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Calc-ulus

Volume 3

4 | DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLES

Figure 4.1 Americans use (and lose) millions of golf balls a year, which keeps golf ball manufacturers in business. In this chapter, we study a profit model and learn methods for calculating optimal production levels for a typical golf ball manufacturing company. (credit: modification of work by oatsy40, Flickr)

Chapter Outline

Introduction

In **Introduction to Applications of Derivatives (http://cnx.org/content/m53602/latest/)** , we studied how to determine the maximum and minimum of a function of one variable over a closed interval. This function might represent the temperature over a given time interval, the position of a car as a function of time, or the altitude of a jet plane as it travels from New York to San Francisco. In each of these examples, the function has one independent variable.

Suppose, however, that we have a quantity that depends on more than one variable. For example, temperature can depend on location and the time of day, or a company's profit model might depend on the number of units sold and the amount of money spent on advertising. In this chapter, we look at a company that produces golf balls. We develop a profit model and, under various restrictions, we find that the optimal level of production and advertising dollars spent determines the maximum possible profit. Depending on the nature of the restrictions, both the method of solution and the solution itself changes (see **Example 4.41**).

When dealing with a function of more than one independent variable, several questions naturally arise. For example, how do we calculate limits of functions of more than one variable? The definition of *derivative* we used before involved a limit. Does the new definition of derivative involve limits as well? Do the rules of differentiation apply in this context? Can we find relative extrema of functions using derivatives? All these questions are answered in this chapter.

4.1 | Functions of Several Variables

Learning Objectives

4.1.1 Recognize a function of two variables and identify its domain and range.
4.1.2 Sketch a graph of a function of two variables.
4.1.3 Sketch several traces or level curves of a function of two variables.
4.1.4 Recognize a function of three or more variables and identify its level surfaces.

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them. We also examine ways to relate the graphs of functions in three dimensions to graphs of more familiar planar functions.

Functions of Two Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition

A **function of two variables** $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane $\mathbb{R}^2$ to a unique real number z. The set D is called the *domain* of the function. The *range* of f is the set of all real numbers

z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in the following figure.

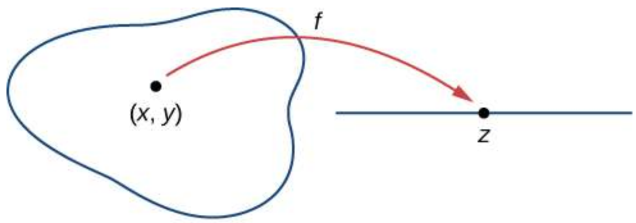


Figure 4.2 The domain of a function of two variables consists of ordered pairs (x, y).

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let's take a look.

Example 4.1

Domains and Ranges for Functions of Two Variables

Find the domain and range of each of the following functions:

a. $f(x, y) = 3x + 5y + 2$

b. $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution

a. This is an example of a linear function in two variables. There are no values or combinations of x and y that cause $f(x, y)$ to be undefined, so the domain of f is $\mathbb{R}^2$. To determine the range, first pick a value for z. We need to find a solution to the equation $f(x, y) = z$, or $3x - 5y + 2 = z$. One such solution can be obtained by first setting $y = 0$, which yields the equation $3x + 2 = z$. The solution to this equation is $x = \frac{z-2}{3}$, which gives the ordered pair $\left(\frac{z-2}{3}, 0\right)$ as a solution to the equation $f(x, y) = z$ for any value of z. Therefore, the range of the function is all real numbers, or $\mathbb{R}$.

b. For the function $g(x, y)$ to have a real value, the quantity under the square root must be nonnegative:

$$9 - x^2 - y^2 \geq 0.$$

This inequality can be written in the form

$$x^2 + y^2 \leq 9.$$

Therefore, the domain of $g(x, y)$ is $\left\{(x, y) \in \mathbb{R}^2 \middle| x^2 + y^2 \leq 9\right\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the following graph.

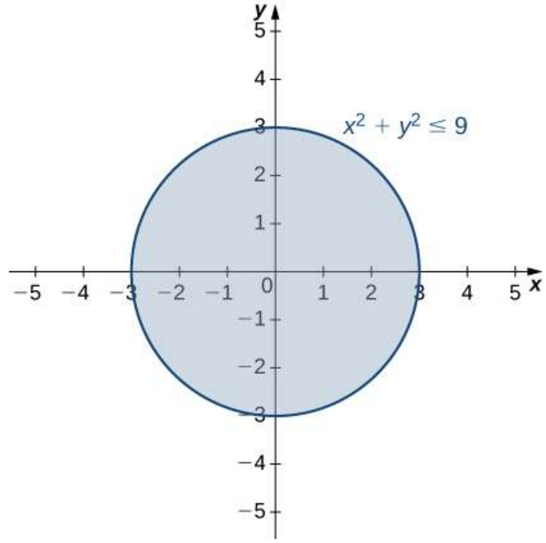


Figure 4.3 The domain of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$ is a closed disk of radius 3.

To determine the range of $g(x, y) = \sqrt{9 - x^2 - y^2}$ we start with a point (x_0, y_0) on the boundary of the domain, which is defined by the relation $x^2 + y^2 = 9.$ It follows that $x_0^2 + y_0^2 = 9$ and

$$g(x_0, y_0) = \sqrt{9 - x_0^2 - y_0^2} = \sqrt{9 - \left(x_0^2 + y_0^2\right)} = \sqrt{9 - 9} = 0.$$

If $x_0^2 + y_0^2 = 0$ (in other words, $x_0 = y_0 = 0$), then

$$g(x_0, y_0) = \sqrt{9 - x_0^2 - y_0^2} = \sqrt{9 - \left(x_0^2 + y_0^2\right)} = \sqrt{9 - 0} = 3.$$

This is the maximum value of the function. Given any value c between 0 and $3,$ we can find an entire set of points inside the domain of g such that $g(x, y) = c$:

$$\begin{aligned} \sqrt{9 - x^2 - y^2} &= c \\ 9 - x^2 - y^2 &= c^2 \\ x^2 + y^2 &= 9 - c^2. \end{aligned}$$

Since $9 - c^2 > 0,$ this describes a circle of radius $\sqrt{9 - c^2}$ centered at the origin. Any point on this circle satisfies the equation $g(x, y) = c.$ Therefore, the range of this function can be written in interval notation as $[0, 3].$

4.1 Find the domain and range of the function $f(x, y) = \sqrt{36 - 9x^2 - 9y^2}.$

Graphing Functions of Two Variables

Suppose we wish to graph the function $z = (x, y)$. This function has two independent variables $(x \text{ and } y)$ and one dependent variable (z). When graphing a function $y = f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x, y) in the plane, and every point in the plane has an ordered pair (x, y) associated with it. With a function of two variables, each ordered pair (x, y) in the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x, y, z). The graph of a function $z = (x, y)$ of two variables is called a **surface**.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x, y) coordinate system laying flat. Then, every point in the domain of the function f has a unique z-value associated with it. If z is positive, then the graphed point is located above the xy-plane, if z is negative, then the graphed point is located below the xy-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

Example 4.2

Graphing Functions of Two Variables

Create a graph of each of the following functions:

a. $g(x, y) = \sqrt{9 - x^2 - y^2}$

b. $f(x, y) = x^2 + y^2$

Solution

a. In Example 4.1, we determined that the domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$ is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 9\}$ and the range is $\{z \in \mathbb{R}^2 \mid 0 \le z \le 3\}$. When $x^2 + y^2 = 9$ we have $g(x, y) = 0$. Therefore any point on the circle of radius 3 centered at the origin in the x, y-plane maps to $z = 0$ in $\mathbb{R}^3$. If $x^2 + y^2 = 8$, then $g(x, y) = 1$, so any point on the circle of radius $2\sqrt{2}$ centered at the origin in the x, y-plane maps to $z = 1$ in $\mathbb{R}^3$. As $x^2 + y^2$ gets closer to zero, the value of z approaches 3. When $x^2 + y^2 = 0$, then $g(x, y) = 3$. This is the origin in the x, y-plane. If $x^2 + y^2$ is equal to any other value between 0 and 9, then $g(x, y)$ equals some other constant between 0 and 3. The surface described by this function is a hemisphere centered at the origin with radius 3 as shown in the following graph.

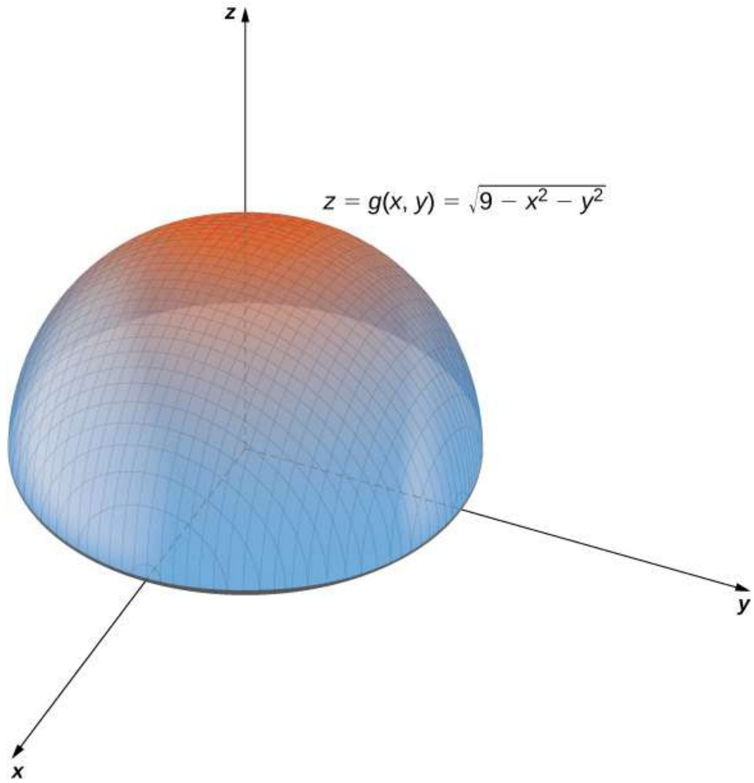


Figure 4.4 Graph of the hemisphere represented by the given function of two variables.

b. This function also contains the expression $x^2 + y^2$. Setting this expression equal to various values starting at zero, we obtain circles of increasing radius. The minimum value of $f(x, y) = x^2 + y^2$ is zero (attained when $x = y = 0.$). When $x = 0,$ the function becomes $z = y^2,$ and when $y = 0,$ then the function becomes $z = x^2.$ These are cross-sections of the graph, and are parabolas. Recall from **Introduction to Vectors in Space** that the name of the graph of $f(x, y) = x^2 + y^2$ is a *paraboloid*. The graph of f appears in the following graph.

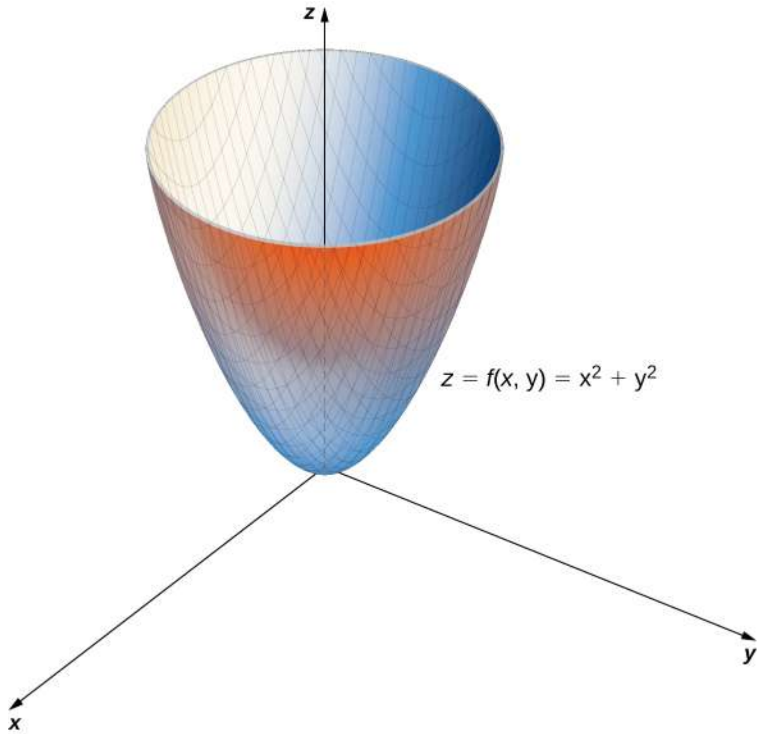


Figure 4.5 A paraboloid is the graph of the given function of two variables.

Example 4.3

Nuts and Bolts

A profit function for a hardware manufacturer is given by

$$f(x,\ y) = 16 - (x-3)^2 - (y-2)^2,$$

where x is the number of nuts sold per month (measured in thousands) and y represents the number of bolts sold per month (measured in thousands). Profit is measured in thousands of dollars. Sketch a graph of this function.

Solution

This function is a polynomial function in two variables. The domain of f consists of $(x,\ y)$ coordinate pairs that yield a nonnegative profit:

$$16 - (x-3)^2 - (y-2)^2 \geq 0$$
$$(x-3)^2 + (y-2)^2 \leq 16.$$

This is a disk of radius 4 centered at $(3,\ 2)$. A further restriction is that both x and y must be nonnegative. When $x = 3$ and $y = 2,$ $f(x,\ y) = 16.$ Note that it is possible for either value to be a noninteger; for example, it is possible to sell 2.5 thousand nuts in a month. The domain, therefore, contains thousands of points, so we can consider all points within the disk. For any $z < 16,$ we can solve the equation $f(x, y) = 16$:

$$\begin{aligned} 16 - (x-3)^2 - (y-2)^2 &= z \\ (x-3)^2 + (y-2)^2 &= 16 - z. \end{aligned}$$

Since $z < 16,$ we know that $16 - z > 0,$ so the previous equation describes a circle with radius $\sqrt{16 - z}$ centered at the point $(3, 2).$ Therefore. the range of $f(x, y)$ is $\{z \in \mathbb{R} | z \le 16\}.$ The graph of $f(x, y)$ is also a paraboloid, and this paraboloid points downward as shown.

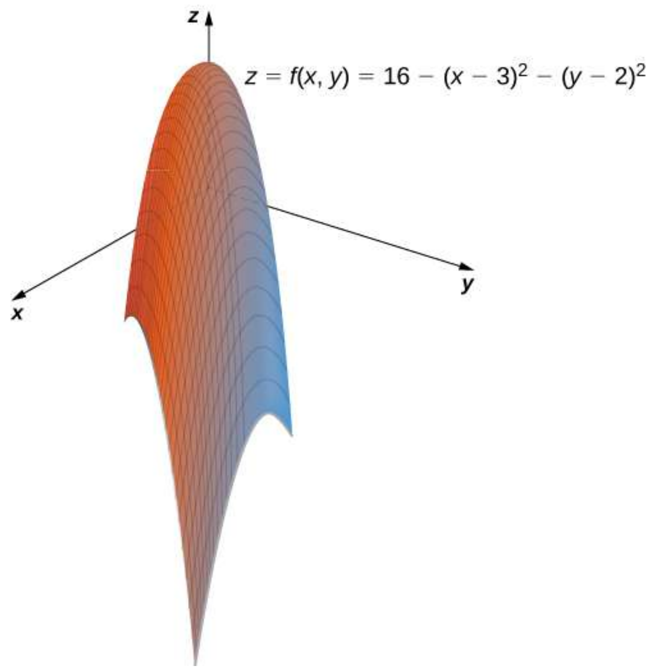
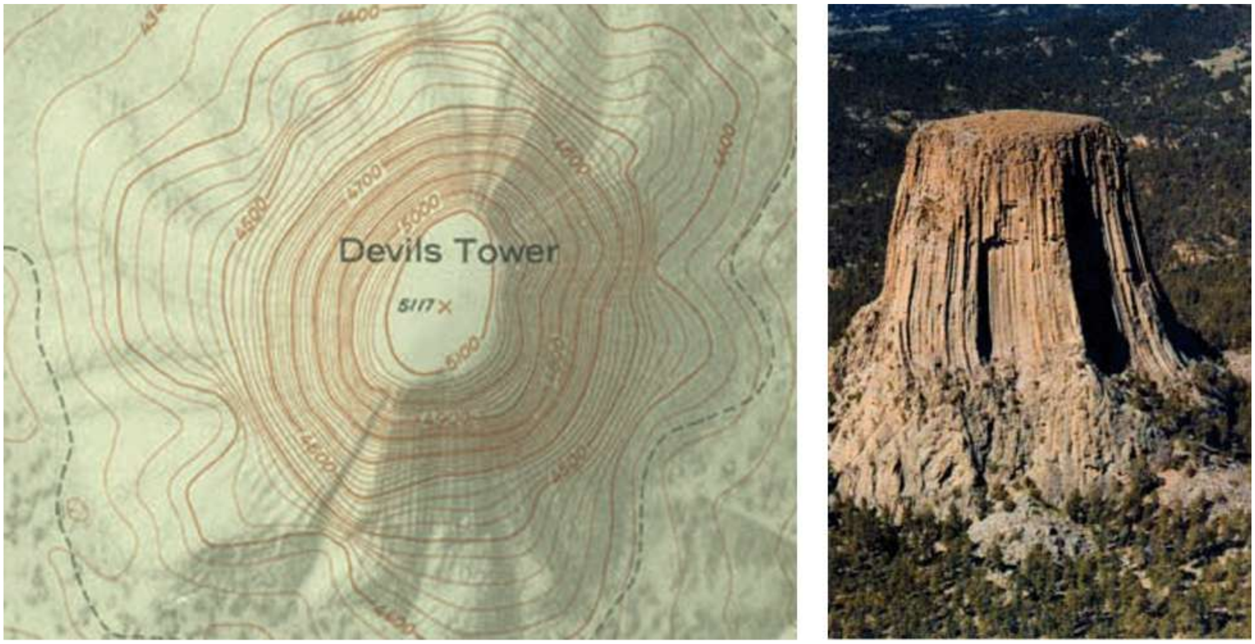


Figure 4.6 The graph of the given function of two variables is also a paraboloid.

Level Curves

If hikers walk along rugged trails, they might use a topographical map that shows how steeply the trails change. A topographical map contains curved lines called *contour lines*. Each contour line corresponds to the points on the map that have equal elevation (**Figure 4.7**). A level curve of a function of two variables $f(x, y)$ is completely analogous to a contour line on a topographical map.



(a) (b)

Figure 4.7 (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

Definition

Given a function $f(x, y)$ and a number c in the range of f, a **level curve of a function of two variables** for the value c is defined to be the set of points satisfying the equation $f(x, y) = c$.

Returning to the function $g(x, y) = \sqrt{9 - x^2 - y^2},$ we can determine the level curves of this function. The range of g is the closed interval $[0, 3]$. First, we choose any number in this closed interval—say, $c = 2$. The level curve corresponding to $c = 2$ is described by the equation

$$\sqrt{9 - x^2 - y^2} = 2.$$

To simplify, square both sides of this equation:

$$9 - x^2 - y^2 = 4.$$

Now, multiply both sides of the equation by -1 and add 9 to each side:

$$x^2 + y^2 = 5.$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$. Using values of c between 0 and 3 yields other circles also centered at the origin. If $c = 3,$ then the circle has radius $0,$ so it consists solely of the origin. **Figure 4.8** is a graph of the level curves of this function corresponding to $c = 0, 1, 2,$ and 3. Note that in the previous derivation it may be possible that we introduced extra solutions by squaring both sides. This is not the case here because the range of the square root function is nonnegative.

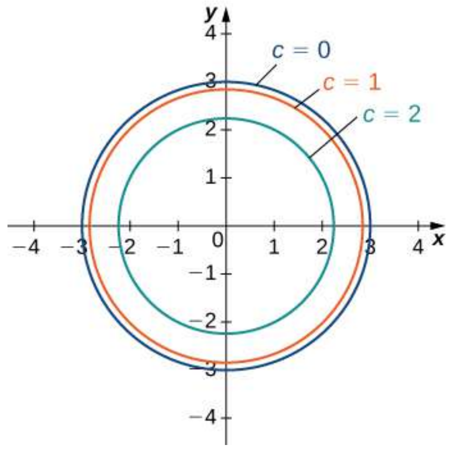


Figure 4.8 Level curves of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, using $c = 0, 1, 2,$ and 3 ($c = 3$ corresponds to the origin).

A graph of the various level curves of a function is called a **contour map**.

Example 4.4

Making a Contour Map

Given the function $f(x, y) = \sqrt{8 + 8x - 4y - 4x^2 - y^2}$, find the level curve corresponding to $c = 0$. Then create a contour map for this function. What are the domain and range of f?

Solution

To find the level curve for $c = 0$, we set $f(x, y) = 0$ and solve. This gives

$$0 = \sqrt{8 + 8x - 4y - 4x^2 - y^2}.$$

We then square both sides and multiply both sides of the equation by -1:

$$4x^2 + y^2 - 8x + 4y - 8 = 0.$$

Now, we rearrange the terms, putting the x terms together and the y terms together, and add 8 to each side:

$$4x^2 - 8x + y^2 + 4y = 8.$$

Next, we group the pairs of terms containing the same variable in parentheses, and factor 4 from the first pair:

$$4\left(x^2 - 2x\right) + \left(y^2 + 4y\right) = 8.$$

Then we complete the square in each pair of parentheses and add the correct value to the right-hand side:

$$4\left(x^2 - 2x + 1\right) + \left(y^2 + 4y + 4\right) = 8 + 4(1) + 4.$$

Next, we factor the left-hand side and simplify the right-hand side:

$$4(x - 1)^2 + (y + 2)^2 = 16.$$

Last, we divide both sides by 16:

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1. \tag{4.1}$$

This equation describes an ellipse centered at $(1, -2)$. The graph of this ellipse appears in the following graph.

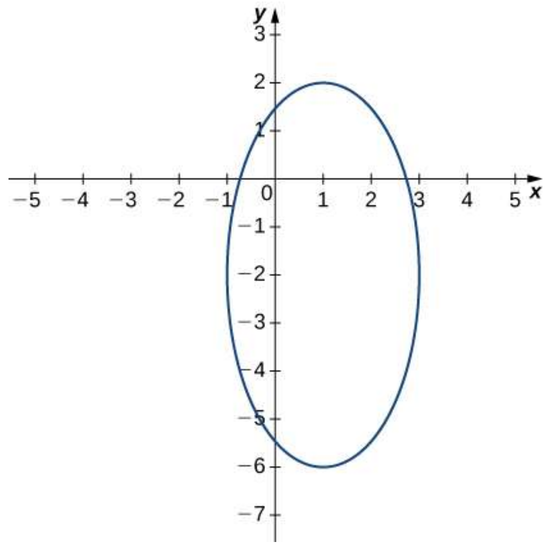


Figure 4.9 Level curve of the function $f(x, y) = \sqrt{8 + 8x - 4y - 4x^2 - y^2}$ corresponding to $c = 0$.

We can repeat the same derivation for values of c less than 4. Then, **Equation 4.1** becomes

$$\frac{4(x-1)^2}{16 - c^2} + \frac{(y+2)^2}{16 - c^2} = 1$$

for an arbitrary value of c. **Figure 4.10** shows a contour map for $f(x, y)$ using the values $c = 0, 1, 2,$ and 3. When $c = 4,$ the level curve is the point $(-1, 2)$.

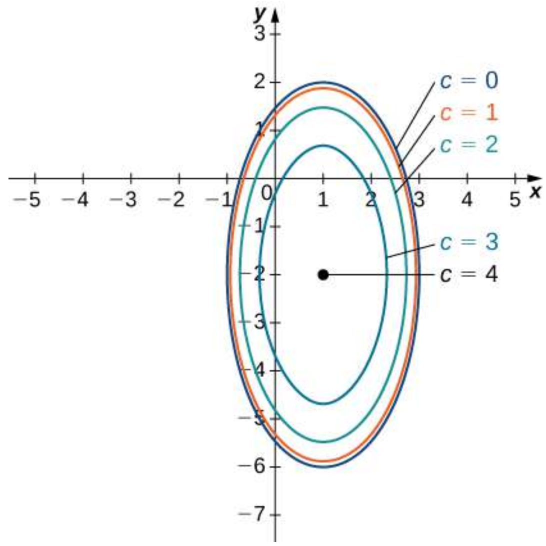


Figure 4.10 Contour map for the function $f(x, y) = \sqrt{8 + 8x - 4y - 4x^2 - y^2}$ using the values $c = 0,\ 1,\ 2,\ 3,$ and $4.$

4.2 Find and graph the level curve of the function $g(x, y) = x^2 + y^2 - 6x + 2y$ corresponding to $c = 15.$

Another useful tool for understanding the **graph of a function of two variables** is called a vertical trace. Level curves are always graphed in the xy-plane, but as their name implies, vertical traces are graphed in the xz- or yz-planes.

Definition

Consider a function $z = f(x, y)$ with domain $D \subseteq \mathbb{R}^2.$ A **vertical trace** of the function can be either the set of points that solves the equation $f(a, y) = z$ for a given constant $x = a$ or $f(x, b) = z$ for a given constant $y = b.$

Example 4.5

Finding Vertical Traces

Find vertical traces for the function $f(x, y) = \sin x \cos y$ corresponding to $x = -\frac{\pi}{4},\ 0,$ and $\frac{\pi}{4},$ and $y = -\frac{\pi}{4},\ 0,$ and $\frac{\pi}{4}.$

Solution

First set $x = -\frac{\pi}{4}$ in the equation $z = \sin x \cos y$:

$$z = \sin\left(-\frac{\pi}{4}\right)\cos y = -\frac{\sqrt{2}\cos y}{2} \approx -0.7071\cos y.$$

This describes a cosine graph in the plane $x = -\frac{\pi}{4}$. The other values of z appear in the following table.

c	***Vertical Trace for*** $x = c$
$-\frac{\pi}{4}$	$z = -\frac{\sqrt{2}\cos y}{2}$
0	$z = 0$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2}\cos y}{2}$

Table 4.1
Vertical Traces Parallel to the xz-Plane for the Function $f(x, y) = \sin x \cos y$

In a similar fashion, we can substitute the y-values in the equation $f(x, y)$ to obtain the traces in the yz-plane, as listed in the following table.

d	***Vertical Trace for*** $y = d$
$-\frac{\pi}{4}$	$z = -\frac{\sqrt{2}\sin x}{2}$
0	$z = \sin x$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2}\sin x}{2}$

Table 4.2
Vertical Traces Parallel to the yz-Plane for the Function $f(x, y) = \sin x \cos y$

The three traces in the xz-plane are cosine functions; the three traces in the yz-plane are sine functions. These curves appear in the intersections of the surface with the planes $x = -\frac{\pi}{4}, x = 0, x = \frac{\pi}{4}$ and $y = -\frac{\pi}{4}, y = 0, y = \frac{\pi}{4}$ as shown in the following figure.

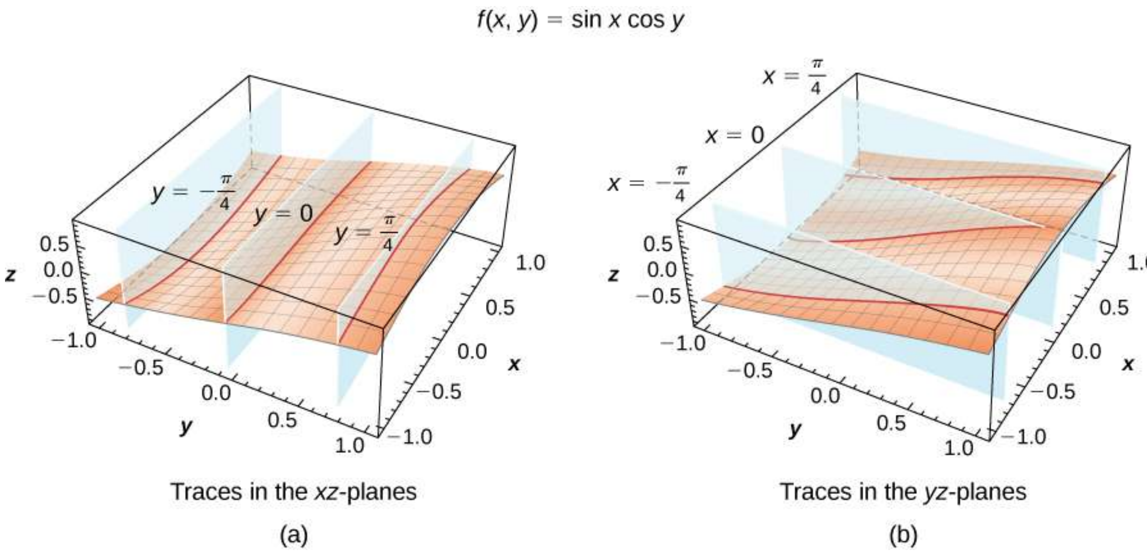


Figure 4.11 Vertical traces of the function $f(x, y)$ are cosine curves in the xz-planes (a) and sine curves in the yz-planes (b).

4.3 Determine the equation of the vertical trace of the function $g(x, y) = -x^2 - y^2 + 2x + 4y - 1$ corresponding to $y = 3,$ and describe its graph.

Functions of two variables can produce some striking-looking surfaces. The following figure shows two examples.

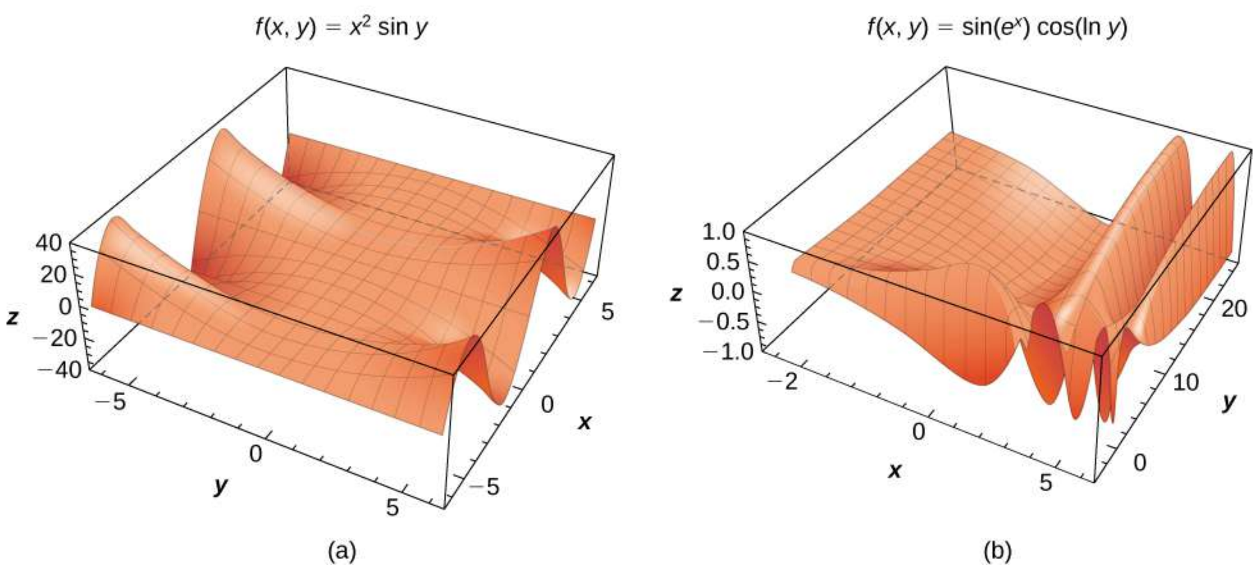


Figure 4.12 Examples of surfaces representing functions of two variables: (a) a combination of a power function and a sine function and (b) a combination of trigonometric, exponential, and logarithmic functions.

Functions of More Than Two Variables

So far, we have examined only functions of two variables. However, it is useful to take a brief look at functions of more than two variables. Two such examples are

$$f(x, y, z) = x^2 - 2xy + y^2 + 3yz - z^2 + 4x - 2y + 3x - 6 \text{ (a polynomial in three variables)}$$

and

$$g(x, y, t) = \left(x^2 - 4xy + y^2\right)\sin t - (3x + 5y)\cos t.$$

In the first function, (x, y, z) represents a point in space, and the function f maps each point in space to a fourth quantity, such as temperature or wind speed. In the second function, (x, y) can represent a point in the plane, and t can represent time. The function might map a point in the plane to a third quantity (for example, pressure) at a given time t. The method for finding the domain of a function of more than two variables is analogous to the method for functions of one or two variables.

Example 4.6

Domains for Functions of Three Variables

Find the domain of each of the following functions:

a. $f(x, y, z) = \dfrac{3x - 4y + 2z}{\sqrt{9 - x^2 - y^2 - z^2}}$

b. $g(x, y, t) = \dfrac{\sqrt{2t - 4}}{x^2 - y^2}$

Solution

a. For the function $f(x, y, z) = \dfrac{3x - 4y + 2z}{\sqrt{9 - x^2 - y^2 - z^2}}$ to be defined (and be a real value), two conditions must hold:

1. The denominator cannot be zero.
2. The radicand cannot be negative.

Combining these conditions leads to the inequality

$$9 - x^2 - y^2 - z^2 > 0.$$

Moving the variables to the other side and reversing the inequality gives the domain as

$$\text{domain}(f) = \left\{(x, y, z) \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 < 9\right\},$$

which describes a ball of radius 3 centered at the origin. (*Note*: The surface of the ball is not included in this domain.)

b. For the function $g(x, y, t) = \dfrac{\sqrt{2t - 4}}{x^2 - y^2}$ to be defined (and be a real value), two conditions must hold:

1. The radicand cannot be negative.
2. The denominator cannot be zero.

Since the radicand cannot be negative, this implies $2t - 4 \geq 0,$ and therefore that $t \geq 2.$ Since the denominator cannot be zero, $x^2 - y^2 \neq 0,$ or $x^2 \neq y^2,$ Which can be rewritten as $y = \pm x,$ which are the equations of two lines passing through the origin. Therefore, the domain of g is

$$\text{domain}(g) = \{(x, y, t) | y \neq \pm x, t \geq 2\}.$$

4.4 Find the domain of the function $h(x, y, t) = (3t - 6)\sqrt{y - 4x^2 + 4}$.

Functions of two variables have level curves, which are shown as curves in the xy-plane. However, when the function has three variables, the curves become surfaces, so we can define level surfaces for functions of three variables.

Definition

Given a function $f(x, y, z)$ and a number c in the range of f, a **level surface of a function of three variables** is defined to be the set of points satisfying the equation $f(x, y, z) = c$.

Example 4.7

Finding a Level Surface

Find the level surface for the function $f(x, y, z) = 4x^2 + 9y^2 - z^2$ corresponding to $c = 1$.

Solution

The level surface is defined by the equation $4x^2 + 9y^2 - z^2 = 1$. This equation describes a hyperboloid of one sheet as shown in the following figure.

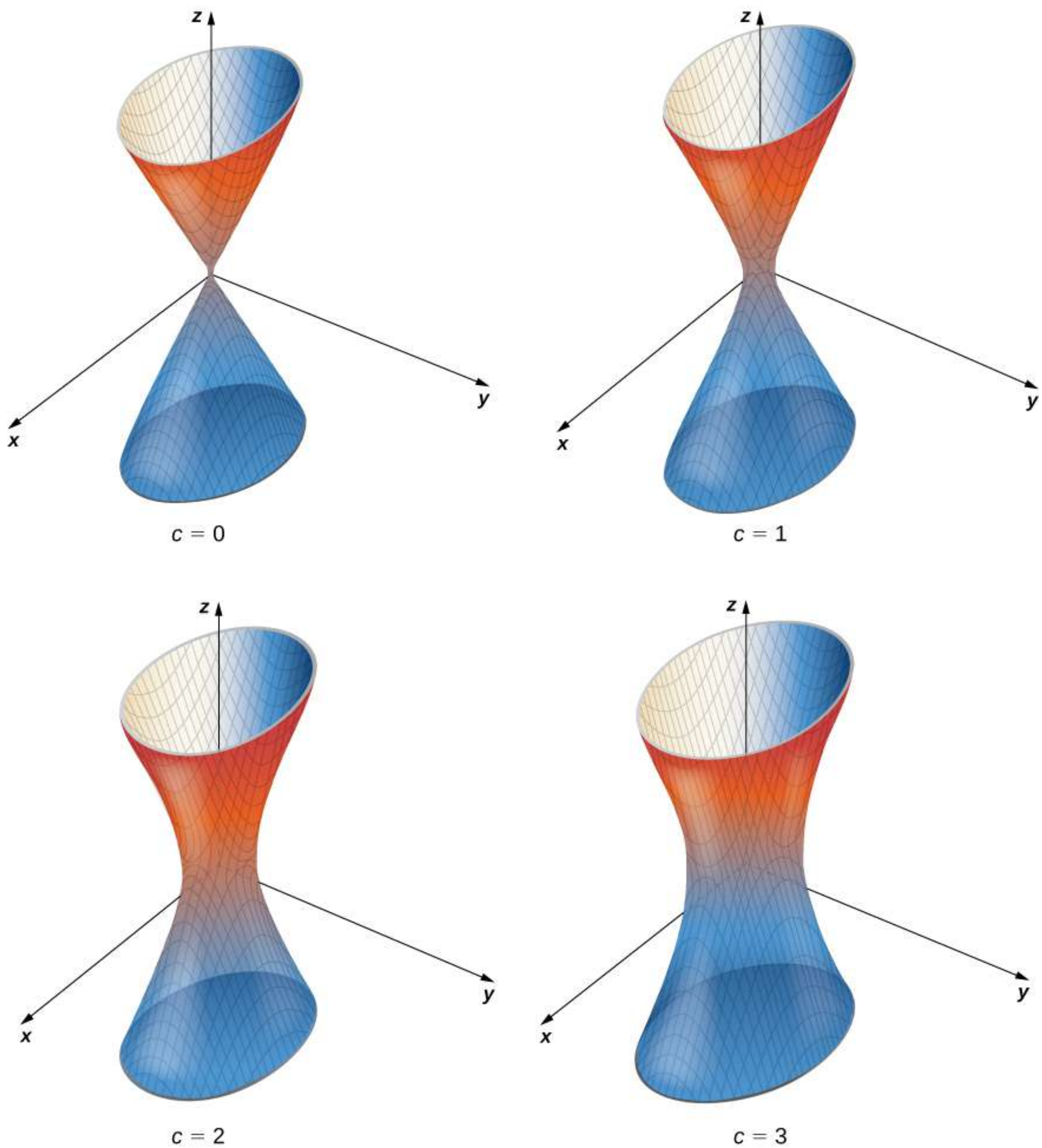


Figure 4.13 A hyperboloid of one sheet with some of its level surfaces.

4.5 Find the equation of the level surface of the function

$$g(x,\ y,\ z) = x^2 + y^2 + z^2 - 2x + 4y - 6z$$

corresponding to $c = 2,$ and describe the surface, if possible.

4.1 EXERCISES

For the following exercises, evaluate each function at the indicated values.

1. $W(x, y) = 4x^2 + y^2$. Find $W(2, -1)$, $W(-3, 6)$.

2. $W(x, y) = 4x^2 + y^2$. Find $W(2 + h, 3 + h)$.

3. The volume of a right circular cylinder is calculated by a function of two variables, $V(x, y) = \pi x^2 y$, where x is the radius of the right circular cylinder and y represents the height of the cylinder. Evaluate $V(2, 5)$ and explain what this means.

4. An oxygen tank is constructed of a right cylinder of height y and radius x with two hemispheres of radius x mounted on the top and bottom of the cylinder. Express the volume of the cylinder as a function of two variables, x and y, find $V(10, 2)$, and explain what this means.

For the following exercises, find the domain of the function.

5. $V(x, y) = 4x^2 + y^2$

6. $f(x, y) = \sqrt{x^2 + y^2 - 4}$

7. $f(x, y) = 4\ln(y^2 - x)$

8. $g(x, y) = \sqrt{16 - 4x^2 - y^2}$

9. $z(x, y) = y^2 - x^2$

10. $f(x, y) = \dfrac{y + 2}{x^2}$

Find the range of the functions.

11. $g(x, y) = \sqrt{16 - 4x^2 - y^2}$

12. $V(x, y) = 4x^2 + y^2$

13. $z = y^2 - x^2$

For the following exercises, find the level curves of each function at the indicated value of c to visualize the given function.

14. $z(x, y) = y^2 - x^2$, $c = 1$

15. $z(x, y) = y^2 - x^2$, $c = 4$

16. $g(x, y) = x^2 + y^2$; $c = 4$, $c = 9$

17. $g(x, y) = 4 - x - y$; $c = 0, 4$

18. $f(x, y) = xy$; $c = 1$; $c = -1$

19. $h(x, y) = 2x - y$; $c = 0, -2, 2$

20. $f(x, y) = x^2 - y$; $c = 1, 2$

21. $g(x, y) = \frac{x}{x + y}$; $c = -1, 0, 2$

22. $g(x, y) = x^3 - y$; $c = -1, 0, 2$

23. $g(x, y) = e^{xy}$; $c = \frac{1}{2}, 3$

24. $f(x, y) = x^2$; $c = 4, 9$

25. $f(x, y) = xy - x$; $c = -2, 0, 2$

26. $h(x, y) = \ln(x^2 + y^2)$; $c = -1, 0, 1$

27. $g(x, y) = \ln\left(\dfrac{y}{x^2}\right)$; $c = -2, 0, 2$

28. $z = f(x, y) = \sqrt{x^2 + y^2}$, $c = 3$

29. $f(x, y) = \dfrac{y + 2}{x^2}$, $c =$ any constant

For the following exercises, find the vertical traces of the functions at the indicated values of x and y, and plot the traces.

30. $z = 4 - x - y$; $x = 2$

31. $f(x, y) = 3x + y^3$, $x = 1$

32. $z = \cos\sqrt{x^2 + y^2}$ $x = 1$

Find the domain of the following functions.

33. $z = \sqrt{100 - 4x^2 - 25y^2}$

34. $z = \ln\left(x - y^2\right)$

35. $f(x, y, z) = \frac{1}{\sqrt{36 - 4x^2 - 9y^2 - z^2}}$

36. $f(x, y, z) = \sqrt{49 - x^2 - y^2 - z^2}$

37. $f(x, y, z) = \sqrt[3]{16 - x^2 - y^2 - z^2}$

38. $f(x, y) = \cos\sqrt{x^2 + y^2}$

For the following exercises, plot a graph of the function.

39. $z = f(x, y) = \sqrt{x^2 + y^2}$

40. $z = x^2 + y^2$

41. Use technology to graph $z = x^2 y$.

Sketch the following by finding the level curves. Verify the graph using technology.

42. $f(x, y) = \sqrt{4 - x^2 - y^2}$

43. $f(x, y) = 2 - \sqrt{x^2 + y^2}$

44. $z = 1 + e^{-x^2 - y^2}$

45. $z = \cos\sqrt{x^2 + y^2}$

46. $z = y^2 - x^2$

47. Describe the contour lines for several values of c for $z = x^2 + y^2 - 2x - 2y$.

Find the level surface for the functions of three variables and describe it.

48. $w(x, y, z) = x - 2y + z, \ c = 4$

49. $w(x, y, z) = x^2 + y^2 + z^2, \ c = 9$

50. $w(x, y, z) = x^2 + y^2 - z^2, \ c = -4$

51. $w(x, y, z) = x^2 + y^2 - z^2, \ c = 4$

52. $w(x, y, z) = 9x^2 - 4y^2 + 36z^2, \ c = 0$

For the following exercises, find an equation of the level curve of f that contains the point P.

53. $f(x, y) = 1 - 4x^2 - y^2, \ P(0, 1)$

54. $g(x, y) = y^2 \arctan x, \ P(1, 2)$

55. $g(x, y) = e^{xy}(x^2 + y^2), \ P(1, 0)$

56. The strength E of an electric field at point (x, y, z) resulting from an infinitely long charged wire lying along the y-axis is given by $E(x, y, z) = k/\sqrt{x^2 + y^2}$, where k is a positive constant. For simplicity, let $k = 1$ and find the equations of the level surfaces for $E = 10$ and $E = 100$.

57. A thin plate made of iron is located in the xy-plane. The temperature T in degrees Celsius at a point $P(x, y)$ is inversely proportional to the square of its distance from the origin. Express T as a function of x and y.

58. Refer to the preceding problem. Using the temperature function found there, determine the proportionality constant if the temperature at point $P(1, 2) \, is \, 50°C$. Use this constant to determine the temperature at point $Q(3, 4)$.

59. Refer to the preceding problem. Find the level curves for $T = 40°C$ and $T = 100°C$, and describe what the level curves represent.

4.2 | Limits and Continuity

Learning Objectives

4.2.1 Calculate the limit of a function of two variables.

4.2.2 Learn how a function of two variables can approach different values at a boundary point, depending on the path of approach.

4.2.3 State the conditions for continuity of a function of two variables.

4.2.4 Verify the continuity of a function of two variables at a point.

4.2.5 Calculate the limit of a function of three or more variables and verify the continuity of the function at a point.

We have now examined functions of more than one variable and seen how to graph them. In this section, we see how to take the limit of a function of more than one variable, and what it means for a function of more than one variable to be continuous at a point in its domain. It turns out these concepts have aspects that just don't occur with functions of one variable.

Limit of a Function of Two Variables

Recall from Section 2.2 the definition of a limit of a function of one variable:

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$ for all x in the domain of f, then

$$|f(x) - L| > \varepsilon.$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Definition

Consider a point $(a, b) \in \mathbb{R}^2$. A $\boldsymbol{\delta}$ **disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b)—that is,

$$\left\{(x, y) \in \mathbb{R}^2 \middle| (x - a)^2 + (y - b)^2 < \delta^2\right\}$$

as shown in the following graph.

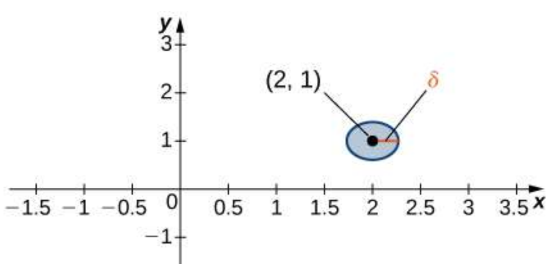


Figure 4.14 A δ disk centered around the point $(2, 1)$.

The idea of a δ disk appears in the definition of the limit of a function of two variables. If δ is small, then all the points (x, y) in the δ disk are close to (a, b). This is completely analogous to x being close to a in the definition of a limit of a function of one variable. In one dimension, we express this restriction as

$$a - \delta < x < a + \delta.$$

In more than one dimension, we use a δ disk.

Definition

Let f be a function of two variables, x and y. The limit of $f(x, y)$ as (x, y) approaches (a, b) is L, written

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x, y) in a δ disk around (a, b), except possibly for (a, b) itself, the value of $f(x, y)$ is no more than ε away from L (**Figure 4.15**). Using symbols, we write the following: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

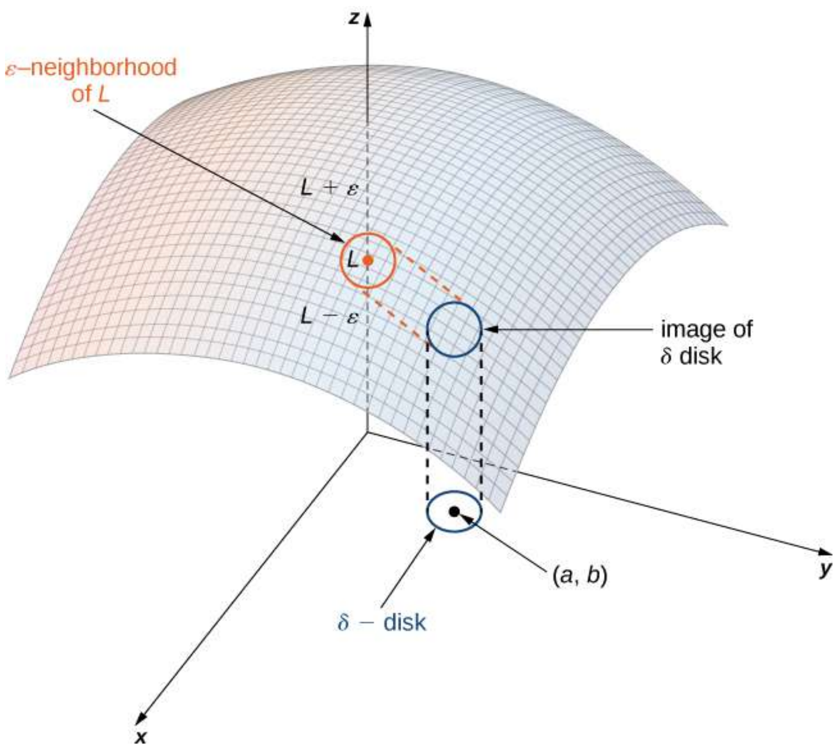


Figure 4.15 The limit of a function involving two variables requires that $f(x, y)$ be within ε of L whenever (x, y) is within δ of (a, b). The smaller the value of ε, the smaller the value of δ.

Proving that a limit exists using the definition of a limit of a function of two variables can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits. The formulas in this theorem are an extension of the formulas in the limit laws theorem in **The Limit Laws (http://cnx.org/content/m53492/latest/)** .

Theorem 4.1: Limit laws for functions of two variables

Let $f(x, y)$ and $g(x, y)$ be defined for all $(x, y) \neq (a, b)$ in a neighborhood around (a, b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that $\lim_{(x, y) \to (a, b)} f(x, y) = L$ and $\lim_{(x, y) \to (a, b)} g(x, y) = M$, and let c be a constant. Then each of the following statements holds:

Constant Law:

$$\lim_{(x,\ y) \to (a,\ b)} c = c \tag{4.2}$$

Identity Laws:

$$\lim_{(x,\ y) \to (a,\ b)} x = a \tag{4.3}$$

$$\lim_{(x,\ y) \to (a,\ b)} y = b \tag{4.4}$$

Sum Law:

$$\lim_{(x,\ y) \to (a,\ b)} (f(x,\ y) + g(x,\ y)) = L + M \tag{4.5}$$

Difference Law:

$$\lim_{(x,\ y) \to (a,\ b)} (f(x,\ y) - g(x,\ y)) = L - M \tag{4.6}$$

Constant Multiple Law:

$$\lim_{(x,\ y) \to (a,\ b)} (cf(x,\ y)) = cL \tag{4.7}$$

Product Law:

$$\lim_{(x,\ y) \to (a,\ b)} (f(x,\ y)g(x,\ y)) = LM \tag{4.8}$$

Quotient Law:

$$\lim_{(x,\ y) \to (a,\ b)} \frac{f(x,\ y)}{g(x,\ y)} = \frac{L}{M} \text{ for } M \neq 0 \tag{4.9}$$

Power Law:

$$\lim_{(x,\ y) \to (a,\ b)} (f(x,\ y))^n = L^n \tag{4.10}$$

for any positive integer n.

Root Law:

$$\lim_{(x,\ y) \to (a,\ b)} \sqrt[n]{f(x,\ y)} = \sqrt[n]{L} \tag{4.11}$$

for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.

The proofs of these properties are similar to those for the limits of functions of one variable. We can apply these laws to finding limits of various functions.

Example 4.8

Finding the Limit of a Function of Two Variables

Find each of the following limits:

a. $\lim_{(x,\ y) \to (2,\ -1)} \left(x^2 - 2xy + 3y^2 - 4x + 3y - 6\right)$

b. $\lim_{(x,\ y) \to (2,\ -1)} \frac{2x + 3y}{4x - 3y}$

Solution

a. First use the sum and difference laws to separate the terms:

$$\begin{aligned}&\lim_{(x,\,y)\to(2,\,-1)}\left(x^2-2xy+3y^2-4x+3y-6\right)\\&=\left(\lim_{(x,\,y)\to(2,\,-1)}x^2\right)-\left(\lim_{(x,\,y)\to(2,\,-1)}2xy\right)+\left(\lim_{(x,\,y)\to(2,\,-1)}3y^2\right)-\left(\lim_{(x,\,y)\to(2,\,-1)}4x\right)\\&\quad+\left(\lim_{(x,\,y)\to(2,\,-1)}3y\right)-\left(\lim_{(x,\,y)\to(2,\,-1)}6\right).\end{aligned}$$

Next, use the constant multiple law on the second, third, fourth, and fifth limits:

$$\begin{aligned}&=\left(\lim_{(x,\,y)\to(2,\,-1)}x^2\right)-2\left(\lim_{(x,\,y)\to(2,\,-1)}xy\right)+3\left(\lim_{(x,\,y)\to(2,\,-1)}y^2\right)-4\left(\lim_{(x,\,y)\to(2,\,-1)}x\right)\\&\quad+3\left(\lim_{(x,\,y)\to(2,\,-1)}y\right)-\lim_{(x,\,y)\to(2,\,-1)}6.\end{aligned}$$

Now, use the power law on the first and third limits, and the product law on the second limit:

$$\begin{aligned}&=\left(\lim_{(x,\,y)\to(2,\,-1)}x\right)^2-2\left(\lim_{(x,\,y)\to(2,\,-1)}x\right)\left(\lim_{(x,\,y)\to(2,\,-1)}y\right)+3\left(\lim_{(x,\,y)\to(2,\,-1)}y\right)^2\\&\quad-4\left(\lim_{(x,\,y)\to(2,\,-1)}x\right)+3\left(\lim_{(x,\,y)\to(2,\,-1)}y\right)-\lim_{(x,\,y)\to(2,\,-1)}6.\end{aligned}$$

Last, use the identity laws on the first six limits and the constant law on the last limit:

$$\begin{aligned}\lim_{(x,\,y)\to(2,\,-1)}\left(x^2-2xy+3y^2-4x+3y-6\right)&=(2)^2-2(2)(-1)+3(-1)^2-4(2)+3(-1)-6\\&=-6.\end{aligned}$$

b. Before applying the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, constant multiple law, and identity law,

$$\begin{aligned}\lim_{(x,\,y)\to(2,\,-1)}(4x-3y)&=\lim_{(x,\,y)\to(2,\,-1)}4x-\lim_{(x,\,y)\to(2,\,-1)}3y\\&=4\left(\lim_{(x,\,y)\to(2,\,-1)}x\right)-3\left(\lim_{(x,\,y)\to(2,\,-1)}y\right)\\&=4(2)-3(-1)=11.\end{aligned}$$

Since the limit of the denominator is nonzero, the quotient law applies. We now calculate the limit of the numerator using the difference law, constant multiple law, and identity law:

$$\begin{aligned}\lim_{(x,\,y)\to(2,\,-1)}(2x+3y)&=\lim_{(x,\,y)\to(2,\,-1)}2x+\lim_{(x,\,y)\to(2,\,-1)}3y\\&=2\left(\lim_{(x,\,y)\to(2,\,-1)}x\right)+3\left(\lim_{(x,\,y)\to(2,\,-1)}y\right)\\&=2(2)+3(-1)\\&=1.\end{aligned}$$

Therefore, according to the quotient law we have

$$\lim_{(x,\, y) \to (2,\, -1)} \frac{2x + 3y}{4x - 3y} = \frac{\lim_{(x,\, y) \to (2,\, -1)} (2x + 3y)}{\lim_{(x,\, y) \to (2,\, -1)} (4x - 3y)} = \frac{1}{11}.$$

4.6 Evaluate the following limit:

$$\lim_{(x,\, y) \to (5,\, -2)} \sqrt[3]{\frac{x^2 - y}{y^2 + x - 1}}.$$

Since we are taking the limit of a function of two variables, the point (a, b) is in $\mathbb{R}^2$, and it is possible to approach this point from an infinite number of directions. Sometimes when calculating a limit, the answer varies depending on the path taken toward (a, b). If this is the case, then the limit fails to exist. In other words, the limit must be unique, regardless of path taken.

Example 4.9

Limits That Fail to Exist

Show that neither of the following limits exist:

a. $\lim_{(x,\, y) \to (0,\, 0)} \frac{2xy}{3x^2 + y^2}$

b. $\lim_{(x,\, y) \to (0,\, 0)} \frac{4xy^2}{x^2 + 3y^4}$

Solution

a. The domain of the function $f(x, y) = \frac{2xy}{3x^2 + y^2}$ consists of all points in the xy-plane except for the point $(0, 0)$ (**Figure 4.16**). To show that the limit does not exist as (x, y) approaches $(0, 0)$, we note that it is impossible to satisfy the definition of a limit of a function of two variables because of the fact that the function takes different values along different lines passing through point $(0, 0)$. First, consider the line $y = 0$ in the xy-plane. Substituting $y = 0$ into $f(x, y)$ gives

$$f(x, 0) = \frac{2x(0)}{3x^2 + 0^2} = 0$$

for any value of x. Therefore the value of f remains constant for any point on the x-axis, and as y approaches zero, the function remains fixed at zero.

Next, consider the line $y = x$. Substituting $y = x$ into $f(x, y)$ gives

$$f(x, x) = \frac{2x(x)}{3x^2 + x^2} = \frac{2x^2}{4x^2} = \frac{1}{2}.$$

This is true for any point on the line $y = x$. If we let x approach zero while staying on this line, the value of the function remains fixed at $\frac{1}{2}$, regardless of how small x is.

Choose a value for ε that is less than $1/2$—say, $1/4$. Then, no matter how small a δ disk we draw around $(0, 0)$, the values of $f(x, y)$ for points inside that δ disk will include both 0 and $\frac{1}{2}$. Therefore, the definition of limit at a point is never satisfied and the limit fails to exist.

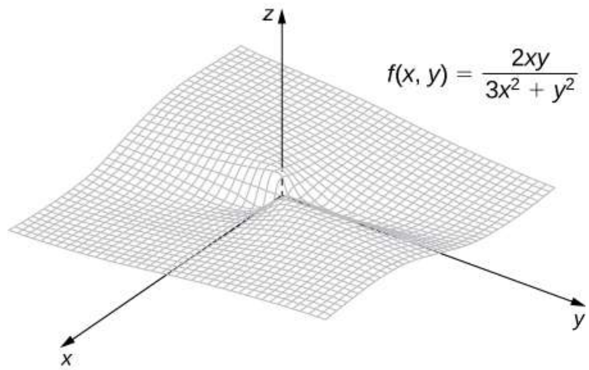


Figure 4.16 Graph of the function $f(x, y) = (2xy)/\left(3x^2 + y^2\right)$. Along the line $y = 0$, the function is equal to zero; along the line $y = x$, the function is equal to $\frac{1}{2}$.

In a similar fashion to a., we can approach the origin along any straight line passing through the origin. If we try the x-axis (i.e., $y = 0$), then the function remains fixed at zero. The same is true for the y-axis. Suppose we approach the origin along a straight line of slope k. The equation of this line is $y = kx$. Then the limit becomes

$$\begin{aligned}
\lim_{(x,\, y) \to (0,\, 0)} \frac{4xy^2}{x^2 + 3y^4} &= \lim_{(x,\, y) \to (0,\, 0)} \frac{4x(kx)^2}{x^2 + 3(kx)^4} \\
&= \lim_{(x,\, y) \to (0,\, 0)} \frac{4k^2x^3}{x^2 + 3k^4x^4} \\
&= \lim_{(x,\, y) \to (0,\, 0)} \frac{4k^2x}{1 + 3k^4x^2} \\
&= \frac{\lim_{(x,\, y) \to (0,\, 0)} \left(4k^2x\right)}{\lim_{(x,\, y) \to (0,\, 0)} \left(1 + 3k^4x^2\right)} \\
&= 0
\end{aligned}$$

regardless of the value of k. It would seem that the limit is equal to zero. What if we chose a curve passing through the origin instead? For example, we can consider the parabola given by the equation $x = y^2$. Substituting y^2 in place of x in $f(x, y)$ gives

$$\begin{aligned}\lim_{(x,\,y) \to (0,\,0)} \frac{4xy^2}{x^2 + 3y^4} &= \lim_{(x,\,y) \to (0,\,0)} \frac{4\left(y^2\right)y^2}{\left(y^2\right)^2 + 3y^4} \\ &= \lim_{(x,\,y) \to (0,\,0)} \frac{4y^4}{y^4 + 3y^4} \\ &= \lim_{(x,\,y) \to (0,\,0)} 1 \\ &= 1.\end{aligned}$$

By the same logic in a., it is impossible to find a δ disk around the origin that satisfies the definition of the limit for any value of $\varepsilon < 1$. Therefore, $\lim_{(x,\,y) \to (0,\,0)} \frac{4xy^2}{x^2 + 3y^4}$ does not exist.

4.7 Show that

$$\lim_{(x,\,y) \to (2,\,1)} \frac{(x-2)(y-1)}{(x-2)^2 + (y-1)^2}$$

does not exist.

Interior Points and Boundary Points

To study continuity and differentiability of a function of two or more variables, we first need to learn some new terminology.

Definition

Let S be a subset of $\mathbb{R}^2$ (**Figure 4.17**).

1. A point P_0 is called an **interior point** of S if there is a δ disk centered around P_0 contained completely in S.
2. A point P_0 is called a **boundary point** of S if every δ disk centered around P_0 contains points both inside and outside S.

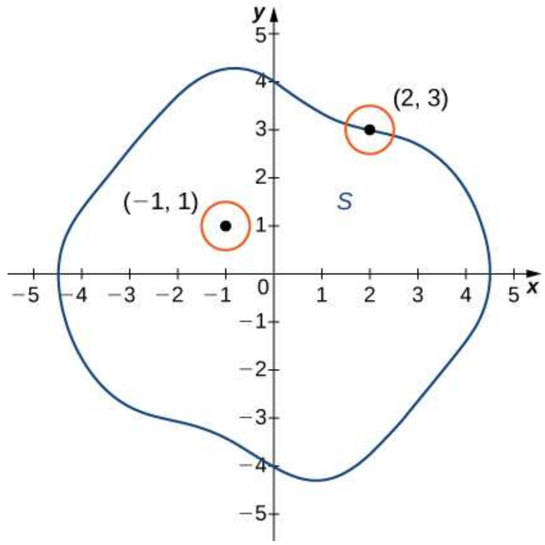


Figure 4.17 In the set S shown, $(-1, 1)$ is an interior point and $(2, 3)$ is a boundary point.

Definition

Let S be a subset of $\mathbb{R}^2$ (**Figure 4.17**).

1. S is called an **open set** if every point of S is an interior point.
2. S is called a **closed set** if it contains all its boundary points.

An example of an open set is a δ disk. If we include the boundary of the disk, then it becomes a closed set. A set that contains some, but not all, of its boundary points is neither open nor closed. For example if we include half the boundary of a δ disk but not the other half, then the set is neither open nor closed.

Definition

Let S be a subset of $\mathbb{R}^2$ (**Figure 4.17**).

1. An open set S is a **connected set** if it cannot be represented as the union of two or more disjoint, nonempty open subsets.
2. A set S is a **region** if it is open, connected, and nonempty.

The definition of a limit of a function of two variables requires the δ disk to be contained inside the domain of the function. However, if we wish to find the limit of a function at a boundary point of the domain, the δ disk is not contained inside the domain. By definition, some of the points of the δ disk are inside the domain and some are outside. Therefore, we need only consider points that are inside both the δ disk and the domain of the function. This leads to the definition of the limit of a function at a boundary point.

Definition

Let f be a function of two variables, x and y, and suppose (a, b) is on the boundary of the domain of f. Then, the limit of $f(x, y)$ as (x, y) approaches (a, b) is L, written

$$\lim_{(x, y) \to (a, b)} f(x, y) = L,$$

if for any $\varepsilon > 0,$ there exists a number $\delta > 0$ such that for any point (x, y) inside the domain of f and within a suitably small distance positive δ of $(a, b),$ the value of $f(x, y)$ is no more than ε away from L (**Figure 4.15**). Using symbols, we can write: For any $\varepsilon > 0,$ there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Example 4.10

Limit of a Function at a Boundary Point

Prove $\lim_{(x, y) \to (4, 3)} \sqrt{25 - x^2 - y^2} = 0.$

Solution

The domain of the function $f(x, y) = \sqrt{25 - x^2 - y^2}$ is $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \le 25\},$ which is a circle of radius 5 centered at the origin, along with its interior as shown in the following graph.

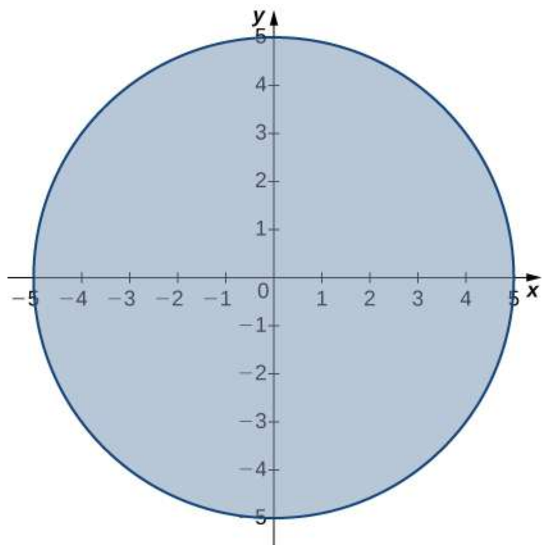


Figure 4.18 Domain of the function $f(x, y) = \sqrt{25 - x^2 - y^2}.$

We can use the limit laws, which apply to limits at the boundary of domains as well as interior points:

$$\begin{aligned}
\lim_{(x, y) \to (4, 3)} \sqrt{25 - x^2 - y^2} &= \sqrt{\lim_{(x, y) \to (4, 3)} \left(25 - x^2 - y^2\right)} \\
&= \sqrt{\lim_{(x, y) \to (4, 3)} 25 - \lim_{(x, y) \to (4, 3)} x^2 - \lim_{(x, y) \to (4, 3)} y^2} \\
&= \sqrt{25 - 4^2 - 3^2} \\
&= 0.
\end{aligned}$$

See the following graph.

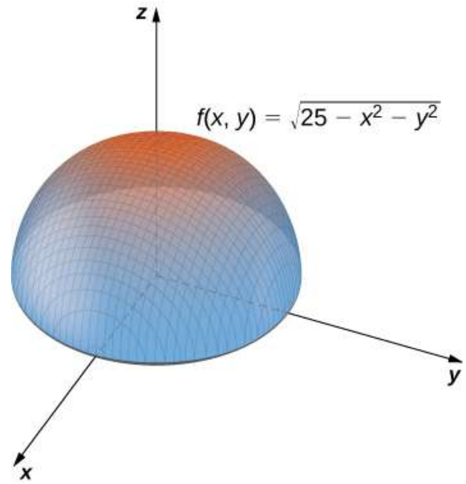


Figure 4.19 Graph of the function $f(x, y) = \sqrt{25 - x^2 - y^2}$.

4.8 Evaluate the following limit:

$$\lim_{(x, y) \to (5, -2)} \sqrt{29 - x^2 - y^2}.$$

Continuity of Functions of Two Variables

In **Continuity (http://cnx.org/content/m53489/latest/)** , we defined the continuity of a function of one variable and saw how it relied on the limit of a function of one variable. In particular, three conditions are necessary for $f(x)$ to be continuous at point $x = a$:

1. $f(a)$ exists.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

These three conditions are necessary for continuity of a function of two variables as well.

Definition

A function $f(x, y)$ is continuous at a point (a, b) in its domain if the following conditions are satisfied:

1. $f(a, b)$ exists.
2. $\lim_{(x, y) \to (a, b)} f(x, y)$ exists.
3. $\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$.

Example 4.11

Demonstrating Continuity for a Function of Two Variables

Show that the function $f(x, y) = \dfrac{3x + 2y}{x + y + 1}$ is continuous at point $(5, -3)$.

Solution

There are three conditions to be satisfied, per the definition of continuity. In this example, $a = 5$ and $b = -3$.

1. $f(a, b)$ exists. This is true because the domain of the function f consists of those ordered pairs for which the denominator is nonzero (i.e., $x + y + 1 \neq 0$). Point $(5, -3)$ satisfies this condition. Furthermore,

$$f(a, b) = f(5, -3) = \frac{3(5) + 2(-3)}{5 + (-3) + 1} = \frac{15 - 6}{2 + 1} = 3.$$

2. $\lim_{(x, y) \to (a, b)} f(x, y)$ exists. This is also true:

$$\begin{aligned}\lim_{(x, y) \to (a, b)} f(x, y) &= \lim_{(x, y) \to (5, -3)} \frac{3x + 2y}{x + y + 1} \\ &= \frac{\lim_{(x, y) \to (5, -3)} (3x + 2y)}{\lim_{(x, y) \to (5, -3)} (x + y + 1)} \\ &= \frac{15 - 6}{5 - 3 + 1} \\ &= 3.\end{aligned}$$

3. $\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$. This is true because we have just shown that both sides of this equation equal three.

4.9 Show that the function $f(x, y) = \sqrt{26 - 2x^2 - y^2}$ is continuous at point $(2, -3)$.

Continuity of a function of any number of variables can also be defined in terms of delta and epsilon. A function of two variables is continuous at a point (x_0, y_0) in its domain if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, whenever $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ it is true, $|f(x, y) - f(a, b)| < \varepsilon$. This definition can be combined with the formal definition (that is, the *epsilon–delta definition*) of continuity of a function of one variable to prove the following theorems:

Theorem 4.2: The Sum of Continuous Functions Is Continuous

If $f(x, y)$ is continuous at (x_0, y_0), and $g(x, y)$ is continuous at (x_0, y_0), then $f(x, y) + g(x, y)$ is continuous at (x_0, y_0).

Theorem 4.3: The Product of Continuous Functions Is Continuous

If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0).

Theorem 4.4: The Composition of Continuous Functions Is Continuous

Let g be a function of two variables from a domain $D \subseteq \mathbb{R}^2$ to a range $R \subseteq \mathbb{R}$. Suppose g is continuous at some point $(x_0, y_0) \in D$ and define $z_0 = g(x_0, y_0)$. Let f be a function that maps $\mathbb{R}$ to $\mathbb{R}$ such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f \circ g$ is continuous at (x_0, y_0) as shown in the following figure.

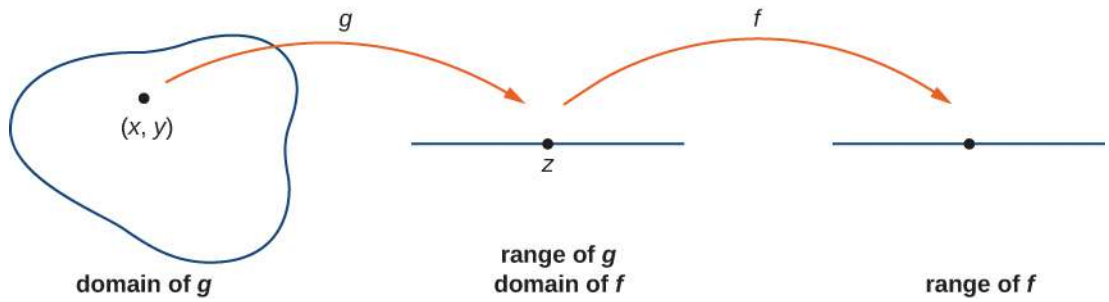


Figure 4.20 The composition of two continuous functions is continuous.

Let's now use the previous theorems to show continuity of functions in the following examples.

Example 4.12

More Examples of Continuity of a Function of Two Variables

Show that the functions $f(x, y) = 4x^3 y^2$ and $g(x, y) = \cos\left(4x^3 y^2\right)$ are continuous everywhere.

Solution

The polynomials $g(x) = 4x^3$ and $h(y) = y^2$ are continuous at every real number, and therefore by the product of continuous functions theorem, $f(x, y) = 4x^3 y^2$ is continuous at every point (x, y) in the xy-plane. Since $f(x, y) = 4x^3 y^2$ is continuous at every point (x, y) in the xy-plane and $g(x) = \cos x$ is continuous at every real number x, the continuity of the composition of functions tells us that $g(x, y) = \cos\left(4x^3 y^2\right)$ is continuous at every point (x, y) in the xy-plane.

4.10 Show that the functions $f(x, y) = 2x^2 y^3 + 3$ and $g(x, y) = \left(2x^2 y^3 + 3\right)^4$ are continuous everywhere.

Functions of Three or More Variables

The limit of a function of three or more variables occurs readily in applications. For example, suppose we have a function $f(x, y, z)$ that gives the temperature at a physical location (x, y, z) in three dimensions. Or perhaps a function $g(x, y, z, t)$ can indicate air pressure at a location (x, y, z) at time t. How can we take a limit at a point in $\mathbb{R}^3$? What does it mean to be continuous at a point in four dimensions?

The answers to these questions rely on extending the concept of a δ disk into more than two dimensions. Then, the ideas of the limit of a function of three or more variables and the continuity of a function of three or more variables are very similar to the definitions given earlier for a function of two variables.

Definition

Let (x_0, y_0, z_0) be a point in $\mathbb{R}^3$. Then, a **δ ball** in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0, y_0, z_0)—that is,

$$\left\{(x, y, z) \in \mathbb{R}^3 \middle| \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta\right\}.$$

To define a δ ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension. For example, given a point $P = (w_0, x_0, y_0, z_0)$ in $\mathbb{R}^4$, a δ ball around P can be described by

$$\left\{(w, x, y, z) \in \mathbb{R}^4 \middle| \sqrt{(w - w_0)^2 + (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta\right\}.$$

To show that a limit of a function of three variables exists at a point (x_0, y_0, z_0), it suffices to show that for any point in a δ ball centered at (x_0, y_0, z_0), the value of the function at that point is arbitrarily close to a fixed value (the limit value). All the limit laws for functions of two variables hold for functions of more than two variables as well.

Example 4.13

Finding the Limit of a Function of Three Variables

Find $\lim_{(x, y, z) \to (4, 1, -3)} \frac{x^2 y - 3z}{2x + 5y - z}$.

Solution

Before we can apply the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, the identity law, and the constant law,

$$\begin{aligned}\lim_{(x, y, z) \to (4, 1, -3)} (2x + 5y - z) &= 2\left(\lim_{(x, y, z) \to (4, 1, -3)} x\right) + 5\left(\lim_{(x, y, z) \to (4, 1, -3)} y\right) - \left(\lim_{(x, y, z) \to (4, 1, -3)} z\right)\\ &= 2(4) + 5(1) - (-3)\\ &= 16.\end{aligned}$$

Since this is nonzero, we next find the limit of the numerator. Using the product law, difference law, constant multiple law, and identity law,

$$\begin{aligned}\lim_{(x, y, z) \to (4, 1, -3)} \left(x^2 y - 3z\right) &= \left(\lim_{(x, y, z) \to (4, 1, -3)} x\right)^2 \left(\lim_{(x, y, z) \to (4, 1, -3)} y\right) - 3 \lim_{(x, y, z) \to (4, 1, -3)} z\\ &= \left(4^2\right)(1) - 3(-3)\\ &= 16 + 9\\ &= 25.\end{aligned}$$

Last, applying the quotient law:

$$\begin{aligned}\lim_{(x, y, z) \to (4, 1, -3)} \frac{x^2 y - 3z}{2x + 5y - z} &= \frac{\lim_{(x, y, z) \to (4, 1, -3)} \left(x^2 y - 3z\right)}{\lim_{(x, y, z) \to (4, 1, -3)} (2x + 5y - z)}\\ &= \frac{25}{16}.\end{aligned}$$

4.11 Find $\lim_{(x,\ y,\ z) \to (4,\ -1,\ 3)} \sqrt{13 - x^2 - 2y^2 + z^2}$.

4.2 EXERCISES

For the following exercises, find the limit of the function.

60. $\lim_{(x, y) \to (1, 2)} x$

61. $\lim_{(x, y) \to (1, 2)} \frac{5x^2 y}{x^2 + y^2}$

62. Show that the limit $\lim_{(x, y) \to (0, 0)} \frac{5x^2 y}{x^2 + y^2}$ exists and is the same along the paths: y-axis and x-axis, and along $y = x$.

For the following exercises, evaluate the limits at the indicated values of x and y. If the limit does not exist, state this and explain why the limit does not exist.

63. $\lim_{(x, y) \to (0, 0)} \frac{4x^2 + 10y^2 + 4}{4x^2 - 10y^2 + 6}$

64. $\lim_{(x, y) \to (11, 13)} \sqrt{\frac{1}{xy}}$

65. $\lim_{(x, y) \to (0, 1)} \frac{y^2 \sin x}{x}$

66. $\lim_{(x, y) \to (0, 0)} \sin\left(\frac{x^8 + y^7}{x - y + 10}\right)$

67. $\lim_{(x, y) \to (\pi/4, 1)} \frac{y \tan x}{y + 1}$

68. $\lim_{(x, y) \to (0, \pi/4)} \frac{\sec x + 2}{3x - \tan y}$

69. $\lim_{(x, y) \to (2, 5)} \left(\frac{1}{x} - \frac{5}{y}\right)$

70. $\lim_{(x, y) \to (4, 4)} x \ln y$

71. $\lim_{(x, y) \to (4, 4)} e^{-x^2 - y^2}$

72. $\lim_{(x, y) \to (0, 0)} \sqrt{9 - x^2 - y^2}$

73. $\lim_{(x, y) \to (1, 2)} \left(x^2 y^3 - x^3 y^2 + 3x + 2y\right)$

74. $\lim_{(x, y) \to (\pi, \pi)} x \sin\left(\frac{x + y}{4}\right)$

75. $\lim_{(x, y) \to (0, 0)} \frac{xy + 1}{x^2 + y^2 + 1}$

76. $\lim_{(x, y) \to (0, 0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

77. $\lim_{(x, y) \to (0, 0)} \ln\left(x^2 + y^2\right)$

For the following exercises, complete the statement.

78. A point (x_0, y_0) in a plane region R is an interior point of R if _________________.

79. A point (x_0, y_0) in a plane region R is called a boundary point of R if ___________.

For the following exercises, use algebraic techniques to evaluate the limit.

80. $\lim_{(x, y) \to (2, 1)} \frac{x - y - 1}{\sqrt{x - y} - 1}$

81. $\lim_{(x, y) \to (0, 0)} \frac{x^4 - 4y^4}{x^2 + 2y^2}$

82. $\lim_{(x, y) \to (0, 0)} \frac{x^3 - y^3}{x - y}$

83. $\lim_{(x, y) \to (0, 0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

For the following exercises, evaluate the limits of the functions of three variables.

84. $\lim_{(x, y, z) \to (1, 2, 3)} \frac{xz^2 - y^2 z}{xyz - 1}$

85. $\lim_{(x, y, z) \to (0, 0, 0)} \frac{x^2 - y^2 - z^2}{x^2 + y^2 - z^2}$

For the following exercises, evaluate the limit of the function by determining the value the function approaches along the indicated paths. If the limit does not exist, explain why not.

86. $\lim_{(x, y) \to (0, 0)} \frac{xy + y^3}{x^2 + y^2}$

a. Along the x-axis $(y = 0)$
b. Along the y-axis $(x = 0)$
c. Along the path $y = 2x$

87. Evaluate $\lim_{(x, y) \to (0, 0)} \frac{xy + y^3}{x^2 + y^2}$ using the results of previous problem.

88. $\lim_{(x, y) \to (0, 0)} \frac{x^2 y}{x^4 + y^2}$

a. Along the x-axis $(y = 0)$
b. Along the y-axis $(x = 0)$
c. Along the path $y = x^2$

89. Evaluate $\lim_{(x, y) \to (0, 0)} \frac{x^2 y}{x^4 + y^2}$ using the results of previous problem.

Discuss the continuity of the following functions. Find the largest region in the xy-plane in which the following functions are continuous.

90. $f(x, y) = \sin(xy)$

91. $f(x, y) = \ln(x + y)$

92. $f(x, y) = e^{3xy}$

93. $f(x, y) = \frac{1}{xy}$

For the following exercises, determine the region in which the function is continuous. Explain your answer.

94. $f(x, y) = \frac{x^2 y}{x^2 + y^2}$

95. $f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ (*Hint*: Show that the function approaches different values along two different paths.)

96. $f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$

97. Determine whether $g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ is continuous at $(0, 0)$.

98. Create a plot using graphing software to determine where the limit does not exist. Determine the region of the coordinate plane in which $f(x, y) = \frac{1}{x^2 - y}$ is continuous.

99. Determine the region of the xy-plane in which the composite function $g(x, y) = \arctan\left(\frac{xy^2}{x + y}\right)$ is continuous. Use technology to support your conclusion.

100. Determine the region of the xy-plane in which $f(x, y) = \ln(x^2 + y^2 - 1)$ is continuous. Use technology to support your conclusion. (*Hint*: Choose the range of values for x and y carefully!)

101. At what points in space is $g(x, y, z) = x^2 + y^2 - 2z^2$ continuous?

102. At what points in space is $g(x, y, z) = \frac{1}{x^2 + z^2 - 1}$ continuous?

103. Show that $\lim_{(x, y) \to (0, 0)} \frac{1}{x^2 + y^2}$ does not exist at $(0, 0)$ by plotting the graph of the function.

104. **[T]** Evaluate $\lim_{(x, y) \to (0, 0)} \frac{-xy^2}{x^2 + y^4}$ by plotting the function using a CAS. Determine analytically the limit along the path $x = y^2$.

105. **[T]**

a. Use a CAS to draw a contour map of $z = \sqrt{9 - x^2 - y^2}$.
b. What is the name of the geometric shape of the level curves?
c. Give the general equation of the level curves.
d. What is the maximum value of z?
e. What is the domain of the function?
f. What is the range of the function?

106. *True or False*: If we evaluate $\lim_{(x, y) \to (0, 0)} f(x)$ along several paths and each time the limit is $1,$ we can conclude that $\lim_{(x, y) \to (0, 0)} f(x) = 1.$

107. Use polar coordinates to find $\lim_{(x, y) \to (0, 0)} \frac{\sin\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}$. You can also find the limit using L'Hôpital's rule.

108. Use polar coordinates to find $\lim_{(x, y) \to (0, 0)} \cos\left(x^2 + y^2\right)$.

109. Discuss the continuity of $f(g(x, y))$ where $f(t) = 1/t$ and $g(x, y) = 2x - 5y$.

110. Given $f(x, y) = x^2 - 4y$, find $\lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$.

111. Given $f(x, y) = x^2 - 4y$, find $\lim_{h \to 0} \frac{f(1 + h, y) - f(1, y)}{h}$.

4.3 | Partial Derivatives

Learning Objectives

- **4.3.1** Calculate the partial derivatives of a function of two variables.
- **4.3.2** Calculate the partial derivatives of a function of more than two variables.
- **4.3.3** Determine the higher-order derivatives of a function of two variables.
- **4.3.4** Explain the meaning of a partial differential equation and give an example.

Now that we have examined limits and continuity of functions of two variables, we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this chapter, with as many applications in mathematics, science, and engineering as differentiation of single-variable functions. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well.

Derivatives of a Function of Two Variables

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition

Let $f(x, y)$ be a function of two variables. Then the **partial derivative** of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}. \tag{4.12}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}. \tag{4.13}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

Example 4.14

Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12.$$

Solution

First, calculate $f(x+h,\ y)$.

$$\begin{aligned} f(x+h,\ y) &= (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h) + 5y - 12 \\ &= x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12. \end{aligned}$$

Next, substitute this into **Equation 4.12** and simplify:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{f(x+h,\ y) - f(x,\ y)}{h} \\ &= \lim_{h \to 0} \frac{\left(x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12\right) - \left(x^2 - 3xy + 2y^2 - 4x + 5y - 12\right)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4x - 4h + 5y - 12 - x^2 + 3xy - 2y^2 + 4x - 5y + 12}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 - 3hy - 4h}{h} \\ &= \lim_{h \to 0} \frac{h(2x + h - 3y - 4)}{h} \\ &= \lim_{h \to 0} (2x + h - 3y - 4) \\ &= 2x - 3y - 4. \end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, first calculate $f(x,\ y+h)$:

$$\begin{aligned} f(x+h,\ y) &= x^2 - 3x(y+h) + 2(y+h)^2 - 4x + 5(y+h) - 12 \\ &= x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12. \end{aligned}$$

Next, substitute this into **Equation 4.13** and simplify:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \lim_{h \to 0} \frac{f(x,\ y+h) - f(x,\ y)}{h} \\ &= \lim_{h \to 0} \frac{\left(x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12\right) - \left(x^2 - 3xy + 2y^2 - 4x + 5y - 12\right)}{h} \\ &= \lim_{h \to 0} \frac{x^2 - 3xy - 3xh + 2y^2 + 4yh + 2h^2 - 4x + 5y + 5h - 12 - x^2 + 3xy - 2y^2 + 4x - 5y + 12}{h} \\ &= \lim_{h \to 0} \frac{-3xh + 4yh + 2h^2 + 5h}{h} \\ &= \lim_{h \to 0} \frac{h(-3x + 4y + 2h + 5)}{h} \\ &= \lim_{h \to 0} (-3x + 4y + 2h + 5) \\ &= -3x + 4y + 5. \end{aligned}$$

4.12 Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,\ y) = 4x^2 + 2xy - y^2 + 3x - 2y + 5.$$

The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. To see why this is true, first fix y and define $g(x) = f(x,\ y)$ as a function of x. Then

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h,\ y) - f(x,\ y)}{h} = \frac{\partial f}{\partial x}.$$

The same is true for calculating the partial derivative of f with respect to y. This time, fix x and define $h(y) = f(x,\ y)$ as a function of y. Then

$$h'(x) = \lim_{k \to 0} \frac{h(x+k) - h(x)}{k} = \lim_{k \to 0} \frac{f(x,\ y+k) - f(x,\ y)}{k} = \frac{\partial f}{\partial y}.$$

All differentiation rules from **Introduction to Derivatives (http://cnx.org/content/m53494/latest/)** apply.

Example 4.15

Calculating Partial Derivatives

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions by holding the opposite variable constant then differentiating:

a. $f(x,\ y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$

b. $g(x,\ y) = \sin\left(x^2 y - 2x + 4\right)$

Solution

a. To calculate $\partial f/\partial x$, treat the variable y as a constant. Then differentiate $f(x,\ y)$ with respect to x using the sum, difference, and power rules:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial x}\left[x^2\right] - \frac{\partial}{\partial x}[3xy] + \frac{\partial}{\partial x}\left[2y^2\right] - \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial x}[5y] - \frac{\partial}{\partial x}[12] \\ &= 2x - 3y + 0 - 4 + 0 - 0 \\ &= 2x - 3y - 4. \end{aligned}$$

The derivatives of the third, fifth, and sixth terms are all zero because they do not contain the variable x, so they are treated as constant terms. The derivative of the second term is equal to the coefficient of x, which is $-3y$. Calculating $\partial f/\partial y$:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial y}\left[x^2\right] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}\left[2y^2\right] - \frac{\partial}{\partial y}[4x] + \frac{\partial}{\partial y}[5y] - \frac{\partial}{\partial y}[12] \\ &= -3x + 4y - 0 + 5 - 0 \\ &= -3x + 4y + 5. \end{aligned}$$

These are the same answers obtained in **Example 4.14**.

b. To calculate $\partial g/\partial x$, treat the variable y as a constant. Then differentiate $g(x,\ y)$ with respect to x using the chain rule and power rule:

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left[\sin\left(x^2 y - 2x + 4\right)\right] \\ &= \cos\left(x^2 y - 2x + 4\right)\frac{\partial}{\partial x}\left[x^2 y - 2x + 4\right] \\ &= (2xy - 2)\cos\left(x^2 y - 2x + 4\right). \end{aligned}$$

To calculate $\partial g/\partial y,$ treat the variable x as a constant. Then differentiate $g(x, y)$ with respect to y using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left[\sin\left(x^2 y - 2x + 4\right)\right]\\ &= \cos\left(x^2 y - 2x + 4\right)\frac{\partial}{\partial y}\left[x^2 y - 2x + 4\right]\\ &= x^2 \cos\left(x^2 y - 2x + 4\right).\end{aligned}$$

4.13 Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function $f(x, y) = \tan\left(x^3 - 3x^2 y^2 + 2y^4\right)$ by holding the opposite variable constant, then differentiating.

How can we interpret these partial derivatives? Recall that the graph of a function of two variables is a surface in $\mathbb{R}^3$. If we remove the limit from the definition of the partial derivative with respect to $x,$ the difference quotient remains:

$$\frac{f(x+h, y) - f(x, y)}{h}.$$

This resembles the difference quotient for the derivative of a function of one variable, except for the presence of the y variable. **Figure 4.21** illustrates a surface described by an arbitrary function $z = f(x, y)$.

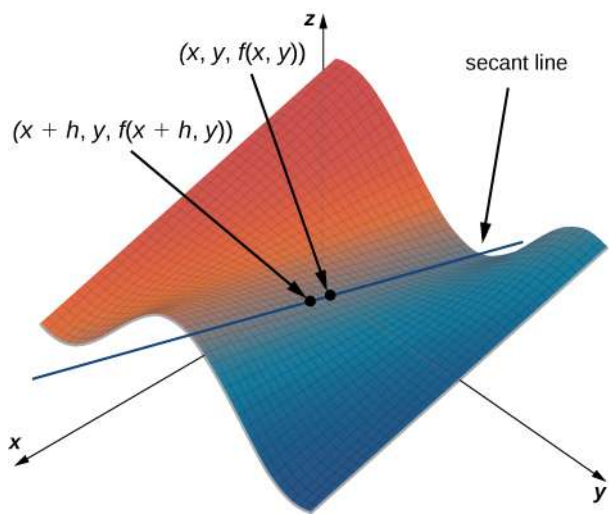


Figure 4.21 Secant line passing through the points $(x, y, f(x, y))$ and $(x+h, y, f(x+h, y))$.

In **Figure 4.21**, the value of h is positive. If we graph $f(x, y)$ and $f(x+h, y)$ for an arbitrary point $(x, y),$ then the slope of the secant line passing through these two points is given by

$$\frac{f(x+h, y) - f(x, y)}{h}.$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As h approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value h, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\partial f/\partial x$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$ parallel to the x-axis and $\partial f/\partial y$ represents the slope of the tangent line passing through the point $(x, y, f(x, y))$ parallel to the y-axis. If we wish to find the slope of a tangent line passing through the same point in any other direction, then we need what are called *directional derivatives*, which we discuss in **Directional Derivatives and the Gradient**.

We now return to the idea of contour maps, which we introduced in **Functions of Several Variables**. We can use a contour map to estimate partial derivatives of a function $g(x, y)$.

Example 4.16

Partial Derivatives from a Contour Map

Use a contour map to estimate $\partial g/\partial x$ at the point $(\sqrt{5}, 0)$ for the function $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution

The following graph represents a contour map for the function $g(x, y) = \sqrt{9 - x^2 - y^2}$.

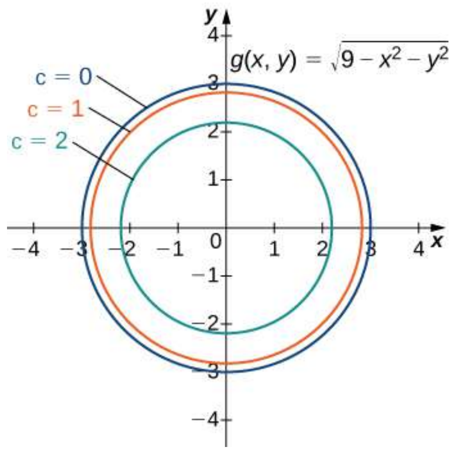


Figure 4.22 Contour map for the function $g(x, y) = \sqrt{9 - x^2 - y^2}$, using $c = 0, 1, 2,$ and 3 ($c = 3$ corresponds to the origin).

The inner circle on the contour map corresponds to $c = 2$ and the next circle out corresponds to $c = 1$. The first circle is given by the equation $2 = \sqrt{9 - x^2 - y^2}$; the second circle is given by the equation $1 = \sqrt{9 - x^2 - y^2}$. The first equation simplifies to $x^2 + y^2 = 5$ and the second equation simplifies to $x^2 + y^2 = 8$. The x-intercept of the first circle is $(\sqrt{5}, 0)$ and the x-intercept of the second circle is $(2\sqrt{2}, 0)$. We can estimate the value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5}, 0)$ using the slope formula:

$$\left.\frac{\partial g}{\partial x}\right|_{(x, y) = (\sqrt{5}, 0)} \approx \frac{g(\sqrt{5}, 0) - g(2\sqrt{2}, 0)}{\sqrt{5} - 2\sqrt{2}} = \frac{2 - 1}{\sqrt{5} - 2\sqrt{2}} = \frac{1}{\sqrt{5} - 2\sqrt{2}} \approx -1.688.$$

To calculate the exact value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5}, 0)$, we start by finding $\partial g/\partial x$ using the chain rule. First, we rewrite the function as $g(x, y) = \sqrt{9 - x^2 - y^2} = \left(9 - x^2 - y^2\right)^{1/2}$ and then differentiate with respect to x while holding y constant:

$$\frac{\partial g}{\partial x} = \frac{1}{2}\left(9 - x^2 - y^2\right)^{-1/2}(-2x) = -\frac{x}{\sqrt{9 - x^2 - y^2}}.$$

Next, we evaluate this expression using $x = \sqrt{5}$ and $y = 0$:

$$\left.\frac{\partial g}{\partial x}\right|_{(x,\ y) = \left(\sqrt{5},\ 0\right)} = -\frac{\sqrt{5}}{\sqrt{9 - \left(\sqrt{5}\right)^2 - (0)^2}} = -\frac{\sqrt{5}}{\sqrt{4}} = -\frac{\sqrt{5}}{2} \approx -1.118.$$

The estimate for the partial derivative corresponds to the slope of the secant line passing through the points $\left(\sqrt{5}, 0, g\left(\sqrt{5}, 0\right)\right)$ and $\left(2\sqrt{2}, 0, g\left(2\sqrt{2}, 0\right)\right)$. It represents an approximation to the slope of the tangent line to the surface through the point $\left(\sqrt{5}, 0, g\left(\sqrt{5}, 0\right)\right)$, which is parallel to the x-axis.

