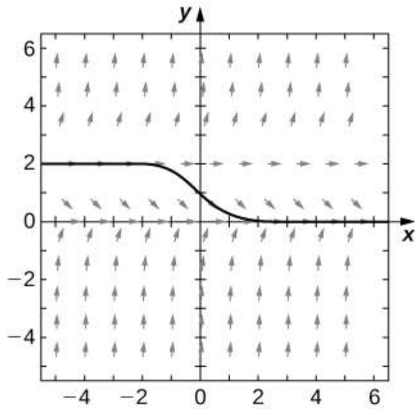
65. $y = \frac{1}{k}\left(e^{kt} - 1\right)$ and $y = x$

67.



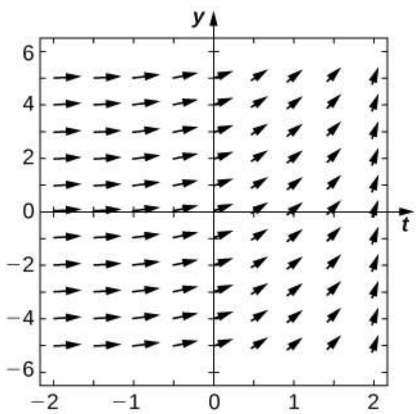
69. $y = 0$ is a stable equilibrium

71.

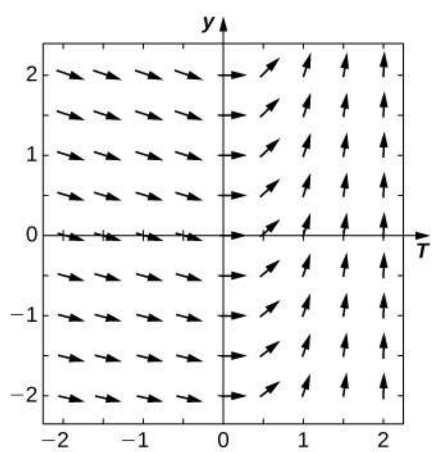


73. $y = 0$ is a stable equilibrium and $y = 2$ is unstable

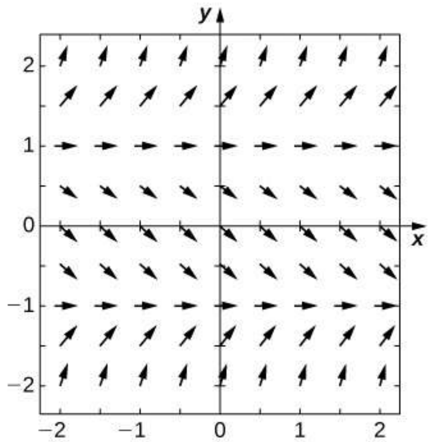
75.



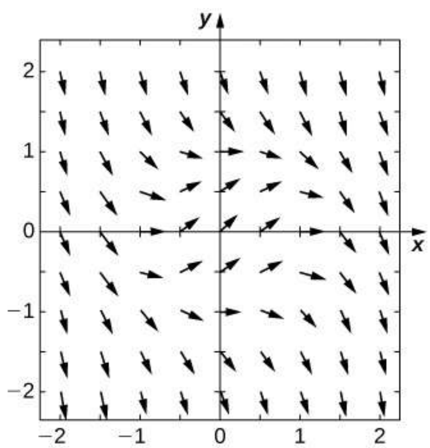
77.



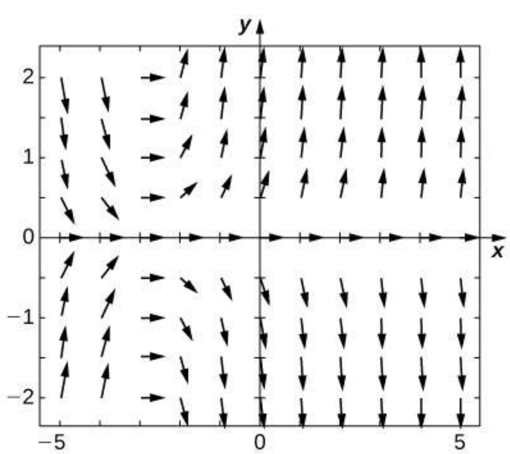
79.



81.



83.

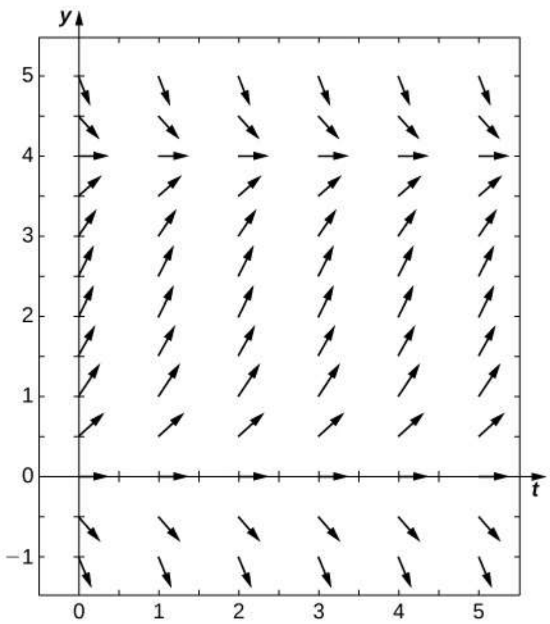


85. E
87. A
89. B
91. A
93. C
95. $2.24, \quad \text{exact: } 3$
97. $7.739364, \quad \text{exact: } 5(e-1)$
99. $-0.2535 \ \text{exact: } 0$

101. $1.345, \quad \text{exact: } \dfrac{1}{\ln(2)}$

103. $-4, \quad \text{exact: } -1/2$

105.



107. $y' = 2e^{t^2/2}$

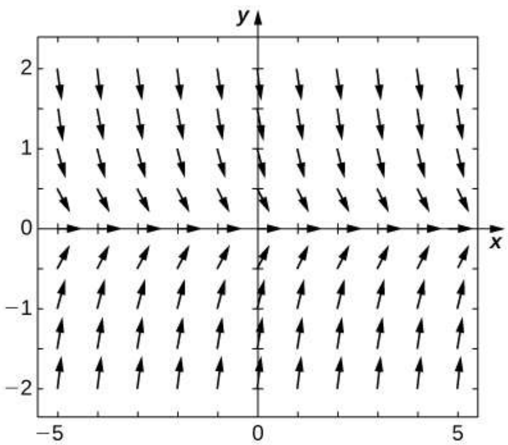
109. 2

111. 3.2756

113. $2\sqrt{e}$

Step Size	Error
$h = 1$	0.3935
$h = 10$	0.06163
$h = 100$	0.006612
$h = 1000$	0.0006661

115.



117. $4.0741e^{-10}$

119. $y = e^t - 1$

121. $y = 1 - e^{-t}$

123. $y = Cxe^{-1/x}$

125. $y = \dfrac{1}{C - x^2}$

127. $y = -\dfrac{2}{C + \ln x}$

129. $y = Ce^x(x + 1) + 1$

131. $y = \sin(\ln t + C)$

133. $y = -\ln(e^{-x})$

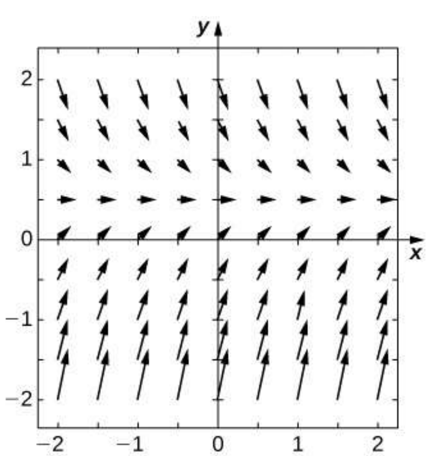
135. $y = \dfrac{1}{\sqrt{2 - e^{x^2}}}$

137. $y = \tanh^{-1}\left(\dfrac{x^2}{2}\right)$

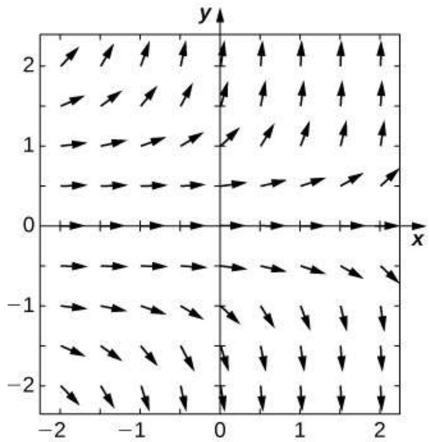
139. $x = -\sin(t - t\ln t)$

141. $y = \ln(\ln(5)) - \ln(2 - 5^x)$

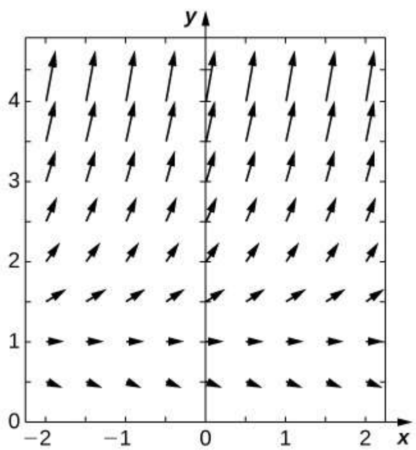
143. $y = Ce^{-2x} + \dfrac{1}{2}$



145. $y = \dfrac{1}{\sqrt{2}\sqrt{C - e^x}}$



147. $y = Ce^{-x} x^x$



149. $y = \frac{r}{d}\left(1 - e^{-dt}\right)$

151. $y(t) = 10 - 9e^{-x/50}$

153. 134.3 kilograms

155. 720 seconds

157. 12 hours 14 minutes

159. $T(t) = 20 + 50e^{-0.125t}$

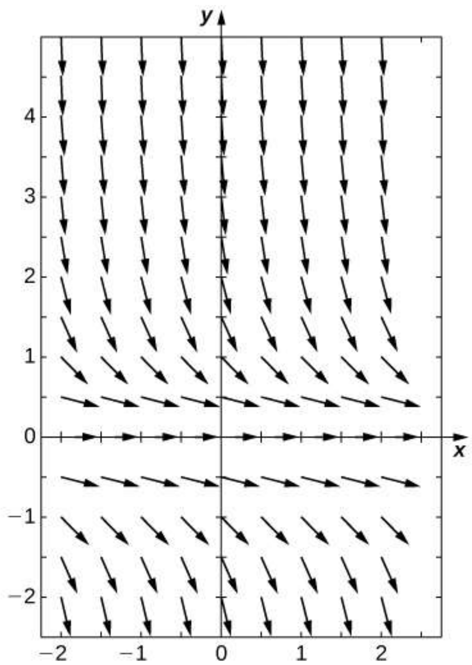
161. $T(t) = 20 + 38.5e^{-0.125t}$

163. $y = \left(c + \frac{b}{a}\right)e^{ax} - \frac{b}{a}$

165. $y(t) = cL + (I - cL)e^{-rt/L}$

167. $y = 40\left(1 - e^{-0.1t}\right)$, 40 g/cm^2

169.



$P = 0$ semi-stable

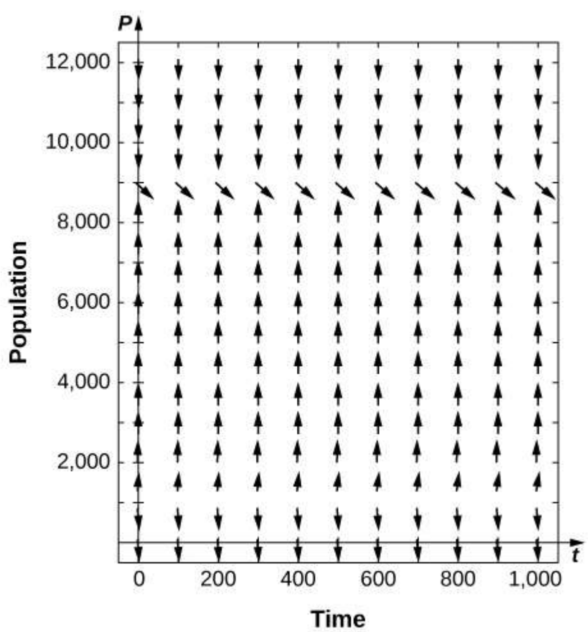
171. $P = \dfrac{10e^{10x}}{e^{10x} + 4}$

173. $P(t) = \dfrac{10000e^{0.02t}}{150 + 50e^{0.02t}}$

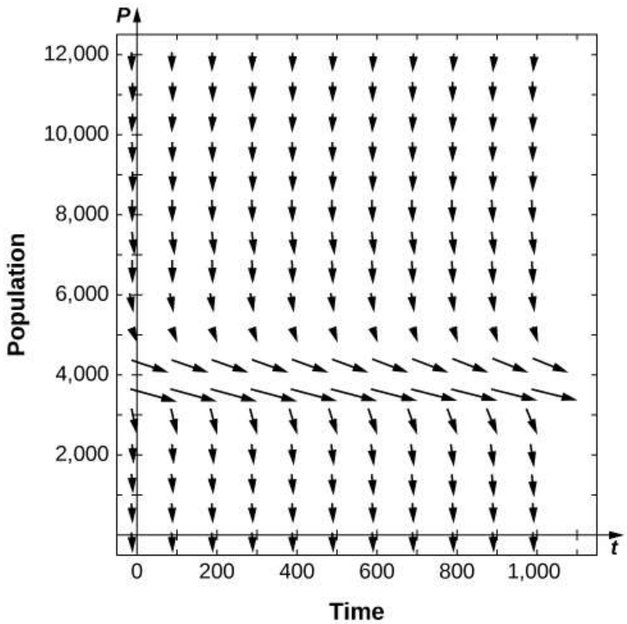
175. 69 hours 5 minutes

177. 7 years 2 months

179.

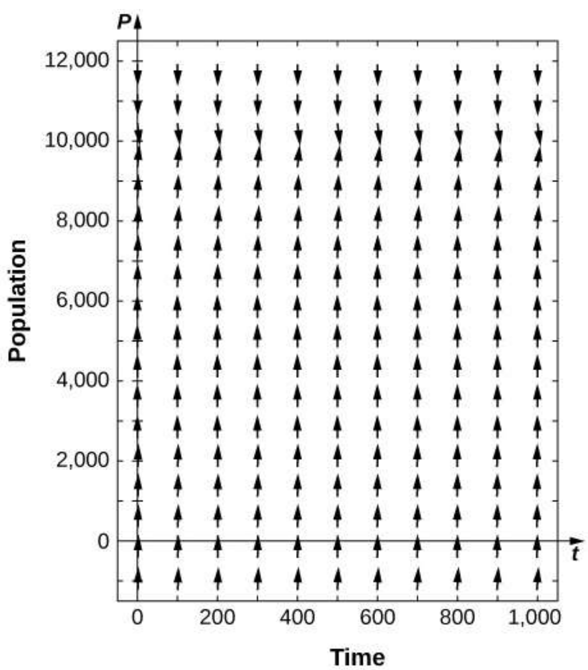


181.



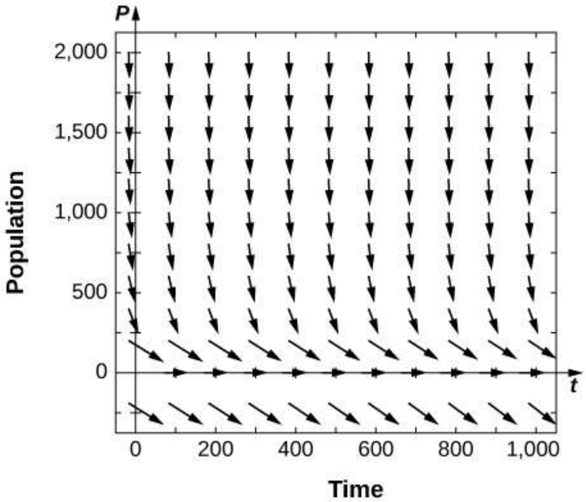
P_1 semi-stable

183.



$P_2 > 0$ stable

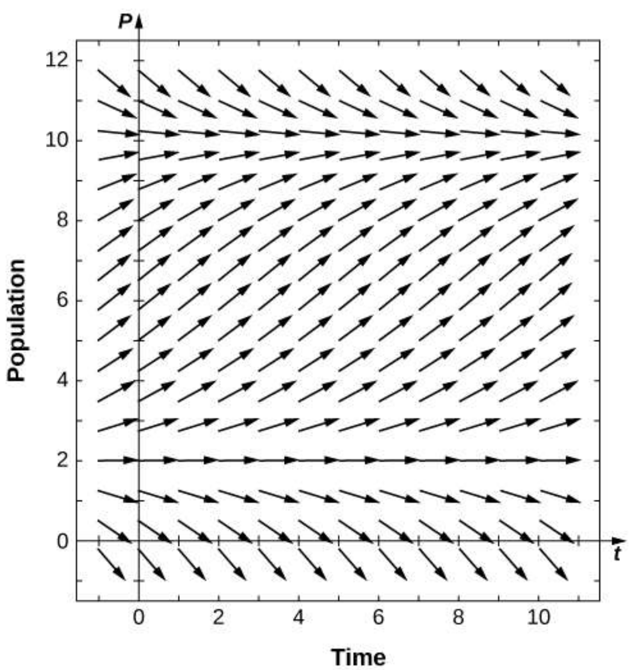
185.



$P_1 = 0$ is semi-stable

187. $y = \dfrac{-20}{4 \times 10^{-6} - 0.002e^{0.01t}}$

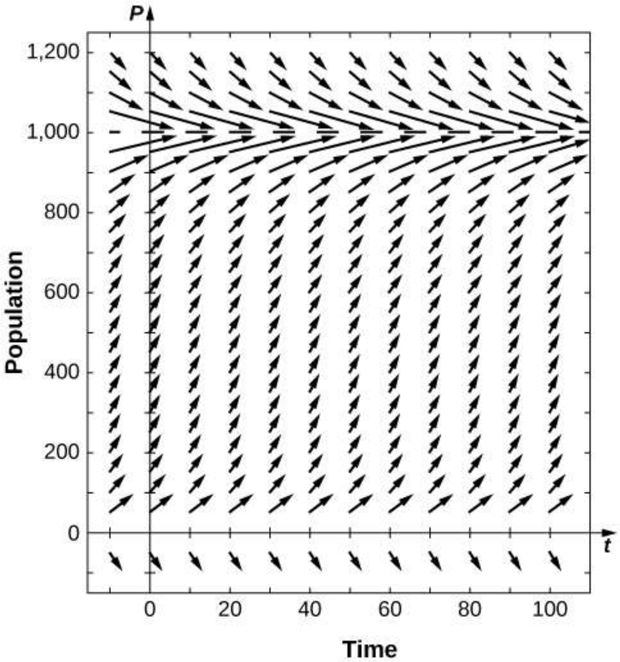
189.



191. $P(t) = \dfrac{850 + 500e^{0.009t}}{85 + 5e^{0.009t}}$

193. 13 years months

195.



