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Calc-ulus

Volume 3

6 | VECTOR CALCULUS

Figure 6.1 Hurricanes form from rotating winds driven by warm temperatures over the ocean. Meteorologists forecast the motion of hurricanes by studying the rotating vector fields of their wind velocity. Shown is Cyclone Catarina in the South Atlantic Ocean in 2004, as seen from the International Space Station. (credit: modification of work by NASA)

Chapter Outline

Introduction

Hurricanes are huge storms that can produce tremendous amounts of damage to life and property, especially when they reach land. Predicting where and when they will strike and how strong the winds will be is of great importance for preparing for protection or evacuation. Scientists rely on studies of rotational vector fields for their forecasts (see **Example 6.3**).

In this chapter, we learn to model new kinds of integrals over fields such as magnetic fields, gravitational fields, or velocity fields. We also learn how to calculate the work done on a charged particle traveling through a magnetic field, the work done on a particle with mass traveling through a gravitational field, and the volume per unit time of water flowing through a net dropped in a river.

All these applications are based on the concept of a vector field, which we explore in this chapter. Vector fields have many applications because they can be used to model real fields such as electromagnetic or gravitational fields. A deep understanding of physics or engineering is impossible without an understanding of vector fields. Furthermore, vector fields have mathematical properties that are worthy of study in their own right. In particular, vector fields can be used to develop several higher-dimensional versions of the Fundamental Theorem of Calculus.

6.1 | Vector Fields

Learning Objectives

- **6.1.1** Recognize a vector field in a plane or in space.
- **6.1.2** Sketch a vector field from a given equation.
- **6.1.3** Identify a conservative field and its associated potential function.

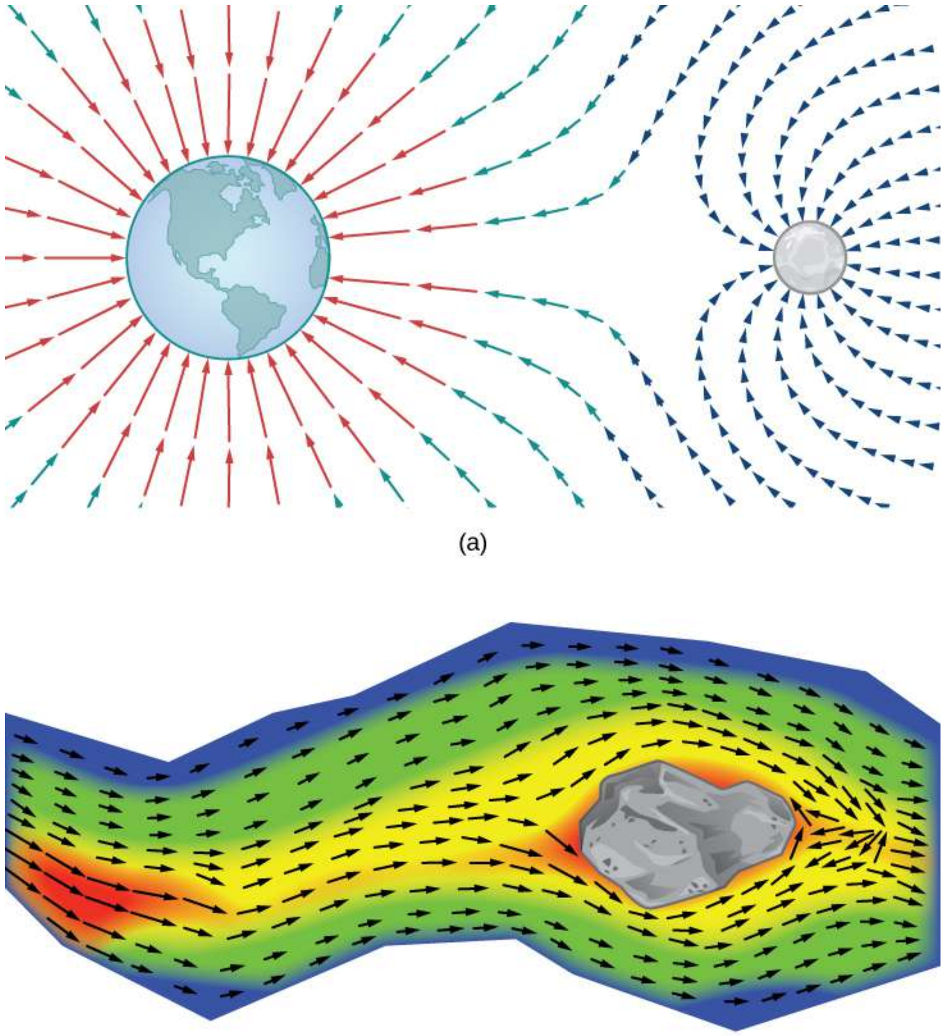
Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism, which affect the behavior of objects over a large region of a plane or of space. They are also useful for dealing with large-scale behavior such as atmospheric storms or deep-sea ocean currents. In this section, we examine the basic definitions and graphs of vector fields so we can study them in more detail in the rest of this chapter.

Examples of Vector Fields

How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river? **Figure 6.2** gives visual representations of such phenomena.

Figure 6.2(a) shows a gravitational field exerted by two astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

Figure 6.2(b) shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.



(b)

Figure 6.2 (a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Each figure illustrates an example of a vector field. Intuitively, a vector field is a map of vectors. In this section, we study vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition

A **vector field** $\mathbf{F}$ in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\mathbf{F}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.

A vector field $\mathbf{F}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\mathbf{F}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.

Vector Fields in $\mathbb{R}^2$

A vector field in $\mathbb{R}^2$ can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle. \tag{6.1}$$

The second way is to use the standard unit vectors:

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}. \tag{6.2}$$

A vector field is said to be *continuous* if its component functions are continuous.

Example 6.1

Finding a Vector Associated with a Given Point

Let $\mathbf{F}(x, y) = (2y^2 + x - 4)\mathbf{i} + \cos(x)\mathbf{j}$ be a vector field in $\mathbb{R}^2$. Note that this is an example of a continuous vector field since both component functions are continuous. What vector is associated with point $(0, -1)$?

Solution

Substitute the point values for *x* and *y*:

$$\begin{aligned} \mathbf{F}(0, 1) &= (2(-1)^2 + 0 - 4)\mathbf{i} + \cos(0)\mathbf{j} \\ &= -2\mathbf{i} + \mathbf{j}. \end{aligned}$$

6.1 Let $\mathbf{G}(x, y) = x^2 y\mathbf{i} - (x + y)\mathbf{j}$ be a vector field in $\mathbb{R}^2$. What vector is associated with the point $(-2, 3)$?

Drawing a Vector Field

We can now represent a vector field in terms of its components of functions or unit vectors, but representing it visually by sketching it is more complex because the domain of a vector field is in $\mathbb{R}^2$, as is the range. Therefore the “graph” of a vector field in $\mathbb{R}^2$ lives in four-dimensional space. Since we cannot represent four-dimensional space visually, we instead draw vector fields in $\mathbb{R}^2$ in a plane itself. To do this, draw the vector associated with a given point at the point in a plane. For example, suppose the vector associated with point $(4, -1)$ is $\langle 3, 1 \rangle$. Then, we would draw vector $\langle 3, 1 \rangle$ at point $(4, -1)$.

We should plot enough vectors to see the general shape, but not so many that the sketch becomes a jumbled mess. If we were to plot the image vector at each point in the region, it would fill the region completely and is useless. Instead, we can choose points at the intersections of grid lines and plot a sample of several vectors from each quadrant of a rectangular coordinate system in $\mathbb{R}^2$.

There are two types of vector fields in $\mathbb{R}^2$ on which this chapter focuses: radial fields and rotational fields. Radial fields model certain gravitational fields and energy source fields, and rotational fields model the movement of a fluid in a vortex. In a **radial field**, all vectors either point directly toward or directly away from the origin. Furthermore, the magnitude of any vector depends only on its distance from the origin. In a radial field, the vector located at point (x, y) is perpendicular to the circle centered at the origin that contains point (x, y), and all other vectors on this circle have the same magnitude.

Example 6.2

Drawing a Radial Vector Field

Sketch the vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$.

Solution

To sketch this vector field, choose a sample of points from each quadrant and compute the corresponding vector. The following table gives a representative sample of points in a plane and the corresponding vectors.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle \frac{1}{2}, 0 \rangle$	$(2, 0)$	$\langle 1, 0 \rangle$	$(1, 1)$	$\langle \frac{1}{2}, \frac{1}{2} \rangle$
$(0, 1)$	$\langle 0, \frac{1}{2} \rangle$	$(0, 2)$	$\langle 0, 1 \rangle$	$(-1, 1)$	$\langle -\frac{1}{2}, \frac{1}{2} \rangle$
$(-1, 0)$	$\langle -\frac{1}{2}, 0 \rangle$	$(-2, 0)$	$\langle -1, 0 \rangle$	$(-1, -1)$	$\langle -\frac{1}{2}, -\frac{1}{2} \rangle$
$(0, -1)$	$\langle 0, -\frac{1}{2} \rangle$	$(0, -2)$	$\langle 0, -1 \rangle$	$(1, -1)$	$\langle \frac{1}{2}, -\frac{1}{2} \rangle$

Figure 6.3(a) shows the vector field. To see that each vector is perpendicular to the corresponding circle, **Figure 6.3**(b) shows circles overlain on the vector field.

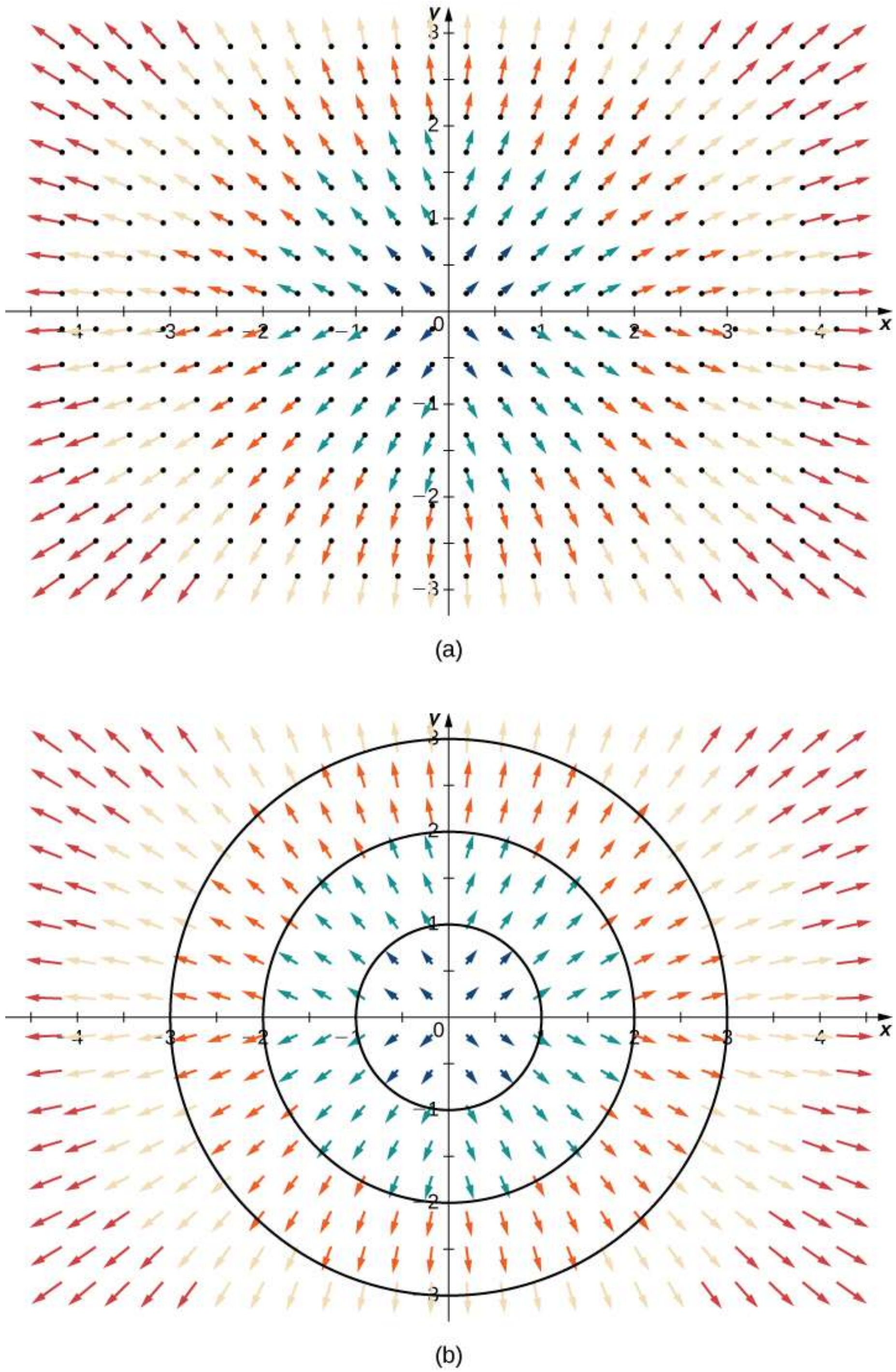


Figure 6.3 (a) A visual representation of the radial vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$. (b) The radial vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$ with overlaid circles. Notice that each vector is perpendicular to the circle on which it is located.

6.2 Draw the radial field $\mathbf{F}(x, y) = -\frac{x}{3}\mathbf{i} - \frac{y}{3}\mathbf{j}$.

In contrast to radial fields, in a **rotational field**, the vector at point (x, y) is tangent (not perpendicular) to a circle with radius $r = \sqrt{x^2 + y^2}$. In a standard rotational field, all vectors point either in a clockwise direction or in a counterclockwise direction, and the magnitude of a vector depends only on its distance from the origin. Both of the following examples are clockwise rotational fields, and we see from their visual representations that the vectors appear to rotate around the origin.

Example 6.3

Chapter Opener: Drawing a Rotational Vector Field

Figure 6.4 (credit: modification of work by NASA)

Sketch the vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$.

Solution

Create a table (see the one that follows) using a representative sample of points in a plane and their corresponding vectors. **Figure 6.6** shows the resulting vector field.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, -1 \rangle$	$(2, 0)$	$\langle 0, -2 \rangle$	$(1, 1)$	$\langle 1, -1 \rangle$
$(0, 1)$	$\langle 1, 0 \rangle$	$(0, 2)$	$\langle 2, 0 \rangle$	$(-1, 1)$	$\langle 1, 1 \rangle$
$(-1, 0)$	$\langle 0, 1 \rangle$	$(-2, 0)$	$\langle 0, 2 \rangle$	$(-1, -1)$	$\langle -1, 1 \rangle$
$(0, -1)$	$\langle -1, 0 \rangle$	$(0, -2)$	$\langle -2, 0 \rangle$	$(1, -1)$	$\langle -1, -1 \rangle$

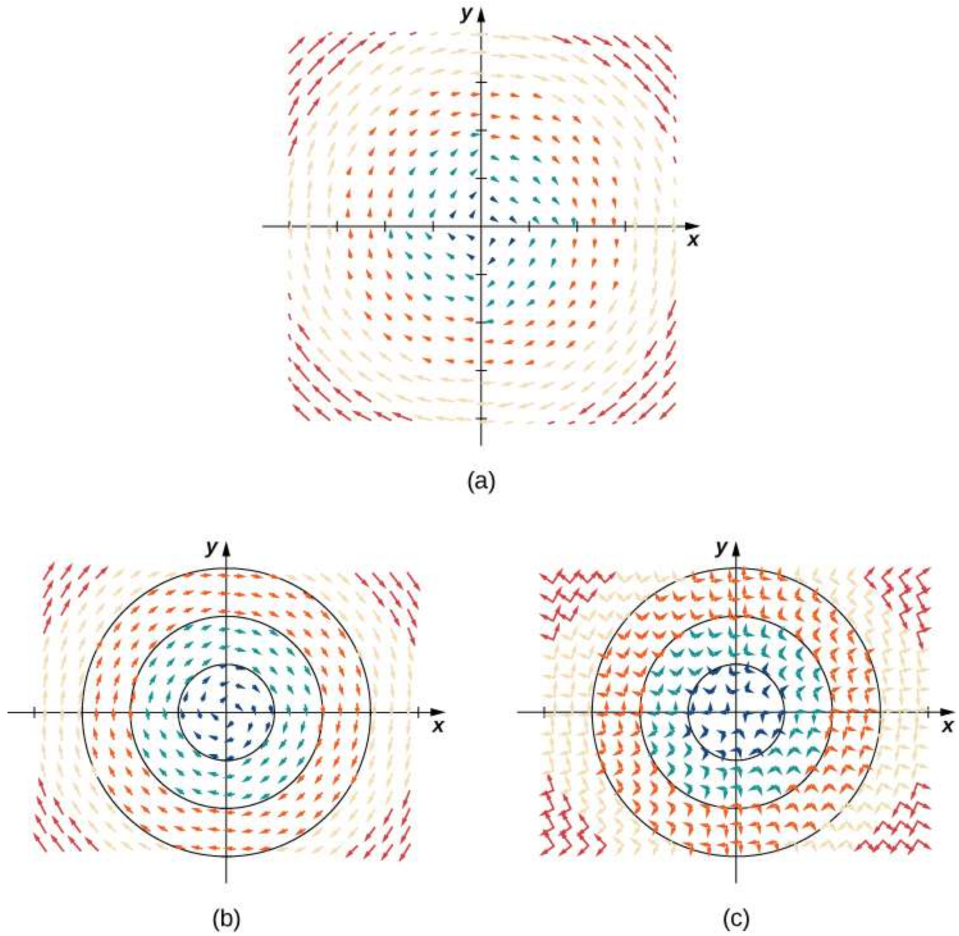


Figure 6.5 (a) A visual representation of vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$. (b) Vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$ with circles centered at the origin. (c) Vector $\mathbf{F}(a, b)$ is perpendicular to radial vector $\langle a, b \rangle$ at point (a, b).

Analysis

Note that vector $\mathbf{F}(a, b) = \langle b, -a \rangle$ points clockwise and is perpendicular to radial vector $\langle a, b \rangle$. (We can verify this assertion by computing the dot product of the two vectors: $\langle a, b \rangle \cdot \langle -b, a \rangle = -ab + ab = 0$.) Furthermore, vector $\langle b, -a \rangle$ has length $r = \sqrt{a^2 + b^2}$. Thus, we have a complete description of this rotational vector field: the vector associated with point (a, b) is the vector with length r tangent to the circle with radius r, and it points in the clockwise direction.

Sketches such as that in **Figure 6.6** are often used to analyze major storm systems, including hurricanes and cyclones. In the northern hemisphere, storms rotate counterclockwise; in the southern hemisphere, storms rotate clockwise. (This is an effect caused by Earth's rotation about its axis and is called the Coriolis Effect.)