4.14 Use a contour map to estimate $\partial f/\partial y$ at point $\left(0, \sqrt{2}\right)$ for the function

$$f(x, y) = x^2 - y^2.$$

Compare this with the exact answer.

Functions of More Than Two Variables

Suppose we have a function of three variables, such as $w = f(x, y, z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

Definition

Let $f(x, y, z)$ be a function of three variables. Then, the *partial derivative of* f *with respect to x*, written as $\partial f/\partial x$, or f_x, is defined to be

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}. \quad \textbf{(4.14)}$$

The *partial derivative of* f *with respect to* y, written as $\partial f/\partial y$, or f_y, is defined to be

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k, z) - f(x, y, z)}{k}. \quad \textbf{(4.15)}$$

The *partial derivative of* f *with respect to* z, written as $\partial f/\partial z$, or f_z, is defined to be

$$\frac{\partial f}{\partial z} = \lim_{m \to 0} \frac{f(x, y, z + m) - f(x, y, z)}{m}. \quad \textbf{(4.16)}$$

We can calculate a partial derivative of a function of three variables using the same idea we used for a function of two variables. For example, if we have a function f of x, y, and z, and we wish to calculate $\partial f/\partial x$, then we treat the other two independent variables as if they are constants, then differentiate with respect to x.

Example 4.17

Calculating Partial Derivatives for a Function of Three Variables

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x, y, z) = x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z.$$

Then, find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Solution

We first calculate $\partial f/\partial x$ using **Equation 4.14**, then we calculate the other two partial derivatives by holding the remaining variables constant. To use the equation to find $\partial f/\partial x,$ we first need to calculate $f(x+h, y, z)$:

$$\begin{aligned} f(x+h, y, z) &= (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h)z + 5yz^2 - 12(x+h) + 4y - 3z \\ &= x^2 + 2xh + h^2 - 3xy - 3xh + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3z \end{aligned}$$

and recall that $f(x, y, z) = x^2 - 3xy + 2y^2 - 4zx + 5yz^2 - 12x + 4y - 3z.$ Next, we substitute these two expressions into the equation:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \to 0}\Bigg[\frac{x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3z}{h} \\ &\quad - \frac{x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z}{h}\Bigg] \\ &= \lim_{h \to 0}\left[\frac{2xh + h^2 - 3hy - 4hz - 12h}{h}\right] \\ &= \lim_{h \to 0}\left[\frac{h(2x + h - 3y - 4z - 12)}{h}\right] \\ &= \lim_{h \to 0}(2x + h - 3y - 4z - 12) \\ &= 2x - 3y - 4z - 12. \end{aligned}$$

Then we find $\partial f/\partial y$ by holding x and z constant. Therefore, any term that does not include the variable y is constant, and its derivative is zero. We can apply the sum, difference, and power rules for functions of one variable:

$$\begin{aligned} &\frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z\right] \\ &= \frac{\partial}{\partial y}\left[x^2\right] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}\left[2y^2\right] - \frac{\partial}{\partial y}[4xz] + \frac{\partial}{\partial y}\left[5yz^2\right] - \frac{\partial}{\partial y}[12x] + \frac{\partial}{\partial y}[4y] - \frac{\partial}{\partial y}[3z] \\ &= 0 - 3x + 4y - 0 + 5z^2 - 0 + 4 - 0 \\ &= -3x + 4y + 5z^2 + 4. \end{aligned}$$

To calculate $\partial f/\partial z,$ we hold x and y constant and apply the sum, difference, and power rules for functions of one variable:

$$\begin{aligned} &\frac{\partial}{\partial z}\left[x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z\right] \\ &= \frac{\partial}{\partial z}\left[x^2\right] - \frac{\partial}{\partial z}[3xy] + \frac{\partial}{\partial z}\left[2y^2\right] - \frac{\partial}{\partial z}[4xz] + \frac{\partial}{\partial z}\left[5yz^2\right] - \frac{\partial}{\partial z}[12x] + \frac{\partial}{\partial z}[4y] - \frac{\partial}{\partial z}[3z] \\ &= 0 - 0 + 0 - 4x + 10yz - 0 + 0 - 3 \\ &= -4x + 10yz - 3. \end{aligned}$$

4.15 Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x, y, z) = 2x^2 - 4x^2 y + 2y^2 + 5xz^2 - 6x + 3z - 8.$$

Then find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Example 4.18

Calculating Partial Derivatives for a Function of Three Variables

Calculate the three partial derivatives of the following functions.

a. $f(x, y, z) = \dfrac{x^2 y - 4xz + y^2}{x - 3yz}$

b. $g(x, y, z) = \sin(x^2 y - z) + \cos(x^2 - yz)$

Solution

In each case, treat all variables as constants except the one whose partial derivative you are calculating.

a.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{x^2 y - 4xz + y^2}{x - 3yz}\right] \\
&= \frac{\frac{\partial}{\partial x}\left(x^2 y - 4xz + y^2\right)(x - 3yz) - \left(x^2 y - 4xz + y^2\right)\frac{\partial}{\partial x}(x - 3yz)}{(x - 3yz)^2} \\
&= \frac{(2xy - 4z)(x - 3yz) - \left(x^2 y - 4xz + y^2\right)(1)}{(x - 3yz)^2} \\
&= \frac{2x^2 y - 6xy^2 z - 4xz + 12yz^2 - x^2 y + 4xz - y^2}{(x - 3yz)^2} \\
&= \frac{x^2 y - 6xy^2 z - 4xz + 12yz^2 + 4xz - y^2}{(x - 3yz)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{x^2 y - 4xz + y^2}{x - 3yz}\right] \\
&= \frac{\frac{\partial}{\partial y}\left(x^2 y - 4xz + y^2\right)(x - 3yz) - \left(x^2 y - 4xz + y^2\right)\frac{\partial}{\partial y}(x - 3yz)}{(x - 3yz)^2} \\
&= \frac{\left(x^2 + 2y\right)(x - 3yz) - \left(x^2 y - 4xz + y^2\right)(-3z)}{(x - 3yz)^2} \\
&= \frac{x^3 - 3x^2 yz + 2xy - 6y^2 z + 3x^2 yz - 12xz^2 + 3y^2 z}{(x - 3yz)^2} \\
&= \frac{x^3 + 2xy - 3y^2 z - 12xz^2}{(x - 3yz)^2}
\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left[\frac{x^2y - 4xz + y^2}{x - 3yz}\right] \\ &= \frac{\frac{\partial}{\partial z}\left(x^2y - 4xz + y^2\right)(x - 3yz) - \left(x^2y - 4xz + y^2\right)\frac{\partial}{\partial z}(x - 3yz)}{(x - 3yz)^2} \\ &= \frac{(-4x)(x - 3yz) - \left(x^2y - 4xz + y^2\right)(-3y)}{(x - 3yz)^2} \\ &= \frac{-4x^2 + 12xyz + 3x^2y^2 - 12xyz + 3y^3}{(x - 3yz)^2} \\ &= \frac{-4x^2 + 3x^2y^2 + 3y^3}{(x - 3yz)^2}\end{aligned}$$

b.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\sin\left(x^2y - z\right) + \cos\left(x^2 - yz\right)\right] \\ &= \left(\cos\left(x^2y - z\right)\right)\frac{\partial}{\partial x}\left(x^2y - z\right) - \left(\sin\left(x^2 - yz\right)\right)\frac{\partial}{\partial x}\left(x^2 - yz\right) \\ &= 2xy\cos\left(x^2y - z\right) - 2x\sin\left(x^2 - yz\right) \\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[\sin\left(x^2y - z\right) + \cos\left(x^2 - yz\right)\right] \\ &= \left(\cos\left(x^2y - z\right)\right)\frac{\partial}{\partial y}\left(x^2y - z\right) - \left(\sin\left(x^2 - yz\right)\right)\frac{\partial}{\partial y}\left(x^2 - yz\right) \\ &= x^2\cos\left(x^2y - z\right) + z\sin\left(x^2 - yz\right) \\ \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left[\sin\left(x^2y - z\right) + \cos\left(x^2 - yz\right)\right] \\ &= \left(\cos\left(x^2y - z\right)\right)\frac{\partial}{\partial z}\left(x^2y - z\right) - \left(\sin\left(x^2 - yz\right)\right)\frac{\partial}{\partial z}\left(x^2 - yz\right) \\ &= -\cos\left(x^2y - z\right) + y\sin\left(x^2 - yz\right)\end{aligned}$$

4.16 Calculate $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ for the function $f(x, y, z) = \sec\left(x^2y\right) - \tan\left(x^3yz^2\right)$.

Higher-Order Partial Derivatives

Consider the function

$$f(x, y) = 2x^3 - 4xy^2 + 5y^3 - 6xy + 5x - 4y + 12.$$

Its partial derivatives are

$$\frac{\partial f}{\partial x} = 6x^2 - 4y^2 - 6y + 5 \text{ and } \frac{\partial f}{\partial y} = -8xy + 15y^2 - 6x - 4.$$

Each of these partial derivatives is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these *second-order derivatives*, *third-order derivatives*, and so on. In general, they are referred to as **higher-order partial derivatives**. There are four second-order partial derivatives for any function (provided they all exist):

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right], \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right], \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right], \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right].$$

An alternative notation for each is $f_{xx}, f_{yx}, f_{xy},$ and $f_{yy},$ respectively. Higher-order partial derivatives calculated with respect to different variables, such as f_{xy} and $f_{yx},$ are commonly called **mixed partial derivatives**.

Example 4.19

Calculating Second Partial Derivatives

Calculate all four second partial derivatives for the function

$$f(x,\ y) = xe^{-3y} + \sin(2x - 5y).$$

Solution

To calculate $\partial^2 f/dx^2$ and $\partial^2 f/\partial y\partial x,$ we first calculate $\partial f/\partial x$:

$$\frac{\partial f}{\partial x} = e^{-3y} + 2\cos(2x - 5y).$$

To calculate $\partial^2 f/dx^2,$ differentiate $\partial f/\partial x$ with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial x}\left[e^{-3y} + 2\cos(2x - 5y)\right] \\ &= -4\sin(2x - 5y).\end{aligned}$$

To calculate $\partial^2 f/\partial y\partial x,$ differentiate $\partial f/\partial x$ with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial y}\left[e^{-3y} + 2\cos(2x - 5y)\right] \\ &= -3e^{-3y} + 10\sin(2x - 5y).\end{aligned}$$

To calculate $\partial^2 f/\partial x\partial y$ and $\partial^2 f/dy^2,$ first calculate $\partial f/\partial y$:

$$\frac{\partial f}{\partial y} = -3xe^{-3y} - 5\cos(2x - 5y).$$

To calculate $\partial^2 f/\partial x\partial y,$ differentiate $\partial f/\partial y$ with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial x}\left[-3xe^{-3y} - 5\cos(2x - 5y)\right] \\ &= -3e^{-3y} + 10\sin(2x - 5y).\end{aligned}$$

To calculate $\partial^2 f/\partial y^2,$ differentiate $\partial f/\partial y$ with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial y}\left[-3xe^{-3y} - 5\cos(2x - 5y)\right] \\ &= 9xe^{-3y} - 25\sin(2x - 5y).\end{aligned}$$

4.17 Calculate all four second partial derivatives for the function

$$f(x, y) = \sin(3x - 2y) + \cos(x + 4y).$$

At this point we should notice that, in both **Example 4.19** and the checkpoint, it was true that $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$. Under certain conditions, this is always true. In fact, it is a direct consequence of the following theorem.

Theorem 4.5: Equality of Mixed Partial Derivatives (Clairaut's Theorem)

Suppose that $f(x, y)$ is defined on an open disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy} = f_{yx}$.

Clairaut's theorem guarantees that as long as mixed second-order derivatives are continuous, the order in which we choose to differentiate the functions (i.e., which variable goes first, then second, and so on) does not matter. It can be extended to higher-order derivatives as well. The proof of Clairaut's theorem can be found in most advanced calculus books.

Two other second-order partial derivatives can be calculated for any function $f(x, y)$. The partial derivative f_{xx} is equal to the partial derivative of f_x with respect to x, and f_{yy} is equal to the partial derivative of f_y with respect to y.

Partial Differential Equations

In **Introduction to Differential Equations (http://cnx.org/content/m53696/latest/)** , we studied differential equations in which the unknown function had one independent variable. A **partial differential equation** is an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives. Examples of partial differential equations are

$$u_t = c^2(u_{xx} + u_{yy}) \tag{4.17}$$

(heat equation in two dimensions)

$$u_{tt} = c^2(u_{xx} + u_{yy}) \tag{4.18}$$

(wave equation in two dimensions)

$$u_{xx} + u_{yy} = 0 \tag{4.19}$$

(Laplace's equation in two dimensions)

In the first two equations, the unknown function u has three independent variables—t, x, and y—and c is an arbitrary constant. The independent variables x and y are considered to be spatial variables, and the variable t represents time. In Laplace's equation, the unknown function u has two independent variables x and y.

Example 4.20

A Solution to the Wave Equation

Verify that

$$u(x, y, t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$$

is a solution to the wave equation

$$u_{tt} = 4(u_{xx} + u_{yy}). \quad \textbf{(4.20)}$$

Solution

First, we calculate u_{tt}, u_{xx}, and u_{yy}:

$$\begin{aligned}
u_{tt} &= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}\right] \\
&= \frac{\partial}{\partial t}[5\sin(3\pi x)\sin(4\pi y)(-10\pi\sin(10\pi t))] \\
&= \frac{\partial}{\partial t}[-50\pi\sin(3\pi x)\sin(4\pi y)\sin(10\pi t)] \\
&= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
u_{xx} &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \\
&= \frac{\partial}{\partial x}[15\pi\cos(3\pi x)\sin(4\pi y)\cos(10\pi t)] \\
&= -45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
u_{yy} &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \\
&= \frac{\partial}{\partial y}[5\sin(3\pi x)(4\pi\cos(4\pi y))\cos(10\pi t)] \\
&= \frac{\partial}{\partial y}[20\pi\sin(3\pi x)\cos(4\pi y)\cos(10\pi t)] \\
&= -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t).
\end{aligned}$$

Next, we substitute each of these into the right-hand side of **Equation 4.20** and simplify:

$$\begin{aligned}
4(u_{xx} + u_{yy}) &= 4\left(-45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) + -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)\right) \\
&= 4\left(-125\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)\right) \\
&= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\
&= u_{tt}.
\end{aligned}$$

This verifies the solution.

4.18 Verify that $u(x, y, t) = 2\sin\left(\frac{x}{3}\right)\sin\left(\frac{y}{4}\right)e^{-25t/16}$ is a solution to the heat equation

$$u_t = 9(u_{xx} + u_{yy}). \quad \textbf{(4.21)}$$

Since the solution to the two-dimensional heat equation is a function of three variables, it is not easy to create a visual representation of the solution. We can graph the solution for fixed values of t, which amounts to snapshots of the heat distributions at fixed times. These snapshots show how the heat is distributed over a two-dimensional surface as time progresses. The graph of the preceding solution at time $t = 0$ appears in the following figure. As time progresses, the extremes level out, approaching zero as t approaches infinity.

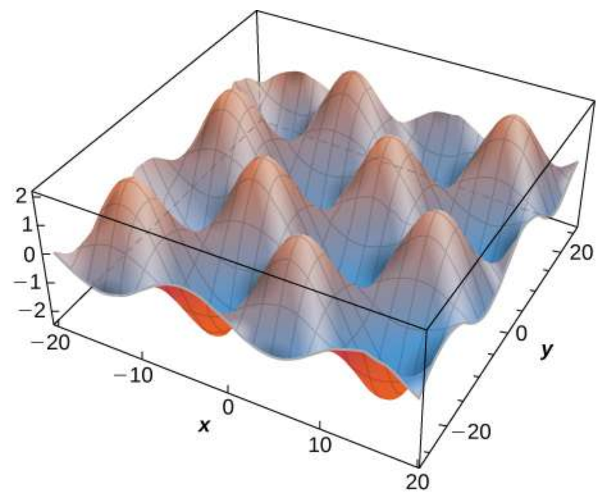


Figure 4.23

If we consider the heat equation in one dimension, then it is possible to graph the solution over time. The heat equation in one dimension becomes

$$u_t = c^2 u_{xx},$$

where c^2 represents the thermal diffusivity of the material in question. A solution of this differential equation can be written in the form

$$u_m(x, t) = e^{-\pi^2 m^2 c^2 t} \sin(m\pi x) \tag{4.22}$$

where m is any positive integer. A graph of this solution using $m = 1$ appears in Figure 4.24, where the initial temperature distribution over a wire of length 1 is given by $u(x, 0) = \sin \pi x.$ Notice that as time progresses, the wire cools off. This is seen because, from left to right, the highest temperature (which occurs in the middle of the wire) decreases and changes color from red to blue.

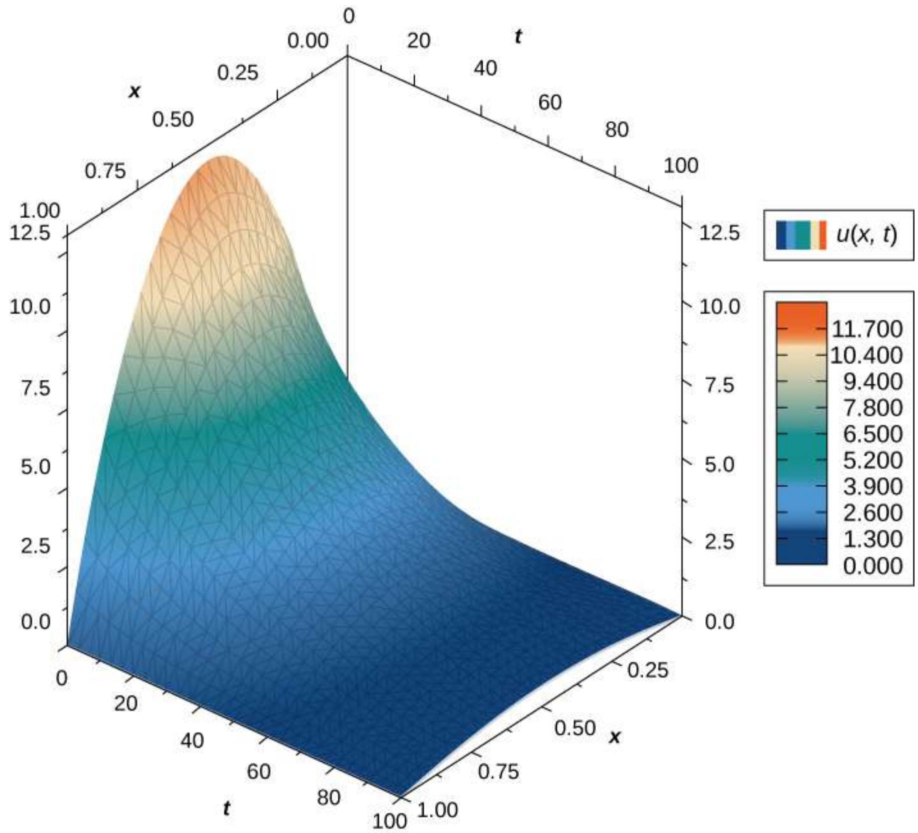


Figure 4.24 Graph of a solution of the heat equation in one dimension over time.

Student PROJECT

Lord Kelvin and the Age of Earth

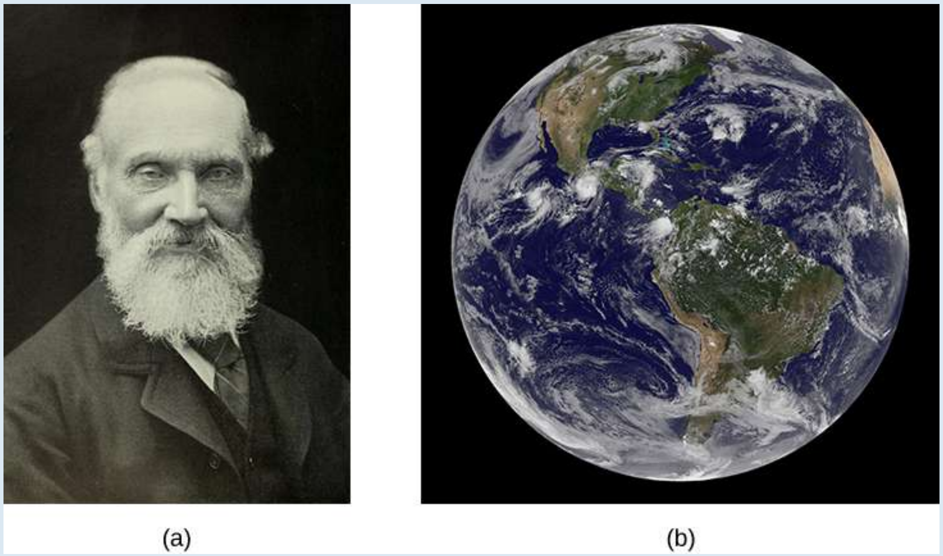


Figure 4.25 (a) William Thomson (Lord Kelvin), 1824-1907, was a British physicist and electrical engineer; (b) Kelvin used the heat diffusion equation to estimate the age of Earth (credit: modification of work by NASA).

During the late 1800s, the scientists of the new field of geology were coming to the conclusion that Earth must be “millions and millions” of years old. At about the same time, Charles Darwin had published his treatise on evolution. Darwin’s view was that evolution needed many millions of years to take place, and he made a bold claim that the Weald chalk fields, where important fossils were found, were the result of 300 million years of erosion.

At that time, eminent physicist William Thomson (Lord Kelvin) used an important partial differential equation, known as the *heat diffusion equation*, to estimate the age of Earth by determining how long it would take Earth to cool from molten rock to what we had at that time. His conclusion was a range of $20 \text{ to } 400$ million years, but most likely about 50 million years. For many decades, the proclamations of this irrefutable icon of science did not sit well with geologists or with Darwin.

Read Kelvin’s **paper (http://www.openstaxcollege.org/l/20_KelEarthAge)** on estimating the age of the Earth.

Kelvin made reasonable assumptions based on what was known in his time, but he also made several assumptions that turned out to be wrong. One incorrect assumption was that Earth is solid and that the cooling was therefore via conduction only, hence justifying the use of the diffusion equation. But the most serious error was a forgivable one—omission of the fact that Earth contains radioactive elements that continually supply heat beneath Earth’s mantle. The discovery of radioactivity came near the end of Kelvin’s life and he acknowledged that his calculation would have to be modified.

Kelvin used the simple one-dimensional model applied only to Earth’s outer shell, and derived the age from graphs and the roughly known temperature gradient near Earth’s surface. Let’s take a look at a more appropriate version of the diffusion equation in radial coordinates, which has the form

$$\frac{\partial T}{\partial t} = K\left[\frac{\partial^2 T}{\partial^2 r} + \frac{2}{r}\frac{\partial T}{\partial r}\right]. \tag{4.23}$$

Here, $T(r, t)$ is temperature as a function of r (measured from the center of Earth) and time t. K is the heat conductivity—for molten rock, in this case. The standard method of solving such a partial differential equation is by separation of variables, where we express the solution as the product of functions containing each variable separately. In this case, we would write the temperature as

$$T(r, t) = R(r)f(t).$$

1. Substitute this form into **Equation 4.13** and, noting that $f(t)$ is constant with respect to distance (r) and $R(r)$ is constant with respect to time (t), show that

$$\frac{1}{f}\frac{\partial f}{\partial t} = \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right].$$

2. This equation represents the separation of variables we want. The left-hand side is only a function of t and the right-hand side is only a function of r, and they must be equal for all values of r and t. Therefore, they both must be equal to a constant. Let's call that constant $-\lambda^2$. (The convenience of this choice is seen on substitution.) So, we have

$$\frac{1}{f}\frac{\partial f}{\partial t} = -\lambda^2 \quad \text{and} \quad \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right] = -\lambda^2.$$

   Now, we can verify through direct substitution for each equation that the solutions are $f(t) = Ae^{-\lambda^2 t}$ and $R(r) = B\left(\frac{\sin \alpha r}{r}\right) + C\left(\frac{\cos \alpha r}{r}\right)$, where $\alpha = \lambda/\sqrt{K}$. Note that $f(t) = Ae^{+\lambda n^2 t}$ is also a valid solution, so we could have chosen $+\lambda^2$ for our constant. Can you see why it would not be valid for this case as time increases?

3. Let's now apply boundary conditions.
   a. The temperature must be finite at the center of Earth, $r = 0$. Which of the two constants, B or C, must therefore be zero to keep R finite at $r = 0$? (Recall that $\sin(\alpha r)/r \to \alpha =$ as $r \to 0$, but $\cos(\alpha r)/r$ behaves very differently.)
   b. Kelvin argued that when magma reaches Earth's surface, it cools very rapidly. A person can often touch the surface within weeks of the flow. Therefore, the surface reached a moderate temperature very early and remained nearly constant at a surface temperature T_s. For simplicity, let's set $T = 0$ at $r = R_E$ and find α such that this is the temperature there for all time t. (Kelvin took the value to be $300\,\text{K} \approx 80°\text{F}$. We can add this $300\,\text{K}$ constant to our solution later.) For this to be true, the sine argument must be zero at $r = R_E$. Note that α has an infinite series of values that satisfies this condition. Each value of α represents a valid solution (each with its own value for A). The total or general solution is the sum of all these solutions.
   c. At $t = 0$, we assume that all of Earth was at an initial hot temperature T_0 (Kelvin took this to be about $7000\,\text{K}$.) The application of this boundary condition involves the more advanced application of Fourier coefficients. As noted in part b. each value of α_n represents a valid solution, and the general solution is a sum of all these solutions. This results in a series solution:

$$T(r, t) = \left(\frac{T_0 R_\text{E}}{\pi}\right)\sum_n \frac{(-1)^{n-1}}{n} e^{-\lambda n^2 t}\frac{\sin(\alpha_n r)}{r}, \text{ where } \alpha_n = n\pi/R_\text{E}.$$

Note how the values of α_n come from the boundary condition applied in part b. The term $\frac{-1^{n-1}}{n}$ is the constant A_n for each term in the series, determined from applying the Fourier method. Letting $\beta = \frac{\pi}{R_\text{E}}$, examine the first few terms of this solution shown here and note how λ^2 in the exponential causes the higher terms to decrease quickly as time progresses:

$$T(r,\ t) = \frac{T_0 R_E}{\pi r}\begin{pmatrix} e^{-K\beta^2 t}(\sin\beta r) - \frac{1}{2}e^{-4K\beta^2 t}(\sin 2\beta r) + \frac{1}{3}e^{-9K\beta^2 t}(\sin 3\beta r) \\ -\frac{1}{4}e^{-16K\beta^2 t}(\sin 4\beta r) + \frac{1}{5}e^{-25K\beta^2 t}(\sin 5\beta r)... \end{pmatrix}.$$

Near time $t = 0,$ many terms of the solution are needed for accuracy. Inserting values for the conductivity K and $\beta = \pi/R_E$ for time approaching merely thousands of years, only the first few terms make a significant contribution. Kelvin only needed to look at the solution near Earth's surface (**Figure 4.26**) and, after a long time, determine what time best yielded the estimated temperature gradient known during his era $(1°F$ increase per 50 ft). He simply chose a range of times with a gradient close to this value. In **Figure 4.26**, the solutions are plotted and scaled, with the $300 - K$ surface temperature added. Note that the center of Earth would be relatively cool. At the time, it was thought Earth must be solid.

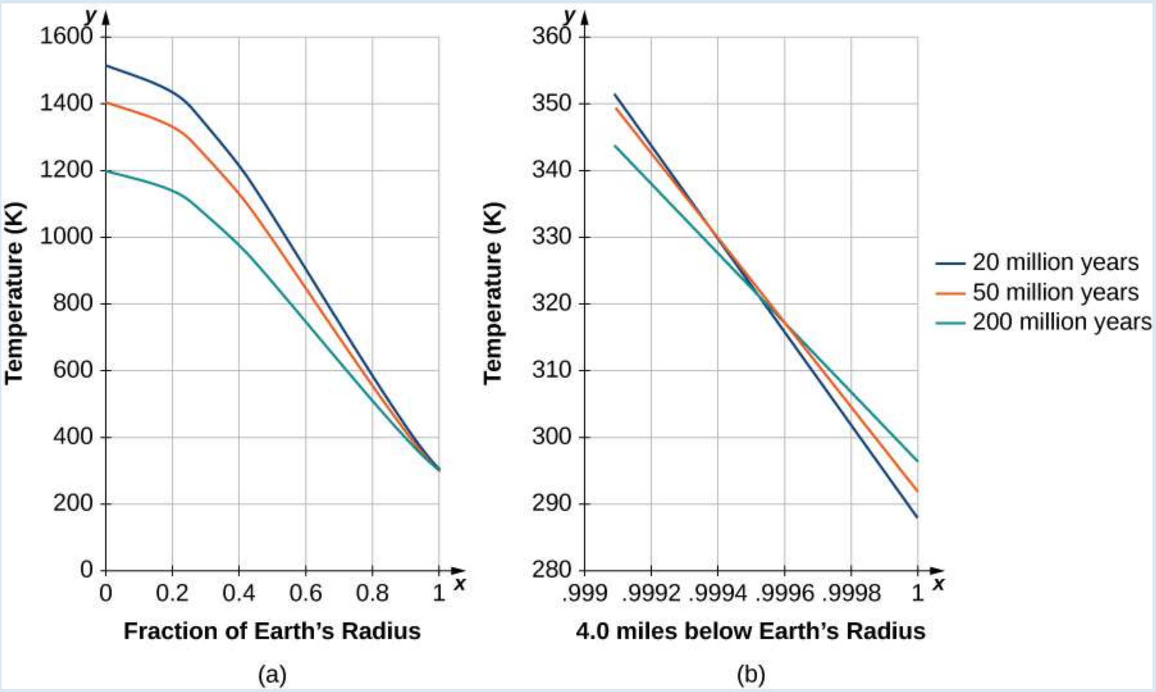


Figure 4.26 Temperature versus radial distance from the center of Earth. (a) Kelvin's results, plotted to scale. (b) A close-up of the results at a depth of 4.0 mi below Earth's surface.

Epilog

On May $20,\ 1904,$ physicist Ernest Rutherford spoke at the Royal Institution to announce a revised calculation that included the contribution of radioactivity as a source of Earth's heat. In Rutherford's own words:

"I came into the room, which was half-dark, and presently spotted Lord Kelvin in the audience, and realised that I was in for trouble at the last part of my speech dealing with the age of the Earth, where my views conflicted with his. To my relief, Kelvin fell fast asleep, but as I came to the important point, I saw the old bird sit up, open an eye and cock a baleful glance at me.

Then a sudden inspiration came, and I said Lord Kelvin had limited the age of the Earth, *provided no new source* [*of heat*] *was discovered*. That prophetic utterance referred to what we are now considering tonight, radium! Behold! The old boy beamed upon me."

Rutherford calculated an age for Earth of about 500 million years. Today's accepted value of Earth's age is about 4.6 billion years.

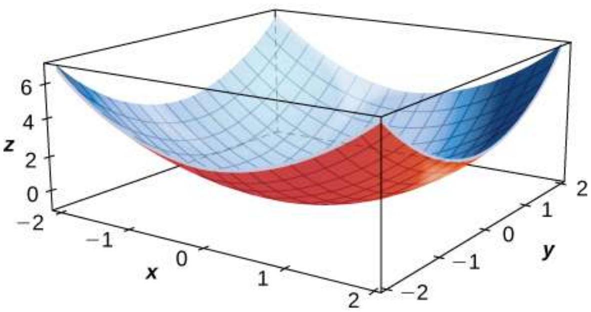
4.3 EXERCISES

For the following exercises, calculate the partial derivative using the limit definitions only.

112. $\frac{\partial z}{\partial x}$ for $z = x^2 - 3xy + y^2$

113. $\frac{\partial z}{\partial y}$ for $z = x^2 - 3xy + y^2$

For the following exercises, calculate the sign of the partial derivative using the graph of the surface.



114. $f_x(1, 1)$

115. $f_x(-1, 1)$

116. $f_y(1, 1)$

117. $f_x(0, 0)$

For the following exercises, calculate the partial derivatives.

118. $\frac{\partial z}{\partial x}$ for $z = \sin(3x)\cos(3y)$

119. $\frac{\partial z}{\partial y}$ for $z = \sin(3x)\cos(3y)$

120. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = x^8 e^{3y}$

121. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = \ln\left(x^6 + y^4\right)$

122. Find $f_y(x, y)$ for $f(x, y) = e^{xy}\cos(x)\sin(y)$.

123. Let $z = e^{xy}$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

124. Let $z = \ln\left(\frac{x}{y}\right)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

125. Let $z = \tan(2x - y)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

126. Let $z = \sinh(2x + 3y)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

127. Let $f(x, y) = \arctan\left(\frac{y}{x}\right)$. Evaluate $f_x(2, -2)$ and $f_y(2, -2)$.

128. Let $f(x, y) = \frac{xy}{x - y}$. Find $f_x(2, -2)$ and $f_y(2, -2)$.

Evaluate the partial derivatives at point $P(0, 1)$.

129. Find $\frac{\partial z}{\partial x}$ at $(0, 1)$ for $z = e^{-x}\cos(y)$.

130. Given $f(x, y, z) = x^3 yz^2$, find $\frac{\partial^2 f}{\partial x \partial y}$ and $f_z(1, 1, 1)$.

131. Given $f(x, y, z) = 2\sin(x + y)$, find $f_x\left(0, \frac{\pi}{2}, -4\right)$, $f_y\left(0, \frac{\pi}{2}, -4\right)$, and $f_z\left(0, \frac{\pi}{2}, -4\right)$.

132. The area of a parallelogram with adjacent side lengths that are a and b, and in which the angle between these two sides is θ, is given by the function $A(a, b, \theta) = ba\sin(\theta)$. Find the rate of change of the area of the parallelogram with respect to the following:

a. Side a
b. Side b
c. Angle θ

133. Express the volume of a right circular cylinder as a function of two variables:

a. its radius r and its height h.
b. Show that the rate of change of the volume of the cylinder with respect to its radius is the product of its circumference multiplied by its height.
c. Show that the rate of change of the volume of the cylinder with respect to its height is equal to the area of the circular base.

134. Calculate $\frac{\partial w}{\partial z}$ for $w = z\sin(xy^2 + 2z)$.

Find the indicated higher-order partial derivatives.

135. f_{xy} for $z = \ln(x - y)$

136. f_{yx} for $z = \ln(x - y)$

137. Let $z = x^2 + 3xy + 2y^2$. Find $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$.

138. Given $z = e^x \tan y$, find $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$.

139. Given $f(x, y, z) = xyz$, find f_{xyy}, f_{yxy}, and f_{yyx}.

140. Given $f(x, y, z) = e^{-2x} \sin\left(z^2 y\right)$, show that $f_{xyy} = f_{yxy}$.

141. Show that $z = \frac{1}{2}\left(e^y - e^{-y}\right)\sin x$ is a solution of the differential equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

142. Find $f_{xx}(x, y)$ for $f(x, y) = \frac{4x^2}{y} + \frac{y^2}{2x}$.

143. Let $f(x, y, z) = x^2 y^3 z - 3xy^2 z^3 + 5x^2 z - y^3 z$. Find f_{xyz}.

144. Let $F(x, y, z) = x^3 yz^2 - 2x^2 yz + 3xz - 2y^3 z$. Find F_{xyz}.

145. Given $f(x, y) = x^2 + x - 3xy + y^3 - 5$, find all points at which $f_x = f_y = 0$ simultaneously.

146. Given $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$, find all points at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously.

147. Given $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$, find all points on f at which $f_x = f_y = 0$ simultaneously.

148. Given $f(x, y) = 15x^3 - 3xy + 15y^3$, find all points at which $f_x(x, y) = f_y(x, y) = 0$ simultaneously.

149. Show that $z = e^x \sin y$ satisfies the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

150. Show that $f(x, y) = \ln\left(x^2 + y^2\right)$ solves Laplace's equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

151. Show that $z = e^{-t} \cos\left(\frac{x}{c}\right)$ satisfies the heat equation $\frac{\partial z}{\partial t} = -e^{-t} \cos\left(\frac{x}{c}\right)$.

152. Find $\lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x, y)}{\Delta x}$ for $f(x, y) = -7x - 2xy + 7y$.

153. Find $\lim\limits_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$ for $f(x, y) = -7x - 2xy + 7y$.

154. Find $\lim\limits_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ for $f(x, y) = x^2 y^2 + xy + y$.

155. Find $\lim\limits_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ for $f(x, y) = \sin(xy)$.

156. The function $P(T, V) = \frac{nRT}{V}$ gives the pressure at a point in a gas as a function of temperature T and volume V. The letters n and R are constants. Find $\frac{\partial P}{\partial V}$ and $\frac{\partial P}{\partial T}$, and explain what these quantities represent.

157. The equation for heat flow in the xy-plane is $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$. Show that $f(x, y, t) = e^{-2t} \sin x \sin y$ is a solution.

158. The basic wave equation is $f_{tt} = f_{xx}$. Verify that $f(x, t) = \sin(x + t)$ and $f(x, t) = \sin(x - t)$ are solutions.

159. The law of cosines can be thought of as a function of three variables. Let x, y, and θ be two sides of any triangle where the angle θ is the included angle between the two sides. Then, $F(x, y, \theta) = x^2 + y^2 - 2xy \cos \theta$ gives the square of the third side of the triangle. Find $\frac{\partial F}{\partial \theta}$ and $\frac{\partial F}{\partial x}$ when $x = 2$, $y = 3$, and $\theta = \frac{\pi}{6}$.

160. Suppose the sides of a rectangle are changing with respect to time. The first side is changing at a rate of 2 in./sec whereas the second side is changing at the rate of 4 in/sec. How fast is the diagonal of the rectangle changing when the first side measures 16 in. and the second side measures 20 in.? (Round answer to three decimal places.)

161. A Cobb-Douglas production function is $f(x, y) = 200x^{0.7}y^{0.3}$, where x and y represent the amount of labor and capital available. Let $x = 500$ and $y = 1000$. Find $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$ at these values, which represent the marginal productivity of labor and capital, respectively.

162. The apparent temperature index is a measure of how the temperature feels, and it is based on two variables: h, which is relative humidity, and t, which is the air temperature. $A = 0.885t - 22.4h + 1.20th - 0.544$. Find $\frac{\partial A}{\partial t}$ and $\frac{\partial A}{\partial h}$ when $t = 20°\text{F}$ and $h = 0.90$.

4.4 | Tangent Planes and Linear Approximations

Learning Objectives

- **4.4.1** Determine the equation of a plane tangent to a given surface at a point.
- **4.4.2** Use the tangent plane to approximate a function of two variables at a point.
- **4.4.3** Explain when a function of two variables is differentiable.
- **4.4.4** Use the total differential to approximate the change in a function of two variables.

In this section, we consider the problem of finding the tangent plane to a surface, which is analogous to finding the equation of a tangent line to a curve when the curve is defined by the graph of a function of one variable, $y = f(x)$. The slope of the tangent line at the point $x = a$ is given by $m = f'(a)$; what is the slope of a tangent plane? We learned about the equation of a plane in **Equations of Lines and Planes in Space**; in this section, we see how it can be applied to the problem at hand.

Tangent Planes

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A tangent plane at a regular point contains all of the lines tangent to that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly.

Definition

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the **tangent plane** to S at P_0 (**Figure 4.27**).

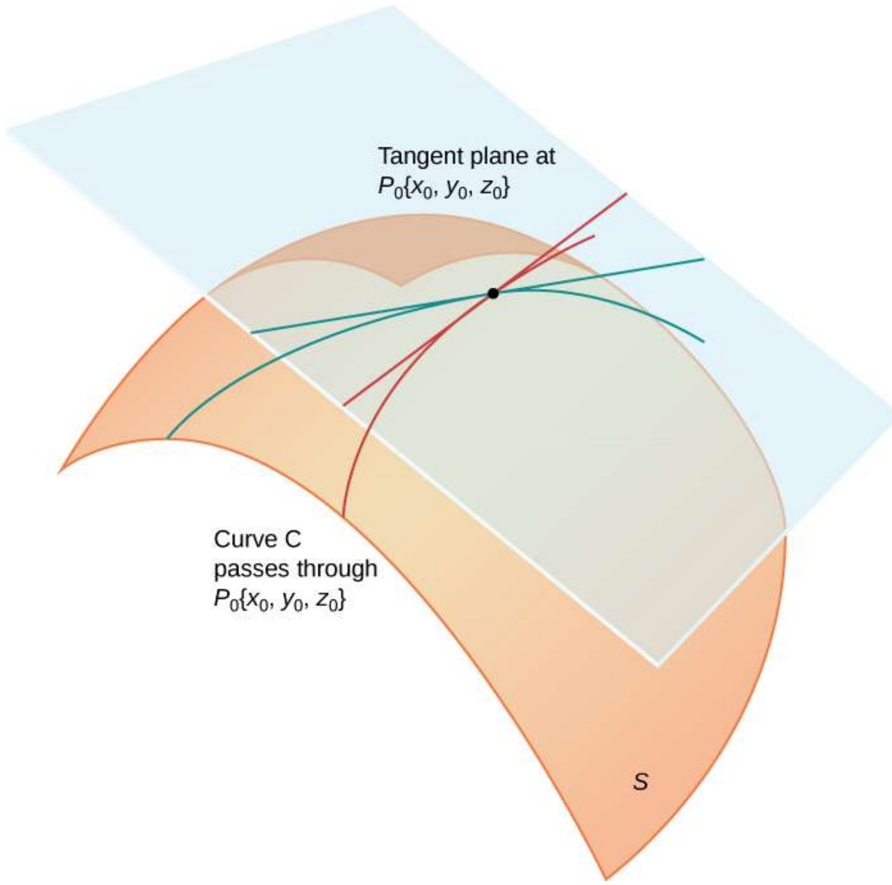


Figure 4.27 The tangent plane to a surface S at a point P_0 contains all the tangent lines to curves in S that pass through P_0.

For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition

Let S be a surface defined by a differentiable function $z = f(x, y),$ and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \quad (4.24)$$

To see why this formula is correct, let's first find two tangent lines to the surface S. The equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $x = x_0$ is $z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Similarly, the equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $y = y_0$ is $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$. A parallel vector to the first tangent line is $\mathbf{a} = \mathbf{j} + f_y(x_0, y_0)\mathbf{k}$; a parallel vector to the second tangent line is $\mathbf{b} = \mathbf{i} + f_x(x_0, y_0)\mathbf{k}$. We can take the cross product of these two vectors:

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \left(\mathbf{j}+f_y(x_0, y_0)\mathbf{k}\right)\times\left(\mathbf{i}+f_x(x_0, y_0)\mathbf{k}\right)\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & f_y(x_0, y_0)\\ 1 & 0 & f_x(x_0, y_0)\end{vmatrix}\\ &= f_x(x_0, y_0)\mathbf{i}+f_y(x_0, y_0)\mathbf{j}-\mathbf{k}.\end{aligned}$$

This vector is perpendicular to both lines and is therefore perpendicular to the tangent plane. We can use this vector as a normal vector to the tangent plane, along with the point $P_0 = \left(x_0, y_0, f(x_0, y_0)\right)$ in the equation for a plane:

$$\begin{aligned}\mathbf{n}\cdot\left((x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+\left(z-f(x_0, y_0)\right)\mathbf{k}\right) &= 0\\ \left(f_x(x_0, y_0)\mathbf{i}+f_y(x_0, y_0)\mathbf{j}\text{-}\mathbf{k}\right)\cdot\left((x-x_0)\mathbf{i}+(y-y_0)\mathbf{j}+\left(z-f(x_0, y_0)\right)\mathbf{k}\right) &= 0\\ f_x(x_0, y_0)(x-x_0)+f_y(x_0, y_0)(y-y_0)-\left(z-f(x_0, y_0)\right) &= 0.\end{aligned}$$

Solving this equation for z gives **Equation 4.24**.

Example 4.21

Finding a Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at point $(2, -1)$.

Solution

First, we must calculate $f_x(x, y)$ and $f_y(x, y)$, then use **Equation 4.24** with $x_0 = 2$ and $y_0 = -1$:

$$\begin{aligned}f_x(x, y) &= 4x - 3y + 2\\ f_y(x, y) &= -3x + 16y - 4\\ f(2, -1) &= 2(2)^2 - 3(2)(-1) + 8(-1)^2 + 2(2) - 4(-1) + 4 = 34.\\ f_x(2, -1) &= 4(2) - 3(-1) + 2 = 13\\ f_y(2, -1) &= -3(2) + 16(-1) - 4 = -26.\end{aligned}$$

Then **Equation 4.24** becomes

$$\begin{aligned}&z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)\\ &z = 34 + 13(x - 2) - 26\left(y - (-1)\right)\\ &z = 34 + 13x - 26 - 26y - 26\\ &z = 13x - 26y - 18.\end{aligned}$$

(See the following figure).

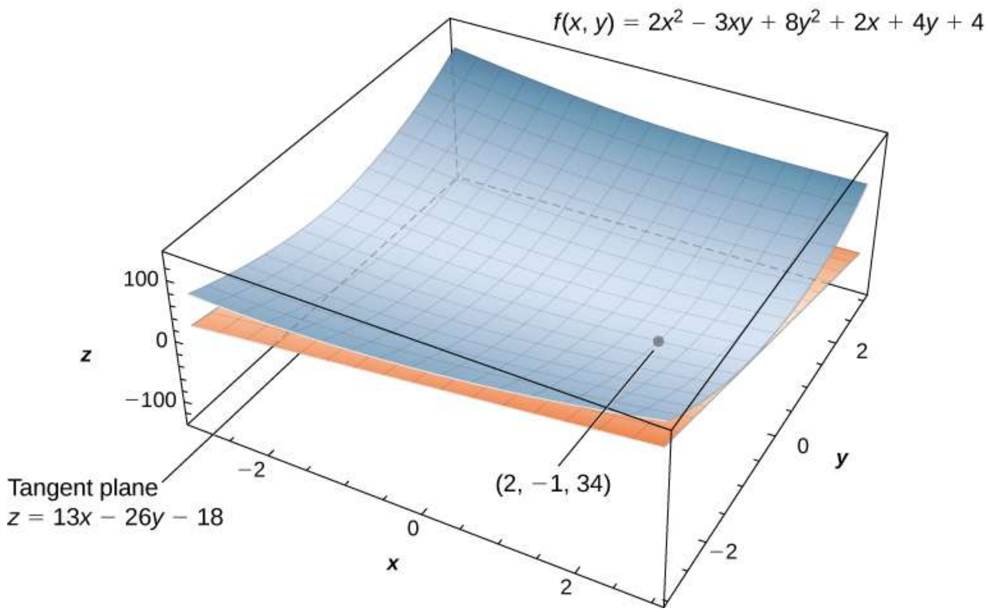


Figure 4.28 Calculating the equation of a tangent plane to a given surface at a given point.

4.19 Find the equation of the tangent plane to the surface defined by the function $f(x,\ y) = x^3 - x^2 y + y^2 - 2x + 3y - 2$ at point $(-1,\ 3)$.

Example 4.22

Finding Another Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x,\ y) = \sin(2x)\cos(3y)$ at the point $(\pi/3,\ \pi/4)$.

Solution

First, calculate $f_x(x,\ y)$ and $f_y(x,\ y)$, then use **Equation 4.24** with $x_0 = \pi/3$ and $y_0 = \pi/4$:

$$\begin{aligned}
f_x(x,\ y) &= 2\cos(2x)\cos(3y) \\
f_y(x,\ y) &= -3\sin(2x)\sin(3y) \\
f\left(\frac{\pi}{3},\ \frac{\pi}{4}\right) &= \sin\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{4} \\
f_x\left(\frac{\pi}{3},\ \frac{\pi}{4}\right) &= 2\cos\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = 2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} \\
f_y\left(\frac{\pi}{3},\ \frac{\pi}{4}\right) &= -3\sin\left(2\left(\frac{\pi}{3}\right)\right)\sin\left(3\left(\frac{\pi}{4}\right)\right) = -3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{6}}{4}.
\end{aligned}$$

Then **Equation 4.24** becomes

$$\begin{aligned} z &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ z &= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{3}\right) - \frac{3\sqrt{6}}{4}\left(y - \frac{\pi}{4}\right) \\ z &= \frac{\sqrt{2}}{2}x - \frac{3\sqrt{6}}{4}y - \frac{\sqrt{6}}{4} - \frac{\pi\sqrt{2}}{6} + \frac{3\pi\sqrt{6}}{16}. \end{aligned}$$

A tangent plane to a surface does not always exist at every point on the surface. Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

The graph of this function follows.

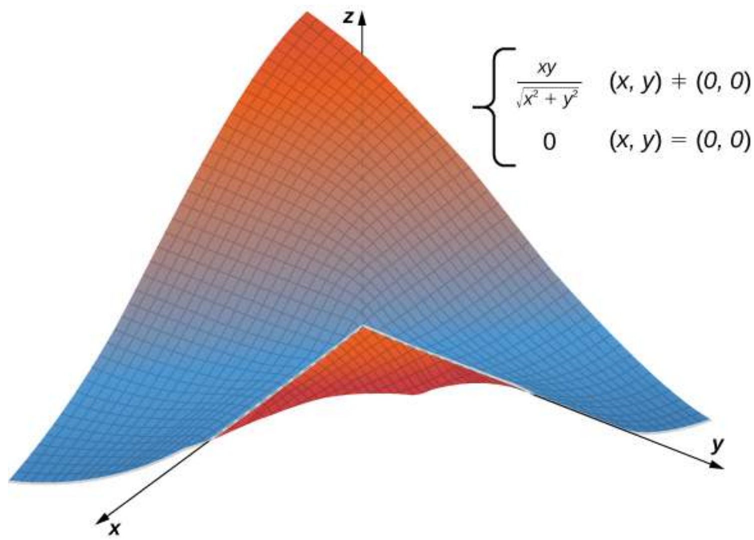


Figure 4.29 Graph of a function that does not have a tangent plane at the origin.

If either $x = 0$ or $y = 0$, then $f(x, y) = 0$, so the value of the function does not change on either the *x*- or *y*-axis. Therefore, $f_x(x, 0) = f_y(0, y) = 0$, so as either $x \, or \, y$ approach zero, these partial derivatives stay equal to zero. Substituting them into **Equation 4.24** gives $z = 0$ as the equation of the tangent line. However, if we approach the origin from a different direction, we get a different story. For example, suppose we approach the origin along the line $y = x$. If we put $y = x$ into the original function, it becomes

$$f(x, x) = \frac{x(x)}{\sqrt{x^2 + (x)^2}} = \frac{x^2}{\sqrt{2x^2}} = \frac{|x|}{\sqrt{2}}.$$

When $x > 0$, the slope of this curve is equal to $\sqrt{2}/2$; when $x < 0$, the slope of this curve is equal to $-\left(\sqrt{2}/2\right)$. This presents a problem. In the definition of *tangent plane*, we presumed that all tangent lines through point P (in this case, the origin) lay in the same plane. This is clearly not the case here. When we study differentiable functions, we will see that this function is not differentiable at the origin.

Linear Approximations

Recall from **Linear Approximations and Differentials (http://cnx.org/content/m53605/latest/)** that the formula for the linear approximation of a function $f(x)$ at the point $x = a$ is given by

$$y \approx f(a) + f'(a)(x - a).$$

The diagram for the linear approximation of a function of one variable appears in the following graph.

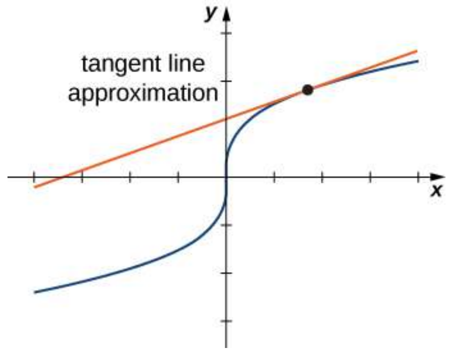


Figure 4.30 Linear approximation of a function in one variable.

The tangent line can be used as an approximation to the function $f(x)$ for values of x reasonably close to $x = a$. When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.

Definition

Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the **linear approximation** of f at the point (x_0, y_0) is given by the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \quad (4.25)$$

Notice that this equation also represents the tangent plane to the surface defined by $z = f(x, y)$ at the point (x_0, y_0). The idea behind using a linear approximation is that, if there is a point (x_0, y_0) at which the precise value of $f(x, y)$ is known, then for values of (x, y) reasonably close to (x_0, y_0), the linear approximation (i.e., tangent plane) yields a value that is also reasonably close to the exact value of $f(x, y)$ (**Figure 4.31**). Furthermore the plane that is used to find the linear approximation is also the tangent plane to the surface at the point (x_0, y_0).

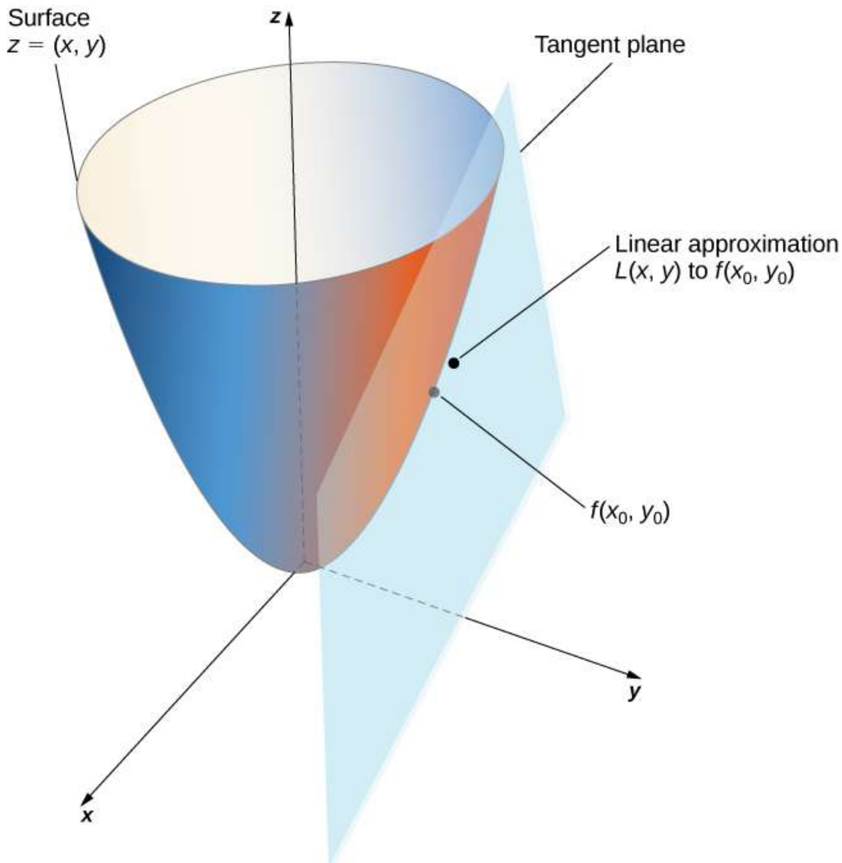


Figure 4.31 Using a tangent plane for linear approximation at a point.

Example 4.23

Using a Tangent Plane Approximation

Given the function $f(x, y) = \sqrt{41 - 4x^2 - y^2}$, approximate $f(2.1, 2.9)$ using point $(2, 3)$ for (x_0, y_0). What is the approximate value of $f(2.1, 2.9)$ to four decimal places?

Solution

To apply **Equation 4.25**, we first must calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = 3$:

$$\begin{aligned} f(x_0, y_0) &= f(2, 3) = \sqrt{41 - 4(2)^2 - (3)^2} = \sqrt{41 - 16 - 9} = \sqrt{16} = 4 \\ f_x(x, y) &= -\frac{4x}{\sqrt{41 - 4x^2 - y^2}} \text{ so } f_x(x_0, y_0) = -\frac{4(2)}{\sqrt{41 - 4(2)^2 - (3)^2}} = -2 \\ f_y(x, y) &= -\frac{y}{\sqrt{41 - 4x^2 - y^2}} \text{ so } f_y(x_0, y_0) = -\frac{3}{\sqrt{41 - 4(2)^2 - (3)^2}} = -\frac{3}{4}. \end{aligned}$$

Now we substitute these values into **Equation 4.25**:

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 4 - 2(x - 2) - \frac{3}{4}(y - 3) \\ &= \frac{41}{4} - 2x - \frac{3}{4}y. \end{aligned}$$

Last, we substitute $x = 2.1$ and $y = 2.9$ into $L(x, y)$:

$$L(2.1, 2.9) = \frac{41}{4} - 2(2.1) - \frac{3}{4}(2.9) = 10.25 - 4.2 - 2.175 = 3.875.$$

The approximate value of $f(2.1, 2.9)$ to four decimal places is

$$f(2.1, 2.9) = \sqrt{41 - 4(2.1)^2 - (2.9)^2} = \sqrt{14.95} \approx 3.8665,$$

which corresponds to a 0.2% error in approximation.

4.20 Given the function $f(x, y) = e^{5 - 2x + 3y}$, approximate $f(4.1, 0.9)$ using point $(4, 1)$ for (x_0, y_0). What is the approximate value of $f(4.1, 0.9)$ to four decimal places?

Differentiability

When working with a function $y = f(x)$ of one variable, the function is said to be differentiable at a point $x = a$ if $f'(a)$ exists. Furthermore, if a function of one variable is differentiable at a point, the graph is “smooth” at that point (i.e., no corners exist) and a tangent line is well-defined at that point.

The idea behind differentiability of a function of two variables is connected to the idea of smoothness at that point. In this case, a surface is considered to be smooth at point P if a tangent plane to the surface exists at that point. If a function is differentiable at a point, then a tangent plane to the surface exists at that point. Recall the formula for a tangent plane at a point (x_0, y_0) is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

For a tangent plane to exist at the point (x_0, y_0), the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness, as was illustrated in **Figure 4.29**. In that case, the partial derivatives existed at the origin, but the function also had a corner on the graph at the origin.

Definition

A function $f(x, y)$ is **differentiable** at a point $P(x_0, y_0)$ if, for all points (x, y) in a δ disk around P, we can write

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y), \tag{4.26}$$

where the error term E satisfies

$$\lim_{(x, y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

The last term in **Equation 4.26** is referred to as the *error term* and it represents how closely the tangent plane comes to the surface in a small neighborhood (δ disk) of point P. For the function f to be differentiable at P, the function must be smooth—that is, the graph of f must be close to the tangent plane for points near P.

Example 4.24

Demonstrating Differentiability

Show that the function $f(x, y) = 2x^2 - 4y$ is differentiable at point $(2, -3)$.

Solution

First, we calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = -3$, then we use **Equation 4.26**:

$$\begin{aligned} f(2, -3) &= 2(2)^2 - 4(-3) = 8 + 12 = 20 \\ f_x(2, -3) &= 4(2) = 8 \\ f_y(2, -3) &= -4. \end{aligned}$$

Therefore $m_1 = 8$ and $m_2 = -4$, and **Equation 4.26** becomes

$$\begin{aligned} f(x, y) &= f(2, -3) + f_x(2, -3)(x-2) + f_y(2, -3)(y+3) + E(x, y) \\ 2x^2 - 4y &= 20 + 8(x-2) - 4(y+3) + E(x, y) \\ 2x^2 - 4y &= 20 + 8x - 16 - 4y - 12 + E(x, y) \\ 2x^2 - 4y &= 8x - 4y - 8 + E(x, y) \\ E(x, y) &= 2x^2 - 8x + 8. \end{aligned}$$

Next, we calculate $\lim_{(x, y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$:

$$\begin{aligned} \lim_{(x, y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} &= \lim_{(x, y) \to (2, -3)} \frac{2x^2 - 8x + 8}{\sqrt{(x-2)^2 + (y+3)^2}} \\ &= \lim_{(x, y) \to (2, -3)} \frac{2\left(x^2 - 4x + 4\right)}{\sqrt{(x-2)^2 + (y+3)^2}} \\ &= \lim_{(x, y) \to (2, -3)} \frac{2(x-2)^2}{\sqrt{(x-2)^2 + (y+3)^2}} \\ &\le \lim_{(x, y) \to (2, -3)} \frac{2\left((x-2)^2 + (y+3)^2\right)}{\sqrt{(x-2)^2 + (y+3)^2}} \\ &= \lim_{(x, y) \to (2, -3)} 2\sqrt{(x-2)^2 + (y+3)^2} \\ &= 0. \end{aligned}$$

Since $E(x, y) \ge 0$ for any value of x or y, the original limit must be equal to zero. Therefore, $f(x, y) = 2x^2 - 4y$ is differentiable at point $(2, -3)$.

4.21 Show that the function $f(x, y) = 3x - 4y^2$ is differentiable at point $(-1, 2)$.

The function $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ is not differentiable at the origin. We can see this by calculating the partial derivatives. This function appeared earlier in the section, where we showed that $f_x(0, 0) = f_y(0, 0) = 0.$ Substituting this information into **Equation 4.26** using $x_0 = 0$ and $y_0 = 0,$ we get

$$\begin{aligned} f(x, y) &= f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) + E(x, y) \\ E(x, y) &= \frac{xy}{\sqrt{x^2 + y^2}}. \end{aligned}$$

Calculating $\lim_{(x, y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}$ gives

$$\begin{aligned} \lim_{(x, y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} &= \lim_{(x, y) \to (0, 0)} \frac{\frac{xy}{\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}} \\ &= \lim_{(x, y) \to (0, 0)} \frac{xy}{x^2 + y^2}. \end{aligned}$$

Depending on the path taken toward the origin, this limit takes different values. Therefore, the limit does not exist and the function f is not differentiable at the origin as shown in the following figure.

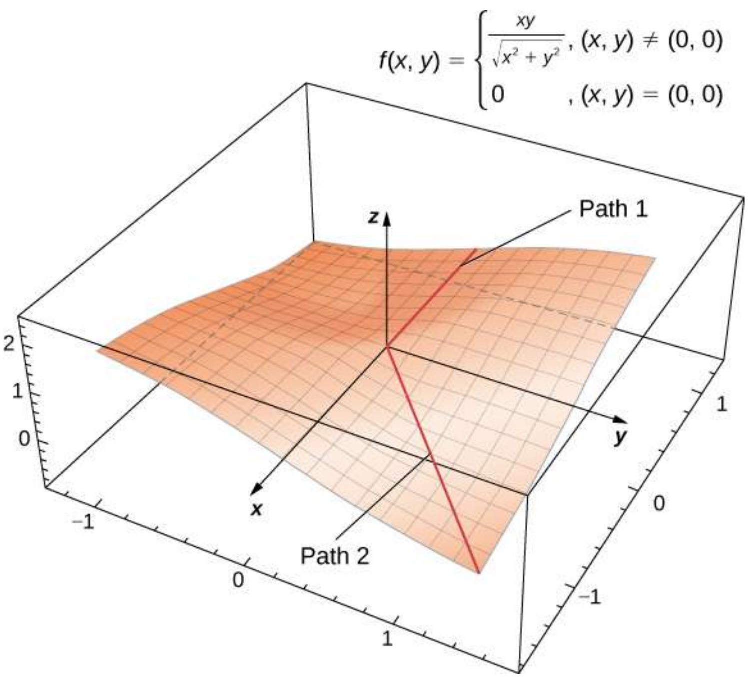


Figure 4.32 This function $f(x, y)$ is not differentiable at the origin.

Differentiability and continuity for functions of two or more variables are connected, the same as for functions of one variable. In fact, with some adjustments of notation, the basic theorem is the same.

Theorem 4.6: Differentiability Implies Continuity

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x, y)$ is differentiable at $(x_0, y_0),$ then $f(x, y)$ is continuous at $(x_0, y_0).$

Differentiability Implies Continuity shows that if a function is differentiable at a point, then it is continuous there. However, if a function is continuous at a point, then it is not necessarily differentiable at that point. For example,

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous at the origin, but it is not differentiable at the origin. This observation is also similar to the situation in single-variable calculus.

Continuity of First Partials Implies Differentiability further explores the connection between continuity and differentiability at a point. This theorem says that if the function and its partial derivatives are continuous at a point, the function is differentiable.

Theorem 4.7: Continuity of First Partials Implies Differentiability

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ all exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0), then $f(x, y)$ is differentiable there.

Recall that earlier we showed that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

was not differentiable at the origin. Let's calculate the partial derivatives f_x and f_y:

$$\frac{\partial f}{\partial x} = \frac{y^3}{\left(x^2 + y^2\right)^{3/2}} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x^3}{\left(x^2 + y^2\right)^{3/2}}.$$

The contrapositive of the preceding theorem states that if a function is not differentiable, then at least one of the hypotheses must be false. Let's explore the condition that $f_x(0, 0)$ must be continuous. For this to be true, it must be true that $\lim_{(x, y) \to (0, 0)} f_x(0, 0) = f_x(0, 0)$:

$$\lim_{(x, y) \to (0, 0)} f_x(x, y) = \lim_{(x, y) \to (0, 0)} \frac{y^3}{\left(x^2 + y^2\right)^{3/2}}.$$

Let $x = ky$. Then

$$\begin{aligned} \lim_{(x, y) \to (0, 0)} \frac{y^3}{\left(x^2 + y^2\right)^{3/2}} &= \lim_{y \to 0} \frac{y^3}{\left((ky)^2 + y^2\right)^{3/2}} \\ &= \lim_{y \to 0} \frac{y^3}{\left(k^2 y^2 + y^2\right)^{3/2}} \\ &= \lim_{y \to 0} \frac{y^3}{|y|^3 \left(k^2 + 1\right)^{3/2}} \\ &= \frac{1}{\left(k^2 + 1\right)^{3/2}} \lim_{y \to 0} \frac{|y|}{y}. \end{aligned}$$

If $y > 0$, then this expression equals $1/\left(k^2 + 1\right)^{3/2}$; if $y < 0$, then it equals $-\left(1/\left(k^2 + 1\right)^{3/2}\right)$. In either case, the value depends on $k,$ so the limit fails to exist.

Differentials

In **Linear Approximations and Differentials (http://cnx.org/content/m53605/latest/)** we first studied the concept of differentials. The differential of $y,$ written $dy,$ is defined as $f'(x)dx.$ The differential is used to approximate $\Delta y = f(x + \Delta x) - f(x),$ where $\Delta x = dx.$ Extending this idea to the linear approximation of a function of two variables at the point (x_0, y_0) yields the formula for the total differential for a function of two variables.

Definition

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of $f,$ and let Δx and Δy be chosen so that $(x_0 + \Delta x, y_0 + \Delta y)$ is also in the domain of $f.$ If f is differentiable at the point $(x_0, y_0),$ then the differentials dx and dy are defined as

$$dx = \Delta x \text{ and } dy = \Delta y.$$

The differential $dz,$ also called the **total differential** of $z = f(x, y)$ at $(x_0, y_0),$ is defined as

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \quad (4.27)$$

Notice that the symbol ∂ is not used to denote the total differential; rather, d appears in front of $z.$ Now, let's define $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$ We use dz to approximate $\Delta z,$ so

$$\Delta z \approx dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

Therefore, the differential is used to approximate the change in the function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and $\Delta y.$ Since $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y),$ this can be used further to approximate $f(x + \Delta x, y + \Delta y)$:

$$\begin{aligned} f(x + \Delta x, y + \Delta y) &= f(x, y) + \Delta z \\ &\approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y. \end{aligned}$$

See the following figure.

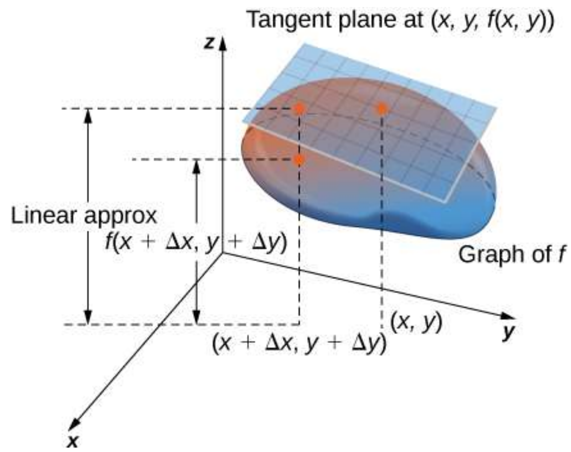


Figure 4.33 The linear approximation is calculated via the formula $f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$

One such application of this idea is to determine error propagation. For example, if we are manufacturing a gadget and are off by a certain amount in measuring a given quantity, the differential can be used to estimate the error in the total volume of the gadget.

Example 4.25

Approximation by Differentials

Find the differential dz of the function $f(x, y) = 3x^2 - 2xy + y^2$ and use it to approximate Δz at point $(2, -3)$. Use $\Delta x = 0.1$ and $\Delta y = -0.05$. What is the exact value of Δz?

Solution

First, we must calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = -3$:

$$\begin{aligned} f(x_0, y_0) &= f(2, -3) = 3(2)^2 - 2(2)(-3) + (-3)^2 = 12 + 12 + 9 = 33 \\ f_x(x, y) &= 6x - 2y \\ f_y(x, y) &= -2x + 2y \\ f_x(x_0, y_0) &= f_x(2, -3) = 6(2) - 2(-3) = 12 + 6 = 18 \\ f_y(x_0, y_0) &= f_y(2, -3) = -2(2) + 2(-3) = -4 - 6 = -10. \end{aligned}$$

Then, we substitute these quantities into **Equation 4.27**:

$$\begin{aligned} dz &= f_x(x_0, y_0)dx + f_y(x_0, y_0)dy \\ dz &= 18(0.1) - 10(-0.05) = 1.8 + 0.5 = 2.3. \end{aligned}$$

This is the approximation to $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$. The exact value of Δz is given by

$$\begin{aligned} \Delta z &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \\ &= f(2 + 0.1, -3 - 0.05) - f(2, -3) \\ &= f(2.1, -3.05) - f(2, -3) \\ &= 2.3425. \end{aligned}$$

4.22 Find the differential dz of the function $f(x, y) = 4y^2 + x^2 y - 2xy$ and use it to approximate Δz at point $(1, -1)$. Use $\Delta x = 0.03$ and $\Delta y = -0.02$. What is the exact value of Δz?

Differentiability of a Function of Three Variables

All of the preceding results for differentiability of functions of two variables can be generalized to functions of three variables. First, the definition:

Definition

A function $f(x, y, z)$ is differentiable at a point $P(x_0, y_0, z_0)$ if for all points (x, y, z) in a δ disk around P we can write

$$\begin{aligned} f(x, y) &= f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) \\ &\quad + f_z(x_0, y_0, z_0)(z - z_0) + E(x, y, z), \end{aligned} \tag{4.28}$$

where the error term E satisfies

$$\lim_{(x,\ y,\ z) \to (x_0,\ y_0,\ z_0)} \frac{E(x,\ y,\ z)}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = 0.$$

If a function of three variables is differentiable at a point $(x_0,\ y_0,\ z_0)$, then it is continuous there. Furthermore, continuity of first partial derivatives at that point guarantees differentiability.

4.4 EXERCISES

For the following exercises, find a unit normal vector to the surface at the indicated point.

163. $f(x, y) = x^3,\ (2, -1, 8)$

164. $\ln\left(\frac{x}{y-z}\right) = 0$ when $x = y = 1$

For the following exercises, as a useful review for techniques used in this section, find a normal vector and a tangent vector at point P.

165. $x^2 + xy + y^2 = 3,\ P(-1, -1)$

166. $\left(x^2 + y^2\right)^2 = 9\left(x^2 - y^2\right),\ P(\sqrt{2}, 1)$

167. $xy^2 - 2x^2 + y + 5x = 6,\ P(4, 2)$

168. $2x^3 - x^2y^2 = 3x - y - 7,\ P(1, -2)$

169. $ze^{x^2 - y^2} - 3 = 0,\quad P(2, 2, 3)$

For the following exercises, find the equation for the tangent plane to the surface at the indicated point. (*Hint:* Solve for z in terms of x and y.)

170. $-8x - 3y - 7z = -19,\ P(1, -1, 2)$

171. $z = -9x^2 - 3y^2,\ P(2, 1, -39)$

172. $x^2 + 10xyz + y^2 + 8z^2 = 0,\ P(-1, -1, -1)$

173. $z = \ln(10x^2 + 2y^2 + 1),\ P(0, 0, 0)$

174. $z = e^{7x^2 + 4y^2},\quad P(0, 0, 1)$

175. $xy + yz + zx = 11,\ P(1, 2, 3)$

176. $x^2 + 4y^2 = z^2,\ P(3, 2, 5)$

177. $x^3 + y^3 = 3xyz,\ P\left(1, 2, \frac{3}{2}\right)$

178. $z = axy,\ P\left(1, \frac{1}{a}, 1\right)$

179. $z = \sin x + \sin y + \sin(x + y),\ P(0, 0, 0)$

180. $h(x, y) = \ln\sqrt{x^2 + y^2},\ P(3, 4)$

181. $z = x^2 - 2xy + y^2,\ P(1, 2, 1)$

For the following exercises, find parametric equations for the normal line to the surface at the indicated point. (Recall that to find the equation of a line in space, you need a point on the line, $P_0(x_0, y_0, z_0)$, and a vector $\mathbf{n} = \langle a, b, c \rangle$ that is parallel to the line. Then the equation of the line is $x - x_0 = at,\ y - y_0 = bt,\ z - z_0 = ct$.)

182. $-3x + 9y + 4z = -4,\ P(1, -1, 2)$

183. $z = 5x^2 - 2y^2,\ P(2, 1, 18)$

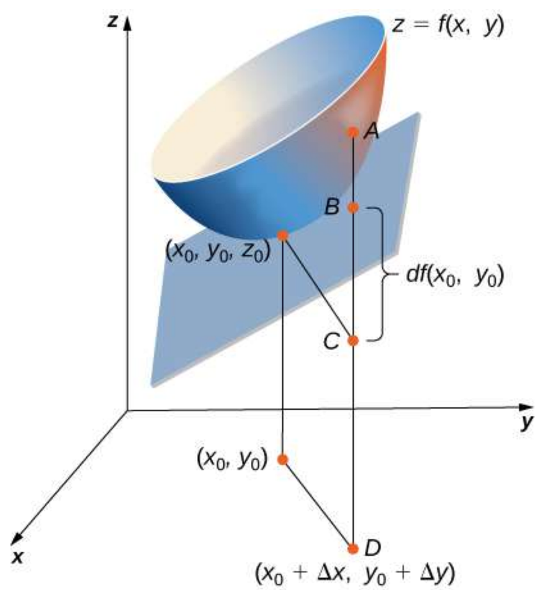
184. $x^2 - 8xyz + y^2 + 6z^2 = 0,\ P(1, 1, 1)$

185. $z = \ln\left(3x^2 + 7y^2 + 1\right),\ P(0, 0, 0)$

186. $z = e^{4x^2 + 6y^2},\ P(0, 0, 1)$

187. $z = x^2 - 2xy + y^2$ at point $P(1, 2, 1)$

For the following exercises, use the figure shown here.



188. The length of line segment AC is equal to what mathematical expression?

189. The length of line segment BC is equal to what mathematical expression?

190. Using the figure, explain what the length of line segment AB represents.

For the following exercises, complete each task.

191. Show that $f(x, y) = e^{xy} x$ is differentiable at point $(1, 0)$.

192. Find the total differential of the function $w = e^y \cos(x) + z^2$.

193. Show that $f(x, y) = x^2 + 3y$ is differentiable at every point. In other words, show that $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where both ε_1 and ε_2 approach zero as $(\Delta x, \Delta y)$ approaches $(0, 0)$.

194. Find the total differential of the function $z = \frac{xy}{y + x}$ where x changes from 10 to 10.5 and y changes from 15 to 13.

195. Let $z = f(x, y) = xe^y$. Compute Δz from $P(1, 2)$ to $Q(1.05, 2.1)$ and then find the approximate change in z from point P to point Q. Recall $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$, and dz and Δz are approximately equal.

196. The volume of a right circular cylinder is given by $V(r, h) = \pi r^2 h$. Find the differential dV. Interpret the formula geometrically.

197. See the preceding problem. Use differentials to estimate the amount of aluminum in an enclosed aluminum can with diameter 8.0 cm and height 12 cm if the aluminum is 0.04 cm thick.

198. Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x, y) moves from point $(1, 1)$ to point $(1.01, 0.97)$. Compare this approximation with the actual change in the function.

199. Let $z = f(x, y) = x^2 + 3xy - y^2$. Find the exact change in the function and the approximate change in the function as x changes from 2.00 to 2.05 and y changes from 3.00 to 2.96.

200. The centripetal acceleration of a particle moving in a circle is given by $a(r, v) = \frac{v^2}{r}$, where v is the velocity and r is the radius of the circle. Approximate the maximum percent error in measuring the acceleration resulting from errors of 3% in v and 2% in r. (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in a is given by $\frac{da}{a}$.)

201. The radius r and height h of a right circular cylinder are measured with possible errors of 4% and 5%, respectively. Approximate the maximum possible percentage error in measuring the volume (Recall that the percentage error is the ratio of the amount of error over the original amount. So, in this case, the percentage error in V is given by $\frac{dV}{V}$.)

202. The base radius and height of a right circular cone are measured as 10 in. and 25 in., respectively, with a possible error in measurement of as much as 0.1 in. each. Use differentials to estimate the maximum error in the calculated volume of the cone.

203. The electrical resistance R produced by wiring resistors R_1 and R_2 in parallel can be calculated from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 and R_2 are measured to be 7Ω and 6Ω, respectively, and if these measurements are accurate to within 0.05Ω, estimate the maximum possible error in computing R. (The symbol Ω represents an ohm, the unit of electrical resistance.)

204. The area of an ellipse with axes of length $2a$ and $2b$ is given by the formula $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

205. The period T of a simple pendulum with small oscillations is calculated from the formula $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the pendulum and g is the acceleration resulting from gravity. Suppose that L and g have errors of, at most, 0.5% and 0.1%, respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.

206. Electrical power P is given by $P = \frac{V^2}{R}$, where V is the voltage and R is the resistance. Approximate the maximum percentage error in calculating power if 120 V is applied to a $2000 - \Omega$ resistor and the possible percent errors in measuring V and R are 3% and 4%, respectively.

For the following exercises, find the linear approximation of each function at the indicated point.

207. $f(x, y) = x\sqrt{y}, \quad P(1, 4)$

208. $f(x, y) = e^x \cos y; \, P(0, 0)$

209. $f(x, y) = \arctan(x + 2y), \, P(1, 0)$

210. $f(x, y) = \sqrt{20 - x^2 - 7y^2}, \quad P(2, 1)$

211. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \quad P(3, 2, 6)$

212. **[T]** Find the equation of the tangent plane to the surface $f(x, y) = x^2 + y^2$ at point $(1, 2, 5)$, and graph the surface and the tangent plane at the point.

213. **[T]** Find the equation for the tangent plane to the surface at the indicated point, and graph the surface and the tangent plane: $z = \ln(10x^2 + 2y^2 + 1), \, P(0, 0, 0)$.

214. **[T]** Find the equation of the tangent plane to the surface $z = f(x, y) = \sin(x + y^2)$ at point $\left(\frac{\pi}{4}, 0, \frac{\sqrt{2}}{2}\right)$, and graph the surface and the tangent plane.

4.5 | The Chain Rule

Learning Objectives

4.5.1 State the chain rules for one or two independent variables.
4.5.2 Use tree diagrams as an aid to understanding the chain rule for several independent and intermediate variables.
4.5.3 Perform implicit differentiation of a function of two or more variables.

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule. In this section, we study extensions of the chain rule and learn how to take derivatives of compositions of functions of more than one variable.

Chain Rules for One or Two Independent Variables

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

$$\frac{d}{dx}\big(f(g(x))\big) = f'\big(g(x)\big)g'(x).$$

In this equation, both $f(x)$ and $g(x)$ are functions of one variable. Now suppose that f is a function of two variables and g is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

Theorem 4.8: Chain Rule for One Independent Variable

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f\big(x(t), y(t)\big)$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}, \tag{4.29}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Proof

The proof of this theorem uses the definition of differentiability of a function of two variables. Suppose that f is differentiable at the point $P(x_0, y_0)$, where $x_0 = g(t_0)$ and $y_0 = h(t_0)$ for a fixed value of t_0. We wish to prove that $z = f\big(x(t), y(t)\big)$ is differentiable at $t = t_0$ and that **Equation 4.29** holds at that point as well.

Since f is differentiable at P, we know that

$$z(t) = f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y), \tag{4.30}$$

where $\lim_{(x, y)\to(x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$ We then subtract $z_0 = f(x_0, y_0)$ from both sides of this equation:

$$\begin{aligned} z(t) - z(t_0) &= f\big(x(t), y(t)\big) - f\big(x(t_0), y(t_0)\big) \\ &= f_x(x_0, y_0)\big(x(t) - x(t_0)\big) + f_y(x_0, y_0)\big(y(t) - y(t_0)\big) + E\big(x(t), y(t)\big). \end{aligned}$$

Next, we divide both sides by $t - t_0$:

$$\frac{z(t) - z(t_0)}{t - t_0} = f_x(x_0, y_0)\left(\frac{x(t) - x(t_0)}{t - t_0}\right) + f_y(x_0, y_0)\left(\frac{y(t) - y(t_0)}{t - t_0}\right) + \frac{E\big(x(t), y(t)\big)}{t - t_0}.$$

Then we take the limit as t approaches t_0:

$$\lim_{t \to t_0} \frac{z(t) - z(t_0)}{t - t_0} = f_x(x_0, y_0) \lim_{t \to t_0} \left(\frac{x(t) - x(t_0)}{t - t_0} \right) + f_y(x_0, y_0) \lim_{t \to t_0} \left(\frac{y(t) - y(t_0)}{t - t_0} \right)$$
$$+ \lim_{t \to t_0} \frac{E(x(t), y(t))}{t - t_0}.$$

The left-hand side of this equation is equal to dz/dt, which leads to

$$\frac{dz}{dt} = f_x(x_0, y_0)\frac{dx}{dt} + f_y(x_0, y_0)\frac{dy}{dt} + \lim_{t \to t_0} \frac{E(x(t), y(t))}{t - t_0}.$$

The last term can be rewritten as

$$\begin{aligned}\lim_{t \to t_0} \frac{E(x(t), y(t))}{t - t_0} &= \lim_{t \to t_0} \left(\frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0} \right) \\ &= \lim_{t \to t_0} \left(\frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right) \lim_{t \to t_0} \left(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0} \right).\end{aligned}$$

As t approaches t_0, $(x(t), y(t))$ approaches $(x(t_0), y(t_0))$, so we can rewrite the last product as

$$\lim_{(x, y) \to (x_0, y_0)} \left(\frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \right) \lim_{(x, y) \to (x_0, y_0)} \left(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0} \right).$$

Since the first limit is equal to zero, we need only show that the second limit is finite:

$$\begin{aligned}\lim_{(x, y) \to (x_0, y_0)} \left(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{t - t_0} \right) &= \lim_{(x, y) \to (x_0, y_0)} \left(\sqrt{\frac{(x - x_0)^2 + (y - y_0)^2}{(t - t_0)^2}} \right) \\ &= \lim_{(x, y) \to (x_0, y_0)} \left(\sqrt{\left(\frac{x - x_0}{t - t_0} \right)^2 + \left(\frac{y - y_0}{t - t_0} \right)^2} \right) \\ &= \sqrt{\left(\lim_{(x, y) \to (x_0, y_0)} \left(\frac{x - x_0}{t - t_0} \right) \right)^2 + \left(\lim_{(x, y) \to (x_0, y_0)} \left(\frac{y - y_0}{t - t_0} \right) \right)^2}.\end{aligned}$$

Since $x(t)$ and $y(t)$ are both differentiable functions of t, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at $t = t_0$; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

□

Closer examination of **Equation 4.29** reveals an interesting pattern. The first term in the equation is $\frac{\partial f}{\partial x} \cdot \frac{dx}{dt}$ and the second term is $\frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$. Recall that when multiplying fractions, cancelation can be used. If we treat these derivatives as fractions, then each product "simplifies" to something resembling $\partial f/dt$. The variables x and y that disappear in this simplification are often called **intermediate variables**: they are independent variables for the function f, but are dependent variables for the variable t. Two terms appear on the right-hand side of the formula, and f is a function of two variables. This pattern works with functions of more than two variables as well, as we see later in this section.

Example 4.26

Using the Chain Rule

Calculate dz/dt for each of the following functions:

a. $z = f(x, y) = 4x^2 + 3y^2, x = x(t) = \sin t, y = y(t) = \cos t$

b. $z = f(x, y) = \sqrt{x^2 - y^2}, x = x(t) = e^{2t}, y = y(t) = e^{-t}$

Solution

a. To use the chain rule, we need four quantities— $\partial z/\partial x, \partial z/\partial y, dx/dt,$ and dy/dt:

$$\frac{\partial z}{\partial x} = 8x \qquad \frac{\partial z}{\partial y} = 6y$$
$$\frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -\sin t$$

Now, we substitute each of these into **Equation 4.29**:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \\ &= (8x)(\cos t) + (6y)(-\sin t) \\ &= 8x\cos t - 6y\sin t.\end{aligned}$$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t) = \sin t$ and $y(t) = \cos t$. We obtain

$$\begin{aligned}\frac{dz}{dt} &= 8x\cos t - 6y\sin t \\ &= 8(\sin t)\cos t - 6(\cos t)\sin t \\ &= 2\sin t\cos t.\end{aligned}$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y),$ then differentiating with respect to t:

$$\begin{aligned}z &= f(x, y) \\ &= f(x(t), y(t)) \\ &= 4(x(t))^2 + 3(y(t))^2 \\ &= 4\sin^2 t + 3\cos^2 t.\end{aligned}$$

Then

$$\begin{aligned}\frac{dz}{dt} &= 2(4\sin t)(\cos t) + 2(3\cos t)(-\sin t) \\ &= 8\sin t\cos t - 6\sin t\cos t \\ &= 2\sin t\cos t,\end{aligned}$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

b. To use the chain rule, we again need four quantities— $\partial z/\partial x, \partial z/dy, dx/dt,$ and dy/dt:

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 - y^2}} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{x^2 - y^2}}$$
$$\frac{dx}{dt} = 2e^{2t} \qquad \frac{dx}{dt} = -e^{-t}.$$

We substitute each of these into **Equation 4.29**:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \\ &= \left(\frac{x}{\sqrt{x^2 - y^2}}\right)\left(2e^{2t}\right) + \left(\frac{-y}{\sqrt{x^2 - y^2}}\right)\left(-e^{-t}\right) \\ &= \frac{2xe^{2t} - ye^{-t}}{\sqrt{x^2 - y^2}}.\end{aligned}$$

To reduce this to one variable, we use the fact that $x(t) = e^{2t}$ and $y(t) = e^{-t}$. Therefore,

$$\begin{aligned}\frac{dz}{dt} &= \frac{2xe^{2t} + ye^{-t}}{\sqrt{x^2 - y^2}} \\ &= \frac{2\left(e^{2t}\right)e^{2t} + \left(e^{-t}\right)e^{-t}}{\sqrt{e^{4t} - e^{-2t}}} \\ &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}.\end{aligned}$$

To eliminate negative exponents, we multiply the top by e^{2t} and the bottom by $\sqrt{e^{4t}}$:

$$\begin{aligned}\frac{dz}{dt} &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}} \cdot \frac{e^{2t}}{\sqrt{e^{4t}}} \\ &= \frac{2e^{6t} + 1}{\sqrt{e^{8t} - e^{2t}}} \\ &= \frac{2e^{6t} + 1}{\sqrt{e^{2t}\left(e^{6t} - 1\right)}} \\ &= \frac{2e^{6t} + 1}{e^t\sqrt{e^{6t} - 1}}.\end{aligned}$$

Again, this derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x, y),$ then differentiating with respect to t:

$$\begin{aligned}z &= f(x, y) \\ &= f(x(t), y(t)) \\ &= \sqrt{(x(t))^2 - (y(t))^2} \\ &= \sqrt{e^{4t} - e^{-2t}} \\ &= \left(e^{4t} - e^{-2t}\right)^{1/2}.\end{aligned}$$

Then

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{2}\left(e^{4t} - e^{-2t}\right)^{-1/2}\left(4e^{4t} + 2e^{-2t}\right) \\ &= \frac{2e^{4t} + e^{-2t}}{\sqrt{e^{4t} - e^{-2t}}}.\end{aligned}$$

This is the same solution.

4.23 Calculate dz/dt given the following functions. Express the final answer in terms of t.

$$z = f(x, y) = x^2 - 3xy + 2y^2, \; x = x(t) = 3\sin 2t, \; y = y(t) = 4\cos 2t$$

It is often useful to create a visual representation of **Equation 4.29** for the chain rule. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula (**Figure 4.34**). This diagram can be expanded for functions of more than one variable, as we shall see very shortly.

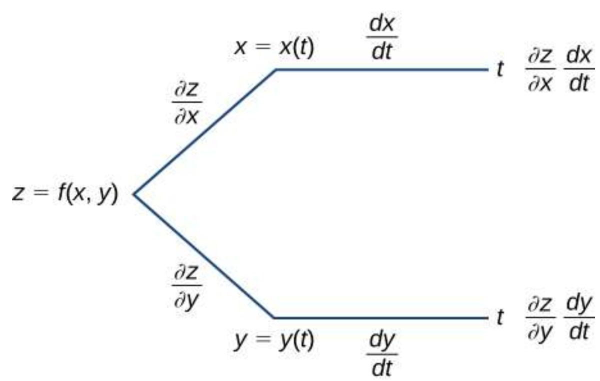


Figure 4.34 Tree diagram for the case $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$.

In this diagram, the leftmost corner corresponds to $z = f(x, y)$. Since f has two independent variables, there are two lines coming from this corner. The upper branch corresponds to the variable x and the lower branch corresponds to the variable y. Since each of these variables is then dependent on one variable t, one branch then comes from x and one branch comes from y. Last, each of the branches on the far right has a label that represents the path traveled to reach that branch. The top branch is reached by following the x branch, then the t branch; therefore, it is labeled $(\partial z/\partial x) \times (dx/dt)$. The bottom branch is similar: first the y branch, then the t branch. This branch is labeled $(\partial z/\partial y) \times (dy/dt)$. To get the formula for dz/dt, add all the terms that appear on the rightmost side of the diagram. This gives us **Equation 4.29**.

In **Chain Rule for Two Independent Variables**, $z = f(x, y)$ is a function of x and y, and both $x = g(u, v)$ and $y = h(u, v)$ are functions of the independent variables u and v.

Theorem 4.9: Chain Rule for Two Independent Variables

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial x}{\partial u} \quad (4.31)$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \quad (4.32)$$

We can draw a tree diagram for each of these formulas as well as follows.

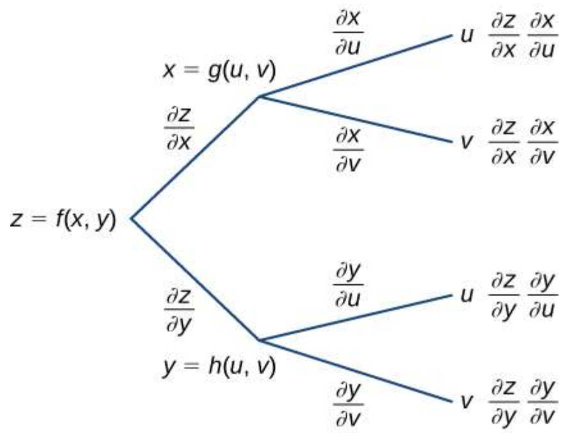


Figure 4.35 Tree diagram for $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$.

To derive the formula for $\partial z/\partial u$, start from the left side of the diagram, then follow only the branches that end with u and add the terms that appear at the end of those branches. For the formula for $\partial z/\partial v$, follow only the branches that end with v and add the terms that appear at the end of those branches.

There is an important difference between these two chain rule theorems. In **Chain Rule for One Independent Variable**, the left-hand side of the formula for the derivative is not a partial derivative, but in **Chain Rule for Two Independent Variables** it is. The reason is that, in **Chain Rule for One Independent Variable**, z is ultimately a function of t alone, whereas in **Chain Rule for Two Independent Variables**, z is a function of both u and v.