197. 31.465 days

199. September 2008

201. $\frac{K+T}{2}$

203. $r = 0.0405$

205. $\alpha = 0.0081$

207. Logistic: 361, Threshold: 436, Gompertz: 309.

209. Yes

211. Yes

213. $y' - x^3 y = \sin x$

215. $y' + \frac{(3x+2)}{x} y = -e^x$

217. $\frac{dy}{dt} - yx(x+1) = 0$

219. e^x

221. $-\ln(\cosh x)$

223. $y = Ce^{3x} - \frac{2}{3}$

225. $y = Cx^3 + 6x^2$

227. $y = Ce^{x^2/2} - 3$

229. $y = C\tan\left(\frac{x}{2}\right) - 2x + 4\tan\left(\frac{x}{2}\right)\ln\left(\sin\left(\frac{x}{2}\right)\right)$

231. $y = Cx^3 - x^2$

233. $y = C(x+2)^2 + \frac{1}{2}$

235. $y = \frac{C}{\sqrt{x}} + 2\sin(3t)$

237. $y = C(x+1)^3 - x^2 - 2x - 1$

239. $y = Ce^{\sinh^{-1} x} - 2$

241. $y = x + 4e^x - 1$

243. $y = -\frac{3x}{2}\left(x^2 - 1\right)$

245. $y = 1 - e^{\tan^{-1} x}$

247. $y = (x+2)\ln\left(\frac{x+2}{2}\right)$

249. $y = 2e^{2\sqrt{x}} - 2x - 2\sqrt{x} - 1$

251. $v(t) = \frac{gm}{k}\left(1 - e^{-kt/m}\right)$

253. 40.451 seconds

255. $\sqrt{\frac{gm}{k}}$

257. $y = Ce^x - a(x+1)$

259. $y = Ce^{x^2/2} - a$

261. $y = \frac{e^{kt} - e^t}{k-1}$

Review Exercises

263. F

265. T

267. $y(x) = \frac{2^x}{\ln(2)} + x\cos^{-1} x - \sqrt{1 - x^2} + C$

269. $y(x) = \ln(C - \cos x)$

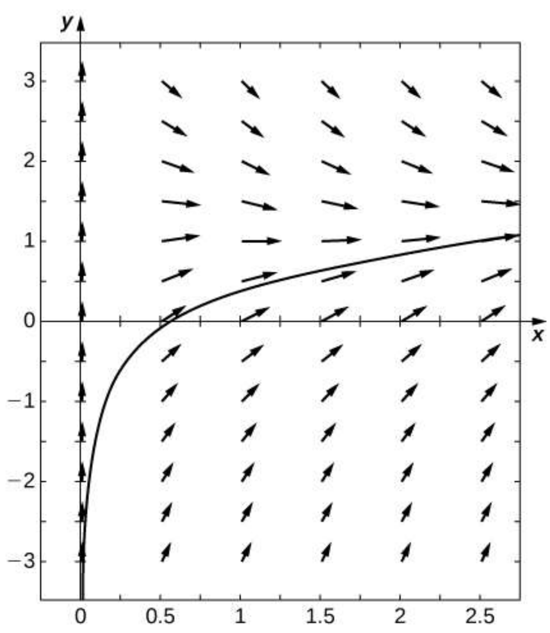
271. $y(x) = e^{e^{C+x}}$

273. $y(x) = 4 + \frac{3}{2}x^2 + 2x - \sin x$

275. $y(x) = -\frac{2}{1 + 3\left(x^2 + 2\sin x\right)}$

277. $y(x) = -2x^2 - 2x - \frac{1}{3} - \frac{2}{3}e^{3x}$

279.



$y(x) = Ce^{-x} + \ln x$

281. Euler: $0.6939,$ exact solution: $y(x) = \dfrac{3^x - e^{-2x}}{2 + \ln(3)}$

283. $\dfrac{40}{49}$ second

285. $x(t) = 5000 + \dfrac{245}{9} - \dfrac{49}{3}t - \dfrac{245}{9}e^{-5/3t},\ t = 307.8$ seconds

287. $T(t) = 200\left(1 - e^{-t/1000}\right)$

289. $P(t) = \dfrac{1600000e^{0.02t}}{9840 + 160e^{0.02t}}$

Chapter 5

Checkpoint

5.1. $a_n = \dfrac{(-1)^{n+1}}{3 + 2n}$

5.2. $a_n = 6n - 10$

5.3. The sequence converges, and its limit is $0.$

5.4. The sequence converges, and its limit is $\sqrt{2/3}.$

5.5. 2

5.6. $0.$

5.7. The series diverges because the kth partial sum $S_k > k.$

5.8. $10.$

5.9. $5/7$

5.10. $475/90$

5.11. $e - 1$

5.12. The series diverges.

5.13. The series diverges.

5.14. The series converges.

5.15. $S_5 \approx 1.09035, \quad R_5 < 0.00267$

5.16. The series converges.

5.17. The series diverges.

5.18. The series converges.

5.19. 0.04762

5.20. The series converges absolutely.

5.21. The series converges.

5.22. The series converges.

5.23. The comparison test because $2^n/(3^n + n) < 2^n/3^n$ for all positive integers $n.$ The limit comparison test could also be used.

Section Exercises

1. $a_n = 0$ if n is odd and $a_n = 2$ if n is even

3. $\{a_n\} = \{1,\ 3,\ 6,\ 10,\ 15,\ 21, \ldots\}$

5. $a_n = \dfrac{n(n+1)}{2}$

7. $a_n = 4n - 7$

9. $a_n = 3.10^{1-n} = 30.10^{-n}$

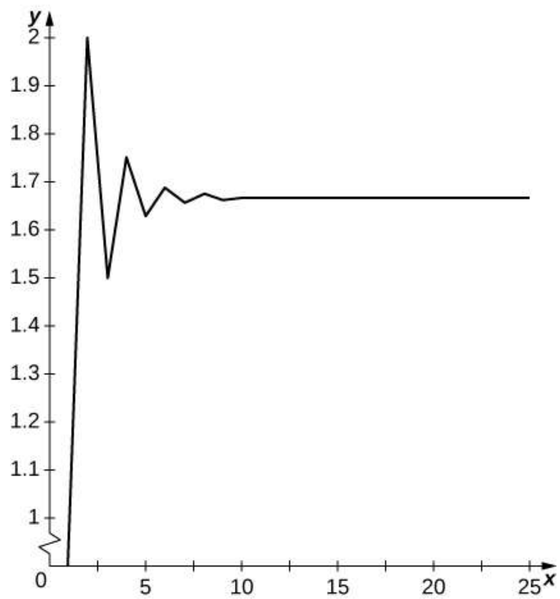
11. $a_n = 2^n - 1$

13. $a_n = \dfrac{(-1)^{n-1}}{2n - 1}$

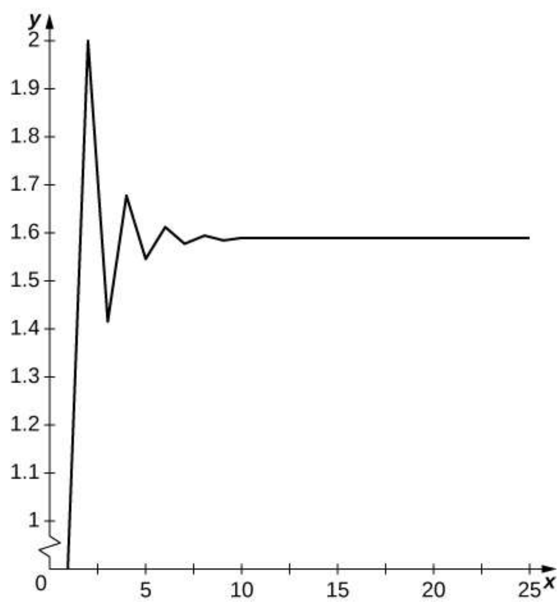
15. $f(n) = 2^n$

17. $f(n) = n!/2^{n-2}$

19. Terms oscillate above and below $5/3$ and appear to converge to $5/3$.



21. Terms oscillate above and below $y \approx 1.57...$ and appear to converge to a limit.



23. 7

25. 0

27. 0

29. 1

31. bounded, decreasing for $n \geq 1$

33. bounded, not monotone

35. bounded, decreasing

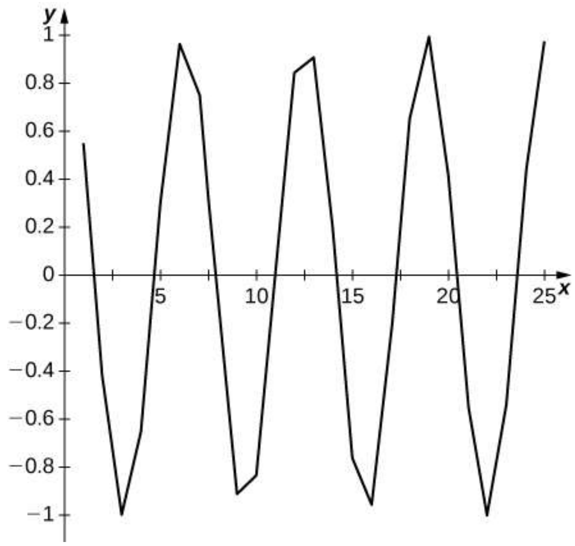
37. not monotone, not bounded

39. a_n is decreasing and bounded below by $2.$ The limit a must satisfy $a = \sqrt{2a}$ so $a = 2,$ independent of the initial value.

41. 0

43. $0 : |\sin x| \le |x|$ and $|\sin x| \le 1$ so $-\frac{1}{n} \le a_n \le \frac{1}{n}$).

45. Graph oscillates and suggests no limit.



47. $n^{1/n} \to 1$ and $2^{1/n} \to 1,$ so $a_n \to 0$

49. Since $(1 + 1/n)^n \to e,$ one has $(1 - 2/n)^n \approx (1 + k)^{-2k} \to e^{-2}$ as $k \to \infty.$

51. $2^n + 3^n \le 2 \cdot 3^n$ and $3^n / 4^n \to 0$ as $n \to \infty,$ so $a_n \to 0$ as $n \to \infty.$

53. $\frac{a_{n+1}}{a_n} = n!/(n+1)(n+2)\cdots(2n) \quad = \frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots(2n)} < 1/2^n.$ In particular, $a_{n+1}/a_n \le 1/2,$ so $a_n \to 0$ as $n \to \infty.$

55. $x_{n+1} = x_n - \left((x_n - 1)^2 - 2\right)/2(x_n - 1);$ $x = 1 + \sqrt{2},$ $x \approx 2.4142,$ $n = 5$

57. $x_{n+1} = x_n - x_n(\ln(x_n) - 1);$ $x = e,$ $x \approx 2.7183,$ $n = 5$

59. a. Without losses, the population would obey $P_n = 1.06P_{n-1}.$ The subtraction of 150 accounts for fish losses. b. After 12 months, we have $P_{12} \approx 1494.$

61. a. The student owes $\$9383$ after 12 months. b. The loan will be paid in full after 139 months or eleven and a half years.

63. $b_1 = 0,$ $x_1 = 2/3,$ $b_2 = 1,$ $x_2 = 4/3 - 1 = 1/3,$ so the pattern repeats, and $1/3 = 0.010101\ldots.$

65. For the starting values $a_1 = 1,$ $a_2 = 2, \ldots,$ $a_1 = 10,$ the corresponding bit averages calculated by the method indicated are $0.5220,$ $0.5000,$ $0.4960,$ $0.4870,$ $0.4860,$ $0.4680,$ $0.5130,$ $0.5210,$ $0.5040,$ and $0.4840.$ Here is an example of ten corresponding averages of strings of 1000 bits generated by a random number generator: $0.4880,$ $0.4870,$ $0.5150,$ $0.5490,$ $0.5130,$ $0.5180,$ $0.4860,$ $0.5030,$ $0.5050,$ $0.4980.$ There is no real pattern in either type of average. The random-number-generated averages range between 0.4860 and $0.5490,$ a range of $0.0630,$ whereas the calculated PRNG bit averages range between 0.4680 and $0.5220,$ a range of $0.0540.$

67. $\sum_{n=1}^{\infty} \frac{1}{n}$

69. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

71. $1, 3, 6, 10$

73. $1, 1, 0, 0$

75. $a_n = S_n - S_{n-1} = \frac{1}{n-1} - \frac{1}{n}.$ Series converges to $S = 1.$

77. $a_n = S_n - S_{n-1} = \sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n-1} + \sqrt{n}}.$ Series diverges because partial sums are unbounded.

79. $S_1 = 1/3$, $S_2 = 1/3 + 2/4 > 1/3 + 1/3 = 2/3$, $S_3 = 1/3 + 2/4 + 3/5 > 3 \cdot (1/3) = 1$. In general $S_k > k/3$. Series diverges.

81. $S_1 = 1/(2.3) = 1/6 = 2/3 - 1/2$,
$S_2 = 1/(2.3) + 1/(3.4) = 2/12 + 1/12 = 1/4 = 3/4 - 1/2$,
$S_3 = 1/(2.3) + 1/(3.4) + 1/(4.5) = 10/60 + 5/60 + 3/60 = 3/10 = 4/5 - 1/2$,
$S_4 = 1/(2.3) + 1/(3.4) + 1/(4.5) + 1/(5.6) = 10/60 + 5/60 + 3/60 + 2/60 = 1/3 = 5/6 - 1/2$. The pattern is $S_k = (k+1)/(k+2) - 1/2$ and the series converges to $1/2$.

83. 0

85. -3

87. diverges, $\sum_{n=1001}^{\infty} \frac{1}{n}$

89. convergent geometric series, $r = 1/10 < 1$

91. convergent geometric series, $r = \pi/e^2 < 1$

93. $\sum_{n=1}^{\infty} 5 \cdot (-1/5)^n$, converges to $-5/6$

95. $\sum_{n=1}^{\infty} 100 \cdot (1/10)^n$, converges to $100/9$

97. $x \sum_{n=0}^{\infty} (-x)^n = \sum_{n=1}^{\infty} (-1)^{n-1} x^n$

99. $\sum_{n=0}^{\infty} (-1)^n \sin^{2n}(x)$

101. $S_k = 2 - 2^{1/(k+1)} \to 1$ as $k \to \infty$.

103. $S_k = 1 - \sqrt{k+1}$ diverges

105. $\sum_{n=1}^{\infty} \ln n - \ln(n+1)$, $S_k = -\ln(k+1)$

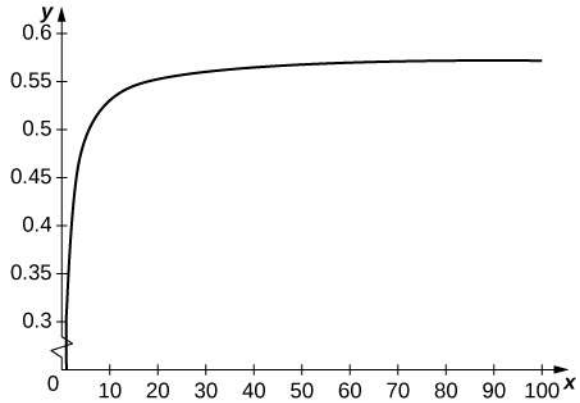
107. $a_n = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$ and $S_k = \frac{1}{\ln(2)} - \frac{1}{\ln(k+1)} \to \frac{1}{\ln(2)}$

109. $\sum_{n=1}^{\infty} a_n = f(1) - f(2)$

111. $c_0 + c_1 + c_2 + c_3 + c_4 = 0$

113. $\frac{2}{n^3 - 1} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$, $S_n = (1 - 1 + 1/3) + (1/2 - 2/3 + 1/4)$ $+ (1/3 - 2/4 + 1/5) + (1/4 - 2/5 + 1/6) + \cdots = 1/2$

115. t_k converges to $0.57721\ldots t_k$ is a sum of rectangles of height $1/k$ over the interval $[k, k+1]$ which lie above the graph of $1/x$.



117. $N=22, \quad S_N=6.1415$

119. $N=3, \quad S_N=1.559877597243667...$

121. a. The probability of any given ordered sequence of outcomes for n coin flips is $1/2^n$. b. The probability of coming up heads for the first time on the n th flip is the probability of the sequence $TT\ldots TH$ which is $1/2^n$. The probability of coming up heads for the first time on an even flip is $\sum_{n=1}^{\infty} 1/2^{2n}$ or $1/3$.

123. $5/9$

125. $E=\sum_{n=1}^{\infty} n/2^{n+1}=1,$ as can be shown using summation by parts

127. The part of the first dose after n hours is $dr^n,$ the part of the second dose is $dr^{n-N},$ and, in general, the part remaining of the mth dose is $dr^{n-mN},$ so $A(n)=\sum_{l=0}^{m} dr^{n-lN}=\sum_{l=0}^{m} dr^{k+(m-l)N}=\sum_{q=0}^{m} dr^{k+qN}=dr^k\sum_{q=0}^{m} r^{Nq}=dr^k\frac{1-r^{(m+1)N}}{1-r^N}, n=k+mN.$

129. $S_{N+1}=a_{N+1}+S_N\ge S_N$

131. Since $S>1,$ $a_2>0,$ and since $k<1,$ $S_2=1+a_2<1+(S-1)=S.$ If $S_n>S$ for some *n*, then there is a smallest *n*. For this *n*, $S>S_{n-1},$ so $S_n=S_{n-1}+k(S-S_{n-1}) \quad =kS+(1-k)S_{n-1}<S,$ a contradiction. Thus $S_n<S$ and $a_{n+1}>0$ for all *n*, so S_n is increasing and bounded by S. Let $S_*=\lim S_n.$ If $S_*<S,$ then $\delta=k(S-S_*)>0,$ but we can find n such that $S_* - S_n<\delta/2,$ which implies that $S_{n+1}=S_n+k(S-S_n) > S_* +\delta/2,$ contradicting that S_n is increasing to S_*. Thus $S_n\to S.$

133. Let $S_k=\sum_{n=1}^{k} a_n$ and $S_k\to L.$ Then S_k eventually becomes arbitrarily close to $L,$ which means that $L-S_N=\sum_{n=N+1}^{\infty} a_n$ becomes arbitrarily small as $N\to\infty.$

135. $L=\left(1+\frac{1}{2}\right)\sum_{n=1}^{\infty} 1/2^n=\frac{3}{2}.$

137. At stage one a square of area $1/9$ is removed, at stage 2 one removes 8 squares of area $1/9^2,$ at stage three one removes 8^2 squares of area $1/9^3,$ and so on. The total removed area after N stages is $\sum_{n=0}^{N-1} 8^N/9^{N+1}=\frac{1}{8}\left(1-(8/9)^N\right)/(1-8/9)\to 1$ as $N\to\infty.$ The total perimeter is $4+4\sum_{n=0} 8^N/3^{N+1}\to\infty.$

139. $\lim_{n\to\infty} a_n=0.$ Divergence test does not apply.

141. $\lim_{n\to\infty} a_n=2.$ Series diverges.

143. $\lim_{n \to \infty} a_n = \infty$ (does not exist). Series diverges.

145. $\lim_{n \to \infty} a_n = 1.$ Series diverges.

147. $\lim_{n \to \infty} a_n$ does not exist. Series diverges.

149. $\lim_{n \to \infty} a_n = 1/e^2.$ Series diverges.

151. $\lim_{n \to \infty} a_n = 0.$ Divergence test does not apply.