Example 6.4

Sketching a Vector Field

Sketch vector field $\mathbf{F}(x,\ y) = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}$.

Solution

To visualize this vector field, first note that the dot product $\mathbf{F}(a,\ b) \cdot (a\mathbf{i} + b\mathbf{j})$ is zero for any point $(a,\ b)$. Therefore, each vector is tangent to the circle on which it is located. Also, as $(a,\ b) \to (0,\ 0)$, the magnitude of $\mathbf{F}(a,\ b)$ goes to infinity. To see this, note that

$$\|\mathbf{F}(a,\ b)\| = \sqrt{\frac{a^2 + b^2}{\left(a^2 + b^2\right)^2}} = \sqrt{\frac{1}{a^2 + b^2}}.$$

Since $\frac{1}{a^2 + b^2} \to \infty$ as $(a,\ b) \to (0,\ 0)$, then $\|\mathbf{F}(a,\ b)\| \to \infty$ as $(a,\ b) \to (0,\ 0)$. This vector field looks similar to the vector field in **Example 6.3**, but in this case the magnitudes of the vectors close to the origin are large. **Table 6.3** shows a sample of points and the corresponding vectors, and **Figure 6.6** shows the vector field. Note that this vector field models the whirlpool motion of the river in **Figure 6.2**(b). The domain of this vector field is all of $\mathbb{R}^2$ except for point $(0,\ 0)$.

$(x,\ y)$	$\mathbf{F}(x,\ y)$	$(x,\ y)$	$\mathbf{F}(x,\ y)$	$(x,\ y)$	$\mathbf{F}(x,\ y)$
$(1,\ 0)$	$\langle 0,\ -1 \rangle$	$(2,\ 0)$	$\langle 0,\ -\frac{1}{2} \rangle$	$(1,\ 1)$	$\langle \frac{1}{2},\ -\frac{1}{2} \rangle$
$(0,\ 1)$	$\langle 1,\ 0 \rangle$	$(0,\ 2)$	$\langle \frac{1}{2},\ 0 \rangle$	$(-1,\ 1)$	$\langle \frac{1}{2},\ \frac{1}{2} \rangle$
$(-1,\ 0)$	$\langle 0,\ 1 \rangle$	$(-2,\ 0)$	$\langle 0,\ \frac{1}{2} \rangle$	$(-1,\ -1)$	$\langle -\frac{1}{2},\ \frac{1}{2} \rangle$
$(0,\ -1)$	$\langle -1,\ 0 \rangle$	$(0,\ -2)$	$\langle -\frac{1}{2},\ 0 \rangle$	$(1,\ -1)$	$\langle -\frac{1}{2},\ -\frac{1}{2} \rangle$

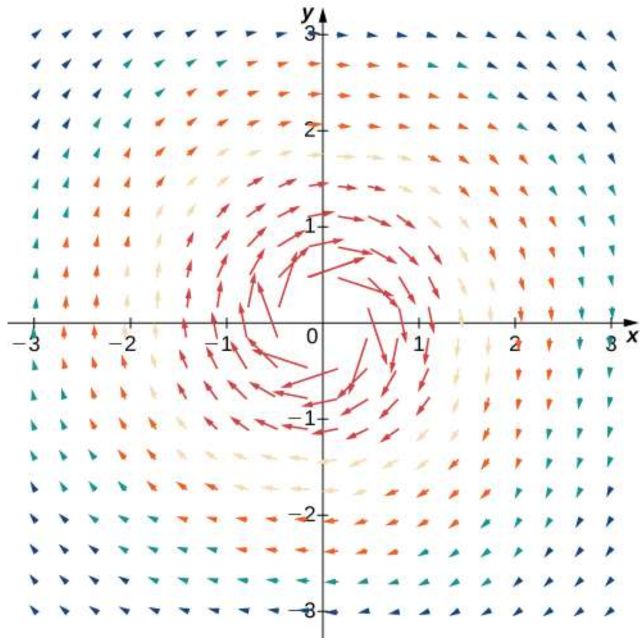


Figure 6.6 A visual representation of vector field $\mathbf{F}(x, y) = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}$. This vector field could be used to model whirlpool motion of a fluid.

6.3 Sketch vector field $\mathbf{F}(x, y) = \langle -2y, 2x \rangle$. Is the vector field radial, rotational, or neither?

Example 6.5

Velocity Field of a Fluid

Suppose that $\mathbf{v}(x, y) = -\frac{2y}{x^2 + y^2}\mathbf{i} + \frac{2x}{x^2 + y^2}\mathbf{j}$ is the velocity field of a fluid. How fast is the fluid moving at point $(1, -1)$? (Assume the units of speed are meters per second.)

Solution

To find the velocity of the fluid at point $(1, -1)$, substitute the point into $\mathbf{v}$:

$$\mathbf{v}(1, -1) = -\frac{2(-1)}{1+1}\mathbf{i} + \frac{2(1)}{1+1}\mathbf{j} = \mathbf{i} + \mathbf{j}.$$

The speed of the fluid at $(1, -1)$ is the magnitude of this vector. Therefore, the speed is $\|\mathbf{i} + \mathbf{j}\| = \sqrt{2}$ m/sec.

6.4 Vector field $v(x, y) = \langle 4|x|, 1 \rangle$ models the velocity of water on the surface of a river. What is the speed of the water at point $(2, 3)$? Use meters per second as the units.

We have examined vector fields that contain vectors of various magnitudes, but just as we have unit vectors, we can also have a unit vector field. A vector field **F** is a **unit vector field** if the magnitude of each vector in the field is 1. In a unit vector field, the only relevant information is the direction of each vector.

Example 6.6

A Unit Vector Field

Show that vector field $\mathbf{F}(x, y) = \left\langle \frac{y}{\sqrt{x^2 + y^2}}, -\frac{x}{\sqrt{x^2 + y^2}} \right\rangle$ is a unit vector field.

Solution

To show that **F** is a unit field, we must show that the magnitude of each vector is 1. Note that

$$\begin{aligned}\sqrt{\left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2 + \left(-\frac{x}{\sqrt{x^2 + y^2}}\right)^2} &= \sqrt{\frac{y^2}{x^2 + y^2} + \frac{x^2}{x^2 + y^2}} \\ &= \sqrt{\frac{x^2 + y^2}{x^2 + y^2}} \\ &= 1.\end{aligned}$$

Therefore, **F** is a unit vector field.

6.5 Is vector field $\mathbf{F}(x, y) = \langle -y, x \rangle$ a unit vector field?

Why are unit vector fields important? Suppose we are studying the flow of a fluid, and we care only about the direction in which the fluid is flowing at a given point. In this case, the speed of the fluid (which is the magnitude of the corresponding velocity vector) is irrelevant, because all we care about is the direction of each vector. Therefore, the unit vector field associated with velocity is the field we would study.

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field, then the corresponding unit vector field is $\left\langle \frac{P}{\|\mathbf{F}\|}, \frac{Q}{\|\mathbf{F}\|}, \frac{R}{\|\mathbf{F}\|} \right\rangle$. Notice that if $\mathbf{F}(x, y) = \langle y, -x \rangle$ is the vector field from **Example 6.3**, then the magnitude of **F** is $\sqrt{x^2 + y^2}$, and therefore the corresponding unit vector field is the field **G** from the previous example.

If **F** is a vector field, then the process of dividing **F** by its magnitude to form unit vector field $\mathbf{F}/\|\mathbf{F}\|$ is called *normalizing* the field **F**.

Vector Fields in $\mathbb{R}^3$

We have seen several examples of vector fields in $\mathbb{R}^2$; let's now turn our attention to vector fields in $\mathbb{R}^3$. These vector fields can be used to model gravitational or electromagnetic fields, and they can also be used to model fluid flow or heat flow in three dimensions. A two-dimensional vector field can really only model the movement of water on a two-dimensional slice of a river (such as the river's surface). Since a river flows through three spatial dimensions, to model the flow of the entire depth of the river, we need a vector field in three dimensions.

The extra dimension of a three-dimensional field can make vector fields in $\mathbb{R}^3$ more difficult to visualize, but the idea is the same. To visualize a vector field in $\mathbb{R}^3$, plot enough vectors to show the overall shape. We can use a similar method to visualizing a vector field in $\mathbb{R}^2$ by choosing points in each octant.

Just as with vector fields in $\mathbb{R}^2$, we can represent vector fields in $\mathbb{R}^3$ with component functions. We simply need an extra component function for the extra dimension. We write either

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle \tag{6.3}$$

or

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}. \tag{6.4}$$

Example 6.7

Sketching a Vector Field in Three Dimensions

Describe vector field $\mathbf{F}(x, y, z) = \langle 1, 1, z \rangle$.

Solution

For this vector field, the *x* and *y* components are constant, so every point in $\mathbb{R}^3$ has an associated vector with *x* and *y* components equal to one. To visualize **F**, we first consider what the field looks like in the *xy*-plane. In the *xy*-plane, $z = 0$. Hence, each point of the form $(a, b, 0)$ has vector $\langle 1, 1, 0 \rangle$ associated with it. For points not in the *xy*-plane but slightly above it, the associated vector has a small but positive *z* component, and therefore the associated vector points slightly upward. For points that are far above the *xy*-plane, the *z* component is large, so the vector is almost vertical. Figure 6.7 shows this vector field.

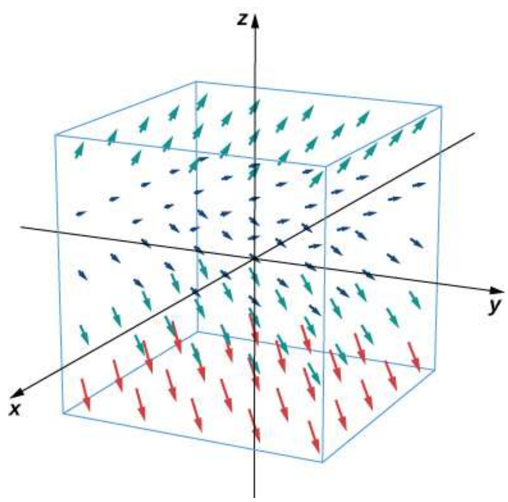


Figure 6.7 A visual representation of vector field $\mathbf{F}(x, y, z) = \langle 1, 1, z \rangle$.

6.6 Sketch vector field $\mathbf{G}(x, y, z) = \langle 2, \frac{z}{2}, 1 \rangle$.

In the next example, we explore one of the classic cases of a three-dimensional vector field: a gravitational field.

Example 6.8

Describing a Gravitational Vector Field

Newton's law of gravitation states that $\mathbf{F} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$, where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1) of mass m_1 located at the origin on another object (object 2) of mass m_2 located at point (x, y, z). Field $\mathbf{F}$ denotes the gravitational force that object 1 exerts on object 2, r is the distance between the two objects, and $\hat{\mathbf{r}}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. Sketch the vector field associated with this equation.

Solution

Since object 1 is located at the origin, the distance between the objects is given by $r = \sqrt{x^2 + y^2 + z^2}$. The unit vector from object 1 to object 2 is $\hat{\mathbf{r}} = \frac{\langle x, y, z \rangle}{\| \langle x, y, z \rangle \|}$, and hence $\hat{\mathbf{r}} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$. Therefore, gravitational vector field $\mathbf{F}$ exerted by object 1 on object 2 is

$$\mathbf{F} = -Gm_1 m_2 \langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle .$$

This is an example of a radial vector field in $\mathbb{R}^3$.

Figure 6.8 shows what this gravitational field looks like for a large mass at the origin. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.

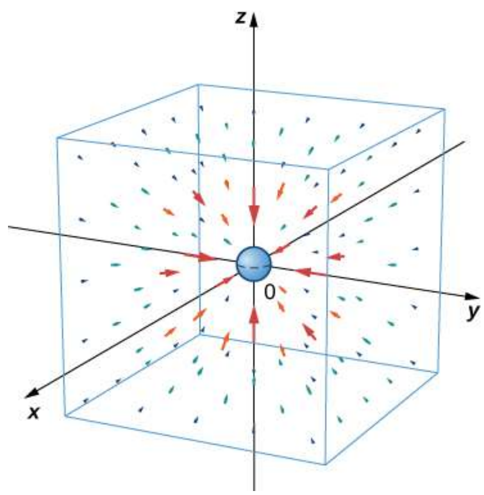


Figure 6.8 A visual representation of gravitational vector field $\mathbf{F} = -Gm_1 m_2 \langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle$ for a large mass at the origin.

6.7 The mass of asteroid 1 is 750,000 kg and the mass of asteroid 2 is 130,000 kg. Assume asteroid 1 is located at the origin, and asteroid 2 is located at $(15, -5, 10)$, measured in units of 10 to the eighth power kilometers. Given that the universal gravitational constant is $G = 6.67384 \times 10^{-11}\ \text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$, find the gravitational force vector that asteroid 1 exerts on asteroid 2.

Gradient Fields

In this section, we study a special kind of vector field called a gradient field or a **conservative field**. These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Recall that if f is a (scalar) function of x and y, then the gradient of f is

$$\text{grad } f = \nabla f = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

We can see from the form in which the gradient is written that ∇f is a vector field in $\mathbb{R}^2$. Similarly, if f is a function of x, y, and z, then the gradient of f is

$$\text{grad } f = \nabla f = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

The gradient of a three-variable function is a vector field in $\mathbb{R}^3$.

A gradient field is a vector field that can be written as the gradient of a function, and we have the following definition.

Definition

A vector field $\mathbf{F}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a **gradient field** if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

Example 6.9

Sketching a Gradient Vector Field

Use technology to plot the gradient vector field of $f(x, y) = x^2y^2$ $f(x, y) = x^2y^2$.

Solution

The gradient of f is $\nabla f = \langle 2xy^2, 2x^2y \rangle$ $\nabla f = \langle 2xy^2, 2x^2y \rangle$. To sketch the vector field, use a computer algebra system such as Mathematica. **Figure 6.9** shows ∇f.

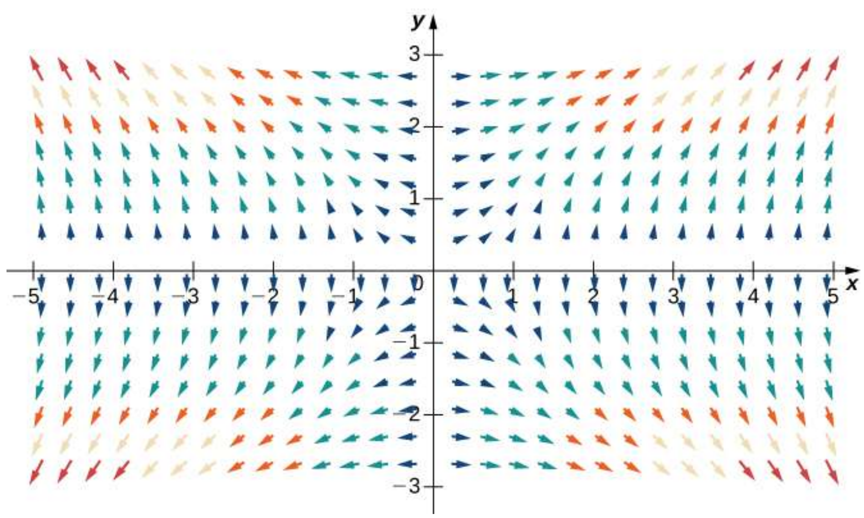


Figure 6.9 The gradient vector field is ∇f, where $f(x, y) = x^2y^2$.

6.8 Use technology to plot the gradient vector field of $f(x, y) = \sin x \cos y$.

Consider the function $f(x, y) = x^2 y^2$ from **Example 6.9**. **Figure 6.11** shows the level curves of this function overlaid on the function's gradient vector field. The gradient vectors are perpendicular to the level curves, and the magnitudes of the vectors get larger as the level curves get closer together, because closely grouped level curves indicate the graph is steep, and the magnitude of the gradient vector is the largest value of the directional derivative. Therefore, you can see the local steepness of a graph by investigating the corresponding function's gradient field.

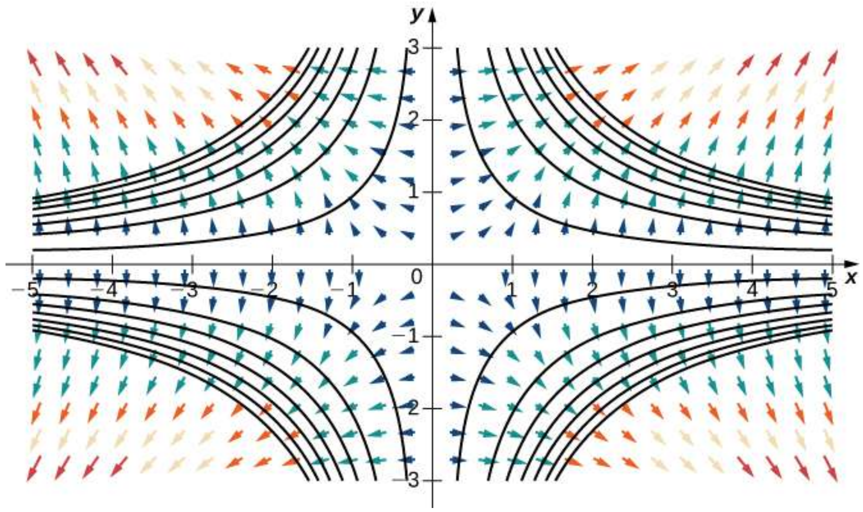


Figure 6.10 The gradient field of $f(x, y) = x^2 y^2$ and several level curves of f. Notice that as the level curves get closer together, the magnitude of the gradient vectors increases.

As we learned earlier, a vector field $\mathbf{F}$ is a conservative vector field, or a gradient field if there exists a scalar function f such that $\nabla f = \mathbf{F}$. In this situation, f is called a **potential function** for $\mathbf{F}$. Conservative vector fields arise in many applications, particularly in physics. The reason such fields are called *conservative* is that they model forces of physical systems in which energy is conserved. We study conservative vector fields in more detail later in this chapter.

You might notice that, in some applications, a potential function f for $\mathbf{F}$ is defined instead as a function such that $-\nabla f = \mathbf{F}$. This is the case for certain contexts in physics, for example.

Example 6.10

Verifying a Potential Function

Is $f(x, y, z) = x^2 yz - \sin(xy)$ a potential function for vector field

$$\mathbf{F}(x, y, z) = \langle 2xyz - y\cos(xy), x^2 z - x\cos(xy), x^2 y \rangle \text{ ?}$$

Solution

We need to confirm whether $\nabla f = \mathbf{F}$. We have

$$f_x = 2xyz - y\cos(xy),\ f_y = x^2 z - x\cos(xy),\ \text{and } f_z = x^2 y.$$

Therefore, $\nabla f = \mathbf{F}$ and f is a potential function for $\mathbf{F}$.