Example 4.27

Using the Chain Rule for Two Variables

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ using the following functions:

$$z = f(x, y) = 3x^2 - 2xy + y^2, \; x = x(u, v) = 3u + 2v, \; y = y(u, v) = 4u - v.$$

Solution

To implement the chain rule for two variables, we need six partial derivatives—$\partial z/\partial x$, $\partial z/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$:

$$\frac{\partial z}{\partial x} = 6x - 2y \qquad \frac{\partial z}{\partial y} = -2x + 2y$$
$$\frac{\partial x}{\partial u} = 3 \qquad \frac{\partial x}{\partial v} = 2$$
$$\frac{\partial y}{\partial u} = 4 \qquad \frac{\partial y}{\partial v} = -1.$$

To find $\partial z/\partial u$, we use **Equation 4.31**:

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \\ &= 3(6x - 2y) + 4(-2x + 2y) \\ &= 10x + 2y.\end{aligned}$$

Next, we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial u} &= 10x + 2y \\ &= 10(3u + 2v) + 2(4u - v) \\ &= 38u + 18v.\end{aligned}$$

To find $\partial z/\partial v$, we use **Equation 4.32**:

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= 2(6x - 2y) + (-1)(-2x + 2y) \\ &= 14x - 6y.\end{aligned}$$

Then we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial v} &= 14x - 6y \\ &= 14(3u + 2v) - 6(4u - v) \\ &= 18u + 34v.\end{aligned}$$

4.24 Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x, y) = \frac{2x - y}{x + 3y}, \; x(u, v) = e^{2u}\cos 3v, \; y(u, v) = e^{2u}\sin 3v.$$

The Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the **generalized chain rule** states.

Theorem 4.10: Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each $i \in \{1, \ldots, m\}$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j} \quad (4.33)$$

for any $j \in \{1, 2, \ldots, n\}$.

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

Example 4.28

Using the Generalized Chain Rule

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ using the following functions:

$$\begin{aligned} w &= f(x, y, z) = 3x^2 - 2xy + 4z^2 \\ x &= x(u, v) = e^u \sin v \\ y &= y(u, v) = e^u \cos v \\ z &= z(u, v) = e^u. \end{aligned}$$

Solution

The formulas for $\partial w/\partial u$ and $\partial w/\partial v$ are

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}.$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\begin{aligned} \frac{\partial w}{\partial x} &= 6x - 2y & \frac{\partial w}{\partial y} &= -2x & \frac{\partial w}{\partial z} &= 8z \\ \frac{\partial x}{\partial u} &= e^u \sin v & \frac{\partial y}{\partial u} &= e^u \cos v & \frac{\partial z}{\partial u} &= e^u \\ \frac{\partial x}{\partial v} &= e^u \cos v & \frac{\partial y}{\partial v} &= -e^u \sin v & \frac{\partial z}{\partial v} &= 0. \end{aligned}$$

Now, we substitute each of them into the first formula to calculate $\partial w/\partial u$:

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \\ &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u, \end{aligned}$$

then substitute $x(u, v) = e^u \sin v$, $y(u, v) = e^u \cos v$, and $z(u, v) = e^u$ into this equation:

$$\begin{aligned} \frac{\partial w}{\partial u} &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u \\ &= (6e^u \sin v - 2e^u \cos v)e^u \sin v - 2(e^u \sin v)e^u \cos v + 8e^{2u} \\ &= 6e^{2u} \sin^2 v - 4e^{2u} \sin v \cos v + 8e^{2u} \\ &= 2e^{2u}\left(3 \sin^2 v - 2 \sin v \cos v + 4\right). \end{aligned}$$

Next, we calculate $\partial w/\partial v$:

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}\\ &= (6x-2y)e^u\cos v-2x(-e^u\sin v)+8z(0),\end{aligned}$$

then we substitute $x(u, v) = e^u \sin v,\ y(u, v) = e^u \cos v,$ and $z(u, v) = e^u$ into this equation:

$$\begin{aligned}\frac{\partial w}{\partial v} &= (6x-2y)e^u\cos v-2x(-e^u\sin v)\\ &= (6e^u\sin v-2e^u\cos v)e^u\cos v+2(e^u\sin v)(e^u\sin v)\\ &= 2e^{2u}\sin^2 v+6e^{2u}\sin v\cos v-2e^{2u}\cos^2 v\\ &= 2e^{2u}\left(\sin^2 v+\sin v\cos v-\cos^2 v\right).\end{aligned}$$

4.25 Calculate $\partial w/\partial u$ and $\partial w/\partial v$ given the following functions:

$$\begin{aligned} w &= f(x, y, z) = \frac{x+2y-4z}{2x-y+3z}\\ x &= x(u, v) = e^{2u}\cos 3v\\ y &= y(u, v) = e^{2u}\sin 3v\\ z &= z(u, v) = e^{2u}.\end{aligned}$$

Example 4.29

Drawing a Tree Diagram

Create a tree diagram for the case when

$$w = f(x, y, z),\ x = x(t, u, v),\ y = y(t, u, v),\ z = z(t, u, v)$$

and write out the formulas for the three partial derivatives of w.

Solution

Starting from the left, the function f has three independent variables: x, y, and z. Therefore, three branches must be emanating from the first node. Each of these three branches also has three branches, for each of the variables t, u, and v.

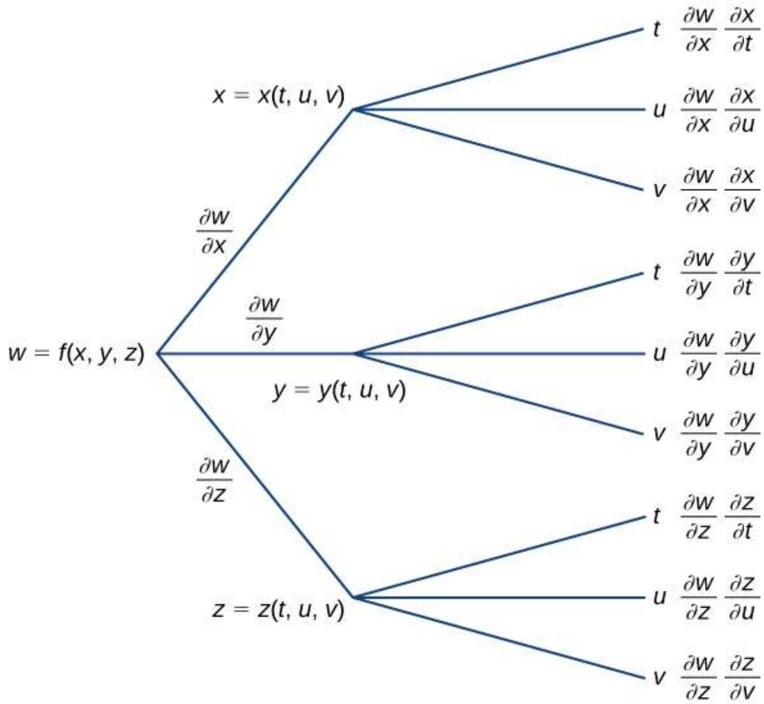


Figure 4.36 Tree diagram for a function of three variables, each of which is a function of three independent variables.

The three formulas are

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

4.26 Create a tree diagram for the case when

$$w = f(x, y),\ x = x(t, u, v),\ y = y(t, u, v)$$

and write out the formulas for the three partial derivatives of w.

Implicit Differentiation

Recall from **Implicit Differentiation (http://cnx.org/content/m53585/latest/)** that implicit differentiation provides a method for finding dy/dx when y is defined implicitly as a function of x. The method involves differentiating both sides of the equation defining the function with respect to x, then solving for dy/dx. Partial derivatives provide an alternative to this method.

Consider the ellipse defined by the equation $x^2 + 3y^2 + 4y - 4 = 0$ as follows.

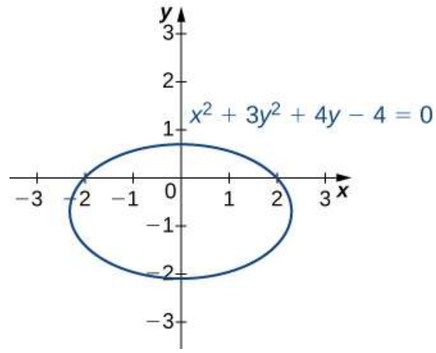


Figure 4.37 Graph of the ellipse defined by $x^2 + 3y^2 + 4y - 4 = 0.$

This equation implicitly defines y as a function of x. As such, we can find the derivative dy/dx using the method of implicit differentiation:

$$\begin{aligned}\frac{d}{dx}\left(x^2 + 3y^2 + 4y - 4\right) &= \frac{d}{dx}(0) \\ 2x + 6y\frac{dy}{dx} + 4\frac{dy}{dx} &= 0 \\ (6y + 4)\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{x}{3y + 2}.\end{aligned}$$

We can also define a function $z = f(x, y)$ by using the left-hand side of the equation defining the ellipse. Then $f(x, y) = x^2 + 3y^2 + 4y - 4.$ The ellipse $x^2 + 3y^2 + 4y - 4 = 0$ can then be described by the equation $f(x, y) = 0.$ Using this function and the following theorem gives us an alternative approach to calculating dy/dx.

Theorem 4.11: Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z = f(x, y)$ defines y implicitly as a function $y = g(x)$ of x via the equation $f(x, y) = 0.$ Then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \tag{4.34}$$

provided $f_y(x, y) \neq 0.$

If the equation $f(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} \tag{4.35}$$

as long as $f_z(x, y, z) \neq 0.$

Equation 4.34 is a direct consequence of **Equation 4.31**. In particular, if we assume that y is defined implicitly as a function of x via the equation $f(x, y) = 0,$ we can apply the chain rule to find dy/dx:

$$\begin{aligned} \frac{d}{dx} f(x,\ y) &= \frac{d}{dx}(0) \\ \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} &= 0 \\ \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} &= 0. \end{aligned}$$

Solving this equation for dy/dx gives **Equation 4.34**. **Equation 4.35** can be derived in a similar fashion.

Let's now return to the problem that we started before the previous theorem. Using **Implicit Differentiation of a Function of Two or More Variables** and the function $f(x,\ y) = x^2 + 3y^2 + 4y - 4,$ we obtain

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x \\ \frac{\partial f}{\partial y} &= 6y + 4. \end{aligned}$$

Then **Equation 4.34** gives

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x}{6y+4} = -\frac{x}{3y+2},$$

which is the same result obtained by the earlier use of implicit differentiation.

Example 4.30

Implicit Differentiation by Partial Derivatives

a. Calculate dy/dx if y is defined implicitly as a function of x via the equation $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0.$ What is the equation of the tangent line to the graph of this curve at point $(2,\ 1)$?

b. Calculate $\partial z/\partial x$ and $\partial z/\partial y,$ given $x^2 e^y - yze^x = 0.$

Solution

a. Set $f(x,\ y) = 3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0,$ then calculate f_x and f_y: $\begin{aligned} f_x &= 6x - 2y + 4 \\ f_y &= -2x + 2y - 6. \end{aligned}$

The derivative is given by

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{6x - 2y + 4}{-2x + 2y - 6} = \frac{3x - y + 2}{x - y + 3}.$$

The slope of the tangent line at point $(2,\ 1)$ is given by

$$\left.\frac{dy}{dx}\right|_{(x,\ y) = (2,\ 1)} = \frac{3(2) - 1 + 2}{2 - 1 + 3} = \frac{7}{4}.$$

To find the equation of the tangent line, we use the point-slope form (**Figure 4.38**):

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= \frac{7}{4}(x - 2) \\ y &= \frac{7}{4}x - \frac{7}{2} + 1 \\ y &= \frac{7}{4}x - \frac{5}{2}. \end{aligned}$$

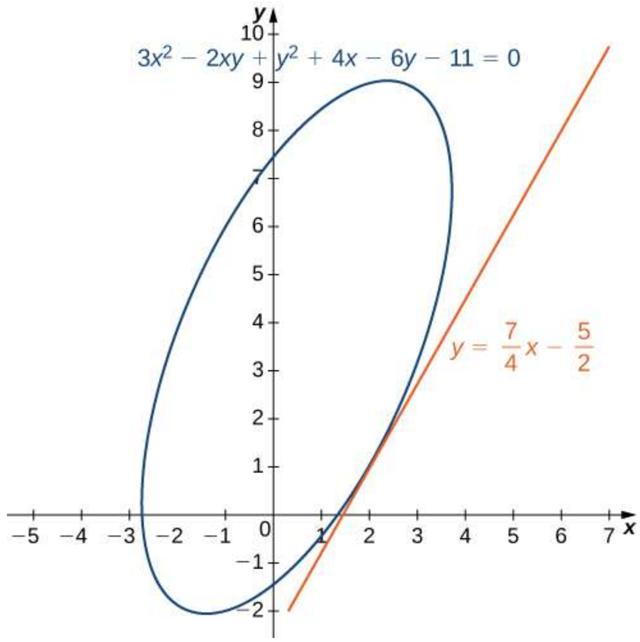


Figure 4.38 Graph of the rotated ellipse defined by $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0.$

b. We have $f(x, y, z) = x^2 e^y - yze^x.$ Therefore,

$$\frac{\partial f}{\partial x} = 2xe^y - yze^x$$
$$\frac{\partial f}{\partial y} = x^2 e^y - ze^x$$
$$\frac{\partial f}{\partial z} = -ye^x.$$

Using **Equation 4.35**,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2xe^y - yze^x}{-ye^x} = \frac{2xe^y - yze^x}{ye^x} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{x^2 e^y - ze^x}{-ye^x} = \frac{x^2 e^y - ze^x}{ye^x}.$$

4.27 Find dy/dx if y is defined implicitly as a function of x by the equation $x^2 + xy - y^2 + 7x - 3y - 26 = 0.$ What is the equation of the tangent line to the graph of this curve at point $(3, -2)$?

4.5 EXERCISES

For the following exercises, use the information provided to solve the problem.

215. Let $w(x, y, z) = xy\cos z$, where $x = t$, $y = t^2$, and $z = \arcsin t$. Find $\frac{dw}{dt}$.

216. Let $w(t, v) = e^{tv}$ where $t = r + s$ and $v = rs$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.

217. If $w = 5x^2 + 2y^2$, $x = -3s + t$, and $y = s - 4t$, find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

218. If $w = xy^2$, $x = 5\cos(2t)$, and $y = 5\sin(2t)$, find $\frac{\partial w}{\partial t}$.

219. If $f(x, y) = xy$, $x = r\cos\theta$, and $y = r\sin\theta$, find $\frac{\partial f}{\partial r}$ and express the answer in terms of r and θ.

220. Suppose $f(x, y) = x + y$, $u = e^x \sin y$, $x = t^2$, and $y = \pi t$, where $x = r\cos\theta$ and $y = r\sin\theta$. Find $\frac{\partial f}{\partial \theta}$.

For the following exercises, find $\frac{df}{dt}$ using the chain rule and direct substitution.

221. $f(x, y) = x^2 + y^2$, $x = t$, $y = t^2$

222. $f(x, y) = \sqrt{x^2 + y^2}$, $y = t^2$, $x = t$

223. $f(x, y) = xy$, $x = 1 - \sqrt{t}$, $y = 1 + \sqrt{t}$

224. $f(x, y) = \frac{x}{y}$, $x = e^t$, $y = 2e^t$

225. $f(x, y) = \ln(x + y)$, $x = e^t$, $y = e^t$

226. $f(x, y) = x^4$, $x = t$, $y = t$

227. Let $w(x, y, z) = x^2 + y^2 + z^2$, $x = \cos t$, $y = \sin t$, and $z = e^t$. Express w as a function of t and find $\frac{dw}{dt}$ directly. Then, find $\frac{dw}{dt}$ using the chain rule.

228. Let $z = x^2 y$, where $x = t^2$ and $y = t^3$. Find $\frac{dz}{dt}$.

229. Let $u = e^x \sin y$, where $x = t^2$ and $y = \pi t$. Find $\frac{du}{dt}$ when $x = \ln 2$ and $y = \frac{\pi}{4}$.

For the following exercises, find $\frac{dy}{dx}$ using partial derivatives.

230. $\sin(6x) + \tan(8y) + 5 = 0$

231. $x^3 + y^2 x - 3 = 0$

232. $\sin(x + y) + \cos(x - y) = 4$

233. $x^2 - 2xy + y^4 = 4$

234. $xe^y + ye^x - 2x^2 y = 0$

235. $x^{2/3} + y^{2/3} = a^{2/3}$

236. $x\cos(xy) + y\cos x = 2$

237. $e^{xy} + ye^y = 1$

238. $x^2 y^3 + \cos y = 0$

239. Find $\frac{dz}{dt}$ using the chain rule where $z = 3x^2 y^3$, $x = t^4$, and $y = t^2$.

240. Let $z = 3\cos x - \sin(xy)$, $x = \frac{1}{t}$, and $y = 3t$. Find $\frac{dz}{dt}$.

241. Let $z = e^{1 - xy}$, $x = t^{1/3}$, and $y = t^3$. Find $\frac{dz}{dt}$.

242. Find $\frac{dz}{dt}$ by the chain rule where $z = \cosh^2(xy)$, $x = \frac{1}{2}t$, and $y = e^t$.

243. Let $z = \frac{x}{y}$, $x = 2\cos u$, and $y = 3\sin v$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

244. Let $z = e^{x^2 y}$, where $x = \sqrt{uv}$ and $y = \frac{1}{v}$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

245. If $z = xye^{x/y}$, $x = r\cos\theta$, and $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ when $r = 2$ and $\theta = \frac{\pi}{6}$.

246. Find $\frac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$, $x = e^{rs^2}$, $y = \ln\left(\frac{r+s}{t}\right)$, and $z = rst^2$.

247. If $w = \sin(xyz)$, $x = 1 - 3t$, $y = e^{1-t}$, and $z = 4t$, find $\frac{\partial w}{\partial t}$.

For the following exercises, use this information: A function $f(x, y)$ is said to be homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$. For all homogeneous functions of degree n, the following equation is true: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$. Show that the given function is homogeneous and verify that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$.

248. $f(x, y) = 3x^2 + y^2$

249. $f(x, y) = \sqrt{x^2 + y^2}$

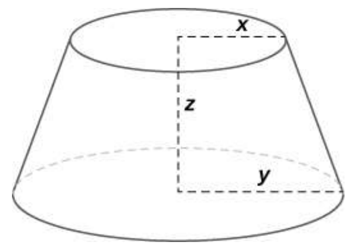
250. $f(x, y) = x^2 y - 2y^3$

251. The volume of a right circular cylinder is given by $V(x, y) = \pi x^2 y$, where x is the radius of the cylinder and y is the cylinder height. Suppose x and y are functions of t given by $x = \frac{1}{2}t$ and $y = \frac{1}{3}t$ so that x and y are both increasing with time. How fast is the volume increasing when $x = 2$ and $y = 5$?

252. The pressure P of a gas is related to the volume and temperature by the formula $PV = kT$, where temperature is expressed in kelvins. Express the pressure of the gas as a function of both V and T. Find $\frac{dP}{dt}$ when $k = 1$, $\frac{dV}{dt} = 2$ cm^3/min, $\frac{dT}{dt} = \frac{1}{2}$ K/min, $V = 20$ cm^3, and $T = 20°$F.

253. The radius of a right circular cone is increasing at 3 cm/min whereas the height of the cone is decreasing at 2 cm/min. Find the rate of change of the volume of the cone when the radius is 13 cm and the height is 18 cm.

254. The volume of a frustum of a cone is given by the formula $V = \frac{1}{3}\pi z\left(x^2 + y^2 + xy\right)$, where x is the radius of the smaller circle, y is the radius of the larger circle, and z is the height of the frustum (see figure). Find the rate of change of the volume of this frustum when $x = 10$ in., $y = 12$ in., and $z = 18$ in.



255. A closed box is in the shape of a rectangular solid with dimensions x, y, and z. (Dimensions are in inches.) Suppose each dimension is changing at the rate of 0.5 in./min. Find the rate of change of the total surface area of the box when $x = 2$ in., $y = 3$ in., and $z = 1$ in.

256. The total resistance in a circuit that has three individual resistances represented by x, y, and z is given by the formula $R(x, y, z) = \frac{xyz}{yz + xz + xy}$. Suppose at a given time the x resistance is 100Ω, the y resistance is 200Ω, and the z resistance is 300Ω. Also, suppose the x resistance is changing at a rate of 2Ω/min, the y resistance is changing at the rate of 1Ω/min, and the z resistance has no change. Find the rate of change of the total resistance in this circuit at this time.

257. The temperature T at a point (x, y) is $T(x, y)$ and is measured using the Celsius scale. A fly crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$ and $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature increasing on the fly's path after 3 sec?

258. The x and y components of a fluid moving in two dimensions are given by the following functions: $u(x, y) = 2y$ and $v(x, y) = -2x$; $x \geq 0$; $y \geq 0$. The speed of the fluid at the point (x, y) is $s(x, y) = \sqrt{u(x, y)^2 + v(x, y)^2}$. Find $\frac{\partial s}{\partial x}$ and $\frac{\partial s}{\partial y}$ using the chain rule.

259. Let $u = u(x, y, z)$, where $x = x(w, t)$, $y = y(w, t)$, $z = z(w, t)$, $w = w(r, s)$, and $t = t(r, s)$. Use a tree diagram and the chain rule to find an expression for $\frac{\partial u}{\partial r}$.

4.6 | Directional Derivatives and the Gradient

Learning Objectives

4.6.1 Determine the directional derivative in a given direction for a function of two variables.
4.6.2 Determine the gradient vector of a given real-valued function.
4.6.3 Explain the significance of the gradient vector with regard to direction of change along a surface.
4.6.4 Use the gradient to find the tangent to a level curve of a given function.
4.6.5 Calculate directional derivatives and gradients in three dimensions.

In **Partial Derivatives** we introduced the partial derivative. A function $z = f(x, y)$ has two partial derivatives: $\partial z/\partial x$ and $\partial z/\partial y$. These derivatives correspond to each of the independent variables and can be interpreted as instantaneous rates of change (that is, as slopes of a tangent line). For example, $\partial z/\partial x$ represents the slope of a tangent line passing through a given point on the surface defined by $z = f(x, y)$, assuming the tangent line is parallel to the *x*-axis. Similarly, $\partial z/\partial y$ represents the slope of the tangent line parallel to the y-axis. Now we consider the possibility of a tangent line parallel to neither axis.

Directional Derivatives

We start with the graph of a surface defined by the equation $z = f(x, y)$. Given a point (a, b) in the domain of f, we choose a direction to travel from that point. We measure the direction using an angle θ, which is measured counterclockwise in the *x*, *y*-plane, starting at zero from the positive *x*-axis (**Figure 4.39**). The distance we travel is h and the direction we travel is given by the unit vector $\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$. Therefore, the *z*-coordinate of the second point on the graph is given by $z = f(a + h\cos\theta, b + h\sin\theta)$.

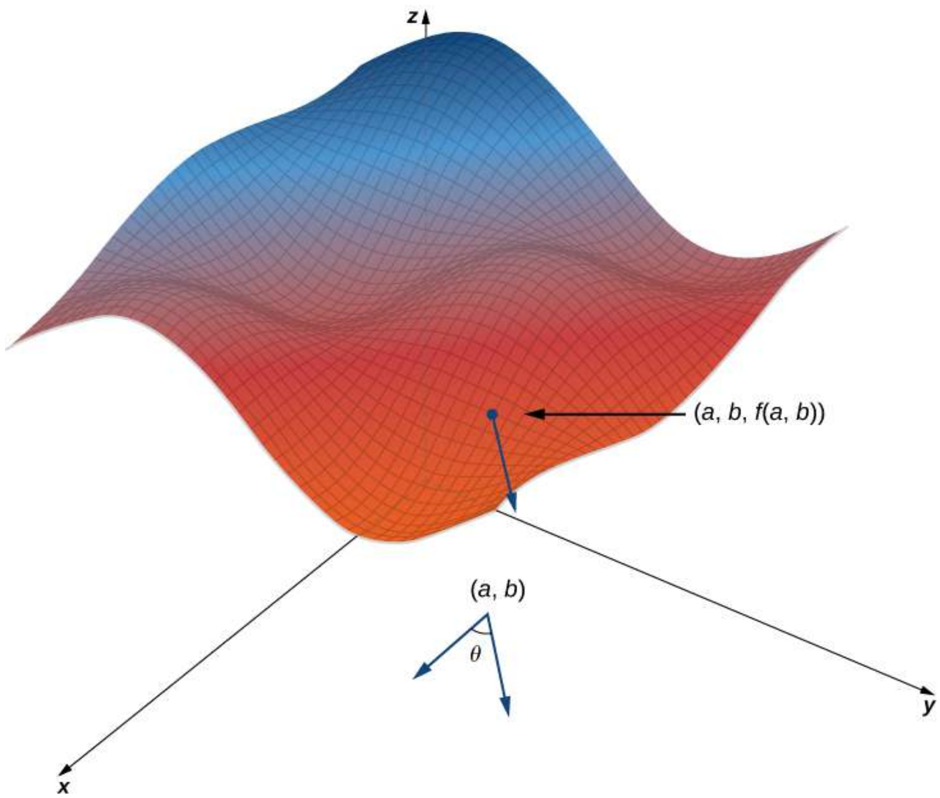


Figure 4.39 Finding the directional derivative at a point on the graph of $z = f(x, y)$. The slope of the black arrow on the graph indicates the value of the directional derivative at that point.

We can calculate the slope of the secant line by dividing the difference in z-values by the length of the line segment connecting the two points in the domain. The length of the line segment is h. Therefore, the slope of the secant line is

$$m_{\text{sec}} = \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h}.$$

To find the slope of the tangent line in the same direction, we take the limit as h approaches zero.

Definition

Suppose $z = f(x, y)$ is a function of two variables with a domain of D. Let $(a, b) \in D$ and define $\text{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. Then the **directional derivative** of f in the direction of u is given by

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h}, \tag{4.36}$$

provided the limit exists.

Equation 4.36 provides a formal definition of the directional derivative that can be used in many cases to calculate a directional derivative.

Example 4.31

Finding a Directional Derivative from the Definition

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\mathbf{u}} f(x, y)$ of $f(x, y) = x^2 - xy + 3y^2$ in the direction of $\mathrm{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$. What is $D_{\mathbf{u}} f(-1, 2)$?

Solution

First of all, since $\cos\theta = 3/5$ and θ is acute, this implies

$$\sin\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Using $f(x, y) = x^2 - xy + 3y^2$, we first calculate $f(x + h\cos\theta,\ y + h\sin\theta)$:

$$\begin{aligned} f(x + h\cos\theta,\ y + h\sin\theta) &= (x + h\cos\theta)^2 - (x + h\cos\theta)(y + h\sin\theta) + 3(y + h\sin\theta)^2 \\ &= x^2 + 2xh\cos\theta + h^2\cos^2\theta - xy - xh\sin\theta - yh\cos\theta \\ &\quad -h^2\sin\theta\cos\theta + 3y^2 + 6yh\sin\theta + 3h^2\sin^2\theta \\ &= x^2 + 2xh\left(\frac{3}{5}\right) + \frac{9h^2}{25} - xy - \frac{4xh}{5} - \frac{3yh}{5} - \frac{12h^2}{25} + 3y^2 \\ &\quad +6yh\left(\frac{4}{5}\right) + 3h^2\left(\frac{16}{25}\right) \\ &= x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}. \end{aligned}$$

We substitute this expression into **Equation 4.36**:

$$\begin{aligned} D_{\mathbf{u}} f(a, b) &= \lim_{h \to 0} \frac{f(a + h\cos\theta,\ b + h\sin\theta) - f(a, b)}{h} \\ &= \lim_{h \to 0} \frac{\left(x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}\right) - \left(x^2 - xy + 3y^2\right)}{h} \\ &= \lim_{h \to 0} \frac{\frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}}{h} \\ &= \lim_{h \to 0} \frac{2x}{5} + \frac{9h}{5} + \frac{21y}{5} \\ &= \frac{2x + 21y}{5}. \end{aligned}$$

To calculate $D_{\mathbf{u}} f(-1, 2)$, we substitute $x = -1$ and $y = 2$ into this answer:

$$\begin{aligned} D_{\mathbf{u}} f(-1, 2) &= \frac{2(-1) + 21(2)}{5} \\ &= \frac{-2 + 42}{5} \\ &= 8. \end{aligned}$$

(See the following figure.)

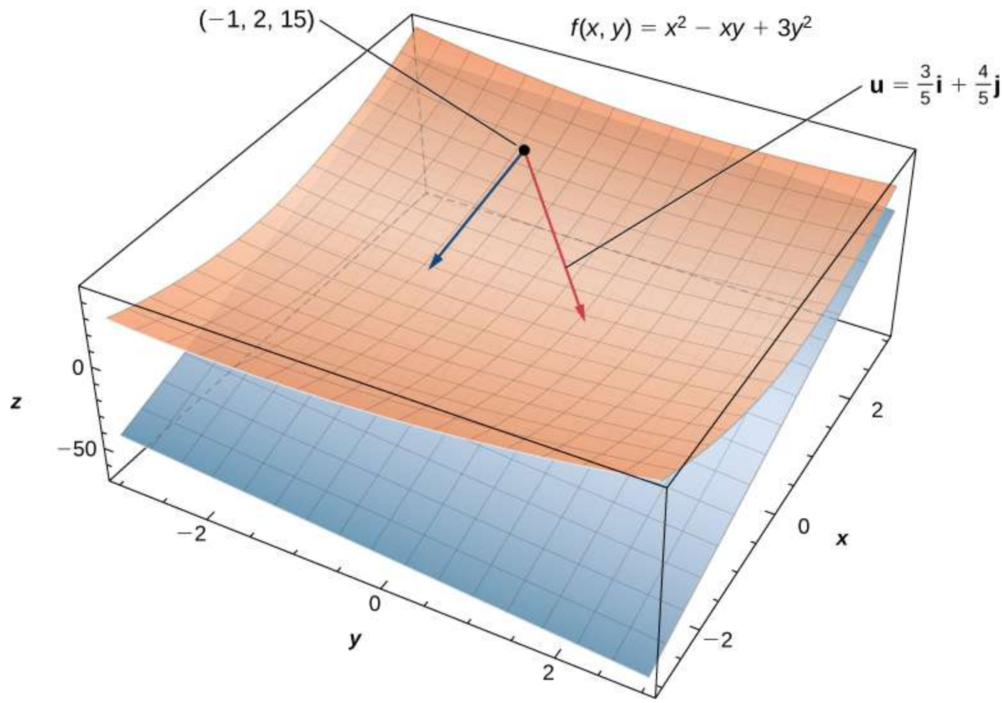


Figure 4.40 Finding the directional derivative in a given direction $\mathbf{u}$ at a given point on a surface. The plane is tangent to the surface at the given point $(-1, 2, 15)$.

Another approach to calculating a directional derivative involves partial derivatives, as outlined in the following theorem.

Theorem 4.12: Directional Derivative of a Function of Two Variables

Let $z = f(x, y)$ be a function of two variables x and y, and assume that f_x and f_y exist. Then the directional derivative of f in the direction of $\text{u} = \cos\theta\text{i} + \sin\theta\text{j}$ is given by

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{4.37}$$

Proof

Equation 4.36 states that the directional derivative of *f* in the direction of $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is given by

$$D_{\mathbf{u}} f(a, b) = \lim_{t \to 0} \frac{f(a + t\cos\theta, b + t\sin\theta) - f(a, b)}{t}.$$

Let $x = a + t\cos\theta$ and $y = b + t\sin\theta$, and define $g(t) = f(x, y)$. Since f_x and f_y both exist, we can use the chain rule for functions of two variables to calculate $g'(t)$:

$$\begin{aligned} g'(t) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \end{aligned}$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta.$$

By the definition of $g'(t)$, it is also true that

$$\begin{aligned} g'(0) &= \lim_{t \to 0} \frac{g(t) - g(0)}{t} \\ &= \lim_{t \to 0} \frac{f\left(x_0 + t\cos\theta,\ y_0 + t\sin\theta\right) - f(x_0,\ y_0)}{t}. \end{aligned}$$

Therefore, $D_{\mathbf{u}} f(x_0,\ y_0) = f_x(x,\ y)\cos\theta + f_y(x,\ y)\sin\theta$.

□

Example 4.32

Finding a Directional Derivative: Alternative Method

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\mathbf{u}} f(x,\ y)$ of $f(x,\ y) = x^2 - xy + 3y^2$ in the direction of $\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$. What is $D_{\mathbf{u}} f(-1,\ 2)$?

Solution

First, we must calculate the partial derivatives of f:

$$\begin{aligned} f_x &= 2x - y \\ f_y &= -x + 6y, \end{aligned}$$

Then we use **Equation 4.37** with $\theta = \arccos(3/5)$:

$$\begin{aligned} D_{\mathbf{u}} f(x,\ y) &= f_x(x,\ y)\cos\theta + f_y(x,\ y)\sin\theta \\ &= (2x - y)\frac{3}{5} + (-x + 6y)\frac{4}{5} \\ &= \frac{6x}{5} - \frac{3y}{5} - \frac{4x}{5} + \frac{24y}{5} \\ &= \frac{2x + 21y}{5}. \end{aligned}$$

To calculate $D_{\mathbf{u}} f(-1,\ 2)$, let $x = -1$ and $y = 2$:

$$D_{\mathbf{u}} f(-1,\ 2) = \frac{2(-1) + 21(2)}{5} = \frac{-2 + 42}{5} = 8.$$

This is the same answer obtained in **Example 4.31**.

4.28 Find the directional derivative $D_{\mathbf{u}} f(x,\ y)$ of $f(x,\ y) = 3x^2 y - 4xy^3 + 3y^2 - 4x$ in the direction of $\mathbf{u} = \left(\cos\frac{\pi}{3}\right)\mathbf{i} + \left(\sin\frac{\pi}{3}\right)\mathbf{j}$ using **Equation 4.37**. What is $D_{\mathbf{u}} f(3,\ 4)$?

If the vector that is given for the direction of the derivative is not a unit vector, then it is only necessary to divide by the norm of the vector. For example, if we wished to find the directional derivative of the function in **Example 4.32** in the direction of the vector $\langle -5,\ 12 \rangle$, we would first divide by its magnitude to get $\mathbf{u}$. This gives us $\mathbf{u} = \langle -(5/13),\ 12/13 \rangle$. Then

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \nabla f(x, y) \cdot \mathbf{u} \\ &= -\frac{5}{13}(2x - y) + \frac{12}{13}(-x + 6y) \\ &= -\frac{22}{13}x + \frac{17}{13}y. \end{aligned}$$

Gradient

The right-hand side of **Equation 4.37** is equal to $f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$, which can be written as the dot product of two vectors. Define the first vector as $\nabla f(x, y) = f_x(x, y)\mathrm{i} + f_y(x, y)\mathrm{j}$ and the second vector as $\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$. Then the right-hand side of the equation can be written as the dot product of these two vectors:

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}. \tag{4.38}$$

The first vector in **Equation 4.38** has a special name: the gradient of the function f. The symbol ∇ is called *nabla* and the vector ∇f is read "del f."

Definition

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. The vector $\nabla f(x, y)$ is called the **gradient** of f and is defined as

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}. \tag{4.39}$$

The vector $\nabla f(x, y)$ is also written as "grad f."

Example 4.33

Finding Gradients

Find the gradient $\nabla f(x, y)$ of each of the following functions:

a. $f(x, y) = x^2 - xy + 3y^2$

b. $f(x, y) = \sin 3x \cos 3y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x and f_y, then use **Equation 4.39**.

a.

$$\begin{aligned} f_x(x, y) &= 2x - y \text{ and } f_y(x, y) = -x + 6y, \text{ so} \\ \nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \\ &= (2x - y)\mathbf{i} + (-x + 6y)\mathbf{j}. \end{aligned}$$

b.

$$\begin{aligned} f_x(x, y) &= 3\cos 3x \cos 3y \text{ and } f_y(x, y) = -3\sin 3x \sin 3y, \text{ so} \\ \nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \\ &= (3\cos 3x \cos 3y)\mathbf{i} - (3\sin 3x \sin 3y)\mathbf{j}. \end{aligned}$$

4.29 Find the gradient $\nabla f(x, y)$ of $f(x, y) = \left(x^2 - 3y^2\right)/(2x + y)$.

The gradient has some important properties. We have already seen one formula that uses the gradient: the formula for the directional derivative. Recall from **The Dot Product** that if the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ is φ, then $\mathbf{a} \cdot \mathrm{b} = \|\mathbf{a}\| \ \|\mathrm{b}\| \cos\varphi$. Therefore, if the angle between $\nabla f(x_0, y_0)$ and $\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$ is φ, we have

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = \|\nabla f(x_0, y_0)\| \ \|\mathbf{u}\| \cos\varphi = \|\nabla f(x_0, y_0)\| \cos\varphi.$$

The $\|\mathbf{u}\|$ disappears because $\mathbf{u}$ is a unit vector. Therefore, the directional derivative is equal to the magnitude of the gradient evaluated at (x_0, y_0) multiplied by $\cos\varphi$. Recall that $\cos\varphi$ ranges from -1 to 1. If $\varphi = 0$, then $\cos\varphi = 1$ and $\nabla f(x_0, y_0)$ and $\mathbf{u}$ both point in the same direction. If $\varphi = \pi$, then $\cos\varphi = -1$ and $\nabla f(x_0, y_0)$ and $\mathbf{u}$ point in opposite directions. In the first case, the value of $D_{\mathrm{u}} f(x_0, y_0)$ is maximized; in the second case, the value of $D_{\mathrm{u}} f(x_0, y_0)$ is minimized. If $\nabla f(x_0, y_0) = 0$, then $D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = 0$ for any vector $\mathbf{u}$. These three cases are outlined in the following theorem.

Theorem 4.13: Properties of the Gradient

Suppose the function $z = f(x, y)$ is differentiable at (x_0, y_0) (**Figure 4.41**).

i. If $\nabla f(x_0, y_0) = \mathbf{0}$, then $D_{\mathbf{u}} f(x_0, y_0) = 0$ for any unit vector $\mathbf{u}$.

ii. If $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $D_{\mathbf{u}} f(x_0, y_0)$ is maximized when $\mathbf{u}$ points in the same direction as $\nabla f(x_0, y_0)$. The maximum value of $D_{\mathbf{u}} f(x_0, y_0)$ is $\|\nabla f(x_0, y_0)\|$.

iii. If $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $D_{\mathbf{u}} f(x_0, y_0)$ is minimized when $\mathbf{u}$ points in the opposite direction from $\nabla f(x_0, y_0)$. The minimum value of $D_{\mathbf{u}} f(x_0, y_0)$ is $-\|\nabla f(x_0, y_0)\|$.

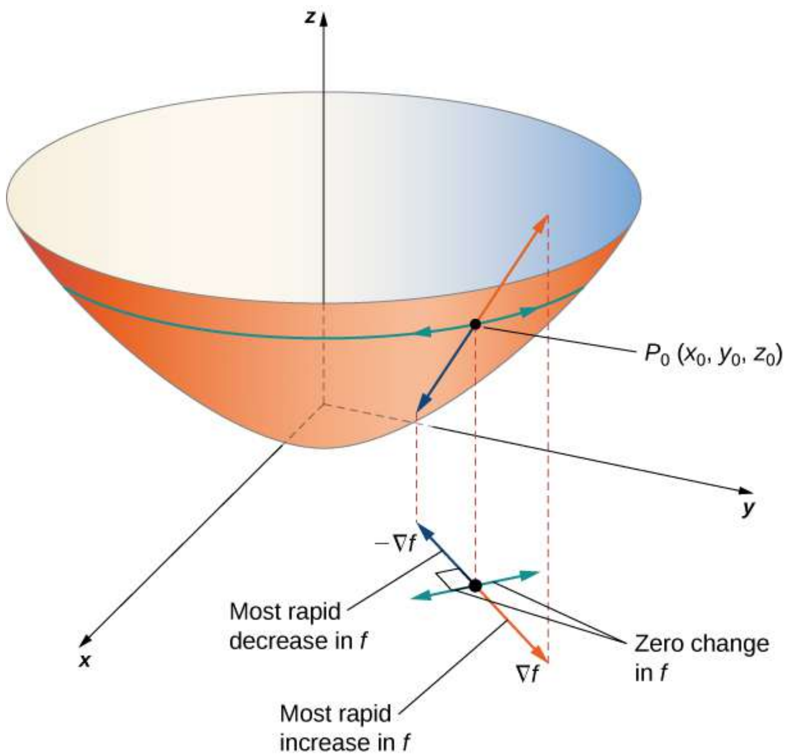


Figure 4.41 The gradient indicates the maximum and minimum values of the directional derivative at a point.

Example 4.34

Finding a Maximum Directional Derivative

Find the direction for which the directional derivative of $f(x, y) = 3x^2 - 4xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Solution

The maximum value of the directional derivative occurs when ∇f and the unit vector point in the same direction. Therefore, we start by calculating $\nabla f(x, y)$:

$$\begin{aligned} f_x(x, y) &= 6x - 4y \text{ and } f_y(x, y) = -4x + 4y, \text{ so} \\ \nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = (6x - 4y)\mathbf{i} + (-4x + 4y)\mathbf{j}. \end{aligned}$$

Next, we evaluate the gradient at $(-2, 3)$:

$$\nabla f(-2, 3) = (6(-2) - 4(3))\mathbf{i} + (-4(-2) + 4(3))\mathbf{j} = -24\mathbf{i} + 20\mathbf{j}.$$

We need to find a unit vector that points in the same direction as $\nabla f(-2, 3),$ so the next step is to divide $\nabla f(-2, 3)$ by its magnitude, which is $\sqrt{(-24)^2 + (20)^2} = \sqrt{976} = 4\sqrt{61}$. Therefore,

$$\frac{\nabla f(-2, 3)}{\| \nabla f(-2, 3) \|} = \frac{-24}{4\sqrt{61}}\mathbf{i} + \frac{20}{4\sqrt{61}}\mathbf{j} = \frac{-6\sqrt{61}}{61}\mathbf{i} + \frac{5\sqrt{61}}{61}\mathbf{j}.$$

This is the unit vector that points in the same direction as $\nabla f(-2, 3)$. To find the angle corresponding to this unit vector, we solve the equations

$$\cos\theta = \frac{-6\sqrt{61}}{61} \text{ and } \sin\theta = \frac{5\sqrt{61}}{61}$$

for θ. Since cosine is negative and sine is positive, the angle must be in the second quadrant. Therefore, $\theta = \pi - \arcsin\left(\left(5\sqrt{61}\right)/61\right) \approx 2.45$ rad.

The maximum value of the directional derivative at $(-2, 3)$ is $\|\nabla f(-2, 3)\| = 4\sqrt{61}$ (see the following figure).

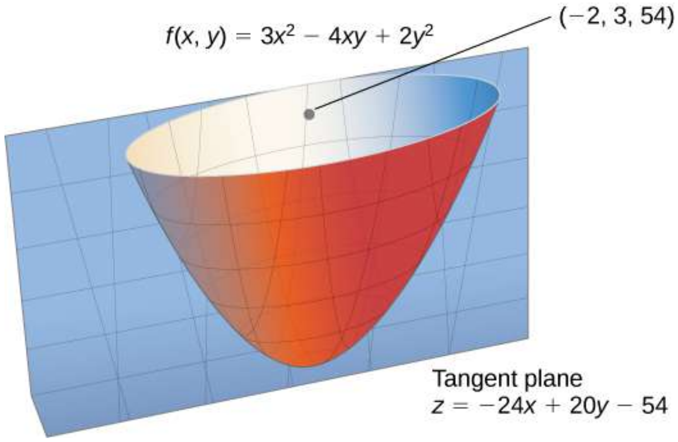


Figure 4.42 The maximum value of the directional derivative at $(-2, 3)$ is in the direction of the gradient.

4.30 Find the direction for which the directional derivative of $g(x, y) = 4x - xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Figure 4.43 shows a portion of the graph of the function $f(x, y) = 3 + \sin x \sin y$. Given a point (a, b) in the domain of f, the maximum value of the gradient at that point is given by $\|\nabla f(a, b)\|$. This would equal the rate of greatest ascent if the surface represented a topographical map. If we went in the opposite direction, it would be the rate of greatest descent.

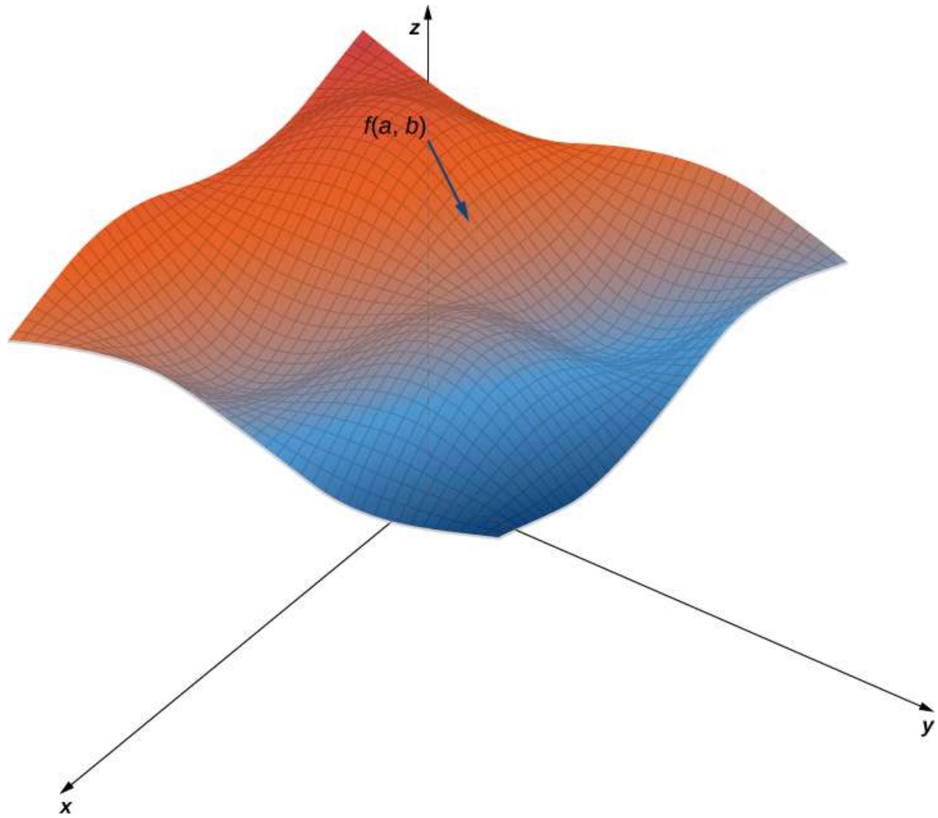


Figure 4.43 A typical surface in $\mathbb{R}^3$. Given a point on the surface, the directional derivative can be calculated using the gradient.

When using a topographical map, the steepest slope is always in the direction where the contour lines are closest together (see **Figure 4.44**). This is analogous to the contour map of a function, assuming the level curves are obtained for equally spaced values throughout the range of that function.

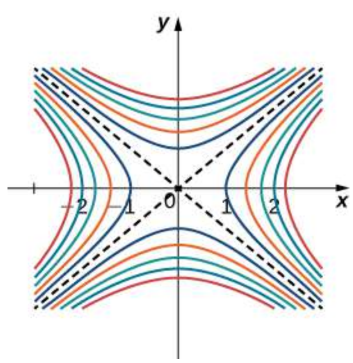


Figure 4.44 Contour map for the function $f(x, y) = x^2 - y^2$ using level values between -5 and 5.

Gradients and Level Curves

Recall that if a curve is defined parametrically by the function pair $(x(t), y(t))$, then the vector $x'(t)\mathbf{i} + y'(t)\mathbf{j}$ is tangent to the curve for every value of t in the domain. Now let's assume $z = f(x, y)$ is a differentiable function of x and y, and (x_0, y_0) is in its domain. Let's suppose further that $x_0 = x(t_0)$ and $y_0 = y(t_0)$ for some value of t, and consider the level curve $f(x, y) = k$. Define $g(t) = f(x(t), y(t))$ and calculate $g'(t)$ on the level curve. By the chain Rule,

$$g'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).$$

But $g'(t) = 0$ because $g(t) = k$ for all t. Therefore, on the one hand,

$$f_x(x(t),\ y(t))x'(t) + f_y(x(t),\ y(t))y'(t) = 0;$$

on the other hand,

$$f_x(x(t),\ y(t))x'(t) + f_y(x(t),\ y(t))y'(t) = \nabla f(x,\ y) \cdot \langle x'(t),\ y'(t) \rangle .$$

Therefore,

$$\nabla f(x,\ y) \cdot \langle x'(t),\ y'(t) \rangle = 0.$$

Thus, the dot product of these vectors is equal to zero, which implies they are orthogonal. However, the second vector is tangent to the level curve, which implies the gradient must be normal to the level curve, which gives rise to the following theorem.

Theorem 4.14: Gradient Is Normal to the Level Curve

Suppose the function $z = f(x,\ y)$ has continuous first-order partial derivatives in an open disk centered at a point $(x_0,\ y_0)$. If $\nabla f(x_0,\ y_0) \neq \mathbf{0}$, then $\nabla f(x_0,\ y_0)$ is normal to the level curve of f at $(x_0,\ y_0)$.

We can use this theorem to find tangent and normal vectors to level curves of a function.

Example 4.35

Finding Tangents to Level Curves

For the function $f(x,\ y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$, find a tangent vector to the level curve at point $(-2,\ 1)$. Graph the level curve corresponding to $f(x,\ y) = 18$ and draw in $\nabla f(-2,\ 1)$ and a tangent vector.

Solution

First, we must calculate $\nabla f(x,\ y)$:

$$f_x(x,\ y) = 4x - 3y + 2 \text{ and } f_y = -3x + 16y - 4 \text{ so } \nabla f(x,\ y) = (4x - 3y + 2)\mathbf{i} + (-3x + 16y - 4)\mathbf{j}.$$

Next, we evaluate $\nabla f(x,\ y)$ at $(-2,\ 1)$:

$$\nabla f(-2,\ 1) = (4(-2) - 3(1) + 2)\mathbf{i} + (-3(-2) + 16(1) - 4)\mathbf{j} = -9\mathbf{i} + 18\mathbf{j}.$$

This vector is orthogonal to the curve at point $(-2,\ 1)$. We can obtain a tangent vector by reversing the components and multiplying either one by -1. Thus, for example, $-18\mathbf{i} - 9\mathbf{j}$ is a tangent vector (see the following graph).

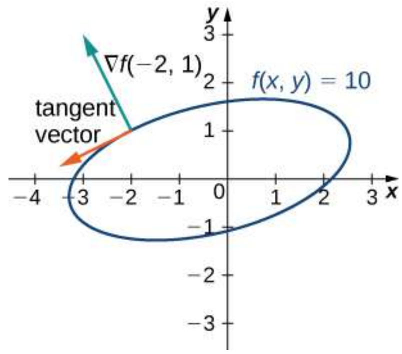


Figure 4.45 Tangent and normal vectors to $2x^2 - 3xy + 8y^2 + 2x - 4y + 4 = 18$ at point $(-2, 1)$.

4.31 For the function $f(x, y) = x^2 - 2xy + 5y^2 + 3x - 2y + 4,$ find the tangent to the level curve at point $(1, 1)$. Draw the graph of the level curve corresponding to $f(x, y) = 8$ and draw $\nabla f(1, 1)$ and a tangent vector.

Three-Dimensional Gradients and Directional Derivatives

The definition of a gradient can be extended to functions of more than two variables.

Definition

Let $w = f(x, y, z)$ be a function of three variables such that $f_x, f_y,$ and f_z exist. The vector $\nabla f(x, y, z)$ is called the gradient of f and is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}. \qquad (4.40)$$

$\nabla f(x, y, z)$ can also be written as $\text{grad} f(x, y, z)$.

Calculating the gradient of a function in three variables is very similar to calculating the gradient of a function in two variables. First, we calculate the partial derivatives $f_x, f_y,$ and $f_z,$ and then we use **Equation 4.40**.

Example 4.36

Finding Gradients in Three Dimensions

Find the gradient $\nabla f(x, y, z)$ of each of the following functions:

a. $f(x, y) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$

b. $f(x, y, z) = e^{-2z} \sin 2x \cos 2y$

Solution

For both parts a. and b., we first calculate the partial derivatives $f_x, f_y,$ and $f_z,$ then use **Equation 4.40**.

a.

$$\begin{aligned} f_z(x, y, z) &= 10x - 2y + 3z, \ f_y(x, y, z) = -2x + 2y - 4z \text{ and } f_z(x, y, z) = 3x - 4y + 2z, \text{ so} \\ \nabla f(x, y, z) &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} \\ &= (10x - 2y + 3z)\mathbf{i} + (-2x + 2y - 4z)\mathbf{j} + (-4x + 3y + 2z)\mathbf{k}. \end{aligned}$$

b.

$$\begin{aligned} f_x(x, y, z) &= -2e^{-2z}\cos 2x \cos 2y, \ f_y(x, y, z) = -2e^{-2z}\sin 2x \sin 2y \text{ and} \\ f_z(x, y, z) &= -2e^{-2z}\sin 2x \cos 2y, \text{ so} \\ \nabla f(x, y, z) &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} \\ &= \left(2e^{-2z}\cos 2x \cos 2y\right)\mathbf{i} + \left(-2e^{-2z}\right)\mathbf{j} + \left(-2e^{-2z}\right) \\ &= 2e^{-2z}(\cos 2x \cos 2y \, \mathrm{i} - \sin 2x \sin 2y \, \mathrm{j} - \sin 2x \cos 2y \, \mathrm{k}). \end{aligned}$$

4.32 Find the gradient $\nabla f(x, y, z)$ of $f(x, y, z) = \dfrac{x^2 - 3y^2 + z^2}{2x + y - 4z}$.

The directional derivative can also be generalized to functions of three variables. To determine a direction in three dimensions, a vector with three components is needed. This vector is a unit vector, and the components of the unit vector are called *directional cosines*. Given a three-dimensional unit vector $\mathbf{u}$ in standard form (i.e., the initial point is at the origin), this vector forms three different angles with the positive $x-$, $y-$, and z-axes. Let's call these angles $\alpha, \beta,$ and γ. Then the directional cosines are given by $\cos\alpha, \cos\beta,$ and $\cos\gamma$. These are the components of the unit vector $\mathbf{u}$; since $\mathbf{u}$ is a unit vector, it is true that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Definition

Suppose $w = f(x, y, z)$ is a function of three variables with a domain of D. Let $(x_0, y_0, z_0) \in D$ and let $\mathrm{u} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$ be a unit vector. Then, the directional derivative of f in the direction of u is given by

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{t \to 0} \frac{f\left(x_0 + t\cos\alpha, \ y_0 + t\cos\beta, \ z_0 + t\cos\gamma\right) - f(x_0, y_0, z_0)}{t}, \tag{4.41}$$

provided the limit exists.

We can calculate the directional derivative of a function of three variables by using the gradient, leading to a formula that is analogous to **Equation 4.38**.

Theorem 4.15: Directional Derivative of a Function of Three Variables

Let $f(x, y, z)$ be a differentiable function of three variables and let $\mathbf{u} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$ be a unit vector. Then, the directional derivative of f in the direction of $\mathbf{u}$ is given by

$$\begin{aligned} D_{\mathbf{u}} f(x, y, z) &= \nabla f(x, y, z) \cdot \mathbf{u} \\ &= f_x(x, y, z)\cos\alpha + f_y(x, y, z)\cos\beta + f_z(x, y, z)\cos\gamma. \end{aligned} \tag{4.42}$$

The three angles $\alpha, \beta,$ and γ determine the unit vector $\mathbf{u}$. In practice, we can use an arbitrary (nonunit) vector, then divide by its magnitude to obtain a unit vector in the desired direction.

Example 4.37

Finding a Directional Derivative in Three Dimensions

Calculate $D_{\mathbf{u}} f(1, -2, 3)$ in the direction of $\text{v} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ for the function

$$f(x, y, z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz.$$

Solution

First, we find the magnitude of $\mathbf{v}$:

$$\| \mathbf{v} \| = \sqrt{(-1)^2 + (2)^2} = 3.$$

Therefore, $\dfrac{\mathbf{v}}{\| \mathbf{v} \|} = \dfrac{-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{3} = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ is a unit vector in the direction of $\mathbf{v}$, so $\cos\alpha = -\frac{1}{3}, \cos\beta = \frac{2}{3},$ and $\cos\gamma = \frac{2}{3}$. Next, we calculate the partial derivatives of f:

$$\begin{aligned} f_x(x, y, z) &= 10x - 2y + 3z \\ f_y(x, y, z) &= -2x + 2y - 4z \\ f_z(x, y, z) &= -4y + 2z + 3x, \end{aligned}$$

then substitute them into **Equation 4.42**:

$$\begin{aligned} D_{\mathbf{u}} f(x, y, z) &= f_x(x, y, z)\cos\alpha + f_y(x, y, z)\cos\beta + f_z(x, y, z)\cos\gamma \\ &= (10x - 2y + 3z)\left(-\frac{1}{3}\right) + (-2x + 2y - 4z)\left(\frac{2}{3}\right) + (-4y + 2z + 3x)\left(\frac{2}{3}\right) \\ &= -\frac{10x}{3} + \frac{2y}{3} - \frac{3z}{3} - \frac{4x}{3} + \frac{4y}{3} - \frac{8z}{3} - \frac{8y}{3} + \frac{4z}{3} + \frac{6x}{3} \\ &= -\frac{8x}{3} - \frac{2y}{3} - \frac{7z}{3}. \end{aligned}$$

Last, to find $D_{\mathbf{u}} f(1, -2, 3)$, we substitute $x = 1, y = -2,$ and $z = 3$:

$$\begin{aligned} D_{\mathbf{u}} f(1, -2, 3) &= -\frac{8(1)}{3} - \frac{2(-2)}{3} - \frac{7(3)}{3} \\ &= -\frac{8}{3} + \frac{4}{3} - \frac{21}{3} \\ &= -\frac{25}{3}. \end{aligned}$$

4.33 Calculate $D_{\mathbf{u}} f(x, y, z)$ and $D_{\mathbf{u}} f(0, -2, 5)$ in the direction of $\text{v} = -3\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$ for the function $f(x, y, z) = 3x^2 + xy - 2y^2 + 4yz - z^2 + 2xz.$

4.6 EXERCISES

For the following exercises, find the directional derivative using the limit definition only.

260. $f(x, y) = 5 - 2x^2 - \frac{1}{2}y^2$ at point $P(3, 4)$ in the direction of $\mathrm{u} = \left(\cos\frac{\pi}{4}\right)\mathbf{i} + \left(\sin\frac{\pi}{4}\right)\mathbf{j}$

261. $f(x, y) = y^2\cos(2x)$ at point $P\left(\frac{\pi}{3}, 2\right)$ in the direction of $\mathrm{u} = \left(\cos\frac{\pi}{4}\right)\mathbf{i} + \left(\sin\frac{\pi}{4}\right)\mathbf{j}$

262. Find the directional derivative of $f(x, y) = y^2\sin(2x)$ at point $P\left(\frac{\pi}{4}, 2\right)$ in the direction of $\mathbf{u} = 5\mathbf{i} + 12\mathbf{j}$.

For the following exercises, find the directional derivative of the function at point P in the direction of $\mathbf{v}$.

263. $f(x, y) = xy$, $P(0, -2)$, $\mathrm{v} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$

264. $h(x, y) = e^x \sin y$, $P\left(1, \frac{\pi}{2}\right)$, $\mathbf{v} = -\mathbf{i}$

265. $h(x, y, z) = xyz$, $P(2, 1, 1)$, $\mathrm{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

266. $f(x, y) = xy$, $P(1, 1)$, $\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

267. $f(x, y) = x^2 - y^2$, $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$, $P(1, 0)$

268. $f(x, y) = 3x + 4y + 7$, $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, $P\left(0, \frac{\pi}{2}\right)$

269. $f(x, y) = e^x \cos y$, $\mathbf{u} = \langle 0, 1 \rangle$, $P = \left(0, \frac{\pi}{2}\right)$

270. $f(x, y) = y^{10}$, $\mathbf{u} = \langle 0, -1 \rangle$, $P = (1, -1)$

271. $f(x, y) = \ln(x^2 + y^2)$, $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, $P(1, 2)$

272. $f(x, y) = x^2 y$, $P(-5, 5)$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

273. $f(x, y) = y^2 + xz$, $P(1, 2, 2)$, $\mathbf{v} = \langle 2, -1, 2 \rangle$

For the following exercises, find the directional derivative of the function in the direction of the unit vector $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

274. $f(x, y) = x^2 + 2y^2$, $\theta = \frac{\pi}{6}$

275. $f(x, y) = \frac{y}{x + 2y}$, $\theta = -\frac{\pi}{4}$

276. $f(x, y) = \cos(3x + y)$, $\theta = \frac{\pi}{4}$

277. $w(x, y) = ye^x$, $\theta = \frac{\pi}{3}$

278. $f(x, y) = x\arctan(y)$, $\theta = \frac{\pi}{2}$

279. $f(x, y) = \ln(x + 2y)$, $\theta = \frac{\pi}{3}$

For the following exercises, find the gradient.

280. Find the gradient of $f(x, y) = \frac{14 - x^2 - y^2}{3}$. Then, find the gradient at point $P(1, 2)$.

281. Find the gradient of $f(x, y, z) = xy + yz + xz$ at point $P(1, 2, 3)$.

282. Find the gradient of $f(x, y, z)$ at P and in the direction of $\mathbf{u}$: $f(x, y, z) = \ln(x^2 + 2y^2 + 3z^2)$, $P(2, 1, 4)$, $\mathbf{u} = \frac{-3}{13}\mathbf{i} - \frac{4}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$.

283. $f(x, y, z) = 4x^5 y^2 z^3$, $P(2, -1, 1)$, $\mathbf{u} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

For the following exercises, find the directional derivative of the function at point P in the direction of Q.

284. $f(x, y) = x^2 + 3y^2$, $P(1, 1)$, $Q(4, 5)$

285. $f(x, y, z) = \frac{y}{x + z}$, $P(2, 1, -1)$, $Q(-1, 2, 0)$

For the following exercises, find the derivative of the function at P in the direction of $\mathbf{u}$.

286. $f(x, y) = -7x + 2y$, $P(2, -4)$, $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$

287. $f(x, y) = \ln(5x + 4y)$, $P(3, 9)$, $\mathbf{u} = 6\mathbf{i} + 8\mathbf{j}$

288. **[T]** Use technology to sketch the level curve of $f(x, y) = 4x - 2y + 3$ that passes through $P(1, 2)$ and draw the gradient vector at P.

289. **[T]** Use technology to sketch the level curve of $f(x, y) = x^2 + 4y^2$ that passes through $P(-2, 0)$ and draw the gradient vector at P.

For the following exercises, find the gradient vector at the indicated point.

290. $f(x, y) = xy^2 - yx^2, P(-1, 1)$

291. $f(x, y) = xe^y - \ln(x), P(-3, 0)$

292. $f(x, y, z) = xy - \ln(z), P(2, -2, 2)$

293. $f(x, y, z) = x\sqrt{y^2 + z^2}, P(-2, -1, -1)$

For the following exercises, find the derivative of the function.

294. $f(x, y) = x^2 + xy + y^2$ at point $(-5, -4)$ in the direction the function increases most rapidly

295. $f(x, y) = e^{xy}$ at point $(6, 7)$ in the direction the function increases most rapidly

296. $f(x, y) = \arctan\left(\frac{y}{x}\right)$ at point $(-9, 9)$ in the direction the function increases most rapidly

297. $f(x, y, z) = \ln(xy + yz + zx)$ at point $(-9, -18, -27)$ in the direction the function increases most rapidly

298. $f(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ at point $(5, -5, 5)$ in the direction the function increases most rapidly

For the following exercises, find the maximum rate of change of f at the given point and the direction in which it occurs.

299. $f(x, y) = xe^{-y}, \quad (1, 0)$

300. $f(x, y) = \sqrt{x^2 + 2y}, \quad (4, 10)$

301. $f(x, y) = \cos(3x + 2y), \left(\frac{\pi}{6}, -\frac{\pi}{8}\right)$

For the following exercises, find equations of

a. the tangent plane and
b. the normal line to the given surface at the given point.

302. The level curve $f(x, y, z) = 12$ for $f(x, y, z) = 4x^2 - 2y^2 + z^2$ at point $(2, 2, 2)$.

303. $f(x, y, z) = xy + yz + xz = 3$ at point $(1, 1, 1)$

304. $f(x, y, z) = xyz = 6$ at point $(1, 2, 3)$

305. $f(x, y, z) = xe^y \cos z - z = 1$ at point $(1, 0, 0)$

For the following exercises, solve the problem.

306. The temperature T in a metal sphere is inversely proportional to the distance from the center of the sphere (the origin: $(0, 0, 0)$). The temperature at point $(1, 2, 2)$ is $120°\text{C}$.

a. Find the rate of change of the temperature at point $(1, 2, 2)$ in the direction toward point $(2, 1, 3)$.
b. Show that, at any point in the sphere, the direction of greatest increase in temperature is given by a vector that points toward the origin.

307. The electrical potential (voltage) in a certain region of space is given by the function $V(x, y, z) = 5x^2 - 3xy + xyz.$

a. Find the rate of change of the voltage at point $(3, 4, 5)$ in the direction of the vector $\langle 1, 1, -1 \rangle$.
b. In which direction does the voltage change most rapidly at point $(3, 4, 5)$?
c. What is the maximum rate of change of the voltage at point $(3, 4, 5)$?

308. If the electric potential at a point (x, y) in the xy-plane is $V(x, y) = e^{-2x}\cos(2y)$, then the electric intensity vector at (x, y) is $\mathbf{E} = -\nabla V(x, y)$.

a. Find the electric intensity vector at $\left(\frac{\pi}{4}, 0\right)$.
b. Show that, at each point in the plane, the electric potential decreases most rapidly in the direction of the vector $\mathbf{E}$.

309. In two dimensions, the motion of an ideal fluid is governed by a velocity potential φ. The velocity components of the fluid u in the x-direction and v in the y-direction, are given by $\langle u, v \rangle = \nabla\varphi$. Find the velocity components associated with the velocity potential $\varphi(x, y) = \sin \pi x \sin 2\pi y$.

4.7 | Maxima/Minima Problems

Learning Objectives

4.7.1 Use partial derivatives to locate critical points for a function of two variables.

4.7.2 Apply a second derivative test to identify a critical point as a local maximum, local minimum, or saddle point for a function of two variables.

4.7.3 Examine critical points and boundary points to find absolute maximum and minimum values for a function of two variables.

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. This application is also important for functions of two or more variables, but as we have seen in earlier sections of this chapter, the introduction of more independent variables leads to more possible outcomes for the calculations. The main ideas of finding critical points and using derivative tests are still valid, but new wrinkles appear when assessing the results.

Critical Points

For functions of a single variable, we defined critical points as the values of the function when the derivative equals zero or does not exist. For functions of two or more variables, the concept is essentially the same, except for the fact that we are now working with partial derivatives.

Definition

Let $z = f(x, y)$ be a function of two variables that is defined on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a **critical point of a function of two variables** f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Example 4.38

Finding Critical Points

Find the critical points of each of the following functions:

a. $f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$

b. $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

Solution

a. First, we calculate $f_x(x, y)$ and $f_y(x, y)$:

$$\begin{aligned} f_x(x, y) &= \frac{1}{2}(-18x + 36)\left(4y^2 - 9x^2 + 24y + 36x + 36\right)^{-1/2} \\ &= \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} \\ f_y(x, y) &= \frac{1}{2}(8y + 24)\left(4y^2 - 9x^2 + 24y + 36x + 36\right)^{-1/2} \\ &= \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}}. \end{aligned}$$

Next, we set each of these expressions equal to zero:

$$\begin{aligned} \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} &= 0 \\ \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} &= 0. \end{aligned}$$

Then, multiply each equation by its common denominator:

$$\begin{aligned} -9x + 18 &= 0 \\ 4y + 12 &= 0. \end{aligned}$$

Therefore, $x = 2$ and $y = -3,$ so $(2, -3)$ is a critical point of f.

We must also check for the possibility that the denominator of each partial derivative can equal zero, thus causing the partial derivative not to exist. Since the denominator is the same in each partial derivative, we need only do this once:

$$4y^2 - 9x^2 + 24y + 36x + 36 = 0.$$

This equation represents a hyperbola. We should also note that the domain of f consists of points satisfying the inequality

$$4y^2 - 9x^2 + 24y + 36x + 36 \geq 0.$$

Therefore, any points on the hyperbola are not only critical points, they are also on the boundary of the domain. To put the hyperbola in standard form, we use the method of completing the square:

$$\begin{aligned} 4y^2 - 9x^2 + 24y + 36x + 36 &= 0 \\ 4y^2 - 9x^2 + 24y + 36x &= -36 \\ 4y^2 + 24y - 9x^2 + 36x &= -36 \\ 4\left(y^2 + 6y\right) - 9\left(x^2 - 4x\right) &= -36 \\ 4\left(y^2 + 6y + 9\right) - 9\left(x^2 - 4x + 4\right) &= -36 + 36 - 36 \\ 4(y + 3)^2 - 9(x - 2)^2 &= -36. \end{aligned}$$

Dividing both sides by -36 puts the equation in standard form:

$$\begin{aligned} \frac{4(y+3)^2}{-36} - \frac{9(x-2)^2}{-36} &= 1 \\ \frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} &= 1. \end{aligned}$$

Notice that point $(2, -3)$ is the center of the hyperbola.

b. First, we calculate $g_x(x, y)$ and $g_y(x, y)$:

$$\begin{aligned} g_x(x, y) &= 2x + 2y + 4 \\ g_y(x, y) &= 2x - 8y - 6. \end{aligned}$$

Next, we set each of these expressions equal to zero, which gives a system of equations in x and y:

$$\begin{aligned} 2x + 2y + 4 &= 0 \\ 2x - 8y - 6 &= 0. \end{aligned}$$

Subtracting the second equation from the first gives $10y + 10 = 0$, so $y = -1$. Substituting this into the first equation gives $2x + 2(-1) + 4 = 0$, so $x = -1$. Therefore $(-1, -1)$ is a critical point of g (**Figure 4.46**). There are no points in $\mathbb{R}^2$ that make either partial derivative not exist.

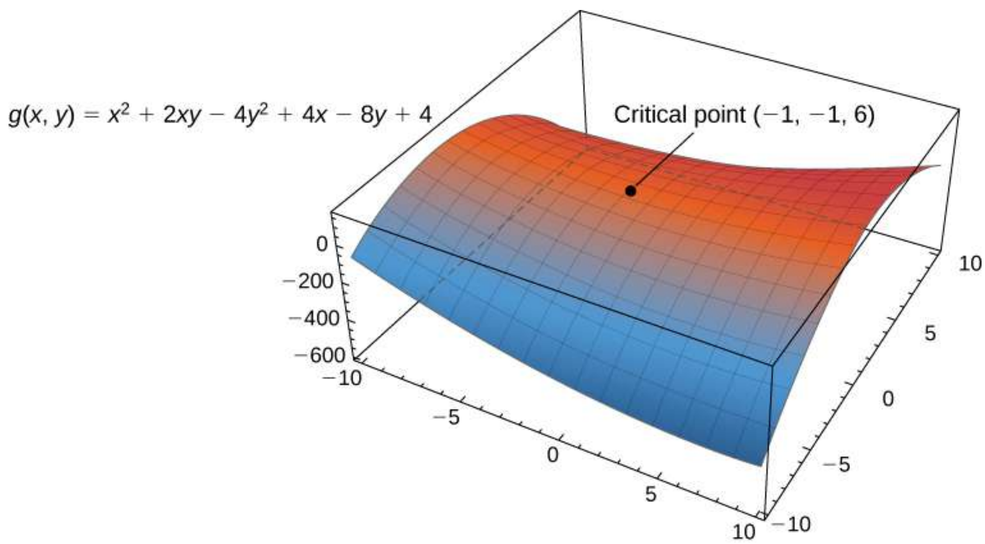


Figure 4.46 The function $g(x, y)$ has a critical point at $(-1, -1, 6)$.

4.34 Find the critical point of the function $f(x, y) = x^3 + 2xy - 2x - 4y$.

The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.

Definition

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Then f has a *local maximum* at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a *local maximum value*. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a *global maximum* (also called an *absolute maximum*) at (x_0, y_0).

The function f has a *local minimum* at (x_0, y_0) if

$$f(x_0, y_0) \leq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a *local minimum value*. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a *global minimum* (also called an *absolute minimum*) at (x_0, y_0).

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a *local extremum* (see the following figure).

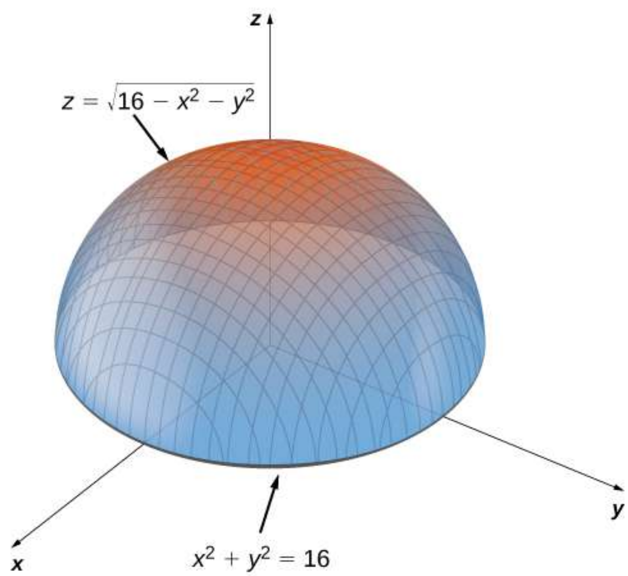


Figure 4.47 The graph of $z = \sqrt{16 - x^2 - y^2}$ has a maximum value when $(x, y) = (0, 0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

In **Maxima and Minima (http://cnx.org/content/m53611/latest/)** , we showed that extrema of functions of one variable occur at critical points. The same is true for functions of more than one variable, as stated in the following theorem.

Theorem 4.16: Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point

(x_0, y_0). Suppose f_x and f_y each exists at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

Second Derivative Test

Consider the function $f(x) = x^3$. This function has a critical point at $x = 0$, since $f'(0) = 3(0)^2 = 0$. However, f does not have an extreme value at $x = 0$. Therefore, the existence of a critical value at $x = x_0$ does not guarantee a local extremum at $x = x_0$. The same is true for a function of two or more variables. One way this can happen is at a **saddle point**. An example of a saddle point appears in the following figure.

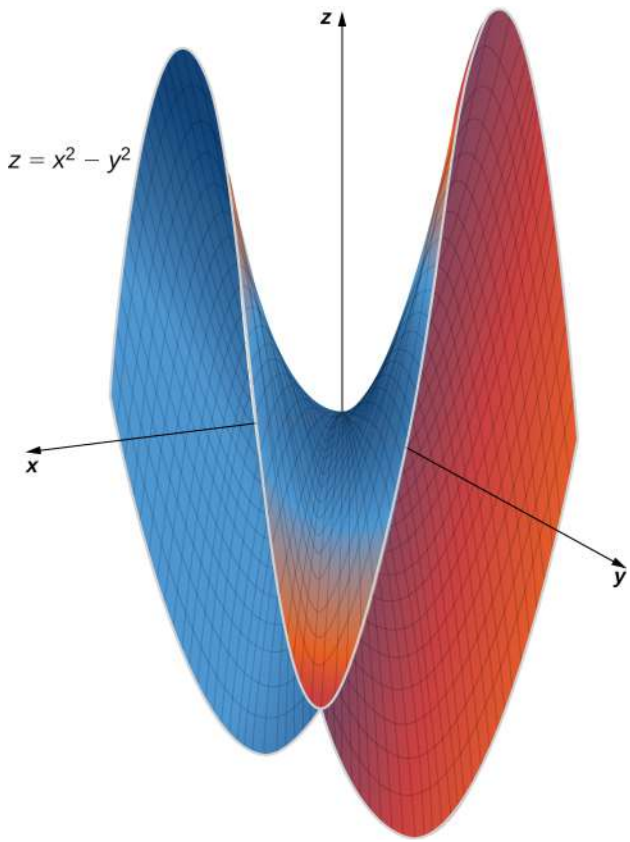


Figure 4.48 Graph of the function $z = x^2 - y^2$. This graph has a saddle point at the origin.

In this graph, the origin is a saddle point. This is because the first partial derivatives of $f(x, y) = x^2 - y^2$ are both equal to zero at this point, but it is neither a maximum nor a minimum for the function. Furthermore the vertical trace corresponding to $y = 0$ is $z = x^2$ (a parabola opening upward), but the vertical trace corresponding to $x = 0$ is $z = -y^2$ (a parabola opening downward). Therefore, it is both a global maximum for one trace and a global minimum for another.

Definition

Given the function $z = f(x, y)$, the point $\left(x_0, y_0, f(x_0, y_0)\right)$ is a saddle point if both $f_0(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

The second derivative test for a function of one variable provides a method for determining whether an extremum occurs at a critical point of a function. When extending this result to a function of two variables, an issue arises related to the fact that there are, in fact, four different second-order partial derivatives, although equality of mixed partials reduces this to three. The second derivative test for a function of two variables, stated in the following theorem, uses a **discriminant** D that replaces $f''(x_0)$ in the second derivative test for a function of one variable.

Theorem 4.17: Second Derivative Test

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2. \quad (4.43)$$

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).

ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).

iii. If $D < 0$, , then f has a saddle point at (x_0, y_0).

iv. If $D = 0$, then the test is inconclusive.

See **Figure 4.49**.

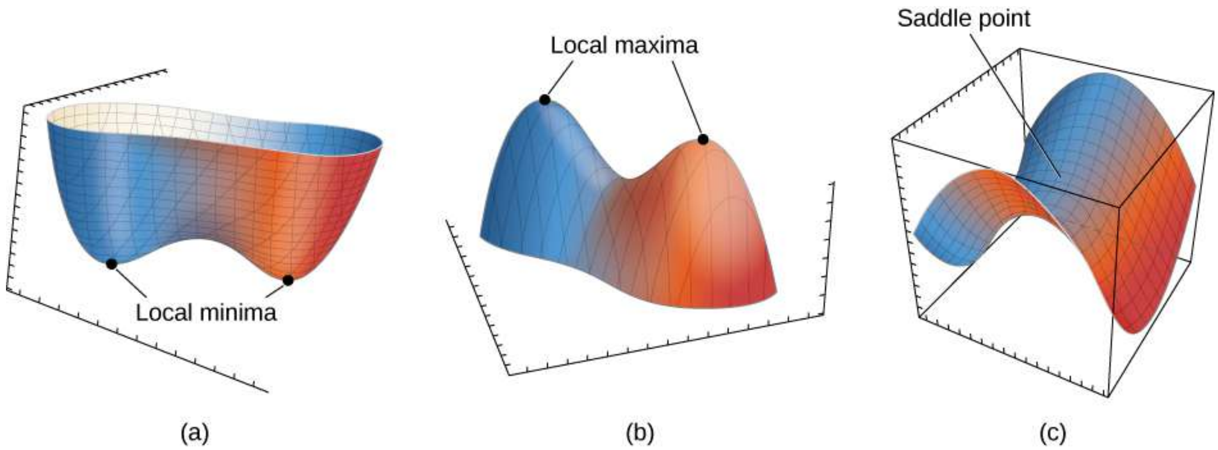


Figure 4.49 The second derivative test can often determine whether a function of two variables has a local minima (a), a local maxima (b), or a saddle point (c).

To apply the second derivative test, it is necessary that we first find the critical points of the function. There are several steps involved in the entire procedure, which are outlined in a problem-solving strategy.

Problem-Solving Strategy: Using the Second Derivative Test for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). To apply the second derivative test to find local extrema, use the following steps:

1. Determine the critical points (x_0, y_0) of the function f where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Discard any points where at least one of the partial derivatives does not exist.
2. Calculate the discriminant $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$ for each critical point of f.

3. Apply **Second Derivative Test** to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the theorem is inconclusive.

Example 4.39

Using the Second Derivative Test

Find the critical points for each of the following functions, and use the second derivative test to find the local extrema:

a. $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$

b. $g(x, y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

Solution

a. Step 1 of the problem-solving strategy involves finding the critical points of f. To do this, we first calculate $f_x(x, y)$ and $f_y(x, y)$, then set each of them equal to zero:

$$\begin{aligned} f_x(x, y) &= 8x + 8 \\ f_y(x, y) &= 18y - 36. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 8x + 8 &= 0 \\ 18y - 36 &= 0. \end{aligned}$$

The solution to this system is $x = -1$ and $y = 2$. Therefore $(-1, 2)$ is a critical point of f.

Step 2 of the problem-solving strategy involves calculating D. To do this, we first calculate the second partial derivatives of f:

$$\begin{aligned} f_{xx}(x, y) &= 8 \\ f_{xy}(x, y) &= 0 \\ f_{yy}(x, y) &= 18. \end{aligned}$$

Therefore, $D = f_{xx}(-1, 2)f_{yy}(-1, 2) - \left(f_{xy}(-1, 2)\right)^2 = (8)(18) - (0)^2 = 144.$

Step 3 states to check **Fermat's Theorem for Functions of Two Variables**. Since $D > 0$ and $f_{xx}(-1, 2) > 0$, this corresponds to case 1. Therefore, f has a local minimum at $(-1, 2)$ as shown in the following figure.

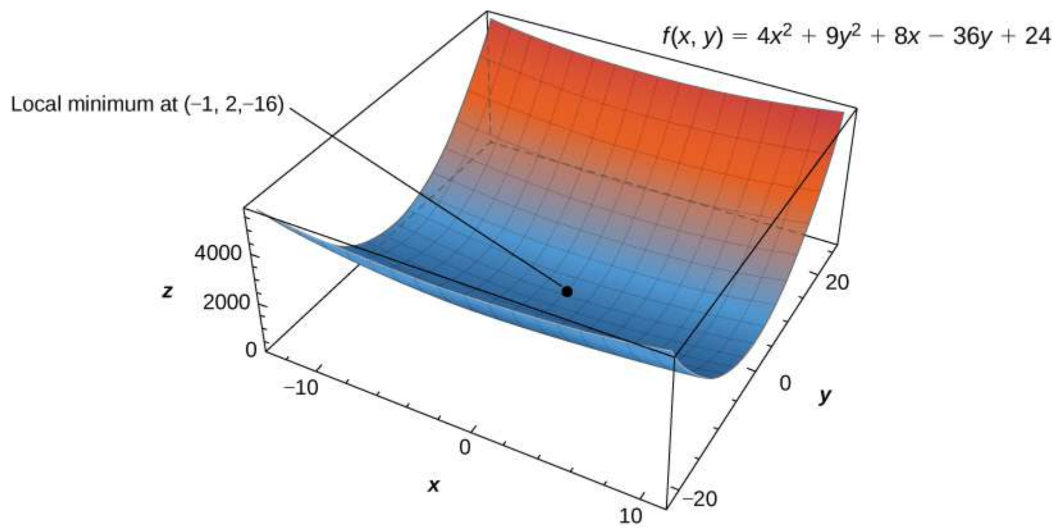


Figure 4.50 The function $f(x, y)$ has a local minimum at $(-1, 2, -16)$.

b. For step 1, we first calculate $g_x(x, y)$ and $g_y(x, y)$, then set each of them equal to zero:

$$\begin{aligned} g_x(x, y) &= x^2 + 2y - 6 \\ g_y(x, y) &= 2y + 2x - 3. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} x^2 + 2y - 6 &= 0 \\ 2y + 2x - 3 &= 0. \end{aligned}$$

To solve this system, first solve the second equation for *y*. This gives $y = \frac{3 - 2x}{2}$. Substituting this into the first equation gives

$$\begin{aligned} x^2 + 3 - 2x - 6 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x - 3)(x + 1) &= 0. \end{aligned}$$

Therefore, $x = -1$ or $x = 3$. Substituting these values into the equation $y = \frac{3 - 2x}{2}$ yields the critical points $\left(-1, \frac{5}{2}\right)$ and $\left(3, -\frac{3}{2}\right)$.

Step 2 involves calculating the second partial derivatives of g:

$$\begin{aligned} g_{xx}(x, y) &= 2x \\ g_{xy}(x, y) &= 2 \\ g_{yy}(x, y) &= 2. \end{aligned}$$

Then, we find a general formula for D:

$$\begin{aligned} D &= g_{xx}(x_0, y_0)g_{yy}(x_0, y_0) - \left(g_{xy}(x_0, y_0)\right)^2 \\ &= (2x_0)(2) - 2^2 \\ &= 4x_0 - 4. \end{aligned}$$

Next, we substitute each critical point into this formula:

$$D\left(-1, \frac{5}{2}\right) = (2(-1))(2) - (2)^2 = -4 - 4 = -8$$
$$D\left(3, -\frac{3}{2}\right) = (2(3))(2) - (2)^2 = 12 - 4 = 8.$$

In step 3, we note that, applying **Fermat's Theorem for Functions of Two Variables** to point $\left(-1, \frac{5}{2}\right)$ leads to case 3, which means that $\left(-1, \frac{5}{2}\right)$ is a saddle point. Applying the theorem to point $\left(3, -\frac{3}{2}\right)$ leads to case 1, which means that $\left(3, -\frac{3}{2}\right)$ corresponds to a local minimum as shown in the following figure.

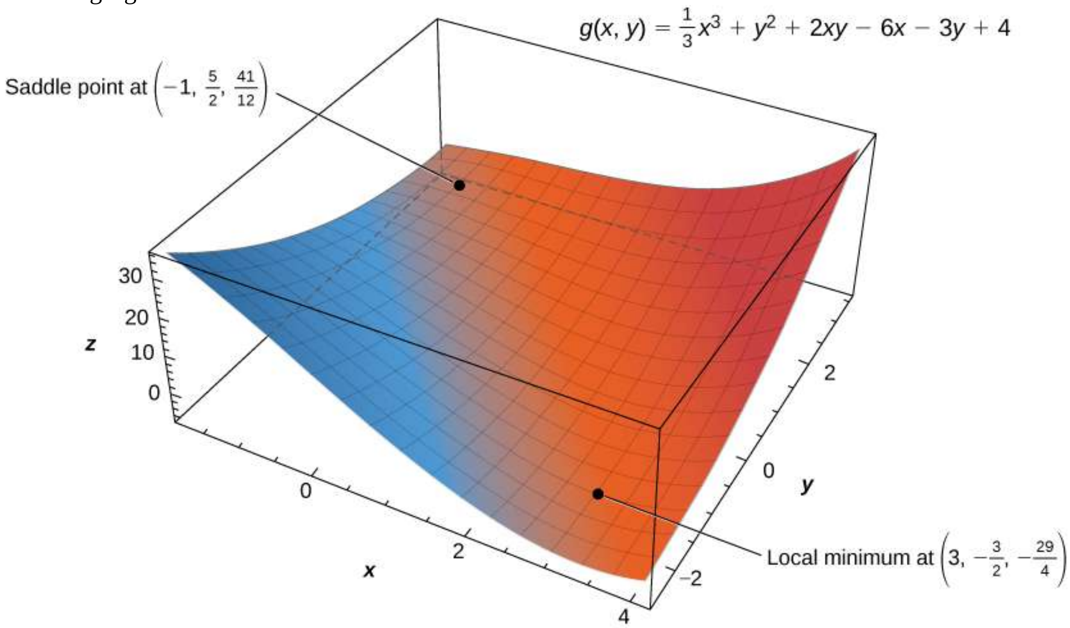


Figure 4.51 The function $g(x, y)$ has a local minimum and a saddle point.

4.35 Use the second derivative to find the local extrema of the function

$$f(x, y) = x^3 + 2xy - 6x - 4y^2.$$

Absolute Maxima and Minima

When finding global extrema of functions of one variable on a closed interval, we start by checking the critical values over that interval and then evaluate the function at the endpoints of the interval. When working with a function of two variables, the closed interval is replaced by a closed, bounded set. A set is *bounded* if all the points in that set can be contained within a ball (or disk) of finite radius. First, we need to find the critical points inside the set and calculate the corresponding critical values. Then, it is necessary to find the maximum and minimum value of the function on the boundary of the set. When we have all these values, the largest function value corresponds to the global maximum and the smallest function value corresponds to the absolute minimum. First, however, we need to be assured that such values exist. The following theorem does this.

Theorem 4.18: Extreme Value Theorem

A continuous function $f(x, y)$ on a closed and bounded set D in the plane attains an absolute maximum value at some point of D and an absolute minimum value at some point of D.

Now that we know any continuous function f defined on a closed, bounded set attains its extreme values, we need to know how to find them.

Theorem 4.19: Finding Extreme Values of a Function of Two Variables

Assume $z = f(x, y)$ is a differentiable function of two variables defined on a closed, bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

i. The values of f at the critical points of f in D.

ii. The values of f on the boundary of D.

The proof of this theorem is a direct consequence of the extreme value theorem and Fermat's theorem. In particular, if either extremum is not located on the boundary of D, then it is located at an interior point of D. But an interior point (x_0, y_0) of D that's an absolute extremum is also a local extremum; hence, (x_0, y_0) is a critical point of f by Fermat's theorem. Therefore the only possible values for the global extrema of f on D are the extreme values of f on the interior or boundary of D.

Problem-Solving Strategy: Finding Absolute Maximum and Minimum Values

Let $z = f(x, y)$ be a continuous function of two variables defined on a closed, bounded set D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps 2 and 3.

Finding the maximum and minimum values of f on the boundary of D can be challenging. If the boundary is a rectangle or set of straight lines, then it is possible to parameterize the line segments and determine the maxima on each of these segments, as seen in **Example 4.40**. The same approach can be used for other shapes such as circles and ellipses.

If the boundary of the set D is a more complicated curve defined by a function $g(x, y) = c$ for some constant c, and the first-order partial derivatives of g exist, then the method of Lagrange multipliers can prove useful for determining the extrema of f on the boundary. The method of Lagrange multipliers is introduced in **Lagrange Multipliers**.

Example 4.40

Finding Absolute Extrema

Use the problem-solving strategy for finding absolute extrema of a function to determine the absolute extrema of each of the following functions:

a. $f(x, y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$ on the domain defined by $0 \le x \le 4$ and $0 \le y \le 2$

b. $g(x, y) = x^2 + y^2 + 4x - 6y$ on the domain defined by $x^2 + y^2 \le 16$

Solution

a. Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x, y)$ and $f_y(x, y)$, then set them each equal to zero:

$$\begin{aligned} f_x(x, y) &= 2x - 2y - 4 \\ f_y(x, y) &= -2x + 8y - 2. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 2x - 2y - 4 &= 0 \\ -2x + 8y - 2 &= 0. \end{aligned}$$

The solution to this system is $x = 3$ and $y = 1$. Therefore $(3, 1)$ is a critical point of f. Calculating $f(3, 1)$ gives $f(3, 1) = 17$.

The next step involves finding the extrema of f on the boundary of its domain. The boundary of its domain consists of four line segments as shown in the following graph:

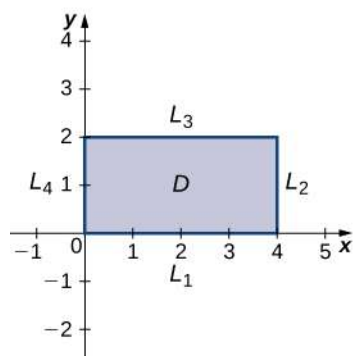


Figure 4.52 Graph of the domain of the function $f(x, y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$.

L_1 is the line segment connecting $(0, 0)$ and $(4, 0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 4$. Define $g(t) = f(x(t), y(t))$. This gives $g(t) = t^2 - 4t + 24$. Differentiating g leads to $g'(t) = 2t - 4$. Therefore, g has a critical value at $t = 2$, which corresponds to the point $(2, 0)$. Calculating $f(2, 0)$ gives the z-value 20.

L_2 is the line segment connecting $(4, 0)$ and $(4, 2)$, and it can be parameterized by the equations $x(t) = 4, y(t) = t$ for $0 \le t \le 2$. Again, define $g(t) = f(x(t), y(t))$. This gives $g(t) = 4t^2 - 10t + 24$. Then, $g'(t) = 8t - 10$. g has a critical value at $t = \frac{5}{4}$, which corresponds to the point $\left(0, \frac{5}{4}\right)$.

Calculating $f\left(0, \frac{5}{4}\right)$ gives the z-value 27.75.