153. Series converges, $p > 1.$

155. Series converges, $p = 4/3 > 1.$

157. Series converges, $p = 2e - \pi > 1.$

159. Series diverges by comparison with $\int_1^\infty \frac{dx}{(x+5)^{1/3}}.$

161. Series diverges by comparison with $\int_1^\infty \frac{x}{1+x^2}dx.$

163. Series converges by comparison with $\int_1^\infty \frac{2x}{1+x^4}dx.$

165. $2^{-\ln n} = 1/n^{\ln 2}.$ Since $\ln 2 < 1,$ diverges by p -series.

167. $2^{-2\ln n} = 1/n^{2\ln 2}.$ Since $2\ln 2 - 1 < 1,$ diverges by p -series.

169. $R_{1000} \le \int_{1000}^\infty \frac{dt}{t^2} = -\frac{1}{t}\Big|_{1000}^\infty = 0.001$

171. $R_{1000} \le \int_{1000}^\infty \frac{dt}{1+t^2} = \tan^{-1}\infty - \tan^{-1}(1000) = \pi/2 - \tan^{-1}(1000) \approx 0.000999$

173. $R_N < \int_N^\infty \frac{dx}{x^2} = 1/N, N > 10^4$

175. $R_N < \int_N^\infty \frac{dx}{x^{1.01}} = 100N^{-0.01}, N > 10^{600}$

177. $R_N < \int_N^\infty \frac{dx}{1+x^2} = \pi/2 - \tan^{-1}(N), N > \tan\left(\pi/2 - 10^{-3}\right) \approx 1000$

179. $R_N < \int_N^\infty \frac{dx}{e^x} = e^{-N}, N > 5\ln(10),$ okay if $N = 12;$ $\sum_{n=1}^{12} e^{-n} = 0.581973....$ Estimate agrees with $1/(e-1)$ to five decimal places.

181. $R_N < \int_N^\infty dx/x^4 = 4/N^3, N > \left(4.10^4\right)^{1/3},$ okay if $N = 35;$ $\sum_{n=1}^{35} 1/n^4 = 1.08231\ldots.$ Estimate agrees with the sum to four decimal places.

183. $\ln(2)$

185. $T = 0.5772...$

187. The expected number of random insertions to get B to the top is $n + n/2 + n/3 + \cdots + n/(n-1).$ Then one more insertion puts B back in at random. Thus, the expected number of shuffles to randomize the deck is $n(1 + 1/2 + \cdots + 1/n).$

189. Set $b_n = a_{n+N}$ and $g(t) = f(t+N)$ such that f is decreasing on $[t, \infty).$

191. The series converges for $p > 1$ by integral test using change of variable.

193. $N = e^{e^{100}} \approx e^{10^{43}}$ terms are needed.

195. Converges by comparison with $1/n^2.$

197. Diverges by comparison with harmonic series, since $2n - 1 \ge n.$

199. $a_n = 1/(n+1)(n+2) < 1/n^2.$ Converges by comparison with p-series, $p = 2.$

201. $\sin(1/n) \le 1/n,$ so converges by comparison with p-series, $p = 2.$

203. $\sin(1/n) \le 1,$ so converges by comparison with *p*-series, $p = 3/2.$

205. Since $\sqrt{n+1} - \sqrt{n} = 1/\left(\sqrt{n+1} + \sqrt{n}\right) \le 2/\sqrt{n},$ series converges by comparison with *p*-series for $p = 1.5.$

207. Converges by limit comparison with *p*-series for $p > 1.$

209. Converges by limit comparison with *p*-series, $p = 2.$

211. Converges by limit comparison with $4^{-n}.$

213. Converges by limit comparison with $1/e^{1.1n}.$

215. Diverges by limit comparison with harmonic series.

217. Converges by limit comparison with *p*-series, $p = 3.$

219. Converges by limit comparison with *p*-series, $p = 3.$

221. Diverges by limit comparison with $1/n.$

223. Converges for $p > 1$ by comparison with a p series for slightly smaller $p.$

225. Converges for all $p > 0.$

227. Converges for all $r > 1.$ If $r > 1$ then $r^n > 4,$ say, once $n > \ln(2)/\ln(r)$ and then the series converges by limit comparison with a geometric series with ratio $1/2.$

229. The numerator is equal to 1 when n is odd and 0 when n is even, so the series can be rewritten $\sum_{n=1}^{\infty} \frac{1}{2n+1},$ which diverges by limit comparison with the harmonic series.

231. $(a-b)^2 = a^2 - 2ab + b^2$ or $a^2 + b^2 \ge 2ab,$ so convergence follows from comparison of $2a_n b_n$ with $a^2{}_n + b^2{}_n.$ Since the partial sums on the left are bounded by those on the right, the inequality holds for the infinite series.

233. $(\ln n)^{-\ln n} = e^{-\ln(n)\ln\ln(n)}.$ If n is sufficiently large, then $\ln\ln n > 2,$ so $(\ln n)^{-\ln n} < 1/n^2,$ and the series converges by comparison to a $p-$ series.

235. $a_n \to 0,$ so $a^2{}_n \le |a_n|$ for large $n.$ Convergence follows from limit comparison. $\sum 1/n^2$ converges, but $\sum 1/n$ does not, so the fact that $\sum_{n=1}^{\infty} a^2{}_n$ converges does not imply that $\sum_{n=1}^{\infty} a_n$ converges.

237. No. $\sum_{n=1}^{\infty} 1/n$ diverges. Let $b_k = 0$ unless $k = n^2$ for some $n.$ Then $\sum_k b_k/k = \sum 1/k^2$ converges.

239. $|\sin t| \le |t|,$ so the result follows from the comparison test.

241. By the comparison test, $x = \sum_{n=1}^{\infty} b_n/2^n \le \sum_{n=1}^{\infty} 1/2^n = 1.$

243. If $b_1 = 0,$ then, by comparison, $x \le \sum_{n=2}^{\infty} 1/2^n = 1/2.$

245. Yes. Keep adding 1-kg weights until the balance tips to the side with the weights. If it balances perfectly, with Robert standing on the other side, stop. Otherwise, remove one of the 1-kg weights, and add 0.1-kg weights one at a time. If it balances after adding some of these, stop. Otherwise if it tips to the weights, remove the last 0.1-kg weight. Start adding 0.01-kg weights. If it balances, stop. If it tips to the side with the weights, remove the last 0.01-kg weight that was added. Continue in this way for the 0.001-kg weights, and so on. After a finite number of steps, one has a finite series of the form $A + \sum_{n=1}^{N} s_n/10^n$ where A is the number of full kg weights and d_n is the number of $1/10^n$-kg weights that were added. If at some state this series is Robert's exact weight, the process will stop. Otherwise it represents the Nth partial sum of an infinite series that gives Robert's exact weight, and the error of this sum is at most $1/10^N.$

247. a. $10^d - 10^{d-1} < 10^d$ b. $h(d) < 9^d$ c. $m(d) = 10^{d-1} + 1$ d. Group the terms in the deleted harmonic series together by number of digits. $h(d)$ bounds the number of terms, and each term is at most $1/m(d).$

$\sum_{d=1}^{\infty} h(d)/m(d) \le \sum_{d=1}^{\infty} 9^d/(10)^{d-1} \le 90.$ One can actually use comparison to estimate the value to smaller than $80.$ The actual value is smaller than $23.$

249. Continuing the hint gives $S_N = \left(1 + 1/N^2\right)\left(1 + 1/(N-1)^2 \ldots (1 + 1/4)\right).$ Then $\ln\left(S_N\right) = \ln\left(1 + 1/N^2\right) + \ln\left(1 + 1/(N-1)^2\right) + \cdots + \ln(1 + 1/4).$ Since $\ln(1 + t)$ is bounded by a constant times $t,$ when $0 < t < 1$ one has $\ln\left(S_N\right) \le C \sum_{n=1}^{N} \frac{1}{n^2},$ which converges by comparison to the *p*-series for $p = 2.$

251. Does not converge by divergence test. Terms do not tend to zero.

253. Converges conditionally by alternating series test, since $\sqrt{n+3}/n$ is decreasing. Does not converge absolutely by comparison with *p*-series, $p = 1/2.$

255. Converges absolutely by limit comparison to $3^n/4^n,$ for example.

257. Diverges by divergence test since $\lim_{n \to \infty} |a_n| = e.$

259. Does not converge. Terms do not tend to zero.

261. $\lim_{n \to \infty} \cos^2(1/n) = 1.$ Diverges by divergence test.

263. Converges by alternating series test.

265. Converges conditionally by alternating series test. Does not converge absolutely by limit comparison with *p*-series, $p = \pi - e$

267. Diverges; terms do not tend to zero.

269. Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

271. Converges absolutely by limit comparison with *p*-series, $p = 3/2,$ after applying the hint.

273. Converges by alternating series test since $n(\tan^{-1}(n+1) - \tan^{-1} n)$ is decreasing to zero for large $n.$ Does not converge absolutely by limit comparison with harmonic series after applying hint.

275. Converges absolutely, since $a_n = \frac{1}{n} - \frac{1}{n+1}$ are terms of a telescoping series.

277. Terms do not tend to zero. Series diverges by divergence test.

279. Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

281. $\ln(N+1) > 10,$ $N + 1 > e^{10},$ $N \ge 22026;$ $S_{22026} = 0.0257\ldots$

283. $2^{N+1} > 10^6$ or $N + 1 > 6\ln(10)/\ln(2) = 19.93.$ or $N \ge 19;$ $S_{19} = 0.333333969\ldots$

285. $(N+1)^2 > 10^6$ or $N > 999;$ $S_{1000} \approx 0.822466.$

287. True. b_n need not tend to zero since if $c_n = b_n - \lim b_n,$ then $c_{2n-1} - c_{2n} = b_{2n-1} - b_{2n}.$

289. True. $b_{3n-1} - b_{3n} \ge 0,$ so convergence of $\sum b_{3n-2}$ follows from the comparison test.

291. True. If one converges, then so must the other, implying absolute convergence.

293. Yes. Take $b_n = 1$ if $a_n \ge 0$ and $b_n = 0$ if $a_n < 0.$ Then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n\,:\,a_n \ge 0} a_n$ converges. Similarly, one can show $\sum_{n\,:\,a_n < 0} a_n$ converges. Since both series converge, the series must converge absolutely.

295. Not decreasing. Does not converge absolutely.

297. Not alternating. Can be expressed as $\sum_{n=1}^{\infty} \left(\frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n}\right),$ which diverges by comparison with $\sum \frac{1}{3n-2}.$

299. Let $a^+{}_n = a_n$ if $a_n \ge 0$ and $a^+{}_n = 0$ if $a_n < 0.$ Then $a^+{}_n \le |a_n|$ for all n so the sequence of partial sums of $a^+{}_n$ is increasing and bounded above by the sequence of partial sums of $|a_n|,$ which converges; hence, $\sum_{n=1}^{\infty} a^+{}_n$ converges.

301. For $N = 5$ one has $\left|R_N\right| b_6 = \theta^{10}/10!.$ When $\theta = 1,$ $R_5 \le 1/10! \approx 2.75 \times 10^{-7}.$ When $\theta = \pi/6,$ $R_5 \le (\pi/6)^{10}/10! \approx 4.26 \times 10^{-10}.$ When $\theta = \pi,$ $R_5 \le \pi^{10}/10! = 0.0258.$

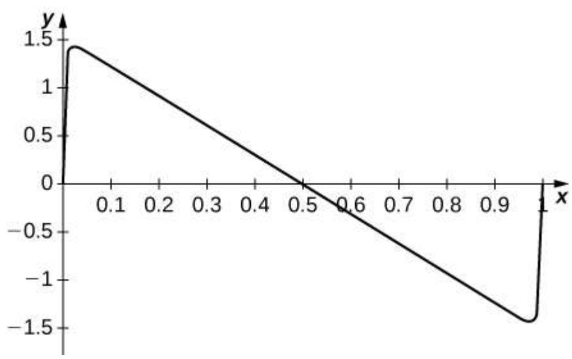
303. Let $b_n = 1/(2n-2)!$. Then $R_N \le 1/(2N)! < 0.00001$ when $(2N)! > 10^5$ or $N = 5$ and $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} = 0.540325\ldots,$ whereas $\cos 1 = 0.5403023\ldots$

305. Let $T = \sum \frac{1}{n^2}.$ Then $T - S = \frac{1}{2}T,$ so $S = T/2.$ $\sqrt{6 \times \sum_{n=1}^{1000} 1/n^2} = 3.140638\ldots;$ $\sqrt{12 \times \sum_{n=1}^{1000} (-1)^{n-1}/n^2} = 3.141591\ldots;$ $\pi = 3.141592\ldots.$ The alternating series is more accurate for 1000 terms.

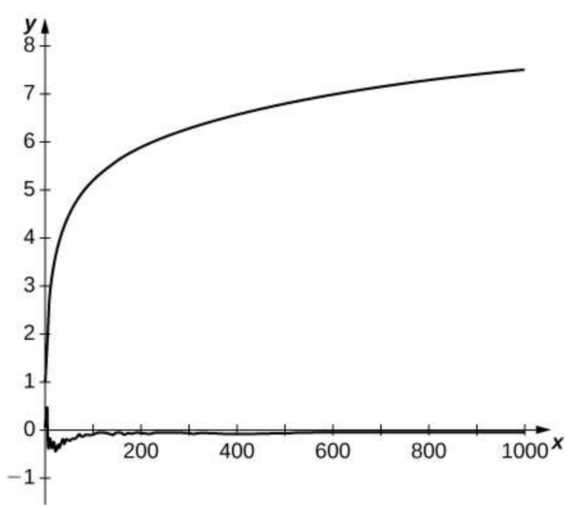
307. $N = 6, \quad S_N = 0.9068$

309. $\ln(2).$ The $3n\text{th}$ partial sum is the same as that for the alternating harmonic series.

311. The series jumps rapidly near the endpoints. For x away from the endpoints, the graph looks like $\pi(1/2 - x).$



313. Here is a typical result. The top curve consists of partial sums of the harmonic series. The bottom curve plots partial sums of a random harmonic series.



315. By the alternating series test, $|S_n - S| \le b_{n+1},$ so one needs 10^4 terms of the alternating harmonic series to estimate $\ln(2)$ to within $0.0001.$ The first 10 partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ are (up to four decimals) $0.5000, 0.6250, 0.6667, 0.6823, 0.6885, 0.6911, 0.6923, 0.6928, 0.6930, 0.6931$ and the tenth partial sum is within 0.0001 of $\ln(2) = 0.6931\ldots.$

317. $a_{n+1}/a_n \to 0.$ Converges.

319. $\frac{a_{n+1}}{a_n} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \to 1/2 < 1.$ Converges.

321. $\frac{a_{n+1}}{a_n} \to 1/27 < 1.$ Converges.

323. $\frac{a_{n+1}}{a_n} \to 4/e^2 < 1.$ Converges.

325. $\frac{a_{n+1}}{a_n} \to 1.$ Ratio test is inconclusive.

327. $\frac{a_n}{a_{n+1}} \to 1/e^2.$ Converges.

329. $(a_k)^{1/k} \to 2 > 1.$ Diverges.

331. $(a_n)^{1/n} \to 1/2 < 1.$ Converges.

333. $(a_k)^{1/k} \to 1/e < 1.$ Converges.

335. $a_n^{1/n} = \frac{1}{e} + \frac{1}{n} \to \frac{1}{e} < 1.$ Converges.

337. $a_n^{1/n} = \frac{(\ln(1 + \ln n))}{(\ln n)} \to 0$ by L'Hôpital's rule. Converges.

339. $\frac{a_{k+1}}{a_k} = \frac{1}{2k+1} \to 0.$ Converges by ratio test.

341. $(a_n)^{1/n} \to 1/e.$ Converges by root test.

343. $a_k^{1/k} \to \ln(3) > 1.$ Diverges by root test.

345. $\frac{a_{n+1}}{a_n} = \frac{3^{2n+1}}{2^{3n^2 + 3n + 1}} \to 0.$ Converge.

347. Converges by root test and limit comparison test since $x_n \to \sqrt{2}.$

349. Converges absolutely by limit comparison with $p-\text{series}, \quad p = 2.$

351. $\lim_{n \to \infty} a_n = 1/e^2 \neq 0.$ Series diverges.

353. Terms do not tend to zero: $a_k \geq 1/2,$ since $\sin^2 x \leq 1.$

355. $a_n = \frac{2}{(n+1)(n+2)},$ which converges by comparison with $p-\text{series}$ for $p = 2.$

357. $a_k = \frac{2^k 1 \cdot 2 \cdots k}{(2k+1)(2k+2)\cdots 3k} \leq (2/3)^k$ converges by comparison with geometric series.

359. $a_k \approx e^{-\ln k^2} = 1/k^2.$ Series converges by limit comparison with $p-\text{series}, \quad p = 2.$

361. If $b_k = c^{1-k}/(c-1)$ and $a_k = k,$ then $b_{k+1} - b_k = -c^{-k}$ and $\sum_{n=1}^{\infty} \frac{k}{c^k} = a_1 b_1 + \frac{1}{c-1} \sum_{k=1}^{\infty} c^{-k} = \frac{c}{(c-1)^2}.$

363. $6 + 4 + 1 = 11$

365. $|x| \leq 1$

367. $|x| < \infty$

369. All real numbers p by the ratio test.

371. $r < 1/p$

373. $0 < r < 1.$ Note that the ratio and root tests are inconclusive. Using the hint, there are $2k$ terms $r^{\sqrt{n}}$ for $k^2 \leq n < (k+1)^2,$ and for $r < 1$ each term is at least $r^k.$ Thus, $\sum_{n=1}^{\infty} r^{\sqrt{n}} = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2 - 1} r^{\sqrt{n}} \geq \sum_{k=1}^{\infty} 2kr^k,$ which converges by the ratio test for $r < 1.$ For $r \geq 1$ the series diverges by the divergence test.

375. One has $a_1 = 1, \quad a_2 = a_3 = 1/2, \ldots a_{2n} = a_{2n+1} = 1/2^n.$ The ratio test does not apply because $a_{n+1}/a_n = 1$ if n is even. However, $a_{n+2}/a_n = 1/2,$ so the series converges according to the previous exercise. Of course, the series is just a duplicated geometric series.

377. $a_{2n}/a_n = \frac{1}{2} \cdot \frac{n+1}{n+1+x} \frac{n+2}{n+2+x} \cdots \frac{2n}{2n+x}.$ The inverse of the $k\text{th}$ factor is $(n+k+x)/(n+k) > 1 + x/(2n)$ so the product is less than $(1 + x/(2n))^{-n} \approx e^{-x/2}$. Thus for $x > 0$, $\frac{a_{2n}}{a_n} \leq \frac{1}{2}e^{-x/2}$. The series converges for $x > 0$.