6.9 Is $f(x, y, z) = x^2 \cos(yz) + y^2 z^2$ a potential function for $\mathbf{F}(x, y, z) = \langle 2x \cos(yz), -x^2 z \sin(yz) + 2yz^2, y^2 \rangle$?

Example 6.11

Verifying a Potential Function

The velocity of a fluid is modeled by field $\mathbf{v}(x, y) = \langle xy, \frac{x^2}{2} - y \rangle$. Verify that $f(x, y) = \frac{x^2 y}{2} - \frac{y^2}{2}$ is a potential function for $\mathbf{v}$.

Solution

To show that f is a potential function, we must show that $\nabla f = \mathbf{v}$. Note that $f_x = xy$ and $f_x = \frac{x^2}{2} - y$. Therefore, $\nabla f = \langle xy, \frac{x^2}{2} - y \rangle$ and f is a potential function for v (**Figure 6.11**).

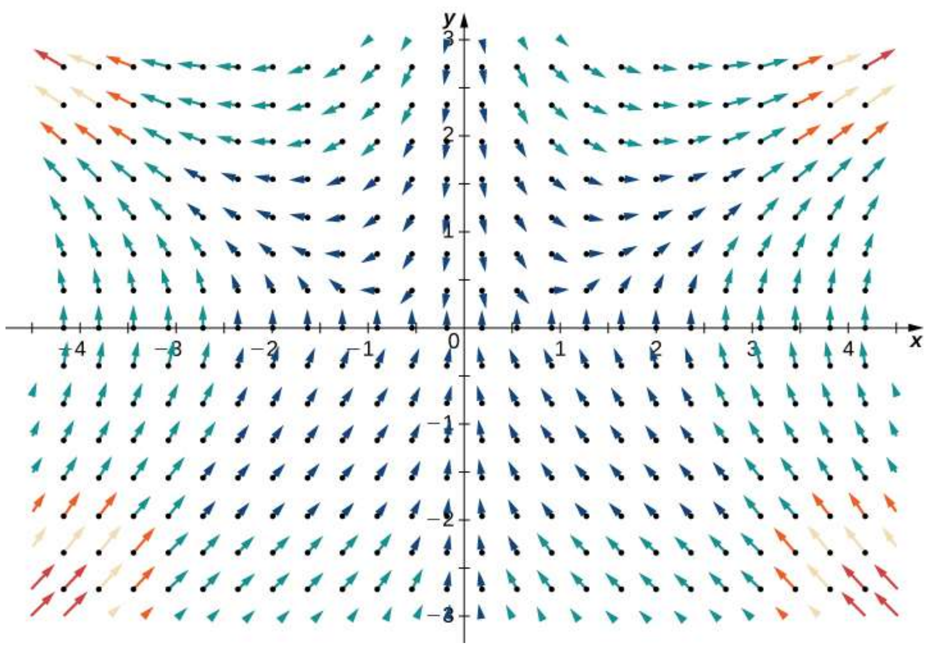


Figure 6.11 Velocity field $\mathbf{v}(x, y)$ has a potential function and is a conservative field.

6.10 Verify that $f(x, y) = x^2 y^2 + x$ is a potential function for velocity field $\mathbf{v}(x, y) = \langle 3x^2 y^2 + 1, 2x^3 y \rangle$.

If $\mathbf{F}$ is a conservative vector field, then there is at least one potential function f such that $\nabla f = \mathbf{F}$. But, could there be more than one potential function? If so, is there any relationship between two potential functions for the same vector field? Before answering these questions, let's recall some facts from single-variable calculus to guide our intuition. Recall that if $k(x)$ is an integrable function, then k has infinitely many antiderivatives. Furthermore, if F and G are both antiderivatives of k, then F and G differ only by a constant. That is, there is some number C such that $F(x) = G(x) + C$.

Now let $\mathbf{F}$ be a conservative vector field and let f and g be potential functions for $\mathbf{F}$. Since the gradient is like a derivative, $\mathbf{F}$ being conservative means that $\mathbf{F}$ is "integrable" with "antiderivatives" f and g. Therefore, if the analogy with single-variable calculus is valid, we expect there is some constant C such that $f(x) = g(x) + C$. The next theorem says that this is indeed the case.

To state the next theorem with precision, we need to assume the domain of the vector field is connected and open. To be connected means if P_1 and P_2 are any two points in the domain, then you can walk from P_1 to P_2 along a path that stays entirely inside the domain.

Theorem 6.1: Uniqueness of Potential Functions

Let $\mathbf{F}$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\nabla f = \mathbf{F}$ and $\nabla g = \mathbf{F}$. Then, there is a constant C such that $f = g + C$.

Proof

Since f and g are both potential functions for $\mathbf{F}$, then $\nabla f = (f - g) = \nabla f - \nabla g = \mathbf{F} - \mathbf{F} = 0$. Let $h = f - g$, then we have $\nabla h = 0$. We would like to show that h is a constant function.

Assume h is a function of x and y (the logic of this proof extends to any number of independent variables). Since $\nabla h = 0$, we have $h_x = 0$ and $h_y = 0$. The expression $h_x = 0$ implies that h is a constant function with respect to x—that is, $h(x, y) = k_1(y)$ for some function k_1. Similarly, $h_y = 0$ implies $h(x, y) = k_2(x)$ for some function k_2. Therefore, function h depends only on y and also depends only on x. Thus, $h(x, y) = C$ for some constant C on the connected domain of $\mathbf{F}$. Note that we really do need connectedness at this point; if the domain of $\mathbf{F}$ came in two separate pieces, then k could be a constant C_1 on one piece but could be a different constant C_2 on the other piece. Since $f - g = h = C$, we have that $f - g + C$, as desired.

□

Conservative vector fields also have a special property called the *cross-partial property*. This property helps test whether a given vector field is conservative.

Theorem 6.2: The Cross-Partial Property of Conservative Vector Fields

Let $\mathbf{F}$ be a vector field in two or three dimensions such that the component functions of $\mathbf{F}$ have continuous second-order mixed-partial derivatives on the domain of $\mathbf{F}$.

If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a conservative vector field in $\mathbb{R}^2$, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. If $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ is a conservative vector field in $\mathbb{R}^3$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

Proof

Since $\mathbf{F}$ is conservative, there is a function $f(x, y)$ such that $\nabla f = \mathbf{F}$. Therefore, by the definition of the gradient, $f_x = P$

and $f_y = Q$. By Clairaut's theorem, $f_{xy} = f_{yx}$, But, $f_{xy} = P_y$ and $f_{yx} = Q_x$, and thus $P_y = Q_x$.

□

Clairaut's theorem gives a fast proof of the cross-partial property of conservative vector fields in $\mathbb{R}^3$, just as it did for vector fields in $\mathbb{R}^2$.

The Cross-Partial Property of Conservative Vector Fields shows that most vector fields are not conservative. The cross-partial property is difficult to satisfy in general, so most vector fields won't have equal cross-partials.

Example 6.12

Showing a Vector Field Is Not Conservative

Show that rotational vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$ is not conservative.

Solution

Let $P(x, y) = y$ and $Q(x, y) = -x$. If $\mathbf{F}$ is conservative, then the cross-partials would be equal—that is, P_y would equal Q_x. Therefore, to show that $\mathbf{F}$ is not conservative, check that $P_y \neq Q_x$. Since $P_y = 1$ and $Q_x = -1$, the vector field is not conservative.

6.11 Show that vector field $\mathbf{F}(x, y)xy\mathbf{i} - x^2 y\mathbf{j}$ is not conservative.

Example 6.13

Showing a Vector Field Is Not Conservative

Is vector field $\mathbf{F}(x, y, z) = \langle 7, -2, x^3 \rangle$ conservative?

Solution

Let $P(x, y, z) = 7$, $Q(x, y, z) = -2$, and $R(x, y, z) = x^3$. If $\mathbf{F}$ is conservative, then all three cross-partial equations will be satisfied—that is, if $\mathbf{F}$ is conservative, then P_y would equal Q_x, Q_z would equal R_y, and R_x would equal P_z. Note that $P_y = Q_x = R_y = Q_z = 0$, so the first two necessary equalities hold. However, $R_x = 3x^3$ and $P_z = 0$ so $R_x \neq P_z$. Therefore, $\mathbf{F}$ is not conservative.

6.12 Is vector field $G(x, y, z) = \langle y, x, xyz \rangle$ conservative?

We conclude this section with a word of warning: **The Cross-Partial Property of Conservative Vector Fields** says that if $\mathbf{F}$ is conservative, then $\mathbf{F}$ has the cross-partial property. The theorem does *not* say that, if $\mathbf{F}$ has the cross-partial property, then $\mathbf{F}$ is conservative (the converse of an implication is not logically equivalent to the original implication). In other words, **The Cross-Partial Property of Conservative Vector Fields** can only help determine that a field is not conservative; it does not let you conclude that a vector field is conservative. For example, consider vector field

$\mathbf{F}(x, y) = \left\langle x^2 y, \frac{x^3}{3} \right\rangle$. This field has the cross-partial property, so it is natural to try to use **The Cross-Partial Property of Conservative Vector Fields** to conclude this vector field is conservative. However, this is a misapplication of the theorem. We learn later how to conclude that **F** is conservative.

6.1 EXERCISES

1. The domain of vector field $\mathbf{F} = \mathbf{F}(x, y)$ is a set of points (x, y) in a plane, and the range of $\mathbf{F}$ is a set of *what* in the plane?

For the following exercises, determine whether the statement is *true or false*.

2. Vector field $\mathbf{F} = \langle 3x^2, 1 \rangle$ is a gradient field for both $\phi_1(x, y) = x^3 + y$ and $\phi_2(x, y) = y + x^3 + 100$.

3. Vector field $\mathbf{F} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is constant in direction and magnitude on a unit circle.

4. Vector field $\mathbf{F} = \dfrac{\langle y, x \rangle}{\sqrt{x^2 + y^2}}$ is neither a radial field nor a rotation.

For the following exercises, describe each vector field by drawing some of its vectors.

5. **[T]** $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

6. **[T]** $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$

7. **[T]** $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$

8. **[T]** $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

9. **[T]** $\mathbf{F}(x, y) = 2x\mathbf{i} + 3y\mathbf{j}$

10. **[T]** $\mathbf{F}(x, y) = 3\mathbf{i} + x\mathbf{j}$

11. **[T]** $\mathbf{F}(x, y) = y\mathbf{i} + \sin x\mathbf{j}$

12. **[T]** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

13. **[T]** $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$

14. **[T]** $\mathbf{F}(x, y, z) = \frac{y}{z}\mathbf{i} - \frac{x}{z}\mathbf{j}$

For the following exercises, find the gradient vector field of each function f.

15. $f(x, y) = x\sin y + \cos y$

16. $f(x, y, z) = ze^{-xy}$

17. $f(x, y, z) = x^2y + xy + y^2z$

18. $f(x, y) = x^2\sin(5y)$

19. $f(x, y) = \ln\left(1 + x^2 + 2y^2\right)$

20. $f(x, y, z) = x\cos\left(\frac{y}{z}\right)$

21. What is vector field $\mathbf{F}(x, y)$ with a value at (x, y) that is of unit length and points toward $(1, 0)$?

For the following exercises, write formulas for the vector fields with the given properties.

22. All vectors are parallel to the x-axis and all vectors on a vertical line have the same magnitude.

23. All vectors point toward the origin and have constant length.

24. All vectors are of unit length and are perpendicular to the position vector at that point.

25. Give a formula $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in a plane that has the properties that $\mathbf{F} = 0$ at $(0, 0)$ and that at any other point (a, b), $\mathbf{F}$ is tangent to circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$.

26. Is vector field $\mathbf{F}(x, y) = (P(x, y), Q(x, y)) = (\sin x + y)\mathbf{i} + (\cos y + x)\mathbf{j}$ a gradient field?

27. Find a formula for vector field $\mathbf{F}(x, y) = \mathrm{M}(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ given the fact that for all points (x, y), $\mathbf{F}$ points toward the origin and $|\mathbf{F}| = \dfrac{10}{x^2 + y^2}$.

For the following exercises, assume that an electric field in the xy-plane caused by an infinite line of charge along the x-axis is a gradient field with potential function $V(x, y) = c\ln\left(\dfrac{r_0}{\sqrt{x^2 + y^2}}\right)$, where $c > 0$ is a constant and r_0 is a reference distance at which the potential is assumed to be zero.

28. Find the components of the electric field in the x- and y-directions, where $\mathbf{E}(x, y) = -\nabla V(x, y)$.

29. Show that the electric field at a point in the xy-plane is directed outward from the origin and has magnitude $\mathbf{E}| = \frac{c}{r}$, where $r = \sqrt{x^2 = y^2}$.

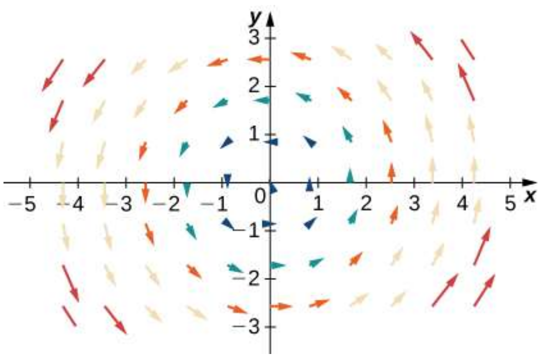
A *flow line* (or *streamline*) of a vector field $\mathbf{F}$ is a curve $\mathbf{r}(t)$ such that $d\mathbf{r}/dt = \mathbf{F}(\mathbf{r}(t))$. If $\mathbf{F}$ represents the velocity field of a moving particle, then the flow lines are paths taken by the particle. Therefore, flow lines are tangent to the vector field. For the following exercises, show that the given curve $\mathbf{c}(t)$ is a flow line of the given velocity vector field $\mathbf{F}(x, y, z)$.

30. $\mathbf{c}(t) = \left(e^{2t}, \ln|t|, \frac{1}{t}\right), t \neq 0; \mathbf{F}(x, y, z) = \langle 2x, z, -z^2 \rangle$

31. $\mathbf{c}(t) = (\sin t, \cos t, e^t); \mathbf{F}(x, y, z) = \langle y, -x, z \rangle$

For the following exercises, let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, $\mathbf{G} = -y\mathbf{i} + x\mathbf{j}$, and $\mathbf{H} = x\mathbf{i} - y\mathbf{j}$. Match $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ with their graphs.

32.



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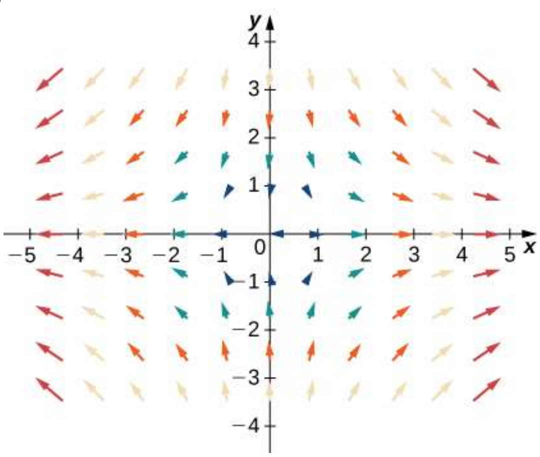

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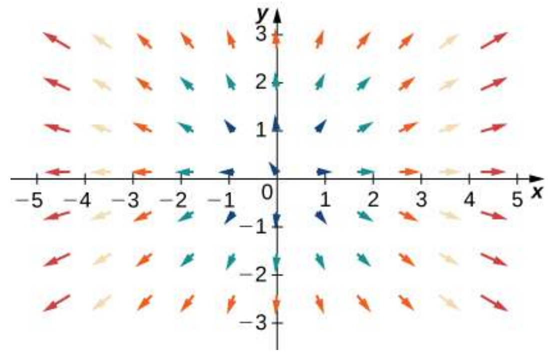


For the following exercises, let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, $\mathbf{G} = -y\mathbf{i} + x\mathbf{j}$, and $\mathbf{H} = -x\mathbf{j} + y\mathbf{j}$. Match the vector fields with their graphs in (I) − (IV).

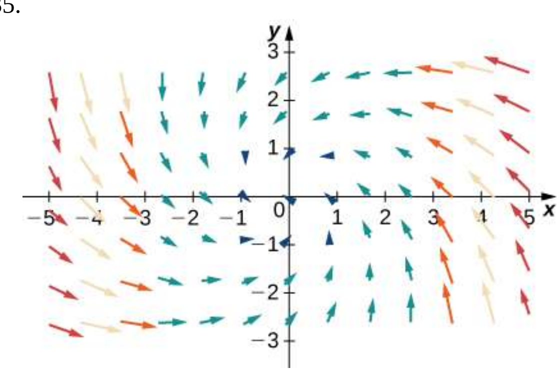
a. $\mathbf{F} + \mathbf{G}$

b. $\mathbf{F} + \mathbf{H}$

c. $\mathbf{G} + \mathbf{H}$

d. $-\mathbf{F} + \mathbf{G}$

35.



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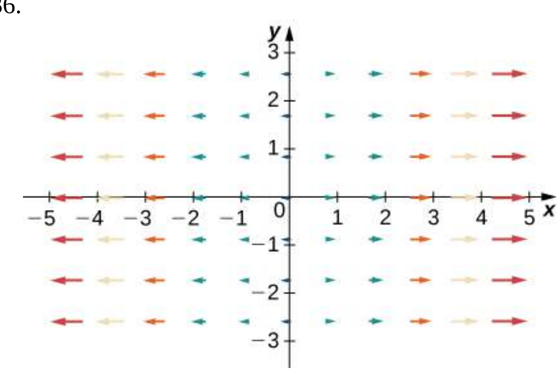

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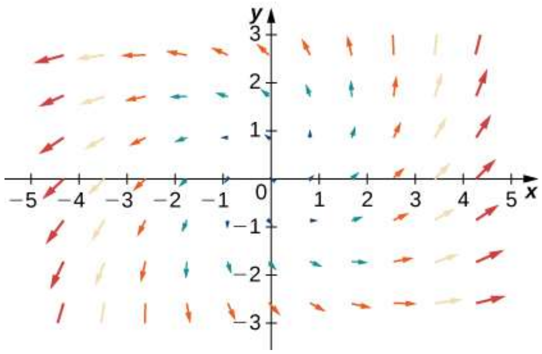

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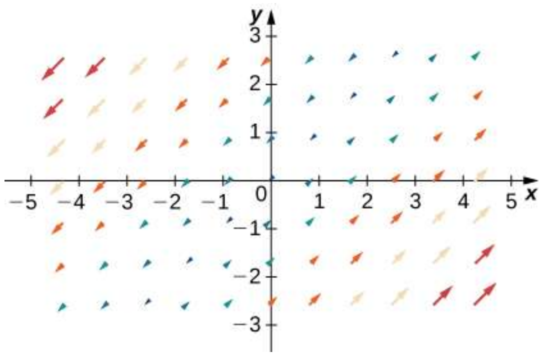


6.2 | Line Integrals

Learning Objectives

6.2.1 Calculate a scalar line integral along a curve.
6.2.2 Calculate a vector line integral along an oriented curve in space.
6.2.3 Use a line integral to compute the work done in moving an object along a curve in a vector field.
6.2.4 Describe the flux and circulation of a vector field.