L_3 is the line segment connecting $(0, 2)$ and $(4, 2),$ and it can be parameterized by the equations $x(t) = t, \ y(t) = 2$ for $0 \leq t \leq 4.$ Again, define $g(t) = f(x(t), \ y(t)).$ This gives $g(t) = t^2 - 8t + 36.$ The critical value corresponds to the point $(4, 2).$ So, calculating $f(4, \ 2)$ gives the z-value $20.$

L_4 is the line segment connecting $(0, 0)$ and $(0, 2),$ and it can be parameterized by the equations $x(t) = 0, \ y(t) = t$ for $0 \leq t \leq 2.$ This time, $g(t) = 4t^2 - 2t + 24$ and the critical value $t = \frac{1}{4}$ correspond to the point $\left(0, \frac{1}{4}\right).$ Calculating $f\left(0, \frac{1}{4}\right)$ gives the z-value $23.75.$

We also need to find the values of $f(x, \ y)$ at the corners of its domain. These corners are located at $(0, 0), \ (4, 0), \ (4, 2)$ and $(0, 2)$:

$$\begin{aligned}
f(0, 0) &= (0)^2 - 2(0)(0) + 4(0)^2 - 4(0) - 2(0) + 24 &= 24 \\
f(4, 0) &= (4)^2 - 2(4)(0) + 4(0)^2 - 4(4) - 2(0) + 24 &= 24 \\
f(4, 2) &= (4)^2 - 2(4)(2) + 4(2)^2 - 4(4) - 2(2) + 24 &= 20 \\
f(0, 2) &= (0)^2 - 2(0)(2) + 4(2)^2 - 4(0) - 2(2) + 24 &= 36.
\end{aligned}$$

The absolute maximum value is $36,$ which occurs at $(0, 2),$ and the global minimum value is $20,$ which occurs at both $(4, 2)$ and $(2, 0)$ as shown in the following figure.

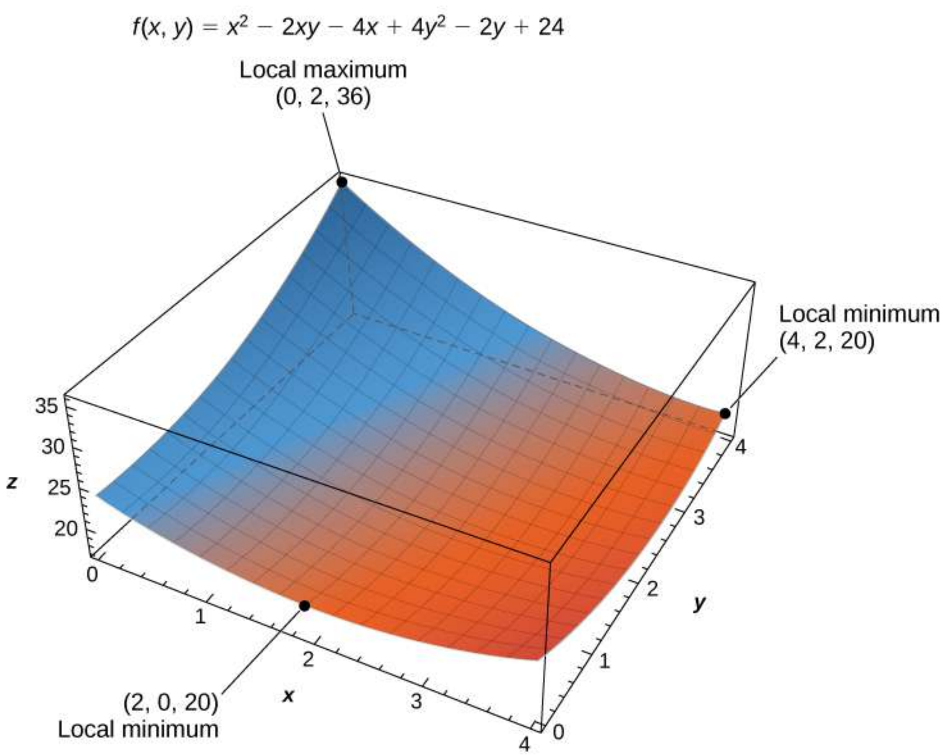


Figure 4.53 The function $f(x, \ y)$ has two global minima and one global maximum over its domain.

b. Using the problem-solving strategy, step 1 involves finding the critical points of g on its domain. Therefore, we first calculate $g_x(x, y)$ and $g_y(x, y)$, then set them each equal to zero:

$$\begin{aligned} g_x(x, y) &= 2x + 4 \\ g_y(x, y) &= 2y - 6. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 2x + 4 &= 0 \\ 2y - 6 &= 0. \end{aligned}$$

The solution to this system is $x = -2$ and $y = 3$. Therefore, $(-2, 3)$ is a critical point of g. Calculating $g(-2, 3)$, we get

$$g(-2, 3) = (-2)^2 + 3^2 + 4(-2) - 6(3) = 4 + 9 - 8 - 18 = -13.$$

The next step involves finding the extrema of g on the boundary of its domain. The boundary of its domain consists of a circle of radius 4 centered at the origin as shown in the following graph.

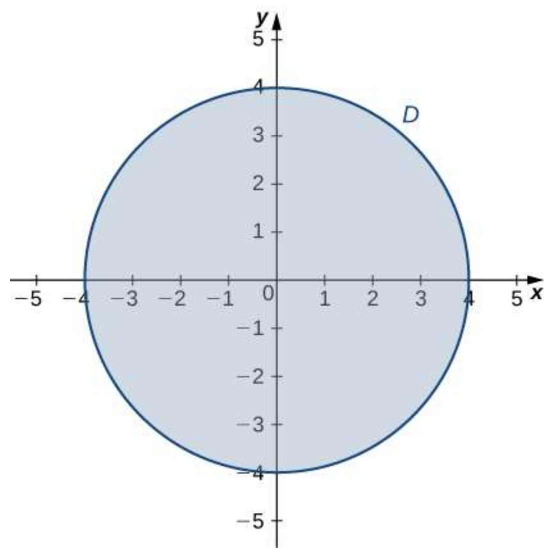


Figure 4.54 Graph of the domain of the function $g(x, y) = x^2 + y^2 + 4x - 6y$.

The boundary of the domain of g can be parameterized using the functions $x(t) = 4\cos t,\ y(t) = 4\sin t$ for $0 \le t \le 2\pi$. Define $h(t) = g(x(t), y(t))$:

$$\begin{aligned} h(t) &= g(x(t), y(t)) \\ &= (4\cos t)^2 + (4\sin t)^2 + 4(4\cos t) - 6(4\sin t) \\ &= 16\cos^2 t + 16\sin^2 t + 16\cos t - 24\sin t \\ &= 16 + 16\cos t - 24\sin t. \end{aligned}$$

Setting $h'(t) = 0$ leads to

$$\begin{aligned} -16 \sin t - 24 \cos t &= 0 \\ -16 \sin t &= 24 \cos t \\ \frac{-16 \sin t}{-16 \cos t} &= \frac{24 \cos t}{-16 \cos t} \\ \tan t &= -\frac{3}{2}. \end{aligned}$$

This equation has two solutions over the interval $0 \le t \le 2\pi$. One is $t = \pi - \arctan\left(\frac{3}{2}\right)$ and the other is $t = 2\pi - \arctan\left(\frac{3}{2}\right)$. For the first angle,

$$\begin{aligned} \sin t &= \sin\left(\pi - \arctan\left(\frac{3}{2}\right)\right) = \sin\left(\arctan\left(\frac{3}{2}\right)\right) = \frac{3\sqrt{13}}{13} \\ \cos t &= \cos\left(\pi - \arctan\left(\frac{3}{2}\right)\right) = -\cos\left(\arctan\left(\frac{3}{2}\right)\right) = -\frac{2\sqrt{13}}{13}. \end{aligned}$$

Therefore, $x(t) = 4 \cos t = -\frac{8\sqrt{13}}{13}$ and $y(t) = 4 \sin t = \frac{12\sqrt{13}}{13}$, so $\left(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13}\right) &= \left(-\frac{8\sqrt{13}}{13}\right)^2 + \left(\frac{12\sqrt{13}}{13}\right)^2 + 4\left(-\frac{8\sqrt{13}}{13}\right) - 6\left(\frac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13} + \frac{64}{13} - \frac{32\sqrt{13}}{13} - \frac{72\sqrt{13}}{13} \\ &= \frac{208 - 104\sqrt{13}}{13} \approx -12.844. \end{aligned}$$

For the second angle,

$$\begin{aligned} \sin t &= \sin\left(2\pi - \arctan\left(\frac{3}{2}\right)\right) = -\sin\left(\arctan\left(\frac{3}{2}\right)\right) = -\frac{3\sqrt{13}}{13} \\ \cos t &= \cos\left(2\pi - \arctan\left(\frac{3}{2}\right)\right) = \cos\left(\arctan\left(\frac{3}{2}\right)\right) = \frac{2\sqrt{13}}{13}. \end{aligned}$$

Therefore, $x(t) = 4 \cos t = \frac{8\sqrt{13}}{13}$ and $y(t) = 4 \sin t = -\frac{12\sqrt{13}}{13}$, so $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right) &= \left(\frac{8\sqrt{13}}{13}\right)^2 + \left(-\frac{12\sqrt{13}}{13}\right)^2 + 4\left(\frac{8\sqrt{13}}{13}\right) - 6\left(-\frac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13} + \frac{64}{13} + \frac{32\sqrt{13}}{13} + \frac{72\sqrt{13}}{13} \\ &= \frac{208 + 104\sqrt{13}}{13} \approx 44.844. \end{aligned}$$

The absolute minimum of g is -13, which is attained at the point $(-2, 3)$, which is an interior point of D. The absolute maximum of g is approximately equal to 44.844, which is attained at the boundary point $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$. These are the absolute extrema of g on D as shown in the following figure.

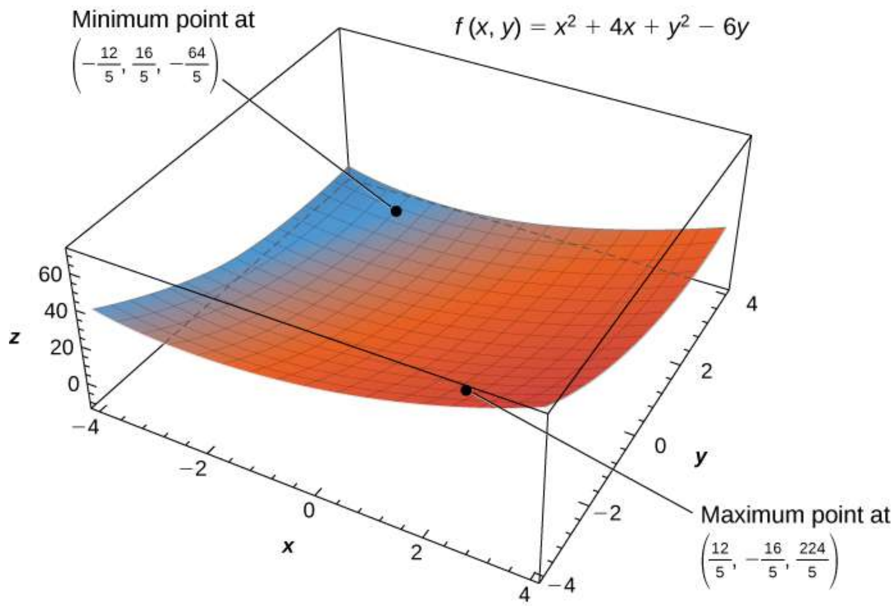


Figure 4.55 The function $f(x, y)$ has a local minimum and a local maximum.

4.36 Use the problem-solving strategy for finding absolute extrema of a function to find the absolute extrema of the function

$$f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

on the domain defined by $0 \le x \le 2$ and $-1 \le y \le 3$.

Example 4.41

Chapter Opener: Profitable Golf Balls

Figure 4.56 (credit: modification of work by oatsy40, Flickr)

Pro- T company has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising *y*, according to the function

$$z = f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The maximum number of golf balls that can be produced and sold is $50{,}000,$ and the maximum number of hours of advertising that can be purchased is $25.$ Find the values of x and y that maximize profit, and find the maximum profit.

Solution

Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x, y)$ and $f_y(x, y),$ then set them each equal to zero:

$$\begin{aligned} f_x(x, y) &= 48 - 2x - 2y \\ f_y(x, y) &= 96 - 2x - 18y. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 48 - 2x - 2y &= 0 \\ 96 - 2x - 18y &= 0. \end{aligned}$$

The solution to this system is $x = 21$ and $y = 3.$ Therefore $(21, 3)$ is a critical point of $f.$ Calculating $f(21, 3)$ gives $f(21, 3) = 48(21) + 96(3) - 21^2 - 2(21)(3) - 9(3)^2 = 648.$

The domain of this function is $0 \le x \le 50$ and $0 \le y \le 25$ as shown in the following graph.

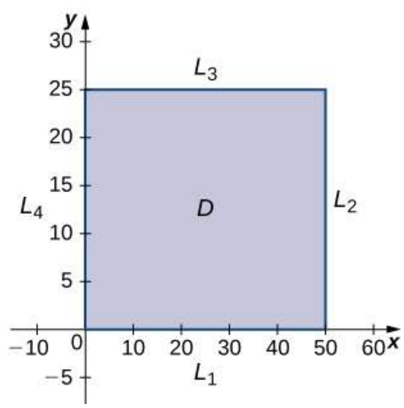


Figure 4.57 Graph of the domain of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2.$

L_1 is the line segment connecting $(0, 0)$ and $(50, 0),$ and it can be parameterized by the equations $x(t) = t,\ y(t) = 0$ for $0 \le t \le 50.$ We then define $g(t) = f(x(t), y(t))$:

$$\begin{aligned} g(t) &= f(x(t), y(t)) \\ &= f(t, 0) \\ &= 48t + 96(0) - y^2 - 2(t)(0) - 9(0)^2 \\ &= 48t - t^2. \end{aligned}$$

Setting $g'(t) = 0$ yields the critical point $t = 24,$ which corresponds to the point $(24, 0)$ in the domain of $f.$

Calculating $f(24, 0)$ gives $576.$

L_2 is the line segment connecting and $(50, 25),$ and it can be parameterized by the equations $x(t) = 50,\ y(t) = t$ for $0 \le t \le 25.$ Once again, we define $g(t) = f(x(t),\ y(t))$:

$$\begin{aligned} g(t) &= f(x(t),\ y(t)) \\ &= f(50,\ t) \\ &= 48(50) + 96t - 50^2 - 2(50)t - 9t^2 \\ &= -9t^2 - 4t - 100. \end{aligned}$$

This function has a critical point at $t = -\frac{2}{9},$ which corresponds to the point $\left(50,\ -\frac{2}{9}\right).$ This point is not in the domain of $f.$

L_3 is the line segment connecting $(0, 25)$ and $(50, 25),$ and it can be parameterized by the equations $x(t) = t,\ y(t) = 25$ for $0 \le t \le 50.$ We define $g(t) = f(x(t),\ y(t))$:

$$\begin{aligned} g(t) &= f(x(t),\ y(t)) \\ &= f(t,\ 25) \\ &= 48t + 96(25) - t^2 - 2t(25) - 9\left(25^2\right) \\ &= -t^2 - 2t - 3225. \end{aligned}$$

This function has a critical point at $t = -1,$ which corresponds to the point $(-1, 25),$ which is not in the domain.

L_4 is the line segment connecting $(0, 0)$ to $(0, 25),$ and it can be parameterized by the equations $x(t) = 0,\ y(t) = t$ for $0 \le t \le 25.$ We define $g(t) = f(x(t),\ y(t))$:

$$\begin{aligned} g(t) &= f(x(t),\ y(t)) \\ &= f(0,\ t) \\ &= 48(0) + 96t - (0)^2 - 2(0)t - 9t^2 \\ &= 96t - t^2. \end{aligned}$$

This function has a critical point at $t = \frac{16}{3},$ which corresponds to the point $\left(0,\ \frac{16}{3}\right),$ which is on the boundary of the domain. Calculating $f\left(0,\ \frac{16}{3}\right)$ gives $256.$

We also need to find the values of $f(x, y)$ at the corners of its domain. These corners are located at $(0, 0), (50, 0), (50, 25)$ and $(0, 25)$:

$$\begin{aligned} f(0, 0) &= 48(0) + 96(0) - (0)^2 - 2(0)(0) - 9(0)^2 = 0 \\ f(50, 0) &= 48(50) + 96(0) - (50)^2 - 2(50)(0) - 9(0)^2 = -100 \\ f(50, 25) &= 48(50) + 96(25) - (50)^2 - 2(50)(25) - 9(25)^2 = -5825 \\ f(0, 25) &= 48(0) + 96(25) - (0)^2 - 2(0)(25) - 9(25)^2 = -3225. \end{aligned}$$

The maximum critical value is $648,$ which occurs at $(21, 3).$ Therefore, a maximum profit of $\$648{,}000$ is realized when $21{,}000$ golf balls are sold and 3 hours of advertising are purchased per month as shown in the following figure.

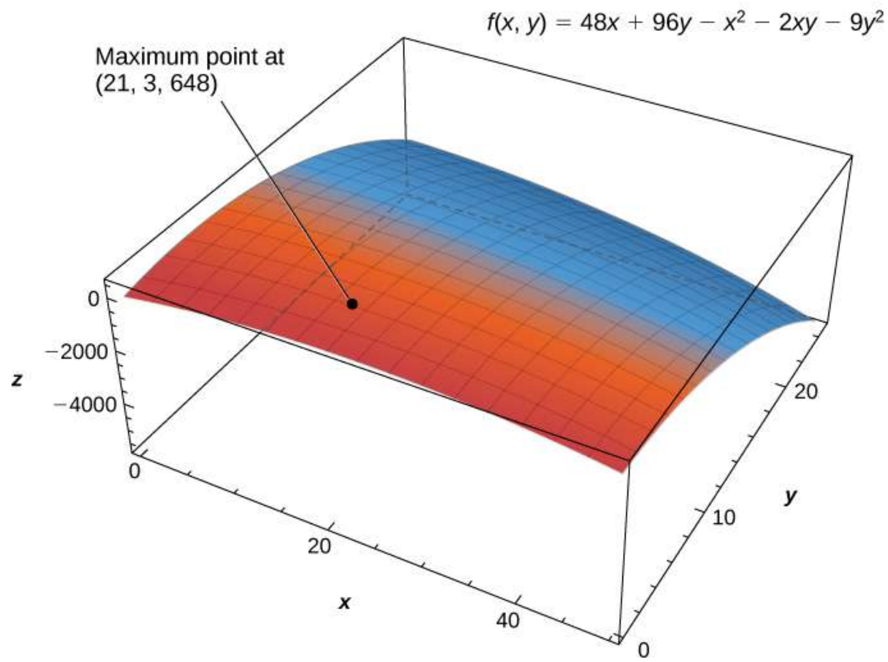


Figure 4.58 The profit function $f(x, y)$ has a maximum at $(21, 3, 648)$.

4.7 EXERCISES

For the following exercises, find all critical points.

310. $f(x, y) = 1 + x^2 + y^2$

311. $f(x, y) = (3x - 2)^2 + (y - 4)^2$

312. $f(x, y) = x^4 + y^4 - 16xy$

313. $f(x, y) = 15x^3 - 3xy + 15y^3$

For the following exercises, find the critical points of the function by using algebraic techniques (completing the square) or by examining the form of the equation. Verify your results using the partial derivatives test.

314. $f(x, y) = \sqrt{x^2 + y^2 + 1}$

315. $f(x, y) = -x^2 - 5y^2 + 8x - 10y - 13$

316. $f(x, y) = x^2 + y^2 + 2x - 6y + 6$

317. $f(x, y) = \sqrt{x^2 + y^2} + 1$

For the following exercises, use the second derivative test to identify any critical points and determine whether each critical point is a maximum, minimum, saddle point, or none of these.

318. $f(x, y) = -x^3 + 4xy - 2y^2 + 1$

319. $f(x, y) = x^2 y^2$

320. $f(x, y) = x^2 - 6x + y^2 + 4y - 8$

321. $f(x, y) = 2xy + 3x + 4y$

322. $f(x, y) = 8xy(x + y) + 7$

323. $f(x, y) = x^2 + 4xy + y^2$

324. $f(x, y) = x^3 + y^3 - 300x - 75y - 3$

325. $f(x, y) = 9 - x^4 y^4$

326. $f(x, y) = 7x^2 y + 9xy^2$

327. $f(x, y) = 3x^2 - 2xy + y^2 - 8y$

328. $f(x, y) = 3x^2 + 2xy + y^2$

329. $f(x, y) = y^2 + xy + 3y + 2x + 3$

330. $f(x, y) = x^2 + xy + y^2 - 3x$

331. $f(x, y) = x^2 + 2y^2 - x^2 y$

332. $f(x, y) = x^2 + y - e^y$

333. $f(x, y) = e^{-(x^2 + y^2 + 2x)}$

334. $f(x, y) = x^2 + xy + y^2 - x - y + 1$

335. $f(x, y) = x^2 + 10xy + y^2$

336. $f(x, y) = -x^2 - 5y^2 + 10x - 30y - 62$

337. $f(x, y) = 120x + 120y - xy - x^2 - y^2$

338. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

339. $f(x, y) = x^2 + x - 3xy + y^3 - 5$

340. $f(x, y) = 2xye^{-x^2 - y^2}$

For the following exercises, determine the extreme values and the saddle points. Use a CAS to graph the function.

341. **[T]** $f(x, y) = ye^x - e^y$

342. **[T]** $f(x, y) = x \sin(y)$

343. **[T]** $f(x, y) = \sin(x)\sin(y),\ x \in (0, 2\pi),\ y \in (0, 2\pi)$

Find the absolute extrema of the given function on the indicated closed and bounded set R.

344. $f(x, y) = xy - x - 3y$; R is the triangular region with vertices $(0, 0), (0, 4),$ and $(5, 0)$.

345. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - 2y + 1$ on the region $R = \{(x, y) | x^2 + y^2 \le 4\}$.

346. $f(x, y) = x^3 - 3xy - y^3$ on $R = \{(x, y): -2 \le x \le 2, -2 \le y \le 2\}$

347. $f(x,\ y) = \dfrac{-2y}{x^2 + y^2 + 1}$ on $R = \{(x,\ y): x^2 + y^2 \le 4\}$

348. Find three positive numbers the sum of which is $27,$ such that the sum of their squares is as small as possible.

349. Find the points on the surface $x^2 - yz = 5$ that are closest to the origin.

350. Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex in the first octant on the plane $x + y + z = 1.$

351. The sum of the length and the girth (perimeter of a cross-section) of a package carried by a delivery service cannot exceed 108 in. Find the dimensions of the rectangular package of largest volume that can be sent.

352. A cardboard box without a lid is to be made with a volume of $4\ \text{ft}^3$. Find the dimensions of the box that requires the least amount of cardboard.

353. Find the point on the surface $f(x,\ y) = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0.$ Identify the point on the plane.

354. Find the point in the plane $2x - y + 2z = 16$ that is closest to the origin.

355. A company manufactures two types of athletic shoes: jogging shoes and cross-trainers. The total revenue from x units of jogging shoes and y units of cross-trainers is given by $R(x,\ y) = -5x^2 - 8y^2 - 2xy + 42x + 102y,$ where x and y are in thousands of units. Find the values of *x* and *y* to maximize the total revenue.

356. A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 in. Find the dimensions of the box that meets this condition and has the largest volume.

357. Find the maximum volume of a cylindrical soda can such that the sum of its height and circumference is 120 cm.

4.8 | Lagrange Multipliers

Learning Objectives

4.8.1 Use the method of Lagrange multipliers to solve optimization problems with one constraint.

4.8.2 Use the method of Lagrange multipliers to solve optimization problems with two constraints.

Solving optimization problems for functions of two or more variables can be similar to solving such problems in single-variable calculus. However, techniques for dealing with multiple variables allow us to solve more varied optimization problems for which we need to deal with additional conditions or constraints. In this section, we examine one of the more common and useful methods for solving optimization problems with constraints.

Lagrange Multipliers

Example 4.41 was an applied situation involving maximizing a profit function, subject to certain **constraints**. In that example, the constraints involved a maximum number of golf balls that could be produced and sold in 1 month (x), and a maximum number of advertising hours that could be purchased per month (y). Suppose these were combined into a budgetary constraint, such as $20x + 4y \le 216$, that took into account the cost of producing the golf balls and the number of advertising hours purchased per month. The goal is, still, be to maximize profit, but now there is a different type of constraint on the values of x and y. This constraint, when combined with the profit function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$, is an example of an **optimization problem**, and the function $f(x, y)$ is called the **objective function**. A graph of various level curves of the function $f(x, y)$ follows.

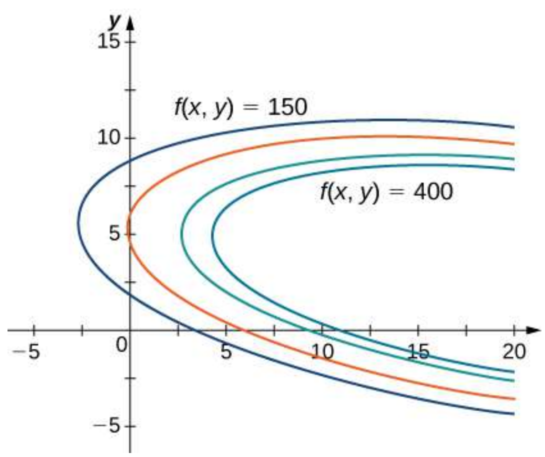


Figure 4.59 Graph of level curves of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and $400.$

In **Figure 4.59**, the value c represents different profit levels (i.e., values of the function f). As the value of c increases, the curve shifts to the right. Since our goal is to maximize profit, we want to choose a curve as far to the right as possible. If there was no restriction on the number of golf balls the company could produce, or the number of units of advertising available, then we could produce as many golf balls as we want, and advertise as much as we want, and there would be not be a maximum profit for the company. Unfortunately, we have a budgetary constraint that is modeled by the inequality $20x + 4y \le 216.$ To see how this constraint interacts with the profit function, **Figure 4.60** shows the graph of the line $20x + 4y = 216$ superimposed on the previous graph.

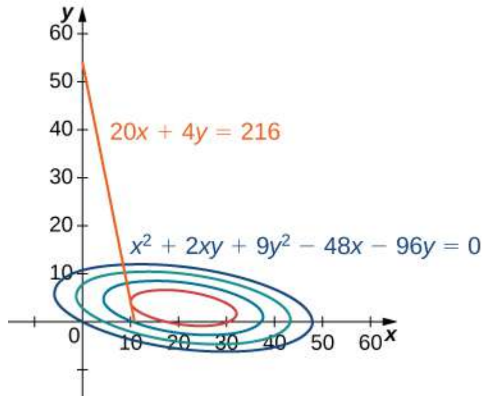


Figure 4.60 Graph of level curves of the function $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and $395.$ The red graph is the constraint function.

As mentioned previously, the maximum profit occurs when the level curve is as far to the right as possible. However, the level of production corresponding to this maximum profit must also satisfy the budgetary constraint, so the point at which this profit occurs must also lie on (or to the left of) the red line in **Figure 4.60**. Inspection of this graph reveals that this point exists where the line is tangent to the level curve of f. Trial and error reveals that this profit level seems to be around $395,$ when x and y are both just less than $5.$ We return to the solution of this problem later in this section. From a theoretical standpoint, at the point where the profit curve is tangent to the constraint line, the gradient of both of the functions evaluated at that point must point in the same (or opposite) direction. Recall that the gradient of a function of more than one variable is a vector. If two vectors point in the same (or opposite) directions, then one must be a constant multiple of the other. This idea is the basis of the **method of Lagrange multipliers**.

Theorem 4.20: Method of Lagrange Multipliers: One Constraint

Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x, y) = 0.$ Suppose that $f,$ when restricted to points on the curve $g(x, y) = 0,$ has a local extremum at the point (x_0, y_0) and that $\nabla g(x_0, y_0) \neq 0.$ Then there is a number λ called a **Lagrange multiplier**, for which

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Proof

Assume that a constrained extremum occurs at the point $(x_0, y_0).$ Furthermore, we assume that the equation $g(x, y) = 0$ can be smoothly parameterized as

$$x = x(s) \text{ and } y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0.$ Therefore, the quantity $z = f(x(s), y(s))$ has a relative maximum or relative minimum at $s = 0,$ and this implies that $\frac{dz}{ds} = 0$ at that point. From the chain rule,

$$\frac{dz}{ds} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \mathbf{j}\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}\right) = 0,$$

where the derivatives are all evaluated at $s = 0.$ However, the first factor in the dot product is the gradient of $f,$ and the second factor is the unit tangent vector $\mathrm{T}(0)$ to the constraint curve. Since the point (x_0, y_0) corresponds to $s = 0,$ it follows from this equation that

$$\nabla f(x_0, y_0) \cdot \mathrm{T}(0) = 0,$$

which implies that the gradient is either $\mathbf{0}$ or is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x, y) = 0$ is a level curve for the function $g(x, y)$ so that if $\nabla g(x_0, y_0) \neq 0$ then $\nabla g(x_0, y_0)$ is normal to this curve at (x_0, y_0) It follows, then, that there is some scalar λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

□

To apply **Method of Lagrange Multipliers: One Constraint** to an optimization problem similar to that for the golf ball manufacturer, we need a problem-solving strategy.

Problem-Solving Strategy: Steps for Using Lagrange Multipliers

1. Determine the objective function $f(x, y)$ and the constraint function $g(x, y)$. Does the optimization problem involve maximizing or minimizing the objective function?
2. Set up a system of equations using the following template:

$$\begin{aligned} \nabla f(x_0, y_0) &= \lambda \nabla g(x_0, y_0) \\ g(x_0, y_0) &= 0. \end{aligned}$$

3. Solve for x_0 and y_0.
4. The largest of the values of f at the solutions found in step 3 maximizes f; the smallest of those values minimizes f.

Example 4.42

Using Lagrange Multipliers

Use the method of Lagrange multipliers to find the minimum value of $f(x, y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7$.

Solution

Let’s follow the problem-solving strategy:

1. The optimization function is $f(x, y) = x^2 + 4y^2 - 2x + 8y$. To determine the constraint function, we must first subtract 7 from both sides of the constraint. This gives $x + 2y - 7 = 0$. The constraint function is equal to the left-hand side, so $g(x, y) = x + 2y - 7$. The problem asks us to solve for the minimum value of f, subject to the constraint (see the following graph).

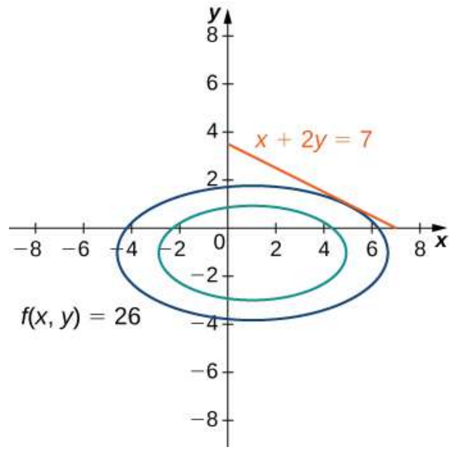


Figure 4.61 Graph of level curves of the function $f(x, y) = x^2 + 4y^2 - 2x + 8y$ corresponding to $c = 10$ and $26.$ The red graph is the constraint function.

2. We then must calculate the gradients of both *f* and *g*:

$$\nabla f(x, y) = (2x - 2)\mathbf{i} + (8y + 8)\mathbf{j}$$
$$\nabla g(x, y) = \mathbf{i} + 2\mathbf{j}.$$

The equation $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ becomes

$$(2x_0 - 2)\mathbf{i} + (8y_0 + 8)\mathbf{j} = \lambda(\mathbf{i} + 2\mathbf{j}),$$

which can be rewritten as

$$(2x_0 - 2)\mathbf{i} + (8y_0 + 8)\mathbf{j} = \lambda\mathbf{i} + 2\lambda\mathbf{j}.$$

Next, we set the coefficients of $\mathbf{i}$ and $\mathbf{j}$ equal to each other:

$$2x_0 - 2 = \lambda$$
$$8y_0 + 8 = 2\lambda.$$

The equation $g(x_0, y_0) = 0$ becomes $x_0 + 2y_0 - 7 = 0.$ Therefore, the system of equations that needs to be solved is

$$\begin{aligned} 2x_0 - 2 &= \lambda \\ 8y_0 + 8 &= 2\lambda \\ x_0 + 2y_0 - 7 &= 0. \end{aligned}$$

3. This is a linear system of three equations in three variables. We start by solving the second equation for λ and substituting it into the first equation. This gives $\lambda = 4y_0 + 4,$ so substituting this into the first equation gives

$$2x_0 - 2 = 4y_0 + 4.$$

Solving this equation for x_0 gives $x_0 = 2y_0 + 3.$ We then substitute this into the third equation:

$$\begin{aligned}(2y_0 + 3) + 2y_0 - 7 &= 0\\ 4y_0 - 4 &= 0\\ y_0 &= 1.\end{aligned}$$

Since $x_0 = 2y_0 + 3,$ this gives $x_0 = 5.$

4. Next, we substitute $(5, 1)$ into $f(x, y) = x^2 + 4y^2 - 2x + 8y,$ gives $f(5, 1) = 5^2 + 4(1)^2 - 2(5) + 8(1) = 27.$ To ensure this corresponds to a minimum value on the constraint function, let's try some other values, such as the intercepts of $g(x, y) = 0,$ Which are $(7, 0)$ and $(0, 3.5)$. We get $f(7, 0) = 35$ and $f(0, 3.5) = 77,$ so it appears f has a minimum at $(5, 1)$.

4.37 Use the method of Lagrange multipliers to find the maximum value of $f(x, y) = 9x^2 + 36xy - 4y^2 - 18x - 8y$ subject to the constraint $3x + 4y = 32.$

Let's now return to the problem posed at the beginning of the section.

Example 4.43

Golf Balls and Lagrange Multipliers

The golf ball manufacturer, Pro-T, has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The budgetary constraint function relating the cost of the production of thousands golf balls and advertising units is given by $20x + 4y = 216.$ Find the values of x and y that maximize profit, and find the maximum profit.

Solution

Again, we follow the problem-solving strategy:

1. The optimization function is $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2.$ To determine the constraint function, we first subtract 216 from both sides of the constraint, then divide both sides by $4,$ which gives $5x + y - 54 = 0.$ The constraint function is equal to the left-hand side, so $g(x, y) = 5x + y - 54.$ The problem asks us to solve for the maximum value of $f,$ subject to this constraint.

2. So, we calculate the gradients of both f and g:

$$\begin{aligned}&\nabla f(x, y) = (48 - 2x - 2y)\mathbf{i} + (96 - 2x - 18y)\mathbf{j}\\ &\nabla g(x, y) = 5\mathbf{i} + \mathbf{j}.\end{aligned}$$

The equation $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ becomes

$$(48 - 2x_0 - 2y_0)\mathbf{i} + (96 - 2x_0 - 18y_0)\mathbf{j} = \lambda(5\mathbf{i} + \mathbf{j}),$$

which can be rewritten as

$$(48 - 2x_0 - 2y_0)\mathbf{i} + (96 - 2x_0 - 18y_0)\mathbf{j} = \lambda 5\mathbf{i} + \lambda\mathbf{j}.$$

We then set the coefficients of $\mathbf{i}$ and $\mathbf{j}$ equal to each other:

$$\begin{aligned} 48 - 2x_0 - 2y_0 &= 5\lambda \\ 96 - 2x_0 - 18y_0 &= \lambda. \end{aligned}$$

The equation $g(x_0, y_0) = 0$ becomes $5x_0 + y_0 - 54 = 0.$ Therefore, the system of equations that needs to be solved is

$$\begin{aligned} 48 - 2x_0 - 2y_0 &= 5\lambda \\ 96 - 2x_0 - 18y_0 &= \lambda \\ 5x_0 + y_0 - 54 &= 0. \end{aligned}$$

3. We use the left-hand side of the second equation to replace λ in the first equation:

$$\begin{aligned} 48 - 2x_0 - 2y_0 &= 5(96 - 2x_0 - 18y_0) \\ 48 - 2x_0 - 2y_0 &= 480 - 10x_0 - 90y_0 \\ 8x_0 &= 432 - 88y_0 \\ x_0 &= 54 - 11y_0. \end{aligned}$$

Then we substitute this into the third equation:

$$\begin{aligned} 5(54 - 11y_0) + y_0 - 54 &= 0 \\ 270 - 55y_0 + y_0 &= 0 \\ 216 - 54y_0 &= 0 \\ y_0 &= 4. \end{aligned}$$

Since $x_0 = 54 - 11y_0,$ this gives $x_0 = 10.$

4. We then substitute $(10, 4)$ into $f(x, y) = 48x + 96y - x^2 - 2xy - 9y^2,$ which gives

$$\begin{aligned} f(10, 4) &= 48(10) + 96(4) - (10)^2 - 2(10)(4) - 9(4)^2 \\ &= 480 + 384 - 100 - 80 - 144 = 540. \end{aligned}$$

Therefore the maximum profit that can be attained, subject to budgetary constraints, is $\$540{,}000$ with a production level of $10{,}000$ golf balls and 4 hours of advertising bought per month. Let's check to make sure this truly is a maximum. The endpoints of the line that defines the constraint are $(10.8, 0)$ and $(0, 54)$ Let's evaluate f at both of these points:

$$\begin{aligned} f(10.8, 0) &= 48(10.8) + 96(0) - 10.8^2 - 2(10.8)(0) - 9(0^2) = 401.76 \\ f(0, 54) &= 48(0) + 96(54) - 0^2 - 2(0)(54) - 9(54^2) = -21,060. \end{aligned}$$

The second value represents a loss, since no golf balls are produced. Neither of these values exceed $540,$ so it seems that our extremum is a maximum value of f.

4.38 A company has determined that its production level is given by the Cobb-Douglas function $f(x, y) = 2.5x^{0.45} y^{0.55}$ where *x* represents the total number of labor hours in 1 year and *y* represents the total capital input for the company. Suppose 1 unit of labor costs $\$40$ and 1 unit of capital costs $\$50.$ Use the method of Lagrange multipliers to find the maximum value of $f(x, y) = 2.5x^{0.45} y^{0.55}$ subject to a budgetary constraint of $\$500{,}000$ per year.

In the case of an optimization function with three variables and a single constraint function, it is possible to use the method of Lagrange multipliers to solve an optimization problem as well. An example of an optimization function with three variables could be the Cobb-Douglas function in the previous example: $f(x, y, z) = x^{0.2} y^{0.4} z^{0.4},$ where x represents the cost of labor, y represents capital input, and z represents the cost of advertising. The method is the same as for the method with a function of two variables; the equations to be solved are

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= 0. \end{aligned}$$

Example 4.44

Lagrange Multipliers with a Three-Variable Optimization Function

Maximize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 1.$

Solution

1. The optimization function is $f(x, y, z) = x^2 + y^2 + z^2.$ To determine the constraint function, we subtract 1 from each side of the constraint: $x + y + z - 1 = 0$ which gives the constraint function as $g(x, y, z) = x + y + z - 1.$
2. Next, we calculate $\nabla f(x, y, z)$ and $\nabla g(x, y, z)$:

$$\begin{aligned} \nabla f(x, y, z) &= \langle 2x, 2y, 2z \rangle \\ \nabla g(x, y, z) &= \langle 1, 1, 1 \rangle . \end{aligned}$$

   This leads to the equations

$$\begin{aligned} \langle 2x_0, 2y_0, 2z_0 \rangle &= \lambda \langle 1, 1, 1 \rangle \\ x_0 + y_0 + z_0 - 1 &= 0 \end{aligned}$$

   which can be rewritten in the following form:

$$\begin{aligned} 2x_0 &= \lambda \\ 2y_0 &= \lambda \\ 2z_0 &= \lambda \\ x_0 + y_0 + z_0 - 1 &= 0. \end{aligned}$$

3. Since each of the first three equations has λ on the right-hand side, we know that $2x_0 = 2y_0 = 2z_0$ and all three variables are equal to each other. Substituting $y_0 = x_0$ and $z_0 = x_0$ into the last equation yields $3x_0 - 1 = 0,$ so $x_0 = \frac{1}{3}$ and $y_0 = \frac{1}{3}$ and $z_0 = \frac{1}{3}$ which corresponds to a critical point on the constraint curve.

4. Then, we evaluate f at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$:

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{3}{9} = \frac{1}{3}.$$

Therefore, an extremum of the function is $\frac{1}{3}$. To verify it is a minimum, choose other points that satisfy the constraint and calculate f at that point. For example,

$$\begin{aligned} f(1, 0, 0) &= 1^2 + 0^2 + 0^2 = 1 \\ f(0, -2, 3) &= 0^2 + + (-2)^2 + 3^2 = 13. \end{aligned}$$

Both of these values are greater than $\frac{1}{3}$, leading us to believe the extremum is a minimum.

4.39 Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x, y, z) = x + y + z$$

subject to the constraint $x^2 + y^2 + z^2 = 1$.

Problems with Two Constraints

The method of Lagrange multipliers can be applied to problems with more than one constraint. In this case the optimization function, w is a function of three variables:

$$w = f(x, y, z)$$

and it is subject to two constraints:

$$g(x, y, z) = 0 \text{ and } h(x, y, z) = 0.$$

There are two Lagrange multipliers, λ_1 and λ_2, and the system of equations becomes

$$\begin{aligned} \nabla f(x_0, y_0, z_0) &= \lambda_1 \nabla g(x_0, y_0, z_0) + \lambda_2 \nabla h(x_0, y_0, z_0) \\ g(x_0, y_0, z_0) &= 0 \\ h(x_0, y_0, z_0) &= 0. \end{aligned}$$

Example 4.45

Lagrange Multipliers with Two Constraints

Find the maximum and minimum values of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints $z^2 = x^2 + y^2$ and $x + y - z + 1 = 0$.

Solution

Let's follow the problem-solving strategy:

1. The optimization function is $f(x, y, z) = x^2 + y^2 + z^2$. To determine the constraint functions, we first subtract z^2 from both sides of the first constraint, which gives $x^2 + y^2 - z^2 = 0$, so $g(x, y, z) = x^2 + y^2 - z^2$. The second constraint function is $h(x, y, z) = x + y - z + 1$.

2. We then calculate the gradients of f, g, and h:

$$\begin{aligned}\nabla f(x, y, z) &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}\\ \nabla g(x, y, z) &= 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}\\ \nabla h(x, y, z) &= \mathbf{i} + \mathbf{j} - \mathbf{k}.\end{aligned}$$

The equation $\nabla f(x_0, y_0, z_0) = \lambda_1 \nabla g(x_0, y_0, z_0) + \lambda_2 \nabla h(x_0, y_0, z_0)$ becomes

$$2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k} = \lambda_1(2x_0\mathbf{i} + 2y_0\mathbf{j} - 2z_0\mathbf{k}) + \lambda_2(\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

which can be rewritten as

$$2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k} = (2\lambda_1 x_0 + \lambda_2)\mathbf{i} + (2\lambda_1 y_0 + \lambda_2)\mathbf{j} - (2\lambda_1 z_0 + \lambda_2)\mathbf{k}.$$

Next, we set the coefficients of $\mathbf{i}$ and $\mathbf{j}$ equal to each other:

$$\begin{aligned}2x_0 &= 2\lambda_1 x_0 + \lambda_2\\ 2y_0 &= 2\lambda_1 y_0 + \lambda_2\\ 2z_0 &= -2\lambda_1 z_0 - \lambda_2.\end{aligned}$$

The two equations that arise from the constraints are $z_0^2 = x_0^2 + y_0^2$ and $x_0 + y_0 - z_0 + 1 = 0$. Combining these equations with the previous three equations gives

$$\begin{aligned}2x_0 &= 2\lambda_1 x_0 + \lambda_2\\ 2y_0 &= 2\lambda_1 y_0 + \lambda_2\\ 2z_0 &= -2\lambda_1 z_0 - \lambda_2\\ z_0^2 &= x_0^2 + y_0^2\\ x_0 + y_0 - z_0 + 1 &= 0.\end{aligned}$$

3. The first three equations contain the variable λ_2. Solving the third equation for λ_2 and replacing into the first and second equations reduces the number of equations to four:

$$\begin{aligned}2x_0 &= 2\lambda_1 x_0 - 2\lambda_1 z_0 - 2z_0\\ 2y_0 &= 2\lambda_1 y_0 - 2\lambda_1 z_0 - 2z_0\\ z_0^2 &= x_0^2 + y_0^2\\ x_0 + y_0 - z_0 + 1 &= 0.\end{aligned}$$

Next, we solve the first and second equation for λ_1. The first equation gives $\lambda_1 = \frac{x_0 + z_0}{x_0 - z_0}$, the second equation gives $\lambda_1 = \frac{y_0 + z_0}{y_0 - z_0}$. We set the right-hand side of each equation equal to each other and cross-multiply:

$$\begin{aligned} \frac{x_0+z_0}{x_0-z_0} &= \frac{y_0+z_0}{y_0-z_0} \\ (x_0+z_0)(y_0-z_0) &= (x_0-z_0)(y_0+z_0) \\ x_0y_0-x_0z_0+y_0z_0-z_0{}^2 &= x_0y_0+x_0z_0-y_0z_0-z_0 2. \\ 2y_0z_0-2x_0z_0 &= 0 \\ 2z_0(y_0-x_0) &= 0. \end{aligned}$$

Therefore, either $z_0=0$ or $y_0=x_0$. If $z_0=0$, then the first constraint becomes $0=x_0{}^2+y_0{}^2$. The only real solution to this equation is $x_0=0$ and $y_0=0$, which gives the ordered triple $(0, 0, 0)$. This point does not satisfy the second constraint, so it is not a solution.
Next, we consider $y_0=x_0$, which reduces the number of equations to three:

$$\begin{aligned} y_0 &= x_0 \\ z_0{}^2 &= x_0{}^2+y_0{}^2 \\ x_0+y_0-z_0+1 &= 0. \end{aligned}$$

We substitute the first equation into the second and third equations:

$$\begin{aligned} z_0{}^2 &= x_0{}^2+x_0{}^2 \\ x_0+x_0-z_0+1 &= 0. \end{aligned}$$

Then, we solve the second equation for z_0, which gives $z_0=2x_0+1$. We then substitute this into the first equation,

$$\begin{aligned} z_0{}^2 &= 2x_0{}^2 \\ (2x_0+1)^2 &= 2x_0{}^2 \\ 4x_0{}^2+4x_0+1 &= 2x_0{}^2 \\ 2x_0{}^2+4x_0+1 &= 0, \end{aligned}$$

and use the quadratic formula to solve for x_0:

$$x_0=\frac{-4\pm\sqrt{4^2-4(2)(1)}}{2(2)}=\frac{-4\pm\sqrt{8}}{4}=\frac{-4\pm 2\sqrt{2}}{4}=-1\pm\frac{\sqrt{2}}{2}.$$

Recall $y_0=x_0$, so this solves for y_0 as well. Then, $z_0=2x_0+1$, so

$$z_0=2x_0+1=2\left(-1\pm\frac{\sqrt{2}}{2}\right)+1=-2+1\pm\sqrt{2}=-1\pm\sqrt{2}.$$

Therefore, there are two ordered triplet solutions:

$$\left(-1+\frac{\sqrt{2}}{2}, -1+\frac{\sqrt{2}}{2}, -1+\sqrt{2}\right) \text{and} \left(-1-\frac{\sqrt{2}}{2}, -1-\frac{\sqrt{2}}{2}, -1-\sqrt{2}\right).$$

4. We substitute $\left(-1+\frac{\sqrt{2}}{2}, -1+\frac{\sqrt{2}}{2}, -1+\sqrt{2}\right)$ into $f(x, y, z)=x^2+y^2+z^2$, which gives

$$\begin{aligned} f\left(-1+\frac{\sqrt{2}}{2}, -1+\frac{\sqrt{2}}{2}, -1+\sqrt{2}\right) &= \left(-1+\frac{\sqrt{2}}{2}\right)^2 + \left(-1+\frac{\sqrt{2}}{2}\right)^2 + \left(-1+\sqrt{2}\right)^2 \\ &= \left(1-\sqrt{2}+\frac{1}{2}\right) + \left(1-\sqrt{2}+\frac{1}{2}\right) + \left(1-2\sqrt{2}+2\right) \\ &= 6-4\sqrt{2}. \end{aligned}$$

Then, we substitute $\left(-1-\frac{\sqrt{2}}{2}, -1-\frac{\sqrt{2}}{2}, -1-\sqrt{2}\right)$ into $f(x, y, z) = x^2 + y^2 + z^2$, which gives

$$\begin{aligned} f\left(-1-\frac{\sqrt{2}}{2}, -1-\frac{\sqrt{2}}{2}, -1-\sqrt{2}\right) &= \left(-1-\frac{\sqrt{2}}{2}\right)^2 + \left(-1-\frac{\sqrt{2}}{2}\right)^2 + (-1-\sqrt{2})^2 \\ &= \left(1+\sqrt{2}+\frac{1}{2}\right) + \left(1+\sqrt{2}+\frac{1}{2}\right) + \left(1+2\sqrt{2}+2\right) \\ &= 6+4\sqrt{2}. \end{aligned}$$

$6+4\sqrt{2}$ is the maximum value and $6-4\sqrt{2}$ is the minimum value of $f(x, y, z)$, subject to the given constraints.

4.40 Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints $2x + y + 2z = 9$ and $5x + 5y + 7z = 29$.

4.8 EXERCISES

For the following exercises, use the method of Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraints.

358. $f(x, y) = x^2 y;\ x^2 + 2y^2 = 6$

359. $f(x, y, z) = xyz,\ x^2 + 2y^2 + 3z^2 = 6$

360. $f(x, y) = xy;\ 4x^2 + 8y^2 = 16$

361. $f(x, y) = 4x^3 + y^2;\ 2x^2 + y^2 = 1$

362. $f(x, y, z) = x^2 + y^2 + z^2,\ x^4 + y^4 + z^4 = 1$

363. $f(x, y, z) = yz + xy,\ xy = 1,\ y^2 + z^2 = 1$

364. $f(x, y) = x^2 + y^2,\ (x-1)^2 + 4y^2 = 4$

365. $f(x, y) = 4xy,\ \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

366. $f(x, y, z) = x + y + z,\ \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$

367. $f(x, y, z) = x + 3y - z,\ x^2 + y^2 + z^2 = 4$

368. $f(x, y, z) = x^2 + y^2 + z^2,\ xyz = 4$

369. Minimize $f(x, y) = x^2 + y^2$ on the hyperbola $xy = 1.$

370. Minimize $f(x, y) = xy$ on the ellipse $b^2 x^2 + a^2 y^2 = a^2 b^2.$

371. Maximize $f(x, y, z) = 2x + 3y + 5z$ on the sphere $x^2 + y^2 + z^2 = 19.$

372. Maximize $\begin{array}{l} f(x, y) = x^2 - y^2;\ x > 0,\ y > 0; \\ g(x, y) = y - x^2 = 0 \end{array}$

373. The curve $x^3 - y^3 = 1$ is asymptotic to the line $y = x.$ Find the point(s) on the curve $x^3 - y^3 = 1$ farthest from the line $y = x.$

374. Maximize $U(x, y) = 8x^{4/5} y^{1/5};\ 4x + 2y = 12$

375. Minimize $f(x, y) = x^2 + y^2,\ x + 2y - 5 = 0.$

376. Maximize $f(x, y) = \sqrt{6 - x^2 - y^2},\ x + y - 2 = 0.$

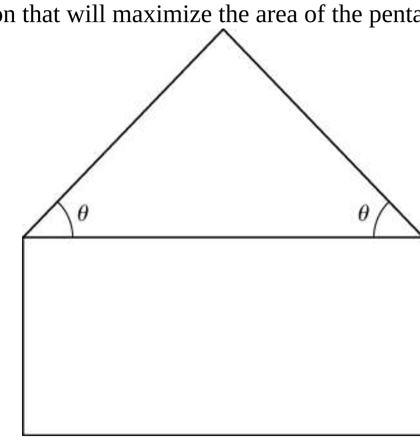
377. Minimize $f(x, y, z) = x^2 + y^2 + z^2,\ x + y + z = 1.$

378. Minimize $f(x, y) = x^2 - y^2$ subject to the constraint $x - 2y + 6 = 0.$

379. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ when $x + y + z = 9$ and $x + 2y + 3z = 20.$

For the next group of exercises, use the method of Lagrange multipliers to solve the following applied problems.

380. A pentagon is formed by placing an isosceles triangle on a rectangle, as shown in the diagram. If the perimeter of the pentagon is 10 in., find the lengths of the sides of the pentagon that will maximize the area of the pentagon.

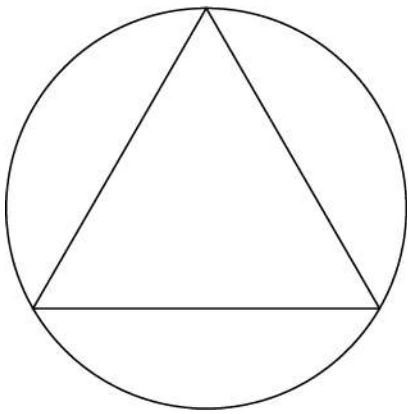


381. A rectangular box without a top (a topless box) is to be made from $12\ \text{ft}^2$ of cardboard. Find the maximum volume of such a box.

382. Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.

383. Find the point on the surface $x^2 - 2xy + y^2 - x + y = 0$ closest to the point $(1, 2, -3).$

384. Show that, of all the triangles inscribed in a circle of radius R (see diagram), the equilateral triangle has the largest perimeter.



385. Find the minimum distance from point $(0, 1)$ to the parabola $x^2 = 4y.$

386. Find the minimum distance from the parabola $y = x^2$ to point $(0, 3).$

387. Find the minimum distance from the plane $x + y + z = 1$ to point $(2, 1, 1).$

388. A large container in the shape of a rectangular solid must have a volume of 480 m^3. The bottom of the container costs $\$5/\text{m}^2$ to construct whereas the top and sides cost $\$3/\text{m}^2$ to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has the minimum cost.

389. Find the point on the line $y = 2x + 3$ that is closest to point $(4, 2).$

390. Find the point on the plane $4x + 3y + z = 2$ that is closest to the point $(1, -1, 1).$

391. Find the maximum value of $f(x, y) = \sin x \sin y,$ where x and y denote the acute angles of a right triangle. Draw the contours of the function using a CAS.

392. A rectangular solid is contained within a tetrahedron with vertices at $(1, 0, 0), (0, 1, 0), (0, 0, 1),$ and the origin. The base of the box has dimensions $x, y,$ and the height of the box is $z.$ If the sum of $x, y,$ and z is 1.0, find the dimensions that maximizes the volume of the rectangular solid.

393. **[T]** By investing x units of labor and y units of capital, a watch manufacturer can produce $P(x, y) = 50x^{0.4} y^{0.6}$ watches. Find the maximum number of watches that can be produced on a budget of $\$20{,}000$ if labor costs \$100/unit and capital costs \$200/unit. Use a CAS to sketch a contour plot of the function.

CHAPTER 4 REVIEW

KEY TERMS

boundary point a point P_0 of R is a boundary point if every δ disk centered around P_0 contains points both inside and outside R

closed set a set S that contains all its boundary points

connected set an open set S that cannot be represented as the union of two or more disjoint, nonempty open subsets

constraint an inequality or equation involving one or more variables that is used in an optimization problem; the constraint enforces a limit on the possible solutions for the problem

contour map a plot of the various level curves of a given function $f(x, y)$

critical point of a function of two variables the point (x_0, y_0) is called a critical point of $f(x, y)$ if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. At least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist

differentiable a function $f(x, y)$ is differentiable at (x_0, y_0) if $f(x, y)$ can be expressed in the form $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$

where the error term $E(x, y)$ satisfies $\lim_{(x, y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$

directional derivative the derivative of a function in the direction of a given unit vector

discriminant the discriminant of the function $f(x, y)$ is given by the formula $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$

function of two variables a function $z = f(x, y)$ that maps each ordered pair (x, y) in a subset D of $\mathbb{R}^2$ to a unique real number z

generalized chain rule the chain rule extended to functions of more than one independent variable, in which each independent variable may depend on one or more other variables

gradient the gradient of the function $f(x, y)$ is defined to be $\nabla f(x, y) = (\partial f/\partial x)\mathbf{i} + (\partial f/\partial y)\mathbf{j},$ which can be generalized to a function of any number of independent variables

graph of a function of two variables a set of ordered triples (x, y, z) that satisfies the equation $z = f(x, y)$ plotted in three-dimensional Cartesian space

higher-order partial derivatives second-order or higher partial derivatives, regardless of whether they are mixed partial derivatives

interior point a point P_0 of R is a boundary point if there is a δ disk centered around P_0 contained completely in R

intermediate variable given a composition of functions (e.g., $f(x(t), y(t))$), the intermediate variables are the variables that are independent in the outer function but dependent on other variables as well; in the function $f(x(t), y(t)),$ the variables x and y are examples of intermediate variables

Lagrange multiplier the constant (or constants) used in the method of Lagrange multipliers; in the case of one constant, it is represented by the variable λ

level curve of a function of two variables the set of points satisfying the equation $f(x, y) = c$ for some real number c in the range of f

level surface of a function of three variables the set of points satisfying the equation $f(x, y, z) = c$ for some real number c in the range of f

linear approximation given a function $f(x, y)$ and a tangent plane to the function at a point (x_0, y_0), we can approximate $f(x, y)$ for points near (x_0, y_0) using the tangent plane formula

method of Lagrange multipliers a method of solving an optimization problem subject to one or more constraints

mixed partial derivatives second-order or higher partial derivatives, in which at least two of the differentiations are with respect to different variables

objective function the function that is to be maximized or minimized in an optimization problem

open set a set S that contains none of its boundary points

optimization problem calculation of a maximum or minimum value of a function of several variables, often using Lagrange multipliers

partial derivative a derivative of a function of more than one independent variable in which all the variables but one are held constant

partial differential equation an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives

region an open, connected, nonempty subset of $\mathbb{R}^2$

saddle point given the function $z = f(x, y)$, the point $\left(x_0, y_0, f(x_0, y_0)\right)$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0)

surface the graph of a function of two variables, $z = f(x, y)$

tangent plane given a function $f(x, y)$ that is differentiable at a point (x_0, y_0), the equation of the tangent plane to the surface $z = f(x, y)$ is given by $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

total differential the total differential of the function $f(x, y)$ at (x_0, y_0) is given by the formula $dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$

tree diagram illustrates and derives formulas for the generalized chain rule, in which each independent variable is accounted for

vertical trace the set of ordered triples (c, y, z) that solves the equation $f(c, y) = z$ for a given constant $x = c$ or the set of ordered triples (x, d, z) that solves the equation $f(x, d) = z$ for a given constant $y = d$

δ ball all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0, y_0, z_0)

δ disk an open disk of radius δ centered at point (a, b)

KEY EQUATIONS

- **Vertical trace**
  $f(a, y) = z$ for $x = a$ or $f(x, b) = z$ for $y = b$
- **Level surface of a function of three variables**
  $f(x, y, z) = c$
- **Partial derivative of f with respect to x**
  $$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

- **Partial derivative of f with respect to y**

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

- **Tangent plane**

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- **Linear approximation**

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- **Total differential**

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

- **Differentiability (two variables)**

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$$

where the error term E satisfies

$$\lim_{(x, y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

- **Differentiability (three variables)**

$$\begin{aligned} f(x, y) &= f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) \\ &\quad + f_z(x_0, y_0, z_0)(z - z_0) + E(x, y, z), \end{aligned}$$

where the error term E satisfies

$$\lim_{(x, y, z) \to (x_0, y_0, z_0)} \frac{E(x, y, z)}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = 0.$$

- **Chain rule, one independent variable**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

- **Chain rule, two independent variables**

$$\frac{dz}{du} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{dz}{dv} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

- **Generalized chain rule**

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2} \frac{\partial x_1}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial t_j}$$

- **directional derivative (two dimensions)**

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h}$$

or

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

- **gradient (two dimensions)**

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

- **gradient (three dimensions)**

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

- **directional derivative (three dimensions)**

$$\begin{aligned} D_{\mathbf{u}} f(x, y, z) &= \nabla f(x, y, z) \cdot \mathbf{u} \\ &= f_x(x, y, z)\cos\alpha + f_y(x, y, z)\cos\beta + f_x(x, y, z)\cos\gamma \end{aligned}$$

- **Discriminant**

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$$

- **Method of Lagrange multipliers, one constraint**

$$\begin{aligned} \nabla f(x_0, y_0) &= \lambda \nabla g(x_0, y_0) \\ g(x_0, y_0) &= 0 \end{aligned}$$

- **Method of Lagrange multipliers, two constraints**

$$\begin{aligned} \nabla f(x_0, y_0, z_0) &= \lambda_1 \nabla g(x_0, y_0, z_0) + \lambda_2 \nabla h(x_0, y_0, z_0) \\ g(x_0, y_0, z_0) &= 0 \\ h(x_0, y_0, z_0) &= 0 \end{aligned}$$

KEY CONCEPTS

4.1 Functions of Several Variables

- The graph of a function of two variables is a surface in $\mathbb{R}^3$ and can be studied using level curves and vertical traces.
- A set of level curves is called a contour map.

4.2 Limits and Continuity

- To study limits and continuity for functions of two variables, we use a δ disk centered around a given point.
- A function of several variables has a limit if for any point in a δ ball centered at a point $P,$ the value of the function at that point is arbitrarily close to a fixed value (the limit value).
- The limit laws established for a function of one variable have natural extensions to functions of more than one variable.
- A function of two variables is continuous at a point if the limit exists at that point, the function exists at that point, and the limit and function are equal at that point.

4.3 Partial Derivatives

- A partial derivative is a derivative involving a function of more than one independent variable.
- To calculate a partial derivative with respect to a given variable, treat all the other variables as constants and use the usual differentiation rules.
- Higher-order partial derivatives can be calculated in the same way as higher-order derivatives.

4.4 Tangent Planes and Linear Approximations

- The analog of a tangent line to a curve is a tangent plane to a surface for functions of two variables.
- Tangent planes can be used to approximate values of functions near known values.
- A function is differentiable at a point if it is ”smooth” at that point (i.e., no corners or discontinuities exist at that point).
- The total differential can be used to approximate the change in a function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and $\Delta y.$

4.5 The Chain Rule

- The chain rule for functions of more than one variable involves the partial derivatives with respect to all the independent variables.
- Tree diagrams are useful for deriving formulas for the chain rule for functions of more than one variable, where each independent variable also depends on other variables.

4.6 Directional Derivatives and the Gradient

- A directional derivative represents a rate of change of a function in any given direction.
- The gradient can be used in a formula to calculate the directional derivative.
- The gradient indicates the direction of greatest change of a function of more than one variable.

4.7 Maxima/Minima Problems

- A critical point of the function $f(x, y)$ is any point (x_0, y_0) where either $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, or at least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist.
- A saddle point is a point (x_0, y_0) where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, but (x_0, y_0) is neither a maximum nor a minimum at that point.
- To find extrema of functions of two variables, first find the critical points, then calculate the discriminant and apply the second derivative test.

4.8 Lagrange Multipliers

- An objective function combined with one or more constraints is an example of an optimization problem.
- To solve optimization problems, we apply the method of Lagrange multipliers using a four-step problem-solving strategy.

CHAPTER 4 REVIEW EXERCISES

For the following exercises, determine whether the statement is *true or false*. Justify your answer with a proof or a counterexample.

394. The domain of $f(x, y) = x^3 \sin^{-1}(y)$ is $x =$ all real numbers, and $-\pi \le y \le \pi$.

395. If the function $f(x, y)$ is continuous everywhere, then $f_{xy} = f_{yx}$.

396. The linear approximation to the function of $f(x, y) = 5x^2 + x\tan(y)$ at $(2, \pi)$ is given by $L(x, y) = 22 + 21(x - 2) + (y - \pi)$.

397. $\left(\frac{3}{4}, \frac{9}{16}\right)$ is a critical point of $g(x, y) = 4x^3 - 2x^2 y + y^2 - 2$.

For the following exercises, sketch the function in one graph and, in a second, sketch several level curves.

398. $f(x, y) = e^{-\left(x^2 + 2y^2\right)}$.

399. $f(x, y) = x + 4y^2$.

For the following exercises, evaluate the following limits, if they exist. If they do not exist, prove it.

400. $\lim\limits_{(x, y) \to (1, 1)} \dfrac{4xy}{x - 2y^2}$

401. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{4xy}{x - 2y^2}$

For the following exercises, find the largest interval of continuity for the function.

402. $f(x, y) = x^3 \sin^{-1}(y)$

403. $g(x, y) = \ln\left(4 - x^2 - y^2\right)$

For the following exercises, find all first partial derivatives.

404. $f(x, y) = \sqrt{x^2 - y^2}$

405. $u(x, y) = x^4 - 3xy + 1,\ x = 2t,\ y = t^3$

For the following exercises, find all second partial derivatives.

406. $g(t, x) = 3t^2 - \sin(x + t)$

407. $h(x, y, z) = \frac{x^3 e^{2y}}{z}$

For the following exercises, find the equation of the tangent plane to the specified surface at the given point.

408. $z = x^3 - 2y^2 + y - 1$ at point $(1, 1, -1)$

409. $3z^3 = e^x + \frac{2}{y}$ at point $(0, 1, 3)$

410. Approximate $f(x, y) = e^{x^2} + \sqrt{y}$ at $(0.1, 9.1)$. Write down your linear approximation function $L(x, y)$. How accurate is the approximation to the exact answer, rounded to four digits?

411. Find the differential dz of $h(x, y) = 4x^2 + 2xy - 3y$ and approximate Δz at the point $(1, -2)$. Let $\Delta x = 0.1$ and $\Delta y = 0.01$.

412. Find the directional derivative of $f(x, y) = x^2 + 6xy - y^2$ in the direction $\text{v} = \mathbf{i} + 4\mathbf{j}$.

413. Find the maximal directional derivative magnitude and direction for the function $f(x, y) = x^3 + 2xy - \cos(\pi y)$ at point $(3, 0)$.

For the following exercises, find the gradient.

414. $c(x, t) = e(t - x)^2 + 3\cos(t)$

415. $f(x, y) = \frac{\sqrt{x} + y^2}{xy}$

For the following exercises, find and classify the critical points.

416. $z = x^3 - xy + y^2 - 1$

For the following exercises, use Lagrange multipliers to find the maximum and minimum values for the functions with the given constraints.

417. $f(x, y) = x^2 y,\ x^2 + y^2 = 4$

418. $f(x, y) = x^2 - y^2,\ x + 6y = 4$

419. A machinist is constructing a right circular cone out of a block of aluminum. The machine gives an error of 5% in height and 2% in radius. Find the maximum error in the volume of the cone if the machinist creates a cone of height 6 cm and radius 2 cm.

420. A trash compactor is in the shape of a cuboid. Assume the trash compactor is filled with incompressible liquid. The length and width are decreasing at rates of 2 ft/sec and 3 ft/sec, respectively. Find the rate at which the liquid level is rising when the length is 14 ft, the width is 10 ft, and the height is 4 ft.

5 | MULTIPLE INTEGRATION

Figure 5.1 The City of Arts and Sciences in Valencia, Spain, has a unique structure along an axis of just two kilometers that was formerly the bed of the River Turia. The l'Hemisfèric has an IMAX cinema with three systems of modern digital projections onto a concave screen of 900 square meters. An oval roof over 100 meters long has been made to look like a huge human eye that comes alive and opens up to the world as the "Eye of Wisdom." (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Chapter Outline

Introduction

In this chapter we extend the concept of a definite integral of a single variable to double and triple integrals of functions of two and three variables, respectively. We examine applications involving integration to compute volumes, masses, and centroids of more general regions. We will also see how the use of other coordinate systems (such as polar, cylindrical, and spherical coordinates) makes it simpler to compute multiple integrals over some types of regions and functions. As an example, we will use polar coordinates to find the volume of structures such as l'Hemisfèric. (See **Example 5.51**.)

In the preceding chapter, we discussed differential calculus with multiple independent variables. Now we examine integral calculus in multiple dimensions. Just as a partial derivative allows us to differentiate a function with respect to one variable while holding the other variables constant, we will see that an iterated integral allows us to integrate a function with respect to one variable while holding the other variables constant.

5.1 | Double Integrals over Rectangular Regions

Learning Objectives

- **5.1.1** Recognize when a function of two variables is integrable over a rectangular region.
- **5.1.2** Recognize and use some of the properties of double integrals.
- **5.1.3** Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- **5.1.4** Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

Volumes and Double Integrals

We begin by considering the space above a rectangular region R. Consider a continuous function $f(x, y) \geq 0$ of two variables defined on the closed rectangle R:

$$R = [a, b] \times [c, d] = \left\{(x, y) \in \mathbb{R}^2 \,|\, a \leq x \leq b, \ c \leq y \leq d\right\}$$

Here $[a, b] \times [c, d]$ denotes the Cartesian product of the two closed intervals $[a, b]$ and $[c, d]$. It consists of rectangular pairs (x, y) such that $a \leq x \leq b$ and $c \leq y \leq d$. The graph of f represents a surface above the xy-plane with equation $z = f(x, y)$ where z is the height of the surface at the point (x, y). Let S be the solid that lies above R and under the graph of f (**Figure 5.2**). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.

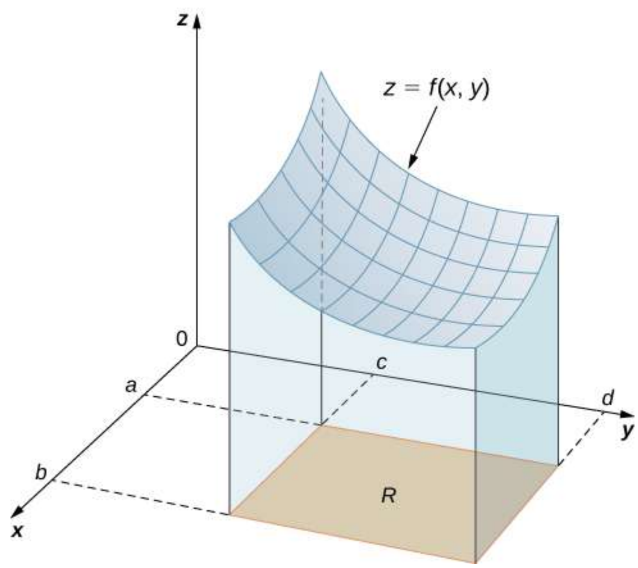


Figure 5.2 The graph of $f(x, y)$ over the rectangle R in the xy-plane is a curved surface.

We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy (**Figure 5.3**). We do this by dividing the interval $[a, b]$ into m subintervals and dividing the interval $[c, d]$ into n subintervals. Hence $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.

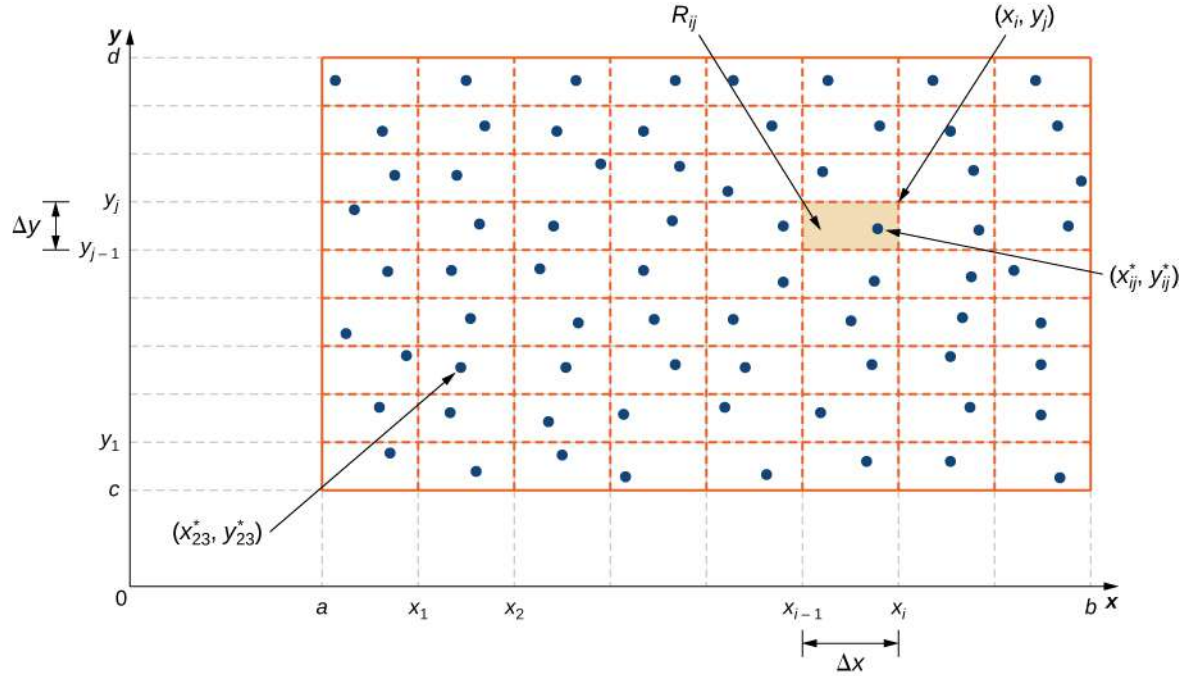


Figure 5.3 Rectangle R is divided into small rectangles R_{ij}, each with area ΔA.

The volume of a thin rectangular box above R_{ij} is $f(x_{ij}^*, y_{ij}^*)\Delta A$, where (x_{ij}^*, y_{ij}^*) is an arbitrary sample point in each R_{ij} as shown in the following figure.

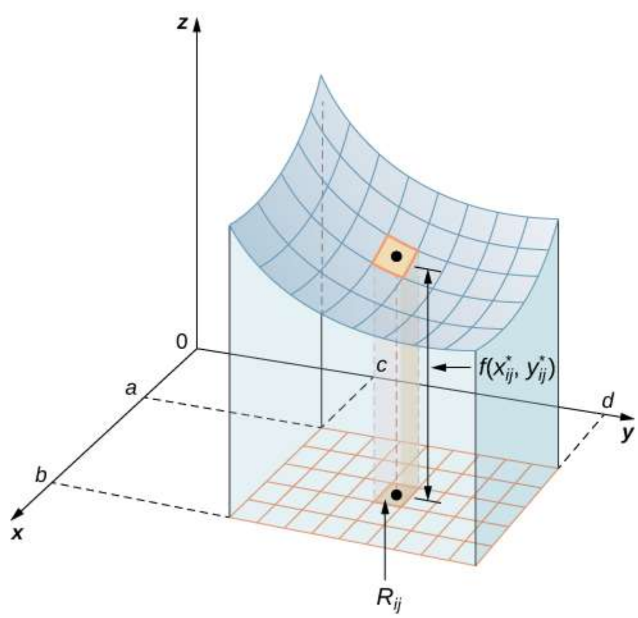


Figure 5.4 A thin rectangular box above R_{ij} with height $f\left(x_{ij}^*, y_{ij}^*\right)$.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$. This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if *m* and *n* become larger.

$$V = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \text{ or } V = \lim_{\Delta x, \Delta y \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base *R*. Now we are ready to define the double integral.

Definition

The **double integral** of the function $f(x, y)$ over the rectangular region R in the xy-plane is defined as

$$\iint_R f(x, y)dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A. \quad (5.1)$$

If $f(x, y) \geq 0,$ then the volume *V* of the solid *S*, which lies above R in the xy-plane and under the graph of *f*, is the double integral of the function $f(x, y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a "signed" volume in a manner similar to the way we defined net signed area in **The Definite Integral (http://cnx.org/content/m53631/latest/)** .

Example 5.1

Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$ (**Figure 5.5**).

a. Set up a double integral for finding the value of the signed volume of the solid *S* that lies above R and "under" the graph of f.

b. Divide *R* into four squares with $m = n = 2,$ and choose the sample point as the upper right corner point of each square $(1, 1), (2, 1), (1, 2),$ and $(2, 2)$ (**Figure 5.6**) to approximate the signed volume of the solid *S* that lies above R and "under" the graph of f.

c. Divide *R* into four squares with $m = n = 2,$ and choose the sample point as the midpoint of each square: $(1/2, 1/2), (3/2, 1/2), (1/2, 3/2),$ and $(3/2, 3/2)$ to approximate the signed volume.

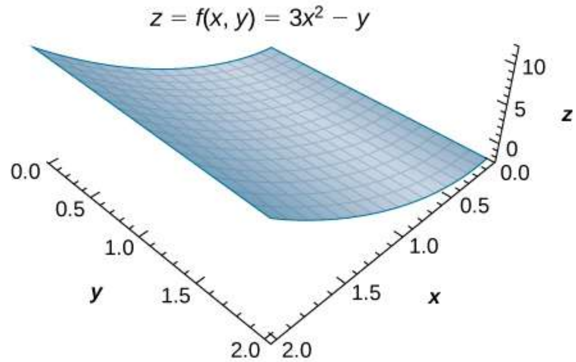


Figure 5.5 The function $z = f(x, y)$ graphed over the rectangular region $R = [0, 2] \times [0, 2]$.

Solution

a. As we can see, the function $z = f(x, y) = 3x^2 - y$ is above the plane. To find the signed volume of S, we need to divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x_{ij}^*, y_{ij}^*) as sample points in each R_{ij}. Hence, a double integral is set up as

$$V = \iint_R (3x^2 - y) dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[3\left(x_{ij}^*\right)^2 - y_{ij}^*\right]\Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in the following figure.

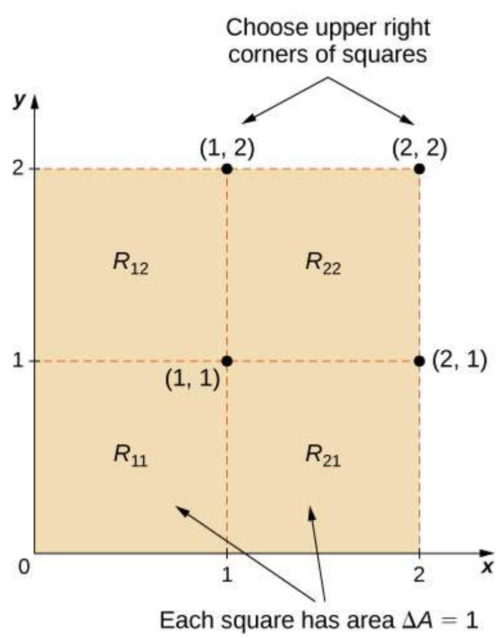


Figure 5.6 Subrectangles for the rectangular region $R = [0, 2] \times [0, 2]$.

Hence,

$$\begin{aligned} V &= \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= \sum_{i=1}^{2} (f(x_{i1}^*, y_{i1}^*) + f(x_{i2}^*, y_{i2}^*))\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1, 1)(1) + f(2, 1)(1) + f(1, 2)(1) + f(2, 2)(1) \\ &= (3 - 1)(1) + (12 - 1)(1) + (3 - 2)(1) + (12 - 2)(1) \\ &= 2 + 11 + 1 + 10 = 24. \end{aligned}$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$, we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).
Hence

$$\begin{aligned} V &= \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1/2, 1/2)(1) + f(3/2, 1/2)(1) + f(1/2, 3/2)(1) + f(3/2, 3/2)(1) \\ &= \left(\tfrac{3}{4} - \tfrac{1}{4}\right)(1) + \left(\tfrac{27}{4} - \tfrac{1}{2}\right)(1) + \left(\tfrac{3}{4} - \tfrac{3}{2}\right)(1) + \left(\tfrac{27}{4} - \tfrac{3}{2}\right)(1) \\ &= \tfrac{2}{4} + \tfrac{25}{4} + \left(-\tfrac{3}{4}\right) + \tfrac{21}{4} = \tfrac{45}{4} = 11. \end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

5.1 Use the same function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square $(0, 1)$, $(1, 1)$, $(0, 2)$, and $(1, 2)$ (**Figure 5.6**) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

Note that we developed the concept of double integral using a rectangular region R. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into R, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region R. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid S approach 0 as m and n approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over R except on a finite number of smooth curves, then the double integral exists and we say that f is integrable over R.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\,dy$ or $dy\,dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint_R f(x, y)\,dA$ can be written as $\iint_R f(x, y)\,dx\,dy$ or

$$\iint_R f(x, y)dy\,dx.$$

Now let's list some of the properties that can be helpful to compute double integrals.

Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x, y)$ is a product of two functions $g(x)$ and $h(y)$.

Theorem 5.1: Properties of Double Integrals

Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region *R*; *S* and *T* are subregions of *R*; and assume that *m* and *M* are real numbers.

i. The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x, y) + g(x, y)]dA = \iint_R f(x, y)dA + \iint_R g(x, y)dA.$$

ii. If *c* is a constant, then $cf(x, y)$ is integrable and

$$\iint_R cf(x, y)dA = c\iint_R f(x, y)dA.$$

iii. If $R = S \cup T$ and $S \cap T = \varnothing$ except an overlap on the boundaries, then

$$\iint_R f(x, y)dA = \iint_S f(x, y)dA + \iint_T f(x, y)dA.$$

iv. If $f(x, y) \geq g(x, y)$ for (x, y) in $R,$ then

$$\iint_R f(x, y)dA \geq \iint_R g(x, y)dA.$$

v. If $m \leq f(x, y) \leq M,$ then

$$m \times A(R) \leq \iint_R f(x, y)dA \leq M \times A(R).$$

vi. In the case where $f(x, y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\},$ the double integral can be written as

$$\iint_R f(x, y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let's get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and $n.$ Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Definition

Assume $a, b, c,$ and d are real numbers. We define an **iterated integral** for a function $f(x, y)$ over the rectangular region $R = [a, b] \times [c, d]$ as

a.

$$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy\right] dx \tag{5.2}$$

b.

$$\int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx\right] dy. \tag{5.3}$$

The notation $\int_a^b \left[\int_c^d f(x, y)\,dy\right] dx$ means that we integrate $f(x, y)$ with respect to *y* while holding *x* constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x, y)\,dx\right] dy$ means that we integrate $f(x, y)$ with respect to *x* while holding *y* constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini's theorem. Think of this theorem as an essential tool for evaluating double integrals.

Theorem 5.2: Fubini's Theorem

Suppose that $f(x, y)$ is a function of two variables that is continuous over a rectangular region $R = \left\{(x, y) \in \mathbb{R}^2 \,|\, a \le x \le b,\ \mathrm{c} \le \mathrm{y} \le \mathrm{d}\right\}$. Then we see from **Figure 5.7** that the double integral of f over the region equals an iterated integral,

$$\iint\limits_R f(x, y)\,dA = \iint\limits_R f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy.$$

More generally, **Fubini's theorem** is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

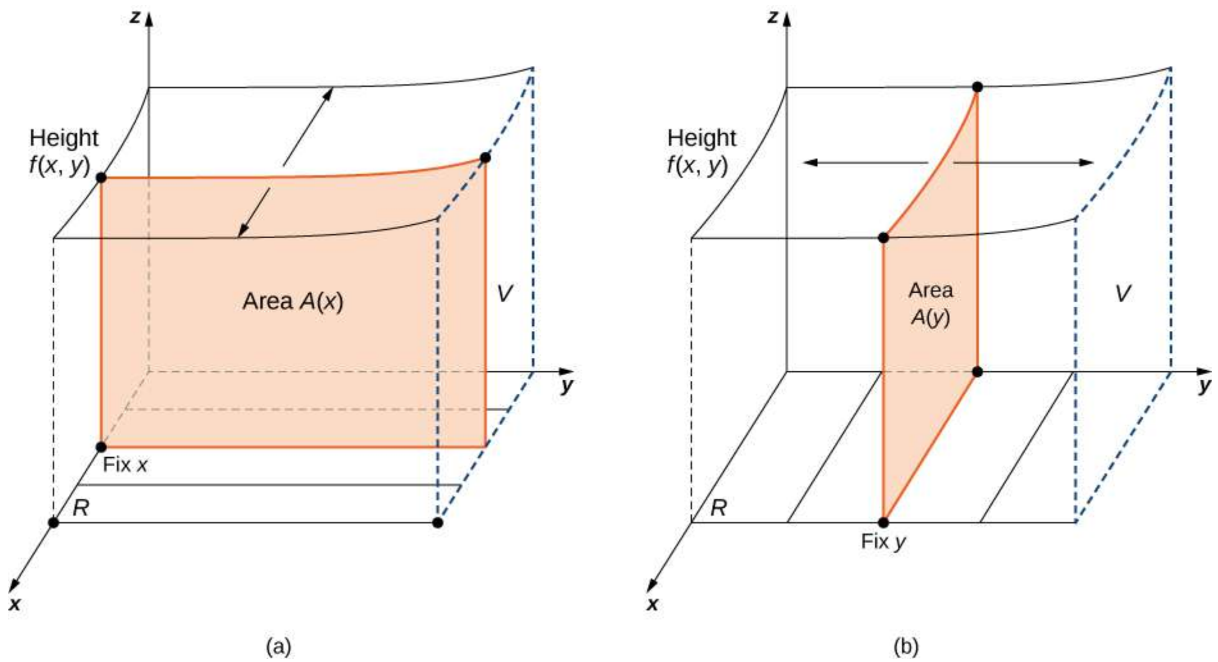


Figure 5.7 (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume *V*; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume *V*.

Example 5.2

Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint_R f(x, y)dA$ where $f(x, y) = x$ and $R = [0, 2] \times [0, 1]$.

Solution

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region *R* become the upper and lower limits of integration.

$$\begin{aligned} \iint_R f(x, y)dA &= \iint_R f(x, y)dx\,dy \\ &= \int_{y=0}^{y=1} \int_{x=0}^{x=2} x\,dx\,dy \\ &= \int_{y=0}^{y=1} \left[\frac{x^2}{2} \bigg|_{x=0}^{x=2} \right] dy \\ &= \int_{y=0}^{y=1} 2dy = 2y\big|_{y=0}^{y=1} = 2. \end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x, y)$ is more complex. Note that the order of integration can be changed (see **Example 5.7**).

Example 5.3

Illustrating Properties i and ii

Evaluate the double integral $\iint_R (xy - 3xy^2)dA$ where $R = \{(x, y)|0 \le x \le 2, 1 \le y \le 2\}$.

Solution

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

$$\iint_R (xy - 3xy^2)dA$$

$$= \iint_R xy\,dA + \iint_R (-3xy^2)dA$$ Property i: Integral of a sum is the sum of the integrals.

$$= \int_{y=1}^{y=2}\int_{x=0}^{x=2} xy\,dx\,dy - \int_{y=1}^{y=2}\int_{x=0}^{x=2} 3xy^2\,dx\,dy$$ Convert double integrals to iterated integrals.

$$= \int_{y=1}^{y=2}\left(\frac{x^2}{2}y\right)\Big|_{x=0}^{x=2} dy - 3\int_{y=1}^{y=2}\left(\frac{x^2}{2}y^2\right)\Big|_{x=0}^{x=2} dy$$ Integrate with respect to x, holding y constant.

$$= \int_{y=1}^{y=2} 2y\,dy - \int_{y=1}^{y=2} 6y^2\,dy$$ Property ii: Placing the constant before the integral.

$$= \int_1^2 y\,dy - 6\int_1^2 y^2\,dy$$ Integrate with respect to y.

$$= 2\frac{y^2}{2}\Big|_1^2 - 6\frac{y^3}{3}\Big|_1^2$$

$$= y^2\Big|_1^2 - 2y^3\Big|_1^2$$

$$= (4 - 1) - 2(8 - 1)$$
$$= 3 - 2(7) = 3 - 14 = -11.$$

Example 5.4

Illustrating Property v.

Over the region $R = \{(x, y)|1 \le x \le 3, 1 \le y \le 2\}$, we have $2 \le x^2 + y^2 \le 13$. Find a lower and an upper bound for the integral $\iint_R (x^2 + y^2)dA$.

Solution

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.

$$\int_1^2 \int_1^3 2dx\,dy = \int_1^2 \left[2x|_1^3\right]dy = \int_1^2 2(2)dy = 4y|_1^2 = 4(2-1) = 4$$
$$\int_1^2 \int_1^3 13dx\,dy = \int_1^2 \left[13x|_1^3\right]dy = \int_1^2 13(2)dy = 26y|_1^2 = 26(2-1) = 26.$$

Hence, we obtain $4 \le \iint_R \left(x^2 + y^2\right)dA \le 26.$

Example 5.5

Illustrating Property vi

Evaluate the integral $\iint_R e^y \cos x\, dA$ over the region $R = \left\{(x,\ y) \middle| 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1\right\}.$

Solution

This is a great example for property vi because the function $f(x,\ y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.

$$\begin{aligned}\iint_R e^y \cos x\, dA &= \int_0^1 \int_0^{\pi/2} e^y \cos x\, dx\, dy \\ &= \left(\int_0^1 e^y\, dy\right)\left(\int_0^{\pi/2} \cos x\, dx\right) \\ &= \left(e^y|_0^1\right)\left(\sin x|_0^{\pi/2}\right) \\ &= e - 1.\end{aligned}$$

5.2 a. Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3 - x + 4y)dy\, dx.$$

b. Show that $0 \le \iint_R \sin \pi x \cos \pi y\, dA \le \frac{1}{32}$ where $R = \left(0,\ \frac{1}{4}\right)\left(\frac{1}{4},\ \frac{1}{2}\right).$

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint_R f(x,\ y)dA$ can be written as $\iint_R f(x,\ y)dx\, dy$ or $\iint_R f(x,\ y)dy\, dx.$ The next example shows that the results are the same regardless of which order of integration we choose.

Example 5.6

Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x, y) = 3x^2 - y$ from **Example 5.1**, this time over the rectangular region $R = [0, 2] \times [0, 3]$. Use Fubini's theorem to evaluate $\iint_R f(x, y)dA$ in two different ways:

a. First integrate with respect to *y* and then with respect to *x*;

b. First integrate with respect to *x* and then with respect to *y*.

Solution

Figure 5.7 shows how the calculation works in two different ways.

a. First integrate with respect to *y* and then integrate with respect to *x*:

$$\begin{aligned}\iint_R f(x, y)dA &= \int_{x=0}^{x=2}\int_{y=0}^{y=3}(3x^2 - y)dy\,dx \\ &= \int_{x=0}^{x=2}\left(\int_{y=0}^{y=3}(3x^2 - y)dy\right)dx = \int_{x=0}^{x=2}\left[3x^2 y - \frac{y^2}{2}\Big|_{y=0}^{y=3}\right]dx \\ &= \int_{x=0}^{x=2}\left(9x^2 - \frac{9}{2}\right)dx = 3x^3 - \frac{9}{2}x\Big|_{x=0}^{x=2} = 15.\end{aligned}$$

b. First integrate with respect to *x* and then integrate with respect to *y*:

$$\begin{aligned}\iint_R f(x, y)dA &= \int_{y=0}^{y=3}\int_{x=0}^{x=2}(3x^2 - y)dx\,dy \\ &= \int_{y=0}^{y=3}\left(\int_{x=0}^{x=2}(3x^2 - y)dx\right)dy = \int_{y=0}^{y=3}\left[x^3 - xy\Big|_{x=0}^{x=2}\right]dy \\ &= \int_{y=0}^{y=3}(8 - 2y)dy = 8y - y^2\Big|_{y=0}^{y=3} = 15.\end{aligned}$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x, y) = 3x^2 - y$ over the region $R = [0, 2] \times [0, 3]$. However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

5.3 Evaluate $\int_{y=-3}^{y=2}\int_{x=3}^{x=5}\left(2 - 3x^2 + y^2\right)dx\,dy.$

In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

Example 5.7

Switching the Order of Integration

Consider the double integral $\iint_R x\sin(xy)dA$ over the region $R = \{(x, y)|0 \le x \le 3,\ 0 \le y \le 2\}$ (**Figure 5.8**).

a. Express the double integral in two different ways.

b. Analyze whether evaluating the double integral in one way is easier than the other and why.

c. Evaluate the integral.

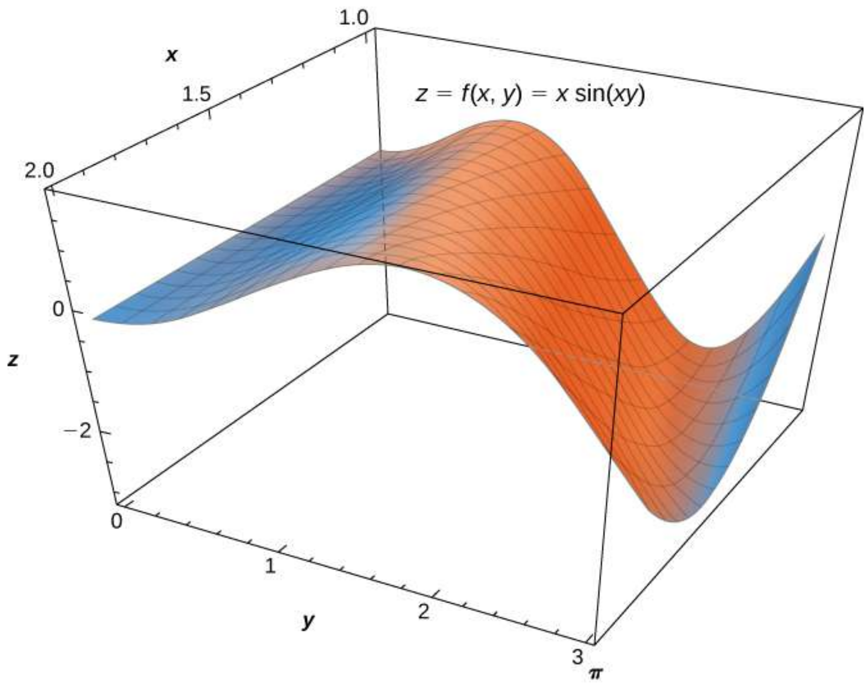


Figure 5.8 The function $z = f(x, y) = x \sin(xy)$ over the rectangular region $R = [0, \pi] \times [1, 2]$.