Review Exercises

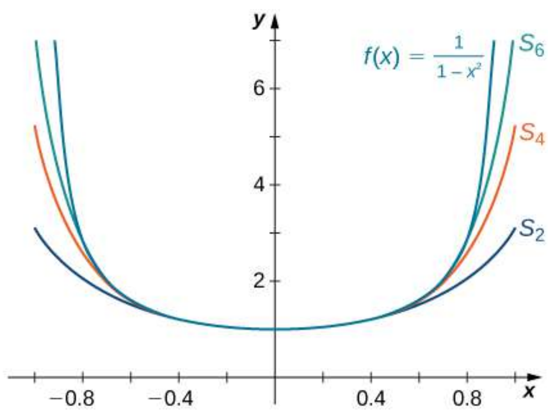
379. false
381. true
383. unbounded, not monotone, divergent
385. bounded, monotone, convergent, 0
387. unbounded, not monotone, divergent
389. diverges
391. converges
393. converges, but not absolutely
395. converges absolutely
397. converges absolutely
399. $\frac{1}{2}$
401. $\infty, \quad 0, \quad x_0$
403. $S_{10} \approx 383, \quad \lim_{n \to \infty} S_n = 400$

Chapter 6

Checkpoint

6.1. The interval of convergence is $[-1, 1)$. The radius of convergence is $R = 1$.

6.2.



6.3. $\sum_{n=0}^{\infty} \frac{x^{n+3}}{2^{n+1}}$ with interval of convergence $(-2, 2)$

6.4. Interval of convergence is $(-2, 2)$.

6.5. $\sum_{n=0}^{\infty} \left(-1 + \frac{1}{2^{n+1}}\right)x^n$. The interval of convergence is $(-1, 1)$.

6.6. $f(x) = \frac{3}{3-x}$. The interval of convergence is $(-3, 3)$.

6.7. $1 + 2x + 3x^2 + 4x^3 + \cdots$

6.8. $\sum_{n=0}^{\infty} (n+2)(n+1)x^n$

6.9. $\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(n-1)}$

6.10.

$p_0(x) = 1$; $p_1(x) = 1 - 2(x-1)$; $p_2(x) = 1 - 2(x-1) + 3(x-1)^2$; $p_3(x) = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3$

6.11.

$p_0(x) = 1;\ p_1(x) = 1 - x;\ p_2(x) = 1 - x + x^2;\ p_3(x) = 1 - x + x^2 - x^3;\ p_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{k=0}^{n} (-1)^k x^k$

6.12.

$p_1(x) = 2 + \frac{1}{4}(x-4);\ p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2;\ p_1(6) = 2.5;\ p_2(6) = 2.4375;$

$|R_1(6)| \le 0.0625;\ |R_2(6)| \le 0.015625$

6.13. 0.96593

6.14. $\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2-x}{2}\right)^n$. The interval of convergence is $(0, 4)$.

6.15. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ By the ratio test, the interval of convergence is $(-\infty, \infty)$. Since $|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$, the series converges to $\cos x$ for all real x.

6.16. $\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

6.17. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!}$

6.18. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n$

6.19. $y = 5e^{2x}$

6.20. $y = a\left(1 - \frac{x^4}{3\cdot 4} + \frac{x^8}{3\cdot 4\cdot 7\cdot 8} - \cdots\right) + b\left(x - \frac{x^5}{4\cdot 5} + \frac{x^9}{4\cdot 5\cdot 8\cdot 9} - \cdots\right)$

6.21. $C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n(2n-2)!}$ The definite integral is approximately 0.514 to within an error of 0.01.

6.22. The estimate is approximately 0.3414. This estimate is accurate to within 0.0000094.

Section Exercises

1. True. If a series converges then its terms tend to zero.

3. False. It would imply that $a_n x^n \to 0$ for $|x| < R$. If $a_n = n^n$, then $a_n x^n = (nx)^n$ does not tend to zero for any $x \ne 0$.

5. It must converge on $(0, 6]$ and hence at: a. $x = 1;$ b. $x = 2;$ c. $x = 3;$ d. $x = 0;$ e. $x = 5.99;$ and f. $x = 0.000001$.

7. $\left|\frac{a_{n+1}2^{n+1}x^{n+1}}{a_n 2^n x^n}\right| = 2|x|\left|\frac{a_{n+1}}{a_n}\right| \to 2|x|$ so $R = \frac{1}{2}$

9. $\left|\frac{a_{n+1}\left(\frac{\pi}{e}\right)^{n+1}x^{n+1}}{a_n\left(\frac{\pi}{e}\right)^n x^n}\right| = \frac{\pi|x|}{e}\left|\frac{a_{n+1}}{a_n}\right| \to \frac{\pi|x|}{e}$ so $R = \frac{e}{\pi}$

11. $\left|\frac{a_{n+1}(-1)^{n+1}x^{2n+2}}{a_n(-1)^n x^{2n}}\right| = |x^2|\left|\frac{a_{n+1}}{a_n}\right| \to |x^2|$ so $R = 1$

13. $a_n = \frac{2^n}{n}$ so $\frac{a_{n+1}x}{a_n} \to 2x$. so $R = \frac{1}{2}$. When $x = \frac{1}{2}$ the series is harmonic and diverges. When $x = -\frac{1}{2}$ the series is alternating harmonic and converges. The interval of convergence is $I = \left[-\frac{1}{2}, \frac{1}{2}\right)$.

15. $a_n = \frac{n}{2^n}$ so $\frac{a_{n+1}x}{a_n} \to \frac{x}{2}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

17. $a_n = \frac{n^2}{2^n}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

19. $a_k = \frac{\pi^k}{k^\pi}$ so $R = \frac{1}{\pi}$. When $x = \pm\frac{1}{\pi}$ the series is an absolutely convergent p-series. The interval of convergence is

$I = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$.

21. $a_n = \frac{10^n}{n!}, \frac{a_{n+1}x}{a_n} = \frac{10x}{n+1} \to 0 < 1$ so the series converges for all x by the ratio test and $I = (-\infty, \infty)$.

23. $a_k = \frac{(k!)^2}{(2k)!}$ so $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4}$ so $R = 4$

25. $a_k = \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ so $\frac{a_{k+1}}{a_k} = \frac{k+1}{2k+1} \to \frac{1}{2}$ so $R = 2$

27. $a_n = \frac{1}{\binom{2n}{n}}$ so $\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{(2n+2)!} \frac{2n!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4}$ so $R = 4$

29. $\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27}$ so $R = 27$

31. $a_n = \frac{n!}{n^n}$ so $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$ so $R = e$

33. $f(x) = \sum_{n=0}^{\infty} (1-x)^n$ on $I = (0, 2)$

35. $\sum_{n=0}^{\infty} x^{2n+1}$ on $I = (-1, 1)$

37. $\sum_{n=0}^{\infty} (-1)^n x^{2n+2}$ on $I = (-1, 1)$

39. $\sum_{n=0}^{\infty} 2^n x^n$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$

41. $\sum_{n=0}^{\infty} 4^n x^{2n+2}$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$

43. $\left|a_n x^n\right|^{1/n} = |a_n|^{1/n}|x| \to |x|r$ as $n \to \infty$ and $|x|r < 1$ when $|x| < \frac{1}{r}$. Therefore, $\sum_{n=1}^{\infty} a_n x^n$ converges when $|x| < \frac{1}{r}$ by the nth root test.

45. $a_k = \left(\frac{k-1}{2k+3}\right)^k$ so $(a_k)^{1/k} \to \frac{1}{2} < 1$ so $R = 2$

47. $a_n = \left(n^{1/n} - 1\right)^n$ so $(a_n)^{1/n} \to 0$ so $R = \infty$

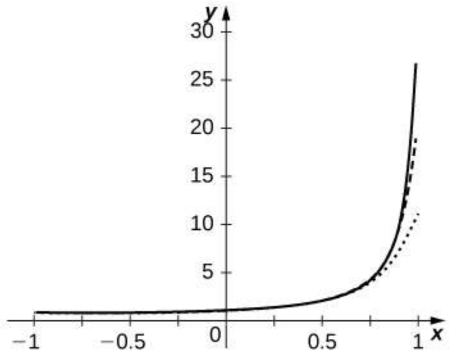
49. We can rewrite $p(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ and $p(x) = p(-x)$ since $x^{2n+1} = -(-x)^{2n+1}$.

51. If $x \in [0, 1]$, then $y = 2x - 1 \in [-1, 1]$ so $p(2x-1) = p(y) = \sum_{n=0}^{\infty} a_n y^n$ converges.

53. Converges on $(-1, 1)$ by the ratio test

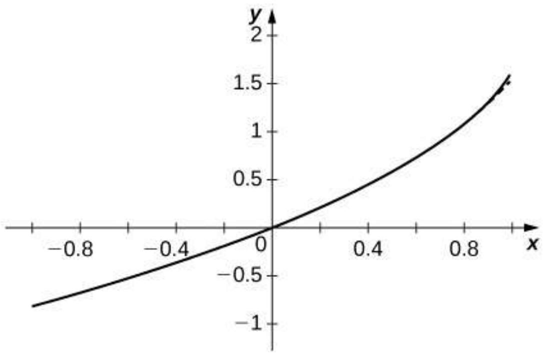
55. Consider the series $\sum b_k x^k$ where $b_k = a_k$ if $k = n^2$ and $b_k = 0$ otherwise. Then $b_k \le a_k$ and so the series converges on $(-1, 1)$ by the comparison test.

57.



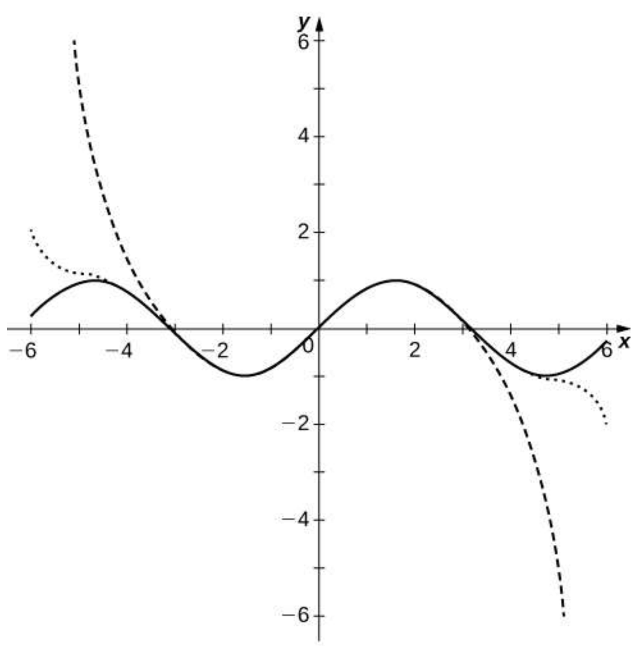
The approximation is more accurate near $x = -1$. The partial sums follow $\frac{1}{1-x}$ more closely as N increases but are never accurate near $x = 1$ since the series diverges there.

59.



The approximation appears to stabilize quickly near both $x = \pm 1$.

61.



The polynomial curves have roots close to those of $\sin x$ up to their degree and then the polynomials diverge from $\sin x$.

63. $\frac{1}{2}(f(x)+g(x))=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ and $\frac{1}{2}(f(x)-g(x))=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$.

65.

$$\frac{4}{(x-3)(x+1)}=\frac{1}{x-3}-\frac{1}{x+1}=-\frac{1}{3\left(1-\frac{x}{3}\right)}-\frac{1}{1-(-x)}=-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n-\sum_{n=0}^{\infty}(-1)^n x^n=\sum_{n=0}^{\infty}\left((-1)^{n+1}-\frac{1}{3^{n+1}}\right)x^n$$

67. $\frac{5}{\left(x^2+4\right)\left(x^2-1\right)}=\frac{1}{x^2-1}-\frac{1}{4}\frac{1}{1+\left(\frac{x}{2}\right)^2}=-\sum_{n=0}^{\infty}x^{2n}-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\left((-1)+(-1)^{n+1}\frac{1}{2^{n+2}}\right)x^{2n}$

69. $\frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{x^n}=\frac{1}{x}\frac{1}{1-\frac{1}{x}}=\frac{1}{x-1}$

71. $\frac{1}{x-3}\frac{1}{1-\frac{1}{(x-3)^2}}=\frac{x-3}{(x-3)^2-1}$

73. $P=P_1+\cdots+P_{20}$ where $P_k=10{,}000\frac{1}{(1+r)^k}$. Then $P=10{,}000\sum_{k=1}^{20}\frac{1}{(1+r)^k}=10{,}000\frac{1-(1+r)^{-20}}{r}$. When $r=0.03$, $P\approx 10{,}000\times 14.8775=148{,}775$. When $r=0.05$, $P\approx 10{,}000\times 12.4622=124{,}622$. When $r=0.07$, $P\approx 105{,}940$.

75. In general, $P=\frac{C\left(1-(1+r)^{-N}\right)}{r}$ for N years of payouts, or $C=\frac{Pr}{1-(1+r)^{-N}}$. For $N=20$ and $P=100{,}000$, one has $C=6721.57$ when $r=0.03$; $C=8024.26$ when $r=0.05$; and $C\approx 9439.29$ when $r=0.07$.

77. In general, $P=\frac{C}{r}$. Thus, $r=\frac{C}{P}=5\times\frac{10^4}{10^6}=0.05$.

79. $\left(x+x^2-x^3\right)\left(1+x^3+x^6+\cdots\right)=\frac{x+x^2-x^3}{1-x^3}$

81. $\left(x-x^2-x^3\right)\left(1+x^3+x^6+\cdots\right)=\frac{x-x^2-x^3}{1-x^3}$

83. $a_n=2,\ b_n=n$ so $c_n=\sum_{k=0}^{n}b_k a_{n-k}=2\sum_{k=0}^{n}k=(n)(n+1)$ and $f(x)g(x)=\sum_{n=1}^{\infty}n(n+1)x^n$

85. $a_n=b_n=2^{-n}$ so $c_n=\sum_{k=1}^{n}b_k a_{n-k}=2^{-n}\sum_{k=1}^{n}1=\frac{n}{2^n}$ and $f(x)g(x)=\sum_{n=1}^{\infty}n\left(\frac{x}{2}\right)^n$

87. The derivative of f is $-\frac{1}{(1+x)^2}=-\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$.

89. The indefinite integral of f is $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

91. $f(x)=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$; $f'\left(\frac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{n}{2^{n-1}}=\frac{d}{dx}(1-x)^{-1}\Big|_{x=1/2}=\frac{1}{(1-x)^2}\Big|_{x=1/2}=4$ so $\sum_{n=1}^{\infty}\frac{n}{2^n}=2$.

93. $f(x)=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$; $f''\left(\frac{1}{2}\right)=\sum_{n=2}^{\infty}\frac{n(n-1)}{2^{n-2}}=\frac{d^2}{dx^2}(1-x)^{-1}\Big|_{x=1/2}=\frac{2}{(1-x)^3}\Big|_{x=1/2}=16$ so $\sum_{n=2}^{\infty}\frac{n(n-1)}{2^n}=4$.

95. $\int\sum(1-x)^n dx=\int\sum(-1)^n(x-1)^n dx=\sum\frac{(-1)^n(x-1)^{n+1}}{n+1}$

97. $-\int_{t=0}^{x^2}\frac{1}{1-t}dt=-\sum_{n=0}^{\infty}\int_0^{x^2}t^n\,dx-\sum_{n=0}^{\infty}\frac{x^{2(n+1)}}{n+1}=-\sum_{n=1}^{\infty}\frac{x^{2n}}{n}$

99. $\int_0^{x^2} \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \int_0^{x^2} t^{2n}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}\Big|_{t=0}^{x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$

101. Term-by-term integration gives $\int_0^x \ln t\,dt = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n} - \frac{1}{n+1}\right)(x-1)^{n+1} = (x-1)\ln x + \sum_{n=2}^{\infty} (-1)^n \frac{(x-1)^n}{n} = x\ln x - x.$

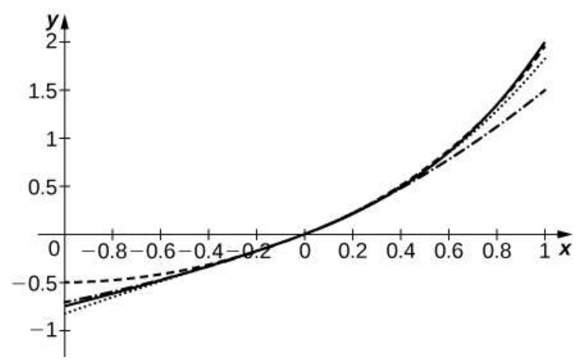
103. We have $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ so $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$ Thus, $\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=1}^{\infty} \left(1+(-1)^{n-1}\right)\frac{x^n}{n} = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}.$ When $x = \frac{1}{3}$ we obtain $\ln(2) = 2\sum_{n=1}^{\infty} \frac{1}{3^{2n-1}(2n-1)}.$ We have $2\sum_{n=1}^{3} \frac{1}{3^{2n-1}(2n-1)} = 0.69300\ldots,$ while $2\sum_{n=1}^{4} \frac{1}{3^{2n-1}(2n-1)} = 0.69313\ldots$ and $\ln(2) = 0.69314\ldots;$ therefore, $N = 4.$

105. $\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x)$ so $\sum_{k=1}^{\infty} \frac{x^{3k}}{6k} = -\frac{1}{6}\ln\left(1-x^3\right).$ The radius of convergence is equal to 1 by the ratio test.

107. If $y = 2^{-x},$ then $\sum_{k=1}^{\infty} y^k = \frac{y}{1-y} = \frac{2^{-x}}{1-2^{-x}} = \frac{1}{2^x - 1}.$ If $a_k = 2^{-kx},$ then $\frac{a_{k+1}}{a_k} = 2^{-x} < 1$ when $x > 0.$ So the series converges for all $x > 0.$

109. Answers will vary.

111.



The solid curve is S_5. The dashed curve is S_2, dotted is S_3, and dash-dotted is S_4

113. When $x = -\frac{1}{2}, -\ln(2) = \ln\left(\frac{1}{2}\right) = -\sum_{n=1}^{\infty} \frac{1}{n2^n}.$ Since $\sum_{n=11}^{\infty} \frac{1}{n2^n} < \sum_{n=11}^{\infty} \frac{1}{2^n} = \frac{1}{2^{10}},$ one has $\sum_{n=1}^{10} \frac{1}{n2^n} = 0.69306\ldots$ whereas $\ln(2) = 0.69314\ldots;$ therefore, $N = 10.$

115. $6S_N\left(\frac{1}{\sqrt{3}}\right) = 2\sqrt{3}\sum_{n=0}^{N} (-1)^n \frac{1}{3^n(2n+1)}.$ One has $\pi - 6S_4\left(\frac{1}{\sqrt{3}}\right) = 0.00101\ldots$ and $\pi - 6S_5\left(\frac{1}{\sqrt{3}}\right) = 0.00028\ldots$ so $N = 5$ is the smallest partial sum with accuracy to within 0.001. Also, $\pi - 6S_7\left(\frac{1}{\sqrt{3}}\right) = 0.00002\ldots$ while $\pi - 6S_8\left(\frac{1}{\sqrt{3}}\right) = -0.000007\ldots$ so $N = 8$ is the smallest N to give accuracy to within 0.00001.

117. $f(-1) = 1;\ f'(-1) = -1;\ f''(-1) = 2;\ f(x) = 1 - (x+1) + (x+1)^2$

119. $f'(x) = 2\cos(2x);\ f''(x) = -4\sin(2x);\ p_2(x) = -2\left(x - \frac{\pi}{2}\right)$

121. $f'(x) = \frac{1}{x};\ f''(x) = -\frac{1}{x^2};\ p_2(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2$

123. $p_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$

125. $\frac{d^2}{dx^2}x^{1/3} = -\frac{2}{9x^{5/3}} \geq -0.00092\ldots$ when $x \geq 28$ so the remainder estimate applies to the linear approximation $x^{1/3} \approx p_1(27) = 3 + \frac{x-27}{27}$, which gives $(28)^{1/3} \approx 3 + \frac{1}{27} = 3.\overline{037}$, while $(28)^{1/3} \approx 3.03658$.