We are familiar with single-variable integrals of the form $\int_a^b f(x)dx,$ where the domain of integration is an interval $[a, b].$ Such an interval can be thought of as a curve in the *xy*-plane, since the interval defines a line segment with endpoints $(a, 0)$ and $(b, 0)$—in other words, a line segment located on the *x*-axis. Suppose we want to integrate over *any* curve in the plane, not just over a line segment on the *x*-axis. Such a task requires a new kind of integral, called a *line integral*.

Line integrals have many applications to engineering and physics. They also allow us to make several useful generalizations of the Fundamental Theorem of Calculus. And, they are closely connected to the properties of vector fields, as we shall see.

Scalar Line Integrals

A **line integral** gives us the ability to integrate multivariable functions and vector fields over arbitrary curves in a plane or in space. There are two types of line integrals: scalar line integrals and vector line integrals. Scalar line integrals are integrals of a scalar function over a curve in a plane or in space. Vector line integrals are integrals of a vector field over a curve in a plane or in space. Let's look at scalar line integrals first.

A scalar line integral is defined just as a single-variable integral is defined, except that for a scalar line integral, the integrand is a function of more than one variable and the domain of integration is a curve in a plane or in space, as opposed to a curve on the *x*-axis.

For a scalar line integral, we let *C* be a smooth curve in a plane or in space and let f be a function with a domain that includes *C*. We chop the curve into small pieces. For each piece, we choose point *P* in that piece and evaluate f at *P*. (We can do this because all the points in the curve are in the domain of $f.$) We multiply $f(P)$ by the arc length of the piece $\Delta s,$ add the product $f(P)\Delta s$ over all the pieces, and then let the arc length of the pieces shrink to zero by taking a limit. The result is the scalar line integral of the function over the curve.

For a formal description of a scalar line integral, let C be a smooth curve in space given by the parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,$ $a \le t \le b.$ Let $f(x, y, z)$ be a function with a domain that includes curve $C.$ To define the line integral of the function f over $C,$ we begin as most definitions of an integral begin: we chop the curve into small pieces. Partition the parameter interval $[a, b]$ into *n* subintervals $[t_{i-l}, t_i]$ of equal width for $1 \le i \le n,$ where $t_0 = a$ and $t_n = b$ (**Figure 6.12**). Let t_i^* be a value in the *i*th interval $[t_{i-1}, t_i].$ Denote the endpoints of $\mathbf{r}(t_0), \mathbf{r}(t_1), \ldots, \mathbf{r}(t_n)$ by $P_0, \ldots, P_n.$ Points P_i divide curve C into n pieces $C_1, C_2, \ldots, C_n,$ with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n,$ respectively. Let P_i^* denote the endpoint of $\mathbf{r}(t_i^*)$ for $1 \le i \le n.$ Now, we evaluate the function f at point P_i^* for $1 \le i \le n.$ Note that P_i^* is in piece $C_1,$ and therefore P_i^* is in the domain of $f.$ Multiply $f(P_i^*)$ by the length Δs_1 of $C_1,$ which gives the area of the "sheet" with base $C_1,$ and height $f(P_i^*).$ This is analogous to using rectangles to approximate area in a single-variable integral. Now, we form the sum $\sum_{i=1}^{n} f(P_i^*)\Delta s_i.$ Note the similarity of this sum versus a Riemann sum; in fact, this definition is a generalization of a Riemann sum to arbitrary curves in space. Just as with Riemann sums and integrals of form $\int_a^b g(x)dx,$ we define an integral by letting the width of the pieces of the curve shrink to zero by taking a limit. The result is the scalar line integral of f along $C.$

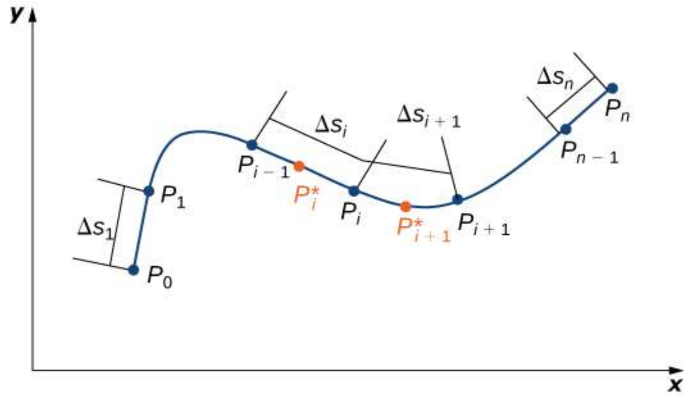


Figure 6.12 Curve C has been divided into n pieces, and a point inside each piece has been chosen.

You may have noticed a difference between this definition of a scalar line integral and a single-variable integral. In this definition, the arc lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$ aren't necessarily the same; in the definition of a single-variable integral, the curve in the x-axis is partitioned into pieces of equal length. This difference does not have any effect in the limit. As we shrink the arc lengths to zero, their values become close enough that any small difference becomes irrelevant.

Definition

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$. The **scalar line integral** of f along C is

$$\int_C f(x, y, z)ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(P_i^*)\Delta s_i \tag{6.5}$$

if this limit exists (t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$, $y = y(t)$, and $a \le t \le b$. If C is smooth and $f(x, y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x, y)ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(P_i^*)\Delta s_i,$$

if this limit exists.

If f is a continuous function on a smooth curve C, then $\int_C f\,ds$ always exists. Since $\int_C f\,ds$ is defined as a limit of Riemann sums, the continuity of f is enough to guarantee the existence of the limit, just as the integral $\int_a^b g(x)dx$ exists if g is continuous over $[a, b]$.

Before looking at how to compute a line integral, we need to examine the geometry captured by these integrals. Suppose that $f(x, y) \ge 0$ for all points (x, y) on a smooth planar curve C. Imagine taking curve C and projecting it "up" to the surface defined by $f(x, y)$, thereby creating a new curve C' that lies in the graph of $f(x, y)$ (**Figure 6.13**). Now we drop a "sheet" from C' down to the xy-plane. The area of this sheet is $\int_C f(x, y)ds$. If $f(x, y) \le 0$ for some points in C, then the value of $\int_C f(x, y)ds$ is the area above the xy-plane less the area below the xy-plane. (Note the similarity with integrals of the form $\int_a^b g(x)dx$.)

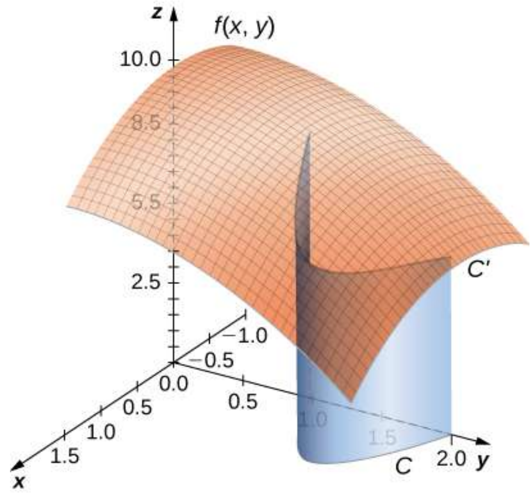


Figure 6.13 The area of the blue sheet is $\int_C f(x,\ y)ds.$

From this geometry, we can see that line integral $\int_C f(x,\ y)ds$ does not depend on the parameterization $\mathbf{r}(t)$ of C. As long as the curve is traversed exactly once by the parameterization, the area of the sheet formed by the function and the curve is the same. This same kind of geometric argument can be extended to show that the line integral of a three-variable function over a curve in space does not depend on the parameterization of the curve.

Example 6.14

Finding the Value of a Line Integral

Find the value of integral $\int_C 2ds,$ where C is the upper half of the unit circle.

Solution

The integrand is $f(x,\ y) = 2.$ **Figure 6.14** shows the graph of $f(x,\ y) = 2,$ curve C, and the sheet formed by them. Notice that this sheet has the same area as a rectangle with width π and length 2. Therefore, $\int_C 2ds = 2\pi.$

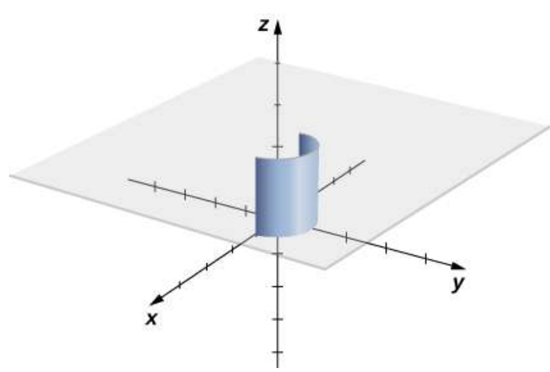


Figure 6.14 The sheet that is formed by the upper half of the unit circle in a plane and the graph of $f(x,\ y) = 2.$

To see that $\int_C 2ds = 2\pi$ using the definition of line integral, we let $\mathbf{r}(t)$ be a parameterization of C. Then, $f(\mathbf{r}(t_i)) = 2$ for any number t_i in the domain of $\mathbf{r}$. Therefore,

$$\begin{aligned}
\int_C f\,ds &= \lim_{n \to \infty} \sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\Delta s_i \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} 2\Delta s_i \\
&= 2 \lim_{n \to \infty} \sum_{i=1}^{n} 2\Delta s_i \\
&= 2(\text{length of C}) \\
&= 2\pi.
\end{aligned}$$

6.13 Find the value of $\int_C (x+y)ds,$ where C is the curve parameterized by $x = t, \quad y = t, \quad 0 \le t \le 1.$

Note that in a scalar line integral, the integration is done with respect to arc length s, which can make a scalar line integral difficult to calculate. To make the calculations easier, we can translate $\int_C f\,ds$ to an integral with a variable of integration that is t.

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$ be a parameterization of C. Since we are assuming that C is smooth, $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ is continuous for all t in $[a, b]$. In particular, $x'(t), y'(t),$ and $z'(t)$ exist for all t in $[a, b]$. According to the arc length formula, we have

$$\text{length}(C_i) = \Delta s_i = \int_{t_{i-1}}^{t_i} \| \mathbf{r}'(t) \| \, dt.$$

If width $\Delta t_i = t_i - t_{i-1}$ is small, then function $\int_{t_{i-1}}^{t_i} \| \mathbf{r}'(t) \| \, dt \approx \| r'(t_i^*) \| \Delta t_i, \quad \| \mathbf{r}'(t) \|$ is almost constant over the interval $[t_{i-1}, t_i]$. Therefore,

$$\int_{t_{i-1}}^{t_i} \| \mathbf{r}'(t) \| \, dt \approx \| \mathbf{r}'(t_i^*) \| \Delta t_i,$$

and we have

$$\sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\Delta s_i = \sum_{i=1}^{n} f(\mathbf{r}(t_i^*)) \| \mathbf{r}'(t_i^*) \| \Delta t_i. \tag{6.6}$$

See **Figure 6.15**.

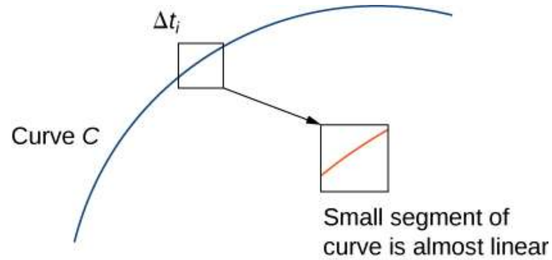


Figure 6.15 If we zoom in on the curve enough by making Δt_i very small, then the corresponding piece of the curve is approximately linear.

Note that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\,\|\,\mathbf{r}'(t_i^*)\,\|\,\Delta t_i = \int_a^b f(\mathbf{r}(t))\,\|\,\mathbf{r}'(t)\,\|\,dt.$$

In other words, as the widths of intervals $[t_{i-1}, t_i]$ shrink to zero, the sum $\sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\,\|\,\mathbf{r}'(t_i^*)\,\|\,\Delta t_i$ converges to the integral $\int_a^b f(\mathbf{r}(t))\,\|\,\mathbf{r}'(t)\,\|\,dt$. Therefore, we have the following theorem.

Theorem 6.3: Evaluating a Scalar Line Integral

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t)$, $a \le t \le b$. Then

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\,\|\,\mathbf{r}'(t)\,\|\,dt. \tag{6.7}$$

Although we have labeled **Equation 6.6** as an equation, it is more accurately considered an approximation because we can show that the left-hand side of **Equation 6.6** approaches the right-hand side as $n \to \infty$. In other words, letting the widths of the pieces shrink to zero makes the right-hand sum arbitrarily close to the left-hand sum. Since

$$\|\,\mathbf{r}'(t)\,\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2},$$

we obtain the following theorem, which we use to compute scalar line integrals.

Theorem 6.4: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$. Then

$$\int_C f(x, y, z)\,ds = \int_a^b f(\mathbf{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt. \tag{6.8}$$

Similarly,

$$\int_C f(x, y)\,ds = \int_a^b f(\mathbf{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

if C is a planar curve and f is a function of two variables.

Note that a consequence of this theorem is the equation $ds = \|\mathbf{r}'(t)\| dt$. In other words, the change in arc length can be viewed as a change in the t domain, scaled by the magnitude of vector $\mathbf{r}'(t)$.

Example 6.15

Evaluating a Line Integral

Find the value of integral $\int_C (x^2 + y^2 + z) ds$, where C is part of the helix parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$.

Solution

To compute a scalar line integral, we start by converting the variable of integration from arc length s to t. Then, we can use **Equation 6.8** to compute the integral with respect to t. Note that $f(\mathbf{r}(t)) = \cos^2 t + \sin^2 t + t = 1 + t$ and

$$\begin{aligned}\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} &= \sqrt{(-\sin(t))^2 + \cos^2(t) + 1} \\ &= \sqrt{2}.\end{aligned}$$

Therefore,

$$\int_C (x^2 + y^2 + z) ds = \int_0^{2\pi} (1 + t)\sqrt{2} dt.$$

Notice that **Equation 6.8** translated the original difficult line integral into a manageable single-variable integral. Since

$$\begin{aligned}\int_0^{2\pi} (1 + t)\sqrt{2} dt &= \left[\sqrt{2}t + \frac{\sqrt{2}t^2}{2}\right]_0^{2\pi} \\ &= 2\sqrt{2}\pi + 2\sqrt{2}\pi^2,\end{aligned}$$

we have

$$\int_C (x^2 + y^2 + z) ds = 2\sqrt{2}\pi + 2\sqrt{2}\pi^2.$$

6.14 Evaluate $\int_C (x^2 + y^2 + z) ds$, where C is the curve with parameterization $\mathbf{r}(t) = \langle \sin(3t), \cos(3t) \rangle$, $0 \le t \le \frac{\pi}{4}$.

Example 6.16

Independence of Parameterization

Find the value of integral $\int_C (x^2 + y^2 + z) ds$, where C is part of the helix parameterized by $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, $0 \le t \le \pi$. Notice that this function and curve are the same as in the previous example; the only difference is that the curve has been reparameterized so that time runs twice as fast.

Solution

As with the previous example, we use **Equation 6.8** to compute the integral with respect to t. Note that $f(\mathbf{r}(t)) = \cos^2(2t) + \sin^2(2t) + 2t = 2t + 1$ and

$$\begin{aligned}\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} &= \sqrt{(-\sin t + \cos t + 4)} \\ &= 2\sqrt{2}\end{aligned}$$

so we have

$$\begin{aligned}\int_C \left(x^2 + y^2 + z\right) ds &= 2\sqrt{2}\int_0^{\pi} (1 + 2t)dt \\ &= 2\sqrt{2}\left[t + t^2\right]_0^{\pi} \\ &= 2\sqrt{2}\left(\pi + \pi^2\right).\end{aligned}$$

Notice that this agrees with the answer in the previous example. Changing the parameterization did not change the value of the line integral. Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.

6.15 Evaluate line integral $\int_C \left(x^2 + yz\right) ds$, where C is the line with parameterization $\mathbf{r}(t) = \langle 2t, 5t, -t \rangle, 0 \le t \le 10.$ Reparameterize C with parameterization $\mathbf{s}(t) = \langle 4t, 10t, -2t \rangle, 0 \le t \le 5,$ recalculate line integral $\int_C \left(x^2 + yz\right) ds,$ and notice that the change of parameterization had no effect on the value of the integral.

Now that we can evaluate line integrals, we can use them to calculate arc length. If $f(x, y, z) = 1,$ then

$$\begin{aligned}\int_C f(x, y, z)ds &= \lim_{n \to \infty} \sum_{i=1}^{n} f\left(t_i^*\right)\Delta s_i \\ &= \lim_{n \to \infty} \sum_{i=1}^{n} \Delta s_i \\ &= \lim_{n \to \infty} \text{length}(C) \\ &= \text{length}(C).\end{aligned}$$

Therefore, $\int_C 1ds$ is the arc length of C.

Example 6.17

Calculating Arc Length

A wire has a shape that can be modeled with the parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{t} \rangle, 0 \le t \le 4\pi.$ Find the length of the wire.

Solution

The length of the wire is given by $\int_C 1ds$, where C is the curve with parameterization $\mathbf{r}$. Therefore,

$$\begin{aligned}\text{The length of the wire } &= \int_C 1ds \\ &= \int_0^{4\pi} \| \mathbf{r}'(t) \| \, dt \\ &= \int_0^{4\pi} \sqrt{(-\sin t)^2 + \cos^2 t + t}dt \\ &= \int_0^{4\pi} \sqrt{1+t}dt \\ &= \left[\frac{2(1+t)^{3/2}}{3}\right]_0^{4\pi} \\ &= \frac{2}{3}\left((1+4\pi)^{3/2} - 1\right).\end{aligned}$$

6.16 Find the length of a wire with parameterization $\mathbf{r}(t) = \langle 3t+1, 4-2t, 5+2t \rangle, \ 0 \le t \le 4.$

Vector Line Integrals

The second type of line integrals are vector line integrals, in which we integrate along a curve through a vector field. For example, let

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

be a continuous vector field in $\mathbb{R}^3$ that represents a force on a particle, and let C be a smooth curve in $\mathbb{R}^3$ contained in the domain of $\mathbf{F}$. How would we compute the work done by $\mathbf{F}$ in moving a particle along C?