Solution

a. We can express $\iint\limits_R x \sin(xy)\,dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.

$$\iint\limits_R x \sin(xy)\,dA$$

$$= \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x \sin(xy)\,dy\,dx \qquad \text{Integrate fir t with respect to } y.$$

$$= \int_{y=1}^{y=2} \int_{x=0}^{x=\pi} x \sin(xy)\,dx\,dy \qquad \text{Integrate fir t with respect to } x.$$

b. If we want to integrate with respect to y first and then integrate with respect to x, we see that we can use the substitution $u = xy$, which gives $du = x\,dy$. Hence the inner integral is simply $\int \sin u\,du$ and we can change the limits to be functions of x,

$$\iint\limits_R x \sin(xy)\,dA = \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x \sin(xy)\,dy\,dx = \int_{x=0}^{x=\pi} \left[\int_{u=x}^{u=2x} \sin(u)\,du \right] dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u = x$ and $dv = \sin(xy)dx$.

Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so

$$\iint_R x\sin(xy)dA = \int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\sin(xy)dx\,dy = \int_{y=1}^{y=2}\left[-\frac{x\cos(xy)}{y}\Big|_{x=0}^{x=\pi} + \frac{1}{y}\int_{x=0}^{x=\pi}\cos(xy)dx\right]dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.

$$\begin{aligned}\iint_R x\sin(xy)dA &= \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\sin(xy)dy\,dx \\ &= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u|_{u=x}^{u=2x}\right]dx = \int_{x=0}^{x=\pi}(-\cos 2x + \cos x)dx \\ &= -\frac{1}{2}\sin 2x + \sin x\Big|_{x=0}^{x=\pi} = 0.\end{aligned}$$

5.4 Evaluate the integral $\iint_R xe^{xy}\,dA$ where $R = [0,\ 1] \times [0,\ \ln 5]$.

Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by integrating the constant function $f(x,\ y) = 1$ over the region R.

Definition

The area of the region R is given by $A(R) = \iint_R 1dA.$

This definition makes sense because using $f(x,\ y) = 1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

Example 5.8

Finding Area Using a Double Integral

Find the area of the region $R = \{(x,\ y)|0 \le x \le 3,\ 0 \le y \le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:

$$A(R) = \int_0^2 \int_0^3 1 dx\, dy = \int_0^2 \left[x|_0^3\right] dy = \int_0^2 3 dy = 3\int_0^2 dy = 3y|_0^2 = 3(2) = 6.$$

We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x, y)$ over a region R provided $f(x, y) \geq 0$ for all (x, y) in $R.$ Here is another example to illustrate this concept.

Example 5.9

Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27,$ the planes $x = 3$ and $y = 3,$ and the three coordinate planes.

Solution

First notice the graph of the surface $z = 27 - 2x^2 - y^2$ in **Figure 5.9**(a) and above the square region $R_1 = [-3, 3] \times [-3, 3].$ However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27,$ the planes $x = 3$ and $y = 3,$ and the three coordinate planes.

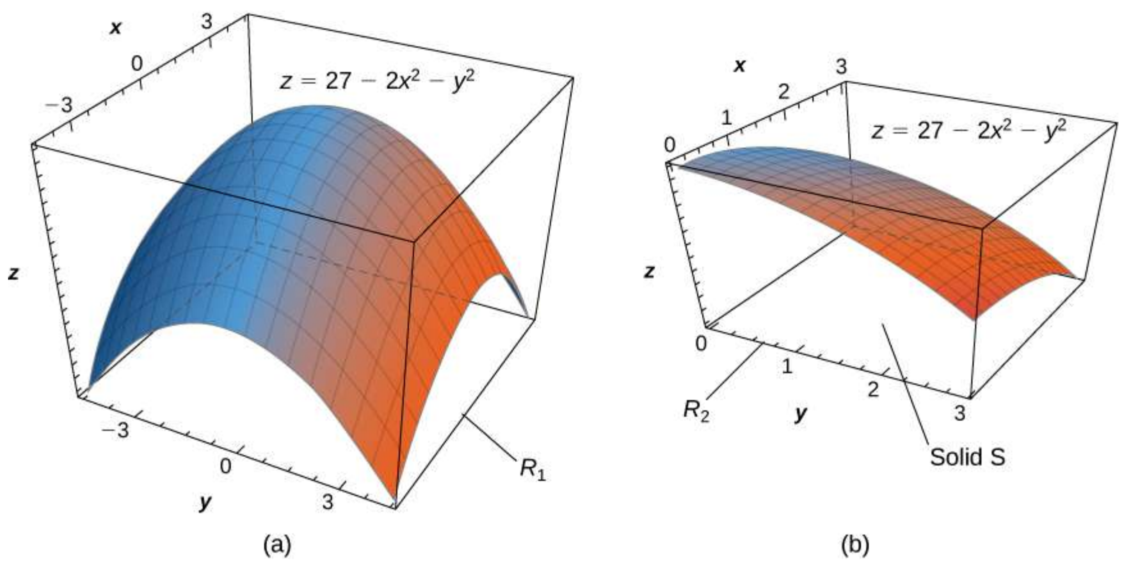


Figure 5.9 (a) The surface $z = 27 - 2x^2 - y^2$ above the square region $R_1 = [-3, 3] \times [-3, 3].$ (b) The solid S lies under the surface $z = 27 - 2x^2 - y^2$ above the square region $R_2 = [0, 3] \times [0, 3].$

Now let's look at the graph of the surface in **Figure 5.9**(b). We determine the volume V by evaluating the double integral over R_2:

$$\begin{aligned} V &= \iint_R z\,dA = \iint_R \left(27 - 2x^2 - y^2\right) dA \\ &= \int_{y=0}^{y=3} \int_{x=0}^{x=3} \left(27 - 2x^2 - y^2\right) dx\,dy && \text{Convert to iterated integral.} \\ &= \int_{y=0}^{y=3} \left[27x - \tfrac{2}{3}x^3 - y^2 x\right]\Big|_{x=0}^{x=3} dy && \text{Integrate with respect to } x. \\ &= \int_{y=0}^{y=3} \left(64 - 3y^2\right) dy = 63y - y^3\Big|_{y=0}^{y=3} = 162. \end{aligned}$$

5.5 Find the volume of the solid bounded above by the graph of $f(x, y) = xy \sin(x^2 y)$ and below by the xy-plane on the rectangular region $R = [0, 1] \times [0, \pi]$.

Recall that we defined the average value of a function of one variable on an interval $[a, b]$ as

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)dx.$$

Similarly, we can define the average value of a function of two variables over a region R. The main difference is that we divide by an area instead of the width of an interval.

Definition

The average value of a function of two variables over a region R is

$$f_{\text{ave}} = \frac{1}{\text{Area } R} \iint_R f(x, y)dA. \tag{5.4}$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

Example 5.10

Calculating Average Storm Rainfall

The weather map in **Figure 5.10** shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.

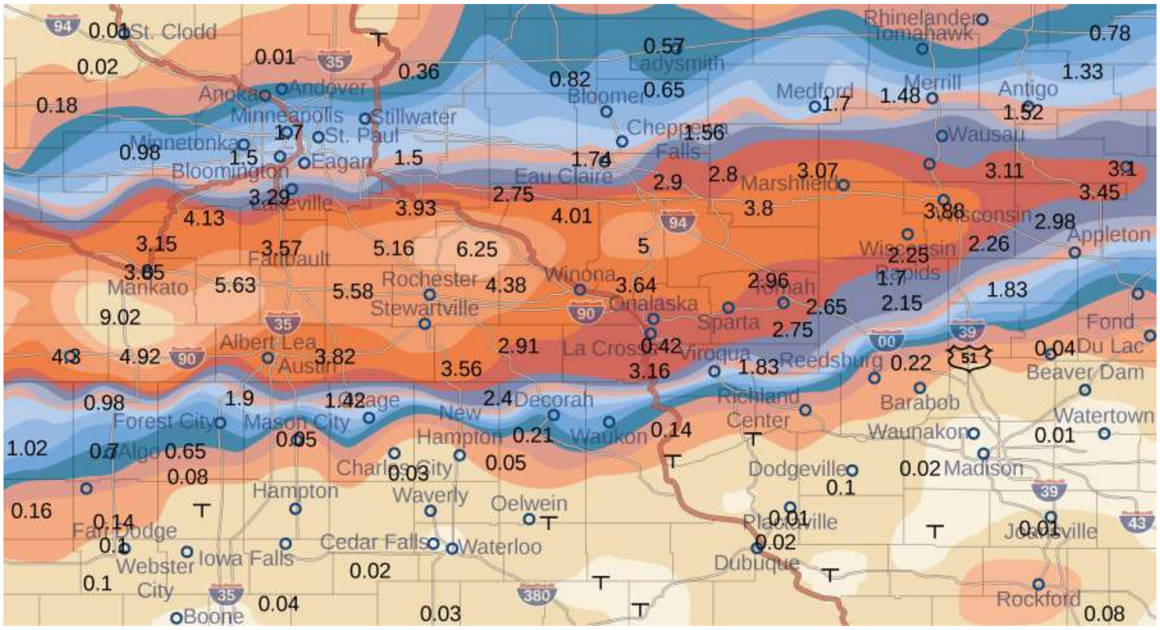


Figure 5.10 Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles $(m = 2 \text{ and } n = 3)$, as shown in **Figure 5.11**. Assume $f(x, y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $250 \times 300 = 75000$ square miles. Then the area of each subrectangle is

$$\Delta A = \frac{1}{6}(75000) = 12500.$$

Assume (x_{ij}^*, y_{ij}^*) are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

At (x_{11}, y_{11}) the rainfall is 0.08.

At (x_{12}, y_{12}) the rainfall is 0.08.

At (x_{13}, y_{13}) the rainfall is 0.01.

At (x_{21}, y_{21}) the rainfall is 1.70.

At (x_{22}, y_{22}) the rainfall is 1.74.

At (x_{23}, y_{23}) the rainfall is 3.00.

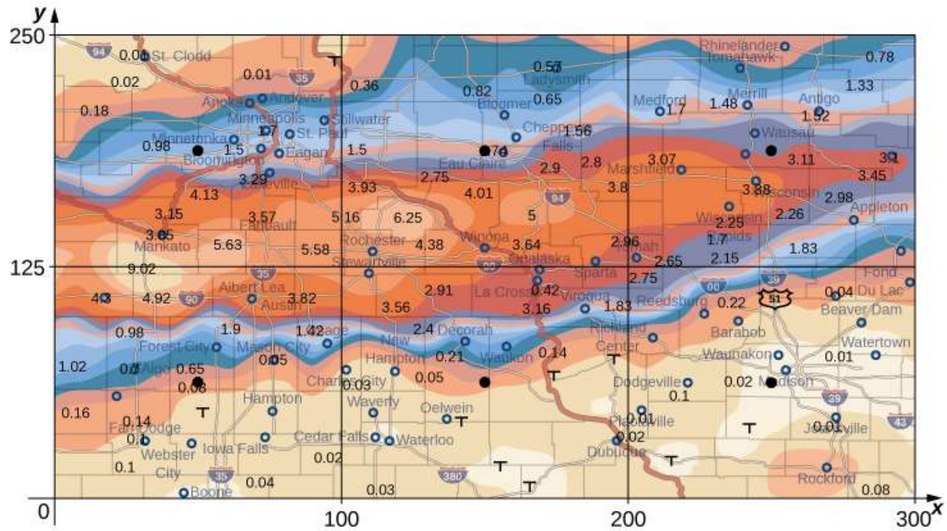


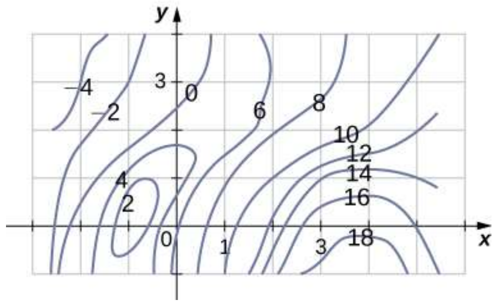
Figure 5.11 Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.

According to our definition, the average storm rainfall in the entire area during those two days was

$$
\begin{aligned}
f_{\text{ave}} &= \frac{1}{\text{Area } R} \iint\limits_R f(x, y)\,dx\,dy = \frac{1}{75000} \iint\limits_R f(x, y)\,dx\,dy \\
&\cong \frac{1}{75{,}000} \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&\cong \frac{1}{75{,}000}\Big[f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A \\
&\quad + f(x_{13}^*, y_{13}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A\Big] \\
&\cong \frac{1}{75{,}000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]\Delta A \\
&\cong \frac{1}{75{,}000}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00]12500 \\
&\cong \frac{5}{30}[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00] \\
&\cong 1.10.
\end{aligned}
$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

5.6 A contour map is shown for a function $f(x, y)$ on the rectangle $R = [-3, 6] \times [-1, 4]$.



a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint\limits_R f(x, y)dA$.

b. Estimate the average value of the function $f(x, y)$.

5.1 EXERCISES

In the following exercises, use the midpoint rule with $m = 4$ and $n = 2$ to estimate the volume of the solid bounded by the surface $z = f(x, y)$, the vertical planes $x = 1$, $x = 2$, $y = 1$, and $y = 2$, and the horizontal plane $z = 0$.

1. $f(x, y) = 4x + 2y + 8xy$

2. $f(x, y) = 16x^2 + \frac{y}{2}$

In the following exercises, estimate the volume of the solid under the surface $z = f(x, y)$ and above the rectangular region R by using a Riemann sum with $m = n = 2$ and the sample points to be the lower left corners of the subrectangles of the partition.

3. $f(x, y) = \sin x - \cos y, \quad R = [0, \pi] \times [0, \pi]$

4. $f(x, y) = \cos x + \cos y, \quad R = [0, \pi] \times \left[0, \frac{\pi}{2}\right]$

5. Use the midpoint rule with $m = n = 2$ to estimate $\iint\limits_R f(x, y)dA,$ where the values of the function f on $R = [8, 10] \times [9, 11]$ are given in the following table.

	y				
x	9	9.5	10	10.5	11
8	9.8	5	6.7	5	5.6
8.5	9.4	4.5	8	5.4	3.4
9	8.7	4.6	6	5.5	3.4
9.5	6.7	6	4.5	5.4	6.7
10	6.8	6.4	5.5	5.7	6.8

6. The values of the function f on the rectangle $R = [0, 2] \times [7, 9]$ are given in the following table. Estimate the double integral $\iint\limits_R f(x, y)dA$ by using a Riemann sum with $m = n = 2$. Select the sample points to be the upper right corners of the subsquares of R.

	$y_0 = 7$	$y_1 = 8$	$y_2 = 9$
$x_0 = 0$	10.22	10.21	9.85
$x_1 = 1$	6.73	9.75	9.63
$x_2 = 2$	5.62	7.83	8.21

7. The depth of a children's 4-ft by 4-ft swimming pool, measured at 1-ft intervals, is given in the following table.
   a. Estimate the volume of water in the swimming pool by using a Riemann sum with $m = n = 2$. Select the sample points using the midpoint rule on $R = [0, 4] \times [0, 4]$.
   b. Find the average depth of the swimming pool.

	y				
x	0	1	2	3	4
0	1	1.5	2	2.5	3
1	1	1.5	2	2.5	3
2	1	1.5	1.5	2.5	3
3	1	1	1.5	2	2.5
4	1	1	1	1.5	2

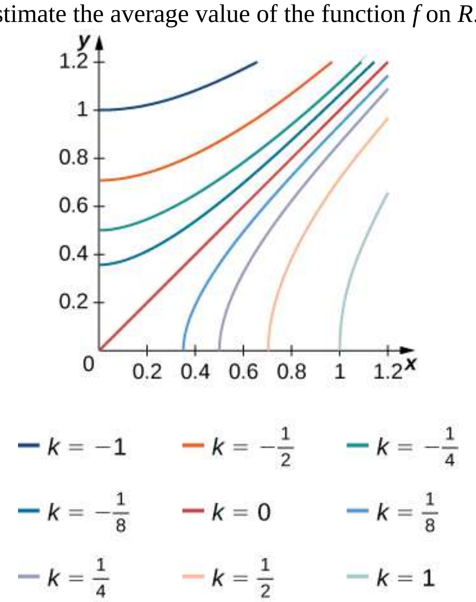
8. The depth of a 3-ft by 3-ft hole in the ground, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of the hole by using a Riemann sum with $m = n = 3$ and the sample points to be the upper left corners of the subsquares of R.
b. Find the average depth of the hole.

	y			
x	0	1	2	3
0	6	6.5	6.4	6
1	6.5	7	7.5	6.5
2	6.5	6.7	6.5	6
3	6	6.5	5	5.6

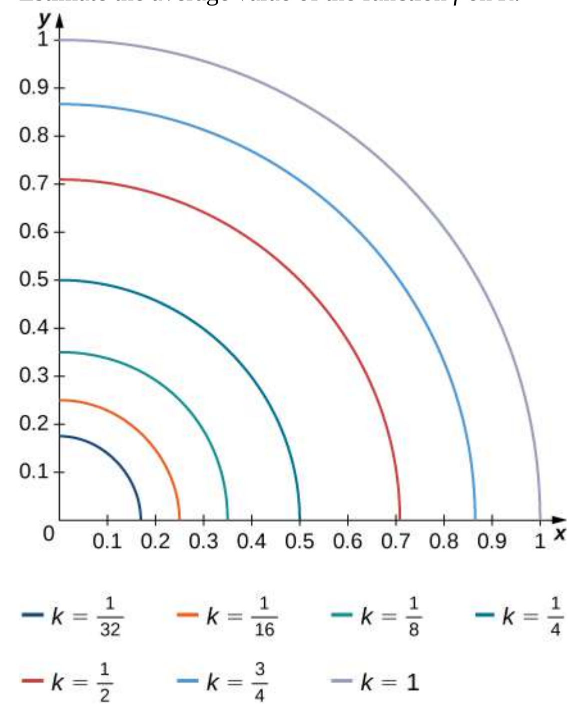
9. The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)dA$, where $R = [0.2, 1] \times [0, 0.8]$.
b. Estimate the average value of the function f on R.

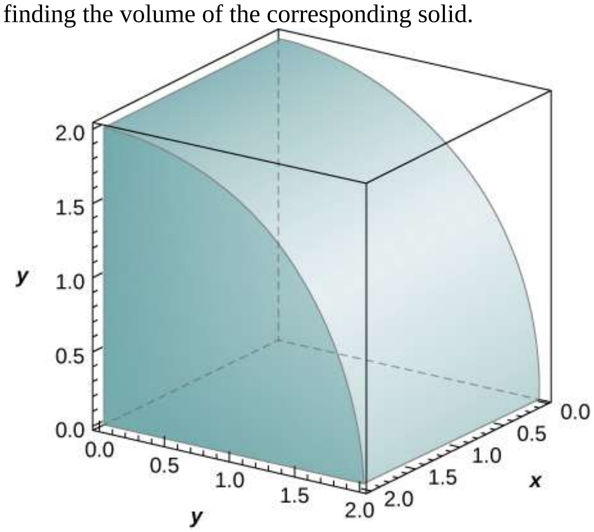


10. The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

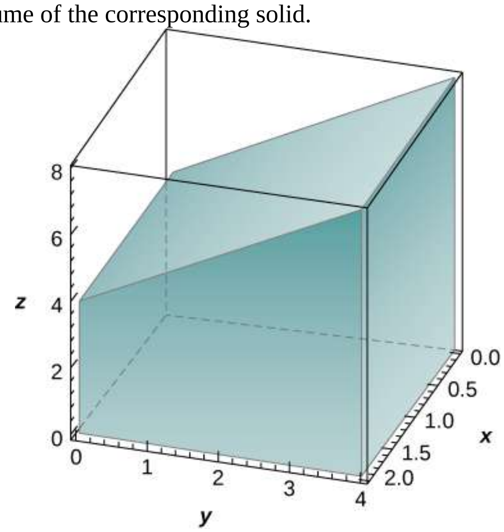
a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)dA$, where $R = [0.1, 0.5] \times [0.1, 0.5]$.
b. Estimate the average value of the function f on R.



11. The solid lying under the surface $z = \sqrt{4 - y^2}$ and above the rectangular region $R = [0, 2] \times [0, 2]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)dA$, where $f(x, y) = \sqrt{4 - y^2}$, by finding the volume of the corresponding solid.



12. The solid lying under the plane $z = y + 4$ and above the rectangular region $R = [0, 2] \times [0, 4]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)dA$, where $f(x, y) = y + 4$, by finding the volume of the corresponding solid.



In the following exercises, calculate the integrals by interchanging the order of integration.

13. $\int_{-1}^{1}\left(\int_{-2}^{2}(2x + 3y + 5)dx\right)dy$

14. $\int_{0}^{2}\left(\int_{0}^{1}(x + 2e^{y} - 3)dx\right)dy$

15. $\int_{1}^{27}\left(\int_{1}^{2}(\sqrt[3]{x} + \sqrt[3]{y})dy\right)dx$

16. $\int_{1}^{16}\left(\int_{1}^{8}(\sqrt[4]{x} + 2\sqrt[3]{y})dy\right)dx$

17. $\int_{\ln 2}^{\ln 3}\left(\int_{0}^{1}e^{x+y}\,dy\right)dx$

18. $\int_{0}^{2}\left(\int_{0}^{1}3^{x+y}\,dy\right)dx$

19. $\int_{1}^{6}\left(\int_{2}^{9}\frac{\sqrt{y}}{x^2}dy\right)dx$

20. $\int_{1}^{9}\left(\int_{4}^{2}\frac{\sqrt{x}}{y^2}dy\right)dx$

In the following exercises, evaluate the iterated integrals by choosing the order of integration.

21. $\int_{0}^{\pi}\int_{0}^{\pi/2}\sin(2x)\cos(3y)dx\,dy$

22. $\int_{\pi/12}^{\pi/8}\int_{\pi/4}^{\pi/3}[\cot x + \tan(2y)]dx\,dy$

23. $\int_{1}^{e}\int_{1}^{e}\left[\frac{1}{x}\sin(\ln x) + \frac{1}{y}\cos(\ln y)\right]dx\,dy$

24. $\int_{1}^{e}\int_{1}^{e}\frac{\sin(\ln x)\cos(\ln y)}{xy}dx\,dy$

25. $\int_{1}^{2}\int_{1}^{2}\left(\frac{\ln y}{x} + \frac{x}{2y + 1}\right)dy\,dx$

26. $\int_{1}^{e}\int_{1}^{2}x^2\ln(x)dy\,dx$

27. $\int_{1}^{\sqrt{3}}\int_{1}^{2}y\arctan\left(\frac{1}{x}\right)dy\,dx$

28. $\int_{0}^{1}\int_{0}^{1/2}(\arcsin x + \arcsin y)dy\,dx$

29. $\int_{0}^{1}\int_{1}^{2}xe^{x+4y}dy\,dx$

30. $\int_{1}^{2}\int_{0}^{1}xe^{x-y}dy\,dx$

31. $\int_{1}^{e}\int_{1}^{e}\left(\frac{\ln y}{\sqrt{y}} + \frac{\ln x}{\sqrt{x}}\right)dy\,dx$

32. $\int_{1}^{e}\int_{1}^{e}\left(\frac{x\ln y}{\sqrt{y}} + \frac{y\ln x}{\sqrt{x}}\right)dy\,dx$

33. $\int_{0}^{1}\int_{1}^{2}\left(\frac{x}{x^2 + y^2}\right)dy\,dx$

34. $\int_{0}^{1}\int_{1}^{2}\frac{y}{x + y^2}dy\,dx$

In the following exercises, find the average value of the function over the given rectangles.

35. $f(x, y) = -x + 2y, \quad R = [0, 1] \times [0, 1]$

36. $f(x, y) = x^4 + 2y^3, \quad R = [1, 2] \times [2, 3]$

37. $f(x, y) = \sinh x + \sinh y, \quad R = [0, 1] \times [0, 2]$

38. $f(x, y) = \arctan(xy), \quad R = [0, 1] \times [0, 1]$

39. Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a, b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c, d]$. Show that the following inequality is true: $m_1 m_2(b-a)(c-d) \le \int_a^b \int_c^d f(x)g(y)dy\,dx \le M_1 M_2(b-a)(c-d)$.

In the following exercises, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

40. $\frac{1}{e^2} \le \iint_R e^{-x^2-y^2}\,dA \le 1$, where $R = [0, 1] \times [0, 1]$

41. $\frac{\pi^2}{144} \le \iint_R \sin x \cos y\,dA \le \frac{\pi^2}{48}$, where $R = \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \times \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

42. $0 \le \iint_R e^{-y} \cos x\,dA \le \frac{\pi}{2}$, where $R = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$

43. $0 \le \iint_R (\ln x)(\ln y)dA \le (e-1)^2$, where $R = [1, e] \times [1, e]$

44. Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a, b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c, d]$. Show that the following inequality is true: $(m_1 + m_2)(b-a)(c-d) \le \int_a^b \int_c^d [f(x) + g(y)]dy\,dx \le (M_1 + M_2)(b-a)(c-d)$.

In the following exercises, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

45. $\frac{2}{e} \le \iint_R \left(e^{-x^2} + e^{-y^2}\right)dA \le 2$, where $R = [0, 1] \times [0, 1]$

46. $\frac{\pi^2}{36} \le \iint_R (\sin x + \cos y)dA \le \frac{\pi^2\sqrt{3}}{36}$, where $R = \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \times \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

47. $\frac{\pi}{2}e^{-\pi/2} \le \iint_R \left(\cos x + e^{-y}\right)dA \le \pi$, where $R = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$

48. $\frac{1}{e} \le \iint_R \left(e^{-y} - \ln x\right)dA \le 2$, where $R = [0, 1] \times [0, 1]$

In the following exercises, the function f is given in terms of double integrals.

a. Determine the explicit form of the function f.
b. Find the volume of the solid under the surface $z = f(x, y)$ and above the region R.
c. Find the average value of the function f on R.
d. Use a computer algebra system (CAS) to plot $z = f(x, y)$ and $z = f_{\text{ave}}$ in the same system of coordinates.

49. **[T]** $f(x, y) = \int_0^y \int_0^x (xs + yt)ds\,dt$, where $(x, y) \in R = [0, 1] \times [0, 1]$

50. **[T]** $f(x, y) = \int_0^x \int_0^y [\cos(s) + \cos(t)]dt\,ds$, where $(x, y) \in R = [0, 3] \times [0, 3]$

51. Show that if f and g are continuous on $[a, b]$ and $[c, d]$, respectively, then

$$\int_a^b \int_c^d [f(x) + g(y)]dy\,dx = (d-c)\int_a^b f(x)dx + \int_a^b \int_c^d g(y)dy\,dx = (b-a)\int_c^d g(y)dy + \int_c^d \int_a^b f(x)dx\,dy.$$

52. Show that $\int_a^b \int_c^d yf(x) + xg(y)dy\,dx = \frac{1}{2}\left(d^2 - c^2\right)\left(\int_a^b f(x)dx\right) + \frac{1}{2}\left(b^2 - a^2\right)\left(\int_c^d g(y)dy\right)$.

53. **[T]** Consider the function $f(x, y) = e^{-x^2 - y^2}$, where $(x, y) \in R = [-1, 1] \times [-1, 1]$.

a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R e^{-x^2 - y^2} dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of f over the region R. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R e^{-x^2 - y^2} dA$ and the plane $z = f_{\text{ave}}$.

54. **[T]** Consider the function $f(x, y) = \sin\left(x^2\right)\cos\left(y^2\right)$, where $(x, y) \in R = [-1, 1] \times [-1, 1]$.

a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R \sin\left(x^2\right)\cos\left(y^2\right)dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of f over the region R. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R \sin\left(x^2\right)\cos\left(y^2\right)dA$ and the plane $z = f_{\text{ave}}$.

In the following exercises, the functions f_n are given, where $n \geq 1$ is a natural number.

a. Find the volume of the solids S_n under the surfaces $z = f_n(x, y)$ and above the region R.

b. Determine the limit of the volumes of the solids S_n as n increases without bound.

55. $f(x, y) = x^n + y^n + xy, \ (x, y) \in R = [0, 1] \times [0, 1]$

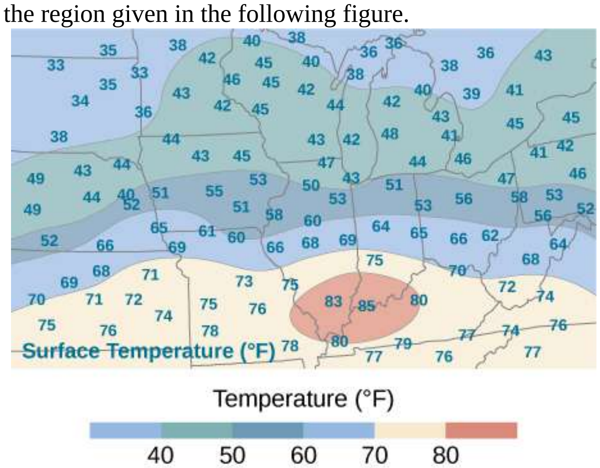
56. $f(x, y) = \dfrac{1}{x^n} + \dfrac{1}{y^n}, \ (x, y) \in R = [1, 2] \times [1, 2]$

57. Show that the average value of a function f on a rectangular region $R = [a, b] \times [c, d]$ is $f_{\text{ave}} \approx \dfrac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)$, where $\left(x_{ij}^*, y_{ij}^*\right)$ are the sample points of the partition of R, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

58. Use the midpoint rule with $m = n$ to show that the average value of a function f on a rectangular region $R = [a, b] \times [c, d]$ is approximated by

$$f_{\text{ave}} \approx \frac{1}{n^2}\sum_{i,\ j=1}^{n} f\left(\frac{1}{2}(x_{i-1} + x_i), \ \frac{1}{2}\left(y_{j-1} + y_j\right)\right).$$

59. An isotherm map is a chart connecting points having the same temperature at a given time for a given period of time. Use the preceding exercise and apply the midpoint rule with $m = n = 2$ to find the average temperature over the region given in the following figure.



5.2 | Double Integrals over General Regions

Learning Objectives

5.2.1 Recognize when a function of two variables is integrable over a general region.

5.2.2 Evaluate a double integral by computing an iterated integral over a region bounded by two vertical lines and two functions of x, or two horizontal lines and two functions of y.

5.2.3 Simplify the calculation of an iterated integral by changing the order of integration.

5.2.4 Use double integrals to calculate the volume of a region between two surfaces or the area of a plane region.

5.2.5 Solve problems involving double improper integrals.

In **Double Integrals over Rectangular Regions**, we studied the concept of double integrals and examined the tools needed to compute them. We learned techniques and properties to integrate functions of two variables over rectangular regions. We also discussed several applications, such as finding the volume bounded above by a function over a rectangular region, finding area by integration, and calculating the average value of a function of two variables.

In this section we consider double integrals of functions defined over a general bounded region D on the plane. Most of the previous results hold in this situation as well, but some techniques need to be extended to cover this more general case.

General Regions of Integration

An example of a general bounded region D on a plane is shown in **Figure 5.12**. Since D is bounded on the plane, there must exist a rectangular region R on the same plane that encloses the region D, that is, a rectangular region R exists such that D is a subset of $R(D \subseteq R)$.

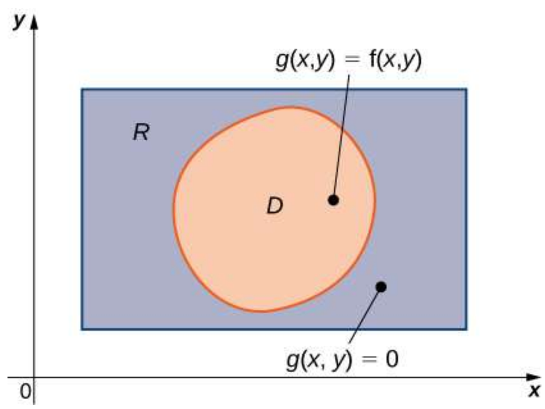


Figure 5.12 For a region D that is a subset of R, we can define a function $g(x, y)$ to equal $f(x, y)$ at every point in D and 0 at every point of R not in D.

Suppose $z = f(x, y)$ is defined on a general planar bounded region D as in **Figure 5.12**. In order to develop double integrals of f over D, we extend the definition of the function to include all points on the rectangular region R and then use the concepts and tools from the preceding section. But how do we extend the definition of f to include all the points on R? We do this by defining a new function $g(x, y)$ on R as follows:

$$g(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Note that we might have some technical difficulties if the boundary of D is complicated. So we assume the boundary to be a piecewise smooth and continuous simple closed curve. Also, since all the results developed in **Double Integrals over Rectangular Regions** used an integrable function $f(x, y)$, we must be careful about $g(x, y)$ and verify that

$g(x, y)$ is an integrable function over the rectangular region R. This happens as long as the region D is bounded by simple closed curves. For now we will concentrate on the descriptions of the regions rather than the function and extend our theory appropriately for integration.

We consider two types of planar bounded regions.

Definition

A region D in the (x, y)-plane is of **Type I** if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is (**Figure 5.13**),

$$D = \{(x, y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

A region D in the xy plane is of **Type II** if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is (**Figure 5.14**),

$$D = \{(x, y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}.$$

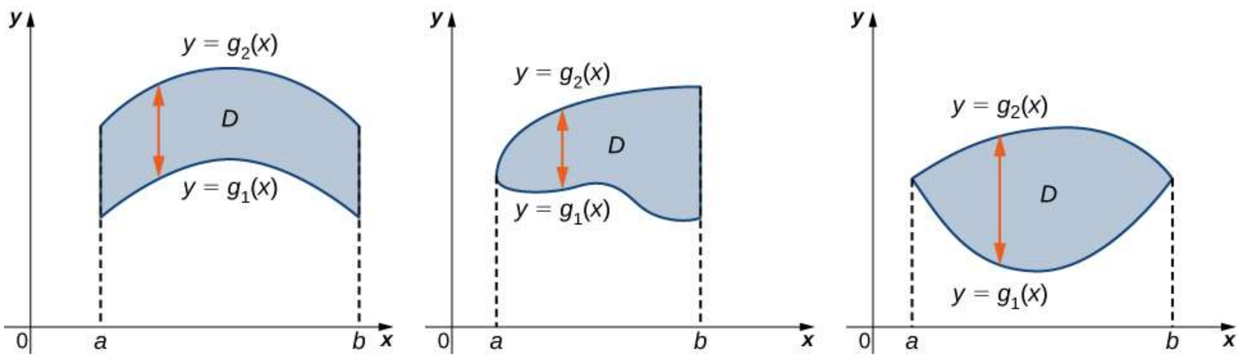


Figure 5.13 A Type I region lies between two vertical lines and the graphs of two functions of x.

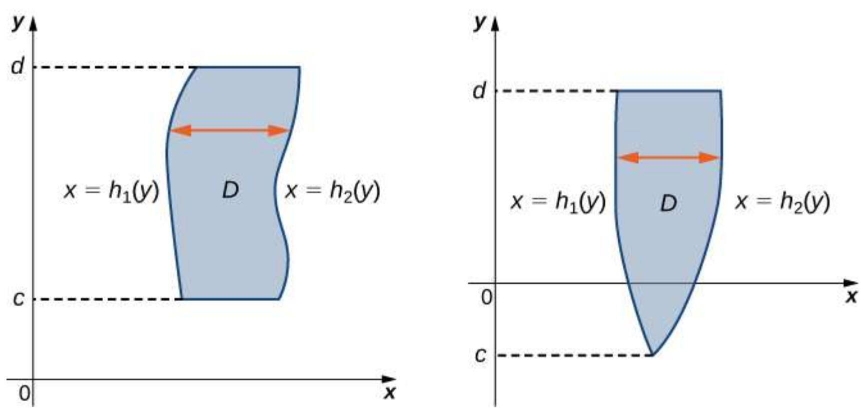


Figure 5.14 A Type II region lies between two horizontal lines and the graphs of two functions of y.

Example 5.11

Describing a Region as Type I and Also as Type II

Consider the region in the first quadrant between the functions $y = \sqrt{x}$ and $y = x^3$ (**Figure 5.15**). Describe the region first as Type I and then as Type II.

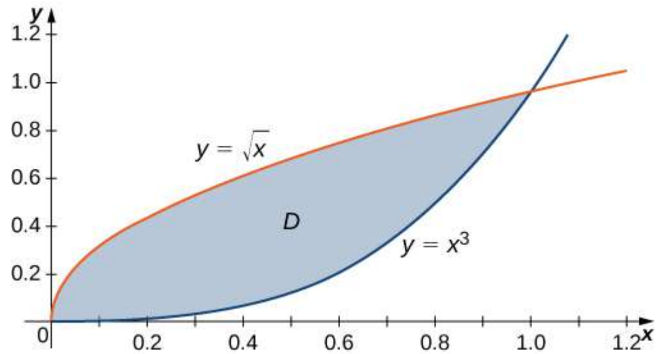


Figure 5.15 Region D can be described as Type I or as Type II.

Solution

When describing a region as Type I, we need to identify the function that lies above the region and the function that lies below the region. Here, region D is bounded above by $y = \sqrt{x}$ and below by $y = x^3$ in the interval for x in $[0, 1]$. Hence, as Type I, D is described as the set $\{(x, y)|0 \le x \le 1, x^3 \le y \le \sqrt{x}\}$.

However, when describing a region as Type II, we need to identify the function that lies on the left of the region and the function that lies on the right of the region. Here, the region D is bounded on the left by $x = y^2$ and on the right by $x = \sqrt[3]{y}$ in the interval for y in $[0, 1]$. Hence, as Type II, D is described as the set $\{(x, y)|0 \le y \le 1, y^2 \le x \le \sqrt[3]{y}\}$.

5.7 Consider the region in the first quadrant between the functions $y = 2x$ and $y = x^2$. Describe the region first as Type I and then as Type II.

Double Integrals over Nonrectangular Regions

To develop the concept and tools for evaluation of a double integral over a general, nonrectangular region, we need to first understand the region and be able to express it as Type I or Type II or a combination of both. Without understanding the regions, we will not be able to decide the limits of integrations in double integrals. As a first step, let us look at the following theorem.

Theorem 5.3: Double Integrals over Nonrectangular Regions

Suppose $g(x, y)$ is the extension to the rectangle R of the function $f(x, y)$ defined on the regions D and R as shown in **Figure 5.12** inside R. Then $g(x, y)$ is integrable and we define the double integral of $f(x, y)$ over D by

$$\iint_D f(x, y)dA = \iint_R g(x, y)dA.$$

The right-hand side of this equation is what we have seen before, so this theorem is reasonable because R is a rectangle and $\iint_R g(x, y)dA$ has been discussed in the preceding section. Also, the equality works because the values of $g(x, y)$ are 0 for any point (x, y) that lies outside D, and hence these points do not add anything to the integral. However, it is important that the rectangle R contains the region D.

As a matter of fact, if the region D is bounded by smooth curves on a plane and we are able to describe it as Type I or Type II or a mix of both, then we can use the following theorem and not have to find a rectangle R containing the region.

Theorem 5.4: Fubini's Theorem (Strong Form)

For a function $f(x, y)$ that is continuous on a region D of Type I, we have

$$\iint_D f(x, y)dA = \iint_D f(x, y)dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)dy\right]dx. \tag{5.5}$$

Similarly, for a function $f(x, y)$ that is continuous on a region D of Type II, we have

$$\iint_D f(x, y)dA = \iint_D f(x, y)dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)dx\right]dy. \tag{5.6}$$

The integral in each of these expressions is an iterated integral, similar to those we have seen before. Notice that, in the inner integral in the first expression, we integrate $f(x, y)$ with x being held constant and the limits of integration being $g_1(x)$ and $g_2(x)$. In the inner integral in the second expression, we integrate $f(x, y)$ with y being held constant and the limits of integration are $h_1(x)$ and $h_2(x)$.

Example 5.12

Evaluating an Iterated Integral over a Type I Region

Evaluate the integral $\iint_D x^2 e^{xy}\,dA$ where D is shown in **Figure 5.16**.

Solution

First construct the region D as a Type I region (**Figure 5.16**). Here $D = \left\{(x, y)|0 \le x \le 2, \frac{1}{2}x \le y \le 1\right\}$. Then we have

$$\iint_D x^2 e^{xy}\,dA = \int_{x=0}^{x=2} \int_{y=1/2x}^{y=1} x^2 e^{xy}\,dy\,dx.$$

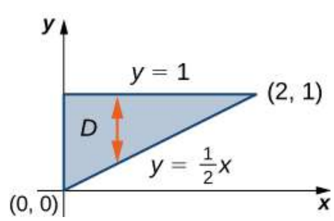


Figure 5.16 We can express region D as a Type I region and integrate from $y = \frac{1}{2}x$ to $y = 1$, between the lines $x = 0$ and $x = 2$.

Therefore, we have

$$\int_{x=0}^{x=2}\int_{y=\frac{1}{2}x}^{y=1} x^2 e^{xy}\,dy\,dx = \int_{x=0}^{x=2}\left[\int_{y=1/2x}^{y=1} x^2 e^{xy}\,dy\right]dx$$

Iterated integral for a Type I region.

$$= \int_{x=0}^{x=2}\left[x^2 \frac{e^{xy}}{x}\right]\Bigg|_{y=1/2x}^{y=1} dx$$

Integrate with respect to y using u-substitution with $u = xy$ where x is held constant.

$$= \int_{x=0}^{x=2}\left[xe^x - xe^{x^2/2}\right]dx$$

Integrate with respect to x using u-substitution with $u = \frac{1}{2}x^2$.

$$= \left[xe^x - e^x - e^{\frac{1}{2}x^2}\right]\Bigg|_{x=0}^{x=2} = 2$$

In **Example 5.12**, we could have looked at the region in another way, such as $D = \{(x, y)|0 \le y \le 1, 0 \le x \le 2y\}$ (**Figure 5.17**).

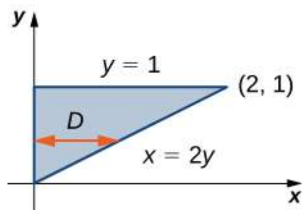


Figure 5.17

This is a Type II region and the integral would then look like

$$\iint_D x^2 e^{xy}\,dA = \int_{y=0}^{y=1}\int_{x=0}^{x=2y} x^2 e^{xy}\,dx\,dy.$$

However, if we integrate first with respect to x, this integral is lengthy to compute because we have to use integration by parts twice.

Example 5.13

Evaluating an Iterated Integral over a Type II Region

Evaluate the integral $\iint_D \left(3x^2 + y^2\right)dA$ where $= \left\{(x, y)| -2 \le y \le 3, y^2 - 3 \le x \le y + 3\right\}$.

Solution

Notice that D can be seen as either a Type I or a Type II region, as shown in **Figure 5.18**. However, in this case describing D as Type I is more complicated than describing it as Type II. Therefore, we use D as a Type II region for the integration.

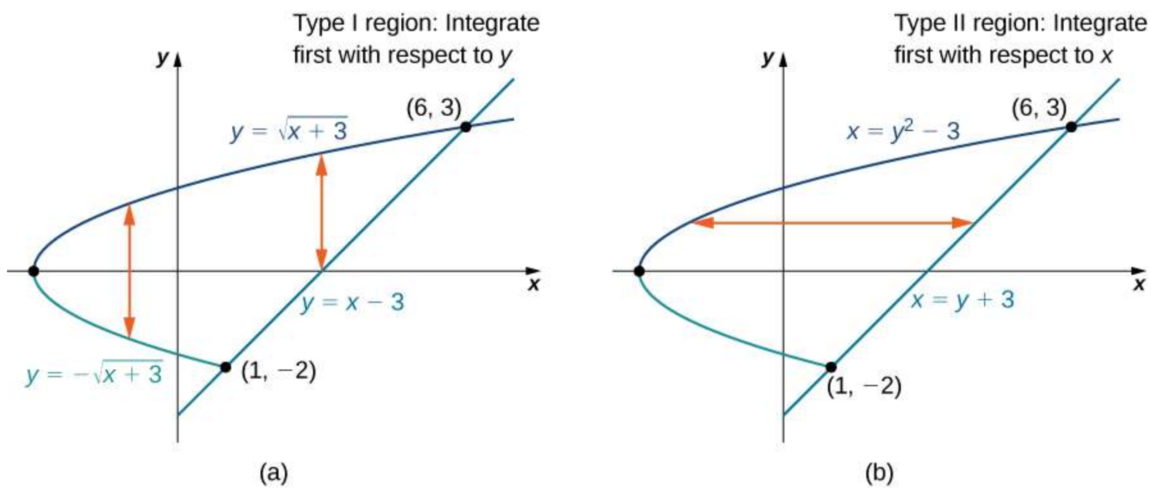


Figure 5.18 The region D in this example can be either (a) Type I or (b) Type II.

Choosing this order of integration, we have

$$\begin{aligned}
\iint_D (3x^2+y^2)\,dA &= \int_{y=-2}^{y=3}\int_{x=y^2-3}^{x=y+3} (3x^2+y^2)\,dx\,dy && \text{Iterated integral, Type II region.}\\
&= \int_{y=-2}^{y=3} \left(x^3+xy^2\right)\Big|_{y^2-3}^{y+3} dy && \text{Integrate with respect to } x.\\
&= \int_{y=-2}^{y=3} \left((y+3)^3+(y+3)y^2-(y^2-3)^3-(y^2-3)y^2\right)dy\\
&= \int_{-2}^{3} (54+27y-12y^2+2y^3+8y^4-y^6)\,dy && \text{Integrate with respect to } y.\\
&= \left[54y+\frac{27y^2}{2}-4y^3+\frac{y^4}{2}+\frac{8y^5}{5}-\frac{y^7}{7}\right]_{-2}^{3}\\
&= \frac{2375}{7}.
\end{aligned}$$

5.8 Sketch the region D and evaluate the iterated integral $\iint_D xy\,dy\,dx$ where D is the region bounded by the curves $y=\cos x$ and $y=\sin x$ in the interval $[-3\pi/4,\ \pi/4]$.

Recall from **Double Integrals over Rectangular Regions** the properties of double integrals. As we have seen from the examples here, all these properties are also valid for a function defined on a nonrectangular bounded region on a plane. In particular, property 3 states:

If $R = S \cup T$ and $S \cap T = \varnothing$ except at their boundaries, then

$$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA.$$

Similarly, we have the following property of double integrals over a nonrectangular bounded region on a plane.

Theorem 5.5: Decomposing Regions into Smaller Regions

Suppose the region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint_D f(x, y)dA = \iint_{D_1} f(x, y)dA + \iint_{D_2} f(x, y)dA. \tag{5.7}$$

This theorem is particularly useful for nonrectangular regions because it allows us to split a region into a union of regions of Type I and Type II. Then we can compute the double integral on each piece in a convenient way, as in the next example.

Example 5.14

Decomposing Regions

Express the region D shown in **Figure 5.19** as a union of regions of Type I or Type II, and evaluate the integral

$$\iint_D (2x + 5y)dA.$$

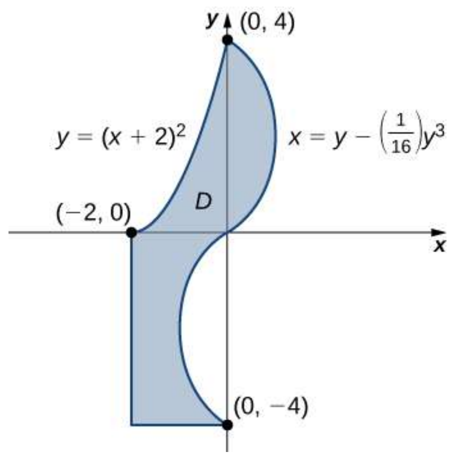


Figure 5.19 This region can be decomposed into a union of three regions of Type I or Type II.

Solution

The region D is not easy to decompose into any one type; it is actually a combination of different types. So we can write it as a union of three regions D_1, D_2, and D_3 where, $D_1 = \left\{(x, y) | -2 \le x \le 0, \ 0 \le y \le (x + 2)^2\right\}$, $D_2 = \left\{(x, y) | 0 \le y \le 4, \ 0 \le x \le \left(y - \frac{1}{16}y^3\right)\right\}$. These regions are illustrated more clearly in **Figure 5.20**.

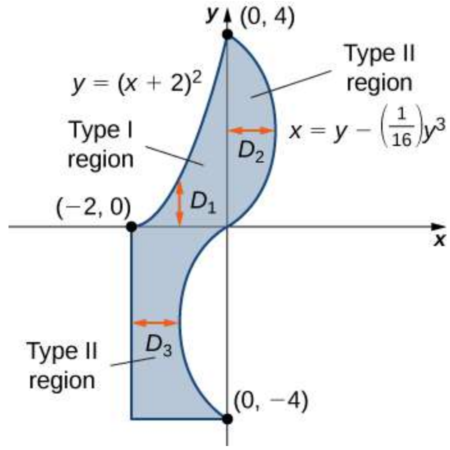


Figure 5.20 Breaking the region into three subregions makes it easier to set up the integration.

Here D_1 is Type I and D_2 and D_3 are both of Type II. Hence,

$$\begin{aligned}
\iint_D (2x+5y)dA &= \iint_{D_1} (2x+5y)dA + \iint_{D_2} (2x+5y)dA + \iint_{D_3} (2x+5y)dA \\
&= \int_{x=-2}^{x=0} \int_{y=0}^{y=(x+2)^2} (2x+5y)dy\,dx + \int_{y=0}^{y=4} \int_{x=0}^{x=y-(1/16)y^3} (2+5y)dx\,dy + \int_{y=-4}^{y=0} \int_{x=-2}^{x=y-(1/16)y^3} (2x+5y)dx\,dy \\
&= \int_{x=-2}^{x=0} \left[\frac{1}{2}(2+x)^2(20+24x+5x^2)\right] + \int_{y=0}^{y=4} \left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2\right] \\
&\quad + \int_{y=-4}^{y=0} \left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2 + 10y - 4\right] \\
&= \frac{40}{3} + \frac{1664}{35} - \frac{1696}{35} = \frac{1304}{105}.
\end{aligned}$$

Now we could redo this example using a union of two Type II regions (see the Checkpoint).

5.9 Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1, 2]$. Decompose the region into smaller regions of Type II.

5.10 Redo **Example 5.14** using a union of two Type II regions.

Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

Example 5.15

Changing the Order of Integration

Reverse the order of integration in the iterated integral $\int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=2-x^2} xe^{x^2}\, dy\, dx$. Then evaluate the new iterated integral.

Solution

The region as presented is of Type I. To reverse the order of integration, we must first express the region as Type II. Refer to **Figure 5.21**.

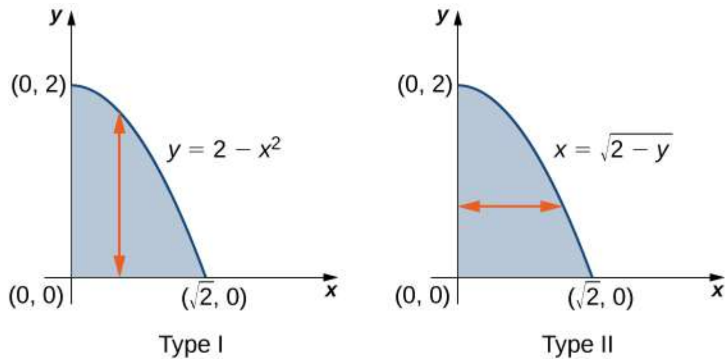


Figure 5.21 Converting a region from Type I to Type II.

We can see from the limits of integration that the region is bounded above by $y = 2 - x^2$ and below by $y = 0,$ where x is in the interval $\left[0, \sqrt{2}\right]$. By reversing the order, we have the region bounded on the left by $x = 0$ and on the right by $x = \sqrt{2 - y}$ where y is in the interval $[0, 2]$. We solved $y = 2 - x^2$ in terms of x to obtain $x = \sqrt{2 - y}$.

Hence

$$\begin{aligned} \int_0^{\sqrt{2}} \int_0^{2-x^2} xe^{x^2}\, dy\, dx &= \int_0^2 \int_0^{\sqrt{2-y}} xe^{x^2}\, dx\, dy \\ &= \int_0^2 \left[\frac{1}{2}ex^2\Big|_0^{\sqrt{2-y}}\right] dy = \int_0^2 \frac{1}{2}\left(e^{2-y} - 1\right) dy = -\frac{1}{2}\left(e^{2-y} + y\right)\Big|_0^2 \\ &= \frac{1}{2}\left(e^2 - 3\right). \end{aligned}$$

Reverse the order of integration then use substitution.

Example 5.16

Evaluating an Iterated Integral by Reversing the Order of Integration

Consider the iterated integral $\iint_R f(x, y)\, dx\, dy$ where $z = f(x, y) = x - 2y$ over a triangular region R that has sides on $x = 0,\ y = 0,$ and the line $x + y = 1$. Sketch the region, and then evaluate the iterated integral by

a. integrating first with respect to y and then

b. integrating first with respect to x.

Solution

A sketch of the region appears in **Figure 5.22**.

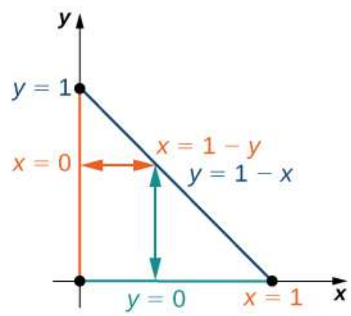


Figure 5.22 A triangular region R for integrating in two ways.

We can complete this integration in two different ways.

a. One way to look at it is by first integrating y from $y=0$ to $y=1-x$ vertically and then integrating x from $x=0$ to $x=1$:

$$\begin{aligned}\iint_R f(x,\,y)dx\,dy &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}(x-2y)dy\,dx = \int_{x=0}^{x=1}\left[xy-2y^2\right]_{y=0}^{y=1-x}dx \\ &= \int_{x=0}^{x=1}\left[x(1-x)-(1-x)^2\right]dx = \int_{x=0}^{x=1}\left[-1+3x-2x^2\right]dx = \left[-x+\frac{3}{2}x^2-\frac{2}{3}x^3\right]_{x=0}^{x=1} = -\frac{1}{6}.\end{aligned}$$

b. The other way to do this problem is by first integrating x from $x=0$ to $x=1-y$ horizontally and then integrating y from $y=0$ to $y=1$:

$$\begin{aligned}\iint_R f(x,\,y)dx\,dy &= \int_{y=0}^{y=1}\int_{x=0}^{x=1-y}(x-2y)dx\,dy = \int_{y=0}^{y=1}\left[\frac{1}{2}x^2-2xy\right]_{x=0}^{x=1-y}dy \\ &= \int_{y=0}^{y=1}\left[\frac{1}{2}(1-y)^2-2y(1-y)\right]dy = \int_{y=0}^{y=1}\left[\frac{1}{2}-3y+\frac{5}{2}y^2\right]dy \\ &= \left[\frac{1}{2}y-\frac{3}{2}y^2+\frac{5}{6}y^3\right]_{y=0}^{y=1} = -\frac{1}{6}.\end{aligned}$$

5.11 Evaluate the iterated integral $\iint_D \left(x^2+y^2\right)dA$ over the region D in the first quadrant between the functions $y=2x$ and $y=x^2$. Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.

Calculating Volumes, Areas, and Average Values

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in **Double Integrals over Rectangular Regions**, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

Example 5.17

Finding the Volume of a Tetrahedron

Find the volume of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0,$ and $2x + 3y + z = 6.$

Solution

The solid is a tetrahedron with the base on the xy-plane and a height $z = 6 - 2x - 3y.$ The base is the region D bounded by the lines, $x = 0,\ y = 0$ and $2x + 3y = 6$ where $z = 0$ (**Figure 5.23**). Note that we can consider the region D as Type I or as Type II, and we can integrate in both ways.

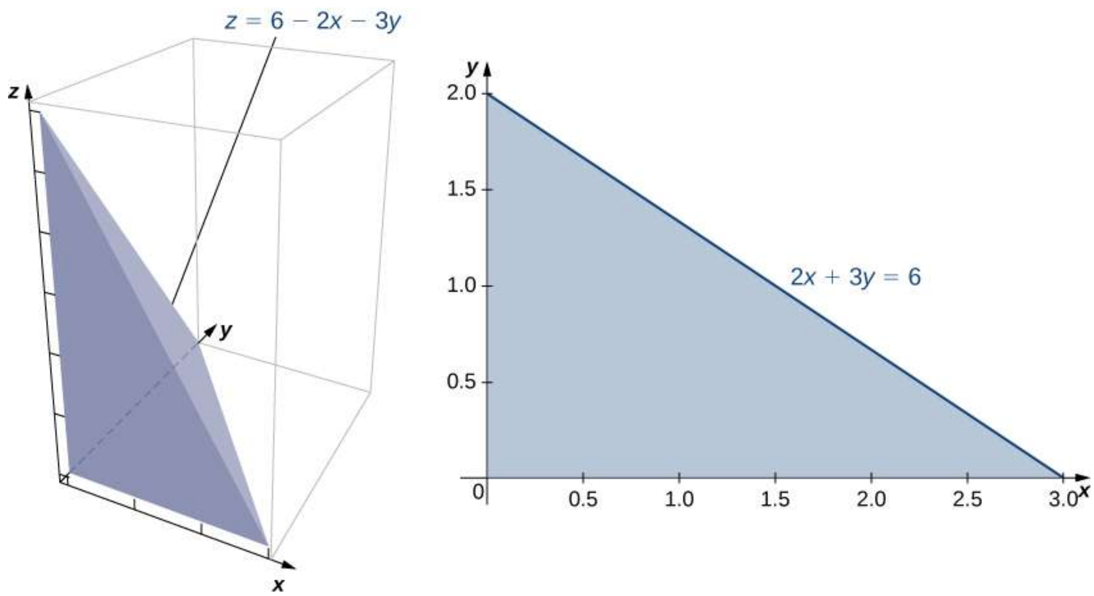


Figure 5.23 A tetrahedron consisting of the three coordinate planes and the plane $z = 6 - 2x - 3y,$ with the base bound by $x = 0,\ y = 0,$ and $2x + 3y = 6.$

First, consider D as a Type I region, and hence $D = \left\{(x, y)|0 \le x \le 3,\ 0 \le y \le 2 - \frac{2}{3}x\right\}.$

Therefore, the volume is

$$\begin{aligned} V &= \int_{x=0}^{x=3} \int_{y=0}^{y=2-(2x/3)} (6 - 2x - 3y)dy\,dx = \int_{x=0}^{x=3} \left[\left(6y - 2xy - \frac{3}{2}y^2\right)\Big|_{y=0}^{y=2-(2x/3)}\right]dx \\ &= \int_{x=0}^{x=3} \left[\frac{2}{3}(x-3)^2\right]dx = 6. \end{aligned}$$

Now consider D as a Type II region, so $D = \left\{(x, y)\middle|0 \le y \le 2,\ 0 \le x \le 3 - \frac{3}{2}y\right\}.$ In this calculation, the volume is

$$V = \int_{y=0}^{y=2} \int_{x=0}^{x=3-(3y/2)} (6 - 2x - 3y)dx\,dy = \int_{y=0}^{y=2} \left[\left(6x - x^2 - 3xy\right)\Big|_{x=0}^{x=3-(3y/2)} \right] dy$$
$$= \int_{y=0}^{y=2} \left[\frac{9}{4}(y-2)^2 \right] dy = 6.$$

Therefore, the volume is 6 cubic units.

5.12 Find the volume of the solid bounded above by $f(x, y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$, where x is in the interval $[0, 1]$.

Finding the area of a rectangular region is easy, but finding the area of a nonrectangular region is not so easy. As we have seen, we can use double integrals to find a rectangular area. As a matter of fact, this comes in very handy for finding the area of a general nonrectangular region, as stated in the next definition.

Definition

The area of a plane-bounded region D is defined as the double integral $\iint_D 1dA$.

We have already seen how to find areas in terms of single integration. Here we are seeing another way of finding areas by using double integrals, which can be very useful, as we will see in the later sections of this chapter.

Example 5.18

Finding the Area of a Region

Find the area of the region bounded below by the curve $y = x^2$ and above by the line $y = 2x$ in the first quadrant (**Figure 5.24**).

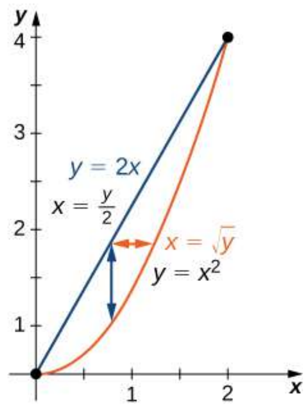


Figure 5.24 The region bounded by $y = x^2$ and $y = 2x$.

Solution

We just have to integrate the constant function $f(x, y) = 1$ over the region. Thus, the area A of the bounded region is $\int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} dy\,dx$ or $\int_{y=0}^{y=4}\int_{x=y/2}^{x=\sqrt{y}} dx\,dy$:

$$A = \iint_D 1\,dx\,dy = \int_{x=0}^{x=2}\int_{y=x^2}^{y=2x} 1\,dy\,dx = \int_{x=0}^{x=2}\left[y\Big|_{y=x^2}^{y=2x}\right]dx = \int_{x=0}^{x=2}\left(2x - x^2\right)dx = x^2 - \frac{x^3}{3}\Bigg|_0^2 = \frac{4}{3}.$$

5.13 Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

We can also use a double integral to find the average value of a function over a general region. The definition is a direct extension of the earlier formula.

Definition

If $f(x, y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is

$$f_{ave} = \frac{1}{A(D)}\iint_D f(x, y)\,dA.$$

Note that the area is $A(D) = \iint_D 1\,dA$.

Example 5.19

Finding an Average Value

Find the average value of the function $f(x, y) = 7xy^2$ on the region bounded by the line $x = y$ and the curve $x = \sqrt{y}$ (Figure 5.25).

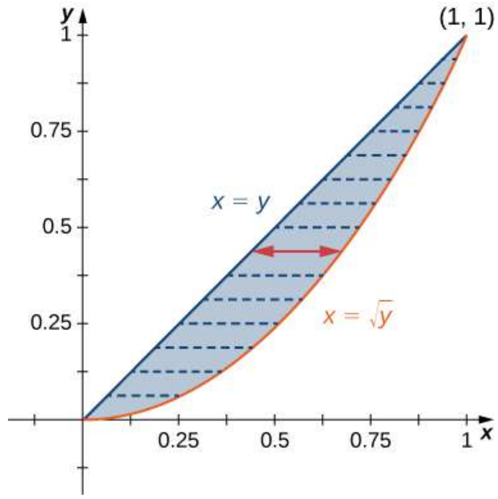


Figure 5.25 The region bounded by $x = y$ and $x = \sqrt{y}$.

Solution

First find the area $A(D)$ where the region D is given by the figure. We have

$$A(D) = \iint_D 1dA = \int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} 1dx\,dy = \int_{y=0}^{y=1}\left[x\Big|_{x=y}^{x=\sqrt{y}}\right]dy = \int_{y=0}^{y=1}(\sqrt{y} - y)dy = \frac{2}{3}y^{3/2} - \frac{y^2}{2}\Big|_0^1 = \frac{1}{6}.$$

Then the average value of the given function over this region is

$$\begin{aligned} f_{ave} &= \frac{1}{A(D)}\iint_D f(x, y)dA = \frac{1}{A(D)}\int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} 7xy^2\,dx\,dy = \frac{1}{1/6}\int_{y=0}^{y=1}\left[\frac{7}{2}x^2 y^2\Big|_{x=y}^{x=\sqrt{y}}\right]dy \\ &= 6\int_{y=0}^{y=1}\left[\frac{7}{2}y^2\left(y - y^2\right)\right]dy = 6\int_{y=0}^{y=1}\left[\frac{7}{2}\left(y^3 - y^4\right)\right]dy = \frac{42}{2}\left(\frac{y^4}{4} - \frac{y^5}{5}\right)\Big|_0^1 = \frac{42}{40} = \frac{21}{20}. \end{aligned}$$

5.14 Find the average value of the function $f(x, y) = xy$ over the triangle with vertices $(0, 0), (1, 0)$ and $(1, 3)$.

Improper Double Integrals

An **improper double integral** is an integral $\iint_D f\,dA$ where either D is an unbounded region or f is an unbounded function. For example, $D = \{(x, y)\,|\,|x - y| \geq 2\}$ is an unbounded region, and the function $f(x, y) = 1/\left(1 - x^2 - 2y^2\right)$ over the ellipse $x^2 + 3y^2 \leq 1$ is an unbounded function. Hence, both of the following integrals are improper integrals:

i. $\iint_D xy\,dA$ where $D = \{(x, y)\,|\,|x - y| \geq 2\}$;

ii. $\iint_D \frac{1}{1 - x^2 - 2y^2}dA$ where $D = \left\{(x, y)\,|\,x^2 + 3y^2 \leq 1\right\}$.

In this section we would like to deal with improper integrals of functions over rectangles or simple regions such that f has only finitely many discontinuities. Not all such improper integrals can be evaluated; however, a form of Fubini's theorem does apply for some types of improper integrals.

Theorem 5.6: Fubini's Theorem for Improper Integrals

If D is a bounded rectangle or simple region in the plane defined by $\{(x, y): a \leq x \leq b, g(x) \leq y \leq h(x)\}$ and also by $\{(x, y): c \leq y \leq d, j(y) \leq x \leq k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D, then

$$\iint_D f\, dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x, y) dy\, dx = \int_{y=c}^{y=d} \int_{x=j(y)}^{x=k(y)} f(x, y) dx\, dy.$$

It is very important to note that we required that the function be nonnegative on D for the theorem to work. We consider only the case where the function has finitely many discontinuities inside D.

Example 5.20

Evaluating a Double Improper Integral

Consider the function $f(x, y) = \frac{e^y}{y}$ over the region $D = \{(x, y): 0 \leq x \leq 1, x \leq y \leq \sqrt{x}\}$.

Notice that the function is nonnegative and continuous at all points on D except $(0, 0)$. Use Fubini's theorem to evaluate the improper integral.

Solution

First we plot the region D (**Figure 5.26**); then we express it in another way.

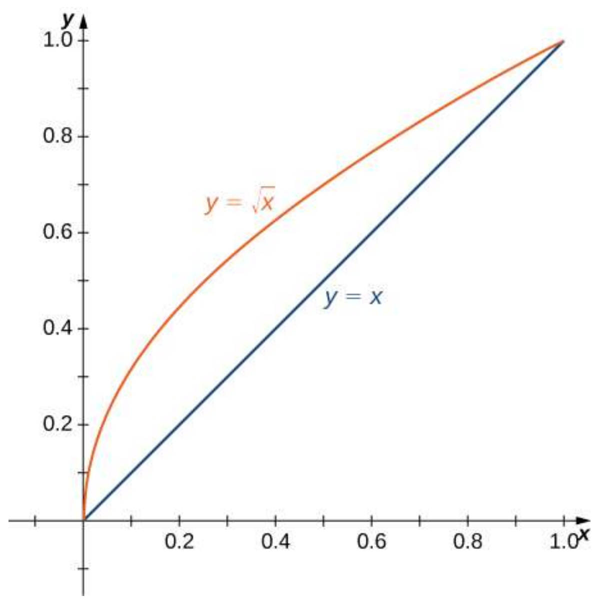


Figure 5.26 The function f is continuous at all points of the region D except $(0, 0)$.

The other way to express the same region D is

$$D = \left\{(x, y): 0 \leq y \leq 1, y^2 \leq x \leq y\right\}.$$

Thus we can use Fubini's theorem for improper integrals and evaluate the integral as

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y} dx\, dy.$$

Therefore, we have

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y} dx\, dy = \int_{y=0}^{y=1} \frac{e^y}{y} x \Big|_{x=y^2}^{x=y} dy = \int_{y=0}^{y=1} \frac{e^y}{y}\left(y - y^2\right) dy = \int_0^1 \left(e^y - ye^y\right) dy = e - 2.$$

As mentioned before, we also have an improper integral if the region of integration is unbounded. Suppose now that the function f is continuous in an unbounded rectangle R.

Theorem 5.7: Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R = \{(x, y) : a \le x \le \infty, c \le y \le \infty\}$, then when the limit exists, we have

$$\iint_R f(x, y) dA = \lim_{(b, d) \to (\infty, \infty)} \int_a^b \left(\int_c^d f(x, y) dy \right) dx = \lim_{(b, d) \to (\infty, \infty)} \int_c^d \left(\int_a^b f(x, y) dy \right) dy.$$

The following example shows how this theorem can be used in certain cases of improper integrals.

Example 5.21

Evaluating a Double Improper Integral

Evaluate the integral $\iint_R xye^{-x^2 - y^2} dA$ where R is the first quadrant of the plane.

Solution

The region R is the first quadrant of the plane, which is unbounded. So

$$\begin{aligned}\iint_R xye^{-x^2 - y^2} dA &= \lim_{(b, d) \to (\infty, \infty)} \int_{x=0}^{x=b} \left(\int_{y=0}^{y=d} xye^{-x^2 - y^2} dy \right) dx = \lim_{(b, d) \to (\infty, \infty)} \int_{y=0}^{y=d} \left(\int_{x=0}^{x=b} xye^{-x^2 - y^2} dy \right) dy \\ &= \lim_{(b, d) \to (\infty, \infty)} \frac{1}{4}\left(1 - e^{-b^2}\right)\left(1 - e^{-d^2}\right) = \frac{1}{4}\end{aligned}$$

Thus, $\iint_R xye^{-x^2 - y^2} dA$ is convergent and the value is $\frac{1}{4}$.

5.15 Evaluate the improper integral $\iint_D \frac{y}{\sqrt{1 - x^2 - y^2}} dA$ where $D = \left\{(x, y) x \ge 0, y \ge 0, x^2 + y^2 \le 1\right\}$.

In some situations in probability theory, we can gain insight into a problem when we are able to use double integrals over general regions. Before we go over an example with a double integral, we need to set a few definitions and become familiar with some important properties.

Definition

Consider a pair of continuous random variables X and Y, such as the birthdays of two people or the number of sunny and rainy days in a month. The joint density function f of X and Y satisfies the probability that (X, Y) lies in a certain region D:

$$P((X, Y) \in D) = \iint\limits_D f(x, y)dA.$$

Since the probabilities can never be negative and must lie between 0 and 1, the joint density function satisfies the following inequality and equation:

$$f(x, y) \geq 0 \text{ and } \iint\limits_{R^2} f(x, y)dA = 1.$$

Definition

The variables X and Y are said to be independent random variables if their joint density function is the product of their individual density functions:

$$f(x, y) = f_1(x)f_2(y).$$

Example 5.22

Application to Probability

At Sydney's Restaurant, customers must wait an average of 15 minutes for a table. From the time they are seated until they have finished their meal requires an additional 40 minutes, on average. What is the probability that a customer spends less than an hour and a half at the diner, assuming that waiting for a table and completing the meal are independent events?

Solution

Waiting times are mathematically modeled by exponential density functions, with m being the average waiting time, as

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{1}{m}e^{-t/m} & \text{if } t \geq 0. \end{cases}$$

If X and Y are random variables for 'waiting for a table' and 'completing the meal,' then the probability density functions are, respectively,

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{15}e^{-x/15} & \text{if } x \geq 0. \end{cases} \text{ and } f_2(y) = \begin{cases} 0 & \text{if } y < 0, \\ \frac{1}{40}e^{-y/40} & \text{if } y \geq 0. \end{cases}$$

Clearly, the events are independent and hence the joint density function is the product of the individual functions

$$f(x, y) = f_1(x)f_2(y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0, \\ \frac{1}{600}e^{-x/15}e^{-y/60} & \text{if } x, y \geq 0. \end{cases}$$

We want to find the probability that the combined time $X + Y$ is less than 90 minutes. In terms of geometry, it means that the region D is in the first quadrant bounded by the line $x + y = 90$ (Figure 5.27).

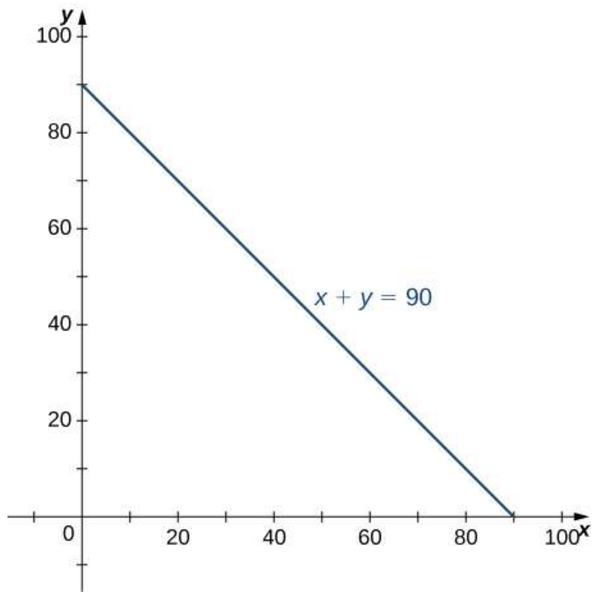


Figure 5.27 The region of integration for a joint probability density function.

Hence, the probability that (X, Y) is in the region D is

$$P(X + Y \leq 90) = P((X, Y) \in D) = \iint_D f(x, y)dA = \iint_D \frac{1}{600}e^{-x/15}e^{-y/40}dA.$$

Since $x + y = 90$ is the same as $y = 90 - x$, we have a region of Type I, so

$$\begin{aligned} D &= \{(x, y) | 0 \leq x \leq 90, 0 \leq y \leq 90 - x\}, \\ P(X + Y \leq 90) &= \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-x/15}e^{-y/40}dx\,dy = \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-x/15}e^{-y/40}dx\,dy \\ &= \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-(x/15 + y/40)}dx\,dy = 0.8328. \end{aligned}$$

Thus, there is an 83.2% chance that a customer spends less than an hour and a half at the restaurant.

Another important application in probability that can involve improper double integrals is the calculation of expected values. First we define this concept and then show an example of a calculation.

Definition

In probability theory, we denote the expected values $E(X)$ and $E(Y)$, respectively, as the most likely outcomes of the events. The expected values $E(X)$ and $E(Y)$ are given by

$$E(X) = \iint_S xf(x, y)dA \text{ and } E(Y) = \iint_S yf(x, y)dA,$$

where S is the sample space of the random variables X and Y.

Example 5.23

Finding Expected Value

Find the expected time for the events 'waiting for a table' and 'completing the meal' in **Example 5.22**.

Solution

Using the first quadrant of the rectangular coordinate plane as the sample space, we have improper integrals for $E(X)$ and $E(Y)$. The expected time for a table is

$$\begin{aligned}
E(X) &= \iint_S x\frac{1}{600}e^{-x/15}e^{-y/40}\,dA = \frac{1}{600}\int_{x=0}^{x=\infty}\int_{y=0}^{y=\infty} xe^{-x/15}e^{-y/40}\,dA \\
&= \frac{1}{600}\lim_{(a,\,b)\to(\infty,\,\infty)}\int_{x=0}^{x=a}\int_{y=0}^{y=b} xe^{-x/15}e^{-y/40}\,dx\,dy \\
&= \frac{1}{600}\left(\lim_{a\to\infty}\int_{x=0}^{x=a} xe^{-x/15}\,dx\right)\left(\lim_{b\to\infty}\int_{y=0}^{y=b} e^{-y/40}\,dy\right) \\
&= \frac{1}{600}\left(\left(\lim_{a\to\infty}\left(-15e^{-x/15}(x+15)\right)\right)\Big|_{x=0}^{x=a}\right)\left(\left(\lim_{b\to\infty}\left(-40e^{-y/40}\right)\right)\Big|_{y=0}^{y=b}\right) \\
&= \frac{1}{600}\left(\lim_{a\to\infty}\left(-15e^{-a/15}(x+15)+225\right)\right)\left(\lim_{b\to\infty}\left(-40e^{-b/40}+40\right)\right) \\
&= \frac{1}{600}(225)(40) \\
&= 15.
\end{aligned}$$

A similar calculation shows that $E(Y) = 40$. This means that the expected values of the two random events are the average waiting time and the average dining time, respectively.

5.16 The joint density function for two random variables X and Y is given by

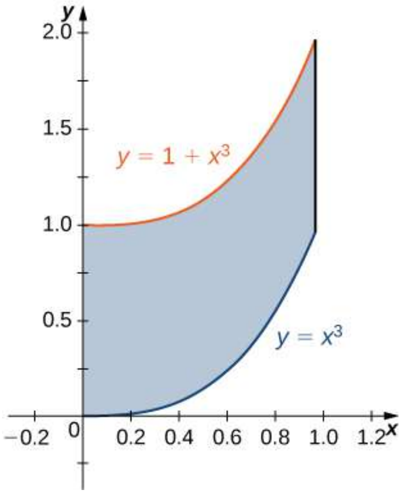
$$f(x,\ y) = \begin{cases} \frac{1}{600}\left(x^2 + y^2\right) & \text{if } 0 \le x \le 15,\ 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that X is at most 10 and Y is at least 5.

5.2 EXERCISES

In the following exercises, specify whether the region is of Type I or Type II.

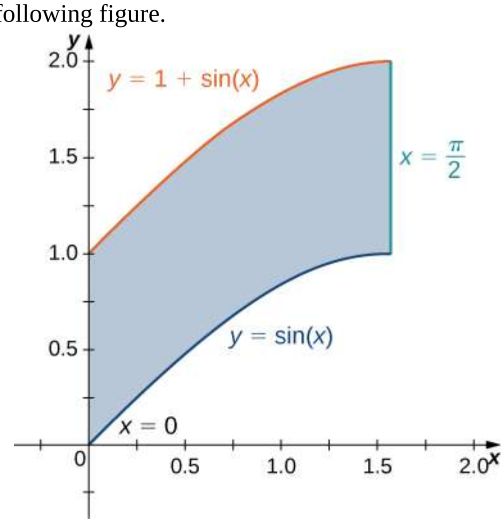
60. The region D bounded by $y = x^3$, $y = x^3 + 1$, $x = 0$, and $x = 1$ as given in the following figure.



61. Find the average value of the function $f(x, y) = 3xy$ on the region graphed in the previous exercise.

62. Find the area of the region D given in the previous exercise.

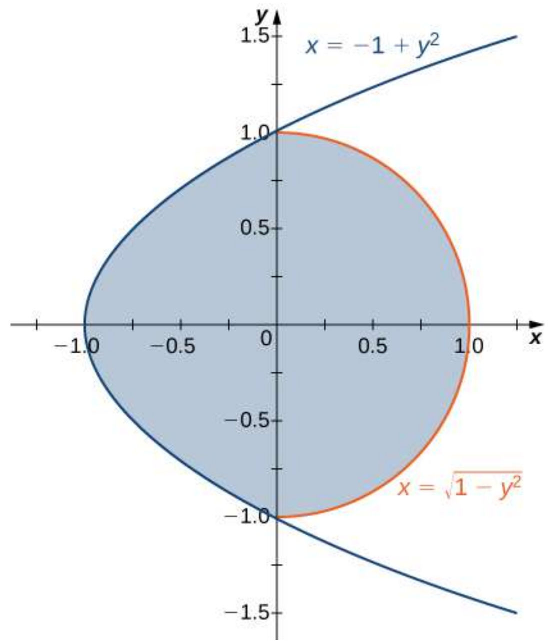
63. The region D bounded by $y = \sin x$, $y = 1 + \sin x$, $x = 0$, and $x = \frac{\pi}{2}$ as given in the following figure.



64. Find the average value of the function $f(x, y) = \cos x$ on the region graphed in the previous exercise.

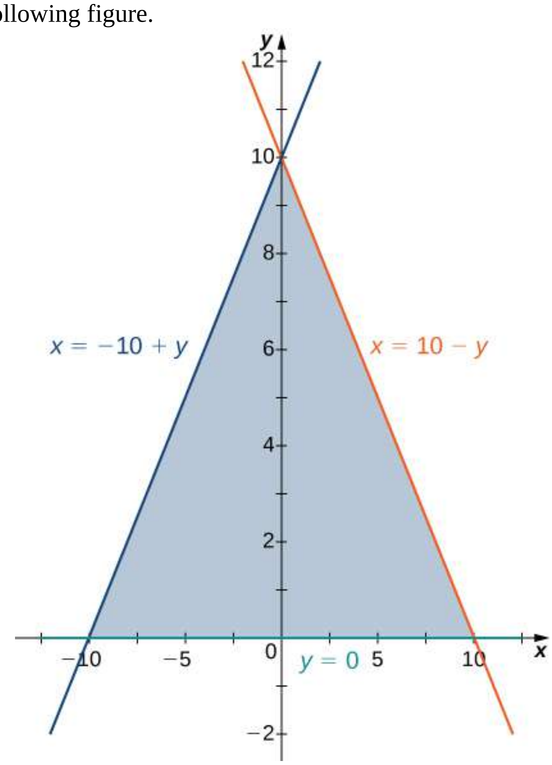
65. Find the area of the region D given in the previous exercise.

66. The region D bounded by $x = y^2 - 1$ and $x = \sqrt{1 - y^2}$ as given in the following figure.



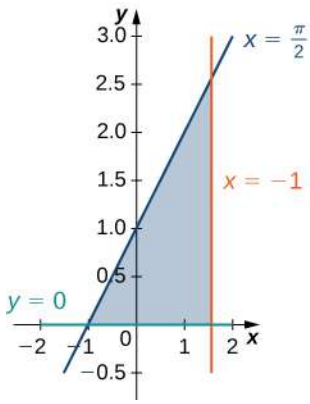
67. Find the volume of the solid under the graph of the function $f(x, y) = xy + 1$ and above the region in the figure in the previous exercise.

68. The region D bounded by $y = 0$, $x = -10 + y$, and $x = 10 - y$ as given in the following figure.

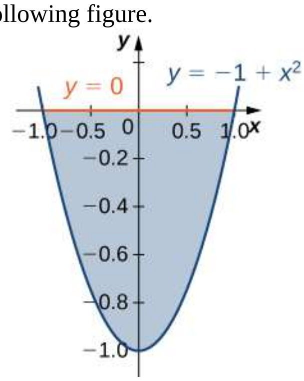


69. Find the volume of the solid under the graph of the function $f(x,\ y) = x + y$ and above the region in the figure from the previous exercise.

70. The region D bounded by $y = 0,\ x = y - 1,\ x = \frac{\pi}{2}$ as given in the following figure.



71. The region D bounded by $y = 0$ and $y = x^2 - 1$ as given in the following figure.



72. Let D be the region bounded by the curves of equations $y = x,\ y = -x,$ and $y = 2 - x^2$. Explain why D is neither of Type I nor II.

73. Let D be the region bounded by the curves of equations $y = \cos x$ and $y = 4 - x^2$ and the x-axis. Explain why D is neither of Type I nor II.

In the following exercises, evaluate the double integral $\iint_D f(x,\ y)dA$ over the region D.

74. $f(x,\ y) = 2x + 5y$ and $D = \left\{(x,\ y) \mid 0 \le x \le 1,\ x^3 \le y \le x^3 + 1\right\}$

75. $f(x,\ y) = 1$ and $D = \left\{(x,\ y) \mid 0 \le x \le \frac{\pi}{2},\ \sin x \le y \le 1 + \sin x\right\}$

76. $f(x,\ y) = 2$ and $D = \{(x,\ y) \mid 0 \le y \le 1,\ y - 1 \le x \le \arccos y\}$

77. $f(x,\ y) = xy$ and $D = \left\{(x,\ y) \mid -1 \le y \le 1,\ y^2 - 1 \le x \le \sqrt{1 - y^2}\right\}$

78. $f(x,\ y) = \sin y$ and D is the triangular region with vertices $(0,\ 0),\ (0,\ 3),$ and $(3,\ 0)$

79. $f(x,\ y) = -x + 1$ and D is the triangular region with vertices $(0,\ 0),\ (0,\ 2),$ and $(2,\ 2)$

Evaluate the iterated integrals.

80. $\displaystyle\int_0^1 \int_{2x}^{3x} \left(x + y^2\right) dy\, dx$

81. $\displaystyle\int_0^1 \int_{2\sqrt{x}}^{2\sqrt{x} + 1} (xy + 1) dy\, dx$

82. $\displaystyle\int_e^{e^2} \int_{\ln u}^{2} (v + \ln u) dv\, du$

83. $\displaystyle\int_1^2 \int_{-u^2 - 1}^{-u} (8uv) dv\, du$

84. $\displaystyle\int_0^1 \int_{-\sqrt{1 - y^2}}^{\sqrt{1 - y^2}} \left(2x + 4x^3\right) dx\, dy$

85. $\displaystyle\int_0^{1/2} \int_{-\sqrt{1 - 4y^2}}^{\sqrt{1 - 4y^2}} 4 dx\, dy$

86. Let D be the region bounded by $y = 1 - x^2,\ y = 4 - x^2,$ and the x- and y-axes.

a. Show that $\displaystyle\iint_D x\, dA = \int_0^1 \int_{1 - x^2}^{4 - x^2} x\, dy\, dx + \int_1^2 \int_0^{4 - x^2} x\, dy\, dx$ by dividing the region D into two regions of Type I.

b. Evaluate the integral $\displaystyle\iint_D x\, dA$.

87. Let D be the region bounded by $y = 1,$ $y = x,$ $y = \ln x,$ and the x-axis.

a. Show that $\iint_D y\,dA = \int_0^1 \int_0^x y\,dy\,dx + \int_1^e \int_{\ln x}^1 y\,dy\,dx$ by dividing D into two regions of Type I.

b. Evaluate the integral $\iint_D y\,dA.$

88.

a. Show that $\iint_D y^2\,dA = \int_{-1}^0 \int_{-x}^{2-x^2} y^2\,dy\,dx + \int_0^1 \int_x^{2-x^2} y^2\,dy\,dx$ by dividing the region D into two regions of Type I, where $D = \{(x, y) | y \ge x, y \ge -x, y \le 2 - x^2\}.$

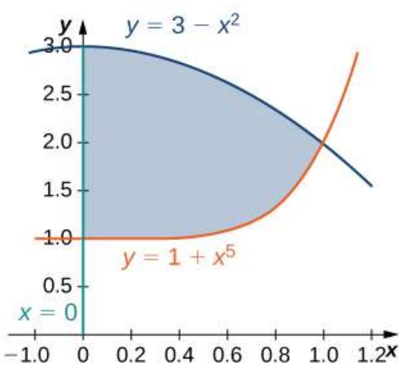
b. Evaluate the integral $\iint_D y^2 dA.$

89. Let D be the region bounded by $y = x^2,$ $y = x + 2,$ and $y = -x.$

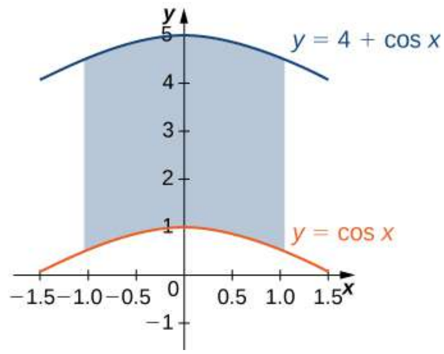
a. Show that $\iint_D x\,dA = \int_0^1 \int_{-y}^{\sqrt{y}} x\,dx\,dy + \int_1^2 \int_{y-2}^{\sqrt{y}} x\,dx\,dy$ by dividing the region D into two regions of Type II, where $D = \{(x, y) | y \ge x^2, y \ge -x, y \le x + 2\}.$

b. Evaluate the integral $\iint_D x\,dA.$

90. The region D bounded by $x = 0,$ $y = x^5 + 1,$ and $y = 3 - x^2$ is shown in the following figure. Find the area $A(D)$ of the region $D.$



91. The region D bounded by $y = \cos x,$ $y = 4\cos x,$ and $x = \pm\frac{\pi}{3}$ is shown in the following figure. Find the area $A(D)$ of the region $D.$



92. Find the area $A(D)$ of the region $D = \{(x, y) | y \ge 1 - x^2, y \le 4 - x^2, y \ge 0, x \ge 0\}.$

93. Let D be the region bounded by $y = 1,$ $y = x,$ $y = \ln x,$ and the x-axis. Find the area $A(D)$ of the region $D.$

94. Find the average value of the function $f(x, y) = \sin y$ on the triangular region with vertices $(0, 0),$ $(0, 3),$ and $(3, 0).$

95. Find the average value of the function $f(x, y) = -x + 1$ on the triangular region with vertices $(0, 0),$ $(0, 2),$ and $(2, 2).$

In the following exercises, change the order of integration and evaluate the integral.

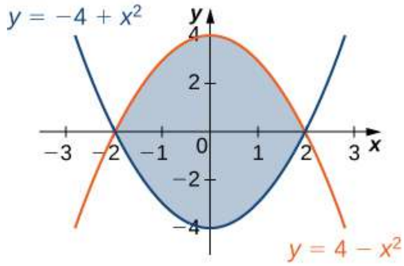
96. $\int_{-1}^{\pi/2} \int_0^{x+1} \sin x\,dy\,dx$

97. $\int_0^1 \int_{x-1}^{1-x} x\,dy\,dx$

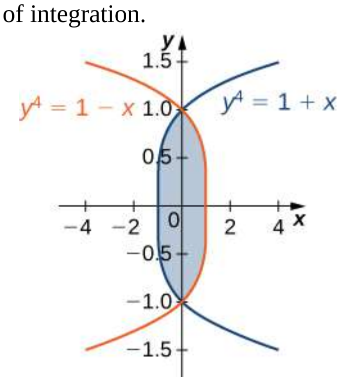
98. $\int_{-1}^0 \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2\,dx\,dy$

99. $\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y\,dx\,dy$

100. The region D is shown in the following figure. Evaluate the double integral $\iint_D (x^2 + y)dA$ by using the easier order of integration.



101. The region D is given in the following figure. Evaluate the double integral $\iint_D (x^2 - y^2)dA$ by using the easier order of integration.



102. Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $y = x^5$ and $y = x$.

103. Find the volume of the solid under the plane $z = 3x + y$ and above the region determined by $y = x^7$ and $y = x$.

104. Find the volume of the solid under the plane $z = x - y$ and above the region bounded by $x = \tan y$, $x = -\tan y$, and $x = 1$.

105. Find the volume of the solid under the surface $z = x^3$ and above the plane region bounded by $x = \sin y$, $x = -\sin y$, and $x = 1$.

106. Let g be a positive, increasing, and differentiable function on the interval $[a, b]$. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = 0$, $y = g(x)$, $x = a$, and $x = b$ is given by $\frac{1}{2}\left(g^2(b) - g^2(a)\right)$.

107. Let g be a positive, increasing, and differentiable function on the interval $[a, b]$, and let k be a positive real number. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = g(x)$, $y = g(x) + k$, $x = a$, and $x = b$ is given by $k(g(b) - g(a))$.

108. Find the volume of the solid situated in the first octant and determined by the planes $z = 2$, $z = 0$, $x + y = 1$, $x = 0$, and $y = 0$.

109. Find the volume of the solid situated in the first octant and bounded by the planes $x + 2y = 1$, $x = 0$, $y = 0$, $z = 4$, and $z = 0$.

110. Find the volume of the solid bounded by the planes $x + y = 1$, $x - y = 1$, $x = 0$, $z = 0$, and $z = 10$.

111. Find the volume of the solid bounded by the planes $x + y = 1$, $x - y = 1$, $x + y = -1$, $x - y = -1$, $z = 1$ and $z = 0$.

112. Let S_1 and S_2 be the solids situated in the first octant under the planes $x + y + z = 1$ and $x + y + 2z = 1$, respectively, and let S be the solid situated between S_1, S_2, $x = 0$, and $y = 0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

113. Let S_1 and S_2 be the solids situated in the first octant under the planes $2x + 2y + z = 2$ and $x + y + z = 1$, respectively, and let S be the solid situated between S_1, S_2, $x = 0$, and $y = 0$.

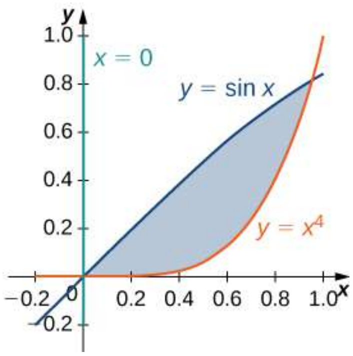
a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

114. Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and under the sphere $x^2 + y^2 + z^2 = 4$, respectively. If the volume of the solid S_2 is $\frac{4\pi}{3}$, determine the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of these solids.