127. Using the estimate $\frac{2^{10}}{10!} < 0.000283$ we can use the Taylor expansion of order 9 to estimate e^x at $x = 2$. as $e^2 \approx p_9(2) = 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} + \cdots + \frac{2^9}{9!} = 7.3887\ldots$ whereas $e^2 \approx 7.3891$.

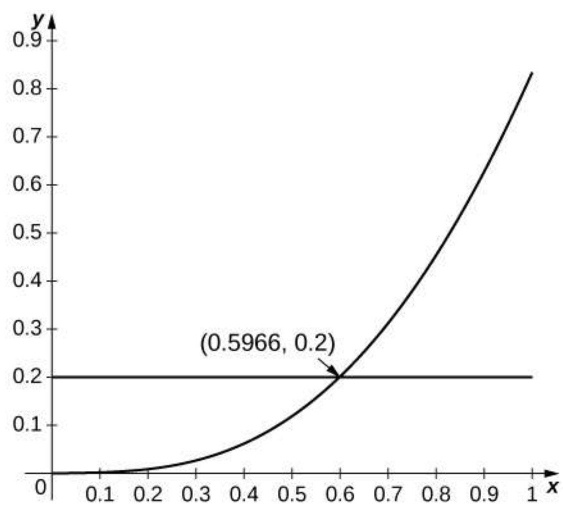
129. Since $\frac{d^n}{dx^n}(\ln x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$, $R_{1000} \approx \frac{1}{1001}$. One has $p_{1000}(1) = \sum_{n=1}^{1000}\frac{(-1)^{n-1}}{n} \approx 0.6936$ whereas $\ln(2) \approx 0.6931\cdots$.

131. $\int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \frac{x^{12}}{720}\right)dx \quad = 1 - \frac{1^3}{3} + \frac{1^5}{10} - \frac{1^7}{42} + \frac{1^9}{9 \cdot 24} - \frac{1^{11}}{120 \cdot 11} + \frac{1^{13}}{720 \cdot 13} \approx 0.74683$ whereas $\int_0^1 e^{-x^2}dx \approx 0.74682$.

133. Since $f^{(n+1)}(z)$ is $\sin z$ or $\cos z$, we have $M = 1$. Since $|x - 0| \leq \frac{\pi}{2}$, we seek the smallest n such that $\frac{\pi^{n+1}}{2^{n+1}(n+1)!} \leq 0.001$. The smallest such value is $n = 7$. The remainder estimate is $R_7 \leq 0.00092$.

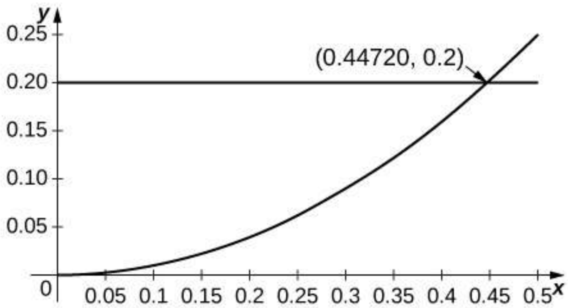
135. Since $f^{(n+1)}(z) = \pm e^{-z}$ one has $M = e^3$. Since $|x - 0| \leq 3$, one seeks the smallest n such that $\frac{3^{n+1}e^3}{(n+1)!} \leq 0.001$. The smallest such value is $n = 14$. The remainder estimate is $R_{14} \leq 0.000220$.

137.



Since $\sin x$ is increasing for small x and since $\sin'' x = -\sin x$, the estimate applies whenever $R^2 \sin(R) \leq 0.2$, which applies up to $R = 0.596$.

139.



Since the second derivative of $\cos x$ is $-\cos x$ and since $\cos x$ is decreasing away from $x = 0,$ the estimate applies when $R^2 \cos R \leq 0.2$ or $R \leq 0.447.$

141. $(x+1)^3 - 2(x+1)^2 + 2(x+1)$

143. Values of derivatives are the same as for $x = 0$ so $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-2\pi)^{2n}}{(2n)!}$

145. $\cos\left(\frac{\pi}{2}\right) = 0,\ -\sin\left(\frac{\pi}{2}\right) = -1$ so $\cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!},$ which is also $-\cos\left(x - \frac{\pi}{2}\right).$

147. The derivatives are $f^{(n)}(1) = e$ so $e^x = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}.$

149. $\frac{1}{(x-1)^3} = -\left(\frac{1}{2}\right)\frac{d^2}{dx^2}\frac{1}{1-x} = -\sum_{n=0}^{\infty} \left(\frac{(n+2)(n+1)x^n}{2}\right)$

151. $2 - x = 1 - (x-1)$

153. $((x-1)-1)^2 = (x-1)^2 - 2(x-1) + 1$

155. $\frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$

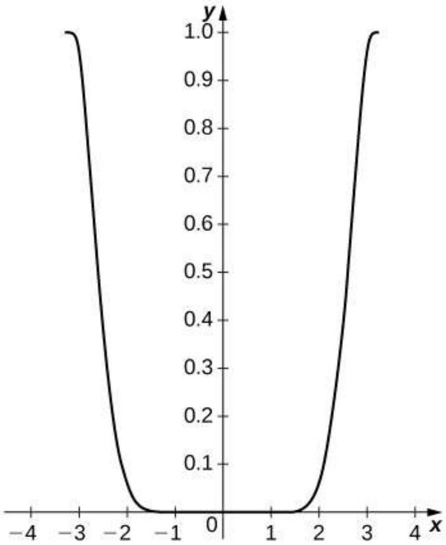
157. $x \sum_{n=0}^{\infty} 2^n (1-x)^{2n} = \sum_{n=0}^{\infty} 2^n (x-1)^{2n+1} + \sum_{n=0}^{\infty} 2^n (x-1)^{2n}$

159. $e^{2x} = e^{2(x-1)+2} = e^2 \sum_{n=0}^{\infty} \frac{2^n (x-1)^n}{n!}$

161. $x = e^2;\ S_{10} = \frac{34{,}913}{4725} \approx 7.3889947$

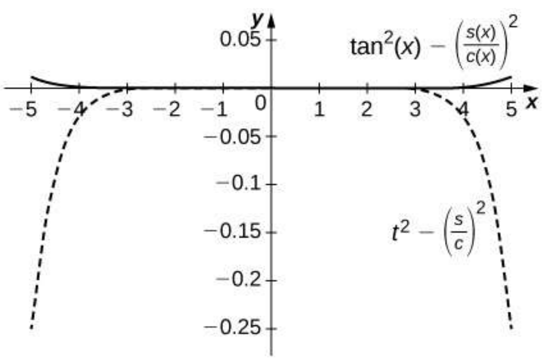
163. $\sin(2\pi) = 0;\ S_{10} = 8.27 \times 10^{-5}$

165.



The difference is small on the interior of the interval but approaches 1 near the endpoints. The remainder estimate is $|R_4| = \frac{\pi^5}{120} \approx 2.552.$

167.



The difference is on the order of 10^{-4} on $[-1,\ 1]$ while the Taylor approximation error is around 0.1 near ± 1. The top curve is a plot of $\tan^2 x - \left(\frac{S_5(x)}{C_4(x)}\right)^2$ and the lower dashed plot shows $t^2 - \left(\frac{S_5}{C_4}\right)^2$.

169. a. Answers will vary. b. The following are the x_n values after 10 iterations of Newton's method to approximation a root of $p_N(x) - 2 = 0$: for $N = 4,\ x = 0.6939...$; for $N = 5,\ x = 0.6932...$; for $N = 6,\ x = 0.69315...$; . (*Note:* $\ln(2) = 0.69314...$) c. Answers will vary.

171. $\frac{\ln\left(1 - x^2\right)}{x^2} \to -1$

173. $\frac{\cos(\sqrt{x}) - 1}{2x} \approx \frac{\left(1 - \frac{x}{2} + \frac{x^2}{4!} - \cdots\right) - 1}{2x} \to -\frac{1}{4}$

175. $\left(1 + x^2\right)^{-1/3} = \sum_{n=0}^{\infty} \binom{-\frac{1}{3}}{n} x^{2n}$

177. $(1 - 2x)^{2/3} = \sum_{n=0}^{\infty} (-1)^n 2^n \binom{\frac{2}{3}}{n} x^n$

179. $\sqrt{2+x^2} = \sum_{n=0}^{\infty} 2^{(1/2)-n}\binom{\frac{1}{2}}{n}x^{2n};\ \left(\left|x^2\right| < 2\right)$

181. $\sqrt{2x-x^2} = \sqrt{1-(x-1)^2}$ so $\sqrt{2x-x^2} = \sum_{n=0}^{\infty} (-1)^n\binom{\frac{1}{2}}{n}(x-1)^{2n}$

183. $\sqrt{x} = 2\sqrt{1+\frac{x-4}{4}}$ so $\sqrt{x} = \sum_{n=0}^{\infty} 2^{1-2n}\binom{\frac{1}{2}}{n}(x-4)^n$

185. $\sqrt{x} = \sum_{n=0}^{\infty} 3^{1-3n}\binom{\frac{1}{2}}{n}(x-9)^n$

187. $10\left(1+\frac{x}{1000}\right)^{1/3} = \sum_{n=0}^{\infty} 10^{1-3n}\binom{\frac{1}{3}}{n}x^n$. Using, for example, a fourth-degree estimate at $x=1$ gives

$$\begin{aligned}(1001)^{1/3} &\approx 10\left(1+\binom{\frac{1}{3}}{1}10^{-3}+\binom{\frac{1}{3}}{2}10^{-6}+\binom{\frac{1}{3}}{3}10^{-9}+\binom{\frac{1}{3}}{4}10^{-12}\right)\\ &= 10\left(1+\frac{1}{3.10^3}-\frac{1}{9.10^6}+\frac{5}{81.10^9}-\frac{10}{243.10^{12}}\right) = 10.00333222...\end{aligned}$$

whereas $(1001)^{1/3} = 10.00332222839093....$ Two terms would suffice for three-digit accuracy.

189. The approximation is 2.3152; the CAS value is $2.23\ldots.$

191. The approximation is $2.583\ldots$; the CAS value is $2.449\ldots.$

193.

$\sqrt{1-x^2} = 1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}+\cdots.$ Thus

$\int_{-1}^{1}\sqrt{1-x^2}dx = x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{7\cdot 16}-\frac{5x^9}{9\cdot 128}+\cdots\Big|_{-1}^{1} \approx 2-\frac{1}{3}-\frac{1}{20}-\frac{1}{56}-\frac{10}{9\cdot 128}+\text{error} = 1.590...$ whereas $\frac{\pi}{2} = 1.570...$

195. $(1+x)^{4/3} = (1+x)\left(1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\cdots\right) = 1+\frac{4x}{3}+\frac{2x^2}{9}-\frac{4x^3}{81}+\frac{5x^4}{243}+\cdots$

197. $\left(1+(x+3)^2\right)^{1/3} = 1+\frac{1}{3}(x+3)^2-\frac{1}{9}(x+3)^4+\frac{5}{81}(x+3)^6-\frac{10}{243}(x+3)^8+\cdots$

199. Twice the approximation is $1.260\ldots$ whereas $2^{1/3} = 1.2599....$

201. $f^{(99)}(0) = 0$

203. $\sum_{n=0}^{\infty} \frac{(\ln(2)x)^n}{n!}$

205. For $x > 0$, $\sin(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n\frac{x^{(2n+1)/2}}{\sqrt{x}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n\frac{x^n}{(2n+1)!}.$

207. $e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$

209. $\sin^2 x = -\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} x^{2k}}{(2k)!}$

211. $\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$

213. $\sin^{-1} x = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n}\frac{x^{2n+1}}{(2n+1)n!}$

215. $F(x) = \sum_{n=0}^{\infty} (-1)^n\frac{x^{n+1}}{(n+1)(2n)!}$

217. $F(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n^2}$

219. $x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

221. $1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$

223. $1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \cdots$

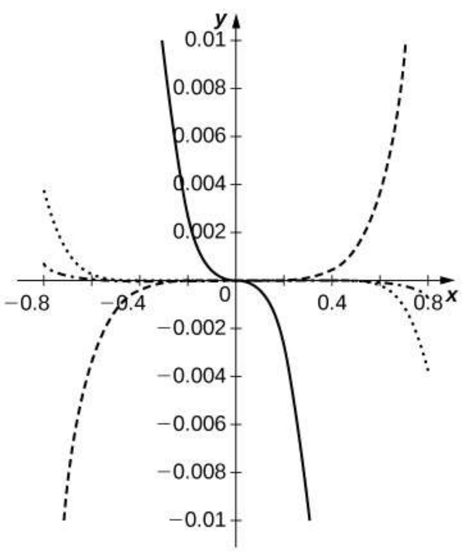
225. Using the expansion for $\tan x$ gives $1 + \frac{x}{3} + \frac{2x^2}{15}$.

227. $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ so $R = 1$ by the ratio test.

229. $\ln\left(1 + x^2\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{2n}$ so $R = 1$ by the ratio test.

231. Add series of e^x and e^{-x} term by term. Odd terms cancel and $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

233.



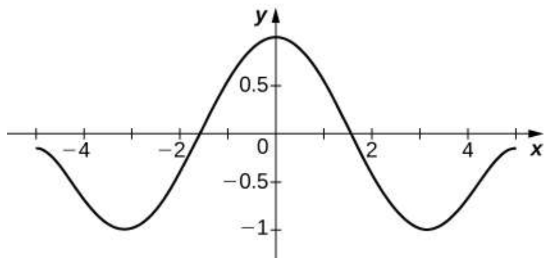
The ratio $\frac{S_n(x)}{C_n(x)}$ approximates $\tan x$ better than does $p_7(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}$ for $N \geq 3$. The dashed curves are $\frac{S_n}{C_n} - \tan$ for $n = 1,\ 2$. The dotted curve corresponds to $n = 3$, and the dash-dotted curve corresponds to $n = 4$. The solid curve is $p_7 - \tan x$.

235. By the term-by-term differentiation theorem, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ so $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} xy' = \sum_{n=1}^{\infty} n a_n x^n$, whereas $y' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ so $xy'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$.

237. The probability is $p = \frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-x^2/2} dx$ where $a = 90$ and $b = 100$, that is,

$$p = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sum_{n=0}^{5} (-1)^n \frac{x^{2n}}{2^n n!} dx = \frac{2}{\sqrt{2\pi}} \sum_{n=0}^{5} (-1)^n \frac{1}{(2n+1)2^n n!} \approx 0.6827.$$

239.

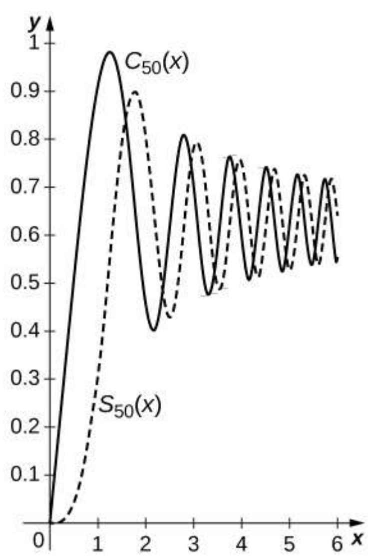


As in the previous problem one obtains $a_n = 0$ if n is odd and $a_n = -(n+2)(n+1)a_{n+2}$ if n is even, so $a_0 = 1$ leads to $a_{2n} = \frac{(-1)^n}{(2n)!}$.

241. $y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$ and $y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$ so $y'' - y' + y = 0$ implies that $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n = 0$ or $a_n = \frac{a_{n-1}}{n} - \frac{a_{n-2}}{n(n-1)}$ for all $n \cdot y(0) = a_0 = 1$ and $y'(0) = a_1 = 0,$ so $a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = 0,$ and $a_5 = -\frac{1}{120}$.

243. a. (Proof) b. We have $R_s \leq \frac{0.1}{(9)!}\pi^9 \approx 0.0082 < 0.01.$ We have $\int_0^{\pi}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}\right)dx = \pi - \frac{\pi^3}{3 \cdot 3!} + \frac{\pi^5}{5 \cdot 5!} - \frac{\pi^7}{7 \cdot 7!} + \frac{\pi^9}{9 \cdot 9!} = 1.852...,$ whereas $\int_0^{\pi} \frac{\sin t}{t}dt = 1.85194...,$ so the actual error is approximately 0.00006.

245.



Since $\cos\left(t^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n}}{(2n)!}$ and $\sin\left(t^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+2}}{(2n+1)!},$ one has $S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!}$ and $C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}$. The sums of the first 50 nonzero terms are plotted below with $C_{50}(x)$ the solid curve and $S_{50}(x)$ the dashed curve.

247. $\int_0^{1/4} \sqrt{x}\left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}\right)dx$

$= \frac{2}{3}2^{-3} - \frac{1}{2}\frac{2}{5}2^{-5} - \frac{1}{8}\frac{2}{7}2^{-7} - \frac{1}{16}\frac{2}{9}2^{-9} - \frac{5}{128}\frac{2}{11}2^{-11} - \frac{7}{256}\frac{2}{13}2^{-13} = 0.0767732...$ whereas $\int_0^{1/4} \sqrt{x - x^2} dx = 0.076773.$

249. $T \approx 2\pi\sqrt{\frac{10}{9.8}}\left(1 + \frac{\sin^2(\theta/12)}{4}\right) \approx 6.453$ seconds. The small angle estimate is $T \approx 2\pi\sqrt{\frac{10}{9.8}} \approx 6.347.$ The relative error is around 2 percent.

251. $\int_0^{\pi/2} \sin^4 \theta d\theta = \frac{3\pi}{16}.$ Hence $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4} + \frac{9}{256}k^4\right).$

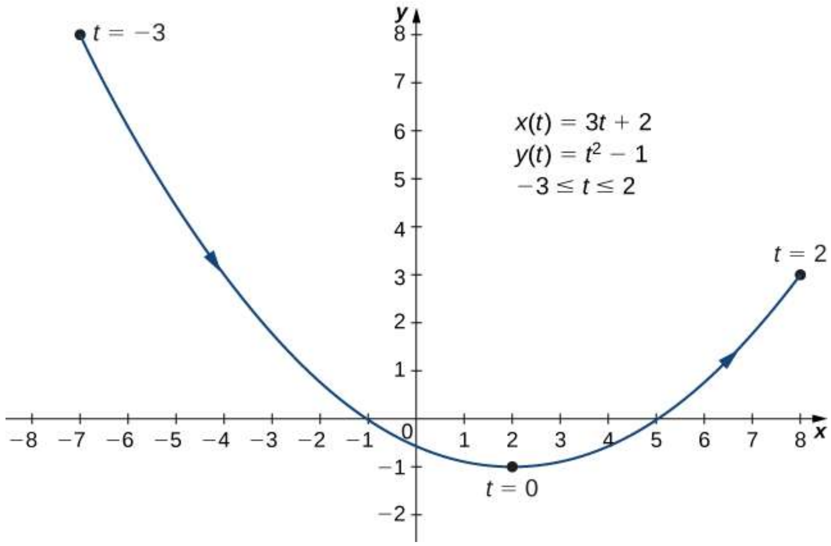
Review Exercises

253. True
255. True
257. ROC: $1;$ IOC: $(0,\ 2)$
259. ROC: $12;$ IOC: $(-16,\ 8)$
261. $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n;$ ROC: $3;$ IOC: $(-3,\ 3)$
263. integration: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(2x)^{2n+1}$
265. $p_4(x) = (x+3)^3 - 11(x+3)^2 + 39(x+3) - 41;$ exact
267. $\sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{2n!}$
269. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}$
271. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^{2n}$
273. $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}$
275. Answers may vary.
277. 2.5%

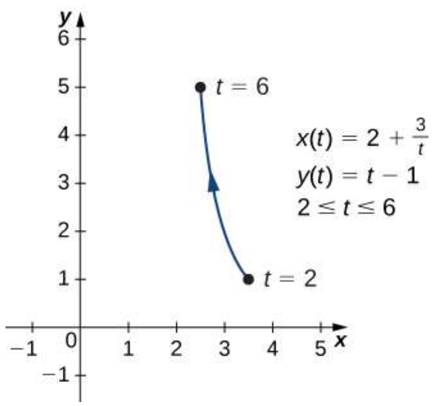
Chapter 7

Checkpoint

7.1.