To answer this question, first note that a particle could travel in two directions along a curve: a forward direction and a backward direction. The work done by the vector field depends on the direction in which the particle is moving. Therefore, we must specify a direction along curve C; such a specified direction is called an **orientation of a curve**. The specified direction is the *positive* direction along C; the opposite direction is the *negative* direction along C. When C has been given an orientation, C is called an *oriented curve* (Figure 6.16). The work done on the particle depends on the direction along the curve in which the particle is moving.

A **closed curve** is one for which there exists a parameterization $\mathbf{r}(t)$, $a \le t \le b$, such that $\mathbf{r}(a) = \mathbf{r}(b)$, and the curve is traversed exactly once. In other words, the parameterization is one-to-one on the domain (a, b).

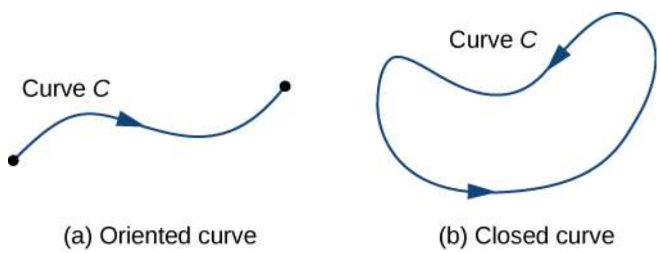


Figure 6.16 (a) An oriented curve between two points. (b) A closed oriented curve.

Let $\mathbf{r}(t)$ be a parameterization of C for $a \le t \le b$ such that the curve is traversed exactly once by the particle and the particle moves in the positive direction along C. Divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$, $0 \leq i \leq n$, of equal width. Denote the endpoints of $\mathbf{r}(t_0), \mathbf{r}(t_1), \ldots, \mathbf{r}(t_n)$ by $P_0, \ldots, P_n$. Points P_i divide C into n pieces. Denote the length of the piece from P_{i-1} to P_i by Δs_i. For each i, choose a value t_i^* in the subinterval $[t_{i-1}, t_i]$. Then, the endpoint of $\mathbf{r}(t_i^*)$ is a point in the piece of C between P_{i-1} and P_i (**Figure 6.17**). If Δs_i is small, then as the particle moves from P_{i-1} to P_i along C, it moves approximately in the direction of $\mathbf{T}(P_i)$, the unit tangent vector at the endpoint of $\mathbf{r}(t_i^*)$. Let P_i^* denote the endpoint of $\mathbf{r}(t_i^*)$. Then, the work done by the force vector field in moving the particle from P_{i-1} to P_i is $\mathbf{F}(P_i^*) \cdot (\Delta s_i \mathbf{T}(P_i^*))$, so the total work done along C is

$$\sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot (\Delta s_i \mathbf{T}(P_i^*)) = \sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*) \Delta s_i.$$

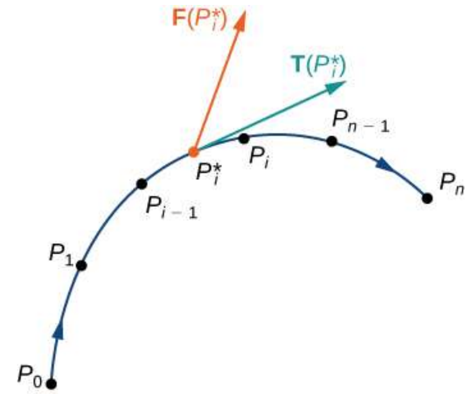


Figure 6.17 Curve C is divided into n pieces, and a point inside each piece is chosen. The dot product of any tangent vector in the ith piece with the corresponding vector $\mathbf{F}$ is approximated by $\mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*)$.

Letting the arc length of the pieces of C get arbitrarily small by taking a limit as $n \to \infty$ gives us the work done by the field in moving the particle along C. Therefore, the work done by $\mathbf{F}$ in moving the particle in the positive direction along C is defined as

$$W = \int_C \mathbf{f} \cdot \leq \mathbf{T} ds,$$

which gives us the concept of a vector line integral.

Definition

The **vector line integral** of vector field $\mathbf{F}$ along oriented smooth curve C is

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot \mathbf{T}(P_i^*) \Delta s_i$$

if that limit exists.

With scalar line integrals, neither the orientation nor the parameterization of the curve matters. As long as the curve is traversed exactly once by the parameterization, the value of the line integral is unchanged. With vector line integrals, the orientation of the curve does matter. If we think of the line integral as computing work, then this makes sense: if you hike up a mountain, then the gravitational force of Earth does negative work on you. If you walk down the mountain by the exact same path, then Earth's gravitational force does positive work on you. In other words, reversing the path changes the work value from negative to positive in this case. Note that if C is an oriented curve, then we let $-C$ represent the same curve but with opposite orientation.

As with scalar line integrals, it is easier to compute a vector line integral if we express it in terms of the parameterization function $\mathbf{r}$ and the variable t. To translate the integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$ in terms of t, note that unit tangent vector $\mathbf{T}$ along C is given by $\mathbf{T} = \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|}$ (assuming $\| \mathbf{r}'(t) \| \neq 0$). Since $ds = \| \mathbf{r}'(t) \| dt$, as we saw when discussing scalar line integrals, we have

$$\mathbf{F} \cdot \mathbf{T} ds = \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|} \| \mathbf{r}'(t) \| dt = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Thus, we have the following formula for computing vector line integrals:

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt. \tag{6.9}$$

Because of **Equation 6.9**, we often use the notation $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the line integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$.

If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then $d\mathbf{r}$ denotes vector $\langle x'(t), y'(t), z'(t) \rangle$.

Example 6.18

Evaluating a Vector Line Integral

Find the value of integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the semicircle parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq \pi$ and $\mathbf{F} = \langle -y, x \rangle$.

Solution

We can use **Equation 6.9** to convert the variable of integration from s to t. We then have

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle \text{ and } \mathbf{r}'(t) = \langle -\sin t, \cos t \rangle.$$

Therefore,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle dt \\ &= \int_0^\pi \sin^2 t + \cos^2 t dt \\ &= \int_0^\pi 1 dt = \pi. \end{aligned}$$

See **Figure 6.18**.

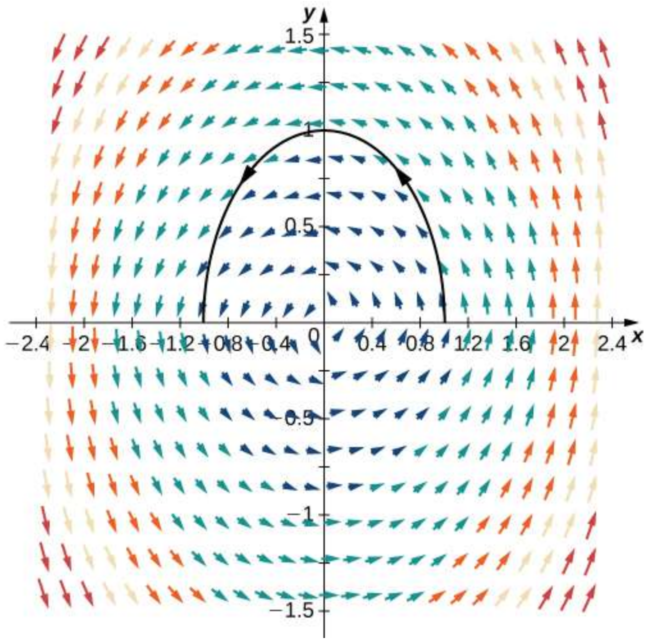


Figure 6.18 This figure shows curve $\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \ 0 \le t \le \pi$ in vector field $\mathbf{F} = \langle -y, x \rangle$.

Example 6.19

Reversing Orientation

Find the value of integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the semicircle parameterized by $\mathbf{r}(t) = \langle \cos t + \pi, \sin t \rangle, \ 0 \le t \le \pi$ and $\mathbf{F} = \langle -y, x \rangle$.

Solution

Notice that this is the same problem as **Example 6.18**, except the orientation of the curve has been traversed. In this example, the parameterization starts at $\mathbf{r}(0) = \langle \pi, 0 \rangle$ and ends at $\mathbf{r}(\pi) = \langle 0, 0 \rangle$. By **Equation 6.9**,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \langle -\sin t, \cos t + \pi \rangle \cdot \langle -\sin t + \pi, \cos t \rangle \, dt \\
&= \int_0^\pi \langle -\sin t, -\cos t \rangle \cdot \langle \sin t, \cos t \rangle \, dt \\
&= \int_0^\pi \left(-\sin^2 t - \cos^2 t \right) dt \\
&= \int_0^\pi -1 \, dt \\
&= -\pi.
\end{aligned}$$

Notice that this is the negative of the answer in **Example 6.18**. It makes sense that this answer is negative because the orientation of the curve goes against the "flow" of the vector field.

Let C be an oriented curve and let $-C$ denote the same curve but with the orientation reversed. Then, the previous two examples illustrate the following fact:

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = -\int_{C} \mathbf{F} \cdot d\mathbf{r}.$$

That is, reversing the orientation of a curve changes the sign of a line integral.

6.17 Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ be a vector field and let C be the curve with parameterization $\langle t, t^2 \rangle$ for $0 \le t \le 2$. Which is greater: $\int_C \mathbf{F} \cdot \mathbf{T}ds$ or $\int_{-C} \mathbf{F} \cdot \mathbf{T}ds$?

Another standard notation for integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is $\int_C Pdx + Qdy + Rdz$. In this notation, P, Q, and R are functions, and we think of $d\mathbf{r}$ as vector $\langle dx, dy, dz \rangle$. To justify this convention, recall that $d\mathbf{r} = \mathbf{T}ds = \mathbf{r}'(t)dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$. Therefore,

$$\mathbf{F} \cdot d\mathbf{r} = \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle = Pdx + Qdy + Rdz.$$

If $d\mathbf{r} = \langle dx, dy, dz \rangle$, then $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$, which implies that $\frac{d\mathbf{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$. Therefore

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C Pdx + Qdy + Rdz \\ &= \int \left(P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt} + R(\mathbf{r}(t))\frac{dz}{dt} \right) dt. \end{aligned} \tag{6.10}$$

Example 6.20

Finding the Value of an Integral of the Form $\int_C Pdx + Qdy + Rdz$

Find the value of integral $\int_C zdx + xdy + ydz$, where C is the curve parameterized by $\mathbf{r}(t) = \langle t^2, \sqrt{t}, t \rangle$, $1 \le t \le 4$.

Solution

As with our previous examples, to compute this line integral we should perform a change of variables to write everything in terms of t. In this case, **Equation 6.10** allows us to make this change:

$$\begin{aligned}\int_C z dx + x dy + y dz &= \int_1^4 \left(t(2t) + t^2\left(\frac{1}{2\sqrt{t}}\right) + \sqrt{t}\right)dt \\ &= \int_1^4 \left(2t^2 + \frac{t^{3/2}}{2} + \sqrt{t}\right)dt \\ &= \left[\frac{2t^3}{3} + \frac{t^{5/2}}{5} + \frac{2t^{3/2}}{3}\right]_{t=1}^{t=4} \\ &= \frac{793}{15}.\end{aligned}$$

6.18 Find the value of $\int_C 4x dx + z dy + 4y^2 dz,$ where C is the curve parameterized by $\mathbf{r}(t) = \langle 4\cos(2t), 2\sin(2t), 3\rangle, 0 \le t \le \frac{\pi}{4}.$

We have learned how to integrate smooth oriented curves. Now, suppose that C is an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves. In this case, we say that C is a **piecewise smooth curve**. To be precise, curve C is piecewise smooth if C can be written as a union of n smooth curves $C_1, C_2, \ldots, C_n$ such that the endpoint of C_i is the starting point of C_{i+1} (**Figure 6.19**). When curves C_i satisfy the condition that the endpoint of C_i is the starting point of $C_{i+1},$ we write their union as $C_1 + C_2 + \cdots + C_n$.

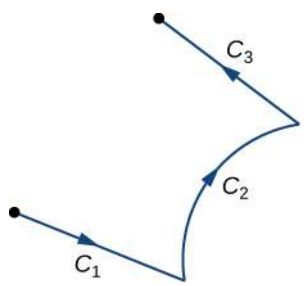


Figure 6.19 The union of C_1, C_2, C_3 is a piecewise smooth curve.

The next theorem summarizes several key properties of vector line integrals.

Theorem 6.5: Properties of Vector Line Integrals

Let $\mathbf{F}$ and $\mathbf{G}$ be continuous vector fields with domains that include the oriented smooth curve C. Then

i. $\int_C (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}$

ii. $\int_C k\mathbf{F} \cdot d\mathbf{r} = k\int_C \mathbf{F} \cdot d\mathbf{r},$ where k is a constant

iii. $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$

iv. Suppose instead that C is a piecewise smooth curve in the domains of $\mathbf{F}$ and $\mathbf{G}$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \ldots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \cdots + \int_{C_n} \mathbf{F} \cdot d\mathbf{s}.$$

Notice the similarities between these items and the properties of single-variable integrals. Properties i. and ii. say that line integrals are linear, which is true of single-variable integrals as well. Property iii. says that reversing the orientation of a curve changes the sign of the integral. If we think of the integral as computing the work done on a particle traveling along C, then this makes sense. If the particle moves backward rather than forward, then the value of the work done has the opposite sign. This is analogous to the equation $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Finally, if $[a_1, a_2], [a_2, a_3], \ldots, [a_{n-1}, a_n]$ are intervals, then

$$\int_{a_1}^{a_n} f(x)dx = \int_{a_1}^{a_2} f(x)dx + \int_{a_1}^{a_3} f(x)dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)dx,$$

which is analogous to property iv.

Example 6.21

Using Properties to Compute a Vector Line Integral

Find the value of integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$, where C is the rectangle (oriented counterclockwise) in a plane with vertices $(0, 0), (2, 0), (2, 1),$ and $(0, 1)$, and where $\mathbf{F} = \langle x - 2y, y - x \rangle$ (**Figure 6.20**).

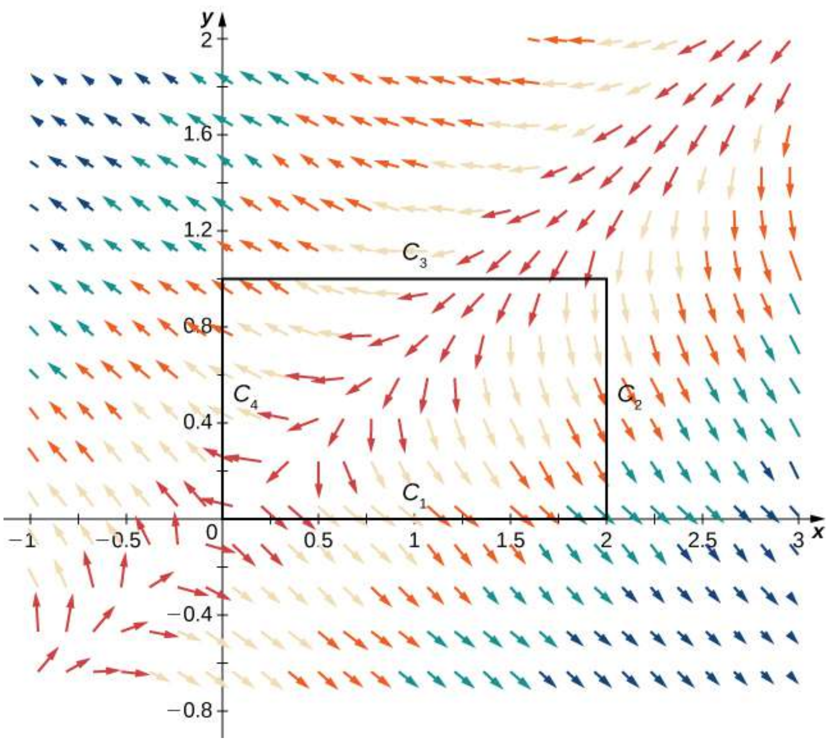


Figure 6.20 Rectangle and vector field for **Example 6.21**.

Solution

Note that curve C is the union of its four sides, and each side is smooth. Therefore C is piecewise smooth. Let C_1 represent the side from $(0, 0)$ to $(2, 0)$, let C_2 represent the side from $(2, 0)$ to $(2, 1)$, let C_3 represent the side from $(2, 1)$ to $(0, 1)$, and let C_4 represent the side from $(0, 1)$ to $(0, 0)$ (**Figure 6.20**). Then,

$$\int_C \mathbf{F} \cdot \mathbf{T} d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T} d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathbf{T} d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot \mathbf{T} d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot \mathbf{T} d\mathbf{r}.$$

We want to compute each of the four integrals on the right-hand side using **Equation 6.8**. Before doing this, we need a parameterization of each side of the rectangle. Here are four parameterizations (note that they traverse C counterclockwise):

$$\begin{aligned} C_1 &: \langle t, 0 \rangle, 0 \le t \le 2 \\ C_2 &: \langle 2, t \rangle, 0 \le t \le 1 \\ C_3 &: \langle 2 - t, 1 \rangle, 0 \le t \le 2 \\ C_4 &: \langle 0, 1 - t \rangle, 0 \le t \le 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot \mathbf{T} d\mathbf{r} &= \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^2 \langle t - 2(0), 0 - t \rangle \cdot \langle 1, 0 \rangle \, dt = \int_0^1 t dt \\ &= \left[\frac{t^2}{2}\right]_0^2 = 2. \end{aligned}$$

Notice that the value of this integral is positive, which should not be surprising. As we move along curve C_1 from left to right, our movement flows in the general direction of the vector field itself. At any point along C_1, the tangent vector to the curve and the corresponding vector in the field form an angle that is less than 90°. Therefore, the tangent vector and the force vector have a positive dot product all along C_1, and the line integral will have positive value.

The calculations for the three other line integrals are done similarly:

$$\begin{aligned} \int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle 2 - 2t, t - 2 \rangle \cdot \langle 0, 1 \rangle \, dt \\ &= \int_0^1 (t - 2) dt \\ &= \left[\frac{t^2}{2} - 2t\right]_0^1 = -\frac{3}{2}, \end{aligned}$$

$$\begin{aligned} \int_{C_3} \mathbf{F} \cdot \mathbf{T} ds &= \int_0^2 \langle (2 - t) - 2, 1 - (2 - t) \rangle \cdot \langle -1, 0 \rangle \, dt \\ &= \int_0^2 t dt = 2, \end{aligned}$$

and

$$\begin{aligned}\int_{C_4} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle -2(1-t), 1-t \rangle \cdot \langle 0, -1 \rangle \, dt \\ &= \int_0^1 (t-1)dt \\ &= \left[\frac{t^2}{2} - t\right]_0^1 = -\frac{1}{2}.\end{aligned}$$

Thus, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = 2.$

6.19 Calculate line integral $\int_C \mathbf{F} \cdot d\mathbf{r},$ where $\mathbf{F}$ is vector field $\langle y^2, 2xy + 1 \rangle$ and C is a triangle with vertices $(0, 0),$ $(4, 0),$ and $(0, 5),$ oriented counterclockwise.