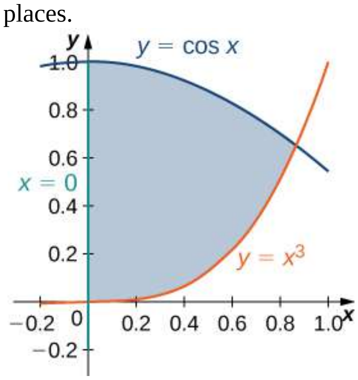
115. Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and bounded by the cylinder $x^2 + y^2 = 4,$ respectively.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of the solids S_1 and S_2.

116. **[T]** The following figure shows the region D bounded by the curves $y = \sin x,$ $x = 0,$ and $y = x^4$. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



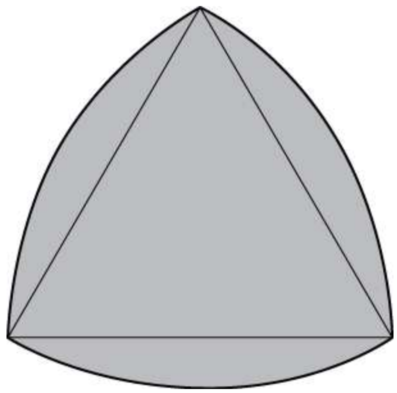
117. **[T]** The region D bounded by the curves $y = \cos x,$ $x = 0,$ and $y = x^3$ is shown in the following figure. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



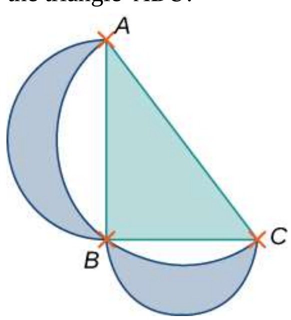
118. Suppose that (X, Y) is the outcome of an experiment that must occur in a particular region S in the xy-plane. In this context, the region S is called the sample space of the experiment and X and Y are random variables. If D is a region included in $S,$ then the probability of (X, Y) being in D is defined as $P[(X, Y) \in D] = \iint_D p(x, y)dx\,dy,$ where $p(x, y)$ is the joint probability density of the experiment. Here, $p(x, y)$ is a nonnegative function for which $\iint_S p(x, y)dx\,dy = 1.$ Assume that a point (X, Y) is chosen arbitrarily in the square $[0, 3] \times [0, 3]$ with the probability density $p(x, y) = \begin{cases} \frac{1}{9} & (x, y) \in [0, 3] \times [0, 3], \\ 0 & \text{otherwise.} \end{cases}$ Find the probability that the point (X, Y) is inside the unit square and interpret the result.

119. Consider X and Y two random variables of probability densities $p_1(x)$ and $p_2(x),$ respectively. The random variables X and Y are said to be independent if their joint density function is given by $p(x, y) = p_1(x)p_2(y).$ At a drive-thru restaurant, customers spend, on average, 3 minutes placing their orders and an additional 5 minutes paying for and picking up their meals. Assume that placing the order and paying for/picking up the meal are two independent events X and Y. If the waiting times are modeled by the exponential probability densities $p_1(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$ and $p_2(y) = \begin{cases} \frac{1}{5}e^{-y/5} & y \geq 0, \\ 0 & \text{otherwise,} \end{cases}$ respectively, the probability that a customer will spend less than 6 minutes in the drive-thru line is given by $P[X + Y \leq 6] = \iint_D p(x, y)dx\,dy,$ where $D = \{(x, y) | x \geq 0, y \geq 0, x + y \leq 6\}.$ Find $P[X + Y \leq 6]$ and interpret the result.

120. **[T]** The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius *s* centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in the following figure of side length s is $\frac{s^2}{2}(\pi - \sqrt{3})$.



121. **[T]** Show that the area of the lunes of Alhazen, the two blue lunes in the following figure, is the same as the area of the right triangle *ABC*. The outer boundaries of the lunes are semicircles of diameters AB and AC, respectively, and the inner boundaries are formed by the circumcircle of the triangle ABC.



5.3 | Double Integrals in Polar Coordinates

Learning Objectives

- **5.3.1** Recognize the format of a double integral over a polar rectangular region.
- **5.3.2** Evaluate a double integral in polar coordinates by using an iterated integral.
- **5.3.3** Recognize the format of a double integral over a general polar region.
- **5.3.4** Use double integrals in polar coordinates to calculate areas and volumes.

Double integrals are sometimes much easier to evaluate if we change rectangular coordinates to polar coordinates. However, before we describe how to make this change, we need to establish the concept of a double integral in a polar rectangular region.

Polar Rectangular Regions of Integration

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a **polar rectangle**, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle as in Figure 5.28(a), with $R = \{(r, \theta)|a \le r \le b, \alpha \le \theta \le \beta\}$.

In this section, we are looking to integrate over polar rectangles. Consider a function $f(r, \theta)$ over a polar rectangle R. We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length $\Delta r = (b - a)/m$ and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_i]$ of width $\Delta\theta = (\beta - \alpha)/n$. This means that the circles $r = r_i$ and rays $\theta = \theta_i$ for $1 \le i \le m$ and $1 \le j \le n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} (Figure 5.28(b)).

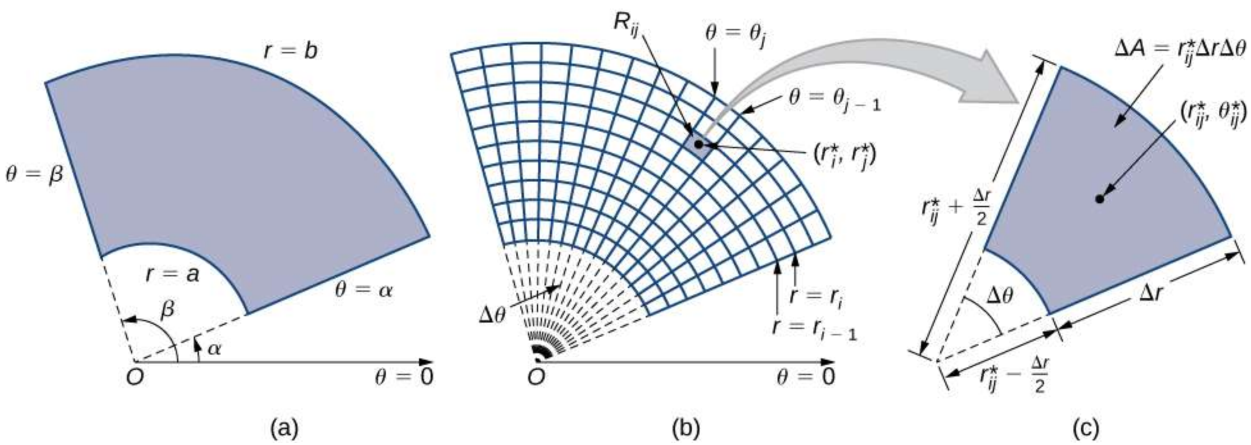


Figure 5.28 (a) A polar rectangle R (b) divided into subrectangles R_{ij}. (c) Close-up of a subrectangle.

As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the "polar" volume of the thin box above R_{ij}. Recall that, in a circle of radius r, the length s of an arc subtended by a central angle of θ radians is $s = r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is

$$\Delta A = \frac{1}{2}\Delta r(r_{i-1}\Delta\theta + r_1\Delta\theta).$$

Simplifying and letting $r_{ij}^* = \frac{1}{2}(r_{i-1} + r_i)$, we have $\Delta A = r_{ij}^*\Delta r\Delta\theta$. Therefore, the polar volume of the thin box above R_{ij} (**Figure 5.29**) is

$$f(r_{ij}^*, \theta_{ij}^*)\Delta A = f(r_{ij}^*, \theta_{ij}^*)r_{ij}^* \Delta r\Delta\theta.$$

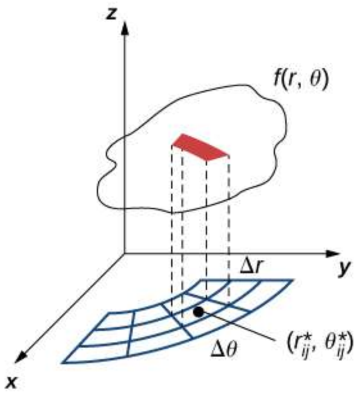


Figure 5.29 Finding the volume of the thin box above polar rectangle R_{ij}.

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*)r_{ij}^* \Delta r\Delta\theta.$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

$$V = \lim_{m,\, n \to \infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*)r_{ij}^* \Delta r\Delta\theta.$$

This becomes the expression for the double integral.

Definition

The double integral of the function $f(r, \theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as

$$\iint_R f(r, \theta)dA = \lim_{m,\, n \to \infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*)\Delta A = \lim_{m,\, n \to \infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*)r_{ij}^* \Delta r\Delta\theta. \quad (5.8)$$

Again, just as in **Double Integrals over Rectangular Regions**, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,

$$\iint_R f(r, \theta)dA = \iint_R f(r, \theta)r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r, \theta)r\,dr\,d\theta.$$

Notice that the expression for dA is replaced by $r\,dr\,d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y, using $x = r\cos\theta,\ y = r\sin\theta,$ and $dA = r\,dr\,d\theta$ changes it to

$$\iint_R f(x, y)dA = \iint_R f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

Note that all the properties listed in **Double Integrals over Rectangular Regions** for the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Example 5.24

Sketching a Polar Rectangular Region

Sketch the polar rectangular region $R = \{(r, \theta)|1 \leq r \leq 3, 0 \leq \theta \leq \pi\}$.

Solution

As we can see from **Figure 5.30**, $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \leq \theta \leq \pi$ covers the entire top half of the plane. Hence the region R looks like a semicircular band.

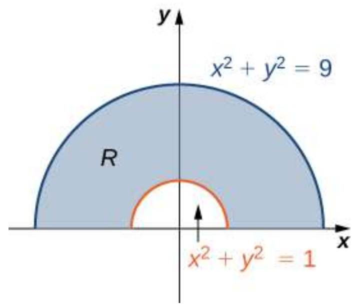


Figure 5.30 The polar region R lies between two semicircles.

Now that we have sketched a polar rectangular region, let us demonstrate how to evaluate a double integral over this region by using polar coordinates.

Example 5.25

Evaluating a Double Integral over a Polar Rectangular Region

Evaluate the integral $\iint_R 3x\,dA$ over the region $R = \{(r, \theta)|1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$.

Solution

First we sketch a figure similar to **Figure 5.30** but with outer radius 2. From the figure we can see that we have

$$\iint_R 3x\,dA = \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} 3r\cos\theta r\,dr\,d\theta$$

Use an iterated integral with correct limits of integration.

$$= \int_{\theta=0}^{\theta=\pi} \cos\theta\left[r^3\Big|_{r=1}^{r=2}\right]d\theta$$

Integrate fir t with respect to r.

$$= \int_{\theta=0}^{\theta=\pi} 7\cos\theta\,d\theta = 7\sin\theta\Big|_{\theta=0}^{\theta=\pi} = 0.$$

5.17 Sketch the region $R = \left\{(r, \theta)|1 \leq r \leq 2, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\}$, and evaluate $\iint_R x\,dA$.

Example 5.26

Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral $\iint_R (1 - x^2 - y^2) dA$ where R is the unit circle on the xy -plane.

Solution

The region R is a unit circle, so we can describe it as $R = \{(r, \theta) | 0 \le r \le 1, 0 \le \theta \le 2\pi\}$.

Using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

$$\begin{aligned}\iint_R (1 - x^2 - y^2) dA &= \int_0^{2\pi}\int_0^1 (1 - r^2) r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r - r^3) dr\,d\theta \\ &= \int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{4} d\theta = \frac{\pi}{2}.\end{aligned}$$

Example 5.27

Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral $\iint_R (x + y) dA$ where $R = \left\{(x, y) | 1 \le x^2 + y^2 \le 4, x \le 0\right\}$.

Solution

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \left\{(r, \theta) | 1 \le r \le 2, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}$ (see the following graph).

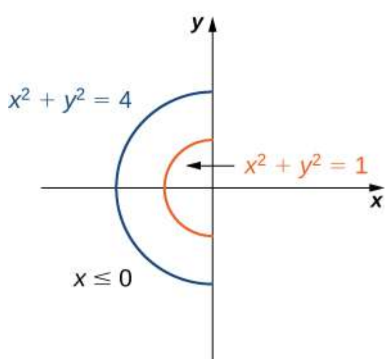


Figure 5.31 The annular region of integration R.

Hence, using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

$$\begin{aligned}\iint_R (x+y)dA &= \int_{\theta=\pi/2}^{\theta=3\pi/2}\int_{r=1}^{r=2}(r\cos\theta + r\sin\theta)r\,dr\,d\theta \\ &= \left(\int_{r=1}^{r=2} r^2 dr\right)\left(\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\theta - \cos\theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

5.18 Evaluate the integral $\iint_R \left(4 - x^2 - y^2\right)dA$ where R is the circle of radius 2 on the xy-plane.

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in **Double Integrals over General Regions**. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as $R = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$ (see the following figure).

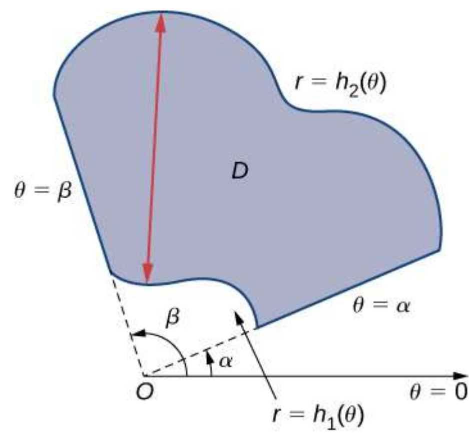


Figure 5.32 A general polar region between $\alpha < \theta < \beta$ and $h_1(\theta) < r < h_2(\theta)$.

Theorem 5.8: Double Integrals over General Polar Regions

If $f(r,\theta)$ is continuous on a general polar region D as described above, then

$$\iint_D f(r,\theta)r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)r\,dr\,d\theta \qquad (5.9)$$

Example 5.28

Evaluating a Double Integral over a General Polar Region

Evaluate the integral $\iint_D r^2 \sin\theta r\, dr\, d\theta$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Solution

We can describe the region D as $\{(r, \theta) | 0 \le \theta \le \pi, 0 \le r \le 1 + \cos\theta\}$ as shown in the following figure.

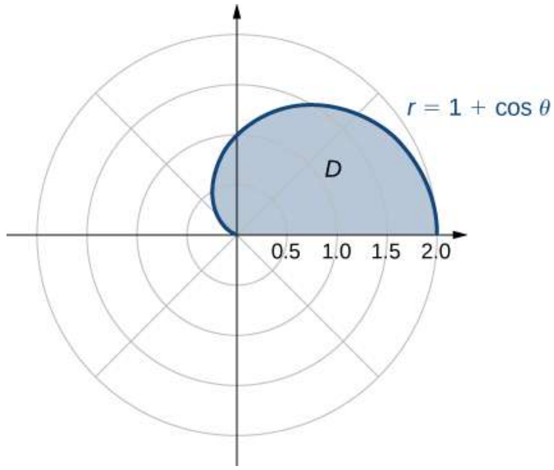


Figure 5.33 The region D is the top half of a cardioid.

Hence, we have

$$\begin{aligned}\iint_D r^2 \sin\theta r\, dr\, d\theta &= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos\theta} (r^2 \sin\theta) r\, dr\, d\theta \\ &= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} [r^4]_{r=0}^{r=1+\cos\theta} \sin\theta\, d\theta \\ &= \frac{1}{4} \int_{\theta=0}^{\theta=\pi} (1+\cos\theta)^4 \sin\theta\, d\theta \\ &= -\frac{1}{4}\left[\frac{(1+\cos\theta)^5}{5}\right]_0^{\pi} = \frac{8}{5}.\end{aligned}$$

5.19 Evaluate the integral

$$\iint_D r^2 \sin^2 2\theta r\, dr\, d\theta \text{ where } D = \left\{(r, \theta) | 0 \le \theta \le \pi, 0 \le r \le 2\sqrt{\cos 2\theta}\right\}.$$

Polar Areas and Volumes

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r, \theta)$, as well as by the surfaces $r = a$, $r = b$, $\theta = \alpha$, and $\theta = \beta$, we can find the volume V of S by double integration, as

$$V = \iint_R f(r, \theta) r \, dr \, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r, \theta) r \, dr \, d\theta.$$

If the base of the solid can be described as $D = \{(r, \theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then the double integral for the volume becomes

$$V = \iint_D f(r, \theta) r \, dr \, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r, \theta) r \, dr \, d\theta.$$

We illustrate this idea with some examples.

Example 5.29

Finding a Volume Using a Double Integral

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (see the following figure).

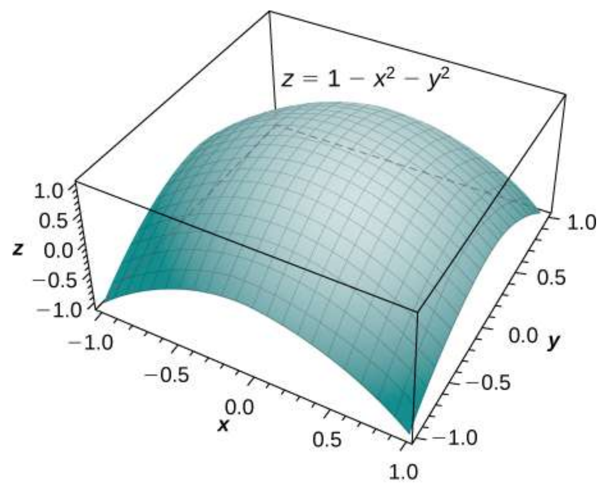


Figure 5.34 The paraboloid $z = 1 - x^2 - y^2$.

Solution

By the method of double integration, we can see that the volume is the iterated integral of the form $\iint_R (1 - x^2 - y^2) dA$ where $R = \{(r, \theta) | 0 \le r \le 1, 0 \le \theta \le 2\pi\}$.

This integration was shown before in **Example 5.26**, so the volume is $\frac{\pi}{2}$ cubic units.

Example 5.30

Finding a Volume Using Double Integration

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk

$(x-1)^2+y^2=1$ on the xy-plane. See the paraboloid in **Figure 5.35** intersecting the cylinder $(x-1)^2+y^2=1$ above the xy-plane.

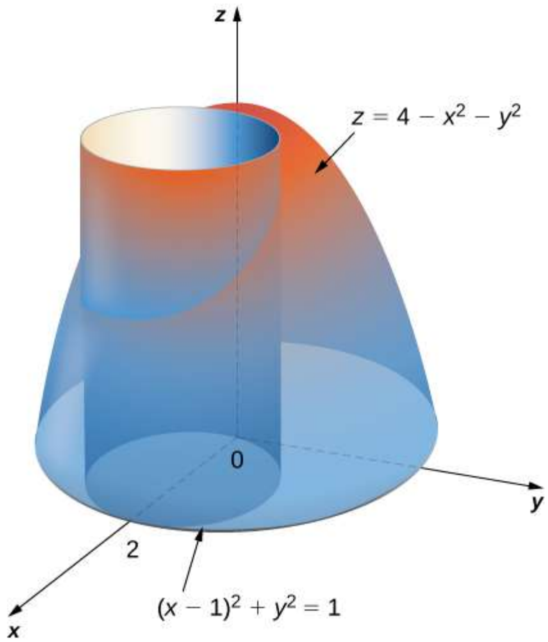


Figure 5.35 Finding the volume of a solid with a paraboloid cap and a circular base.

Solution

First change the disk $(x-1)^2+y^2=1$ to polar coordinates. Expanding the square term, we have $x^2-2x+1+y^2=1$. Then simplify to get $x^2+y^2=2x$, which in polar coordinates becomes $r^2=2r\cos\theta$ and then either $r=0$ or $r=2\cos\theta$. Similarly, the equation of the paraboloid changes to $z=4-r^2$. Therefore we can describe the disk $(x-1)^2+y^2=1$ on the xy-plane as the region

$$D=\{(r,\,\theta)|0\le\theta\le\pi,\ 0\le r\le 2\cos\theta\}.$$

Hence the volume of the solid bounded above by the paraboloid $z=4-x^2-y^2$ and below by $r=2\cos\theta$ is

$$\begin{aligned} V &= \iint_D f(r,\theta)r\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\cos\theta}\left(4-r^2\right)r\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=\pi}\left[4\frac{r^2}{2}-\frac{r^4}{4}\Big|_0^{2\cos\theta}\right]d\theta \\ &= \int_0^{\pi}\left[8\cos^2\theta-4\cos^2\theta\right]d\theta = \left[\frac{5}{2}\theta+\frac{5}{2}\sin\theta\cos\theta-\sin\theta\cos^3\theta\right]_0^{\pi} = \frac{5}{2}\pi. \end{aligned}$$

Notice in the next example that integration is not always easy with polar coordinates. Complexity of integration depends on the function and also on the region over which we need to perform the integration. If the region has a more natural expression in polar coordinates or if f has a simpler antiderivative in polar coordinates, then the change in polar coordinates is appropriate; otherwise, use rectangular coordinates.

Example 5.31

Finding a Volume Using a Double Integral

Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane (**Figure 5.36**).

Solution

First examine the region over which we need to set up the double integral and the accompanying paraboloid.

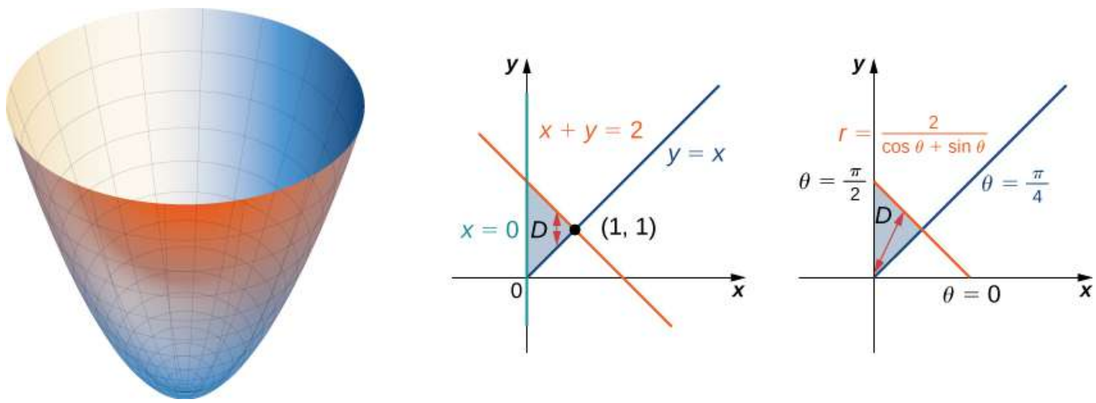


Figure 5.36 Finding the volume of a solid under a paraboloid and above a given triangle.

The region D is $\{(x, y)|0 \le x \le 1, x \le y \le 2 - x\}$. Converting the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane to functions of r and θ, we have $\theta = \pi/4$, $\theta = \pi/2$, and $r = 2/(\cos\theta + \sin\theta)$, respectively. Graphing the region on the xy-plane, we see that it looks like $D = \{(r, \theta)|\pi/4 \le \theta \le \pi/2, 0 \le r \le 2/(\cos\theta + \sin\theta)\}$. Now converting the equation of the surface gives $z = x^2 + y^2 = r^2$. Therefore, the volume of the solid is given by the double integral

$$\begin{aligned} V &= \iint_D f(r, \theta) r\,dr\,d\theta = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\theta+\sin\theta)} r^2\,r\,dr\,d\theta = \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\theta+\sin\theta)} d\theta \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\theta + \sin\theta}\right)^4 d\theta = \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta + \sin\theta}\right)^4 d\theta = 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta + \sin\theta}\right)^4 d\theta. \end{aligned}$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\,dy\,dx.$$

Evaluating gives

$$\begin{aligned} V &= \int_0^1 \int_x^{2-x} \left(x^2 + y^2\right) dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} dx \\ &= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3} dx \\ &= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3} \right]_0^1 = \frac{4}{3}. \end{aligned}$$

To answer the question of how the formulas for the volumes of different standard solids such as a sphere, a cone, or a cylinder are found, we want to demonstrate an example and find the volume of an arbitrary cone.

Example 5.32

Finding a Volume Using a Double Integral

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2 + y^2}$ and above the xy-plane.

Solution

The region D for the integration is the base of the cone, which appears to be a circle on the xy-plane (see the following figure).

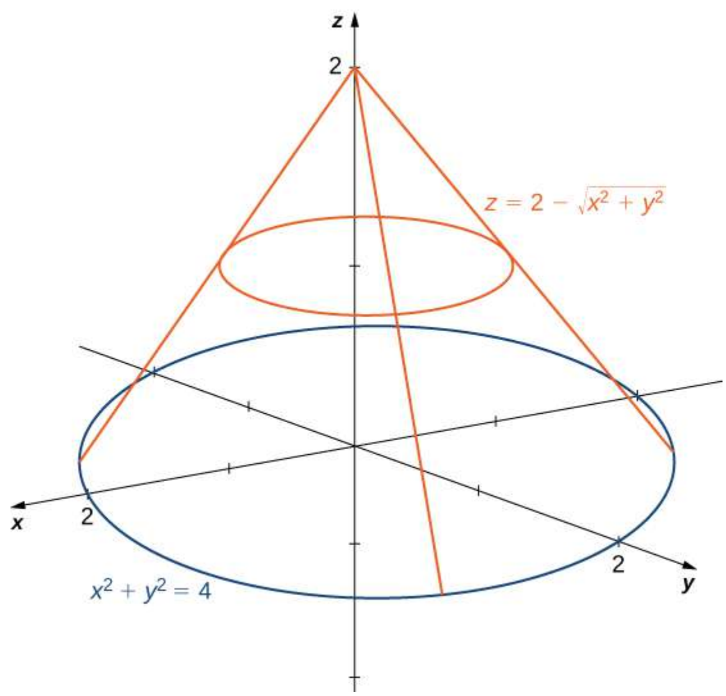


Figure 5.37 Finding the volume of a solid inside the cone and above the xy-plane.

We find the equation of the circle by setting $z = 0$:

$$\begin{aligned} 0 &= 2 - \sqrt{x^2 + y^2} \\ 2 &= \sqrt{x^2 + y^2} \\ x^2 + y^2 &= 4. \end{aligned}$$

This means the radius of the circle is $2,$ so for the integration we have $0 \le \theta \le 2\pi$ and $0 \le r \le 2.$ Substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $z = 2 - \sqrt{x^2 + y^2}$ we have $z = 2 - r.$ Therefore, the volume of the cone is

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (2 - r) r \, dr \, d\theta = 2\pi \frac{4}{3} = \frac{8\pi}{3} \text{ cubic units.}$$

Analysis

Note that if we were to find the volume of an arbitrary cone with radius a units and height h units, then the equation of the cone would be $z = h - \frac{h}{a}\sqrt{x^2 + y^2}.$

We can still use **Figure 5.37** and set up the integral as $\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(h - \frac{h}{a} r\right) r \, dr \, d\theta.$

Evaluating the integral, we get $\frac{1}{3}\pi a^2 h.$

5.20 Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2.$

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. Sketching a graph and identifying the region can be helpful to realize the limits of integration. Generally, the area formula in double integration will look like

$$\text{Area } A = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} 1 r \, dr \, d\theta.$$

Example 5.33

Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos 4\theta.$

Solution

Sketching the graph of the function $r = \cos 4\theta$ reveals that it is a polar rose with eight petals (see the following figure).

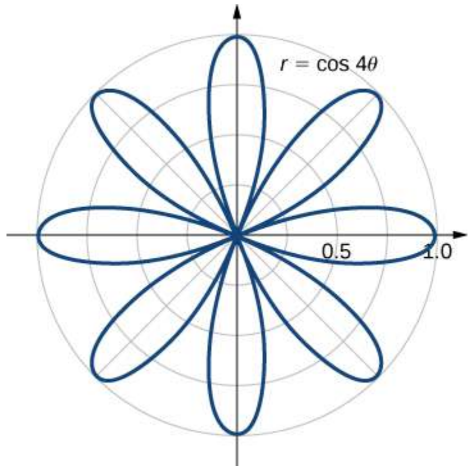


Figure 5.38 Finding the area of a polar rose with eight petals.

Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function $\cos 4\theta$, and these are odd multiples of $\pi/8$. Thus, one of the petals corresponds to the values of θ in the interval $[-\pi/8, \pi/8]$. Therefore, the area bounded by the curve $r = \cos 4\theta$ is

$$\begin{aligned} A &= 8 \int_{\theta = -\pi/8}^{\theta = \pi/8} \int_{r=0}^{r = \cos 4\theta} 1 r \, dr \, d\theta \\ &= 8 \int_{-\pi/8}^{\pi/8} \left[\frac{1}{2} r^2 \Big|_0^{\cos 4\theta} \right] d\theta = 8 \int_{-\pi/8}^{\pi/8} \frac{1}{2} \cos^2 4\theta \, d\theta = 8 \left[\frac{1}{4}\theta + \frac{1}{16} \sin 4\theta \cos 4\theta \Big|_{-\pi/8}^{\pi/8} \right] = 8 \left[\frac{\pi}{16} \right] = \frac{\pi}{2}. \end{aligned}$$

Example 5.34

Finding Area Between Two Polar Curves

Find the area enclosed by the circle $r = 3 \cos \theta$ and the cardioid $r = 1 + \cos \theta$.

Solution

First and foremost, sketch the graphs of the region (**Figure 5.39**).

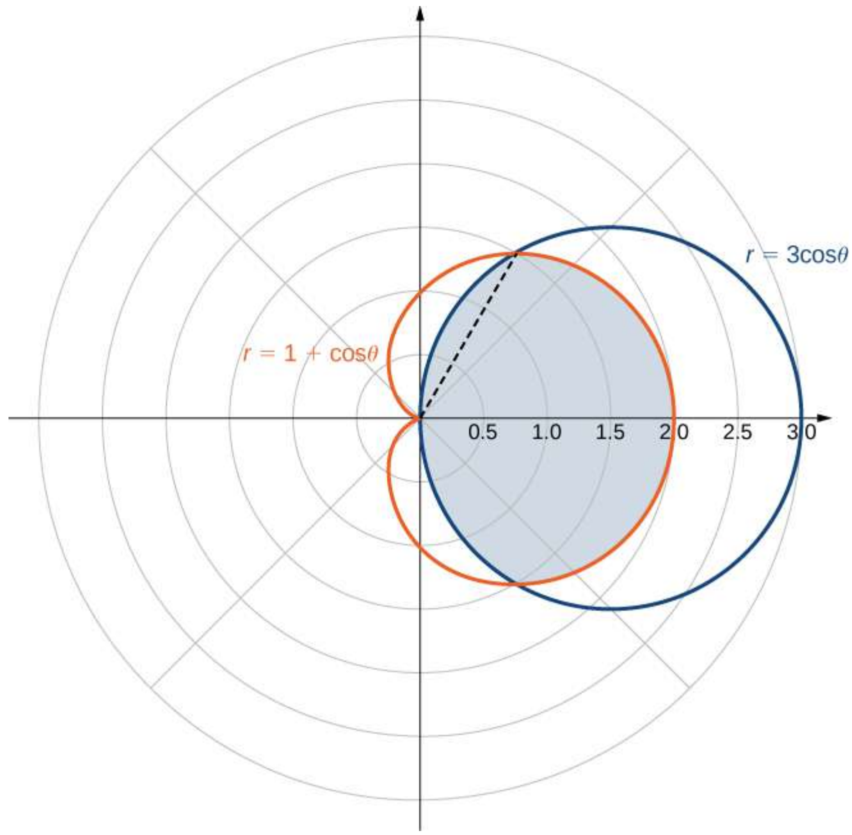


Figure 5.39 Finding the area enclosed by both a circle and a cardioid.

We can from see the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

$$3\cos\theta = 1 + \cos\theta.$$

One of the points of intersection is $\theta = \pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid from $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle from $\theta = \pi/3$ to $\theta = \pi/2$. By symmetry, the total area is twice the area above the polar axis. Thus, we have

$$A = 2\left[\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1+\cos\theta} 1r\,dr\,d\theta + \int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=0}^{r=3\cos\theta} 1r\,dr\,d\theta\right].$$

Evaluating each piece separately, we find that the area is

$$A = 2\left(\frac{1}{4}\pi + \frac{9}{16}\sqrt{3} + \frac{3}{8}\pi - \frac{9}{16}\sqrt{3}\right) = 2\left(\frac{5}{8}\pi\right) = \frac{5}{4}\pi \text{ square units.}$$

5.21 Find the area enclosed inside the cardioid $r = 3 - 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Example 5.35

Evaluating an Improper Double Integral in Polar Coordinates

Evaluate the integral $\iint_{\mathbb{R}^2} e^{-10\left(x^2+y^2\right)} dx\, dy.$

Solution

This is an improper integral because we are integrating over an unbounded region $\mathbb{R}^2$. In polar coordinates, the entire plane $\mathbb{R}^2$ can be seen as $0 \le \theta \le 2\pi, \quad 0 \le r \le \infty.$

Using the changes of variables from rectangular coordinates to polar coordinates, we have

$$\begin{aligned}
\iint_{\mathbb{R}^2} e^{-10\left(x^2+y^2\right)} dx\, dy &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\infty} e^{-10r^2}\, r\, dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} \left(\lim_{a \to \infty} \int_{r=0}^{r=a} e^{-10r^2}\, r\, dr \right) d\theta \\
&= \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) \left(\lim_{a \to \infty} \int_{r=0}^{r=a} e^{-10r^2}\, r\, dr \right) \\
&= 2\pi \left(\lim_{a \to \infty} \int_{r=0}^{r=a} e^{-10r^2}\, r\, dr \right) \\
&= 2\pi \lim_{a \to \infty} \left(-\frac{1}{20} \right) \left(e^{-10r^2} \Big|_0^a \right) \\
&= 2\pi \left(-\frac{1}{20} \right) \lim_{a \to \infty} \left(e^{-10a^2} - 1 \right) \\
&= \frac{\pi}{10}.
\end{aligned}$$

5.22 Evaluate the integral $\iint_{\mathbb{R}^2} e^{-4\left(x^2+y^2\right)} dx\, dy.$

5.3 EXERCISES

In the following exercises, express the region D in polar coordinates.

122. D is the region of the disk of radius 2 centered at the origin that lies in the first quadrant.

123. D is the region between the circles of radius 4 and radius 5 centered at the origin that lies in the second quadrant.

124. D is the region bounded by the y-axis and $x = \sqrt{1 - y^2}$.

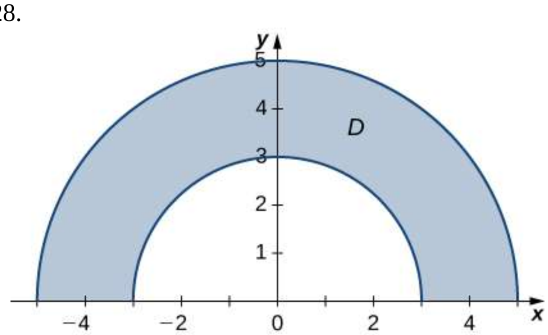
125. D is the region bounded by the x-axis and $y = \sqrt{2 - x^2}$.

126. $D = \{(x, y) | x^2 + y^2 \le 4x\}$

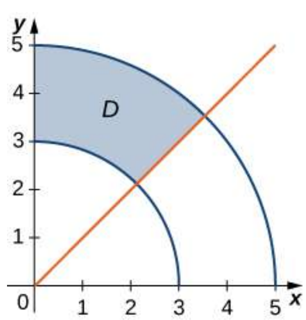
127. $D = \{(x, y) | x^2 + y^2 \le 4y\}$

In the following exercises, the graph of the polar rectangular region D is given. Express D in polar coordinates.

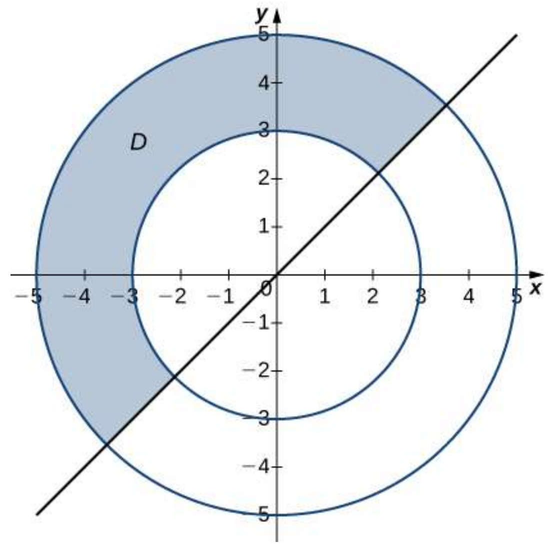
128.



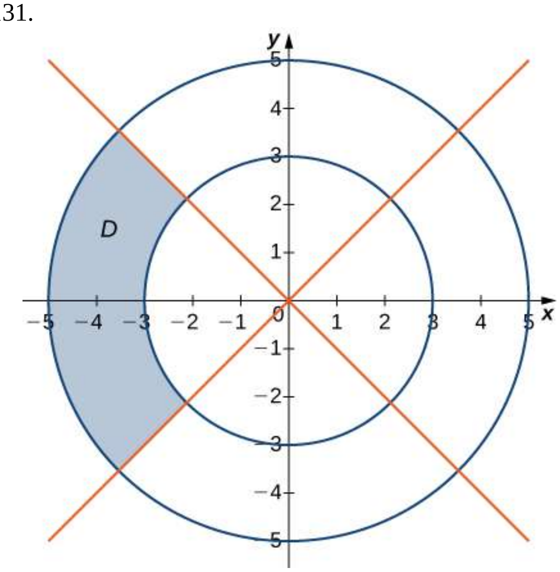
129.



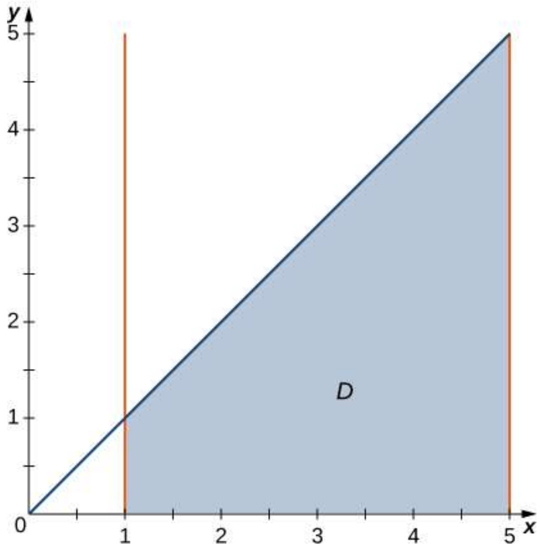
130.



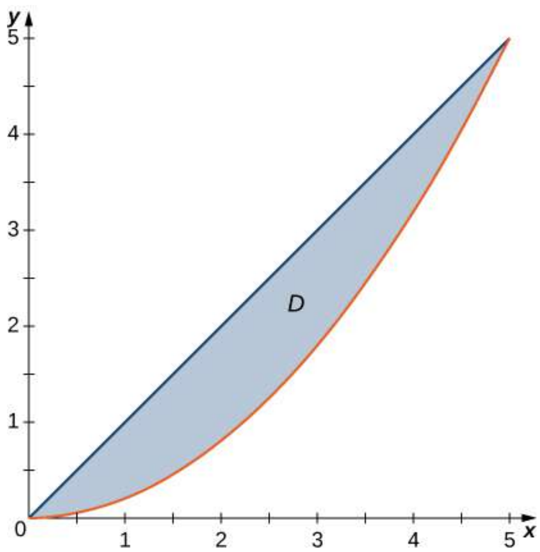
131.



132. In the following graph, the region D is situated below $y = x$ and is bounded by $x = 1,\, x = 5,$ and $y = 0.$



133. In the following graph, the region D is bounded by $y = x$ and $y = x^2.$



In the following exercises, evaluate the double integral $\iint\limits_R f(x, y)dA$ over the polar rectangular region D.

134. $f(x, y) = x^2 + y^2,\, D = \{(r, \theta) | 3 \le r \le 5,\, 0 \le \theta \le 2\pi\}$

135. $f(x, y) = x + y,\quad D = \{(r, \theta) | 3 \le r \le 5,\, 0 \le \theta \le 2\pi\}$

136. $f(x, y) = x^2 + xy,\, D = \{(r, \theta) | 1 \le r \le 2,\, \pi \le \theta \le 2\pi\}$

137. $f(x, y) = x^4 + y^4,\, D = \left\{(r, \theta) | 1 \le r \le 2,\, \frac{3\pi}{2} \le \theta \le 2\pi\right\}$

138. $f(x, y) = \sqrt[3]{x^2 + y^2},$ where $D = \left\{(r, \theta) | 0 \le r \le 1,\, \frac{\pi}{2} \le \theta \le \pi\right\}.$

139. $f(x, y) = x^4 + 2x^2y^2 + y^4,$ where $D = \left\{(r, \theta) | 3 \le r \le 4,\, \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}\right\}.$

140. $f(x, y) = \sin\left(\arctan\frac{y}{x}\right),$ where $D = \left\{(r, \theta) \middle| 1 \le r \le 2,\, \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right\}$

141. $f(x, y) = \arctan\left(\frac{y}{x}\right),$ where $D = \left\{(r, \theta) \middle| 2 \le r \le 3,\, \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\right\}$

142. $\iint\limits_D e^{x^2 + y^2}\left[1 + 2\arctan\left(\frac{y}{x}\right)\right]dA,\, D = \left\{(r, \theta) | 1 \le r \le 2,\, \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right\}$

143. $\iint\limits_D \left(e^{x^2 + y^2} + x^4 + 2x^2y^2 + y^4\right)\arctan\left(\frac{y}{x}\right)dA,\, D = \left\{(r, \theta) | 1 \le r \le 2,\, \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\right\}$

In the following exercises, the integrals have been converted to polar coordinates. Verify that the identities are true and choose the easiest way to evaluate the integrals, in rectangular or polar coordinates.

144. $\displaystyle\int_1^2 \int_0^x \left(x^2 + y^2\right)dy\,dx = \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\sec\theta} r^3\,dr\,d\theta$

145. $\displaystyle\int_2^3 \int_0^x \frac{x}{\sqrt{x^2 + y^2}}dy\,dx = \int_0^{\pi/4} \int_0^{\tan\theta\sec\theta} r\cos\theta\,dr\,d\theta$

146. $\displaystyle\int_0^1 \int_{x^2}^x \frac{1}{\sqrt{x^2 + y^2}}dy\,dx = \int_0^{\pi/4} \int_0^{\tan\theta\sec\theta} dr\,d\theta$

147. $\displaystyle\int_0^1 \int_{x^2}^x \frac{y}{\sqrt{x^2 + y^2}}dy\,dx = \int_0^{\pi/4} \int_0^{\tan\theta\sec\theta} r\sin\theta\,dr\,d\theta$

In the following exercises, convert the integrals to polar coordinates and evaluate them.

148. $\int_0^3 \int_0^{\sqrt{9-y^2}} (x^2+y^2)\,dx\,dy$

149. $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2+y^2)^2\,dx\,dy$

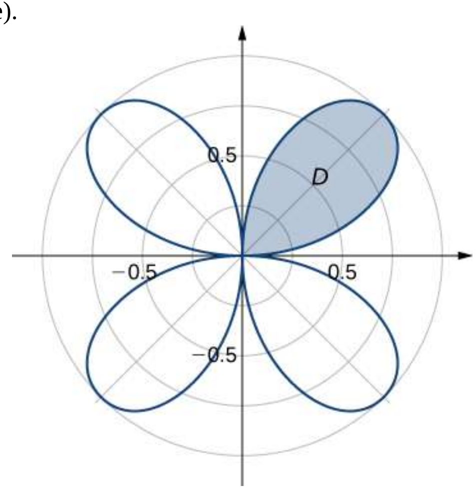
150. $\int_0^1 \int_0^{\sqrt{1-x^2}} (x+y)\,dy\,dx$

151. $\int_0^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sin(x^2+y^2)\,dy\,dx$

152. Evaluate the integral $\iint_D r\,dA$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

153. Find the area of the region D bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

154. Evaluate the integral $\iint_D r\,dA$, where D is the region bounded by the part of the four-leaved rose $r = \sin 2\theta$ situated in the first quadrant (see the following figure).

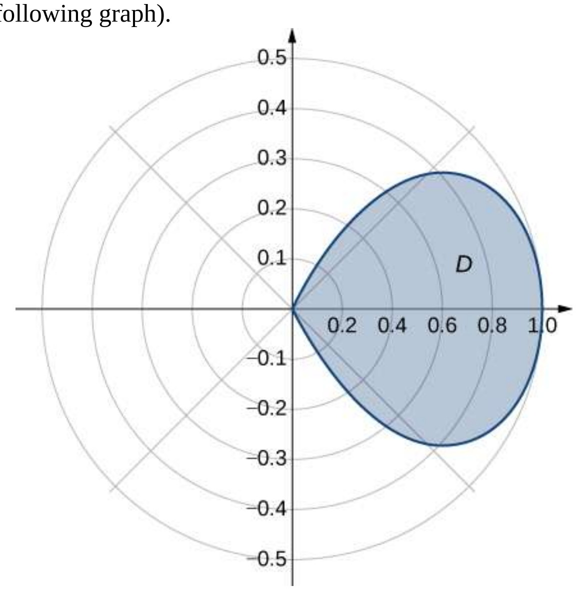


155. Find the total area of the region enclosed by the four-leaved rose $r = \sin 2\theta$ (see the figure in the previous exercise).

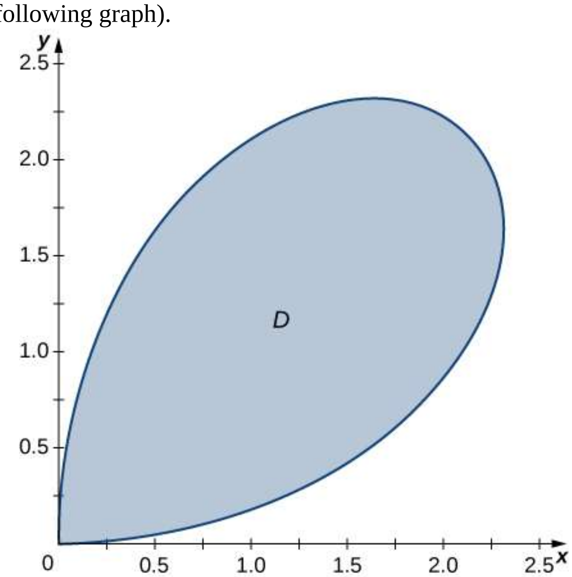
156. Find the area of the region D, which is the region bounded by $y = \sqrt{4-x^2}$, $x = \sqrt{3}$, $x = 2$, and $y = 0$.

157. Find the area of the region D, which is the region inside the disk $x^2 + y^2 \le 4$ and to the right of the line $x = 1$.

158. Determine the average value of the function $f(x, y) = x^2 + y^2$ over the region D bounded by the polar curve $r = \cos 2\theta$, where $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (see the following graph).



159. Determine the average value of the function $f(x, y) = \sqrt{x^2 + y^2}$ over the region D bounded by the polar curve $r = 3\sin 2\theta$, where $0 \le \theta \le \frac{\pi}{2}$ (see the following graph).



160. Find the volume of the solid situated in the first octant and bounded by the paraboloid $z = 1 - 4x^2 - 4y^2$ and the planes $x = 0$, $y = 0$, and $z = 0$.

161. Find the volume of the solid bounded by the paraboloid $z = 2 - 9x^2 - 9y^2$ and the plane $z = 1$.

162.

a. Find the volume of the solid S_1 bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.

b. Find the volume of the solid S_2 outside the double cone $z^2 = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$, and above the plane $z = 0$.

c. Find the volume of the solid inside the cone $z^2 = x^2 + y^2$ and below the plane $z = 1$ by subtracting the volumes of the solids S_1 and S_2.

163.

a. Find the volume of the solid S_1 inside the unit sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.

b. Find the volume of the solid S_2 inside the double cone $(z-1)^2 = x^2 + y^2$ and above the plane $z = 0$.

c. Find the volume of the solid outside the double cone $(z-1)^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

For the following two exercises, consider a spherical ring, which is a sphere with a cylindrical hole cut so that the axis of the cylinder passes through the center of the sphere (see the following figure).

164. If the sphere has radius 4 and the cylinder has radius 2, find the volume of the spherical ring.

165. A cylindrical hole of diameter 6 cm is bored through a sphere of radius 5 cm such that the axis of the cylinder passes through the center of the sphere. Find the volume of the resulting spherical ring.

166. Find the volume of the solid that lies under the double cone $z^2 = 4x^2 + 4y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.

167. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.

168. Find the volume of the solid that lies under the plane $x + y + z = 10$ and above the disk $x^2 + y^2 = 4x$.

169. Find the volume of the solid that lies under the plane $2x + y + 2z = 8$ and above the unit disk $x^2 + y^2 = 1$.

170. A radial function f is a function whose value at each point depends only on the distance between that point and the origin of the system of coordinates; that is, $f(x, y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. Show that if f is a continuous radial function, then $\iint_D f(x, y)dA = (\theta_2 - \theta_1)[G(R_2) - G(R_1)]$, where $G'(r) = rg(r)$ and $(x, y) \in D = \{(r, \theta) | R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

171. Use the information from the preceding exercise to calculate the integral $\iint_D \left(x^2 + y^2\right)^3 dA$, where D is the unit disk.

172. Let $f(x, y) = \frac{F'(r)}{r}$ be a continuous radial function defined on the annular region $D = \{(r, \theta) | R_1 \le r \le R_2,\ 0 \le \theta \le 2\pi\}$, where $r = \sqrt{x^2 + y^2}$, $0 < R_1 < R_2$, and F is a differentiable function. Show that $\iint_D f(x, y)dA = 2\pi[F(R_2) - F(R_1)]$.

173. Apply the preceding exercise to calculate the integral $\iint_D \frac{e^{\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}} dx\,dy$, where D is the annular region between the circles of radii 1 and 2 situated in the third quadrant.

174. Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x, y) = h(\theta)$, where $(x, y) \in D = \{(r, \theta) | R_1 \le r \le R_2, \theta_1 \le \theta \le \theta_2\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$. Show that $\iint_D f(x, y)dA = \frac{1}{2}\left(R_2^2 - R_1^2\right)[H(\theta_2) - H(\theta_1)]$, where H is an antiderivative of h.

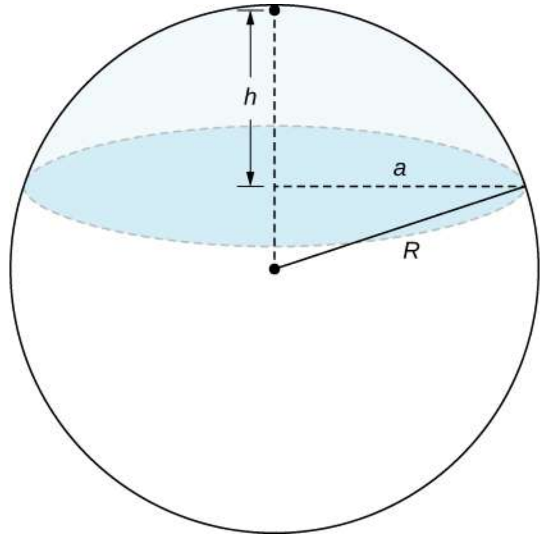
175. Apply the preceding exercise to calculate the integral $\iint_D \frac{y^2}{x^2}dA$, where $D = \left\{(r, \theta) \middle| 1 \le r \le 2, \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right\}$.

176. Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x, y) = g(r)h(\theta)$, where $(x, y) \in D = \{(r, \theta) | R_1 \le r \le R_2, \theta_1 \le \theta \le \theta_2\}$ with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$. Show that $\iint_D f(x, y)dA = [G(R_2) - G(R_1)][H(\theta_2) - H(\theta_1)]$, where G and H are antiderivatives of g and h, respectively.

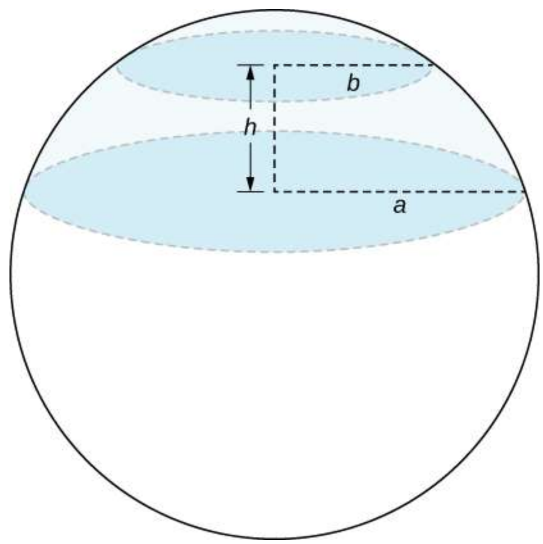
177. Evaluate $\iint_D \arctan\left(\frac{y}{x}\right)\sqrt{x^2 + y^2}dA$, where $D = \left\{(r, \theta) \middle| 2 \le r \le 3, \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\right\}$.

178. A spherical cap is the region of a sphere that lies above or below a given plane.

a. Show that the volume of the spherical cap in the figure below is $\frac{1}{6}\pi h\left(3a^2 + h^2\right)$.



b. A spherical segment is the solid defined by intersecting a sphere with two parallel planes. If the distance between the planes is h, show that the volume of the spherical segment in the figure below is $\frac{1}{6}\pi h\left(3a^2 + 3b^2 + h^2\right)$.



179. In statistics, the joint density for two independent, normally distributed events with a mean $\mu = 0$ and a standard distribution σ is defined by $p(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2 + y^2}{2\sigma^2}}$. Consider (X, Y), the Cartesian coordinates of a ball in the resting position after it was released from a position on the *z*-axis toward the xy-plane. Assume that the coordinates of the ball are independently normally distributed with a mean $\mu = 0$ and a standard deviation of σ (in feet). The probability that the ball will stop no more than a feet from the origin is given by $P\left[X^2 + Y^2 \le a^2\right] = \iint_D p(x, y) dy\, dx$, where D is the disk of radius *a* centered at the origin. Show that $P\left[X^2 + Y^2 \le a^2\right] = 1 - e^{-a^2/2\sigma^2}$.

180. The double improper integral $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\left(-x^2 + y^2/2\right)} dy\, dx$ may be defined as the limit value of the double integrals $\iint_{D_a} e^{\left(-x^2 + y^2/2\right)} dA$ over disks D_a of radii *a* centered at the origin, as *a* increases without bound; that is, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\left(-x^2 + y^2/2\right)} dy\, dx = \lim_{a \to \infty} \iint_{D_a} e^{\left(-x^2 + y^2/2\right)} dA$.

a. Use polar coordinates to show that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\left(-x^2 + y^2/2\right)} dy\, dx = 2\pi$.

b. Show that $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$, by using the relation $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\left(-x^2 + y^2/2\right)} dy\, dx = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2} dy\right)$.

5.4 | Triple Integrals

Learning Objectives

5.4.1 Recognize when a function of three variables is integrable over a rectangular box.
5.4.2 Evaluate a triple integral by expressing it as an iterated integral.
5.4.3 Recognize when a function of three variables is integrable over a closed and bounded region.
5.4.4 Simplify a calculation by changing the order of integration of a triple integral.
5.4.5 Calculate the average value of a function of three variables.

In **Double Integrals over Rectangular Regions**, we discussed the double integral of a function $f(x, y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x, y, z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. Later in this section we extend the definition to more general regions in $\mathbb{R}^3$.

Integrable Functions of Three Variables

We can define a rectangular box B in $\mathbb{R}^3$ as $B = \{(x, y, z) | a \le x \le b, c \le y \le d, e \le z \le f\}$. We follow a similar procedure to what we did in **Double Integrals over Rectangular Regions**. We divide the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length $\Delta x = \frac{x_i - x_{i-1}}{l}$, divide the interval $[c, d]$ into m subintervals $[y_{i-1}, y_i]$ of equal length $\Delta y = \frac{y_j - y_{j-1}}{m}$, and divide the interval $[e, f]$ into n subintervals $[z_{i-1}, z_i]$ of equal length $\Delta z = \frac{z_k - z_{k-1}}{n}$. Then the rectangular box B is subdivided into lmn subboxes $B_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i]$, as shown in **Figure 5.40**.

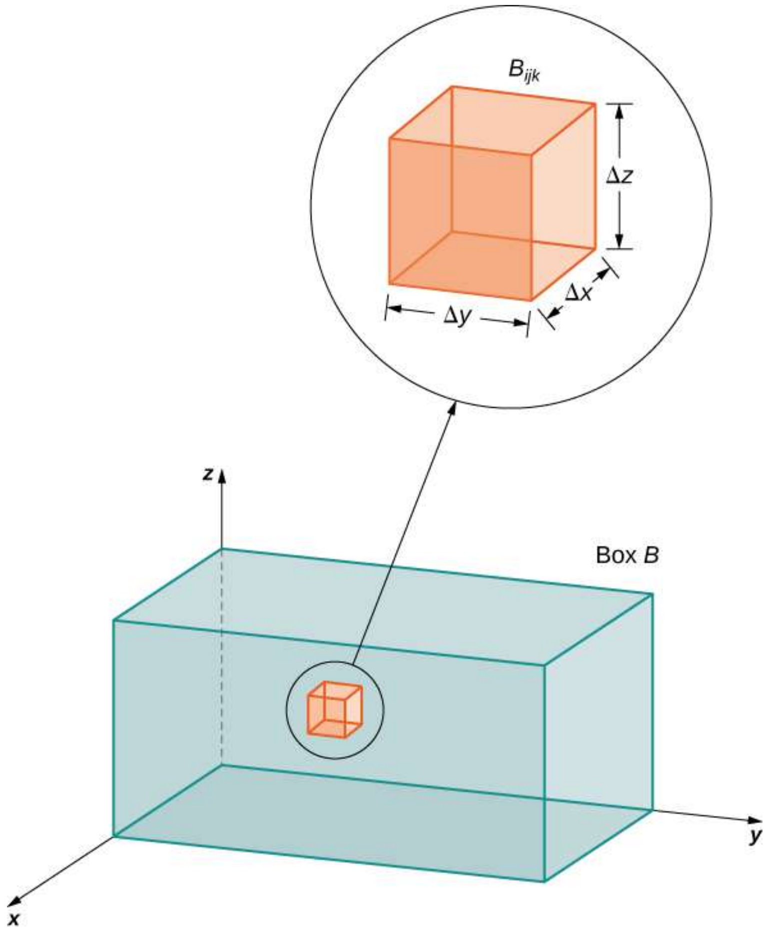


Figure 5.40 A rectangular box in $\mathbb{R}^3$ divided into subboxes by planes parallel to the coordinate planes.

For each $i, j,$ and $k,$ consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z.$ Form the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta x \Delta y \Delta z.$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Definition

The **triple integral** of a function $f(x, y, z)$ over a rectangular box B is defined as

$$\lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta x \Delta y \Delta z = \iiint_B f(x, y, z) dV \tag{5.10}$$

if this limit exists.

When the triple integral exists on $B,$ the function $f(x, y, z)$ is said to be integrable on B. Also, the triple integral exists if $f(x, y, z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words, f is bounded on B and continuous except possibly on the boundary of B.

The sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ can be any point in the rectangular sub-box B_{ijk} and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of Fubini's thereom for triple integrals exists.

Theorem 5.9: Fubini's Theorem for Triple Integrals

If $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x, y, z)\,dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\,dx\,dy\,dz.$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

For $a, b, c, d, e,$ and f real numbers, the iterated triple integral can be expressed in six different orderings:

$$\begin{aligned}
\int_e^f \int_c^d \int_a^b f(x, y, z)\,dx\,dy\,dz &= \int_e^f \left(\int_c^d \left(\int_a^b f(x, y, z)\,dx\right)dy\right)dz = \int_c^d \left(\int_e^f \left(\int_a^b f(x, y, z)\,dx\right)dz\right)dy \\
&= \int_a^b \left(\int_e^f \left(\int_c^d f(x, y, z)\,dy\right)dz\right)dx = \int_e^f \left(\int_a^b \left(\int_c^d f(x, y, z)\,dy\right)dx\right)dz \\
&= \int_c^e \left(\int_a^b \left(\int_e^f f(x, y, z)\,dz\right)dx\right)dy = \int_a^b \left(\int_c^e \left(\int_e^f f(x, y, z)\,dz\right)dy\right)dx.
\end{aligned}$$

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example 5.36

Evaluating a Triple Integral

Evaluate the triple integral $\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz.$

Solution

The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.

$$\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2) dx\, dy\, dz$$

$$= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[\frac{x^2}{2} + xyz^2 \bigg|_{x=-1}^{x=5} \right] dy\, dz \quad \text{Integrate with respect to } x.$$

$$= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[12 + 6yz^2 \right] dy\, dz \quad \text{Evaluate.}$$

$$= \int_{z=0}^{z=1} \left[12y + 6\frac{y^2}{2} z^2 \bigg|_{y=2}^{y=4} \right] dz \quad \text{Integrate with respect to } y.$$

$$= \int_{z=0}^{z=1} \left[24 + 36z^2 \right] dz \quad \text{Evaluate.}$$

$$= \left[24z + 36\frac{z^3}{3} \right]_{z=0}^{z=1} = 36. \quad \text{Integrate with respect to } z.$$

Example 5.37

Evaluating a Triple Integral

Evaluate the triple integral $\iiint\limits_B x^2 yz\, dV$ where $B = \{(x, y, z) | -2 \le x \le 1,\ 0 \le y \le 3,\ 1 \le z \le 5\}$ as shown in the following figure.

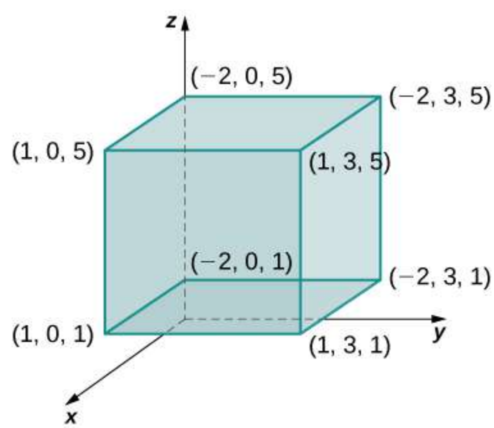


Figure 5.41 Evaluating a triple integral over a given rectangular box.

Solution

The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate y first, then x, and then z.

$$\iiint_B x^2 yz\, dV = \int_1^5 \int_{-2}^1 \int_0^3 \left[x^2 yz\right] dy\, dx\, dz = \int_1^5 \int_{-2}^1 \left[x^2 \frac{y^2}{2} z \Big|_0^3\right] dx\, dz$$
$$= \int_1^5 \int_{-2}^1 \frac{9}{2} x^2 z\, dx\, dz = \int_1^5 \left[\frac{9}{2}\frac{x^3}{3} z \Big|_{-2}^1\right] dz = \int_1^5 \frac{27}{2} z\, dz = \frac{27}{2}\frac{z^2}{2}\Big|_1^5 = 162.$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, and then y.

$$\iiint_B x^2 yz\, dV = \int_0^3 \int_1^5 \int_{-2}^1 \left[x^2 yz\right] dx\, dz\, dy = \int_0^3 \int_1^5 \left[\frac{x^3}{3} yz \Big|_{-2}^1\right] dz\, dy$$
$$= \int_0^3 \int_1^5 3yz\, dz\, dy = \int_0^3 \left[3y \frac{z^2}{2}\Big|_1^5\right] dy = \int_0^3 36y\, dy = 36\frac{y^2}{2}\Big|_0^3 = 18(9 - 0) = 162.$$

5.23 Evaluate the triple integral $\iiint_B z \sin x \cos y\, dV$ where $B = \left\{(x, y, z) \Big| 0 \le x \le \pi, \frac{3\pi}{2} \le y \le 2\pi, 1 \le z \le 3\right\}.$

Triple Integrals over a General Bounded Region

We now expand the definition of the triple integral to compute a triple integral over a more general bounded region E in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}.$$

For two functions $z = u_1(x, y)$ and $z = u_2(x, y)$, such that $u_1(x, y) \le u_2(x, y)$ for all (x, y) in D as shown in the following figure.

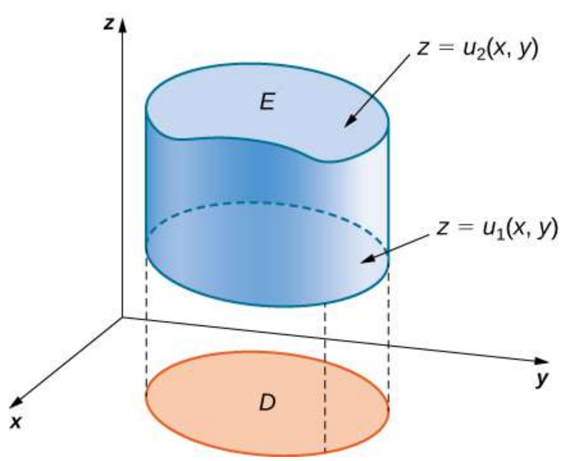


Figure 5.42 We can describe region E as the space between $u_1(x, y)$ and $u_2(x, y)$ above the projection D of E onto the xy-plane.

Theorem 5.10: Triple Integral over a General Region

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)dz \right] dA.$$

Similarly, we can consider a general bounded region D in the xy-plane and two functions $y = u_1(x, z)$ and $y = u_2(x, z)$ such that $u_1(x, z) \le u_2(x, z)$ for all (x, z) in D. Then we can describe the solid region E in $\mathbb{R}^3$ as

$$E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)dy \right] dA.$$

Finally, if D is a general bounded region in the yz-plane and we have two functions $x = u_1(y, z)$ and $x = u_2(y, z)$ such that $u_1(y, z) \le u_2(y, z)$ for all (y, z) in D, then the solid region E in $\mathbb{R}^3$ can be described as

$$E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \le x \le u_2(y, z)\}$$

where D is the projection of E onto the yz-plane and the triple integral is

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)dx \right] dA.$$

Note that the region D in any of the planes may be of Type I or Type II as described in **Double Integrals over General Regions**. If D in the xy-plane is of Type I (**Figure 5.43**), then

$$E = \{(x, y, z) | a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x, y) \le z \le u_2(x, y)\}.$$

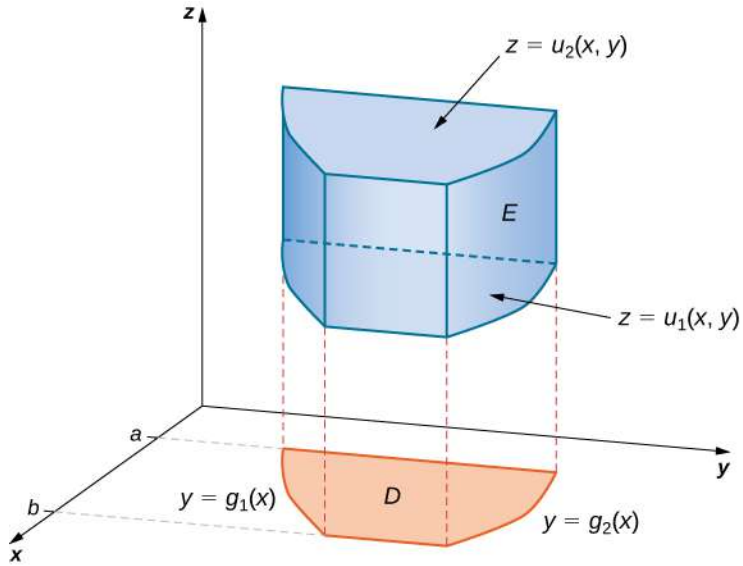


Figure 5.43 A box E where the projection D in the xy -plane is of Type I.

Then the triple integral becomes

$$\iiint_E f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)dz\,dy\,dx.$$

If D in the xy -plane is of Type II (**Figure 5.44**), then

$$E = \{(x, y, z)|c \le x \le d, h_1(x) \le y \le h_2(x), u_1(x, y) \le z \le u_2(x, y)\}.$$

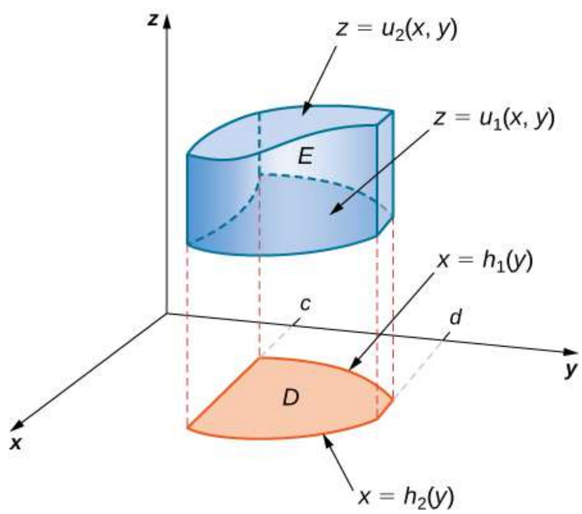


Figure 5.44 A box E where the projection D in the xy -plane is of Type II.

Then the triple integral becomes

$$\iiint_E f(x, y, z)dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x, y)}^{z=u_2(x, y)} f(x, y, z)dz\,dx\,dy.$$

Example 5.38

Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x, y, z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0, y = 0, z = 0,$ and $x + y + z = 1.$

Solution

Figure 5.45 shows the solid tetrahedron E and its projection D on the xy -plane.

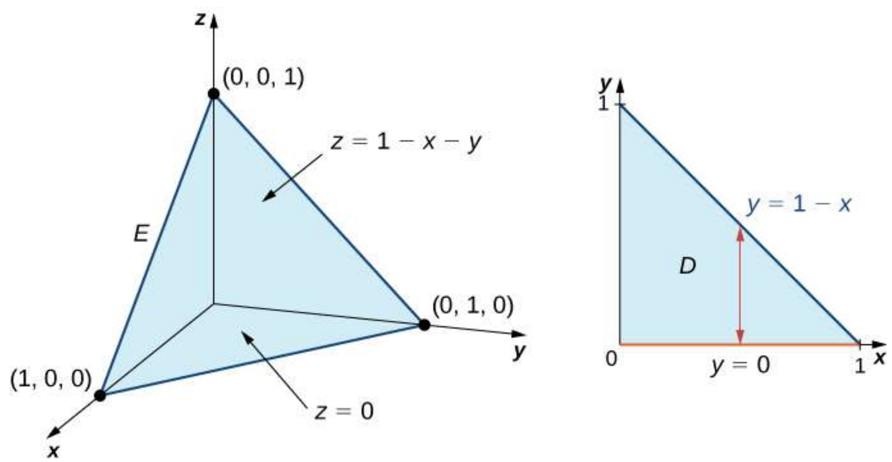


Figure 5.45 The solid E has a projection D on the xy -plane of Type I.

We can describe the solid region tetrahedron as

$$E = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}.$$

Hence, the triple integral is

$$\iiint_E f(x, y, z)dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)dz\,dy\,dx.$$

To simplify the calculation, first evaluate the integral $\int_{z=0}^{z=1-x-y} (5x - 3y)dz.$ We have

$$\int_{z=0}^{z=1-x-y} (5x - 3y)dz = (5x - 3y)(1 - x - y).$$

Now evaluate the integral $\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)dy,$ obtaining

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)dy = \tfrac{1}{2}(x - 1)^2(6x - 1).$$

Finally, evaluate

$$\int_{x=0}^{x=1} \tfrac{1}{2}(x - 1)^2(6x - 1)dx = \tfrac{1}{12}.$$

Putting it all together, we have

$$\iiint_E f(x, y, z)dV = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y}(5x-3y)dz\,dy\,dx = \frac{1}{12}.$$

Just as we used the double integral $\iint_D 1dA$ to find the area of a general bounded region $D,$ we can use $\iiint_E 1dV$ to find the volume of a general solid bounded region $E.$ The next example illustrates the method.

Example 5.39

Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1, 1] \times [-1, 1]$ and vertex at the point $(0, 0, 1)$ as shown in the following figure.

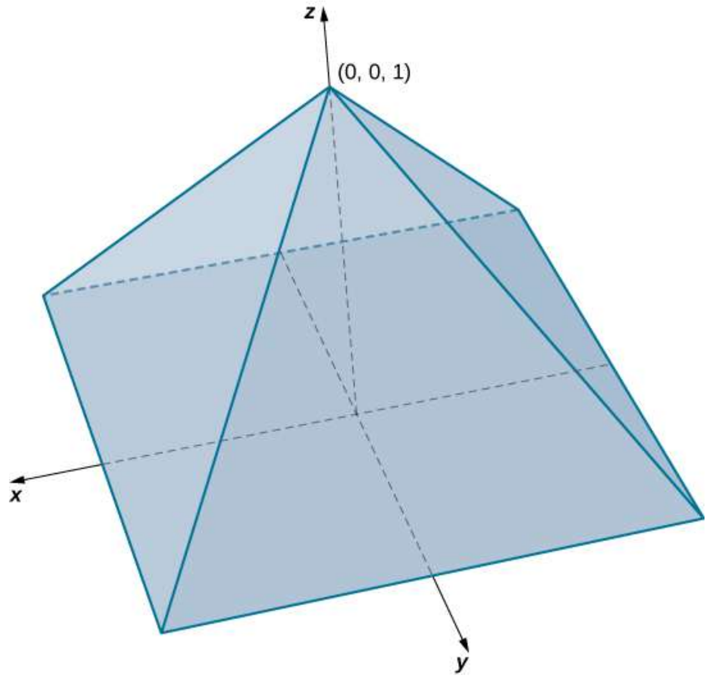


Figure 5.46 Finding the volume of a pyramid with a square base.

Solution

In this pyramid the value of z changes from 0 to $1,$ and at each height $z,$ the cross section of the pyramid for any value of z is the square $[-1+z, 1-z] \times [-1+z, 1-z].$ Hence, the volume of the pyramid is $\iiint_E 1dV$ where

$$E = \{(x, y, z)|0 \le z \le 1, -1+z \le y \le 1-z, -1+z \le x \le 1-z\}.$$

Thus, we have

$$\iiint_E 1dV = \int_{z=0}^{z=1} \int_{y=1+z}^{y=1-z} \int_{x=1+z}^{x=1-z} 1dx\,dy\,dz = \int_{z=0}^{z=1} \int_{y=1+z}^{y=1-z} (2-2z)dy\,dz = \int_{z=0}^{z=1} (2-2z)^2\,dz = \frac{4}{3}.$$

Hence, the volume of the pyramid is $\frac{4}{3}$ cubic units.

5.24 Consider the solid sphere $E = \{(x, y, z) | x^2 + y^2 + z^2 = 9\}$. Write the triple integral $\iiint_E f(x, y, z)dV$ for an arbitrary function f as an iterated integral. Then evaluate this triple integral with $f(x, y, z) = 1$. Notice that this gives the volume of a sphere using a triple integral.

Changing the Order of Integration

As we have already seen in double integrals over general bounded regions, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

Example 5.40

Changing the Order of Integration

Consider the iterated integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=y} f(x, y, z)dz\,dy\,dx.$$

The order of integration here is first with respect to z, then y, and then x. Express this integral by changing the order of integration to be first with respect to x, then z, and then y. Verify that the value of the integral is the same if we let $f(x, y, z) = xyz$.

Solution

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

$$E = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le y\}.$$

and

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=y^2} f(x, y, z)dz\,dy\,dx = \iiint_E f(x, y, z)dV.$$

We need to express this triple integral as

$$\int_{y=c}^{y=d} \int_{z=v_1(y)}^{z=v_2(y)} \int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x, y, z)\,dx\,dz\,dy.$$

Knowing the region E we can draw the following projections (**Figure 5.47**):

on the xy-plane is $D_1 = \{(x, y) | 0 \le x \le 1, 0 \le y \le x^2\} = \{(x, y) | 0 \le y \le 1, \sqrt{y} \le x \le 1\}$,

on the yz-plane is $D_2 = \{(y, z) | 0 \le y \le 1, 0 \le z \le y^2\}$, and

on the xz-plane is $D_3 = \{(x, z) | 0 \le x \le 1, 0 \le z \le x^2\}$.

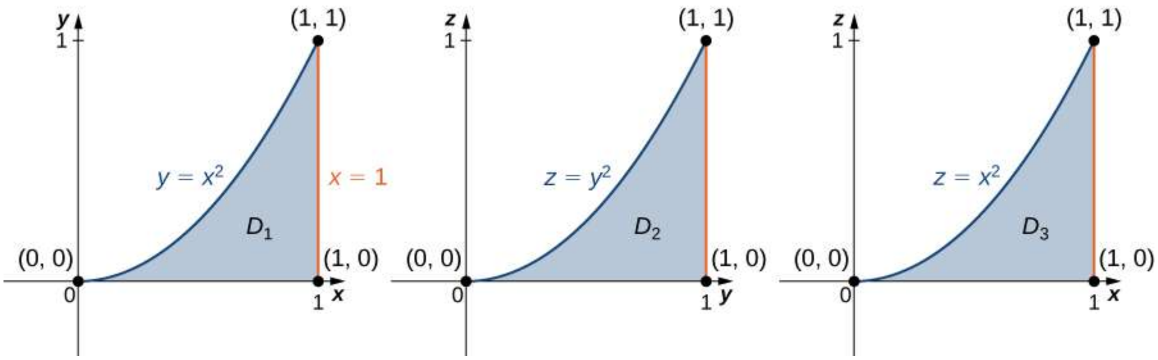


Figure 5.47 The three cross sections of E on the three coordinate planes.

Now we can describe the same region E as $\{(x, y, z) | 0 \le y \le 1, 0 \le z \le y^2, \sqrt{y} \le x \le 1\}$, and consequently, the triple integral becomes

$$\int_{y=c}^{y=d} \int_{z=v_1(y)}^{z=v_2(y)} \int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x, y, z)\,dx\,dz\,dy = \int_{y=0}^{y=1} \int_{z=0}^{z=x^2} \int_{x=\sqrt{y}}^{x=1} f(x, y, z)\,dx\,dz\,dy.$$

Now assume that $f(x, y, z) = xyz$ in each of the integrals. Then we have

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y^2} xyz\,dz\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left[xy\frac{z^2}{2}\Big|_{z=0}^{z=y^2}\right]dy\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left(x\frac{y^5}{2}\right)dy\,dx = \int_{x=0}^{x=1}\left[x\frac{y^6}{12}\Big|_{y=0}^{y=x^2}\right]dx = \int_{x=0}^{x=1}\frac{x^{13}}{12}dx = \frac{1}{168},$$

$$\int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\int_{x=\sqrt{y}}^{x=1} xyz\,dx\,dz\,dy$$

$$= \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left[yz\frac{x^2}{2}\Big|_{\sqrt{y}}^{1}\right]dz\,dy$$

$$= \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left(\frac{yz}{2}-\frac{y^2z}{2}\right)dz\,dy = \int_{y=0}^{y=1}\left[\frac{yz^2}{4}-\frac{y^2z^2}{4}\Big|_{z=0}^{z=y^2}\right]dy = \int_{y=0}^{y=1}\left(\frac{y^5}{4}-\frac{y^6}{4}\right)dy = \frac{1}{168}.$$

The answers match.

5.25 Write five different iterated integrals equal to the given integral

$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f(x,\ y,\ z)dx\,dy\,dz.$$

Example 5.41

Changing Integration Order and Coordinate Systems

Evaluate the triple integral $\iiint_E \sqrt{x^2+z^2}dV,$ where E is the region bounded by the paraboloid $y = x^2 + z^2$ (**Figure 5.48**) and the plane $y = 4.$

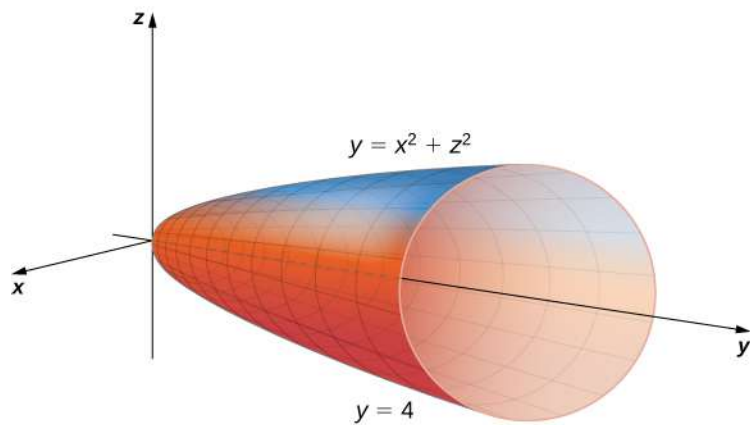


Figure 5.48 Integrating a triple integral over a paraboloid.

Solution

The projection of the solid region E onto the xy -plane is the region bounded above by $y = 4$ and below by the parabola $y = x^2$ as shown.

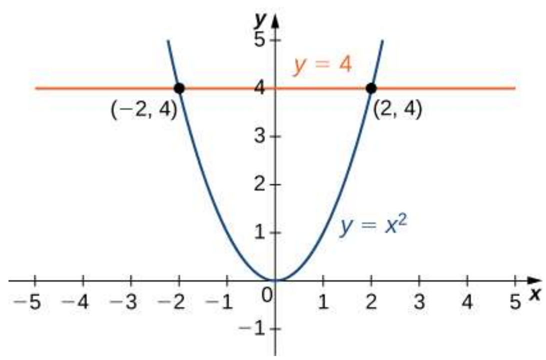


Figure 5.49 Cross section in the xy -plane of the paraboloid in **Figure 5.48**.

Thus, we have

$$E = \left\{(x,\ y,\ z) | -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\right\}.$$

The triple integral becomes

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx.$$

This expression is difficult to compute, so consider the projection of E onto the xz -plane. This is a circular disc $x^2 + z^2 \le 4.$ So we obtain

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx.$$

Here the order of integration changes from being first with respect to $z,$ then $y,$ and then x to being first with respect to $y,$ then to $z,$ and then to $x.$ It will soon be clear how this change can be beneficial for computation. We have

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\,dy\,dz\,dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \left(4-x^2-z^2\right)\sqrt{x^2+z^2}\,dz\,dx.$$

Now use the polar substitution $x = r\cos\theta,\ z = r\sin\theta,$ and $dz\,dx = r\,dr\,d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by $z.$ Consequently the limits of integration change and we have, by using $r^2 = x^2 + z^2,$

$$\begin{aligned} \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \left(4-x^2-z^2\right)\sqrt{x^2+z^2}\,dz\,dx &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \left(4-r^2\right) r r\,dr\,d\theta \\ &= \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5}\Big|_0^2\right] d\theta = \int_0^{2\pi} \frac{64}{15}\,d\theta = \frac{128\pi}{15}. \end{aligned}$$

Average Value of a Function of Three Variables

Recall that we found the average value of a function of two variables by evaluating the double integral over a region on the plane and then dividing by the area of the region. Similarly, we can find the average value of a function in three variables by evaluating the triple integral over a solid region and then dividing by the volume of the solid.

Theorem 5.11: Average Value of a Function of Three Variables

If $f(x, y, z)$ is integrable over a solid bounded region E with positive volume $V(E),$ then the average value of the function is

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z)\,dV.$$

Note that the volume is $V(E) = \iiint_E 1\,dV.$

Example 5.42

Finding an Average Temperature

The temperature at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 1$ is $T(x, y, z) = (xy + 8z + 20)$°C. Find the average temperature over the solid.

Solution

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division. Notice that the plane $x + y + z = 1$ has intercepts $(1, 0, 0), (0, 1, 0),$ and $(0, 0, 1)$. The region E looks like

$$E = \{(x, y, z) | 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}.$$

Hence the triple integral of the temperature is

$$\iiint_E f(x, y, z)dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (xy + 8z + 20)dz\,dy\,dx = \frac{147}{40}.$$

The volume evaluation is $V(E) = \iiint_E 1dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 1dz\,dy\,dx = \frac{1}{6}.$

Hence the average value is $T_{\text{ave}} = \frac{147/40}{1/6} = \frac{6(147)}{40} = \frac{441}{20}$ degrees Celsius.

5.26 Find the average value of the function $f(x, y, z) = xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

5.4 EXERCISES

In the following exercises, evaluate the triple integrals over the rectangular solid box B.

181. $\iiint_B (2x + 3y^2 + 4z^3)dV,$ where $B = \{(x, y, z)|0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3\}$

182. $\iiint_B (xy + yz + xz)dV,$ where $B = \{(x, y, z)|1 \le x \le 2, 0 \le y \le 2, 1 \le z \le 3\}$

183. $\iiint_B (x\cos y + z)dV,$ where $B = \{(x, y, z)|0 \le x \le 1, 0 \le y \le \pi, -1 \le z \le 1\}$

184. $\iiint_B (z\sin x + y^2)dV,$ where $B = \{(x, y, z)|0 \le x \le \pi, 0 \le y \le 1, -1 \le z \le 2\}$

In the following exercises, change the order of integration by integrating first with respect to $z,$ then $x,$ then $y.$

185. $\int_0^1 \int_1^2 \int_2^3 (x^2 + \ln y + z)dx\,dy\,dz$

186. $\int_0^1 \int_{-1}^1 \int_0^3 (ze^x + 2y)dx\,dy\,dz$

187. $\int_{-1}^2 \int_1^3 \int_0^4 \left(x^2 z + \frac{1}{y}\right)dx\,dy\,dz$

188. $\int_1^2 \int_{-2}^{-1} \int_0^1 \frac{x+y}{z}dx\,dy\,dz$

189. Let $F, G,$ and H be continuous functions on $[a, b], [c, d],$ and $[e, f],$ respectively, where $a, b, c, d, e,$ and f are real numbers such that $a < b, c < d,$ and $e < f.$ Show that

$$\int_a^b \int_c^d \int_e^f F(x)G(y)H(z)dz\,dy\,dx = \left(\int_a^b F(x)dx\right)\left(\int_c^d G(y)dy\right)\left(\int_e^f H(z)dz\right).$$

190. Let $F, G,$ and H be differential functions on $[a, b], [c, d],$ and $[e, f],$ respectively, where $a, b, c, d, e,$ and f are real numbers such that $a < b, c < d,$ and $e < f.$ Show that

$$\int_a^b \int_c^d \int_e^f F'(x)G'(y)H'(z)dz\,dy\,dx = [F(b) - F(a)][G(d) - G(c)][H(f) - H(e)].$$

In the following exercises, evaluate the triple integrals over the bounded region $E = \{(x, y, z)|a \le x \le b, h_1(x) \le y \le h_2(x), e \le z \le f\}.$

191. $\iiint_E (2x + 5y + 7z)dV,$ where $E = \{(x, y, z)|0 \le x \le 1, 0 \le y \le -x + 1, 1 \le z \le 2\}$

192. $\iiint_E (y\ln x + z)dV,$ where $E = \{(x, y, z)|1 \le x \le e, 0 \le y \le \ln x, 0 \le z \le 1\}$

193. $\iiint_E (\sin x + \sin y)dV,$ where $E = \left\{(x, y, z)|0 \le x \le \frac{\pi}{2}, -\cos x \le y \le \cos x, -1 \le z \le 1\right\}$

194. $\iiint_E (xy + yz + xz)dV,$ where $E = \left\{(x, y, z)|0 \le x \le 1, -x^2 \le y \le x^2, 0 \le z \le 1\right\}$

In the following exercises, evaluate the triple integrals over the indicated bounded region $E.$

195. $\iiint_E (x + 2yz)dV,$ where $E = \{(x, y, z)|0 \le x \le 1, 0 \le y \le x, 0 \le z \le 5 - x - y\}$

196. $\iiint_E (x^3 + y^3 + z^3)dV,$ where $E = \{(x, y, z)|0 \le x \le 2, 0 \le y \le 2x, 0 \le z \le 4 - x - y\}$

197. $\iiint_E y\,dV,$ where $E = \left\{(x, y, z)\middle| -1 \le x \le 1, -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}, 0 \le z \le 1 - x^2 - y^2\right\}$

198. $\iiint_E x\,dV,$ where $E = \left\{(x, y, z)\middle| -2 \le x \le 2, -4\sqrt{1 - x^2} \le y \le \sqrt{4 - x^2}, 0 \le z \le 4 - x^2 - y^2\right\}$

In the following exercises, evaluate the triple integrals over the bounded region E of the form $E = \{(x, y, z)|g_1(y) \le x \le g_2(y), c \le y \le d, e \le z \le f\}.$

199. $\iiint_E x^2\,dV,$ where $E = \{(x, y, z) | 1 - y^2 \le x \le y^2 - 1, -1 \le y \le 1, 1 \le z \le 2\}$

200. $\iiint_E (\sin x + y)dV,$ where $E = \{(x, y, z) | -y^4 \le x \le y^4, 0 \le y \le 2, 0 \le z \le 4\}$

201. $\iiint_E (x - yz)dV,$ where $E = \{(x, y, z) | -y^6 \le x \le \sqrt{y}, 0 \le y \le 1x, -1 \le z \le 1\}$

202. $\iiint_E z dV,$ where $E = \{(x, y, z) | 2 - 2y \le x \le 2 + \sqrt{y}, 0 \le y \le 1x, 2 \le z \le 3\}$

In the following exercises, evaluate the triple integrals over the bounded region

$E = \{(x, y, z) | g_1(y) \le x \le g_2(y), c \le y \le d, u_1(x, y) \le z \le u_2(x, y)\}.$

203. $\iiint_E z dV,$ where $E = \{(x, y, z) | -y \le x \le y, 0 \le y \le 1, 0 \le z \le 1 - x^4 - y^4\}$

204. $\iiint_E (xz + 1)dV,$ where $E = \{(x, y, z) | 0 \le x \le \sqrt{y}, 0 \le y \le 2, 0 \le z \le 1 - x^2 - y^2\}$

205. $\iiint_E (x - z)dV,$ where $E = \{(x, y, z) | -\sqrt{1 - y^2} \le x \le y, 0 \le y \le \frac{1}{2}x, 0 \le z \le 1 - x^2 - y^2\}$

206. $\iiint_E (x + y)dV,$ where $E = \{(x, y, z) | 0 \le x \le \sqrt{1 - y^2}, 0 \le y \le 1x, 0 \le z \le 1 - x\}$

In the following exercises, evaluate the triple integrals over the bounded region

$E = \{(x, y, z) | (x, y) \in D, u_1(x, y)x \le z \le u_2(x, y)\},$

where D is the projection of E onto the xy-plane.

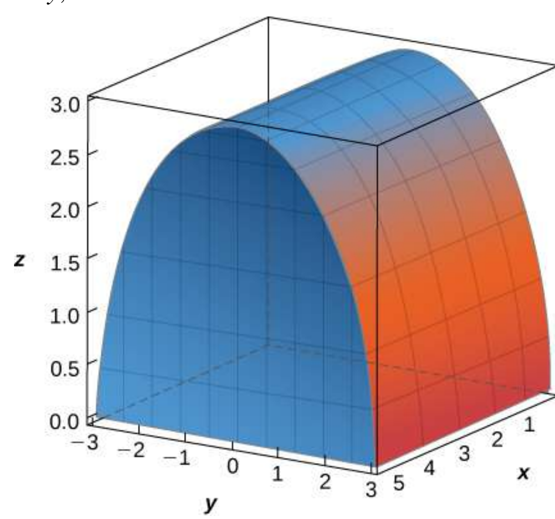
207. $\iint_D \left(\int_1^2 (x + z)dz\right)dA,$ where $D = \{(x, y) | x^2 + y^2 \le 1\}$

208. $\iint_D \left(\int_1^3 x(z + 1)dz\right)dA,$ where $D = \{(x, y) | x^2 - y^2 \ge 1, x \le \sqrt{5}\}$

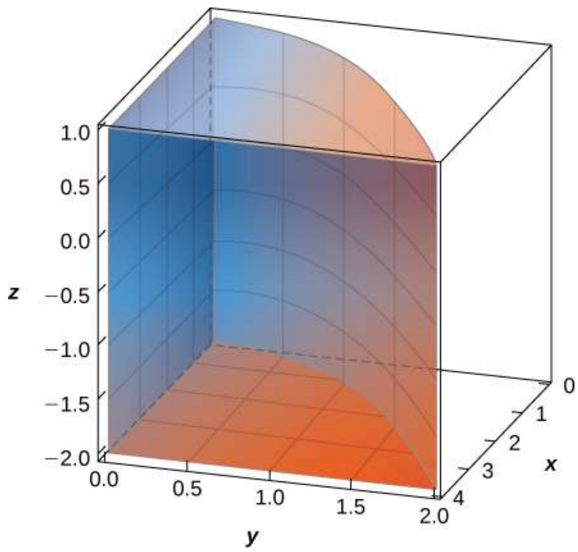
209. $\iint_D \left(\int_0^{10 - x - y} (x + 2z)dz\right)dA,$ where $D = \{(x, y) | y \ge 0, x \ge 0, x + y \le 10\}$

210. $\iint_D \left(\int_0^{4x^2 + 4y^2} y\,dz\right)dA,$ where $D = \{(x, y) | x^2 + y^2 \le 4, y \ge 1, x \ge 0\}$

211. The solid E bounded by $y^2 + z^2 = 9,$ $z = 0,$ and $x = 5$ is shown in the following figure. Evaluate the integral $\iiint_E z\,dV$ by integrating first with respect to $z,$ then $y,$ and then $x.$



212. The solid E bounded by $y = \sqrt{x}$, $x = 4$, $y = 0$, and $z = 1$ is given in the following figure. Evaluate the integral $\iiint_E xyz\, dV$ by integrating first with respect to x, then y, and then z.