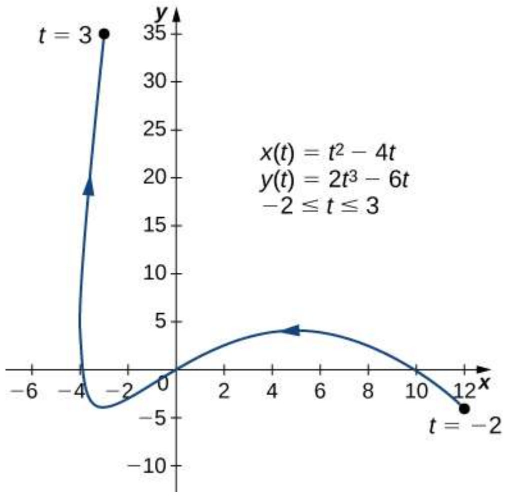
7.2. $x = 2 + \frac{3}{y+1},$ or $y = -1 + \frac{3}{x-2}.$ This equation describes a portion of a rectangular hyperbola centered at $(2, -1)$.



7.3. One possibility is $x(t) = t, \quad y(t) = t^2 + 2t.$ Another possibility is $x(t) = 2t - 3, \quad y(t) = (2t-3)^2 + 2(2t-3) = 4t^2 - 8t + 3.$ There are, in fact, an infinite number of possibilities.

7.4. $x'(t) = 2t - 4$ and $y'(t) = 6t^2 - 6,$ so $\frac{dy}{dx} = \frac{6t^2 - 6}{2t - 4} = \frac{3t^2 - 3}{t - 2}.$

This expression is undefined when $t = 2$ and equal to zero when $t = \pm 1.$



7.5. The equation of the tangent line is $y = 24x + 100.$

7.6. $\frac{d^2 y}{dx^2} = \frac{3t^2 - 12t + 3}{2(t-2)^3}.$ Critical points $(5, 4), (-3, -4),$ and $(-4, 6).$

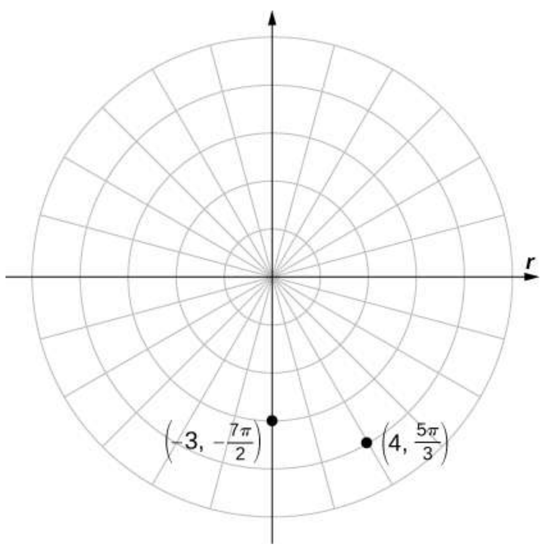
7.7. $A = 3\pi$ (Note that the integral formula actually yields a negative answer. This is due to the fact that $x(t)$ is a decreasing function over the interval $[0, 2\pi];$ that is, the curve is traced from right to left.)

7.8. $s = 2\left(10^{3/2} - 2^{3/2}\right) \approx 57.589$

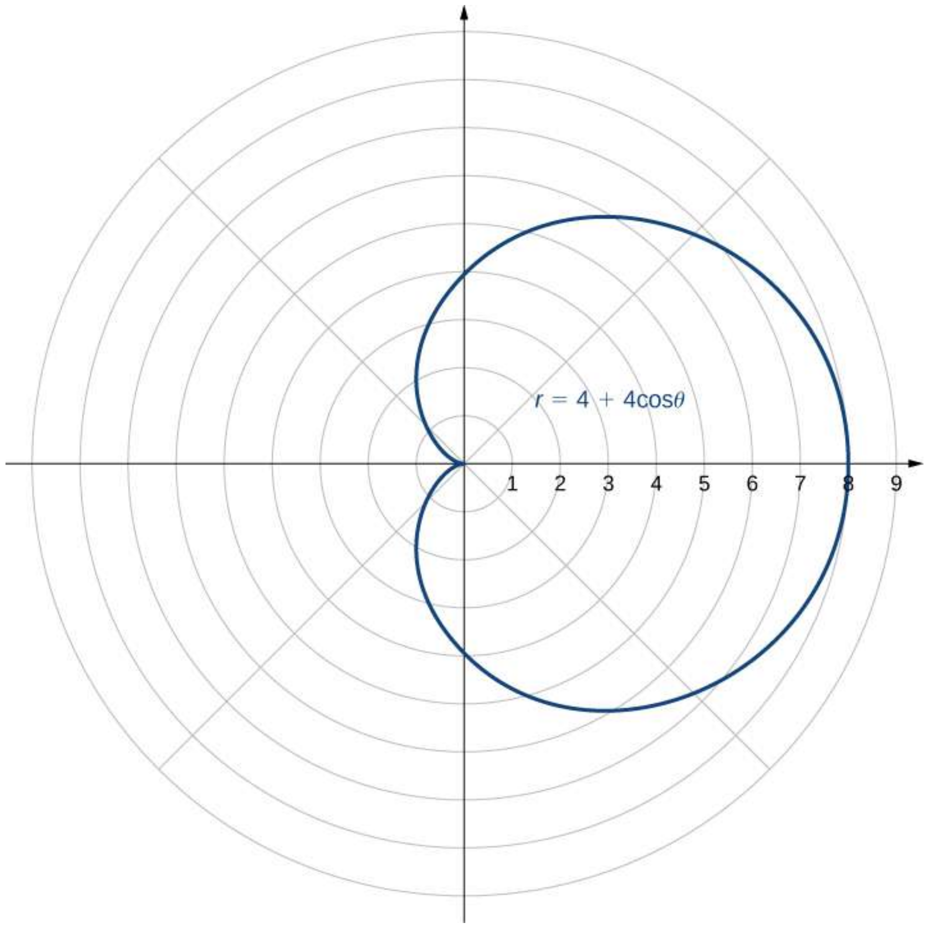
7.9. $A = \frac{\pi\left(494\sqrt{13} + 128\right)}{1215}$

7.10. $\left(8\sqrt{2}, \frac{5\pi}{4}\right)$ and $\left(-2, 2\sqrt{3}\right)$

7.11.



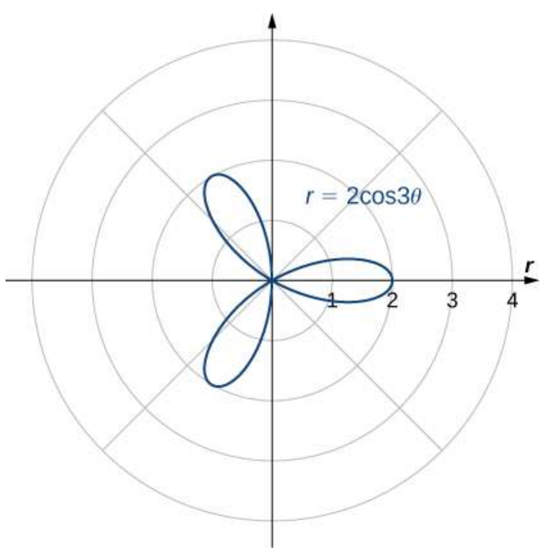
7.12.



The name of this shape is a cardioid, which we will study further later in this section.

7.13. $y = x^2$, which is the equation of a parabola opening upward.

7.14. Symmetric with respect to the polar axis.

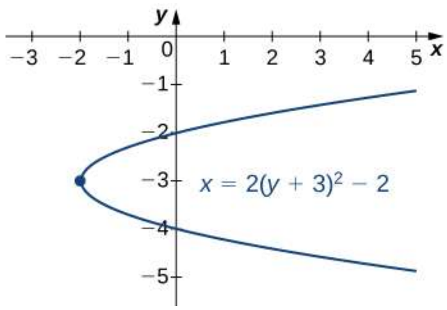


7.15. $A = 3\pi/2$

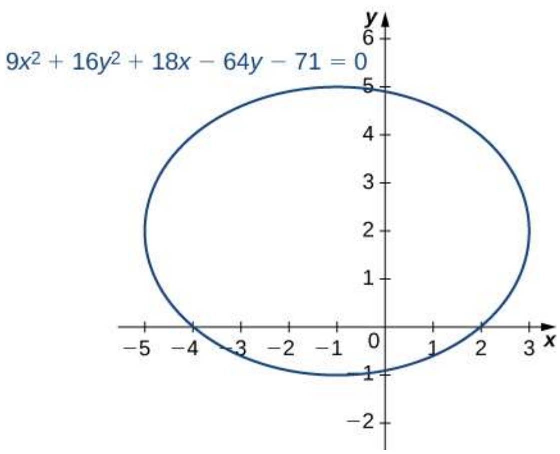
7.16. $A = \frac{4\pi}{3} + 4\sqrt{3}$

7.17. $s = 3\pi$

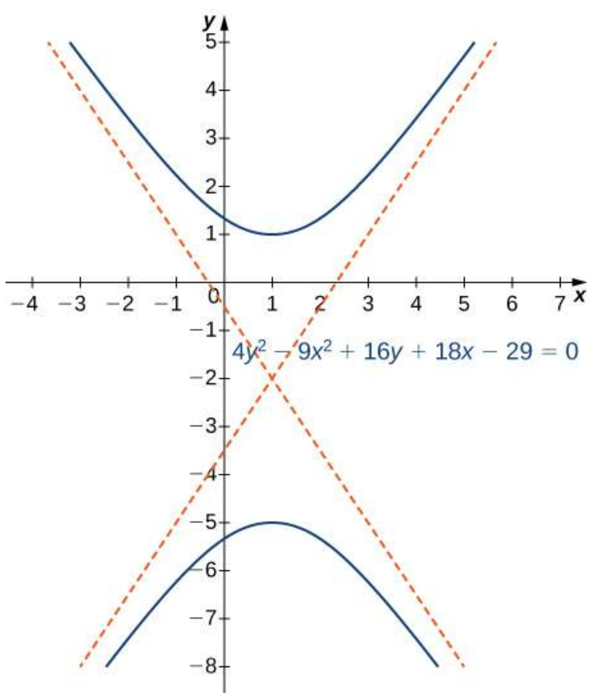
7.18. $x = 2(y + 3)^2 - 2$



7.19. $\frac{(x + 1)^2}{16} + \frac{(y - 2)^2}{9} = 1$

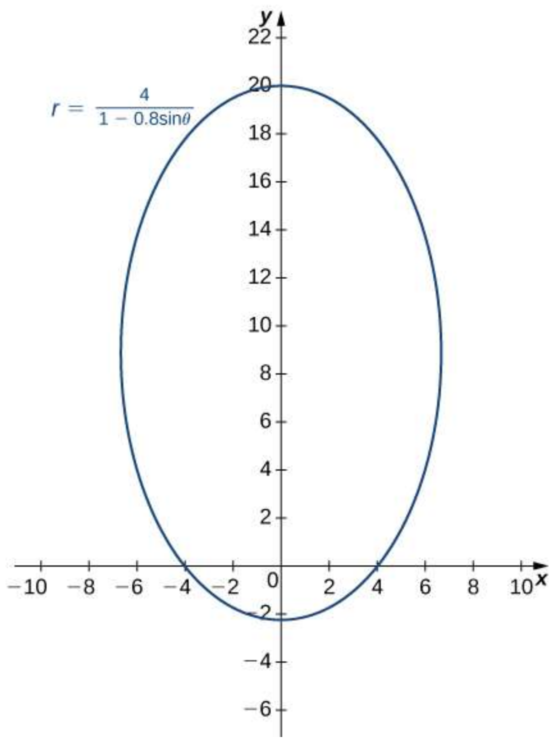


7.20. $\frac{(y + 2)^2}{9} - \frac{(x - 1)^2}{4} = 1$. This is a vertical hyperbola. Asymptotes $y = -2 \pm \frac{3}{2}(x - 1)$.



7.21. $e = \frac{c}{a} = \frac{\sqrt{74}}{7} \approx 1.229$

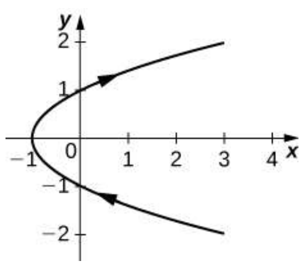
7.22. Here $e = 0.8$ and $p = 5.$ This conic section is an ellipse.



7.23. The conic is a hyperbola and the angle of rotation of the axes is $\theta = 22.5°.$

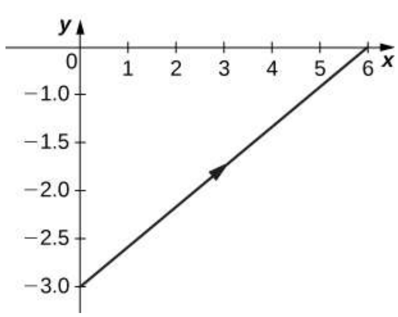
Section Exercises

1.



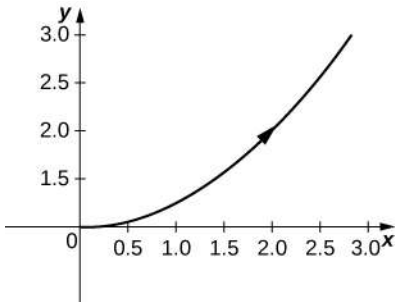
orientation: bottom to top

3.



orientation: left to right

5. $y = \frac{x^2}{4} + 1$



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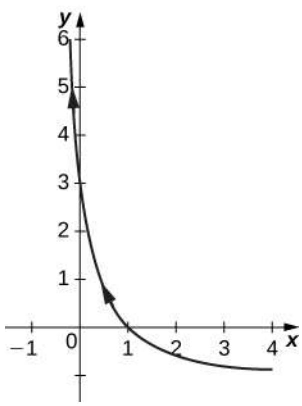

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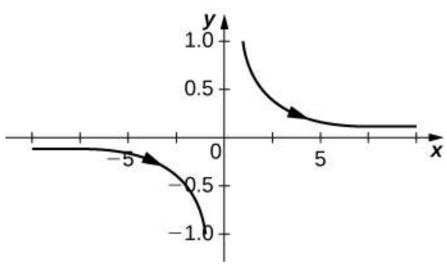

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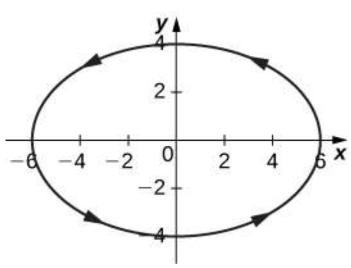

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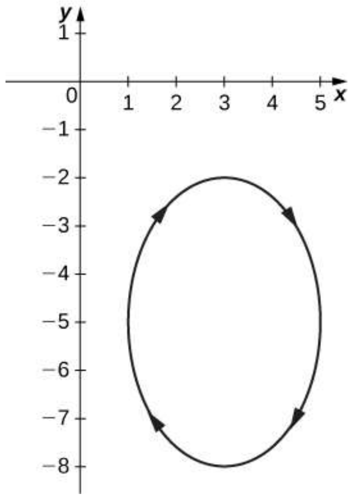

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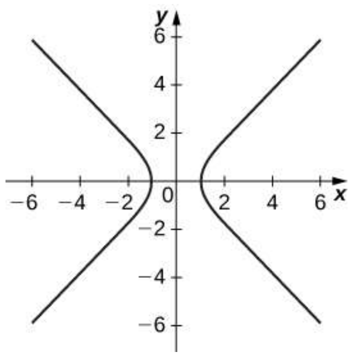
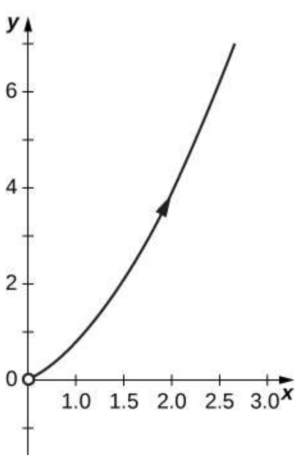


Asymptotes are $y = x$ and $y = -x$

17.



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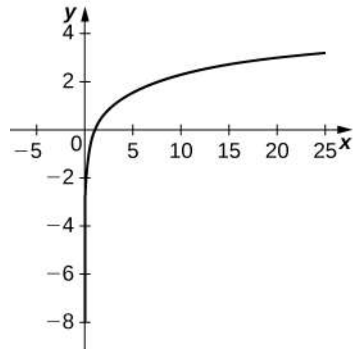


21. $x = 4y^2 - 1;$ domain: $x \in [1,\ \infty).$

23. $\frac{x^2}{16} + \frac{y^2}{9} = 1;$ domain $x \in [-4,\ 4].$

25. $y = 3x + 2;$ domain: all real numbers.

27. $(x - 1)^2 + (y - 3)^2 = 1;$ domain: $x \in [0,\ 2].$

29. $y = \sqrt{x^2 - 1};$ domain: $x \in [-1,\ 1].$

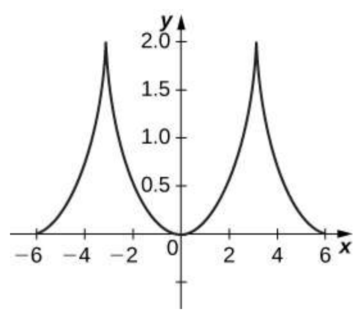
31. $y^2 = \frac{1 - x}{2};$ domain: $x \in [2,\ \infty) \cup (-\infty,\ -2].$

33. $y = \ln x;$ domain: $x \in (0,\ \infty).$

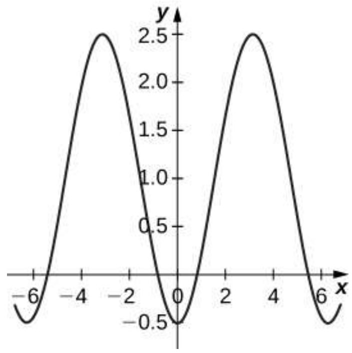
35. $y = \ln x;$ domain: $x \in (0,\ \infty).$

37. $x^2 + y^2 = 4;$ domain: $x \in [-2,\ 2].$

39. line
41. parabola
43. circle
45. ellipse
47. hyperbola
51. The equations represent a cycloid.