Applications of Line Integrals

Scalar line integrals have many applications. They can be used to calculate the length or mass of a wire, the surface area of a sheet of a given height, or the electric potential of a charged wire given a linear charge density. Vector line integrals are extremely useful in physics. They can be used to calculate the work done on a particle as it moves through a force field, or the flow rate of a fluid across a curve. Here, we calculate the mass of a wire using a scalar line integral and the work done by a force using a vector line integral.

Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function $\rho(x, y, z).$ We can calculate the total mass of the wire using the scalar line integral $\int_C \rho(x, y, z)ds.$

The reason is that mass is density multiplied by length, and therefore the density of a small piece of the wire can be approximated by $\rho(x^*, y^*, z^*)\Delta s$ for some point (x^*, y^*, z^*) in the piece. Letting the length of the pieces shrink to zero with a limit yields the line integral $\int_C \rho(x, y, z)ds.$

Example 6.22

Calculating the Mass of a Wire

Calculate the mass of a spring in the shape of a curve parameterized by $\langle t, 2\cos t, 2\sin t \rangle,$ $0 \le t \le \frac{\pi}{2},$ with a density function given by $\rho(x, y, z) = e^x + yz$ kg/m (**Figure 6.21**).

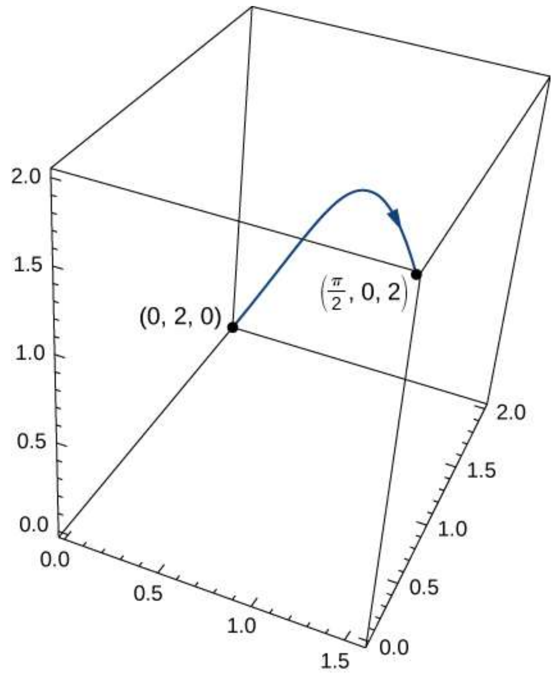


Figure 6.21 The wire from **Example 6.22**.

Solution

To calculate the mass of the spring, we must find the value of the scalar line integral $\int_C (e^x + yz)ds,$ where C is the given helix. To calculate this integral, we write it in terms of t using **Equation 6.8**:

$$\begin{aligned}
\int_C e^x + yzds &= \int_0^{\pi/2} \left((e^t + 4\cos t \sin t)\sqrt{1 + (-2\cos t)^2 + (2\sin t)^2} \right) dt \\
&= \int_0^{\pi/2} \left((e^t + 4\cos t \sin t)\sqrt{5} \right) dt \\
&= \sqrt{5}\left[e^t + 2\sin^2 t \right]_{t=0}^{t=\pi/2} \\
&= \sqrt{5\left(e^{\pi/2} + 1 \right)}.
\end{aligned}$$

Therefore, the mass is $\sqrt{5}\left(e^{\pi/2} + 1\right)$ kg.

6.20 Calculate the mass of a spring in the shape of a helix parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle,\ 0 \le t \le 6\pi,$ with a density function given by $\rho(x, y, z) = x + y + z$ kg/m.

When we first defined vector line integrals, we used the concept of work to motivate the definition. Therefore, it is not surprising that calculating the work done by a vector field representing a force is a standard use of vector line integrals. Recall that if an object moves along curve C in force field $\mathbf{F}$, then the work required to move the object is given by $\int_C \mathbf{F} \cdot d\mathbf{r}.$

Example 6.23

Calculating Work

How much work is required to move an object in vector force field $\mathbf{F} = \langle yz, xy, xz \rangle$ along path $\mathbf{r}(t) = \langle t^2, t, t^4 \rangle, \quad 0 \le t \le 1$? See **Figure 6.22**.

Solution

Let C denote the given path. We need to find the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$. To do this, we use **Equation 6.9**:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \left(\langle t^5, t^3, t^6 \rangle \cdot \langle 2t, 1, 4t^3 \rangle \right) dt \\ &= \int_0^1 \left(2t^6 + t^3 + 4t^9\right) dt \\ &= \left[\frac{2t^7}{7} + \frac{t^4}{4} + \frac{2t^{10}}{5}\right]_{t=0}^{t=1} = \frac{131}{140}.\end{aligned}$$

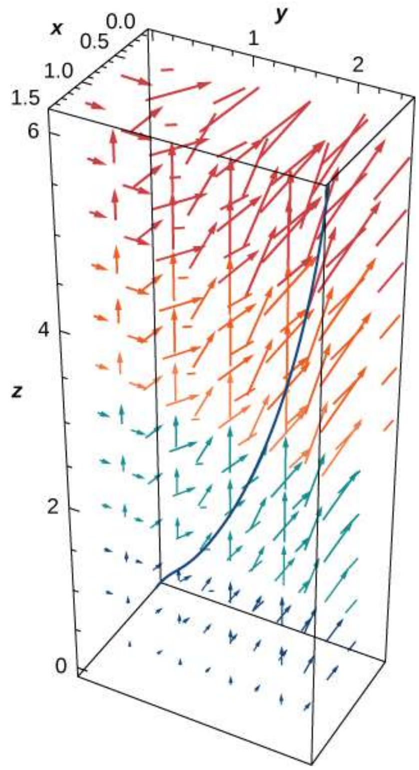


Figure 6.22 The curve and vector field for **Example 6.23**.

Flux and Circulation

We close this section by discussing two key concepts related to line integrals: flux across a plane curve and circulation along a plane curve. Flux is used in applications to calculate fluid flow across a curve, and the concept of circulation is important for characterizing conservative gradient fields in terms of line integrals. Both these concepts are used heavily throughout the rest of this chapter. The idea of flux is especially important for Green's theorem, and in higher dimensions for Stokes' theorem and the divergence theorem.

Let C be a plane curve and let $\mathbf{F}$ be a vector field in the plane. Imagine C is a membrane across which fluid flows, but C does not impede the flow of the fluid. In other words, C is an idealized membrane invisible to the fluid. Suppose $\mathbf{F}$ represents the velocity field of the fluid. How could we quantify the rate at which the fluid is crossing C?

Recall that the line integral of $\mathbf{F}$ along C is $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$—in other words, the line integral is the dot product of the vector field with the unit tangential vector with respect to arc length. If we replace the unit tangential vector with unit normal vector $\mathbf{N}(t)$ and instead compute integral $\int_C \mathbf{F} \cdot \mathbf{N}\,ds$, we determine the flux across C. To be precise, the definition of integral $\int_C \mathbf{F} \cdot \mathbf{N}\,ds$ is the same as integral $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$, except the $\mathbf{T}$ in the Riemann sum is replaced with $\mathbf{N}$. Therefore, the flux across C is defined as

$$\int_C \mathbf{F} \cdot \mathbf{N}\,ds = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot \mathbf{N}(P_i^*) \Delta s_i,$$

where P_i^* and Δs_i are defined as they were for integral $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$. Therefore, a flux integral is an integral that is *perpendicular* to a vector line integral, because $\mathbf{N}$ and $\mathbf{T}$ are perpendicular vectors.

If $\mathbf{F}$ is a velocity field of a fluid and C is a curve that represents a membrane, then the flux of $\mathbf{F}$ across C is the quantity of fluid flowing across C per unit time, or the rate of flow.

More formally, let C be a plane curve parameterized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$. Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$ be the vector that is normal to C at the endpoint of $\mathbf{r}(t)$ and points to the right as we traverse C in the positive direction (**Figure 6.23**). Then, $\mathbf{N}(t) = \dfrac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}$ is the unit normal vector to C at the endpoint of $\mathbf{r}(t)$ that points to the right as we traverse C.

Definition

The **flux** of $\mathbf{F}$ across C is line integral $\int_C \mathbf{F} \cdot \dfrac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|}\,ds.$

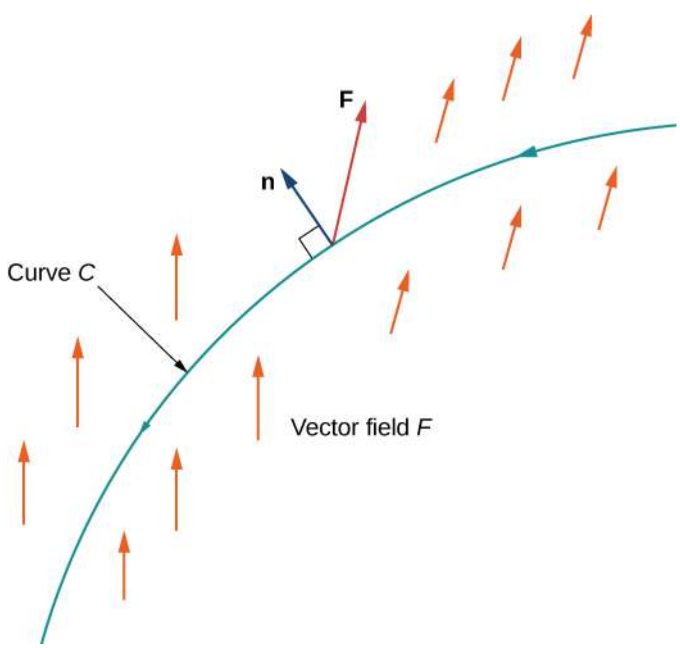


Figure 6.23 The flux of vector field $\mathbf{F}$ across curve C is computed by an integral similar to a vector line integral.

We now give a formula for calculating the flux across a curve. This formula is analogous to the formula used to calculate a vector line integral (see **Equation 6.9**).

Theorem 6.6: Calculating Flux across a Curve

Let $\mathbf{F}$ be a vector field and let C be a smooth curve with parameterization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$. Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$. The flux of $\mathbf{F}$ across C is

$$\int_C \mathbf{F} \cdot \mathbf{N} ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t) dt \tag{6.11}$$

Proof

The proof of **Equation 6.11** is similar to the proof of **Equation 6.8**. Before deriving the formula, note that $\| \mathbf{n}(t) \| = \| \langle y'(t), -x'(t) \rangle \| = \sqrt{(y'(t))^2 + (x'(t))^2} = \| \mathbf{r}'(t) \|$. Therefore,

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{N} ds &= \int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\| \mathbf{n}(t) \|} ds \\ &= \int_a^b \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\| \mathbf{n}(t) \|} \| \mathbf{r}'(t) \| dt \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t) dt. \end{aligned}$$

□

Example 6.24

Flux across a Curve

Calculate the flux of $\mathbf{F} = \langle 2x, 2y \rangle$ across a unit circle oriented counterclockwise (**Figure 6.24**).

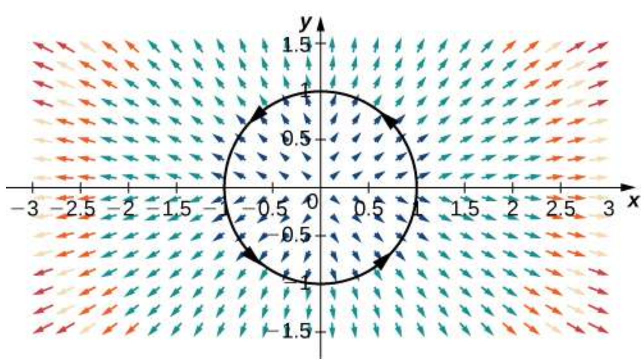


Figure 6.24 A unit circle in vector field $\mathbf{F} = \langle 2x, 2y \rangle$.

Solution

To compute the flux, we first need a parameterization of the unit circle. We can use the standard parameterization $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$. The normal vector to a unit circle is $\langle \cos t, \sin t \rangle$. Therefore, the flux is

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{N}ds &= \int_0^{2\pi} \langle 2\cos t, 2\sin t \rangle \cdot \langle \cos t, \sin t \rangle\, dt \\ &= \int_0^{2\pi} \left(2\cos^2 t + 2\sin^2 t\right) dt = 2\int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right) dt \\ &= 2\int_0^{2\pi} dt = 4\pi.\end{aligned}$$

6.21 Calculate the flux of $\mathbf{F} = \langle x + y, 2y \rangle$ across the line segment from $(0, 0)$ to $(2, 3)$, where the curve is oriented from left to right.

Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be a two-dimensional vector field. Recall that integral $\int_C \mathbf{F}\cdot\mathbf{T}ds$ is sometimes written as $\int_C Pdx + Qdy$. Analogously, flux $\int_C \mathbf{F}\cdot\mathbf{N}ds$ is sometimes written in the notation $\int_C -Qdx + Pdy$, because the unit normal vector **N** is perpendicular to the unit tangent **T**. Rotating the vector $d\mathbf{r} = \langle dx, dy \rangle$ by 90° results in vector $\langle dy, -dx \rangle$. Therefore, the line integral in **Example 6.21** can be written as $\int_C -2ydx + 2xdy$.

Now that we have defined flux, we can turn our attention to circulation. The line integral of vector field **F** along an oriented closed curve is called the **circulation** of **F** along C. Circulation line integrals have their own notation: $\oint_C \mathbf{F}\cdot\mathbf{T}ds$. The circle on the integral symbol denotes that C is "circular" in that it has no endpoints. **Example 6.18** shows a calculation of circulation.

To see where the term *circulation* comes from and what it measures, let **v** represent the velocity field of a fluid and let C be an oriented closed curve. At a particular point P, the closer the direction of $\mathbf{v}(P)$ is to the direction of $\mathbf{T}(P)$, the larger the value of the dot product $\mathbf{v}(P)\cdot\mathbf{T}(P)$. The maximum value of $\mathbf{v}(P)\cdot\mathbf{T}(P)$ occurs when the two vectors are pointing in the exact same direction; the minimum value of $\mathbf{v}(P)\cdot\mathbf{T}(P)$ occurs when the two vectors are pointing in opposite directions. Thus, the value of the circulation $\oint_C \mathbf{v}\cdot\mathbf{T}ds$ measures the tendency of the fluid to move in the direction of C.

Example 6.25

Calculating Circulation

Let $\mathbf{F} = \langle -y, x \rangle$ be the vector field from **Example 6.16** and let C represent the unit circle oriented counterclockwise. Calculate the circulation of **F** along C.

Solution

We use the standard parameterization of the unit circle: $\mathbf{r}(t) = \langle \cos t, \sin t \rangle, 0 \le t \le 2\pi$. Then, $\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle$ and $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$. Therefore, the circulation of **F** along C is

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{T}ds &= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt \\ &= \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Notice that the circulation is positive. The reason for this is that the orientation of C "flows" with the direction of $\mathbf{F}$. At any point along the circle, the tangent vector and the vector from $\mathbf{F}$ form an angle of less than 90°, and therefore the corresponding dot product is positive.

In **Example 6.25**, what if we had oriented the unit circle clockwise? We denote the unit circle oriented clockwise by $-C$. Then

$$\oint_{-C} \mathbf{F}\cdot\mathbf{T}ds = -\oint_C \mathbf{F}\cdot\mathbf{T}ds = -2\pi.$$

Notice that the circulation is negative in this case. The reason for this is that the orientation of the curve flows against the direction of $\mathbf{F}$.

6.22 Calculate the circulation of $\mathbf{F}(x, y) = \left\langle -\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ along a unit circle oriented counterclockwise.

Example 6.26

Calculating Work

Calculate the work done on a particle that traverses circle C of radius 2 centered at the origin, oriented counterclockwise, by field $\mathbf{F}(x, y) = \langle -2, y \rangle$. Assume the particle starts its movement at $(1, 0)$.

Solution

The work done by $\mathbf{F}$ on the particle is the circulation of $\mathbf{F}$ along C: $\oint_C \mathbf{F}\cdot\mathbf{T}ds$. We use the parameterization $\mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$, $0 \le t \le 2\pi$ for C. Then, $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t \rangle$ and $\mathbf{F}(\mathbf{r}(t)) = \langle -2, 2\sin t \rangle$. Therefore, the circulation of $\mathbf{F}$ along C is

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{T}ds &= \int_0^{2\pi} \langle -2, 2\sin t \rangle \cdot \langle -2\sin t, 2\cos t \rangle\, dt \\ &= \int_0^{2\pi} (4\sin t + 4\sin t\cos t)dt \\ &= \left[-4\cos t + 4\sin^2 t\right]_0^{2\pi} \\ &= \left(-4\cos(2\pi) + 2\sin^2(2\pi)\right) - \left(-4\cos(0) + 4\sin^2(0)\right) \\ &= -4 + 4 = 0.\end{aligned}$$

The force field does zero work on the particle.

Notice that the circulation of **F** along C is zero. Furthermore, notice that since **F** is the gradient of $f(x, y) = -2x + \frac{y^2}{2}$, **F** is conservative. We prove in a later section that under certain broad conditions, the circulation of a conservative vector field along a closed curve is zero.

6.23 Calculate the work done by field $\mathbf{F}(x, y) = \langle 2x, 3y \rangle$ on a particle that traverses the unit circle. Assume the particle begins its movement at $(-1, 0)$.

6.2 EXERCISES

39. *True or False?* Line integral $\int_C f(x, y)ds$ is equal to a definite integral if C is a smooth curve defined on $[a, b]$ and if function f is continuous on some region that contains curve C.

40. *True or False?* Vector functions $\mathbf{r}_1 = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{r}_2 = (1-t)\mathbf{i} + (1-t)^2\mathbf{j}$, $0 \le t \le 1$, define the same oriented curve.

41. *True or False?* $\int_{-C}(Pdx + Qdy) = \int_C(Pdx - Qdy)$

42. *True or False?* A piecewise smooth curve C consists of a finite number of smooth curves that are joined together end to end.

43. *True or False?* If C is given by $x(t) = t, y(t) = t, 0 \le \mathrm{t} \le 1,$ then $\int_C xyds = \int_0^1 t^2\,dt.$

For the following exercises, use a computer algebra system (CAS) to evaluate the line integrals over the indicated path.

44. **[T]** $\int_C (x+y)ds$ $C: x = t,\ y = (1-t), z = 0$ from $(0, 1, 0)$ to $(1, 0, 0)$

45. **[T]** $\int_C (x-y)ds$ $C: \mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j}$ when $0 \le t \le 2$

46. **[T]** $\int_C (x^2 + y^2 + z^2)ds$ $C: \mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + 8t\mathbf{k}$ when $0 \le t \le \frac{\pi}{2}$

47. **[T]** Evaluate $\int_C xy^4\,ds,$ where C is the right half of circle $x^2 + y^2 = 16$ and is traversed in the clockwise direction.