213. **[T]** The volume of a solid E is given by the integral $\int_{-2}^{0} \int_{x}^{0} \int_{0}^{x^2+y^2} dz\, dy\, dx$. Use a computer algebra system (CAS) to graph E and find its volume. Round your answer to two decimal places.

214. **[T]** The volume of a solid E is given by the integral $\int_{-1}^{0} \int_{-x^2}^{0} \int_{0}^{1+\sqrt{x^2+y^2}} dz\, dy\, dx$. Use a CAS to graph E and find its volume V. Round your answer to two decimal places.

In the following exercises, use two circular permutations of the variables $x, y,$ and z to write new integrals whose values equal the value of the original integral. A circular permutation of $x, y,$ and z is the arrangement of the numbers in one of the following orders: $y, z,$ and x or $z, x,$ and y.

215. $\int_0^1 \int_1^3 \int_2^4 \left(x^2 z^2 + 1\right) dx\, dy\, dz$

216. $\int_1^3 \int_0^1 \int_0^{-x+1} (2x + 5y + 7z) dy\, dx\, dz$

217. $\int_0^1 \int_{-y}^{y} \int_0^{1-x^4-y^4} \ln x\, dz\, dx\, dy$

218. $\int_{-1}^{1} \int_0^1 \int_{-y^6}^{\sqrt{y}} (x + yz) dx\, dy\, dz$

219. Set up the integral that gives the volume of the solid E bounded by $y^2 = x^2 + z^2$ and $y = a^2$, where $a > 0$.

220. Set up the integral that gives the volume of the solid E bounded by $x = y^2 + z^2$ and $x = a^2$, where $a > 0$.

221. Find the average value of the function $f(x, y, z) = x + y + z$ over the parallelepiped determined by $x = 0$, $x = 1$, $y = 0$, $y = 3$, $z = 0$, and $z = 5$.

222. Find the average value of the function $f(x, y, z) = xyz$ over the solid $E = [0, 1] \times [0, 1] \times [0, 1]$ situated in the first octant.

223. Find the volume of the solid E that lies under the plane $x + y + z = 9$ and whose projection onto the xy-plane is bounded by $x = \sqrt{y - 1}$, $x = 0$, and $x + y = 7$.

224. Find the volume of the solid E that lies under the plane $2x + y + z = 8$ and whose projection onto the xy-plane is bounded by $x = \sin^{-1} y$, $y = 0$, and $x = \frac{\pi}{2}$.

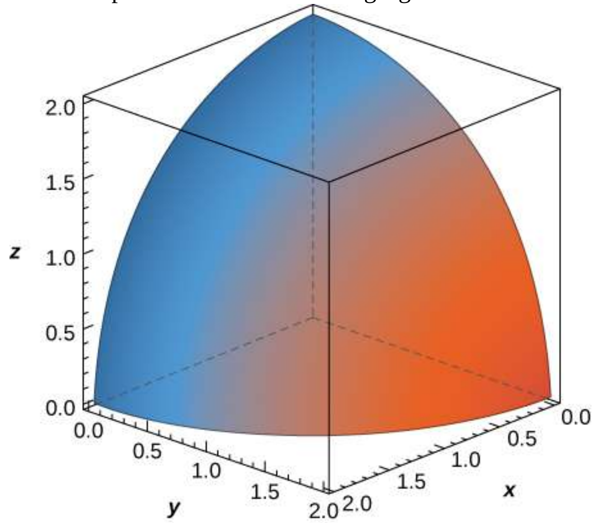
225. Consider the pyramid with the base in the xy-plane of $[-2, 2] \times [-2, 2]$ and the vertex at the point $(0, 0, 8)$.

a. Show that the equations of the planes of the lateral faces of the pyramid are $4y + z = 8$, $4y - z = -8$, $4x + z = 8$, and $-4x + z = 8$.
b. Find the volume of the pyramid.

226. Consider the pyramid with the base in the xy-plane of $[-3, 3] \times [-3, 3]$ and the vertex at the point $(0, 0, 9)$.

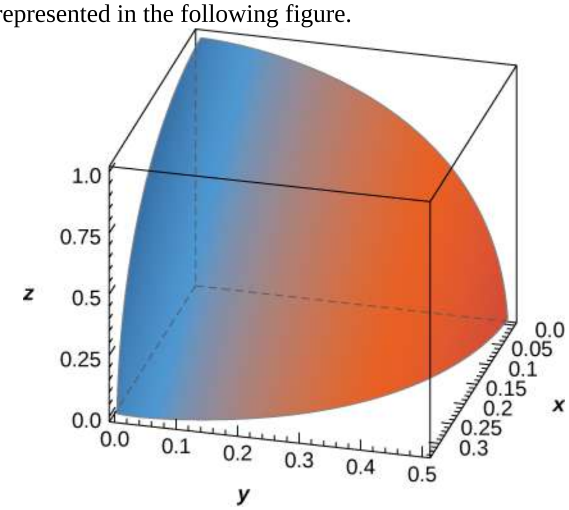
a. Show that the equations of the planes of the side faces of the pyramid are $3y + z = 9$, $3y + z = 9$, $y = 0$ and $x = 0$.
b. Find the volume of the pyramid.

227. The solid E bounded by the sphere of equation $x^2 + y^2 + z^2 = r^2$ with $r > 0$ and located in the first octant is represented in the following figure.



a. Write the triple integral that gives the volume of E by integrating first with respect to $z,$ then with $y,$ and then with $x.$
b. Rewrite the integral in part a. as an equivalent integral in five other orders.

228. The solid E bounded by the equation $9x^2 + 4y^2 + z^2 = 1$ and located in the first octant is represented in the following figure.

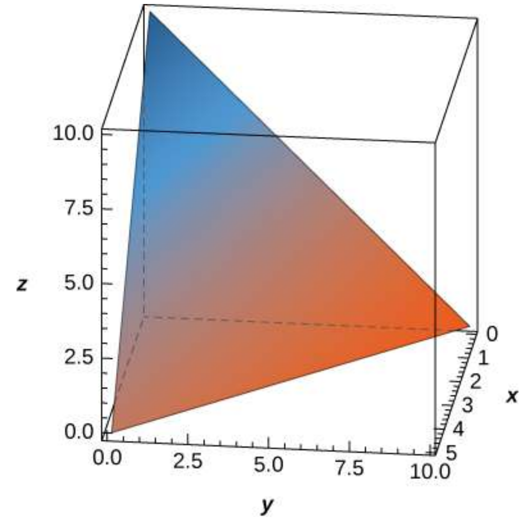


a. Write the triple integral that gives the volume of E by integrating first with respect to $z,$ then with $y,$ and then with $x.$
b. Rewrite the integral in part a. as an equivalent integral in five other orders.

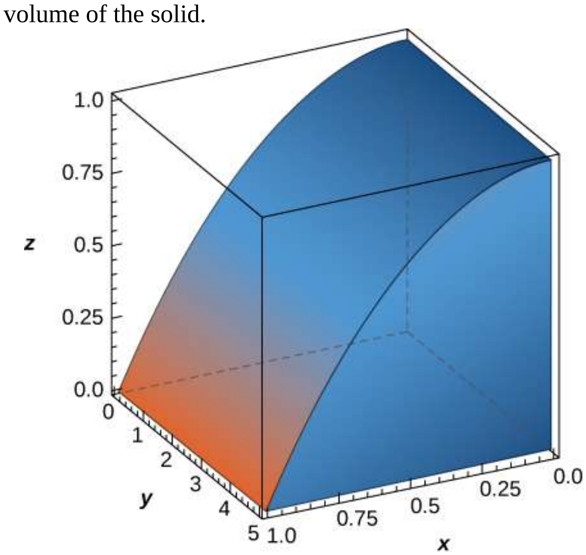
229. Find the volume of the prism with vertices $(0,\ 0,\ 0),\ (2,\ 0,\ 0),\ (2,\ 3,\ 0),$ $(0,\ 3,\ 0),\ (0,\ 0,\ 1),$ and $(2,\ 0,\ 1).$

230. Find the volume of the prism with vertices $(0,\ 0,\ 0),\ (4,\ 0,\ 0),\ (4,\ 6,\ 0),$ $(0,\ 6,\ 0),\ (0,\ 0,\ 1),$ and $(4,\ 0,\ 1).$

231. The solid E bounded by $z = 10 - 2x - y$ and situated in the first octant is given in the following figure. Find the volume of the solid.



232. The solid E bounded by $z = 1 - x^2$ and situated in the first octant is given in the following figure. Find the volume of the solid.



233. The midpoint rule for the triple integral $\iiint_B f(x, y, z)dV$ over the rectangular solid box B is a generalization of the midpoint rule for double integrals. The region B is divided into subboxes of equal sizes and the integral is approximated by the triple Riemann sum $\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(\bar{x}_i, \bar{y}_j, \bar{z}_k\right)\Delta V,$ where $\left(\bar{x}_i, \bar{y}_j, \bar{z}_k\right)$ is the center of the box B_{ijk} and ΔV is the volume of each subbox. Apply the midpoint rule to approximate $\iiint_B x^2\,dV$ over the solid $B = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

234. **[T]**

a. Apply the midpoint rule to approximate $\iiint_B e^{-x^2}\,dV$ over the solid $B = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

b. Use a CAS to improve the above integral approximation in the case of a partition of n^3 cubes of equal size, where $n = 3, 4,\ldots, 10.$

235. Suppose that the temperature in degrees Celsius at a point (x, y, z) of a solid E bounded by the coordinate planes and $x + y + z = 5$ is $T(x, y, z) = xz + 5z + 10.$ Find the average temperature over the solid.

236. Suppose that the temperature in degrees Fahrenheit at a point (x, y, z) of a solid E bounded by the coordinate planes and $x + y + z = 5$ is $T(x, y, z) = x + y + xy.$ Find the average temperature over the solid.

237. Show that the volume of a right square pyramid of height h and side length a is $v = \frac{ha^2}{3}$ by using triple integrals.

238. Show that the volume of a regular right hexagonal prism of edge length a is $\frac{3a^3\sqrt{3}}{2}$ by using triple integrals.

239. Show that the volume of a regular right hexagonal pyramid of edge length a is $\frac{a^3\sqrt{3}}{2}$ by using triple integrals.

240. If the charge density at an arbitrary point (x, y, z) of a solid E is given by the function $\rho(x, y, z),$ then the total charge inside the solid is defined as the triple integral $\iiint_E \rho(x, y, z)dV.$ Assume that the charge density of the solid E enclosed by the paraboloids $x = 5 - y^2 - z^2$ and $x = y^2 + z^2 - 5$ is equal to the distance from an arbitrary point of E to the origin. Set up the integral that gives the total charge inside the solid E.

5.5 | Triple Integrals in Cylindrical and Spherical Coordinates

Learning Objectives

5.5.1 Evaluate a triple integral by changing to cylindrical coordinates.
5.5.2 Evaluate a triple integral by changing to spherical coordinates.

Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter opener, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2,$ a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\cos\theta,\quad y = r\sin\theta,\quad r^2 = x^2 + y^2$ and $\tan\theta = \left(\frac{y}{x}\right)$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3,$ a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the xy-plane as shown in the following figure.

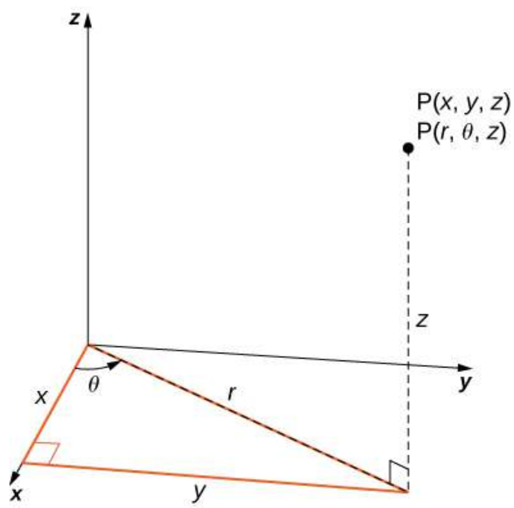


Figure 5.50 Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion $x = r\cos\theta$ and $y = r\sin\theta.$ To convert from cylindrical to rectangular coordinates, we use $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right).$ The z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis.

With the polar coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z), the equations $x = k, y = l,$ and $z = m,$ where $k, l,$ and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c, \theta = \alpha,$ and $z = m,$ where $c, \alpha,$ and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively. This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates. (Refer to **Cylindrical and Spherical Coordinates** for more review.)

Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in **Table 5.1**. These equations will become handy as we proceed with solving problems using triple integrals.

	Circular cylinder	Circular cone	Sphere	Paraboloid
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2(x^2 + y^2)$	$x^2 + y^2 + z^2 = c^2$	$z = c(x^2 + y^2)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

Table 5.1 Equations of Some Common Shapes

As before, we start with the simplest bounded region B in $\mathbb{R}^3$, to describe in cylindrical coordinates, in the form of a cylindrical box, $B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$ (**Figure 5.51**). Suppose we divide each interval into l, m and n subdivisions such that $\Delta r = \frac{b-a}{l}, \Delta\theta = \frac{\beta - \alpha}{m},$ and $\Delta z = \frac{d-c}{n}.$ Then we can state the following definition for a triple integral in cylindrical coordinates.

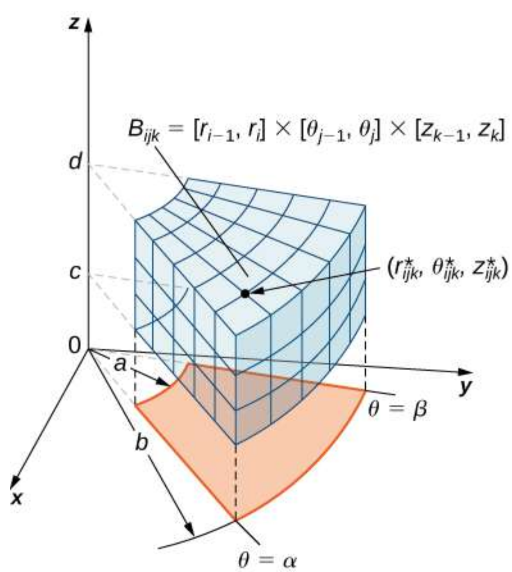


Figure 5.51 A cylindrical box B described by cylindrical coordinates.

Definition

Consider the cylindrical box (expressed in cylindrical coordinates)

$$B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}.$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r_{ijk}^*, \theta_{ijk}^*, z_{ijk}^*)$ is any sample point in the cylindrical subbox $B_{ijk} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j] \times [z_{k-1}, z_k]$ (**Figure 5.51**), then we can define the **triple integral in cylindrical coordinates** as the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r_{ijk}^*, \theta_{ijk}^*, z_{ijk}^*) r_{ijk}^* \Delta r \Delta\theta \Delta z.$$

Note that if $g(x, y, z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral $\iiint_B g(x, y, z) dV$ is equal to the triple integral $\iiint_B g(r\cos\theta, r\sin\theta, z) r\, dr\, d\theta\, dz$ and we have

$$\iiint_B g(x, y, z) dV = \iiint_B g(r\cos\theta, r\sin\theta, z) r\, dr\, d\theta\, dz = \iiint_B f(r, \theta, z) r\, dr\, d\theta\, dz. \quad (5.11)$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Theorem 5.12: Fubini's Theorem in Cylindrical Coordinates

Suppose that $g(x, y, z)$ is continuous on a rectangular box B, which when described in cylindrical coordinates looks like $B = \{(r, \theta, z) | a \le r \le b, \alpha \le \theta \le \beta, c \le z \le d\}$.

Then $g(x, y, z) = g(r\cos\theta, r\sin\theta, z) = f(r, \theta, z)$ and

$$\iiint_B g(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b f(r, \theta, z) r\, dr\, d\theta\, dz.$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example 5.43

Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral $\iiint_B (zr\sin\theta) r\, dr\, d\theta\, dz$ where the cylindrical box B is $B = \{(r, \theta, z) | 0 \le r \le 2, 0 \le \theta \le \pi/2, 0 \le z \le 4\}$.

Solution

As stated in Fubini's theorem, we can write the triple integral as the iterated integral

$$\iiint\limits_B (zr \sin \theta) r\, dr\, d\theta\, dz = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr \sin \theta) r\, dz\, dr\, d\theta.$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

$$\int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr \sin \theta) r\, dz\, dr\, d\theta$$
$$= \left(\int_0^{\pi/2} \sin \theta\, d\theta\right)\left(\int_0^2 r^2\, dr\right)\left(\int_0^4 z\, dz\right) = \left(-\cos \theta\big|_0^{\pi/2}\right)\left(\frac{r^3}{3}\bigg|_0^2\right)\left(\frac{z^2}{2}\bigg|_0^4\right) = \frac{64}{3}.$$

5.27 Evaluate the triple integral $\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=4} rz \sin \theta r\, dz\, dr\, d\theta.$

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function $f(r, \theta, z)$ over a general solid region $E = \{(r, \theta, z) | (r, \theta) \in D, u_1(r, \theta) \le z \le u_2(r, \theta)\}$ in $\mathbb{R}^3$, where D is the projection of E onto the $r\theta$-plane, is

$$\iiint\limits_E f(r, \theta, z) r\, dr\, d\theta\, dz = \iint\limits_D \left[\int_{u_1(r, \theta)}^{u_2(r, \theta)} f(r, \theta, z) dz\right] r\, dr\, d\theta.$$

In particular, if $D = \{(r, \theta) | g_1(\theta) \le r \le g_2(\theta), \alpha \le \theta \le \beta\}$, then we have

$$\iiint\limits_E f(r, \theta, z) r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r, \theta)}^{z=u_2(r, \theta)} f(r, \theta, z) r\, dz\, dr\, d\theta.$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

Example 5.44

Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r = 2 \sin \theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin (**Figure 5.52**). Set up a triple integral over this region with a function $f(r, \theta, z)$ in cylindrical coordinates.

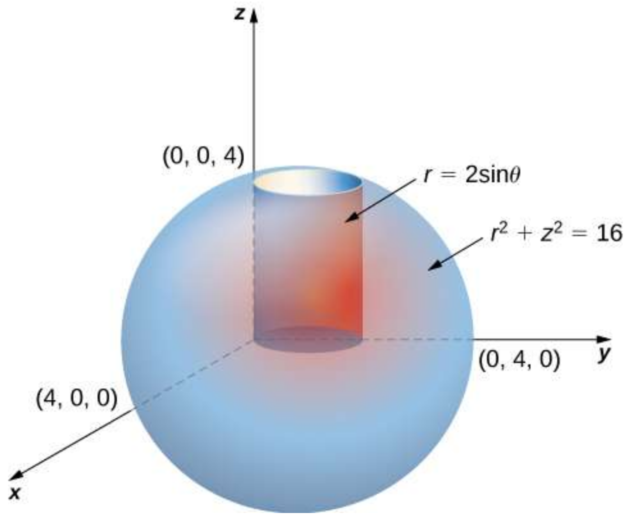


Figure 5.52 Setting up a triple integral in cylindrical coordinates over a cylindrical region.

Solution

First, identify that the equation for the sphere is $r^2 + z^2 = 16$. We can see that the limits for z are from 0 to $z = \sqrt{16 - r^2}$. Then the limits for r are from 0 to $r = 2\sin\theta$. Finally, the limits for θ are from 0 to π. Hence the region is

$$E = \left\{(r, \theta, z) | 0 \le \theta \le \pi, 0 \le r \le 2\sin\theta, 0 \le z \le \sqrt{16 - r^2}\right\}.$$

Therefore, the triple integral is

$$\iiint_E f(r, \theta, z) r\, dz\, dr\, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=\sqrt{16-r^2}} f(r, \theta, z) r\, dz\, dr\, d\theta.$$

5.28 Consider the region E inside the right circular cylinder with equation $r = 2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r, \theta, z)$ in cylindrical coordinates.

Example 5.45

Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. (**Figure 5.53**). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz\, dr\, d\theta$

b. $dr\, dz\, d\theta$.

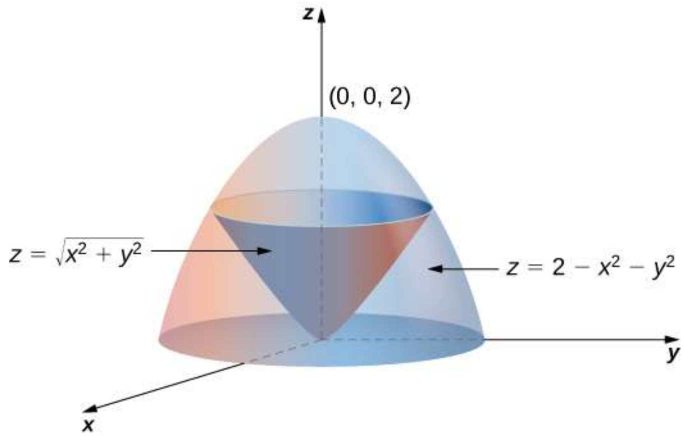


Figure 5.53 Setting up a triple integral in cylindrical coordinates over a conical region.

Solution

a. The cone is of radius 1 where it meets the paraboloid. Since $z = 2 - x^2 - y^2 = 2 - r^2$ and $z = \sqrt{x^2 + y^2} = r$ (assuming r is nonnegative), we have $2 - r^2 = r$. Solving, we have $r^2 + r - 2 = (r + 2)(r - 1) = 0$. Since $r \geq 0$, we have $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

Thus, we can describe the region as

$$E = \left\{(r, \theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r \leq z \leq 2 - r^2\right\}.$$

Hence the integral for the volume is

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r \, dz \, dr \, d\theta.$$

b. We can also write the cone surface as $r = z$ and the paraboloid as $r^2 = 2 - z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z = 1$ divides the region into two regions. Then the region can be described as

$$\begin{aligned} E &= \{(r, \theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1, 0 \leq r \leq z\} \\ &\cup \left\{(r, \theta, z) | 0 \leq \theta \leq 2\pi, 1 \leq z \leq 2, 0 \leq r \leq \sqrt{2 - z}\right\}. \end{aligned}$$

Now the integral for the volume becomes

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r \, dr \, dz \, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r \, dr \, dz \, d\theta.$$

5.29 Redo the previous example with the order of integration $d\theta \, dz \, dr$.

Example 5.46

Finding a Volume with Triple Integrals in Two Ways

Let E be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4,$ and on the sides by the cylinder $x^2 + y^2 = 1$ (**Figure 5.54**). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\, dr\, d\theta$

b. $dr\, dz\, d\theta.$

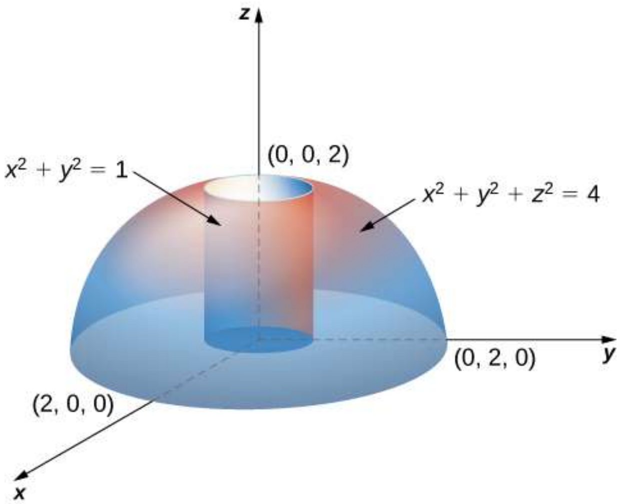


Figure 5.54 Finding a cylindrical volume with a triple integral in cylindrical coordinates.

Solution

a. Note that the equation for the sphere is

$$x^2 + y^2 + z^2 = 4 \text{ or } r^2 + z^2 = 4$$

and the equation for the cylinder is

$$x^2 + y^2 = 1 \text{ or } r^2 = 1.$$

Thus, we have for the region E

$$E = \left\{(r,\ \theta,\ z)|0 \le z \le \sqrt{4 - r^2},\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\right\}$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left[rz|_{z=0}^{z=\sqrt{4-r^2}}\right]dr\,d\theta = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left(r\sqrt{4-r^2}\right)dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{8}{3}-\sqrt{3}\right)d\theta = 2\pi\left(\frac{8}{3}-\sqrt{3}\right)\text{cubic units.} \end{aligned}$$

b. Since the sphere is $x^2+y^2+z^2=4,$ which is $r^2+z^2=4,$ and the cylinder is $x^2+y^2=1,$ which is $r^2=1,$ we have $1+z^2=4,$ that is, $z^2=3.$ Thus we have two regions, since the sphere and the cylinder intersect at $\left(1,\sqrt{3}\right)$ in the rz-plane

$$E_1 = \left\{(r,\theta,z)|0 \le r \le \sqrt{4-r^2}, \sqrt{3} \le z \le 2, 0 \le \theta \le 2\pi\right\}$$

and

$$E_2 = \{(r,\theta,z)|0 \le r \le 1, 0 \le z \le \sqrt{3}, 0 \le \theta \le 2\pi\}.$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi}\int_{z=\sqrt{3}}^{z=2}\int_{r=0}^{r=\sqrt{4-r^2}} r\,dr\,dz\,d\theta + \int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=\sqrt{3}}\int_{r=0}^{r=1} r\,dr\,dz\,d\theta \\ &= \sqrt{3}\pi + \left(\frac{16}{3}-3\sqrt{3}\right)\pi = 2\pi\left(\frac{8}{3}-\sqrt{3}\right)\text{cubic units.} \end{aligned}$$

5.30 Redo the previous example with the order of integration $d\theta\,dz\,dr$.

Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP (**Figure 5.55**). Note that $\rho \ge 0$ and $0 \le \varphi \le \pi$. (Refer to **Cylindrical and Spherical Coordinates** for a review.) Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.

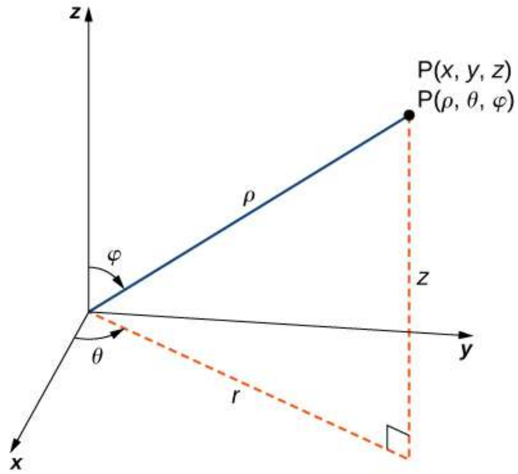


Figure 5.55 The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:

$$x = \rho \sin \varphi \cos \theta,\ y = \rho \sin \varphi \sin \theta,\ \text{ and } z = \rho \cos \varphi.$$

From rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan \theta = \frac{y}{x},\ \varphi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).$$

Other relationships that are important to know for conversions are

- $r = \rho \sin \varphi$
- $\theta = \theta$
- $z = \rho \cos \varphi$

These equations are used to convert from spherical coordinates to cylindrical coordinates

and

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$

These equations are used to convert from cylindrical coordinates to spherical coordinates.

The following figure shows a few solid regions that are convenient to express in spherical coordinates.

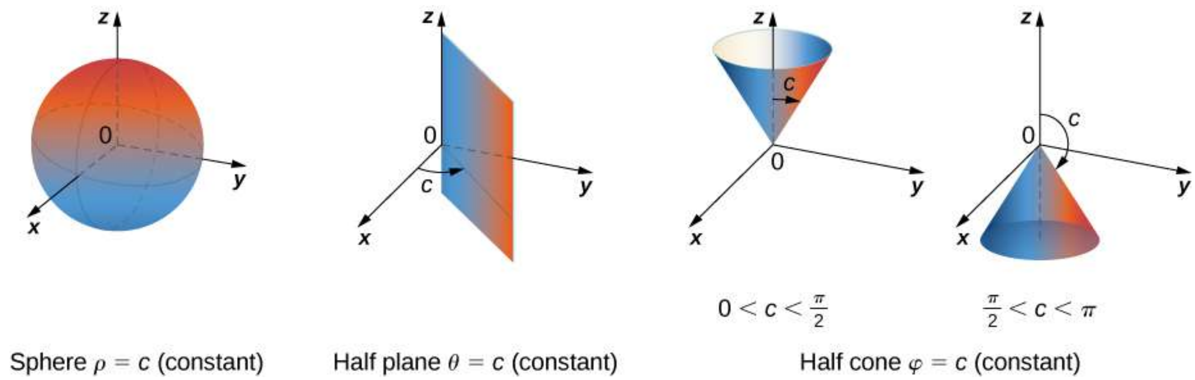


Figure 5.56 Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function $f(\rho, \theta, \varphi)$ be continuous in a bounded spherical box, $B = \{(\rho, \theta, \varphi)|a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \varphi \le \psi\}$. We then divide each interval into l, m and n subdivisions such that $\Delta\rho = \frac{b-a}{l}, \Delta\theta = \frac{\beta - \alpha}{m}, \Delta\varphi = \frac{\psi - \gamma}{n}$.

Now we can illustrate the following theorem for triple integrals in spherical coordinates with $(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})$ being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have $\Delta V = (\Delta\rho)(\rho\Delta\varphi)(\rho \sin \varphi\Delta\theta)$, , as shown in the following figure.

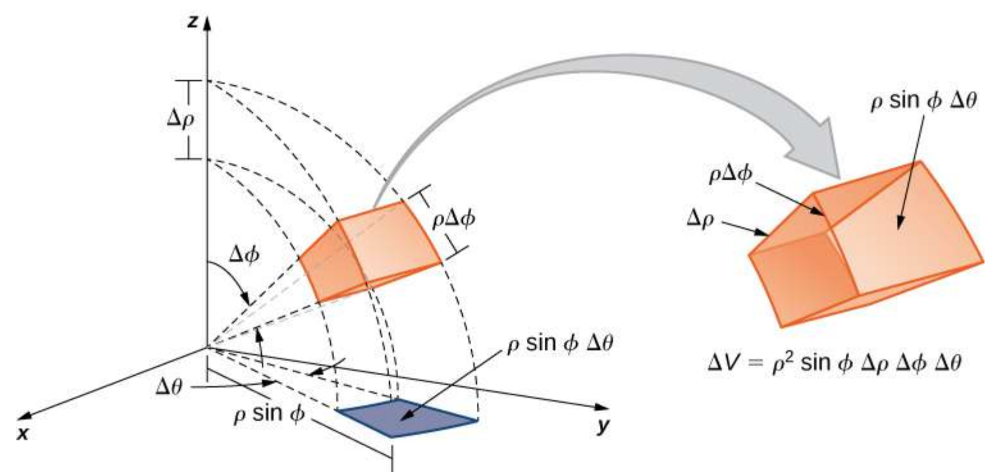


Figure 5.57 The volume element of a box in spherical coordinates.

Definition

The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,

$$\lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})(\rho^*_{ijk})^2 \sin \varphi \Delta\rho\Delta\theta\Delta\varphi$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini's theorem takes the following form.

Theorem 5.13: Fubini's Theorem for Spherical Coordinates

If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then

$$\iiint_B f(\rho, \theta, \varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta. \tag{5.12}$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example 5.47

Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral $\int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/2} \int_{p=0}^{\rho=1} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$

Solution

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi \, d\varphi \int_0^1 \rho^2 \, d\rho = (2\pi)(1)\left(\frac{1}{3}\right) = \frac{2\pi}{3}.$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to express the integrals.

The triple integral of a continuous function $f(\rho, \theta, \varphi)$ over a general solid region

$$E = \{(\rho, \theta, \varphi) | (\rho, \theta) \in D, u_1(\rho, \theta) \le \varphi \le u_2(\rho, \theta)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is

$$\iiint_E f(\rho, \theta, \varphi) dV = \iint_D \left[\int_{u_1(\rho, \theta)}^{u_2(\rho, \theta)} f(\rho, \theta, \varphi) d\varphi \right] dA.$$

In particular, if $D = \{(\rho, \theta) | g_1(\theta) \le \rho \le g_2(\theta), \alpha \le \theta \le \beta\}$, then we have

$$\iiint_E f(\rho, \theta, \varphi) dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(\rho, \theta)}^{u_2(\rho, \theta)} f(\rho, \theta, \varphi)\rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta.$$

Similar formulas occur for projections onto the other coordinate planes.

Example 5.48

Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z = \sqrt{3(x^2 + y^2)}$ and the hemisphere $z = \sqrt{4 - x^2 - y^2}$ (see the figure below).

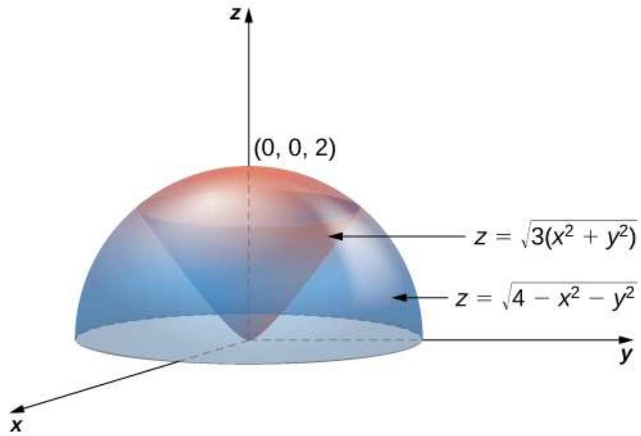


Figure 5.58 A region bounded below by a cone and above by a hemisphere.

Solution

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z = \sqrt{3(x^2 + y^2)}$ or $\rho \cos \varphi = \sqrt{3}\rho \sin \varphi$ or $\tan \varphi = \frac{1}{\sqrt{3}}$ or $\varphi = \frac{\pi}{6}$.

For the sphere: $z = \sqrt{4 - x^2 - y^2}$ or $z^2 + x^2 + y^2 = 4$ or $\rho^2 = 4$ or $\rho = 2$.

Thus, the triple integral for the volume is $V(E) = \int_{\theta = 0}^{\theta = 2\pi} \int_{\phi = 0}^{\varphi = \pi/6} \int_{\rho = 0}^{\rho = 2} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.$

5.31 Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho = 2$ and bounded below by the cone $\varphi = \pi/3$.

Example 5.49

Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $z = x^2 + y^2 + z^2$ (**Figure 5.59**). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho \, d\phi \, d\theta$,

b. $d\varphi\, d\rho\, d\theta.$

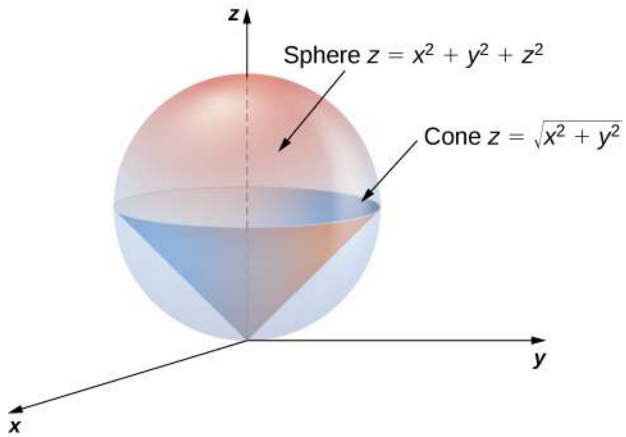


Figure 5.59 A region bounded below by a cone and above by a sphere.

Solution

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.
For the sphere:

$$\begin{aligned} x^2 + y^2 + z^2 &= z \\ \rho^2 &= \rho\cos\varphi \\ \rho &= \cos\varphi. \end{aligned}$$

For the cone:

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \\ \rho\cos\varphi &= \sqrt{\rho^2\sin^2\varphi\cos^2\phi + \rho^2\sin^2\varphi\sin^2\phi} \\ \rho\cos\varphi &= \sqrt{\rho^2\sin^2\varphi\left(\cos^2\phi + \sin^2\phi\right)} \\ \rho\cos\varphi &= \rho\sin\varphi \\ \cos\varphi &= \sin\varphi \\ \varphi &= \pi/4. \end{aligned}$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/4}\int_{\rho=0}^{\rho=\cos\varphi} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta.$$

b. Consider the $\varphi\rho$ -plane. Note that the ranges for φ and ρ (from part a.) are

$$\begin{aligned} 0 \le \varphi \le \pi/4 \\ 0 \le \rho \le \cos\varphi. \end{aligned}$$

The curve $\rho = \cos\varphi$ meets the line $\varphi = \pi/4$ at the point $\left(\pi/4, \sqrt{2}/2\right)$. Thus, to change the order of integration, we need to use two pieces:

$$\begin{aligned} 0 \le \rho \le \sqrt{2}/2 \\ 0 \le \varphi \le \pi/4 \end{aligned} \quad \text{and} \quad \begin{aligned} \sqrt{2}/2 &\le \rho \le 1 \\ 0 &\le \varphi \le \cos^{-1}\rho. \end{aligned}$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=\sqrt{2}/2} \int_{\varphi=0}^{\varphi=\pi/4} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\sqrt{2}/2}^{\rho=1} \int_{\varphi=0}^{\varphi=\cos^{-1}\rho} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta.$$

In each case, the integration results in $V(E) = \frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example 5.50

Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy.$$

Solution

The ranges of the variables are

$$\begin{aligned} -1 &\le y \le 1 \\ 0 &\le x \le \sqrt{1-y^2} \\ x^2+y^2 &\le z \le \sqrt{x^2+y^2}. \end{aligned}$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \text{ and } 0 \le r \le 1.$$

The limits of z are $r^2 \le z \le r,$ hence

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy = \int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=0}^{r=1} \int_{z=r^2}^{z=r} r(r\cos\theta)(r\sin\theta)z \, dz \, dr \, d\theta.$$

Example 5.51

Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

$$\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} \left(x^2+y^2+z^2\right) dz\, dx\, dy.$$

Solution

The ranges of the variables are

$$\begin{aligned} 0 &\le y \le 3 \\ 0 &\le x \le \sqrt{9-y^2} \\ \sqrt{x^2+y^2} &\le z \le \sqrt{18-x^2-y^2}. \end{aligned}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy -plane. Hence the range for θ is $0 \le \theta \le \frac{\pi}{2}$.

The lower bound $z = \sqrt{x^2+y^2}$ is the upper half of a cone and the upper bound $z = \sqrt{18-x^2-y^2}$ is the upper half of a sphere. Therefore, we have $0 \le \rho \le \sqrt{18}$, which is $0 \le \rho \le 3\sqrt{2}$.

For the ranges of φ, we need to find where the cone and the sphere intersect, so solve the equation

$$\begin{aligned} r^2 + z^2 &= 18 \\ \left(\sqrt{x^2+y^2}\right)^2 + z^2 &= 18 \\ z^2 + z^2 &= 18 \\ 2z^2 &= 18 \\ z^2 &= 9 \\ z &= 3. \end{aligned}$$

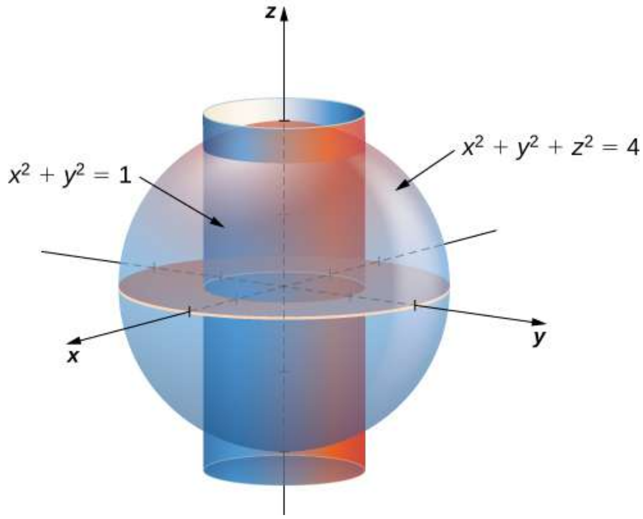
This gives

$$\begin{aligned} 3\sqrt{2}\cos\varphi &= 3 \\ \cos\varphi &= \frac{1}{\sqrt{2}} \\ \varphi &= \frac{\pi}{4}. \end{aligned}$$

Putting this together, we obtain

$$\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} \left(x^2+y^2+z^2\right) dz\, dx\, dy = \int_{\varphi=0}^{\varphi=\pi/4} \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=3\sqrt{2}} \rho^4 \sin\varphi\, d\rho\, d\theta\, d\varphi.$$

5.32 Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ but outside the cylinder $x^2 + y^2 = 1$.



Now that we are familiar with the spherical coordinate system, let's find the volume of some known geometric figures, such as spheres and ellipsoids.

Example 5.52

Chapter Opener: Finding the Volume of l'Hemisphèric

Find the volume of the spherical planetarium in l'Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2 + y^2 + z^2 = r^2$.

Figure 5.60 (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution

We calculate the volume of the ball in the first octant, where $x \geq 0$, $y \geq 0$, and $z \geq 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2}, \; 0 \leq \rho \leq r, \; 0 \leq \theta \leq \frac{\pi}{2}.$$

Therefore,

$$\begin{aligned} V &= \iiint_D dx\,dy\,dz = 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=\pi} \int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin\theta \, d\varphi \, d\rho \, d\theta \\ &= 8 \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=r} \rho^2 \, d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\theta \, d\theta \\ &= 8\left(\frac{\pi}{2}\right)\left(\frac{r^3}{3}\right)(1) \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is $\frac{4}{3}\pi(50)^3 \approx 523{,}600\,\text{ft}^3$.

For the next example we find the volume of an ellipsoid.

Example 5.53

Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Solution

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \geq 0,\ y \geq 0,$ and $z \geq 0$ and then multiply the result by 8.

In this case the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2},\ 0 \leq \rho \leq \frac{\pi}{2},\ 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

Also, we need to change the rectangular to spherical coordinates in this way:

$$x = a\rho \cos\varphi \sin\theta,\ y = b\rho \sin\varphi \sin\theta, \text{ and } z = c\rho \cos\theta.$$

Then the volume of the ellipsoid becomes

$$\begin{aligned} V &= \iiint_D dx\,dy\,dz \\ &= 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\pi/2} abc\rho^2 \sin\theta \, d\varphi \, d\rho \, d\theta \\ &= 8abc \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=1} \rho^2 \, d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\theta \, d\theta \\ &= 8abc\left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right)(1) \\ &= \frac{4}{3}\pi abc. \end{aligned}$$

Example 5.54

Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$.

Solution

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a=75$ ft, $b=80$ ft, and $c=90$ ft in the result from **Example 5.53**. Hence the volume of the ellipsoid is

$$V_{\text{ellipsoid}}=\frac{4}{3}\pi(75)(80)(90)\approx 2{,}262{,}000\text{ ft}^3.$$

From **Example 5.52**, the volume of the sphere is

$$V_{\text{sphere}}\approx 523{,}600\text{ ft}^3.$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$ is approximately

$$V_{\text{Hemisferic}}=V_{\text{ellipsoid}}-V_{\text{sphere}}=1{,}738{,}400\text{ ft}^3.$$

Student PROJECT

Hot air balloons

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.

Figure 5.61 Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (credit: David Herrera, Flickr)

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28 feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 6 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.

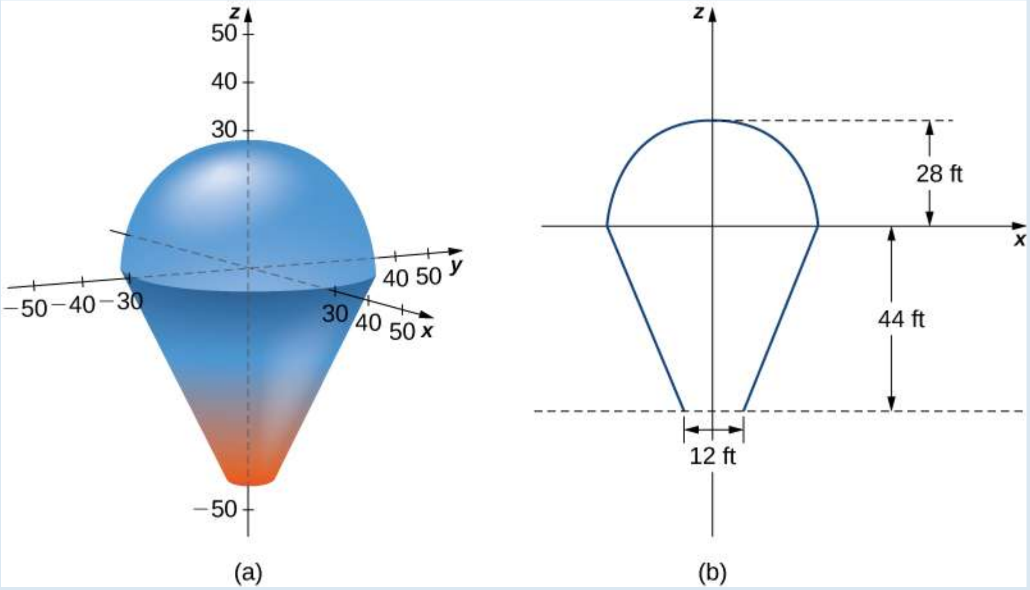


Figure 5.62 (a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.
    a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)
    b. Verify the answer using the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3,$ and for the volume of a cone, $V = \frac{1}{3}\pi r^2 h.$

In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)
Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure.

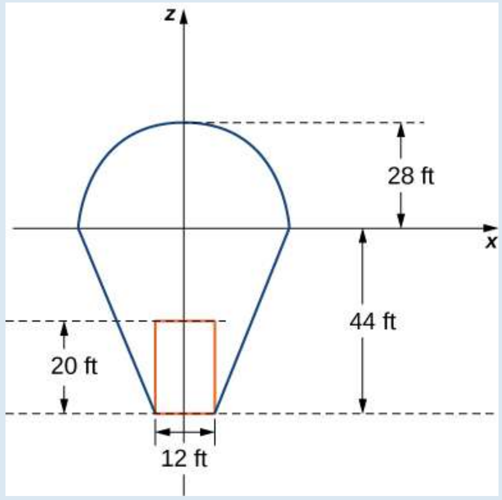


Figure 5.63 Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

$$H(r, \theta, z) = -2z - 48.$$

Then the temperature of the air in the column is given by

$$T_1(r, \theta, z) = \frac{z - r}{10} + 210 + (-2z - 48),$$

while the temperature in the remainder of the balloon is still given by

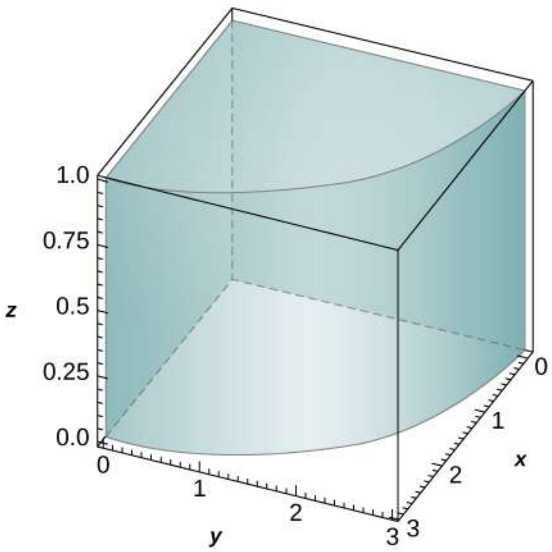
$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

5.5 EXERCISES

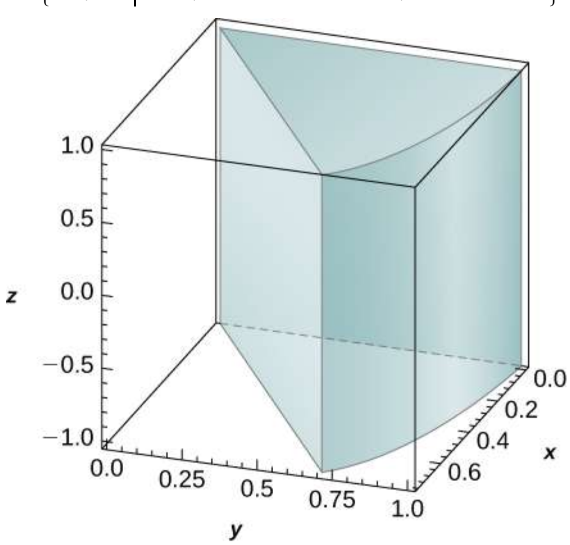
In the following exercises, evaluate the triple integrals $\iiint_E f(x, y, z)dV$ over the solid E.

241. $f(x, y, z) = z,$ $B = \{(x, y, z)|x^2 + y^2 \leq 9, x \geq 0, y \geq 0, 0 \leq z \leq 1\}$



242. $f(x, y, z) = xz^2,$ $B = \{(x, y, z)|x^2 + y^2 \leq 16, x \geq 0, y \leq 0, -1 \leq z \leq 1\}$

243. $f(x, y, z) = xy,$ $B = \{(x, y, z)|x^2 + y^2 \leq 1, x \geq 0, x \geq y, -1 \leq z \leq 1\}$



244. $f(x, y, z) = x^2 + y^2,$ $B = \{(x, y, z)|x^2 + y^2 \leq 4, x \geq 0, x \leq y, 0 \leq z \leq 3\}$

245. $f(x, y, z) = e^{\sqrt{x^2 + y^2}},$ $B = \{(x, y, z)|1 \leq x^2 + y^2 \leq 4, y \leq 0, x \leq y\sqrt{3}, 2 \leq z \leq 3\}$

246. $f(x, y, z) = \sqrt{x^2 + y^2},$ $B = \{(x, y, z)|1 \leq x^2 + y^2 \leq 9, y \leq 0, 0 \leq z \leq 1\}$

247.

a. Let B be a cylindrical shell with inner radius $a,$ outer radius $b,$ and height $c,$ where $0 < a < b$ and $c > 0.$ Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x, y, z) = f(r) + h(z),$ where f and h are differentiable functions. If $\int_a^b \tilde{f}(r)dr = 0$ and $\tilde{h}(0) = 0,$ where $\tilde{f}$ and $\tilde{h}$ are antiderivatives of f and $h,$ respectively, show that

$$\iiint_B F(x, y, z)dV = 2\pi c\left(b\tilde{f}(b) - a\tilde{f}(a)\right) + \pi\left(b^2 - a^2\right)\tilde{h}(c).$$

b. Use the previous result to show that $\iiint_B \left(z + \sin\sqrt{x^2 + y^2}\right)dx\,dy\,dz = 6\pi^2(\pi - 2),$ where B is a cylindrical shell with inner radius $\pi,$ outer radius $2\pi,$ and height $2.$

248.

a. Let B be a cylindrical shell with inner radius $a,$ outer radius $b,$ and height $c,$ where $0 < a < b$ and $c > 0.$ Assume that a function F defined on B can be expressed in cylindrical coordinates as $F(x, y, z) = f(r)g(\theta)h(z),$ where $f, g,$ and h are differentiable functions. If $\int_a^b \tilde{f}(r)dr = 0,$ where $\tilde{f}$ is an antiderivative of $f,$ show that

$$\iiint_B F(x, y, z)dV = \left[b\tilde{f}(b) - a\tilde{f}(a)\right]\left[\tilde{g}(2\pi) - \tilde{g}(0)\right]\left[\tilde{h}(c) - \tilde{h}(0)\right],$$

where $\tilde{g}$ and $\tilde{h}$ are antiderivatives of g and $h,$ respectively.

b. Use the previous result to show that $\iiint_B z\sin\sqrt{x^2 + y^2}dx\,dy\,dz = -12\pi^2,$ where B is a cylindrical shell with inner radius $\pi,$ outer radius $2\pi,$ and height $2.$

In the following exercises, the boundaries of the solid E are given in cylindrical coordinates.

a. Express the region E in cylindrical coordinates.

b. Convert the integral $\iiint\limits_E f(x, y, z)dV$ to cylindrical coordinates.

249. E is bounded by the right circular cylinder $r = 4\sin\theta$, the $r\theta$-plane, and the sphere $r^2 + z^2 = 16$.

250. E is bounded by the right circular cylinder $r = \cos\theta$, the $r\theta$-plane, and the sphere $r^2 + z^2 = 9$.

251. E is located in the first octant and is bounded by the circular paraboloid $z = 9 - 3r^2$, the cylinder $r = \sqrt{3}$, and the plane $r(\cos\theta + \sin\theta) = 20 - z$.

252. E is located in the first octant outside the circular paraboloid $z = 10 - 2r^2$ and inside the cylinder $r = \sqrt{5}$ and is bounded also by the planes $z = 20$ and $\theta = \frac{\pi}{4}$.

In the following exercises, the function f and region E are given.

a. Express the region E and the function f in cylindrical coordinates.

b. Convert the integral $\iiint\limits_B f(x, y, z)dV$ into cylindrical coordinates and evaluate it.

253. $f(x, y, z) = \frac{1}{x+3}$, $E = \left\{(x, y, z)\middle|0 \le x^2 + y^2 \le 9,\ x \ge 0,\ y \ge 0,\ 0 \le z \le x + 3\right\}$

254. $f(x, y, z) = x^2 + y^2$, $E = \left\{(x, y, z)\middle|0 \le x^2 + y^2 \le 4,\ y \ge 0,\ 0 \le z \le 3 - x\right\}$

255. $f(x, y, z) = x$, $E = \left\{(x, y, z)\middle|1 \le y^2 + z^2 \le 9,\ 0 \le x \le 1 - y^2 - z^2\right\}$

256. $f(x, y, z) = y$, $E = \left\{(x, y, z)\middle|1 \le x^2 + z^2 \le 9,\ 0 \le y \le 1 - x^2 - z^2\right\}$

In the following exercises, find the volume of the solid E whose boundaries are given in rectangular coordinates.

257. E is above the xy-plane, inside the cylinder $x^2 + y^2 = 1$, and below the plane $z = 1$.

258. E is below the plane $z = 1$ and inside the paraboloid $z = x^2 + y^2$.

259. E is bounded by the circular cone $z = \sqrt{x^2 + y^2}$ and $z = 1$.

260. E is located above the xy-plane, below $z = 1$, outside the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$, and inside the cylinder $x^2 + y^2 = 2$.

261. E is located inside the cylinder $x^2 + y^2 = 1$ and between the circular paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2$.

262. E is located inside the sphere $x^2 + y^2 + z^2 = 1$, above the xy-plane, and inside the circular cone $z = \sqrt{x^2 + y^2}$.

263. E is located outside the circular cone $x^2 + y^2 = (z-1)^2$ and between the planes $z = 0$ and $z = 2$.

264. E is located outside the circular cone $z = 1 - \sqrt{x^2 + y^2}$, above the xy-plane, below the circular paraboloid, and between the planes $z = 0$ and $z = 2$.

265. **[T]** Use a computer algebra system (CAS) to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^{r} r\,dz\,dr\,d\theta$. Find the volume V of the solid. Round your answer to four decimal places.

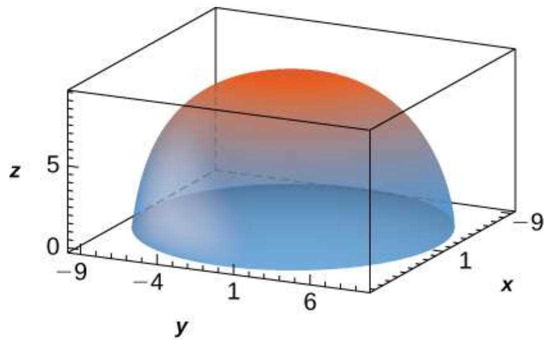
266. **[T]** Use a CAS to graph the solid whose volume is given by the iterated integral in cylindrical coordinates $\int_0^{\pi/2} \int_0^1 \int_{r^4}^{r} r\,dz\,dr\,d\theta$. Find the volume V of the solid Round your answer to four decimal places.

267. Convert the integral $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xz\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

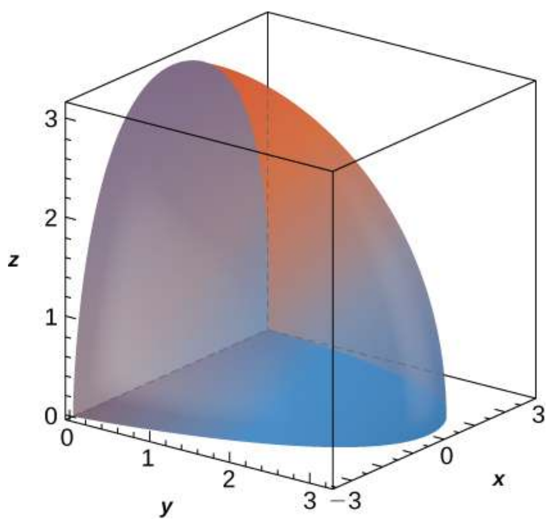
268. Convert the integral $\int_0^2 \int_0^x \int_0^1 (xy+z)\,dz\,dx\,dy$ into an integral in cylindrical coordinates.

In the following exercises, evaluate the triple integral $\iiint_B f(x, y, z)dV$ over the solid B.

269. $f(x, y, z) = 1$, $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 90, z \ge 0\}$



270. $f(x, y, z) = 1 - \sqrt{x^2 + y^2 + z^2}$, $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 9, y \ge 0, z \ge 0\}$



271. $f(x, y, z) = \sqrt{x^2 + y^2}$, B is bounded above by the half-sphere $x^2 + y^2 + z^2 = 9$ with $z \ge 0$ and below by the cone $2z^2 = x^2 + y^2$.

272. $f(x, y, z) = z$, B is bounded above by the half-sphere $x^2 + y^2 + z^2 = 16$ with $z \ge 0$ and below by the cone $2z^2 = x^2 + y^2$.

273. Show that if $F(\rho, \theta, \varphi) = f(\rho)g(\theta)h(\varphi)$ is a continuous function on the spherical box $B = \{(\rho, \theta, \varphi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \varphi \le \psi\}$, then $\iiint_B F\,dV = \left(\int_a^b \rho^2 f(\rho)dr\right)\left(\int_\alpha^\beta g(\theta)d\theta\right)\left(\int_\gamma^\psi h(\varphi)\sin\varphi\,d\varphi\right)$.

274.

a. A function F is said to have spherical symmetry if it depends on the distance to the origin only, that is, it can be expressed in spherical coordinates as $F(x, y, z) = f(\rho)$, where $\rho = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\iiint_B F(x, y, z)dV = 2\pi \int_a^b \rho^2 f(\rho)d\rho,$$

where B is the region between the upper concentric hemispheres of radii a and b centered at the origin, with $0 < a < b$ and F a spherical function defined on B.

b. Use the previous result to show that $\iiint_B (x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}\,dV = 21\pi$, where $B = \{(x, y, z) \mid 1 \le x^2 + y^2 + z^2 \le 2, z \ge 0\}$.

275.

a. Let B be the region between the upper concentric hemispheres of radii a and b centered at the origin and situated in the first octant, where $0 < a < b$. Consider F a function defined on B whose form in spherical coordinates (ρ, θ, φ) is $F(x, y, z) = f(\rho)\cos\varphi$. Show that if $g(a) = g(b) = 0$ and $\int_a^b h(\rho)d\rho = 0$, then

$$\iiint_B F(x, y, z)dV = \frac{\pi^2}{4}[ah(a) - bh(b)],$$

where g is an antiderivative of f and h is an antiderivative of g.

b. Use the previous result to show that $\iiint_B \frac{z\cos\sqrt{x^2 + y^2 + z^2}}{\sqrt{x^2 + y^2 + z^2}}dV = \frac{3\pi^2}{2}$, where B is the region between the upper concentric hemispheres of radii π and 2π centered at the origin and situated in the first octant.

In the following exercises, the function f and region E are given.

a. Express the region E and function f in cylindrical coordinates.
b. Convert the integral $\iiint_B f(x, y, z)dV$ into cylindrical coordinates and evaluate it.

276. $f(x, y, z) = z;$ $E = \left\{(x, y, z) \middle| 0 \le x^2 + y^2 + z^2 \le 1, z \ge 0\right\}$

277. $f(x, y, z) = x + y;$ $E = \left\{(x, y, z) \middle| 1 \le x^2 + y^2 + z^2 \le 2, z \ge 0, y \ge 0\right\}$

278. $f(x, y, z) = 2xy;$ $E = \left\{(x, y, z) \middle| \sqrt{x^2 + y^2} \le z \le \sqrt{1 - x^2 - y^2}, x \ge 0, y \ge 0\right\}$

279. $f(x, y, z) = z;$ $E = \left\{(x, y, z) \middle| x^2 + y^2 + z^2 - 2z \le 0, \sqrt{x^2 + y^2} \le z\right\}$

In the following exercises, find the volume of the solid E whose boundaries are given in rectangular coordinates.

280. $E = \left\{(x, y, z) \middle| \sqrt{x^2 + y^2} \le z \le \sqrt{16 - x^2 - y^2}, x \ge 0, y \ge 0\right\}$

281. $E = \left\{(x, y, z) \middle| x^2 + y^2 + z^2 - 2z \le 0, \sqrt{x^2 + y^2} \le z\right\}$

282. Use spherical coordinates to find the volume of the solid situated outside the sphere $\rho = 1$ and inside the sphere $\rho = \cos \varphi$, with $\varphi \in \left[0, \frac{\pi}{2}\right]$.

283. Use spherical coordinates to find the volume of the ball $\rho \le 3$ that is situated between the cones $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{3}$.

284. Convert the integral $\int_{-4}^{4} \int_{-\sqrt{16 - y^2}}^{\sqrt{16 - y^2}} \int_{-\sqrt{16 - x^2 - y^2}}^{\sqrt{16 - x^2 - y^2}} \left(x^2 + y^2 + z^2\right) dz\, dx\, dy$ into an integral in spherical coordinates.

285. Convert the integral $\int_{0}^{4} \int_{0}^{\sqrt{16 - x^2}} \int_{-\sqrt{16 - x^2 - y^2}}^{\sqrt{16 - x^2 - y^2}} \left(x^2 + y^2 + z^2\right)^2 dz\, dy\, dx$ into an integral in spherical coordinates.

286. Convert the integral $\int_{-2}^{2} \int_{-\sqrt{4 - x^2}}^{\sqrt{4 - x^2}} \int_{\sqrt{x^2 + y^2}}^{\sqrt{16 - x^2 - y^2}} dz\, dy\, dx$ into an integral in spherical coordinates and evaluate it.

287. **[T]** Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates $\int_{\pi/2}^{\pi} \int_{5\pi/6}^{\pi/6} \int_{0}^{2} \rho^2 \sin \varphi\, d\rho\, d\varphi\, d\theta.$ Find the volume V of the solid. Round your answer to three decimal places.

288. **[T]** Use a CAS to graph the solid whose volume is given by the iterated integral in spherical coordinates as $\int_{0}^{2\pi} \int_{3\pi/4}^{\pi/4} \int_{0}^{1} \rho^2 \sin \varphi\, d\rho\, d\varphi\, d\theta.$ Find the volume V of the solid. Round your answer to three decimal places.

289. **[T]** Use a CAS to evaluate the integral $\iiint_E \left(x^2 + y^2\right) dV$ where E lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 3y.$

290. **[T]**

a. Evaluate the integral $\iiint_E e^{\sqrt{x^2 + y^2 + z^2}} dV,$ where E is bounded by the spheres $4x^2 + 4y^2 + 4z^2 = 1$ and $x^2 + y^2 + z^2 = 1.$
b. Use a CAS to find an approximation of the previous integral. Round your answer to two decimal places.

291. Express the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$ as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

292. Express the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$ that is located in the first octant as triple integrals in cylindrical coordinates and spherical coordinates, respectively.

293. The power emitted by an antenna has a power density per unit volume given in spherical coordinates by $p(\rho, \theta, \varphi) = \frac{P_0}{\rho^2}\cos^2\theta \sin^4\varphi,$ where P_0 is a constant with units in watts. The total power within a sphere B of radius r meters is defined as $P = \iiint_B p(\rho, \theta, \varphi)dV.$ Find the total power $P.$

294. Use the preceding exercise to find the total power within a sphere B of radius 5 meters when the power density per unit volume is given by $p(\rho, \theta, \varphi) = \frac{30}{\rho^2}\cos^2\theta \sin^4\varphi.$

295. A charge cloud contained in a sphere B of radius r centimeters centered at the origin has its charge density given by $q(x, y, z) = k\sqrt{x^2 + y^2 + z^2}\frac{\mu\text{C}}{\text{cm}^3},$ where $k > 0.$ The total charge contained in B is given by $Q = \iiint_B q(x, y, z)dV.$ Find the total charge $Q.$

296. Use the preceding exercise to find the total charge cloud contained in the unit sphere if the charge density is $q(x, y, z) = 20\sqrt{x^2 + y^2 + z^2}\frac{\mu\text{C}}{\text{cm}^3}.$

5.6 | Calculating Centers of Mass and Moments of Inertia

Learning Objectives

- **5.6.1** Use double integrals to locate the center of mass of a two-dimensional object.
- **5.6.2** Use double integrals to find the moment of inertia of a two-dimensional object.
- **5.6.3** Use triple integrals to locate the center of mass of a three-dimensional object.

We have already discussed a few applications of multiple integrals, such as finding areas, volumes, and the average value of a function over a bounded region. In this section we develop computational techniques for finding the center of mass and moments of inertia of several types of physical objects, using double integrals for a lamina (flat plate) and triple integrals for a three-dimensional object with variable density. The density is usually considered to be a constant number when the lamina or the object is homogeneous; that is, the object has uniform density.

Center of Mass in Two Dimensions

The center of mass is also known as the center of gravity if the object is in a uniform gravitational field. If the object has uniform density, the center of mass is the geometric center of the object, which is called the centroid. **Figure 5.64** shows a point P as the center of mass of a lamina. The lamina is perfectly balanced about its center of mass.

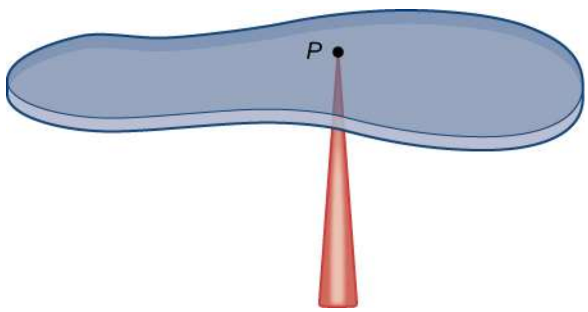


Figure 5.64 A lamina is perfectly balanced on a spindle if the lamina's center of mass sits on the spindle.

To find the coordinates of the center of mass $P(\bar{x}, \bar{y})$ of a lamina, we need to find the moment M_x of the lamina about the x-axis and the moment M_y about the y-axis. We also need to find the mass m of the lamina. Then

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}.$$

Refer to **Moments and Centers of Mass (http://cnx.org/content/m53649/latest/)** for the definitions and the methods of single integration to find the center of mass of a one-dimensional object (for example, a thin rod). We are going to use a similar idea here except that the object is a two-dimensional lamina and we use a double integral.

If we allow a constant density function, then $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$ give the *centroid* of the lamina.

Suppose that the lamina occupies a region R in the xy-plane, and let $\rho(x, y)$ be its density (in units of mass per unit area) at any point (x, y). Hence, $\rho(x, y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$, where Δm and ΔA are the mass and area of a small rectangle containing the point (x, y) and the limit is taken as the dimensions of the rectangle go to 0 (see the following figure).

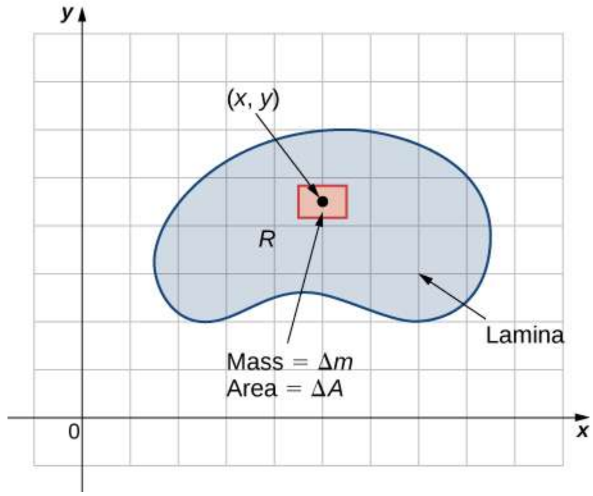


Figure 5.65 The density of a lamina at a point is the limit of its mass per area in a small rectangle about the point as the area goes to zero.

Just as before, we divide the region R into tiny rectangles R_{ij} with area ΔA and choose $\left(x_{ij}^*, y_{ij}^*\right)$ as sample points. Then the mass m_{ij} of each R_{ij} is equal to $\rho\left(x_{ij}^*, y_{ij}^*\right)\Delta A$ (**Figure 5.66**). Let k and l be the number of subintervals in x and $y,$ respectively. Also, note that the shape might not always be rectangular but the limit works anyway, as seen in previous sections.

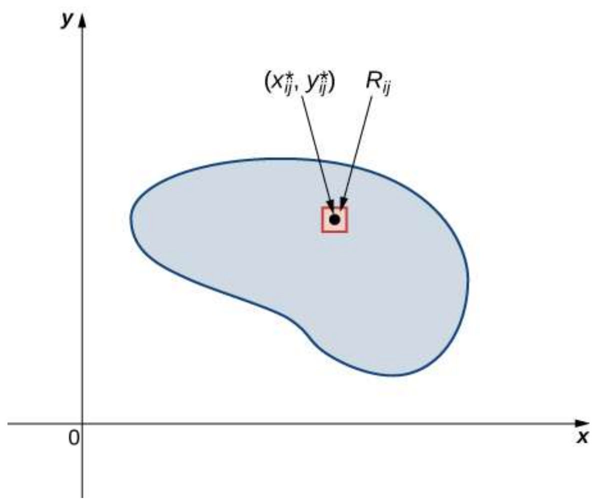


Figure 5.66 Subdividing the lamina into tiny rectangles $R_{ij},$ each containing a sample point (x_{ij}^*, y_{ij}^*).

Hence, the mass of the lamina is

$$m = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_R \rho(x, y)dA. \tag{5.13}$$

Let’s see an example now of finding the total mass of a triangular lamina.

Example 5.55

Finding the Total Mass of a Lamina

Consider a triangular lamina R with vertices $(0, 0), (0, 3), \ (3, 0)$ and with density $\rho(x, y) = xy \text{ kg/m}^2$. Find the total mass.

Solution

A sketch of the region R is always helpful, as shown in the following figure.

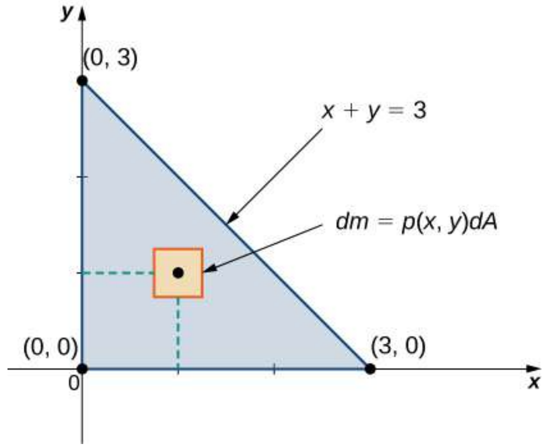


Figure 5.67 A lamina in the xy-plane with density $\rho(x, y) = xy$.

Using the expression developed for mass, we see that

$$\begin{aligned} m &= \iint_R dm = \iint_R \rho(x, y)dA = \int_{x=0}^{x=3} \int_{y=0}^{y=3-x} xy\,dy\,dx = \int_{x=0}^{x=3} \left[x\frac{y^2}{2}\bigg|_{y=0}^{y=3-x} \right] dx \\ &= \int_{x=0}^{x=3} \frac{1}{2}x(3-x)^2\,dx = \left[\frac{9x^2}{4} - x^3 + \frac{x^4}{8}\right]\Bigg|_{x=0}^{x=3} \\ &= \frac{27}{8}. \end{aligned}$$

The computation is straightforward, giving the answer $m = \frac{27}{8}$ kg.

5.33 Consider the same region R as in the previous example, and use the density function $\rho(x, y) = \sqrt{xy}$. Find the total mass.

Now that we have established the expression for mass, we have the tools we need for calculating moments and centers of mass. The moment M_x about the x-axis for R is the limit of the sums of moments of the regions R_{ij} about the x-axis. Hence

$$M_x = \lim_{k, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^*\right) m_{ij} = \lim_{k, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^*\right) \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \iint_R y\rho(x, y)dA. \quad (5.14)$$

Similarly, the moment M_y about the y-axis for R is the limit of the sums of moments of the regions R_{ij} about the y-axis. Hence

$$M_y = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(x_{ij}^*\right) m_{ij} = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^*\right) \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \iint_R x\rho(x, y)\,dA. \tag{5.15}$$

Example 5.56

Finding Moments

Consider the same triangular lamina R with vertices $(0, 0), (0, 3), (3, 0)$ and with density $\rho(x, y) = xy$. Find the moments M_x and M_y.

Solution

Use double integrals for each moment and compute their values:

$$M_x = \iint_R y\rho(x, y)\,dA = \int_{x=0}^{x=3} \int_{y=0}^{y=3-x} xy^2\, dy\, dx = \frac{81}{20},$$

$$M_y = \iint_R x\rho(x, y)\,dA = \int_{x=0}^{x=3} \int_{y=0}^{y=3-x} x^2 y\, dy\, dx = \frac{81}{20}.$$

The computation is quite straightforward.

5.34 Consider the same lamina R as above, and use the density function $\rho(x, y) = \sqrt{xy}$. Find the moments M_x and M_y.

Finally we are ready to restate the expressions for the center of mass in terms of integrals. We denote the x-coordinate of the center of mass by $\bar{x}$ and the y-coordinate by $\bar{y}$. Specifically,

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x, y)\,dA}{\iint_R \rho(x, y)\,dA} \text{ and } \bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x, y)\,dA}{\iint_R \rho(x, y)\,dA}. \tag{5.16}$$

Example 5.57

Finding the Center of Mass

Again consider the same triangular region R with vertices $(0, 0), (0, 3), (3, 0)$ and with density function $\rho(x, y) = xy$. Find the center of mass.

Solution

Using the formulas we developed, we have

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x, y)dA}{\iint_R \rho(x, y)dA} = \frac{81/20}{27/8} = \frac{6}{5},$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x, y)dA}{\iint_R \rho(x, y)dA} = \frac{81/20}{27/8} = \frac{6}{5}.$$

Therefore, the center of mass is the point $\left(\frac{6}{5}, \frac{6}{5}\right)$.

Analysis

If we choose the density $\rho(x, y)$ instead to be uniform throughout the region (i.e., constant), such as the value 1 (any constant will do), then we can compute the centroid,

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{9/2}{9/2} = 1,$$

$$y_c = \frac{M_x}{m} \frac{\iint_R y\,dA}{\iint_R dA} = \frac{9/2}{9/2} = 1.$$

Notice that the center of mass $\left(\frac{6}{5}, \frac{6}{5}\right)$ is not exactly the same as the centroid $(1, 1)$ of the triangular region. This is due to the variable density of R. If the density is constant, then we just use $\rho(x, y) = c$ (constant). This value cancels out from the formulas, so for a constant density, the center of mass coincides with the centroid of the lamina.

5.35 Again use the same region R as above and the density function $\rho(x, y) = \sqrt{xy}$. Find the center of mass.

Once again, based on the comments at the end of **Example 5.57**, we have expressions for the centroid of a region on the plane:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \text{ and } y_c = \frac{M_x}{m} \frac{\iint_R y\,dA}{\iint_R dA}.$$

We should use these formulas and verify the centroid of the triangular region R referred to in the last three examples.

Example 5.58

Finding Mass, Moments, and Center of Mass

Find the mass, moments, and the center of mass of the lamina of density $\rho(x, y) = x + y$ occupying the region R under the curve $y = x^2$ in the interval $0 \leq x \leq 2$ (see the following figure).

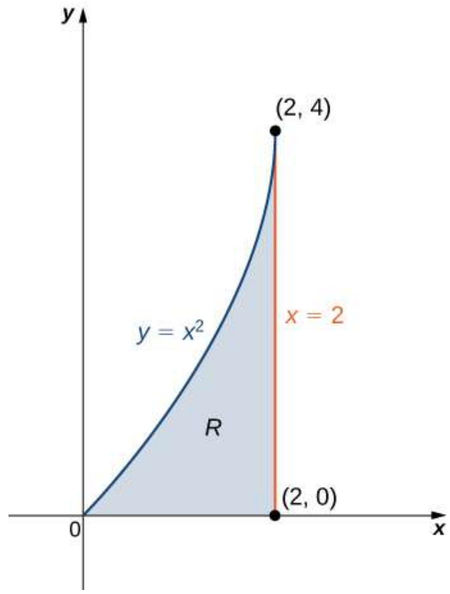


Figure 5.68 Locating the center of mass of a lamina R with density $\rho(x, y) = x + y$.

Solution

First we compute the mass m. We need to describe the region between the graph of $y = x^2$ and the vertical lines $x = 0$ and $x = 2$:

$$\begin{aligned} m &= \iint_R dm = \iint_R \rho(x, y)dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} (x+y)dy\,dx = \int_{x=0}^{x=2} \left[xy + \frac{y^2}{2} \Big|_{y=0}^{y=x^2} \right] dx \\ &= \int_{x=0}^{x=2} \left[x^3 + \frac{x^4}{2} \right] dx = \left[\frac{x^4}{4} + \frac{x^5}{10} \right] \Bigg|_{x=0}^{x=2} = \frac{36}{5}. \end{aligned}$$

Now compute the moments M_x and M_y:

$$M_x = \iint_R y\rho(x, y)dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} y(x+y)dy\,dx = \frac{80}{7},$$

$$M_y = \iint_R x\rho(x, y)dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} x(x+y)dy\,dx = \frac{176}{15}.$$

Finally, evaluate the center of mass,

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x, y)dA}{\iint_R \rho(x, y)dA} = \frac{176/15}{36/5} = \frac{44}{27},$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x, y)dA}{\iint_R \rho(x, y)dA} = \frac{80/7}{36/5} = \frac{100}{63}.$$

Hence the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{44}{27}, \frac{100}{63}\right)$.

5.36 Calculate the mass, moments, and the center of mass of the region between the curves $y = x$ and $y = x^2$ with the density function $\rho(x, y) = x$ in the interval $0 \le x \le 1$.

Example 5.59

Finding a Centroid

Find the centroid of the region under the curve $y = e^x$ over the interval $1 \le x \le 3$ (see the following figure).

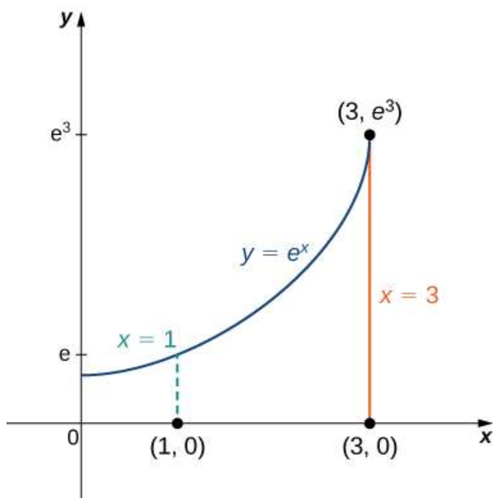


Figure 5.69 Finding a centroid of a region below the curve $y = e^x$.

Solution

To compute the centroid, we assume that the density function is constant and hence it cancels out:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \text{ and } y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA},$$

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} x\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} xe^x\,dx}{\int_{x=1}^{x=3} e^x\,dx} = \frac{2e^3}{e^3 - e} = \frac{2e^2}{e^2 - 1},$$

$$y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} y\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} \frac{e^{2x}}{2}dx}{\int_{x=1}^{x=3} e^x\,dx} = \frac{\frac{1}{4}e^2\left(e^4 - 1\right)}{e\left(e^2 - 1\right)} = \frac{1}{4}e\left(e^2 + 1\right).$$

Thus the centroid of the region is

$$(x_c, y_c) = \left(\frac{2e^2}{e^2 - 1}, \frac{1}{4}e\left(e^2 + 1\right)\right).$$

5.37 Calculate the centroid of the region between the curves $y = x$ and $y = \sqrt{x}$ with uniform density in the interval $0 \le x \le 1$.

Moments of Inertia

For a clear understanding of how to calculate moments of inertia using double integrals, we need to go back to the general definition in Section 6.6. The moment of inertia of a particle of mass m about an axis is mr^2, where r is the distance of the particle from the axis. We can see from **Figure 5.66** that the moment of inertia of the subrectangle R_{ij} about the x-axis is $(y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A$. Similarly, the moment of inertia of the subrectangle R_{ij} about the y-axis is $(x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A$. The moment of inertia is related to the rotation of the mass; specifically, it measures the tendency of the mass to resist a change in rotational motion about an axis.

The moment of inertia I_x about the x-axis for the region R is the limit of the sum of moments of inertia of the regions R_{ij} about the x-axis. Hence

$$I_x = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^*\right)^2 m_{ij} = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right)\Delta A = \iint_R y^2 \rho(x, y)dA.$$

Similarly, the moment of inertia I_y about the y-axis for R is the limit of the sum of moments of inertia of the regions R_{ij} about the y-axis. Hence

$$I_y = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(x_{ij}^*\right)^2 m_{ij} = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(x_{ij}^*\right)^2 \rho\left(x_{ij}^*, y_{ij}^*\right)\Delta A = \iint_R x^2 \rho(x, y)dA.$$

Sometimes, we need to find the moment of inertia of an object about the origin, which is known as the polar moment of inertia. We denote this by I_0 and obtain it by adding the moments of inertia I_x and I_y. Hence

$$I_0 = I_x + I_y = \iint\limits_R \left(x^2 + y^2\right)\rho(x, y)dA.$$

All these expressions can be written in polar coordinates by substituting $x = r\cos\theta, \quad y = r\sin\theta,$ and $dA = r\,dr\,d\theta.$ For example, $I_0 = \iint\limits_R r^2\rho(r\cos\theta, r\sin\theta)dA.$

Example 5.60

Finding Moments of Inertia for a Triangular Lamina

Use the triangular region R with vertices $(0, 0), (2, 2),$ and $(2, 0)$ and with density $\rho(x, y) = xy$ as in previous examples. Find the moments of inertia.

Solution

Using the expressions established above for the moments of inertia, we have

$$\begin{aligned} I_x &= \iint\limits_R y^2\rho(x, y)dA = \int_{x=0}^{x=2}\int_{y=0}^{y=x} xy^3\,dy\,dx = \frac{8}{3}, \\ I_y &= \iint\limits_R x^2\rho(x, y)dA = \int_{x=0}^{x=2}\int_{y=0}^{y=x} x^3y\,dy\,dx = \frac{16}{3}, \\ I_0 &= \iint\limits_R \left(x^2 + y^2\right)\rho(x, y)dA = \int_0^2\int_0^x \left(x^2 + y^2\right)xy\,dy\,dx \\ &= I_x + I_y = 8. \end{aligned}$$

5.38 Again use the same region R as above and the density function $\rho(x, y) = \sqrt{xy}$. Find the moments of inertia.

As mentioned earlier, the moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis, also known as the **radius of gyration**.

Hence the radii of gyration with respect to the x-axis, the y-axis, and the origin are

$$R_x = \sqrt{\frac{I_x}{m}},\ R_y = \sqrt{\frac{I_y}{m}},\ \text{and}\ R_0 = \sqrt{\frac{I_0}{m}},$$

respectively. In each case, the radius of gyration tells us how far (perpendicular distance) from the axis of rotation the entire mass of an object might be concentrated. The moments of an object are useful for finding information on the balance and torque of the object about an axis, but radii of gyration are used to describe the distribution of mass around its centroidal axis. There are many applications in engineering and physics. Sometimes it is necessary to find the radius of gyration, as in the next example.

Example 5.61

Finding the Radius of Gyration for a Triangular Lamina

Consider the same triangular lamina R with vertices $(0, 0), (2, 2),$ and $(2, 0)$ and with density $\rho(x, y) = xy$ as in previous examples. Find the radii of gyration with respect to the x-axis, the y-axis, and the origin.

Solution

If we compute the mass of this region we find that $m = 2.$ We found the moments of inertia of this lamina in **Example 5.58**. From these data, the radii of gyration with respect to the x-axis, y-axis, and the origin are, respectively,

$$\begin{aligned} R_x &= \sqrt{\frac{I_x}{m}} = \sqrt{\frac{8/3}{2}} = \sqrt{\frac{8}{6}} = \frac{2\sqrt{3}}{3}, \\ R_y &= \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16/3}{2}} = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}, \\ R_0 &= \sqrt{\frac{I_0}{m}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2. \end{aligned}$$

5.39 Use the same region R from **Example 5.61** and the density function $\rho(x, y) = \sqrt{xy}.$ Find the radii of gyration with respect to the x-axis, the y-axis, and the origin.

Center of Mass and Moments of Inertia in Three Dimensions

All the expressions of double integrals discussed so far can be modified to become triple integrals.

Definition

If we have a solid object Q with a density function $\rho(x, y, z)$ at any point (x, y, z) in space, then its mass is

$$m = \iiint_Q \rho(x, y, z)dV.$$

Its moments about the xy-plane, the xz-plane, and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x, y, z)dV, \; M_{xz} = \iiint_Q y\rho(x, y, z)dV,$$
$$M_{yz} = \iiint_Q x\rho(x, y, z)dV.$$

If the center of mass of the object is the point $(\bar{x}, \bar{y}, \bar{z}),$ then

$$\bar{x} = \frac{M_{yz}}{m}, \; \bar{y} = \frac{M_{xz}}{m}, \; \bar{z} = \frac{M_{xy}}{m}.$$

Also, if the solid object is homogeneous (with constant density), then the center of mass becomes the centroid of the solid. Finally, the moments of inertia about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q \left(y^2 + z^2\right)\rho(x, y, z)dV,$$
$$I_y = \iiint_Q \left(x^2 + z^2\right)\rho(x, y, z)dV,$$
$$I_z = \iiint_Q \left(x^2 + y^2\right)\rho(x, y, z)dV.$$

Example 5.62

Finding the Mass of a Solid

Suppose that Q is a solid region bounded by $x + 2y + 3z = 6$ and the coordinate planes and has density $\rho(x, y, z) = x^2 yz$. Find the total mass.

Solution

The region Q is a tetrahedron (**Figure 5.70**) meeting the axes at the points $(6, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 2)$. To find the limits of integration, let $z = 0$ in the slanted plane $z = \frac{1}{3}(6 - x - 2y)$. Then for x and y find the projection of Q onto the xy-plane, which is bounded by the axes and the line $x + 2y = 6$. Hence the mass is

$$m = \iiint\limits_Q \rho(x, y, z)dV = \int_{x=0}^{x=6} \int_{y=0}^{y=1/2(6-x)} \int_{z=0}^{z=1/3(6-x-2y)} x^2 yz\, dz\, dy\, dx = \frac{108}{35} \approx 3.086.$$

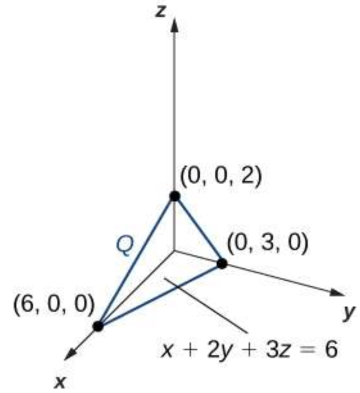


Figure 5.70 Finding the mass of a three-dimensional solid Q.

5.40 Consider the same region Q (**Figure 5.70**), and use the density function $\rho(x, y, z) = xy^2 z$. Find the mass.

Example 5.63

Finding the Center of Mass of a Solid

Suppose Q is a solid region bounded by the plane $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2 yz$ (see **Figure 5.70**). Find the center of mass using decimal approximation.

Solution

We have used this tetrahedron before and know the limits of integration, so we can proceed to the computations right away. First, we need to find the moments about the xy-plane, the xz-plane, and the yz-plane:

$$M_{xy} = \iiint_Q z\rho(x, y, z)dV = \int_{x=0}^{x=6} \int_{y=0}^{y=1/2(6-x)} \int_{z=0}^{z=1/3(6-x-2y)} x^2 yz^2\, dz\, dy\, dx = \frac{54}{35} \approx 1.543,$$
$$M_{xz} = \iiint_Q y\rho(x, y, z)dV = \int_{x=0}^{x=6} \int_{y=0}^{y=1/2(6-x)} \int_{z=0}^{z=1/3(6-x-2y)} x^2 y^2 z\, dz\, dy\, dx = \frac{81}{35} \approx 2.314,$$
$$M_{yz} = \iiint_Q x\rho(x, y, z)dV = \int_{x=0}^{x=6} \int_{y=0}^{y=1/2(6-x)} \int_{z=0}^{z=1/3(6-x-2y)} x^3 yz\, dz\, dy\, dx = \frac{243}{35} \approx 6.943.$$

Hence the center of mass is

$$\bar{x} = \frac{M_{yz}}{m}, \ \bar{y} = \frac{M_{xz}}{m}, \ \bar{z} = \frac{M_{xy}}{m},$$
$$\bar{x} = \frac{M_{yz}}{m} = \frac{243/35}{108/35} = \frac{243}{108} = 2.25,$$
$$\bar{y} = \frac{M_{xz}}{m} = \frac{81/35}{108/35} = \frac{81}{108} = 0.75,$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{54/35}{108/35} = \frac{54}{108} = 0.5.$$

The center of mass for the tetrahedron Q is the point $(2.25, 0.75, 0.5)$.

5.41 Consider the same region Q (**Figure 5.70**) and use the density function $\rho(x, y, z) = xy^2 z$. Find the center of mass.

We conclude this section with an example of finding moments of inertia $I_x, I_y,$ and I_z.

Example 5.64

Finding the Moments of Inertia of a Solid

Suppose that Q is a solid region and is bounded by $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2 yz$ (see **Figure 5.70**). Find the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane.

Solution

Once again, we can almost immediately write the limits of integration and hence we can quickly proceed to evaluating the moments of inertia. Using the formula stated before, the moments of inertia of the tetrahedron Q about the xy-plane, the xz-plane, and the yz-plane are

$$I_x = \iiint_Q \left(y^2 + z^2\right)\rho(x, y, z)dV,$$
$$I_y = \iiint_Q \left(x^2 + z^2\right)\rho(x, y, z)dV,$$

and

$$I_z = \iiint_Q \left(x^2 + y^2\right)\rho(x, y, z)dV \text{ with } \rho(x, y, z) = x^2 yz.$$

Proceeding with the computations, we have

$$I_x = \iiint_Q \left(y^2 + z^2\right)x^2 yz\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} \left(y^2 + z^2\right)x^2 yz\, dz\, dy\, dx = \frac{117}{35} \approx 3.343,$$
$$I_y = \iiint_Q \left(x^2 + z^2\right)x^2 yz\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} \left(x^2 + z^2\right)x^2 yz\, dz\, dy\, dx = \frac{684}{35} \approx 19.543,$$
$$I_z = \iiint_Q \left(x^2 + y^2\right)x^2 yz\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} \left(x^2 + y^2\right)x^2 yz\, dz\, dy\, dx = \frac{729}{35} \approx 20.829.$$

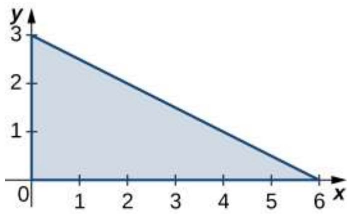
Thus, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $117/35,\ 684/35,$ and $729/35,$ respectively.

5.42 Consider the same region Q (**Figure 5.70**), and use the density function $\rho(x, y, z) = xy^2 z.$ Find the moments of inertia about the three coordinate planes.

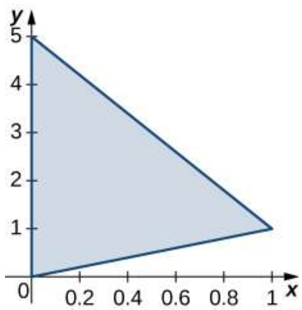
5.6 EXERCISES

In the following exercises, the region R occupied by a lamina is shown in a graph. Find the mass of R with the density function ρ.

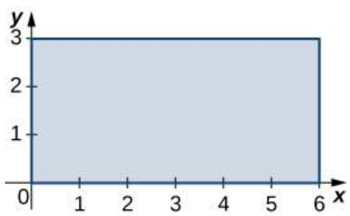
297. R is the triangular region with vertices $(0, 0), (0, 3),$ and $(6, 0)$; $\rho(x, y) = xy$.