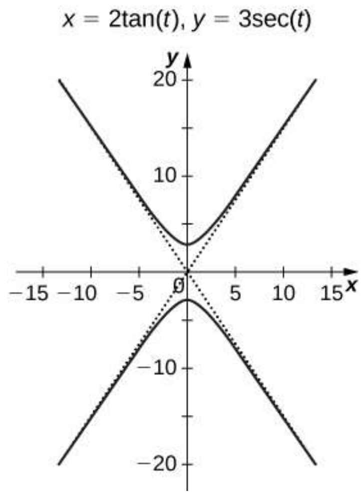


53.



55. 22,092 meters at approximately 51 seconds.

57.



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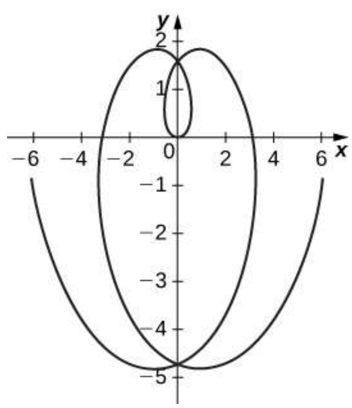
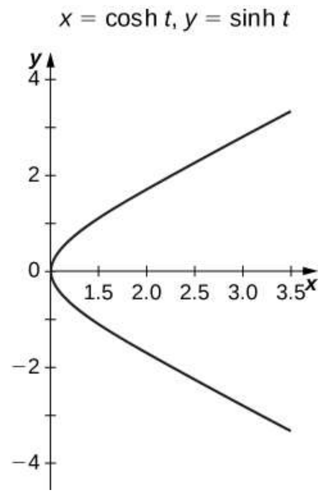


61.



63. 0

65. $\frac{-3}{5}$

67. $\text{Slope} = 0; \ \ y = 8.$

69. Slope is undefined; $x = 2.$

71. $t = \arctan(-2); \ \left(\frac{4}{\sqrt{5}}, \frac{-8}{\sqrt{5}}\right).$

73. No points possible; undefined expression.

75. $y = -\left(\frac{2}{e}\right)x + 3$

77. $y = 2x - 7$

79. $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

81. $\frac{dy}{dx} = -\tan(t)$

83. $\frac{dy}{dx} = \frac{3}{4}$ and $\frac{d^2y}{dx^2} = 0,$ so the curve is neither concave up nor concave down at $t = 3.$ Therefore the graph is linear and has a constant slope but no concavity.

85. $\frac{dy}{dx} = 4, \ \frac{d^2y}{dx^2} = -6\sqrt{3};$ the curve is concave down at $\theta = \frac{\pi}{6}.$

87. No horizontal tangents. Vertical tangents at $(1, 0), (-1, 0).$

89. $-\sec^3(\pi t)$

91. Horizontal $(0, -9);$ vertical $(\pm 2, -6).$

93. 1

95. 0

97. 4

99. Concave up on $t > 0.$

101. 1

103. $\frac{3\pi}{2}$

105. $6\pi a^2$

107. $2\pi ab$

109. $\frac{1}{3}(2\sqrt{2} - 1)$

111. 7.075

113. $6a$

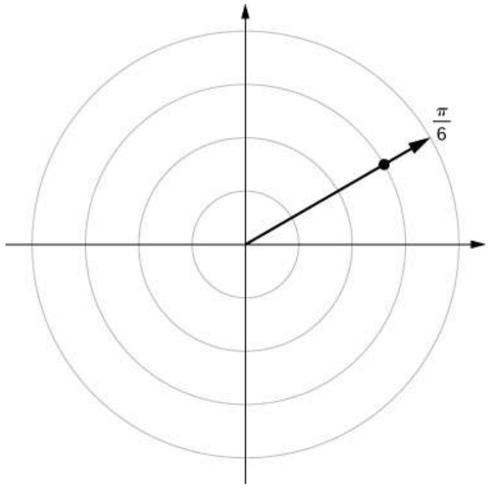
115. $6\sqrt{2}$

119. $\dfrac{2\pi\left(247\sqrt{13}+64\right)}{1215}$

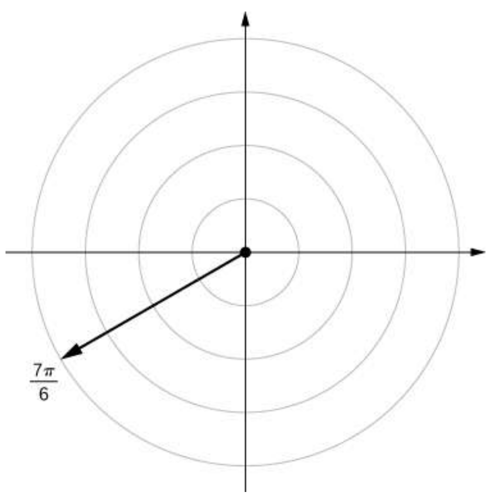
121. 59.101

123. $\dfrac{8\pi}{3}(17\sqrt{17}-1)$

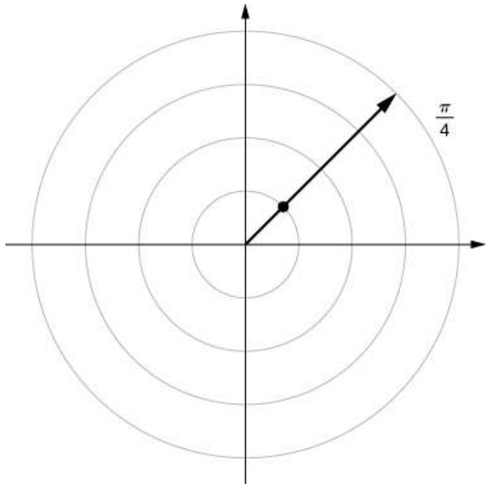
125.



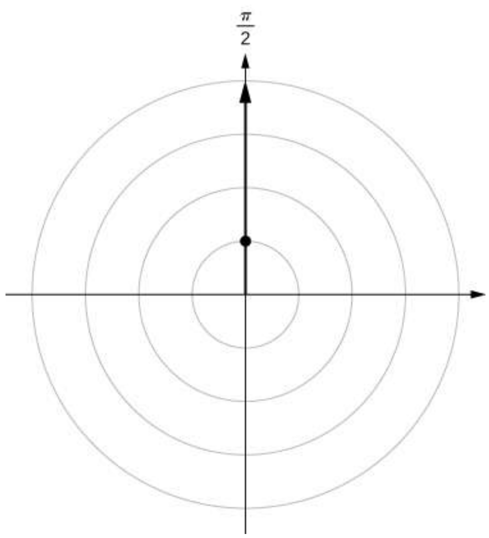
127.



129.



131.



133. $B\left(3, \frac{-\pi}{3}\right)\ B\left(-3, \frac{2\pi}{3}\right)$

135. $D\left(5, \frac{7\pi}{6}\right)D\left(-5, \frac{\pi}{6}\right)$

137. $(5, -0.927)\ (-5, -0.927 + \pi)$

139. $(10, -0.927)(-10, -0.927 + \pi)$

141. $\left(2\sqrt{3}, -0.524\right)\left(-2\sqrt{3}, -0.524 + \pi\right)$

143. $\left(-\sqrt{3},\quad -1\right)$

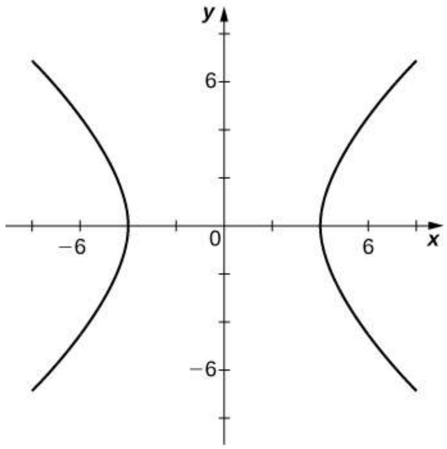
145. $\left(-\frac{\sqrt{3}}{2},\quad \frac{-1}{2}\right)$

147. $(0,\quad 0)$

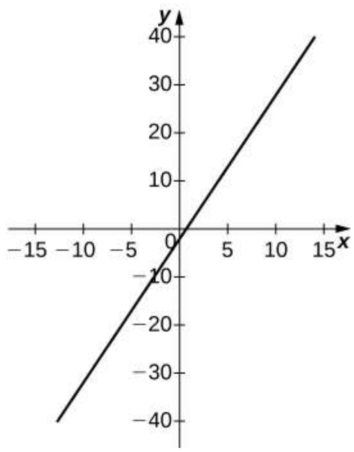
149. Symmetry with respect to the *x*-axis, *y*-axis, and origin.
151. Symmetric with respect to *x*-axis only.
153. Symmetry with respect to *x*-axis only.
155. Line $y = x$

157. $y = 1$

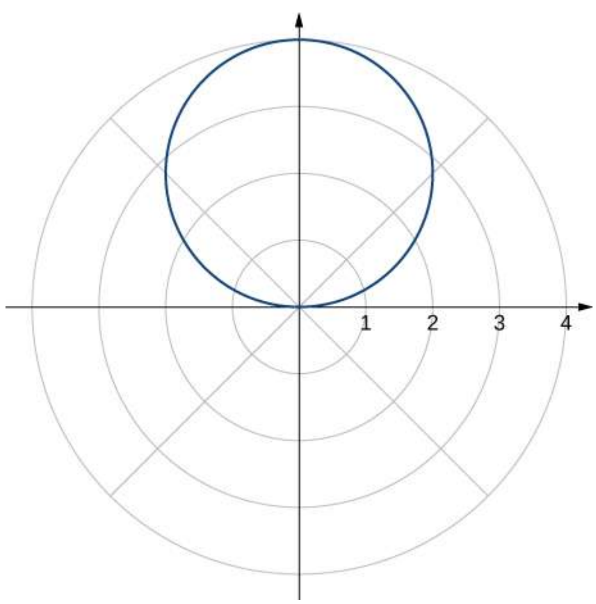
159. Hyperbola; polar form $r^2 \cos(2\theta) = 16$ or $r^2 = 16 \sec\theta$.



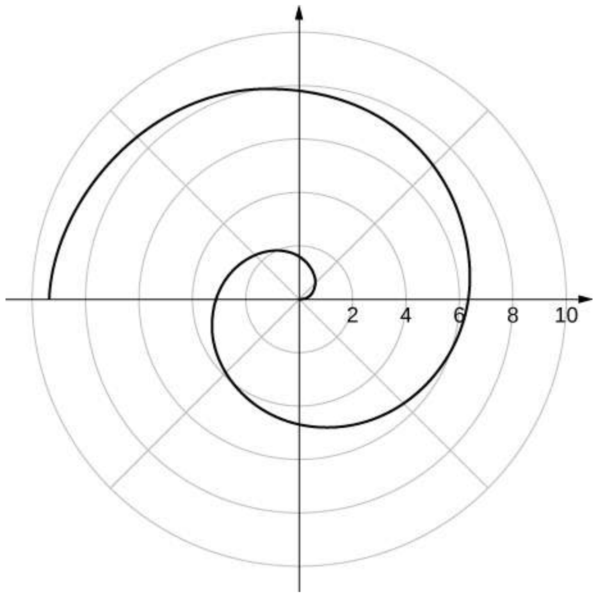
161. $r = \dfrac{2}{3\cos\theta - \sin\theta}$



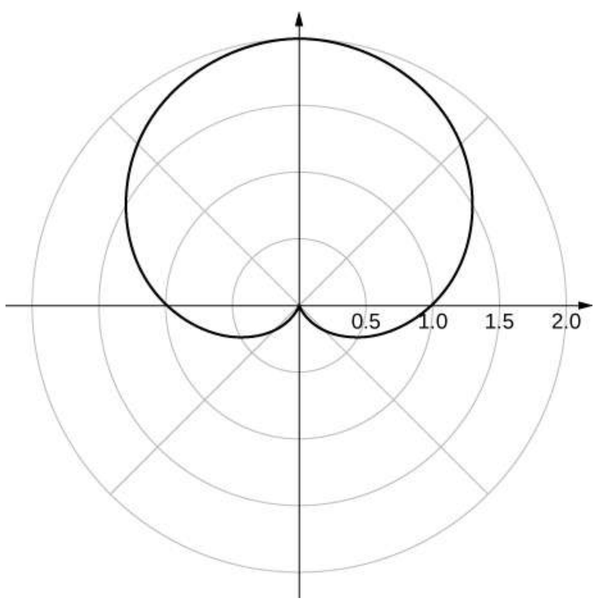
163. $x^2 + y^2 = 4y$



165. $x \tan\sqrt{x^2 + y^2} = y$

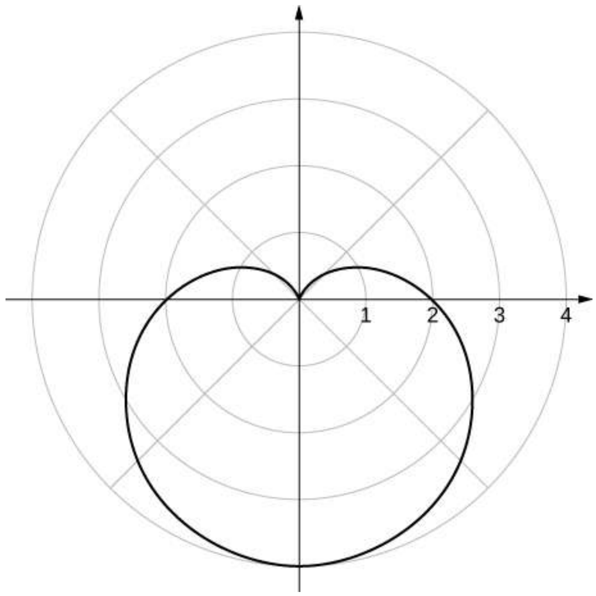


167.



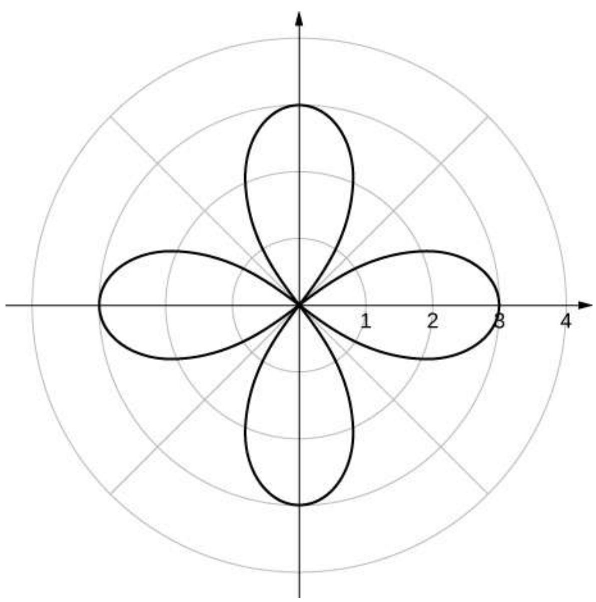
y-axis symmetry

169.



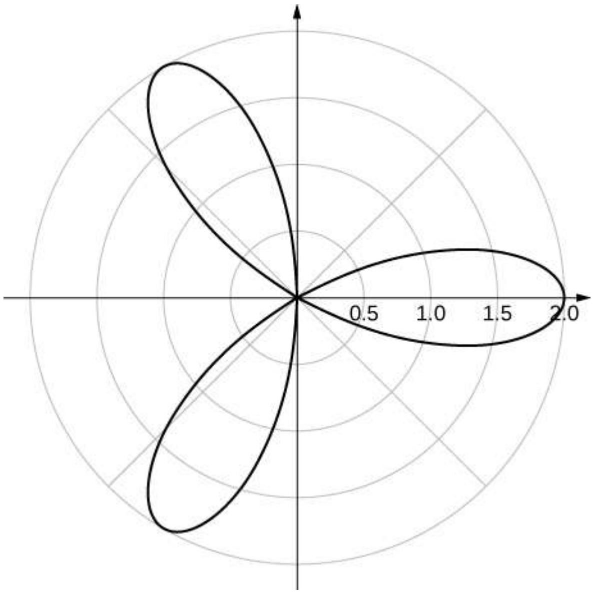
y-axis symmetry

171.



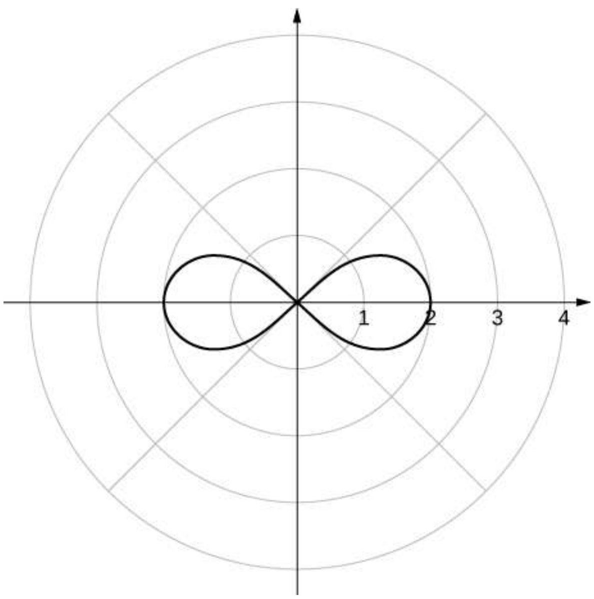
x- and *y*-axis symmetry and symmetry about the pole

173.



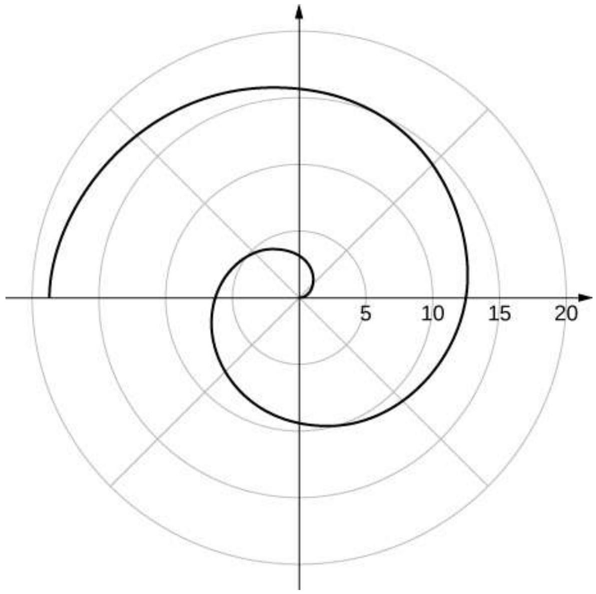
x-axis symmetry

175.