48. **[T]** Evaluate $\int_C 4x^3\,ds,$ where C is the line segment from $(-2, -1)$ to $(1, 2)$.

For the following exercises, find the work done.

49. Find the work done by vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + 3xy\mathbf{j} - (x+z)\mathbf{k}$ on a particle moving along a line segment that goes from $(1, 4, 2)$ to $(0, 5, 1)$.

50. Find the work done by a person weighing 150 lb walking exactly one revolution up a circular, spiral staircase of radius 3 ft if the person rises 10 ft.

51. Find the work done by force field $\mathbf{F}(x, y, z) = -\frac{1}{2}x\mathbf{i} - \frac{1}{2}y\mathbf{j} + \frac{1}{4}\mathbf{k}$ on a particle as it moves along the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from point $(1, 0, 0)$ to point $(-1, 0, 3\pi)$.

52. Find the work done by vector field $\mathbf{F}(x, y) = y\mathbf{i} + 2x\mathbf{j}$ in moving an object along path C, which joins points (1, 0) and (0, 1).

53. Find the work done by force $\mathbf{F}(x, y) = 2y\mathbf{i} + 3x\mathbf{j} + (x+y)\mathbf{k}$ in moving an object along curve $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \frac{1}{6}\mathbf{k},$ where $0 \le t \le 2\pi.$

54. Find the mass of a wire in the shape of a circle of radius 2 centered at (3, 4) with linear mass density $\rho(x, y) = y^2.$

For the following exercises, evaluate the line integrals.

55. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r},$ where $\mathbf{F}(x, y) = -1\mathbf{j},$ and C is the part of the graph of $y = \frac{1}{2}x^3 - x$ from $(2, 2)$ to $(-2, -2)$.

56. Evaluate $\int_\gamma \left(x^2 + y^2 + z^2\right)^{-1} ds,$ where γ is the helix $x = \cos t,\ y = \sin t,\ z = t(0 \le t \le T).$

57. Evaluate $\int_C yz\,dx + xz\,dy + xy\,dz$ over the line segment from $(1, 1, 1)$ to $(3, 2, 0)$.

58. Let C be the line segment from point (0, 1, 1) to point (2, 2, 3). Evaluate line integral $\int_C yds.$

59. **[T]** Use a computer algebra system to evaluate the line integral $\int_C y^2\,dx + xdy,$ where C is the arc of the parabola $x = 4 - y^2$ from (−5, −3) to (0, 2).

60. **[T]** Use a computer algebra system to evaluate the line integral $\int_C (x + 3y^2)dy$ over the path C given by $x = 2t, y = 10t,$ where $0 \le t \le 1$.

61. **[T]** Use a CAS to evaluate line integral $\int_C xydx + ydy$ over path C given by $x = 2t, y = 10t,$ where $0 \le t \le 1$.

62. Evaluate line integral $\int_C (2x - y)dx + (x + 3y)dy,$ where C lies along the x-axis from $x = 0$ to $x = 5$.

63. **[T]** Use a CAS to evaluate $\int_C \frac{y}{2x^2 - y^2}ds,$ where C is $x = t, y = t, 1 \le t \le 5$.

64. **[T]** Use a CAS to evaluate $\int_C xyds,$ where C is $x = t^2,\ y = 4t,\ 0 \le t \le 1$.

In the following exercises, find the work done by force field **F** on an object moving along the indicated path.

65. $\mathbf{F}(x,\ y) = -x\mathbf{i} - 2y\mathbf{j}$
$C: y = x^3$ from (0, 0) to (2, 8)

66. $\mathbf{F}(x, y) = 2xi + y\mathbf{j}$ C: counterclockwise around the triangle with vertices (0, 0), (1, 0), and (1, 1)

67. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$
$C: \mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, 0 \le t \le 2\pi$

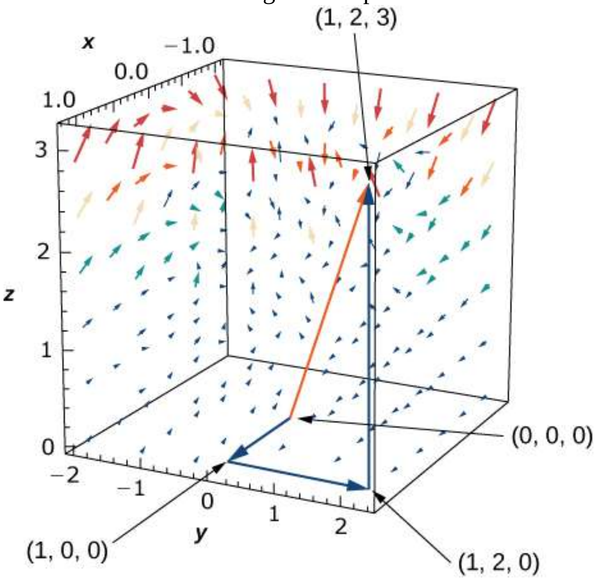
68. Let **F** be vector field $\mathbf{F}(x,\ y) = (y^2 + 2xe^y + 1)\mathbf{i} + (2xy + x^2e^y + 2y)\mathbf{j}$. Compute the work of integral $\int_C \mathbf{F} \cdot d\mathbf{r},$ where C is the path $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j}, 0 \le t \le \frac{\pi}{2}$.

69. Compute the work done by force $\mathbf{F}(x,\ y,\ z) = 2x\mathbf{i} + 3y\mathbf{j} - z\mathbf{k}$ along path $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},$ where $0 \le t \le 1$.

70. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r},$ where $\mathbf{F}(x,\ y) = \frac{1}{x + y}\mathbf{i} + \frac{1}{x + y}\mathbf{j}$ and C is the segment of the unit circle going counterclockwise from $(1,\ 0)$ to (0, 1).

71. Force $\mathbf{F}(x,\ y,\ z) = zy\mathbf{i} + x\mathbf{j} + z^2x\mathbf{k}$ acts on a particle that travels from the origin to point (1, 2, 3). Calculate the work done if the particle travels:

a. along the path $(0,\ 0,\ 0) \to (1,\ 0,\ 0) \to (1,\ 2,\ 0) \to (1,\ 2,\ 3)$ along straight-line segments joining each pair of endpoints;
b. along the straight line joining the initial and final points.
c. Is the work the same along the two paths?



72. Find the work done by vector field $\mathbf{F}(x,\ y,\ z) = x\mathbf{i} + 3xy\mathbf{j} - (x + z)\mathbf{k}$ on a particle moving along a line segment that goes from (1, 4, 2) to (0, 5, 1).

73. How much work is required to move an object in vector field $\mathbf{F}(x,\ y) = y\mathbf{i} + 3x\mathbf{j}$ along the upper part of ellipse $\frac{x^2}{4} + y^2 = 1$ from (2, 0) to $(-2,\ 0)$?

74. A vector field is given by $\mathbf{F}(x,\ y) = (2x + 3y)\mathbf{i} + (3x + 2y)\mathbf{j}$. Evaluate the line integral of the field around a circle of unit radius traversed in a clockwise fashion.

75. Evaluate the line integral of scalar function xy along parabolic path $y = x^2$ connecting the origin to point (1, 1).

76. Find $\int_C y^2\,dx + (xy - x^2)dy$ along C: $y = 3x$ from (0, 0) to (1, 3).

77. Find $\int_C y^2\,dx + (xy - x^2)dy$ along C: $y^2 = 9x$ from (0, 0) to (1, 3).

For the following exercises, use a CAS to evaluate the given line integrals.

78. **[T]** Evaluate $\mathbf{F}(x, y, z) = x^2 z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$, where C is represented by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, 1 \le t \le 3$.

79. **[T]** Evaluate line integral $\int_\gamma xe^y \, ds$ where, γ is the arc of curve $x = e^y$ from $(1, 0)$ to $(e, 1)$.

80. **[T]** Evaluate the integral $\int_\gamma xy^2 \, ds$, where γ is a triangle with vertices (0, 1, 2), (1, 0, 3), and $(0, -1, 0)$.

81. **[T]** Evaluate line integral $\int_\gamma (y^2 - xy) dx$, where γ is curve $y = \ln x$ from (1, 0) toward $(e, 1)$.

82. **[T]** Evaluate line integral $\int_\gamma xy^4 \, ds$, where γ is the right half of circle $x^2 + y^2 = 16$.

83. **[T]** Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2 y\mathbf{i} + (x - z)\mathbf{j} + xyz\mathbf{k}$ and C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}, 0 \le t \le 1$.

84. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = 2x \sin(y)\mathbf{i} + \left(x^2 \cos(y) - 3y^2\right)\mathbf{j}$ and C is any path from $(-1, 0)$ to (5, 1).

85. Find the line integral of $\mathbf{F}(x, y, z) = 12x^2\mathbf{i} - 5xy\mathbf{j} + xz\mathbf{k}$ over path C defined by $y = x^2$, $z = x^3$ from point (0, 0, 0) to point (2, 4, 8).

86. Find the line integral of $\int_C \left(1 + x^2 y\right) ds$, where C is ellipse $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 3 \sin t\mathbf{j}$ from $0 \le t \le \pi$.

For the following exercises, find the flux.

87. Compute the flux of $\mathbf{F} = x^2\mathbf{i} + y\mathbf{j}$ across a line segment from (0, 0) to (1, 2).

88. Let $\mathbf{F} = 5\mathbf{i}$ and let C be curve $y = 0, 0 \le x \le 4$. Find the flux across C.

89. Let $\mathbf{F} = 5\mathbf{j}$ and let C be curve $y = 0, 0 \le x \le 4$. Find the flux across C.

90. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ and let C: $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ $(0 \le t \le 2\pi)$. Calculate the flux across C.

91. Let $\mathbf{F} = \left(x^2 + y^3\right)\mathbf{i} + (2xy)\mathbf{j}$. Calculate flux $\mathbf{F}$ orientated counterclockwise across curve C: $x^2 + y^2 = 9$.

92. Find the line integral of $\int_C z^2 \, dx + y dy + 2y dz$, where C consists of two parts: C_1 and C_2. C_1 is the intersection of cylinder $x^2 + y^2 = 16$ and plane $z = 3$ from (0, 4, 3) to $(-4, 0, 3)$. C_2 is a line segment from $(-4, 0, 3)$ to (0, 1, 5).

93. A spring is made of a thin wire twisted into the shape of a circular helix $x = 2 \cos t, y = 2 \sin t, z = t$. Find the mass of two turns of the spring if the wire has constant mass density.

94. A thin wire is bent into the shape of a semicircle of radius a. If the linear mass density at point P is directly proportional to its distance from the line through the endpoints, find the mass of the wire.

95. An object moves in force field $\mathbf{F}(x, y, z) = y^2\mathbf{i} + 2(x + 1)y\mathbf{j}$ counterclockwise from point (2, 0) along elliptical path $x^2 + 4y^2 = 4$ to $(-2, 0)$, and back to point (2, 0) along the x-axis. How much work is done by the force field on the object?

96. Find the work done when an object moves in force field $\mathbf{F}(x, y, z) = 2x\mathbf{i} - (x + z)\mathbf{j} + (y - x)\mathbf{k}$ along the path given by $\mathbf{r}(t) = t^2\mathbf{i} + \left(t^2 - t\right)\mathbf{j} + 3\mathbf{k}$, $0 \le t \le 1$.

97. If an inverse force field $\mathbf{F}$ is given by $\mathbf{F}(x, y, z) = \dfrac{\mathbf{k}}{\| \mathbf{r} \|^3}\mathbf{r}$, where k is a constant, find the work done by $\mathbf{F}$ as its point of application moves along the x-axis from $A(1, 0, 0)$ to $B(2, 0, 0)$.

98. David and Sandra plan to evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path in the xy-plane from (0, 0) to (1, 1). The force field is $\mathbf{F}(x, y) = (x + 2y)\mathbf{i} + (-x + y^2)\mathbf{j}$. David chooses the path that runs along the x-axis from (0, 0) to (1, 0) and then runs along the vertical line $x = 1$ from (1, 0) to the final point (1, 1). Sandra chooses the direct path along the diagonal line $y = x$ from (0, 0) to (1, 1). Whose line integral is larger and by how much?

6.3 | Conservative Vector Fields

Learning Objectives

- **6.3.1** Describe simple and closed curves; define connected and simply connected regions.
- **6.3.2** Explain how to find a potential function for a conservative vector field.
- **6.3.3** Use the Fundamental Theorem for Line Integrals to evaluate a line integral in a vector field.
- **6.3.4** Explain how to test a vector field to determine whether it is conservative.

In this section, we continue the study of conservative vector fields. We examine the Fundamental Theorem for Line Integrals, which is a useful generalization of the Fundamental Theorem of Calculus to line integrals of conservative vector fields. We also discover show how to test whether a given vector field is conservative, and determine how to build a potential function for a vector field known to be conservative.

Curves and Regions

Before continuing our study of conservative vector fields, we need some geometric definitions. The theorems in the subsequent sections all rely on integrating over certain kinds of curves and regions, so we develop the definitions of those curves and regions here.

We first define two special kinds of curves: closed curves and simple curves. As we have learned, a closed curve is one that begins and ends at the same point. A simple curve is one that does not cross itself. A curve that is both closed and simple is a simple closed curve (**Figure 6.25**).

Definition

Curve C is a **closed curve** if there is a parameterization $\mathbf{r}(t),\ a \le t \le b$ of C such that the parameterization traverses the curve exactly once and $\mathbf{r}(a) = \mathbf{r}(b)$. Curve C is a **simple curve** if C does not cross itself. That is, C is simple if there exists a parameterization $\mathbf{r}(t),\ a \le t \le b$ of C such that $\mathbf{r}$ is one-to-one over (a, b). It is possible for $\mathbf{r}(a) = \mathbf{r}(b)$, meaning that the simple curve is also closed.

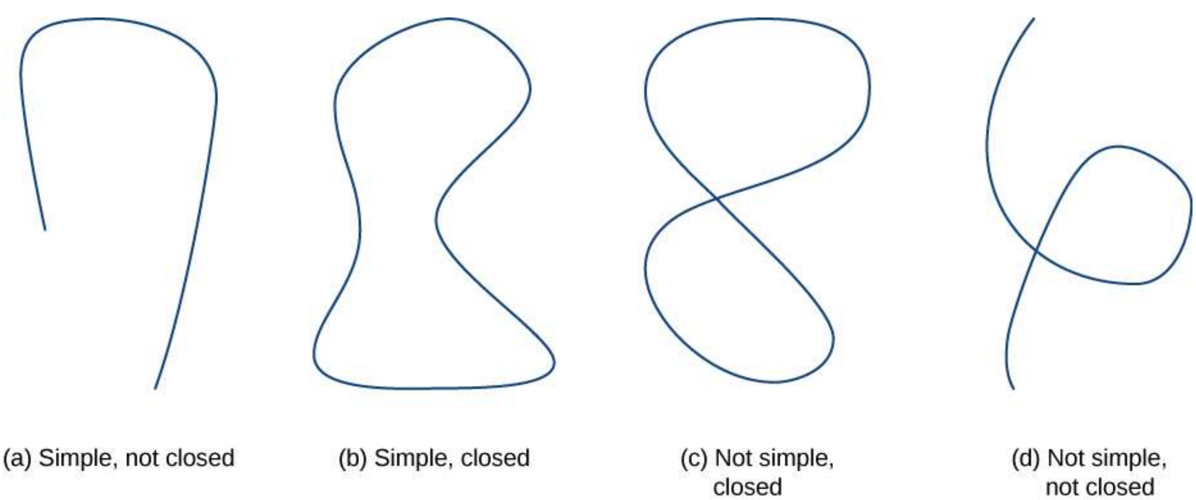


Figure 6.25 Types of curves that are simple or not simple and closed or not closed.

Example 6.27

Determining Whether a Curve Is Simple and Closed

Is the curve with parameterization $\mathbf{r}(t) = \langle \cos t, \frac{\sin(2t)}{2} \rangle, 0 \le t \le 2\pi$ a simple closed curve?

Solution

Note that $\mathbf{r}(0) = \langle 1, 0 \rangle = \mathbf{r}(2\pi);$ therefore, the curve is closed. The curve is not simple, however. To see this, note that $\mathbf{r}\left(\frac{\pi}{2}\right) = \langle 0, 0 \rangle = \mathbf{r}\left(\frac{3\pi}{2}\right),$ and therefore the curve crosses itself at the origin (**Figure 6.26**).

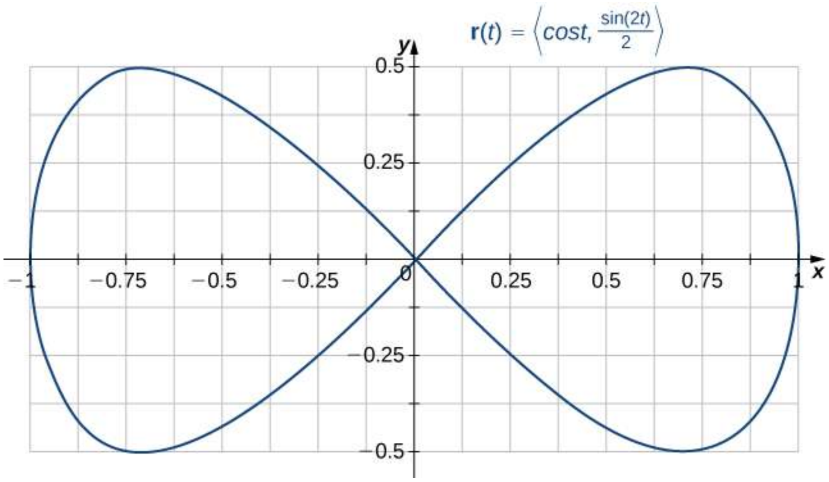


Figure 6.26 A curve that is closed but not simple.

6.24 Is the curve given by parameterization $\mathbf{r}(t) = \langle 2\cos t, 3\sin t \rangle, 0 \le t \le 6\pi,$ a simple closed curve?

Many of the theorems in this chapter relate an integral over a region to an integral over the boundary of the region, where the region's boundary is a simple closed curve or a union of simple closed curves. To develop these theorems, we need two geometric definitions for regions: that of a connected region and that of a simply connected region. A connected region is one in which there is a path in the region that connects any two points that lie within that region. A simply connected region is a connected region that does not have any holes in it. These two notions, along with the notion of a simple closed curve, allow us to state several generalizations of the Fundamental Theorem of Calculus later in the chapter. These two definitions are valid for regions in any number of dimensions, but we are only concerned with regions in two or three dimensions.