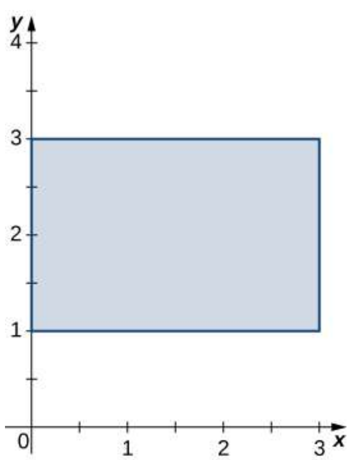
298. R is the triangular region with vertices $(0, 0), (1, 1),$ $(0, 5)$; $\rho(x, y) = x + y$.



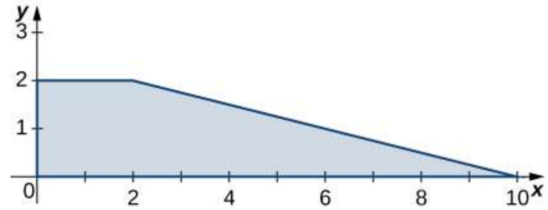
299. R is the rectangular region with vertices $(0, 0), (0, 3), (6, 3),$ and $(6, 0)$; $\rho(x, y) = \sqrt{xy}$.



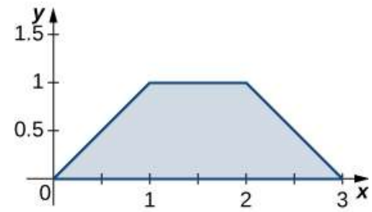
300. R is the rectangular region with vertices $(0, 1), (0, 3), (3, 3),$ and $(3, 1)$; $\rho(x, y) = x^2 y$.



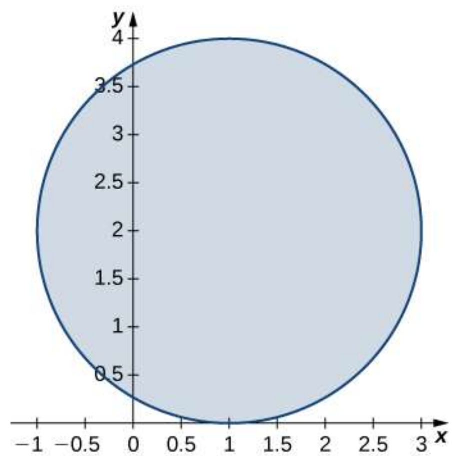
301. R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2},\ y = 0,\ y = 2,$ and $x = 0$; $\rho(x, y) = 3xy$.



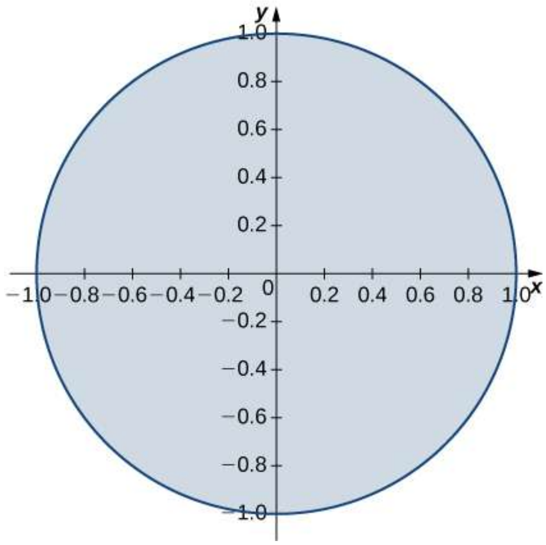
302. R is the trapezoidal region determined by the lines $y = 0,\ y = 1,\ y = x,$ and $y = -x + 3$; $\rho(x, y) = 2x + y$.



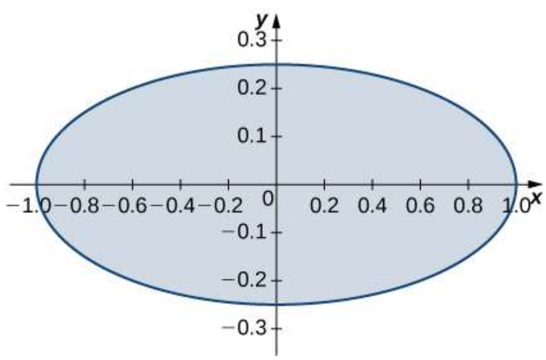
303. R is the disk of radius 2 centered at $(1, 2)$; $\rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.



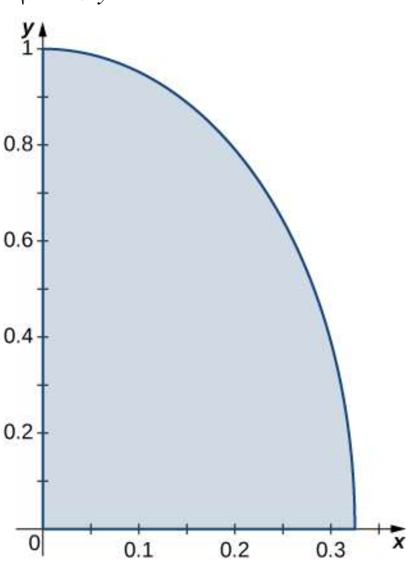
304. R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.



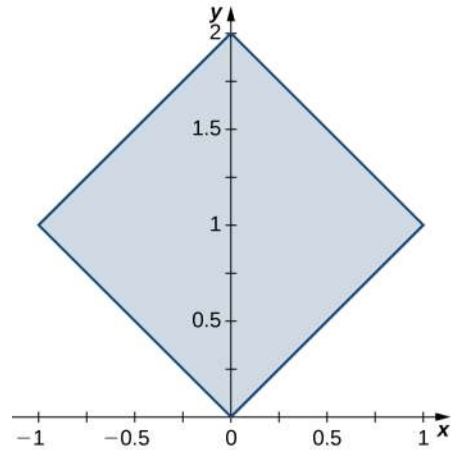
305. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1$; $\rho(x, y) = 1$.



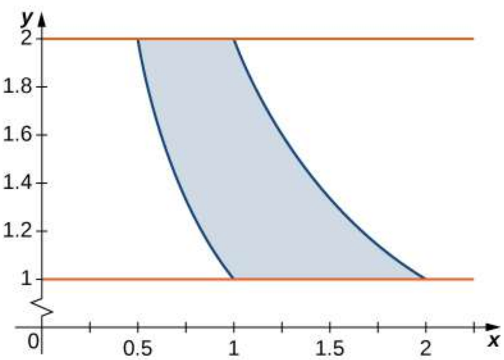
306. $R = \{(x, y) | 9x^2 + y^2 \le 1, x \ge 0, y \ge 0\}$; $\rho(x, y) = \sqrt{9x^2 + y^2}$.



307. R is the region bounded by $y = x$, $y = -x$, $y = x + 2$, $y = -x + 2$; $\rho(x, y) = 1$.



308. R is the region bounded by $y = \frac{1}{x}$, $y = \frac{2}{x}$, $y = 1$, and $y = 2$; $\rho(x, y) = 4(x + y)$.



In the following exercises, consider a lamina occupying the region R and having the density function ρ given in the preceding group of exercises. Use a computer algebra system (CAS) to answer the following questions.

a. Find the moments M_x and M_y about the x-axis and y-axis, respectively.

b. Calculate and plot the center of mass of the lamina.

c. **[T]** Use a CAS to locate the center of mass on the graph of R.

309. **[T]** R is the triangular region with vertices $(0, 0)$, $(0, 3)$, and $(6, 0)$; $\rho(x, y) = xy$.

310. **[T]** R is the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(0, 5)$; $\rho(x, y) = x + y$.

311. **[T]** R is the rectangular region with vertices $(0, 0)$, $(0, 3)$, $(6, 3)$, and $(6, 0)$; $\rho(x, y) = \sqrt{xy}$.

312. **[T]** R is the rectangular region with vertices $(0, 1)$, $(0, 3)$, $(3, 3)$, and $(3, 1)$; $\rho(x, y) = x^2y$.

313. **[T]** R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2}$, $y = 0$, $y = 2$, and $x = 0$; $\rho(x, y) = 3xy$.

314. **[T]** R is the trapezoidal region determined by the lines $y = 0$, $y = 1$, $y = x$, and $y = -x + 3$; $\rho(x, y) = 2x + y$.

315. **[T]** R is the disk of radius 2 centered at $(1, 2)$; $\rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

316. **[T]** R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.

317. **[T]** R is the region enclosed by the ellipse $x^2 + 4y^2 = 1$; $\rho(x, y) = 1$.

318. **[T]** $R = \{(x, y)|9x^2 + y^2 \le 1, x \ge 0, y \ge 0\}$; $\rho(x, y) = \sqrt{9x^2 + y^2}$.

319. **[T]** R is the region bounded by $y = x$, $y = -x$, $y = x + 2$, and $y = -x + 2$; $\rho(x, y) = 1$.

320. **[T]** R is the region bounded by $y = \frac{1}{x}$, $y = \frac{2}{x}$, $y = 1$, and $y = 2$; $\rho(x, y) = 4(x + y)$.

In the following exercises, consider a lamina occupying the region R and having the density function ρ given in the first two groups of Exercises.

a. Find the moments of inertia I_x, I_y, and I_0 about the x-axis, y-axis, and origin, respectively.

b. Find the radii of gyration with respect to the x-axis, y-axis, and origin, respectively.

321. R is the triangular region with vertices $(0, 0)$, $(0, 3)$, and $(6, 0)$; $\rho(x, y) = xy$.

322. R is the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(0, 5)$; $\rho(x, y) = x + y$.

323. R is the rectangular region with vertices $(0, 0)$, $(0, 3)$, $(6, 3)$, and $(6, 0)$; $\rho(x, y) = \sqrt{xy}$.

324. R is the rectangular region with vertices $(0, 1)$, $(0, 3)$, $(3, 3)$, and $(3, 1)$; $\rho(x, y) = x^2y$.

325. R is the trapezoidal region determined by the lines $y = -\frac{1}{4}x + \frac{5}{2}$, $y = 0$, $y = 2$, and $x = 0$; $\rho(x, y) = 3xy$.

326. R is the trapezoidal region determined by the lines $y = 0$, $y = 1$, $y = x$, and $y = -x + 3$; $\rho(x, y) = 2x + y$.

327. R is the disk of radius 2 centered at $(1, 2)$; $\rho(x, y) = x^2 + y^2 - 2x - 4y + 5$.

328. R is the unit disk; $\rho(x, y) = 3x^4 + 6x^2y^2 + 3y^4$.

329. R is the region enclosed by the ellipse $x^2 + 4y^2 = 1$; $\rho(x, y) = 1$.

330. $R = \{(x, y)|9x^2 + y^2 \le 1, x \ge 0, y \ge 0\}$; $\rho(x, y) = \sqrt{9x^2 + y^2}$.

331. R is the region bounded by $y = x$, $y = -x$, $y = x + 2$, and $y = -x + 2$; $\rho(x, y) = 1$.

332. R is the region bounded by $y = \frac{1}{x}$, $y = \frac{2}{x}$, $y = 1$, and $y = 2$; $\rho(x, y) = 4(x + y)$.

333. Let Q be the solid unit cube. Find the mass of the solid if its density ρ is equal to the square of the distance of an arbitrary point of Q to the xy-plane.

334. Let Q be the solid unit hemisphere. Find the mass of the solid if its density ρ is proportional to the distance of an arbitrary point of Q to the origin.

335. The solid Q of constant density 1 is situated inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the sphere $x^2 + y^2 + z^2 = 1$. Show that the center of mass of the solid is not located within the solid.

336. Find the mass of the solid $Q = \{(x, y, z)|1 \le x^2 + z^2 \le 25, y \le 1 - x^2 - z^2\}$ whose density is $\rho(x, y, z) = k$, where $k > 0$.

337. **[T]** The solid $Q = \left\{(x, y, z) \mid x^2 + y^2 \le 9,\ 0 \le z \le 1,\ x \ge 0,\ y \ge 0\right\}$ has density equal to the distance to the xy-plane. Use a CAS to answer the following questions.

a. Find the mass of Q.
b. Find the moments M_{xy}, M_{xz}, and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.
c. Find the center of mass of Q.
d. Graph Q and locate its center of mass.

338. Consider the solid $Q = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$ with the density function $\rho(x, y, z) = x + y + 1$.

a. Find the mass of Q.
b. Find the moments M_{xy}, M_{xz}, and M_{yz} about the xy-plane, xz-plane, and yz-plane, respectively.
c. Find the center of mass of Q.

339. **[T]** The solid Q has the mass given by the triple integral $\int_{-1}^{1}\int_{0}^{\pi/4}\int_{0}^{1} r^2\,dr\,d\theta\,dz$. Use a CAS to answer the following questions.

a. Show that the center of mass of Q is located in the xy-plane.
b. Graph Q and locate its center of mass.

340. The solid Q is bounded by the planes $x + 4y + z = 8$, $x = 0$, $y = 0$, and $z = 0$. Its density at any point is equal to the distance to the xz-plane. Find the moments of inertia I_y of the solid about the xz-plane.

341. The solid Q is bounded by the planes $x + y + z = 3$, $x = 0$, $y = 0$, and $z = 0$. Its density is $\rho(x, y, z) = x + ay$, where $a > 0$. Show that the center of mass of the solid is located in the plane $z = \frac{3}{5}$ for any value of a.

342. Let Q be the solid situated outside the sphere $x^2 + y^2 + z^2 = z$ and inside the upper hemisphere $x^2 + y^2 + z^2 = R^2$, where $R > 1$. If the density of the solid is $\rho(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, find R such that the mass of the solid is $\frac{7\pi}{2}$.

343. The mass of a solid Q is given by $\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} \left(x^2 + y^2 + z^2\right)^n dz\,dy\,dx$, where n is an integer. Determine n such the mass of the solid is $(2 - \sqrt{2})\pi$.

344. Let Q be the solid bounded above the cone $x^2 + y^2 = z^2$ and below the sphere $x^2 + y^2 + z^2 - 4z = 0$. Its density is a constant $k > 0$. Find k such that the center of mass of the solid is situated 7 units from the origin.

345. The solid $Q = \left\{(x, y, z) \mid 0 \le x^2 + y^2 \le 16,\ x \ge 0,\ y \ge 0,\ 0 \le z \le x\right\}$ has the density $\rho(x, y, z) = k$. Show that the moment M_{xy} about the xy-plane is half of the moment M_{yz} about the yz-plane.

346. The solid Q is bounded by the cylinder $x^2 + y^2 = a^2$, the paraboloid $b^2 - z = x^2 + y^2$, and the xy-plane, where $0 < a < b$. Find the mass of the solid if its density is given by $\rho(x, y, z) = \sqrt{x^2 + y^2}$.

347. Let Q be a solid of constant density k, where $k > 0$, that is located in the first octant, inside the circular cone $x^2 + y^2 = 9(z - 1)^2$, and above the plane $z = 0$. Show that the moment M_{xy} about the xy-plane is the same as the moment M_{yz} about the xz-plane.

348. The solid Q has the mass given by the triple integral $\int_{0}^{1}\int_{0}^{\pi/2}\int_{0}^{r^2}\left(r^4 + r\right)dz\,d\theta\,dr.$

a. Find the density of the solid in rectangular coordinates.
b. Find the moment M_{xy} about the xy-plane.

349. The solid Q has the moment of inertia I_x about the yz-plane given by the triple integral $\int_{0}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\frac{1}{2}\left(x^2+y^2\right)}^{\sqrt{x^2+y^2}}\left(y^2 + z^2\right)\left(x^2 + y^2\right)dz\,dx\,dy.$

a. Find the density of Q.
b. Find the moment of inertia I_z about the xy-plane.

350. The solid Q has the mass given by the triple integral $\int_0^{\pi/4} \int_0^{2 \sec \theta} \int_0^1 \left(r^3 \cos \theta \sin \theta + 2r\right) dz\, dr\, d\theta.$

a. Find the density of the solid in rectangular coordinates.
b. Find the moment M_{xz} about the xz-plane.

351. Let Q be the solid bounded by the xy-plane, the cylinder $x^2 + y^2 = a^2$, and the plane $z = 1$, where $a > 1$ is a real number. Find the moment M_{xy} of the solid about the xy-plane if its density given in cylindrical coordinates is $\rho(r, \theta, z) = \frac{d^2 f}{dr^2}(r)$, where f is a differentiable function with the first and second derivatives continuous and differentiable on $(0, a)$.

352. A solid Q has a volume given by $\iint_D \int_a^b dA\, dz$, where D is the projection of the solid onto the xy-plane and $a < b$ are real numbers, and its density does not depend on the variable z. Show that its center of mass lies in the plane $z = \frac{a + b}{2}$.

353. Consider the solid enclosed by the cylinder $x^2 + z^2 = a^2$ and the planes $y = b$ and $y = c$, where $a > 0$ and $b < c$ are real numbers. The density of Q is given by $\rho(x, y, z) = f'(y)$, where f is a differential function whose derivative is continuous on (b, c). Show that if $f(b) = f(c)$, then the moment of inertia about the xz-plane of Q is null.

354. **[T]** The average density of a solid Q is defined as $\rho_{ave} = \frac{1}{V(Q)} \iiint_Q \rho(x, y, z) dV = \frac{m}{V(Q)}$, where $V(Q)$ and m are the volume and the mass of Q, respectively. If the density of the unit ball centered at the origin is $\rho(x, y, z) = e^{-x^2 - y^2 - z^2}$, use a CAS to find its average density. Round your answer to three decimal places.

355. Show that the moments of inertia I_x, I_y, and I_z about the yz-plane, xz-plane, and xy-plane, respectively, of the unit ball centered at the origin whose density is $\rho(x, y, z) = e^{-x^2 - y^2 - z^2}$ are the same. Round your answer to two decimal places.

5.7 | Change of Variables in Multiple Integrals

Learning Objectives

- **5.7.1** Determine the image of a region under a given transformation of variables.
- **5.7.2** Compute the Jacobian of a given transformation.
- **5.7.3** Evaluate a double integral using a change of variables.
- **5.7.4** Evaluate a triple integral using a change of variables.

Recall from **Substitution Rule (http://cnx.org/content/m53634/latest/)** the method of integration by substitution. When evaluating an integral such as $\int_2^3 x(x^2-4)^5\,dx,$ we substitute $u = g(x) = x^2 - 4.$ Then $du = 2x\,dx$ or $x\,dx = \frac{1}{2}du$ and the limits change to $u = g(2) = 2^2 - 4 = 0$ and $u = g(3) = 9 - 4 = 5.$ Thus the integral becomes $\int_0^5 \frac{1}{2}u^5\,du$ and this integral is much simpler to evaluate. In other words, when solving integration problems, we make appropriate substitutions to obtain an integral that becomes much simpler than the original integral.

We also used this idea when we transformed double integrals in rectangular coordinates to polar coordinates and transformed triple integrals in rectangular coordinates to cylindrical or spherical coordinates to make the computations simpler. More generally,

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du,$$

Where $x = g(u),\ dx = g'(u)du,$ and $u = c$ and $u = d$ satisfy $c = g(a)$ and $d = g(b).$

A similar result occurs in double integrals when we substitute $x = f(r, \theta) = r\cos\theta,$ $y = g(r, \theta) = r\sin\theta,$ and $dA = dx\,dy = r\,dr\,d\theta.$ Then we get

$$\iint_R f(x, y)dA = \iint_S f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

where the domain R is replaced by the domain S in polar coordinates. Generally, the function that we use to change the variables to make the integration simpler is called a **transformation** or mapping.

Planar Transformations

A **planar transformation** T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of $R^2.$ For example, **Figure 5.71** shows a region G in the uv-plane transformed into a region R in the xy-plane by the change of variables $x = g(u, v)$ and $y = h(u, v),$ or sometimes we write $x = x(u, v)$ and $y = y(u, v).$ We shall typically assume that each of these functions has continuous first partial derivatives, which means $g_u, g_v, h_u,$ and h_v exist and are also continuous. The need for this requirement will become clear soon.

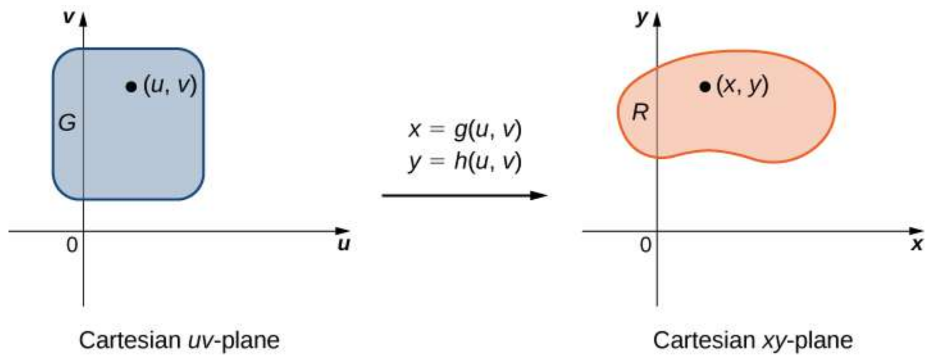


Figure 5.71 The transformation of a region G in the uv-plane into a region R in the xy-plane.

Definition

A transformation $T: G \to R,$ defined as $T(u, v) = (x, y),$ is said to be a **one-to-one transformation** if no two points map to the same image point.

To show that T is a one-to-one transformation, we assume $T(u_1, v_1) = T(u_2, v_2)$ and show that as a consequence we obtain $(u_1, v_1) = (u_2, v_2)$. If the transformation T is one-to-one in the domain $G,$ then the inverse T^{-1} exists with the domain R such that $T^{-1} \circ T$ and $T \circ T^{-1}$ are identity functions.

Figure 5.71 shows the mapping $T(u, v) = (x, y)$ where x and y are related to u and v by the equations $x = g(u, v)$ and $y = h(u, v)$. The region G is the domain of T and the region R is the range of $T,$ also known as the *image* of G under the transformation T.

Example 5.65

Determining How the Transformation Works

Suppose a transformation T is defined as $T(r, \theta) = (x, y)$ where $x = r\cos\theta, y = r\sin\theta$. Find the image of the polar rectangle $G = \{(r, \theta)|0 < r \leq 1, 0 \leq \theta \leq \pi/2\}$ in the $r\theta$-plane to a region R in the xy-plane. Show that T is a one-to-one transformation in G and find $T^{-1}(x, y)$.

Solution

Since r varies from 0 to 1 in the $r\theta$-plane, we have a circular disc of radius 0 to 1 in the xy-plane. Because θ varies from 0 to $\pi/2$ in the $r\theta$-plane, we end up getting a quarter circle of radius 1 in the first quadrant of the xy-plane (**Figure 5.72**). Hence R is a quarter circle bounded by $x^2 + y^2 = 1$ in the first quadrant.

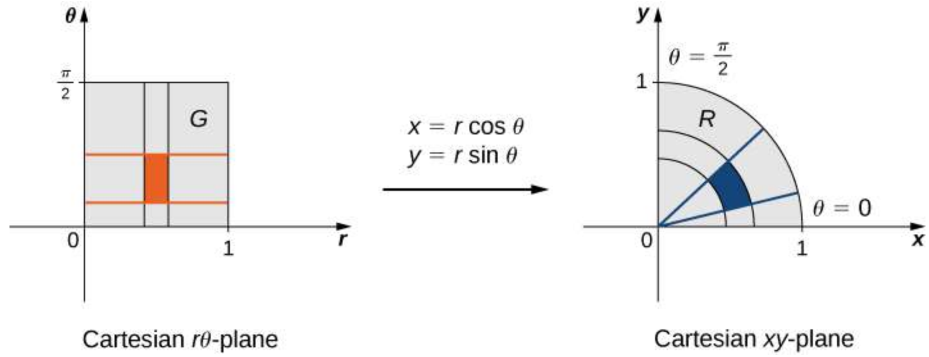


Figure 5.72 A rectangle in the $r\theta$-plane is mapped into a quarter circle in the xy-plane.

In order to show that T is a one-to-one transformation, assume $T(r_1, \theta_1) = T(r_2, \theta_2)$ and show as a consequence that $(r_1, \theta_1) = (r_2, \theta_2)$. In this case, we have

$$\begin{aligned} T(r_1, \theta_1) &= T(r_2, \theta_2), \\ (x_1, y_1) &= (x_1, y_1), \\ (r_1 \cos\theta_1, r_1 \sin\theta_1) &= (r_2 \cos\theta_2, r_2 \sin\theta_2), \\ r_1 \cos\theta_1 &= r_2 \cos\theta_2, \ r_1 \sin\theta_1 = r_2 \sin\theta_2. \end{aligned}$$

Dividing, we obtain

$$\begin{aligned} \frac{r_1 \cos\theta_1}{r_1 \sin\theta_1} &= \frac{r_2 \cos\theta_2}{r_2 \sin\theta_2} \\ \frac{\cos\theta_1}{\sin\theta_1} &= \frac{\cos\theta_2}{\sin\theta_2} \\ \tan\theta_1 &= \tan\theta_2 \\ \theta_1 &= \theta_2 \end{aligned}$$

since the tangent function is one-one function in the interval $0 \le \theta \le \pi/2$. Also, since $0 < r \le 1$, we have $r_1 = r_2, \theta_1 = \theta_2$. Therefore, $(r_1, \theta_1) = (r_2, \theta_2)$ and T is a one-to-one transformation from G into R.

To find $T^{-1}(x, y)$ solve for r, θ in terms of x, y. We already know that $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. Thus $T^{-1}(x, y) = (r, \theta)$ is defined as $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

Example 5.66

Finding the Image under T

Let the transformation T be defined by $T(u, v) = (x, y)$ where $x = u^2 - v^2$ and $y = uv$. Find the image of the triangle in the uv-plane with vertices $(0, 0), (0, 1),$ and $(1, 1)$.

Solution

The triangle and its image are shown in **Figure 5.73**. To understand how the sides of the triangle transform, call the side that joins $(0, 0)$ and $(0, 1)$ side $A,$ the side that joins $(0, 0)$ and $(1, 1)$ side $B,$ and the side that joins $(1, 1)$ and $(0, 1)$ side $C.$

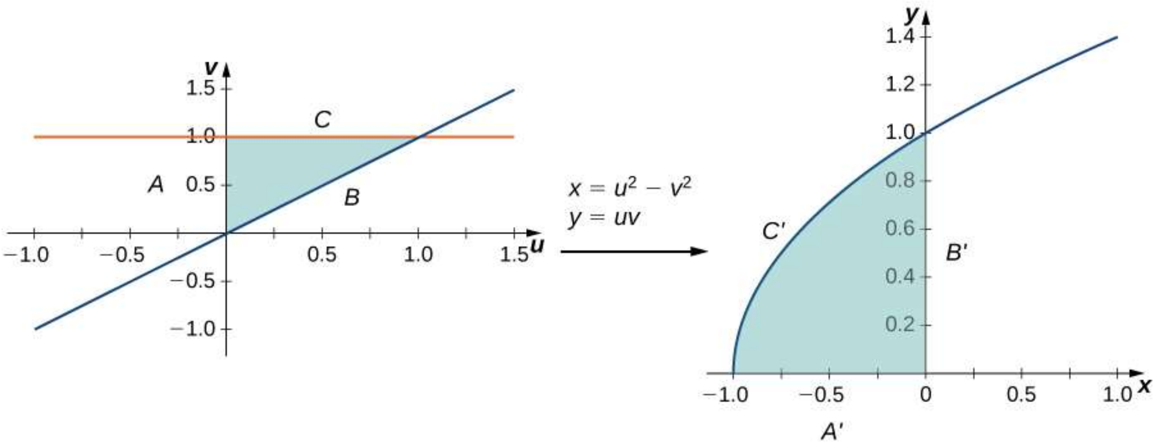


Figure 5.73 A triangular region in the uv-plane is transformed into an image in the xy-plane.

For the side $A: u = 0, 0 \leq v \leq 1$ transforms to $x = -v^2, y = 0$ so this is the side A' that joins $(-1, 0)$ and $(0, 0).$

For the side $B: u = v, 0 \leq u \leq 1$ transforms to $x = 0, y = u^2$ so this is the side B' that joins $(0, 0)$ and $(0, 1).$

For the side $C: 0 \leq u \leq 1, v = 1$ transforms to $x = u^2 - 1, y = u$ (hence $x = y^2 - 1$) so this is the side C' that makes the upper half of the parabolic arc joining $(-1, 0)$ and $(0, 1).$

All the points in the entire region of the triangle in the uv-plane are mapped inside the parabolic region in the xy-plane.

5.43 Let a transformation T be defined as $T(u, v) = (x, y)$ where $x = u + v, y = 3v.$ Find the image of the rectangle $G = \{(u, v): 0 \leq u \leq 1, 0 \leq v \leq 2\}$ from the uv-plane after the transformation into a region R in the xy-plane. Show that T is a one-to-one transformation and find $T^{-1}(x, y).$

Jacobians

Recall that we mentioned near the beginning of this section that each of the component functions must have continuous first partial derivatives, which means that $g_u, g_v, h_u,$ and h_v exist and are also continuous. A transformation that has this property is called a C^1 transformation (here C denotes continuous). Let $T(u, v) = (g(u, v), h(u, v)),$ where $x = g(u, v)$ and $y = h(u, v),$ be a one-to-one C^1 transformation. We want to see how it transforms a small rectangular region $S,$ Δu units by Δv units, in the uv-plane (see the following figure).

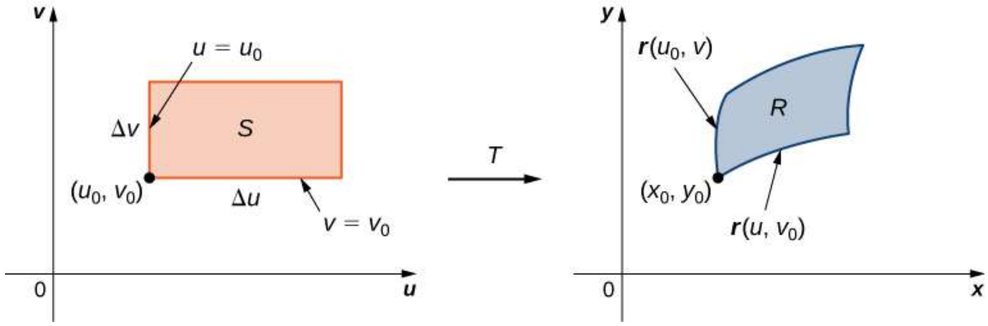


Figure 5.74 A small rectangle S in the uv-plane is transformed into a region R in the xy-plane.

Since $x = g(u, v)$ and $y = h(u, v)$, we have the position vector $\mathbf{r}(u, v) = g(u, v)\mathbf{i} + h(u, v)\mathbf{j}$ of the image of the point (u, v). Suppose that (u_0, v_0) is the coordinate of the point at the lower left corner that mapped to $(x_0, y_0) = T(u_0, v_0)$. The line $v = v_0$ maps to the image curve with vector function $\mathrm{r}(u, v_0)$, and the tangent vector at (x_0, y_0) to the image curve is

$$\mathbf{r}_u = g_u(u_0, v_0)\mathbf{i} + h_u(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}.$$

Similarly, the line $u = u_0$ maps to the image curve with vector function $\mathbf{r}(u_0, v)$, and the tangent vector at (x_0, y_0) to the image curve is

$$\mathbf{r}_v = g_v(u_0, v_0)\mathbf{i} + h_v(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}.$$

Now, note that

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u} \text{ so } \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \mathbf{r}_u.$$

Similarly,

$$\mathbf{r}_v = \lim_{\Delta v \to 0} \frac{\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)}{\Delta v} \text{ so } \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \mathbf{r}_v.$$

This allows us to estimate the area ΔA of the image R by finding the area of the parallelogram formed by the sides $\Delta v \mathbf{r}_v$ and $\Delta u \mathbf{r}_u$. By using the cross product of these two vectors by adding the **k**th component as 0, the area ΔA of the image R (refer to **The Cross Product**) is approximately $|\Delta u \mathbf{r}_u \times \Delta v \mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$. In determinant form, the cross product is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\mathbf{k}.$$

Since $|\mathbf{k}| = 1$, we have $\Delta A \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\Delta u \Delta v.$

Definition

The **Jacobian** of the C^1 transformation $T(u, v) = (g(u, v), h(u, v))$ is denoted by $J(u, v)$ and is defined by the 2×2 determinant

$$J(u,\ v) = \left|\frac{\partial(x,\ y)}{\partial(u,\ v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right).$$

Using the definition, we have

$$\Delta A \approx J(u,\ v)\Delta u \Delta v = \left|\frac{\partial(x,\ y)}{\partial(u,\ v)}\right|\Delta u \Delta v.$$

Note that the Jacobian is frequently denoted simply by

$$J(u,\ v) = \frac{\partial(x,\ y)}{\partial(u,\ v)}.$$

Note also that

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Hence the notation $J(u,\ v) = \frac{\partial(x,\ y)}{\partial(u,\ v)}$ suggests that we can write the Jacobian determinant with partials of x in the first row and partials of y in the second row.

Example 5.67

Finding the Jacobian

Find the Jacobian of the transformation given in **Example 5.65**.

Solution

The transformation in the example is $T(r,\ \theta) = (r\cos\theta,\ r\sin\theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Thus the Jacobian is

$$\begin{aligned} J(r,\ \theta) &= \frac{\partial(x,\ y)}{\partial(r,\ \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r. \end{aligned}$$

Example 5.68

Finding the Jacobian

Find the Jacobian of the transformation given in **Example 5.66**.

Solution

The transformation in the example is $T(u,\ v) = \left(u^2 - v^2,\ uv\right)$ where $x = u^2 - v^2$ and $y = uv$. Thus the Jacobian is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & v \\ -2v & u \end{vmatrix} = 2u^2 + 2v^2.$$

5.44 Find the Jacobian of the transformation given in the previous checkpoint: $T(u, v) = (u + v, 2v)$.

Change of Variables for Double Integrals

We have already seen that, under the change of variables $T(u, v) = (x, y)$ where $x = g(u, v)$ and $y = h(u, v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u \Delta v$ in the uv-plane by the approximation

$$\Delta A \approx J(u, v)\Delta u, \Delta v.$$

Now let's go back to the definition of double integral for a minute:

$$\iint_R f(x, y)dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A.$$

Referring to **Figure 5.75**, observe that we divided the region S in the uv-plane into small subrectangles S_{ij} and we let the subrectangles R_{ij} in the xy-plane be the images of S_{ij} under the transformation $T(u, v) = (x, y)$.

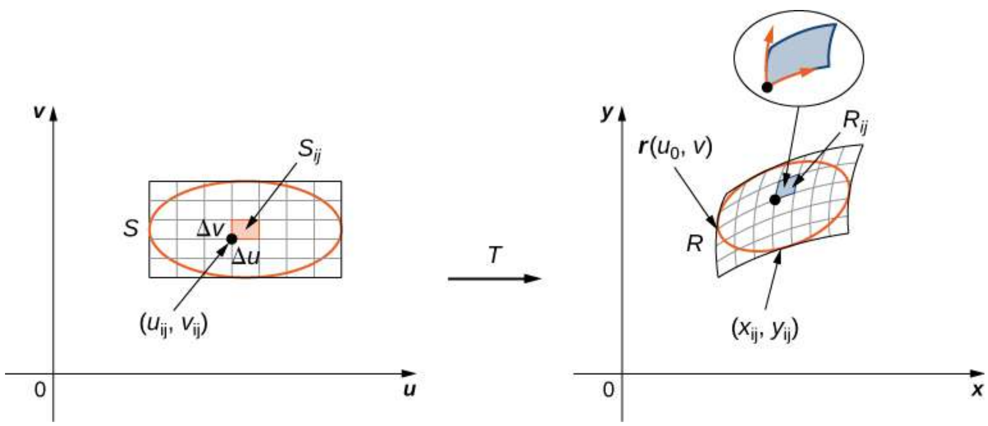


Figure 5.75 The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Then the double integral becomes

$$\iint_R f(x, y)dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_{ij}, v_{ij}), h(u_{ij}, v_{ij}))\left|J(u_{ij}, v_{ij})\right|\Delta u \Delta v.$$

Notice this is exactly the double Riemann sum for the integral

$$\iint_S f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv.$$

Theorem 5.14: Change of Variables for Double Integrals

Let $T(u, v) = (x, y)$ where $x = g(u, v)$ and $y = h(u, v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane; it maps S into the region R in the xy-plane. If f is continuous on R, then

$$\iint_R f(x, y)dA = \iint_S f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right|du\,dv.$$

With this theorem for double integrals, we can change the variables from (x, y) to (u, v) in a double integral simply by replacing

$$dA = dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right|du\,dv$$

when we use the substitutions $x = g(u, v)$ and $y = h(u, v)$ and then change the limits of integration accordingly. This change of variables often makes any computations much simpler.

Example 5.69

Changing Variables from Rectangular to Polar Coordinates

Consider the integral

$$\int_0^2 \int_0^{\sqrt{2x - x^2}} \sqrt{x^2 + y^2}\,dy\,dx.$$

Use the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, and find the resulting integral.

Solution

First we need to find the region of integration. This region is bounded below by $y = 0$ and above by $y = \sqrt{2x - x^2}$ (see the following figure).

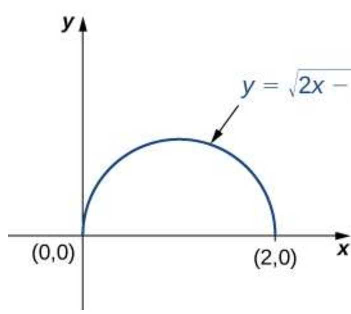


Figure 5.76 Changing a region from rectangular to polar coordinates.

Squaring and collecting terms, we find that the region is the upper half of the circle $x^2 + y^2 - 2x = 0$, that is, $y^2 + (x - 1)^2 = 1$. In polar coordinates, the circle is $r = 2\cos\theta$ so the region of integration in polar coordinates is bounded by $0 \le r \le \cos\theta$ and $0 \le \theta \le \frac{\pi}{2}$.

The Jacobian is $J(r, \theta) = r$, as shown in **Example 5.67**. Since $r \geq 0$, we have $|J(r, \theta)| = r$.

The integrand $\sqrt{x^2 + y^2}$ changes to r in polar coordinates, so the double iterated integral is

$$\int_0^2 \int_0^{\sqrt{2x - x^2}} \sqrt{x^2 + y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r|J(r, \theta)|\, dr\, d\theta = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta.$$

5.45 Considering the integral $\int_0^1 \int_0^{\sqrt{1 - x^2}} \left(x^2 + y^2\right) dy\, dx$, use the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, and find the resulting integral.

Notice in the next example that the region over which we are to integrate may suggest a suitable transformation for the integration. This is a common and important situation.

Example 5.70

Changing Variables

Consider the integral $\iint_R (x - y)\, dy\, dx$, where R is the parallelogram joining the points $(1, 2)$, $(3, 4)$, $(4, 3)$, and $(6, 5)$ (**Figure 5.77**). Make appropriate changes of variables, and write the resulting integral.

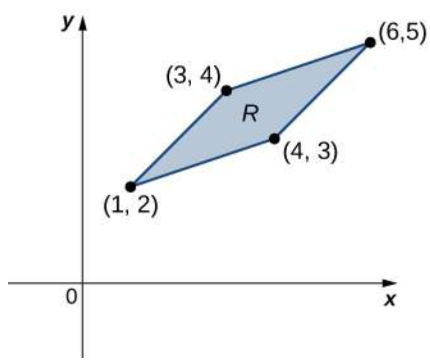


Figure 5.77 The region of integration for the given integral.

Solution

First, we need to understand the region over which we are to integrate. The sides of the parallelogram are $x - y + 1 = 0$, $x - y - 1 = 0$, $x - 3y + 5 = 0$, and $x - 3y + 9 = 0$ (**Figure 5.78**). Another way to look at them is $x - y = -1$, $x - y = 1$, $x - 3y = -5$, and $x - 3y = 9$.

Clearly the parallelogram is bounded by the lines $y = x + 1$, $y = x - 1$, $y = \frac{1}{3}(x + 5)$, and $y = \frac{1}{3}(x + 9)$.

Notice that if we were to make $u = x - y$ and $v = x - 3y$, then the limits on the integral would be

$-1 \le u \le 1$ and $-9 \le v \le -5$.

To solve for x and y, we multiply the first equation by 3 and subtract the second equation, $3u - v = (3x - 3y) - (x - 3y) = 2x$. Then we have $x = \frac{3u - v}{2}$. Moreover, if we simply subtract the second equation from the first, we get $u - v = (x - y) - (x - 3y) = 2y$ and $y = \frac{u - v}{2}$.

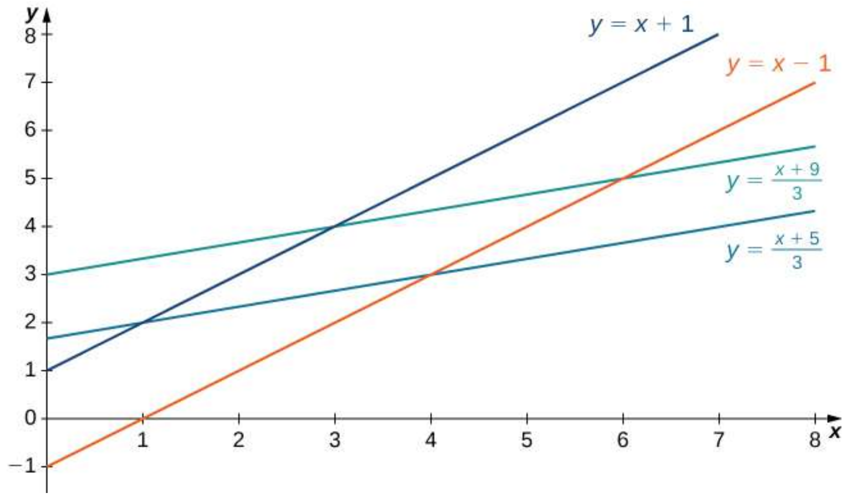


Figure 5.78 A parallelogram in the xy-plane that we want to transform by a change in variables.

Thus, we can choose the transformation

$$T(u, v) = \left(\frac{3u - v}{2}, \frac{u - v}{2}\right)$$

and compute the Jacobian $J(u, v)$. We have

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3/2 & -1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}.$$

Therefore, $|J(u, v)| = \frac{1}{2}$. Also, the original integrand becomes

$$x - y = \frac{1}{2}[3u - v - u + v] = \frac{1}{2}[3u - u] = \frac{1}{2}[2u] = u.$$

Therefore, by the use of the transformation T, the integral changes to

$$\iint_R (x - y)dy\,dx = \int_{-9}^{-5}\int_{-1}^{1} J(u, v)u\,du\,dv = \int_{-9}^{-5}\int_{-1}^{1}\left(\frac{1}{2}\right)u\,du\,dv,$$

which is much simpler to compute.

5.46 Make appropriate changes of variables in the integral $\iint_R \frac{4}{(x - y)^2}dy\,dx$, where R is the trapezoid bounded by the lines $x - y = 2$, $x - y = 4$, $x = 0$, and $y = 0$. Write the resulting integral.

We are ready to give a problem-solving strategy for change of variables.

Problem-Solving Strategy: Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x = g(u, v)$ and $y = h(u, v)$.
3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J(u, v)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\,dx$ or $dx\,dy$, whichever occurs, by $J(u, v)du\,dv$.

In the next example, we find a substitution that makes the integrand much simpler to compute.

Example 5.71

Evaluating an Integral

Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral

$$\iint_R (x - y)e^{x^2 - y^2}\,dA,$$

where R is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$ (see the first region in **Figure 5.79**).

Solution

As before, first find the region R and picture the transformation so it becomes easier to obtain the limits of integration after the transformations are made (**Figure 5.79**).

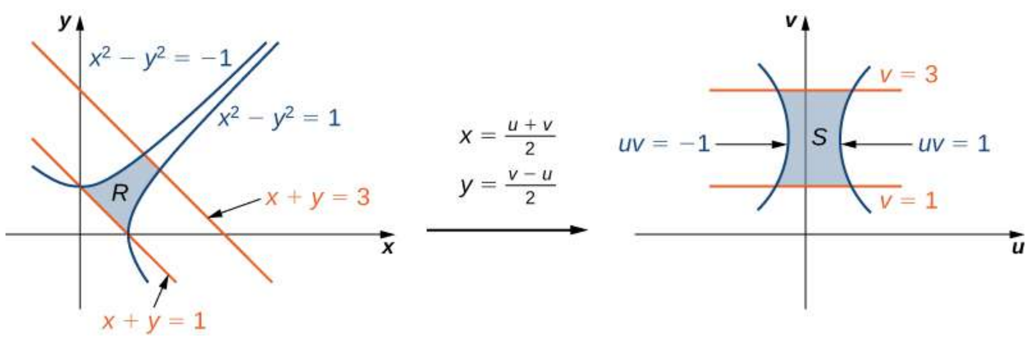


Figure 5.79 Transforming the region R into the region S to simplify the computation of an integral.

Given $u = x - y$ and $v = x + y$, we have $x = \frac{u + v}{2}$ and $y = \frac{v - u}{2}$ and hence the transformation to use is $T(u, v) = \left(\frac{u + v}{2}, \frac{v - u}{2}\right)$. The lines $x + y = 1$ and $x + y = 3$ become $v = 1$ and $v = 3$, respectively. The curves $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ become $uv = 1$ and $uv = -1$, respectively.

Thus we can describe the region S (see the second region Figure 5.79) as

$$S = \left\{(u, v) | 1 \le v \le 3, \frac{-1}{v} \le u \le \frac{1}{v}\right\}.$$

The Jacobian for this transformation is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Therefore, by using the transformation T, the integral changes to

$$\iint_R (x - y)e^{x^2 - y^2} dA = \frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\, du\, dv.$$

Doing the evaluation, we have

$$\frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\, du\, dv = \frac{4}{3e} \approx 0.490.$$

5.47 Using the substitutions $x = v$ and $y = \sqrt{u + v}$, evaluate the integral $\iint_R y \sin\left(y^2 - x\right) dA$ where R is the region bounded by the lines $y = \sqrt{x}$, $x = 2$, and $y = 0$.

Change of Variables for Triple Integrals

Changing variables in triple integrals works in exactly the same way. Cylindrical and spherical coordinate substitutions are special cases of this method, which we demonstrate here.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space (Figure 5.80) by a one-to-one C^1 transformation $T(u, v, w) = (x, y, z)$ where $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$.

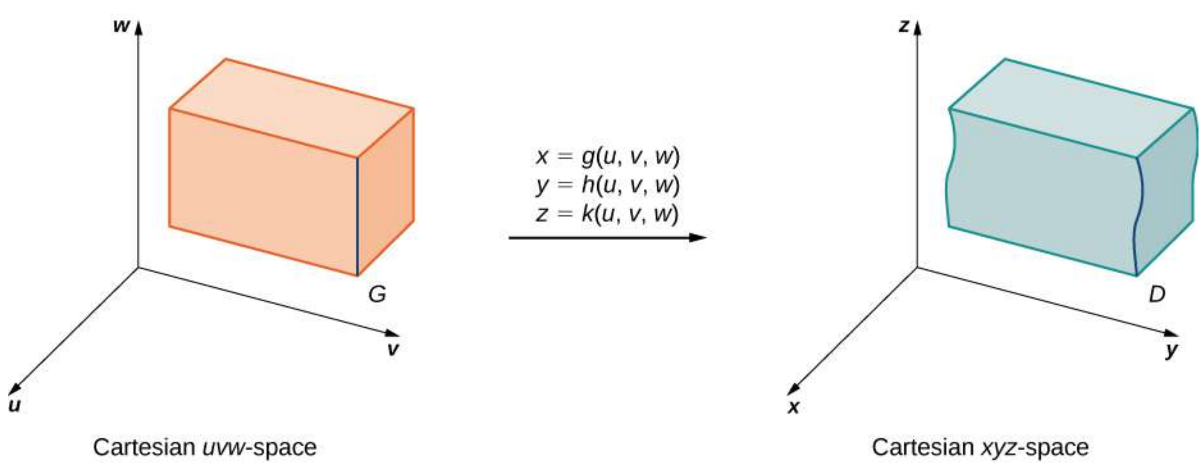


Figure 5.80 A region G in uvw-space mapped to a region D in xyz-space.

Then any function $F(x, y, z)$ defined on D can be thought of as another function $H(u, v, w)$ that is defined on G:

$$F(x, y, z) = F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w).$$

Now we need to define the Jacobian for three variables.

Definition

The Jacobian determinant $J(u, v, w)$ in three variables is defined as follows:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This is also the same as

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The Jacobian can also be simply denoted as $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.

With the transformations and the Jacobian for three variables, we are ready to establish the theorem that describes change of variables for triple integrals.

Theorem 5.15: Change of Variables for Triple Integrals

Let $T(u, v, w) = (x, y, z)$ where $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-plane into the region D in the xyz-plane. As in the two-dimensional case, if F is continuous on D, then

$$\begin{aligned} \iiint_R F(x, y, z)dV &= \iiint_G F(g(u, v, w), h(u, v, w), k(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw \\ &= \iiint_G H(u, v, w)|J(u, v, w)|du\, dv\, dw. \end{aligned}$$

Let us now see how changes in triple integrals for cylindrical and spherical coordinates are affected by this theorem. We expect to obtain the same formulas as in **Triple Integrals in Cylindrical and Spherical Coordinates**.

Example 5.72

Obtaining Formulas in Triple Integrals for Cylindrical and Spherical Coordinates

Derive the formula in triple integrals for

a. cylindrical and
b. spherical coordinates.

Solution

a. For cylindrical coordinates, the transformation is $T(r, \theta, z) = (x, y, z)$ from the Cartesian $r\theta z$-plane to the Cartesian xyz-plane (**Figure 5.81**). Here $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. The Jacobian for the transformation is

$$J(r, \theta, z) = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

We know that $r \geq 0$, so $|J(r, \theta, z)| = r$. Then the triple integral is

$$\iiint_D f(x, y, z)dV = \iiint_G f(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz.$$

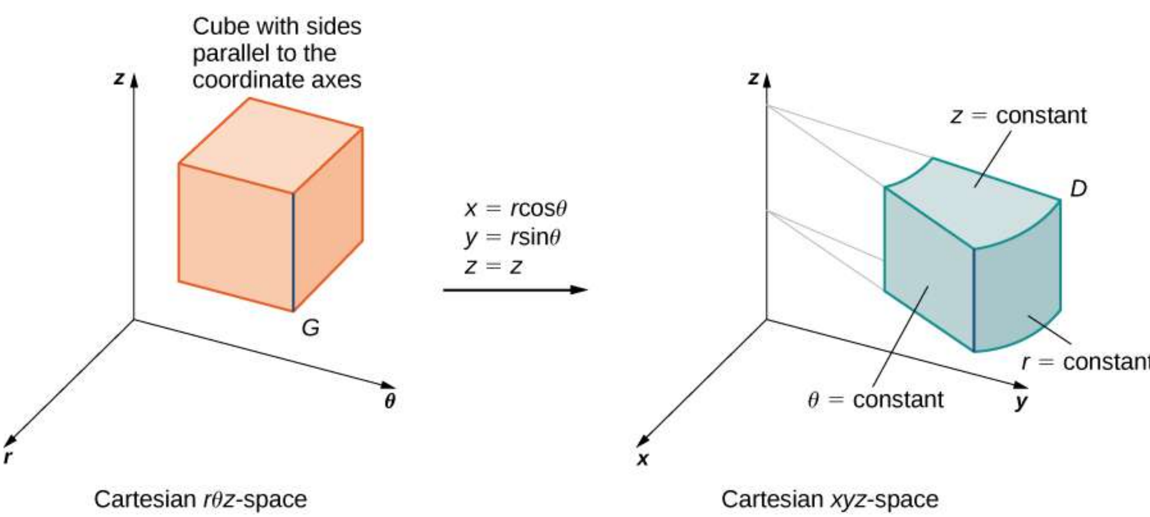


Figure 5.81 The transformation from rectangular coordinates to cylindrical coordinates can be treated as a change of variables from region G in $r\theta z$-space to region D in xyz-space.

b. For spherical coordinates, the transformation is $T(\rho, \theta, \varphi) = (x, y, z)$ from the Cartesian $p\theta\varphi$-plane to the Cartesian xyz-plane (**Figure 5.82**). Here $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$. The Jacobian for the transformation is

$$J(\rho, \theta, \varphi) = \frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & -\rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\theta & 0 & -\rho\sin\varphi \end{vmatrix}.$$

Expanding the determinant with respect to the third row:

$$= \cos\varphi \begin{vmatrix} -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \rho\sin\varphi\sin\theta & \rho\cos\varphi\sin\theta \end{vmatrix} - \rho\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix}$$

$$= \cos\varphi\left(-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta\right)$$
$$-\rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right)$$

$$= -\rho^2\sin\varphi\cos^2\varphi\left(\sin^2\theta + \cos^2\theta\right) - \rho^2\sin\varphi\sin^2\varphi\left(\sin^2\theta + \cos^2\theta\right)$$

$$= -\rho^2\sin\varphi\cos^2\varphi - \rho^2\sin\varphi\sin^2\varphi$$

$$= -\rho^2\sin\varphi\left(\cos^2\varphi + \sin^2\varphi\right) = -\rho^2\sin\varphi.$$

Since $0 \le \varphi \le \pi$, we must have $\sin\varphi \ge 0$. Thus $|J(\rho, \theta, \varphi)| = \left|-\rho^2\sin\varphi\right| = \rho^2\sin\varphi$.

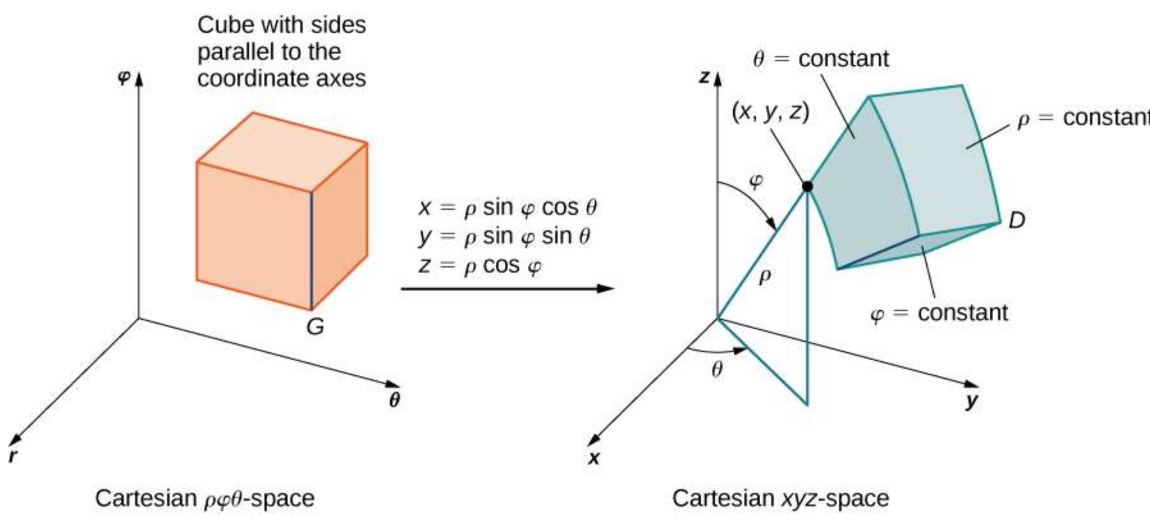


Figure 5.82 The transformation from rectangular coordinates to spherical coordinates can be treated as a change of variables from region G in $\rho\theta\varphi$-space to region D in xyz-space.

Then the triple integral becomes

$$\iiint\limits_D f(x, y, z)dV = \iiint\limits_G f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Let's try another example with a different substitution.

Example 5.73

Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$$

in xyz-space by using the transformation

$$u = (2x - y)/2,\ v = y/2,\ \text{and}\ w = z/3.$$

Then integrate over an appropriate region in uvw-space.

Solution

As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space (**Figure 5.83**). Clearly G in xyz-space is bounded by the planes $x = y/2,\ x = (y/2) + 1,\ y = 0,\quad y = 4,\quad z = 0,$ and $z = 4$. We also know that we have to use $u = (2x - y)/2,\ v = y/2,$ and $w = z/3$ for the transformations. We need to solve for $x,\ y,$ and z. Here we find that $x = u + v,\quad y = 2v,$ and $z = 3w$.

Using elementary algebra, we can find the corresponding surfaces for the region G and the limits of integration in uvw-space. It is convenient to list these equations in a table.

Equations in xyz for the region D	**Corresponding equations in uvw for the region G**	**Limits for the integration in uvw**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = y/2$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

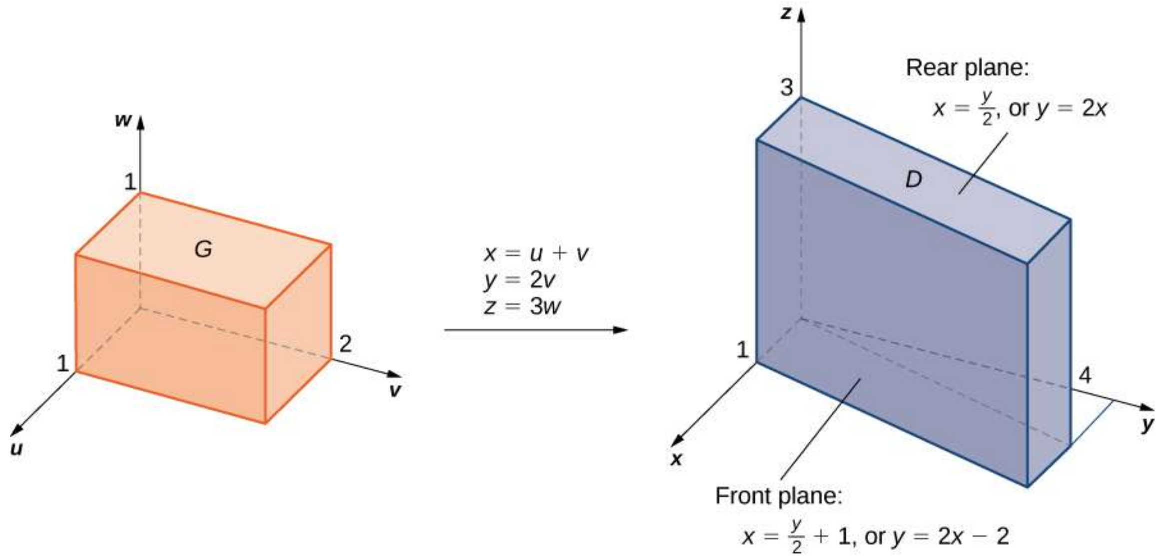


Figure 5.83 The region G in uvw-space is transformed to region D in xyz-space.

Now we can calculate the Jacobian for the transformation:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

The function to be integrated becomes

$$f(x, y, z) = x + \frac{z}{3} = u + v + \frac{3w}{3} = u + v + w.$$

We are now ready to put everything together and complete the problem.

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$$

$$= \int_0^1 \int_0^2 \int_0^1 (u + v + w)|J(u, v, w)|du\, dv\, dw = \int_0^1 \int_0^2 \int_0^1 (u + v + w)|6|du\, dv\, dw$$

$$= 6\int_0^1 \int_0^2 \int_0^1 (u + v + w)du\, dv\, dw = 6\int_0^1 \int_0^2 \left[\frac{u^2}{2} + vu + wu\right]_0^1 dv\, dw$$

$$= 6\int_0^1 \int_0^2 \left(\frac{1}{2} + v + w\right)dv\, dw = 6\int_0^1 \left[\frac{1}{2}v + \frac{v^2}{2} + wv\right]_0^2 dw$$

$$= 6\int_0^1 (3 + 2w)dw = 6\left[3w + w^2\right]_0^1 = 24.$$

5.48 Let D be the region in xyz-space defined by $1 \leq x \leq 2,\ 0 \leq xy \leq 2,$ and $0 \leq z \leq 1.$

Evaluate $\iiint_D \left(x^2 y + 3xyz\right) dx\, dy\, dz$ by using the transformation $u = x,\ v = xy,$ and $w = 3z.$

5.7 EXERCISES

In the following exercises, the function $T : S \to R,\ T(u, v) = (x, y)$ on the region $S = \{(u, v) | 0 \le u \le 1,\ 0 \le v \le 1\}$ bounded by the unit square is given, where $R \subset \mathbb{R}^2$ is the image of S under T.

a. Justify that the function T is a C^1 transformation.
b. Find the images of the vertices of the unit square S through the function T.
c. Determine the image R of the unit square S and graph it.

356. $x = 2u,\ y = 3v$

357. $x = \frac{u}{2},\ y = \frac{v}{3}$

358. $x = u - v,\ y = u + v$

359. $x = 2u - v,\ y = u + 2v$

360. $x = u^2,\ y = v^2$

361. $x = u^3,\ y = v^3$

In the following exercises, determine whether the transformations $T : S \to R$ are one-to-one or not.

362. $x = u^2,\ y = v^2$, where S is the rectangle of vertices $(-1, 0), (1, 0), (1, 1)$, and $(-1, 1)$.

363. $x = u^4,\ y = u^2 + v$, where S is the triangle of vertices $(-2, 0), (2, 0)$, and $(0, 2)$.

364. $x = 2u,\ y = 3v$, where S is the square of vertices $(-1, 1), (-1, -1), (1, -1)$, and $(1, 1)$.

365. $T(u, v) = (2u - v, u)$, where S is the triangle of vertices $(-1, 1), (-1, -1)$, and $(1, -1)$.

366. $x = u + v + w,\ y = u + v,\ z = w$, where $S = R = \mathbb{R}^3$.

367. $x = u^2 + v + w,\ y = u^2 + v,\ z = w$, where $S = R = \mathbb{R}^3$.

In the following exercises, the transformations $T : S \to R$ are one-to-one. Find their related inverse transformations $T^{-1} : R \to S$.

368. $x = 4u,\ y = 5v$, where $S = R = \mathbb{R}^2$.

369. $x = u + 2v,\ y = -u + v$, where $S = R = \mathbb{R}^2$.

370. $x = e^{2u + v},\ y = e^{u - v}$, where $S = \mathbb{R}^2$ and $R = \{(x, y) | x > 0,\ y > 0\}$

371. $x = \ln u,\ y = \ln(uv)$, where $S = \{(u, v) | u > 0,\ v > 0\}$ and $R = \mathbb{R}^2$.

372. $x = u + v + w,\ y = 3v,\ z = 2w$, where $S = R = \mathbb{R}^3$.

373. $x = u + v,\ y = v + w,\ z = u + w$, where $S = R = \mathbb{R}^3$.

In the following exercises, the transformation $T : S \to R,\ T(u, v) = (x, y)$ and the region $R \subset \mathbb{R}^2$ are given. Find the region $S \subset \mathbb{R}^2$.

374. $x = au,\ y = bv,\ R = \left\{(x, y) | x^2 + y^2 \le a^2 b^2\right\}$, where $a, b > 0$

375. $x = au,\ y = bv,\ R = \left\{(x, y) | \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}$, where $a, b > 0$

376. $x = \frac{u}{a},\ y = \frac{v}{b},\ z = \frac{w}{c}$, $R = \left\{(x, y) | x^2 + y^2 + z^2 \le 1\right\}$, where $a, b, c > 0$

377. $x = au,\ y = bv,\ z = cw,\ R = \left\{(x, y) | \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} \le 1,\ z > 0\right\}$, where $a, b, c > 0$

In the following exercises, find the Jacobian J of the transformation.

378. $x = u + 2v,\ y = -u + v$

379. $x = \frac{u^3}{2},\ y = \frac{v}{u^2}$

380. $x = e^{2u - v},\ y = e^{u + v}$

381. $x = ue^v,\ y = e^{-v}$

382. $x = u\cos(e^v),\ y = u\sin(e^v)$

383. $x = v\sin\left(u^2\right),\ y = v\cos\left(u^2\right)$

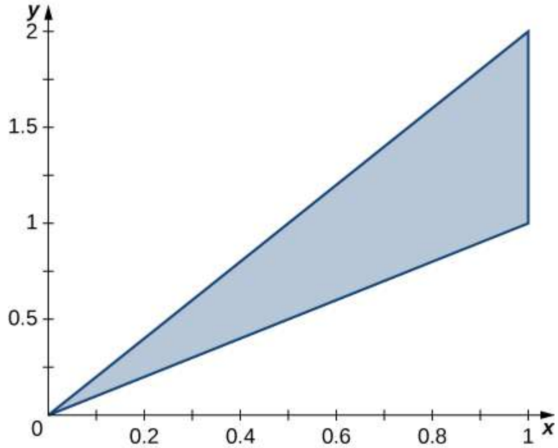
384. $x = u\cosh v,\ y = u\sinh v,\ z = w$

385. $x = v\cosh\left(\frac{1}{u}\right),\ y = v\sinh\left(\frac{1}{u}\right),\ z = u + w^2$

386. $x = u + v,\ y = v + w,\ z = u$

387. $x = u - v,\ y = u + v,\ z = u + v + w$

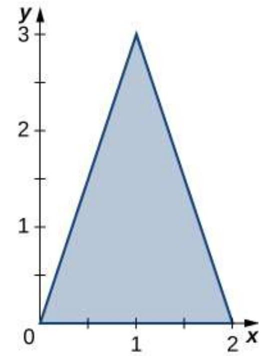
388. The triangular region R with the vertices $(0,\ 0),\ (1,\ 1)$, and $(1,\ 2)$ is shown in the following figure.



a. Find a transformation $T : S \to R$, $T(u,\ v) = (x,\ y) = (au + bv,\ cu + dv)$, where $a,\ b,\ c$, and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,\ 0) = (0,\ 0)$, $T^{-1}(1,\ 1) = (1,\ 0)$, and $T^{-1}(1,\ 2) = (0,\ 1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

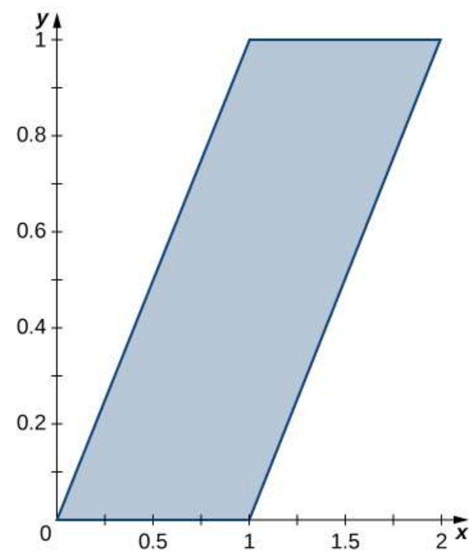
389. The triangular region R with the vertices $(0,\ 0),\ (2,\ 0)$, and $(1,\ 3)$ is shown in the following figure.



a. Find a transformation $T : S \to R$, $T(u,\ v) = (x,\ y) = (au + bv,\ cu + dv)$, where $a,\ b,\ c$ and d are real numbers with $ad - bc \neq 0$ such that $T^{-1}(0,\ 0) = (0,\ 0)$, $T^{-1}(2,\ 0) = (1,\ 0)$, and $T^{-1}(1,\ 3) = (0,\ 1)$.

b. Use the transformation T to find the area $A(R)$ of the region R.

In the following exercises, use the transformation $u = y - x,\ v = y$, to evaluate the integrals on the parallelogram R of vertices $(0,\ 0),\ (1,\ 0),\ (2,\ 1)$, and $(1,\ 1)$ shown in the following figure.

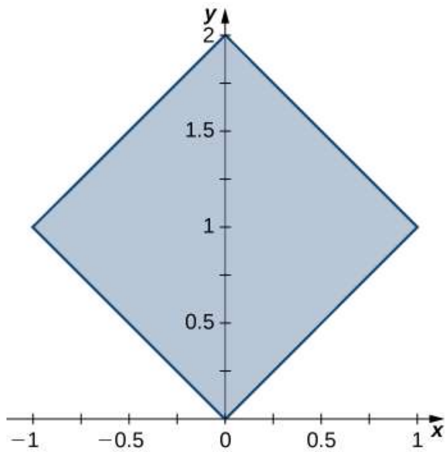


390. $\iint\limits_R (y - x)dA$

391. $\iint\limits_R \left(y^2 - xy\right)dA$

In the following exercises, use the transformation

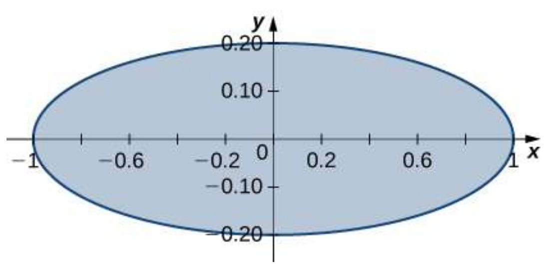
$y - x = u,\ x + y = v$ to evaluate the integrals on the square R determined by the lines $y = x,\ y = -x + 2,\ y = x + 2,$ and $y = -x$ shown in the following figure.



392. $\iint_R e^{x+y}\, dA$

393. $\iint_R \sin(x - y) dA$

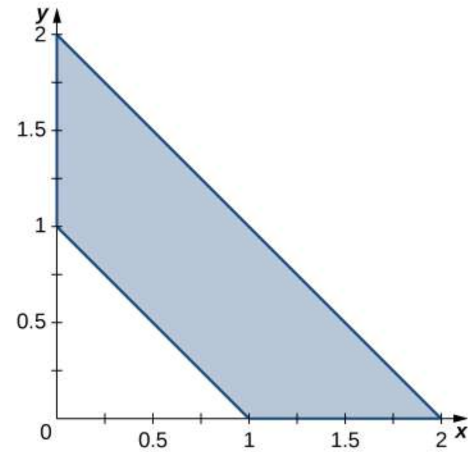
In the following exercises, use the transformation $x = u,\ 5y = v$ to evaluate the integrals on the region R bounded by the ellipse $x^2 + 25y^2 = 1$ shown in the following figure.



394. $\iint_R \sqrt{x^2 + 25y^2}\, dA$

395. $\iint_R \left(x^2 + 25y^2\right)^2 dA$

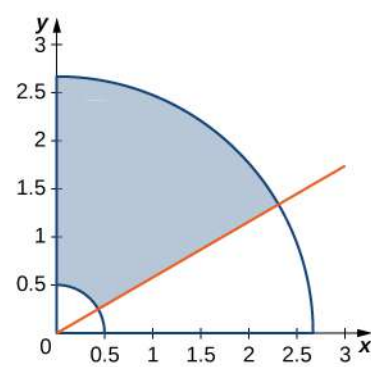
In the following exercises, use the transformation $u = x + y,\ v = x - y$ to evaluate the integrals on the trapezoidal region R determined by the points $(1, 0), (2, 0), (0, 2),$ and $(0, 1)$ shown in the following figure.



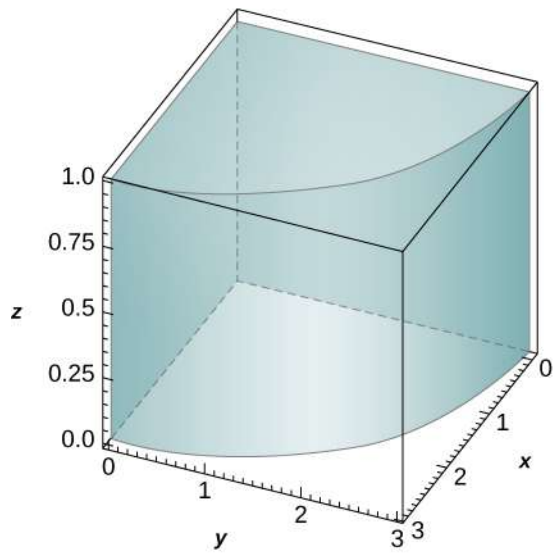
396. $\iint_R \left(x^2 - 2xy + y^2\right)e^{x+y}\, dA$

397. $\iint_R \left(x^3 + 3x^2 y + 3xy^2 + y^3\right) dA$

398. The circular annulus sector R bounded by the circles $4x^2 + 4y^2 = 1$ and $9x^2 + 9y^2 = 64,$ the line $x = y\sqrt{3},$ and the y-axis is shown in the following figure. Find a transformation T from a rectangular region S in the $r\theta$-plane to the region R in the xy-plane. Graph S.



399. The solid R bounded by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0, z = 1,$ $x = 0,$ and $y = 0$ is shown in the following figure. Find a transformation T from a cylindrical box S in $r\theta z$-space to the solid R in xyz-space.



400. Show that $\iint_R f\left(\sqrt{\frac{x^2}{3} + \frac{y^2}{3}}\right) dA = 2\pi\sqrt{15}\int_0^1 f(\rho)\rho\, d\rho,$ where f is a continuous function on $[0, 1]$ and R is the region bounded by the ellipse $5x^2 + 3y^2 = 15.$

401. Show that $\iiint_R f\left(\sqrt{16x^2 + 4y^2 + z^2}\right) dV = \frac{\pi}{2}\int_0^1 f(\rho)\rho^2\, d\rho,$ where f is a continuous function on $[0, 1]$ and R is the region bounded by the ellipsoid $16x^2 + 4y^2 + z^2 = 1.$

402. **[T]** Find the area of the region bounded by the curves $xy = 1,\ xy = 3,\ y = 2x,$ and $y = 3x$ by using the transformation $u = xy$ and $v = \frac{y}{x}.$ Use a computer algebra system (CAS) to graph the boundary curves of the region $R.$

403. **[T]** Find the area of the region bounded by the curves $x^2 y = 2,\ x^2 y = 3,\ y = x,$ and $y = 2x$ by using the transformation $u = x^2 y$ and $v = \frac{y}{x}.$ Use a CAS to graph the boundary curves of the region $R.$

404. Evaluate the triple integral $\int_0^1 \int_1^2 \int_z^{z+1} (y + 1)dx\, dy\, dz$ by using the transformation $u = x - z,$ $v = 3y,$ and $w = \frac{z}{2}.$

405. Evaluate the triple integral $\int_0^2 \int_4^6 \int_{3z}^{3z+2} (5 - 4y)dx\, dz\, dy$ by using the transformation $u = x - 3z,\ v = 4y,$ and $w = z.$

406. A transformation $T : \mathrm{R}^2 \to \mathrm{R}^2,\ T(u, v) = (x, y)$ of the form $x = au + bv,\ y = cu + dv,$ where $a, b, c,$ and d are real numbers, is called linear. Show that a linear transformation for which $ad - bc \neq 0$ maps parallelograms to parallelograms.

407. The transformation $T_\theta : \mathrm{R}^2 \to \mathrm{R}^2,\ T_\theta(u, v) = (x, y),$ where $x = u\cos\theta - v\sin\theta,$ $y = u\sin\theta + v\cos\theta,$ is called a rotation of angle $\theta.$ Show that the inverse transformation of T_θ satisfies $T_\theta{}^{-1} = T_{-\theta},$ where $T_{-\theta}$ is the rotation of angle $-\theta.$

408. **[T]** Find the region S in the uv-plane whose image through a rotation of angle $\frac{\pi}{4}$ is the region R enclosed by the ellipse $x^2 + 4y^2 = 1.$ Use a CAS to answer the following questions.

a. Graph the region $S.$
b. Evaluate the integral $\iint_S e^{-2uv}\, du\, dv.$ Round your answer to two decimal places.

409. **[T]** The transformations $T_i : \mathbb{R}^2 \to \mathbb{R}^2,$ $i = 1,\ldots, 4,$ defined by $T_1(u, v) = (u, -v),$ $T_2(u, v) = (-u, v),\ T_3(u, v) = (-u, -v),$ and $T_4(u, v) = (v, u)$ are called reflections about the x-axis, y-axis, origin, and the line $y = x,$ respectively.

a. Find the image of the region $S = \left\{(u, v) \mid u^2 + v^2 - 2u - 4v + 1 \leq 0\right\}$ in the xy-plane through the transformation $T_1 \circ T_2 \circ T_3 \circ T_4.$
b. Use a CAS to graph $R.$
c. Evaluate the integral $\iint_S \sin(u^2)du\, dv$ by using a CAS. Round your answer to two decimal places.

410. **[T]** The transformation $T_{k,\,1,\,1} : \mathbb{R}^3 \to \mathbb{R}^3,\ T_{k,\,1,\,1}(u, v, w) = (x, y, z)$ of the form $x = ku$, $y = v, z = w$, where $k \neq 1$ is a positive real number, is called a stretch if $k > 1$ and a compression if $0 < k < 1$ in the x-direction. Use a CAS to evaluate the integral $\iiint_S e^{-(4x^2 + 9y^2 + 25z^2)}dx\,dy\,dz$ on the solid $S = \{(x, y, z) | 4x^2 + 9y^2 + 25z^2 \leq 1\}$ by considering the compression $T_{2,\,3,\,5}(u, v, w) = (x, y, z)$ defined by $x = \frac{u}{2}, y = \frac{v}{3},$ and $z = \frac{w}{5}$. Round your answer to four decimal places.

411. **[T]** The transformation $T_{a,\,0} : \mathbb{R}^2 \to \mathbb{R}^2,\ T_{a,\,0}(u, v) = (u + av, v)$, where $a \neq 0$ is a real number, is called a shear in the x-direction. The transformation, $T_{b,\,0} : \mathrm{R}^2 \to \mathrm{R}^2,\ T_{o,\,b}(u, v) = (u, bu + v)$, where $b \neq 0$ is a real number, is called a shear in the y-direction.

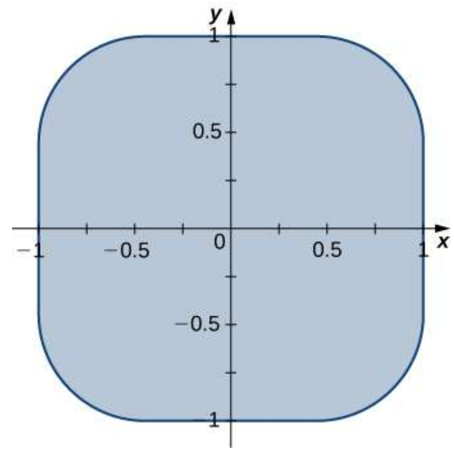
a. Find transformations $T_{0,\,2} \circ T_{3,\,0}$.
b. Find the image R of the trapezoidal region S bounded by $u = 0, v = 0, v = 1,$ and $v = 2 - u$ through the transformation $T_{0,\,2} \circ T_{3,\,0}$.
c. Use a CAS to graph the image R in the xy-plane.
d. Find the area of the region R by using the area of region S.

412. Use the transformation, $x = au, y = av, z = cw$ and spherical coordinates to show that the volume of a region bounded by the spheroid $\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi a^2 c}{3}$.

413. Find the volume of a football whose shape is a spheroid $\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$ whose length from tip to tip is 11 inches and circumference at the center is 22 inches. Round your answer to two decimal places.

414. **[T]** Lamé ovals (or superellipses) are plane curves of equations $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, where a, b, and n are positive real numbers.

a. Use a CAS to graph the regions R bounded by Lamé ovals for $a = 1, b = 2, n = 4$ and $n = 6$, respectively.
b. Find the transformations that map the region R bounded by the Lamé oval $x^4 + y^4 = 1$, also called a squircle and graphed in the following figure, into the unit disk.



c. Use a CAS to find an approximation of the area $A(R)$ of the region R bounded by $x^4 + y^4 = 1$. Round your answer to two decimal places.

415. **[T]** Lamé ovals have been consistently used by designers and architects. For instance, Gerald Robinson, a Canadian architect, has designed a parking garage in a shopping center in Peterborough, Ontario, in the shape of a superellipse of the equation $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$ with $\frac{a}{b} = \frac{9}{7}$ and $n = e$. Use a CAS to find an approximation of the area of the parking garage in the case $a = 900$ yards, $b = 700$ yards, and $n = 2.72$ yards.

CHAPTER 5 REVIEW

KEY TERMS

double integral of the function $f(x, y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum, $\iint_R f(x, y)dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$

double Riemann sum of the function $f(x, y)$ over a rectangular region R is $\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$ where R is divided into smaller subrectangles R_{ij} and (x_{ij}^*, y_{ij}^*) is an arbitrary point in R_{ij}

Fubini's theorem if $f(x, y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x, y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$, then the double integral of f over the region equals an iterated integral, $\iint_R f(x, y)dy\,dx = \int_a^b \int_c^d f(x, y)dx\,dy = \int_c^d \int_a^b f(x, y)dx\,dy$

improper double integral a double integral over an unbounded region or of an unbounded function

iterated integral for a function $f(x, y)$ over the region R is

a. $\int_a^b \int_c^d f(x, y)dx\,dy = \int_a^b \left[\int_c^d f(x, y)dy\right]dx,$

b. $\int_c^d \int_b^a f(x, y)dx\,dy = \int_c^d \left[\int_a^b f(x, y)dx\right]dy,$

where $a, b, c,$ and d are any real numbers and $R = [a, b] \times [c, d]$

Jacobian the Jacobian $J(u, v)$ in two variables is a 2×2 determinant:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix};$$

the Jacobian $J(u, v, w)$ in three variables is a 3×3 determinant:

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

one-to-one transformation a transformation $T : G \to R$ defined as $T(u, v) = (x, y)$ is said to be one-to-one if no two points map to the same image point

planar transformation a function T that transforms a region G in one plane into a region R in another plane by a change of variables

polar rectangle the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r, \theta) | a \le r \le b, \alpha \le \theta \le \beta\}$

radius of gyration the distance from an object's center of mass to its axis of rotation

transformation a function that transforms a region G in one plane into a region R in another plane by a change of variables

triple integral the triple integral of a continuous function $f(x, y, z)$ over a rectangular solid box B is the limit of a Riemann sum for a function of three variables, if this limit exists

triple integral in cylindrical coordinates the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r_{ijk}^*, \theta_{ijk}^*, z_{ijk}^*) r_{ijk}^* \Delta r \Delta\theta \Delta z$$

triple integral in spherical coordinates the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho_{ijk}^*, \theta_{ijk}^*, \varphi_{ijk}^*)(\rho_{ijk}^*)^2 \sin\varphi \Delta\rho \Delta\theta \Delta\varphi$$

Type I a region D in the xy-plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

Type II a region D in the xy-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$

KEY EQUATIONS

- **Double integral**

$$\iint_R f(x, y)dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

- **Iterated integral**

$$\int_a^b \int_c^d f(x, y)dx\,dy = \int_a^b \left[\int_c^d f(x, y)dy\right]dx$$

or

$$\int_c^d \int_b^a f(x, y)dx\,dy = \int_c^d \left[\int_a^b f(x, y)dx\right]dy$$

- **Average value of a function of two variables**

$$f_{\text{ave}} = \frac{1}{\text{Area } R} \iint_R f(x, y)dx\,dy$$

- **Iterated integral over a Type I region**

$$\iint_D f(x, y)dA = \iint_D f(x, y)dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)dy\right]dx$$

- **Iterated integral over a Type II region**

$$\iint_D f(x, y)dA = \iint_D f(x, y)dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)dx\right]dy$$

- **Double integral over a polar rectangular region R**

$$\iint_R f(r, \theta)dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(r_{ij}^*, \theta_{ij}^*\right)\Delta A = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(r_{ij}^*, \theta_{ij}^*\right)r_{ij}^* \Delta r \Delta\theta$$

- **Double integral over a general polar region**

$$\iint_D f(r, \theta) r \, dr \, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r, \theta) r \, dr \, d\theta$$

- **Triple integral**

$$\lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta x \Delta y \Delta z = \iiint_B f(x, y, z) dV$$

- **Triple integral in cylindrical coordinates**

$$\iiint_B g(x, y, z) dV = \iiint_B g(r\cos\theta, r\sin\theta, z) r \, dr \, d\theta \, dz = \iiint_B f(r, \theta, z) r \, dr \, d\theta \, dz$$

- **Triple integral in spherical coordinates**

$$\iiint_B f(\rho, \theta, \varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

- **Mass of a lamina**

$$m = \lim_{k, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} = \lim_{k, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_R \rho(x, y) dA$$

- **Moment about the *x*-axis**

$$M_x = \lim_{k, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^*\right) m_{ij} = \lim_{k, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^*\right) \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_R y\rho(x, y) dA$$

- **Moment about the *y*-axis**

$$M_y = \lim_{k, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(x_{ij}^*\right) m_{ij} = \lim_{k, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(x_{ij}^*\right) \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_R x\rho(x, y) dA$$

- **Center of mass of a lamina**

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x, y) dA}{\iint_R \rho(x, y) dA} \text{ and } \bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x, y) dA}{\iint_R \rho(x, y) dA}$$

KEY CONCEPTS

5.1 Double Integrals over Rectangular Regions

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini's theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

5.2 Double Integrals over General Regions

- A general bounded region D on the plane is a region that can be enclosed inside a rectangular region. We can use this idea to define a double integral over a general bounded region.
- To evaluate an iterated integral of a function over a general nonrectangular region, we sketch the region and express it as a Type I or as a Type II region or as a union of several Type I or Type II regions that overlap only on their boundaries.

- We can use double integrals to find volumes, areas, and average values of a function over general regions, similarly to calculations over rectangular regions.
- We can use Fubini's theorem for improper integrals to evaluate some types of improper integrals.

5.3 Double Integrals in Polar Coordinates

- To apply a double integral to a situation with circular symmetry, it is often convenient to use a double integral in polar coordinates. We can apply these double integrals over a polar rectangular region or a general polar region, using an iterated integral similar to those used with rectangular double integrals.
- The area dA in polar coordinates becomes $r\,dr\,d\theta$.
- Use $x = r\cos\theta,\quad y = r\sin\theta,\quad \text{and } dA = r\,dr\,d\theta$ to convert an integral in rectangular coordinates to an integral in polar coordinates.
- Use $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ to convert an integral in polar coordinates to an integral in rectangular coordinates, if needed.
- To find the volume in polar coordinates bounded above by a surface $z = f(r, \theta)$ over a region on the xy-plane, use a double integral in polar coordinates.

5.4 Triple Integrals

- To compute a triple integral we use Fubini's theorem, which states that if $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f],$ then

$$\iiint_B f(x, y, z)dV = \int_e^f \int_c^d \int_a^b f(x, y, z)dx\,dy\,dz$$

  and is also equal to any of the other five possible orderings for the iterated triple integral.
- To compute the volume of a general solid bounded region E we use the triple integral

$$V(E) = \iiint_E 1dV.$$

- Interchanging the order of the iterated integrals does not change the answer. As a matter of fact, interchanging the order of integration can help simplify the computation.
- To compute the average value of a function over a general three-dimensional region, we use

$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z)dV.$$

5.5 Triple Integrals in Cylindrical and Spherical Coordinates

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)} f(r, \theta, z)r\,dz\,dr\,d\theta.$$

- To evaluate a triple integral in spherical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)} \int_{\varphi=u_1(r,\theta)}^{\varphi=u_2(r,\theta)} f(\rho, \theta, \varphi)\rho^2 \sin\varphi\,d\varphi\,d\rho\,d\theta.$$

5.6 Calculating Centers of Mass and Moments of Inertia

Finding the mass, center of mass, moments, and moments of inertia in double integrals:

- For a lamina R with a density function $\rho(x, y)$ at any point (x, y) in the plane, the mass is $m = \iint_R \rho(x, y)dA$.
- The moments about the x-axis and y-axis are

$$M_x = \iint_R y\rho(x, y)dA \text{ and } M_y = \iint_R x\rho(x, y)dA.$$

- The center of mass is given by $\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m}$.
- The center of mass becomes the centroid of the plane when the density is constant.
- The moments of inertia about the $x-$ axis, $y-$ axis, and the origin are

$$I_x = \iint_R y^2\rho(x, y)dA, \quad I_y = \iint_R x^2\rho(x, y)dA, \text{ and } I_0 = I_x + I_y = \iint_R \left(x^2 + y^2\right)\rho(x, y)dA.$$

Finding the mass, center of mass, moments, and moments of inertia in triple integrals:

- For a solid object Q with a density function $\rho(x, y, z)$ at any point (x, y, z) in space, the mass is $m = \iiint_Q \rho(x, y, z)dV$.
- The moments about the xy-plane, the xz-plane, and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x, y, z)dV, \quad M_{xz} = \iiint_Q y\rho(x, y, z)dV, \quad M_{yz} = \iiint_Q x\rho(x, y, z)dV.$$

- The center of mass is given by $\bar{x} = \frac{M_{yz}}{m}, \bar{y} = \frac{M_{xz}}{m}, \bar{z} = \frac{M_{xy}}{m}$.
- The center of mass becomes the centroid of the solid when the density is constant.
- The moments of inertia about the yz-plane, the xz-plane, and the xy-plane are

$$I_x = \iiint_Q \left(y^2 + z^2\right)\rho(x, y, z)dV, \quad I_y = \iiint_Q \left(x^2 + z^2\right)\rho(x, y, z)dV,$$
$$I_z = \iiint_Q \left(x^2 + y^2\right)\rho(x, y, z)dV.$$

5.7 Change of Variables in Multiple Integrals

- A transformation T is a function that transforms a region G in one plane (space) into a region R in another plane (space) by a change of variables.
- A transformation $T : G \to R$ defined as $T(u, v) = (x, y)$ (or $T(u, v, w) = (x, y, z)$) is said to be a one-to-one transformation if no two points map to the same image point.
- If f is continuous on R, then $\iint_R f(x, y)dA = \iint_S f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right|du\,dv$.
- If F is continuous on R, then

$$\begin{aligned}\iiint_R F(x, y, z)dV &= \iiint_G F(g(u, v, w), h(u, v, w), k(u, v, w))\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right|du\,dv\,dw \\ &= \iiint_G H(u, v, w)|J(u, v, w)|du\,dv\,dw.\end{aligned}$$

CHAPTER 5 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample.

416. $\int_a^b \int_c^d f(x, y)dy\,dx = \int_c^d \int_a^b f(x, y)dy\,dx$

417. Fubini's theorem can be extended to three dimensions, as long as f is continuous in all variables.

418. The integral $\int_0^{2\pi}\int_0^1\int_r^1 dz\,dr\,d\theta$ represents the volume of a right cone.

419. The Jacobian of the transformation for $x = u^2 - 2v,\ y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v.$

Evaluate the following integrals.

420. $\iint_R \left(5x^3y^2 - y^2\right)dA,\ R = \{(x,\ y)|0 \le x \le 2,\ 1 \le y \le 4\}$

421. $\iint_D \frac{y}{3x^2+1}dA,\ D = \{(x,\ y)|0 \le x \le 1,\ -x \le y \le x\}$

422. $\iint_D \sin\left(x^2 + y^2\right)dA$ where D is a disk of radius 2 centered at the origin

423. $\int_0^1\int_y^1 xye^{x^2}\,dx\,dy$

424. $\int_{-1}^1\int_0^z\int_0^{x-z} 6dy\,dx\,dz$

425. $\iiint_R 3y\,dV,$ where $R = \left\{(x,\ y,\ z)|0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le \sqrt{9 - y^2}\right\}$

426. $\int_0^2\int_0^{2\pi}\int_r^1 r\,dz\,d\theta\,dr$

427. $\int_0^{2\pi}\int_0^{\pi/2}\int_1^3 \rho^2\sin(\varphi)d\rho\,d\varphi\,d\theta$

428. $\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$

For the following problems, find the specified area or volume.

429. The area of region enclosed by one petal of $r = \cos(4\theta).$

430. The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8.$

431. The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2.$

432. The volume of the intersection between two spheres of radius 1, the top whose center is $(0,\ 0,\ 0.25)$ and the bottom, which is centered at $(0,\ 0,\ 0).$

For the following problems, find the center of mass of the region.

433. $\rho(x,\ y) = xy$ on the circle with radius 1 in the first quadrant only.

434. $\rho(x,\ y) = (y + 1)\sqrt{x}$ in the region bounded by $y = e^x,\quad y = 0,$ and $x = 1.$

435. $\rho(x,\ y,\ z) = z$ on the inverted cone with radius 2 and height $2.$

436. The volume an ice cream cone that is given by the solid above $z = \sqrt{\left(x^2 + y^2\right)}$ and below $z^2 + x^2 + y^2 = z.$

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

437. If the compacted trash used to build Mount Holly on average has a density $400\,\text{lb/ft}^3,$ find the amount of work required to build the mountain.

438. In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density $400\,\text{lb/ft}^3$ and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth's layers.

439. [T] The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Layer	Depth from center (km)	Temperature $°C$
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Source: http://www.enchantedlearning.com/subjects/astronomy/planets/earth/Inside.shtml

440. [T] The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Layer	Depth from center (km)	Density (g/cm3)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

Source: http://hyperphysics.phy-astr.gsu.edu/hbase/geophys/earthstruct.html

The following problems concern the Theorem of Pappus (see **Moments and Centers of Mass (http://cnx.org/content/m53649/latest/)** for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x = 2\pi A\,\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\,\bar{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

441. Find the volume when you revolve the region around the x-axis.

442. Find the volume when you revolve the region around the y-axis.