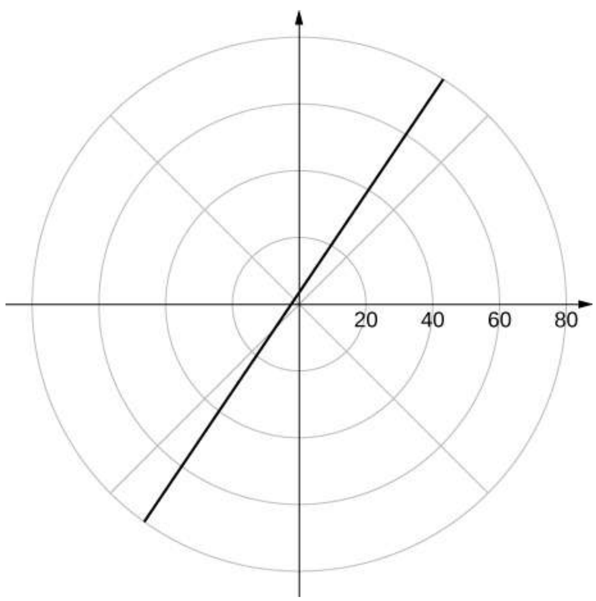
x- and *y*-axis symmetry and symmetry about the pole

177.



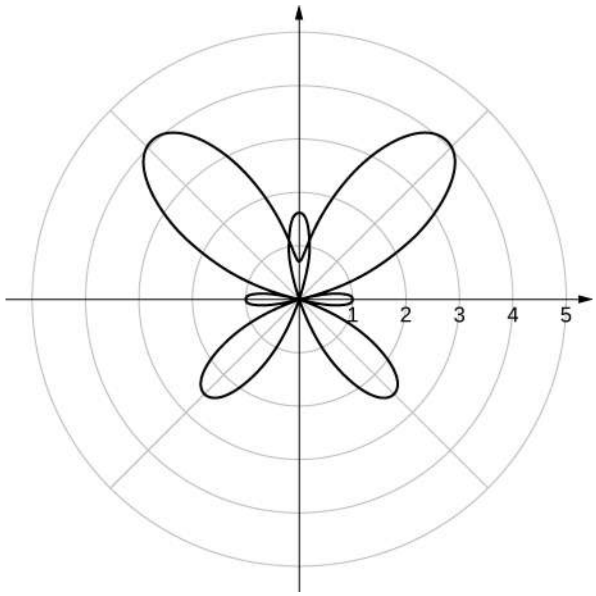
no symmetry

179.



a line

181.



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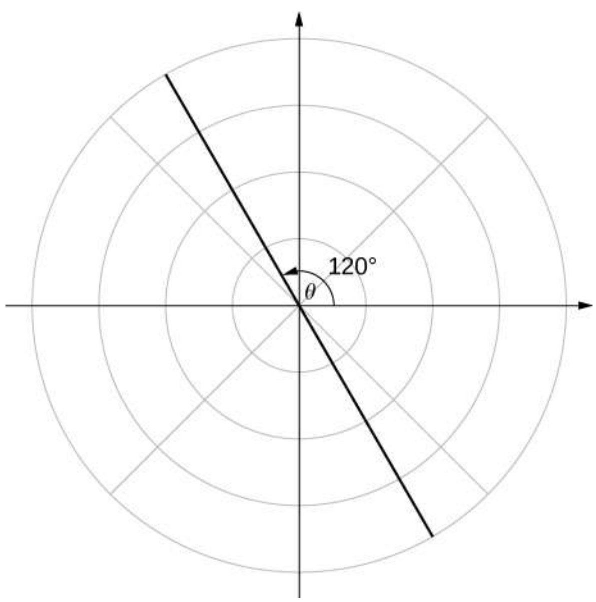
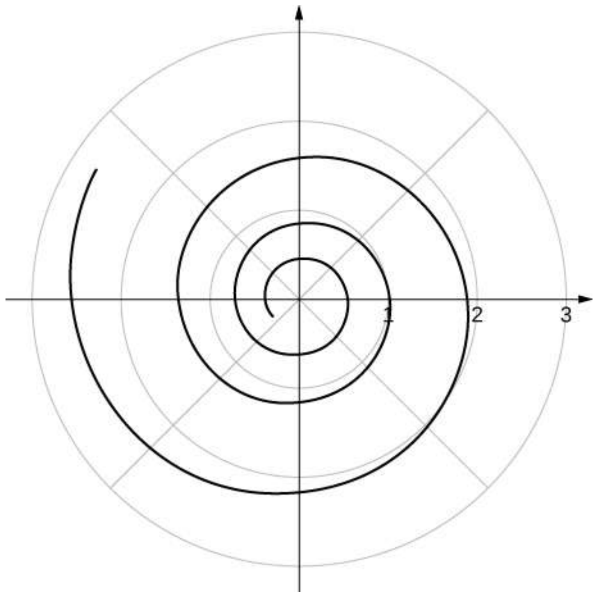


185.



187. Answers vary. One possibility is the spiral lines become closer together and the total number of spirals increases.

189. $\frac{9}{2}\int_0^{\pi}\sin^2\theta\,d\theta$

191. $32\int_0^{\pi/2}\sin^2(2\theta)d\theta$

193. $\frac{1}{2}\int_{\pi}^{2\pi}(1-\sin\theta)^2\,d\theta$

195. $\int_{\sin^{-1}(2/3)}^{\pi/2}(2-3\sin\theta)^2 d\theta$

197. $\int_0^{\pi}(1-2\cos\theta)^2\,d\theta-\int_0^{\pi/3}(1-2\cos\theta)^2 d\theta$

199. $4\int_0^{\pi/3}d\theta+16\int_{\pi/3}^{\pi/2}\left(\cos^2\theta\right)d\theta$

201. 9π

203. $\frac{9\pi}{4}$

205. $\frac{9\pi}{8}$

207. $\frac{18\pi-27\sqrt{3}}{2}$

209. $\frac{4}{3}\left(4\pi-3\sqrt{3}\right)$

211. $\frac{3}{2}\left(4\pi-3\sqrt{3}\right)$

213. $2\pi-4$

215. $\int_0^{2\pi}\sqrt{(1+\sin\theta)^2+\cos^2\theta}d\theta$

217. $\sqrt{2}\int_0^1 e^{\theta}\,d\theta$

219. $\frac{\sqrt{10}}{3}\left(e^6-1\right)$

221. 32

223. 6.238
225. 2
227. 4.39

229. $A = \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{2}$ and $\frac{1}{2}\int_0^{\pi}(1 + 2\sin\theta\cos\theta)d\theta = \frac{\pi}{2}$

231. $C = 2\pi\left(\frac{3}{2}\right) = 3\pi$ and $\int_0^{\pi} 3d\theta = 3\pi$

233. $C = 2\pi(5) = 10\pi$ and $\int_0^{\pi} 10\, d\theta = 10\pi$

235. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

237. The slope is $\frac{1}{\sqrt{3}}$.

239. The slope is 0.

241. At $(4, 0),$ the slope is undefined. At $\left(-4, \frac{\pi}{2}\right),$ the slope is 0.

243. The slope is undefined at $\theta = \frac{\pi}{4}$.

245. Slope = −1.

247. Slope is $\frac{-2}{\pi}$.

249. Calculator answer: −0.836.

251. Horizontal tangent at $\left(\pm\sqrt{2}, \frac{\pi}{6}\right),$ $\left(\pm\sqrt{2}, -\frac{\pi}{6}\right)$.

253. Horizontal tangents at $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$. Vertical tangents at $\frac{\pi}{6}, \frac{5\pi}{6}$ and also at the pole $(0, 0)$.

255. $y^2 = 16x$

257. $x^2 = 2y$

259. $x^2 = -4(y - 3)$

261. $(x + 3)^2 = 8(y - 3)$

263. $\frac{x^2}{16} + \frac{y^2}{12} = 1$

265. $\frac{x^2}{13} + \frac{y^2}{4} = 1$

267. $\frac{(y-1)^2}{16} + \frac{(x+3)^2}{12} = 1$

269. $\frac{x^2}{16} + \frac{y^2}{12} = 1$

271. $\frac{x^2}{25} - \frac{y^2}{11} = 1$

273. $\frac{x^2}{7} - \frac{y^2}{9} = 1$

275. $\frac{(y+2)^2}{4} - \frac{(x+2)^2}{32} = 1$

277. $\frac{x^2}{4} - \frac{y^2}{32} = 1$

279. $e = 1,$ parabola

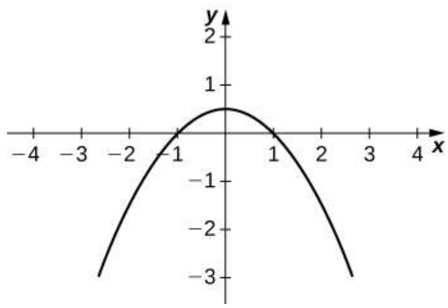
281. $e = \frac{1}{2},$ ellipse

283. $e = 3,$ hyperbola

285. $r = \dfrac{4}{5 + \cos\theta}$

287. $r = \dfrac{4}{1 + 2\sin\theta}$

289.



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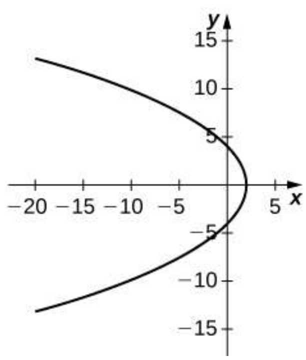

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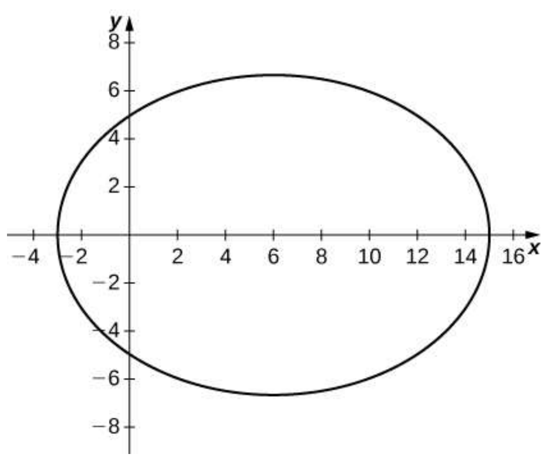

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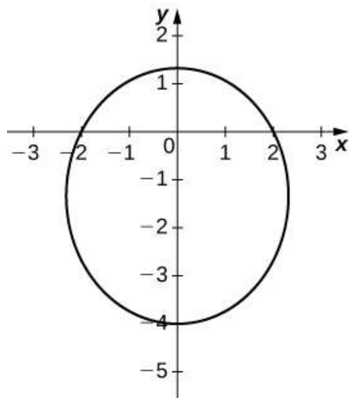
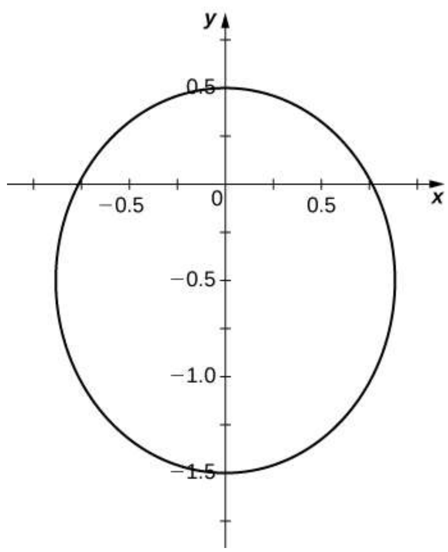
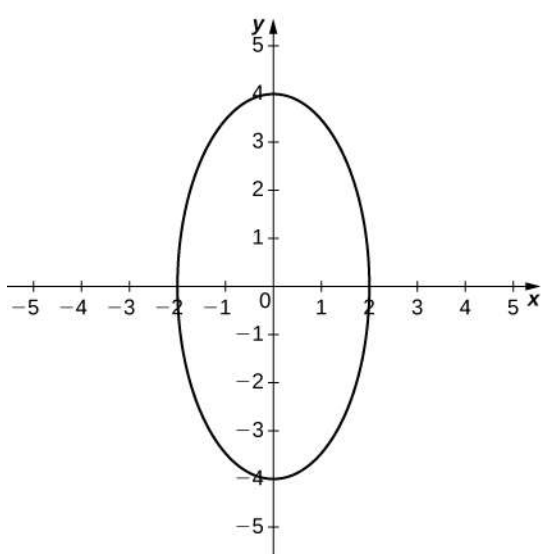

y
2
1
−3
−2
−1
0
1
2
3
x
−1
−2
−3
−4
−5

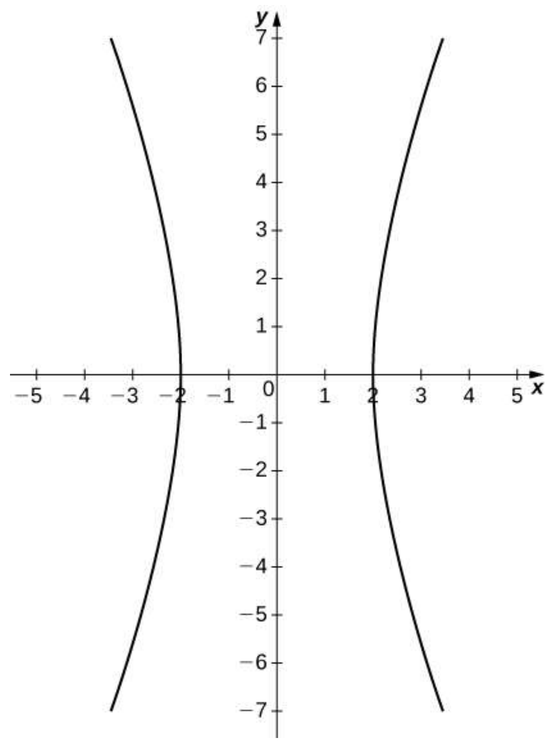
297.


y
0.5
−0.5
0
0.5
x
−0.5
−1.0
−1.5

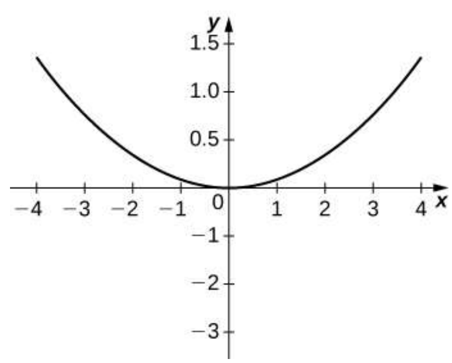
299.


y
5
4
3
2
1
−5
−4
−3
−2
−1
0
1
2
3
4
5
x
−1
−2
−3
−4
−5

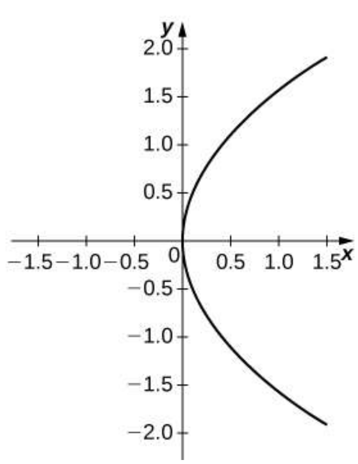
301.



303.



305.

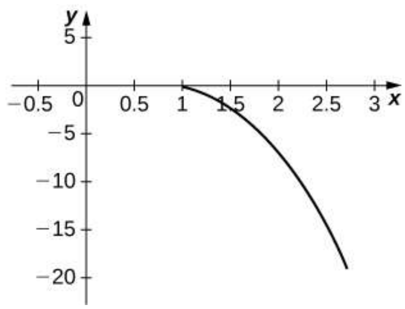


307. Hyperbola
309. Ellipse

311. Ellipse
313. At the point 2.25 feet above the vertex.
315. 0.5625 feet
317. Length is 96 feet and height is approximately 26.53 feet.
319. $r = \dfrac{2.616}{1 + 0.995 \cos \theta}$
321. $r = \dfrac{5.192}{1 + 0.0484 \cos \theta}$

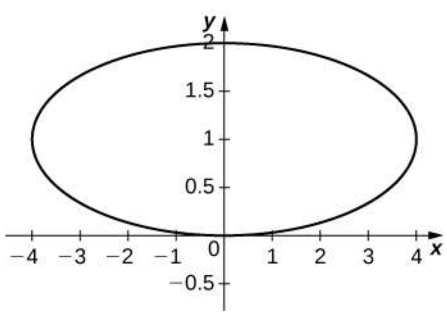
Review Exercises

323. True.
325. False. Imagine $y = t + 1, \quad x = -t + 1.$
327.



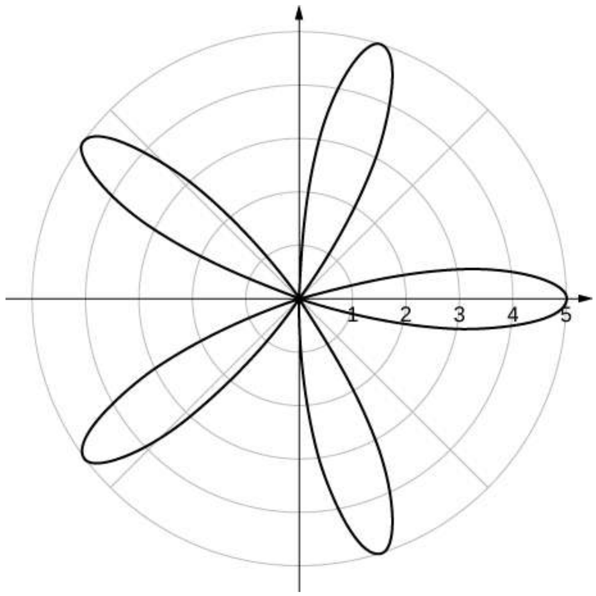
$y = 1 - x^3$

329.



$\dfrac{x^2}{16} + (y - 1)^2 = 1$

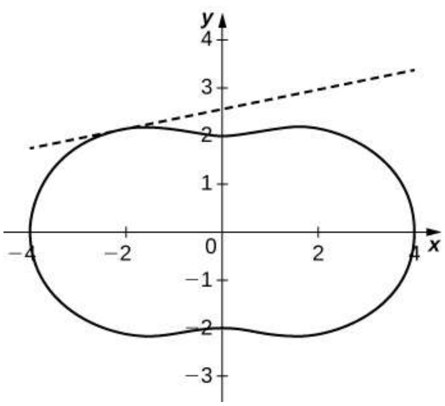
331.



Symmetric about polar axis

333. $r^2 = \dfrac{4}{\sin^2\theta - \cos^2\theta}$

335.



$y = \dfrac{3\sqrt{2}}{2} + \dfrac{1}{5}\left(x + \dfrac{3\sqrt{2}}{2}\right)$

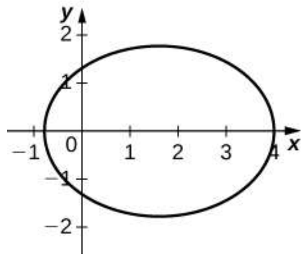
337. $\dfrac{e^2}{2}$

339. $9\sqrt{10}$

341. $(y+5)^2 = -8x + 32$

343. $\dfrac{(y+1)^2}{16} - \dfrac{(x+2)^2}{9} = 1$

345. $e = \dfrac{2}{3},$ ellipse



347. $\frac{y^2}{19.03^2}+\frac{x^2}{19.63^2}=1, \quad e=0.2447$

INDEX