Definition

A region D is a **connected region** if, for any two points P_1 and $P_2,$ there is a path from P_1 to P_2 with a trace contained entirely inside D. A region D is a **simply connected region** if D is connected for any simple closed curve C that lies inside D, and curve C can be shrunk continuously to a point while staying entirely inside D. In two dimensions, a region is simply connected if it is connected and has no holes.

All simply connected regions are connected, but not all connected regions are simply connected (**Figure 6.27**).

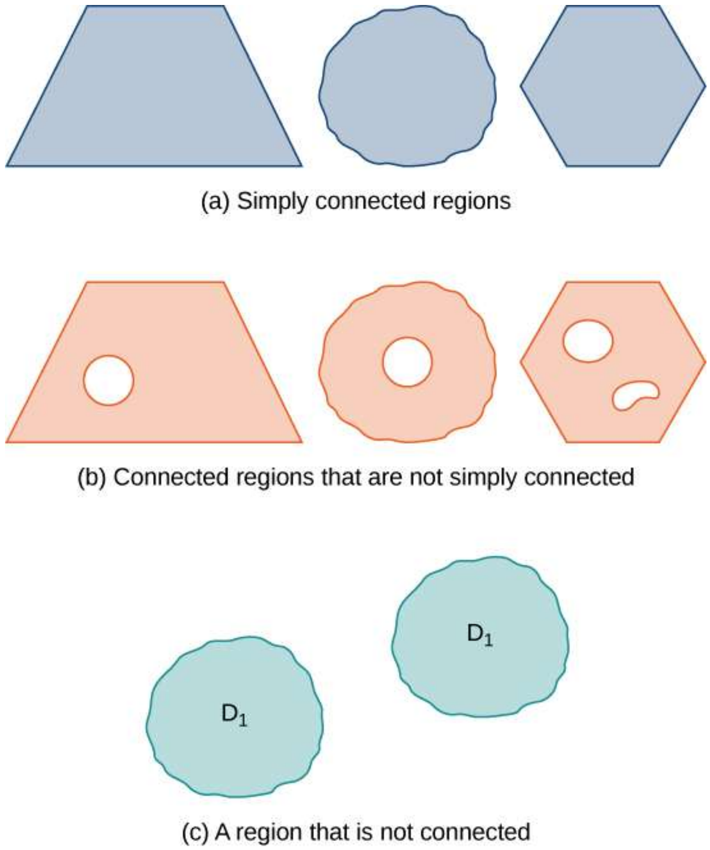
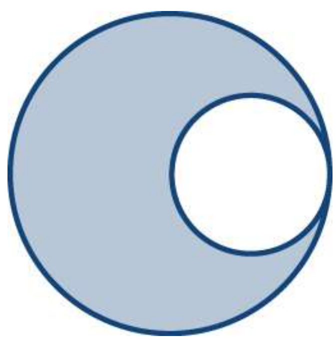


Figure 6.27 Not all connected regions are simply connected. (a) Simply connected regions have no holes. (b) Connected regions that are not simply connected may have holes but you can still find a path in the region between any two points. (c) A region that is not connected has some points that cannot be connected by a path in the region.

6.25 Is the region in the below image connected? Is the region simply connected?



Fundamental Theorem for Line Integrals

Now that we understand some basic curves and regions, let's generalize the Fundamental Theorem of Calculus to line integrals. Recall that the Fundamental Theorem of Calculus says that if a function f has an antiderivative F, then the integral of f from a to b depends only on the values of F at a and at b—that is,

$$\int_a^b f(x)dx = \mathrm{F}(b) - \mathrm{F}(a).$$

If we think of the gradient as a derivative, then the same theorem holds for vector line integrals. We show how this works using a motivational example.

Example 6.28

Evaluating a Line Integral and the Antiderivatives of the Endpoints

Let $\mathbf{F}(x, y) = \langle 2x, 4y \rangle$. Calculate $\int_C \mathbf{F} \bullet d\mathbf{r}$, where C is the line segment from (0,0) to (2,2)(**Figure 6.28**).

Solution

We use **Equation 6.9** to calculate $\int_C \mathbf{F} \bullet d\mathbf{r}$. Curve C can be parameterized by $\mathbf{r}(t) = \langle 2t, 2t \rangle, \ 0 \le t \le 1$. Then, $\mathbf{F}(\mathbf{r}(t)) = \langle 4t, 8t \rangle$ and $\mathbf{r}'(t) = \langle 2, 2 \rangle$, which implies that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle 4t, 8t \rangle \cdot \langle 2, 2 \rangle \, dt \\ &= \int_0^1 (8t + 16t)dt = \int_0^1 24t \, dt \\ &= \left[12t^2\right]_0^1 = 12.\end{aligned}$$

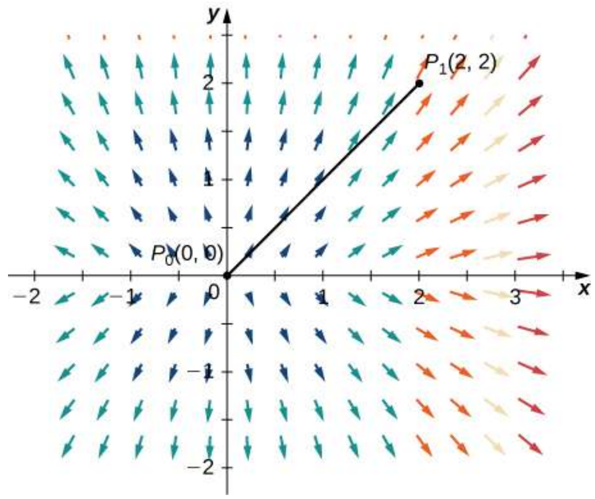


Figure 6.28 The value of line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ depends only on the value of the potential function of **F** at the endpoints of the curve.

Notice that $F = \nabla f$, where $f(x, y) = x^2 + 2y^2$. If we think of the gradient as a derivative, then f is an "antiderivative" of **F**. In the case of single-variable integrals, the integral of derivative $g'(x)$ is $g(b) - g(a)$, where a is the start point of the interval of integration and b is the endpoint. If vector line integrals work like single-variable integrals, then we would expect integral **F** to be $f(P_1) - f(P_0)$, where P_1 is the endpoint of the curve of integration and P_0 is the start point. Notice that this is the case for this example:

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla f \bullet d\mathbf{r} = 12$$

and

$$f(2, 2) - f(0, 0) = 4 + 8 - 0 = 12.$$

In other words, the integral of a “derivative” can be calculated by evaluating an “antiderivative” at the endpoints of the curve and subtracting, just as for single-variable integrals.

The following theorem says that, under certain conditions, what happened in the previous example holds for any gradient field. The same theorem holds for vector line integrals, which we call the **Fundamental Theorem for Line Integrals.**

Theorem 6.7: The Fundamental Theorem for Line Integrals

Let C be a piecewise smooth curve with parameterization $\mathbf{r}(t)$, $a \le t \le b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,

$$\int_C \nabla f \bullet d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{6.12}$$

Proof

By **Equation 6.9**,

$$\int_C \nabla f \bullet d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt.$$

By the chain rule,

$$\frac{d}{dt}(f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t).$$

Therefore, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_C \nabla f \bullet d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt \\ &= \int_a^b \frac{d}{dt}(f(\mathbf{r}(t))dt \\ &= [f(\mathbf{r}(t))]_{t=a}^{t=b} \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)).\end{aligned}$$

□

We know that if $\mathbf{F}$ is a conservative vector field, there are potential functions f such that $\nabla f = \mathbf{F}$. Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. In other words, just as with the Fundamental Theorem of Calculus, computing the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is conservative, is a two-step process: (1) find a potential function (“antiderivative”) f for $\mathbf{F}$ and (2) compute the value of f at the endpoints of C and calculate their difference $f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. Keep in mind, however, there is one major difference between the Fundamental Theorem of Calculus and the Fundamental Theorem for Line Integrals. *A function of one variable that is continuous must have an antiderivative. However, a vector field, even if it is continuous, does not need to have a potential function.*

Example 6.29

Applying the Fundamental Theorem

Calculate integral $\int_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle 2x \ln y, \frac{x^2}{y} + z^2, 2yz \rangle$ and C is a curve with parameterization $\mathbf{r}(t) = \langle t^2, t, t \rangle$, $1 \le t \le e$

a. without using the Fundamental Theorem of Line Integrals and
b. using the Fundamental Theorem of Line Integrals.

Solution

a. First, let's calculate the integral without the Fundamental Theorem for Line Integrals and instead use **Equation 6.9**:

$$\begin{aligned}
\int_C \mathbf{F} \bullet dr &= \int_1^e \mathbf{F}(r(t)) \bullet r'(t)dt \\
&= \int_1^e \langle 2t^2 \ln t, \frac{t^4}{t} + t^2, 2t^2 \rangle \bullet \langle 2t, 1, 1 \rangle dt \\
&= \int_1^e \left(4t^3 \ln t + t^3 + 3t^2\right)dt \\
&= \int_1^e 4t^3 \ln t dt + \int_1^e \left(t^3 + 3t^2\right)dt \\
&= \int_1^e 4t^3 \ln t dt + \left[\frac{t^4}{4} + t^3\right]_1^e \\
&= r\int_1^e t^3 \ln t dt + \frac{e^4}{4} + e^3 - \frac{5}{4}.
\end{aligned}$$

Integral $\int_1^e t^3 \ln t dt$ requires integration by parts. Let $u = \ln t$ and $dv = t^3$. Then $u = \ln t, dv = t^3$ and

$$du = \frac{1}{t}dt, \ v = \frac{t^4}{4}.$$

Therefore,

$$\begin{aligned}
\int_1^e t^3 \ln t dt &= \left[\frac{t^4}{4}\ln t\right]_1^e - \frac{1}{4}\int_1^e t^3\, dt \\
&= \frac{e^4}{4} - \frac{1}{r}\left(\frac{e^4}{4} - \frac{1}{4}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_C \mathbf{F} \bullet dr &= 4\int_1^e t^3 \ln t dt + \frac{e^4}{4} + e^3 - \frac{5}{4} \\
&= 4\left(\frac{e^4}{4} - \frac{1}{4}\left(\frac{e^4}{4} - \frac{1}{4}\right)\right) + \frac{e^4}{4} + e^3 - \frac{5}{4} \\
&= e^4 - \frac{e^4}{4} + \frac{1}{4} + \frac{e^4}{4} + e^3 - \frac{5}{4} \\
&= e^4 + e^3 - 1.
\end{aligned}$$

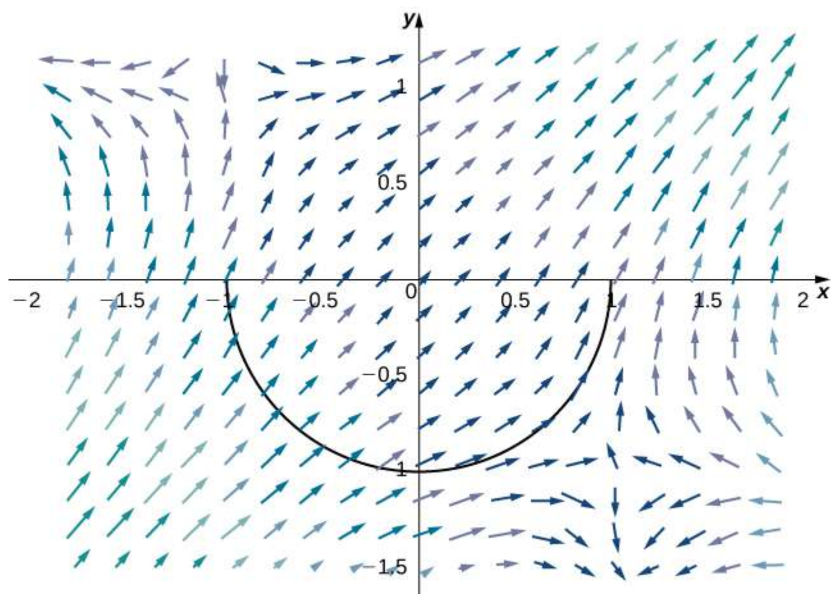
b. Given that $f(x, y, z) = x^2 \ln y + yz^2$ is a potential function for $\mathbf{F}$, let's use the Fundamental Theorem for Line Integrals to calculate the integral. Note that

$$\begin{aligned}\int_C \mathbf{F} \bullet d\mathbf{r} &= \int_C \nabla f \bullet d\mathbf{r} \\ &= f(\mathbf{r}(e)) - f(\mathbf{r}(1)) \\ &= f\left(e^2, e, e\right) - f(1, 1, 1) \\ &= e^4 + e^3 - 1.\end{aligned}$$

This calculation is much more straightforward than the calculation we did in (a). As long as we have a potential function, calculating a line integral using the Fundamental Theorem for Line Integrals is much easier than calculating without the theorem.

Example 6.29 illustrates a nice feature of the Fundamental Theorem of Line Integrals: it allows us to calculate more easily many vector line integrals. As long as we have a potential function, calculating the line integral is only a matter of evaluating the potential function at the endpoints and subtracting.

6.26 Given that $f(x, y) = (x-1)^2 y + (y+1)^2 x$ is a potential function for $\mathbf{F} = \langle 2xy - 2y + (y+1)^2, (x-1)^2 + 2yx + 2x \rangle$, calculate integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the lower half of the unit circle oriented counterclockwise.



The Fundamental Theorem for Line Integrals has two important consequences. The first consequence is that if **F** is conservative and C is a closed curve, then the circulation of **F** along C is zero—that is, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$ To see why this is true, let f be a potential function for **F**. Since C is a closed curve, the terminal point **r**(b) of C is the same as the initial point **r**(a) of C—that is, $\mathbf{r}(a) = \mathbf{r}(b).$ Therefore, by the Fundamental Theorem for Line Integrals,

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C \nabla f \cdot d\mathbf{r} \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(b)) \\ &= 0.\end{aligned}$$

Recall that the reason a conservative vector field **F** is called "conservative" is because such vector fields model forces in which energy is conserved. We have shown gravity to be an example of such a force. If we think of vector field **F** in integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ as a gravitational field, then the equation $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ follows. If a particle travels along a path that starts and ends at the same place, then the work done by gravity on the particle is zero.

The second important consequence of the Fundamental Theorem for Line Integrals is that line integrals of conservative vector fields are independent of path—meaning, they depend only on the endpoints of the given curve, and do not depend on the path between the endpoints.

Definition

Let **F** be a vector field with domain D. The vector field **F** is **independent of path** (or **path independent**) if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any paths C_1 and C_2 in D with the same initial and terminal points.

The second consequence is stated formally in the following theorem.

Theorem 6.8: Path Independence of Conservative Fields

If **F** is a conservative vector field, then **F** is independent of path.

Proof

Let D denote the domain of **F** and let C_1 and C_2 be two paths in D with the same initial and terminal points (**Figure 6.29**). Call the initial point P_1 and the terminal point P_2. Since **F** is conservative, there is a potential function f for **F**. By the Fundamental Theorem for Line Integrals,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = f(P_2) - f(P_1) = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Therefore, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ and **F** is independent of path.

□

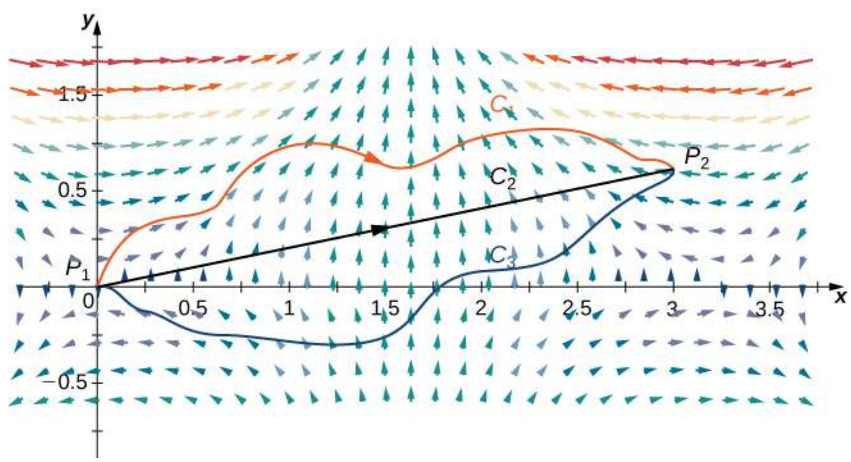


Figure 6.29 The vector field is conservative, and therefore independent of path.

To visualize what independence of path means, imagine three hikers climbing from base camp to the top of a mountain. Hiker 1 takes a steep route directly from camp to the top. Hiker 2 takes a winding route that is not steep from camp to the top. Hiker 3 starts by taking the steep route but halfway to the top decides it is too difficult for him. Therefore he returns to camp and takes the non-steep path to the top. All three hikers are traveling along paths in a gravitational field. Since gravity is a force in which energy is conserved, the gravitational field is conservative. By independence of path, the total amount of work done by gravity on each of the hikers is the same because they all started in the same place and ended in the same place. The work done by the hikers includes other factors such as friction and muscle movement, so the total amount of energy each one expended is not the same, but the net energy expended against gravity is the same for all three hikers.

We have shown that if **F** is conservative, then **F** is independent of path. It turns out that if the domain of **F** is open and connected, then the converse is also true. That is, if **F** is independent of path and the domain of **F** is open and connected, then **F** is conservative. Therefore, the set of conservative vector fields on open and connected domains is precisely the set of vector fields independent of path.

Theorem 6.9: The Path Independence Test for Conservative Fields

If **F** is a continuous vector field that is independent of path and the domain *D* of **F** is open and connected, then **F** is conservative.

Proof

We prove the theorem for vector fields in $\mathbb{R}^2$. The proof for vector fields in $\mathbb{R}^3$ is similar. To show that $\mathbf{F} = \langle P, Q \rangle$ is conservative, we must find a potential function f for **F**. To that end, let *X* be a fixed point in *D*. For any point (x, y) in *D*, let *C* be a path from *X* to (x, y). Define $f(x, y)$ by $f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r}$. (Note that this definition of f makes sense only because **F** is independent of path. If **F** was not independent of path, then it might be possible to find another path C' from *X* to (x, y) such that $\int_C \mathbf{F} \cdot d\mathbf{r} \neq \int_{C'} \mathbf{F} \cdot d\mathbf{r}$, and in such a case $f(x, y)$ would not be a function.) We want to show that f has the property $\nabla f = \mathbf{F}$.

Since domain *D* is open, it is possible to find a disk centered at (x, y) such that the disk is contained entirely inside *D*. Let (a, y) with $a < x$ be a point in that disk. Let *C* be a path from *X* to (x, y) that consists of two pieces: C_1 and C_2. The first piece, C_1, is any path from *C* to (a, y) that stays inside *D*; C_2 is the horizontal line segment from (a, y) to (x, y) (**Figure 6.30**). Then

$$f(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

The first integral does not depend on *x*, so

$$f_x = \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \bullet d\mathbf{r}.$$

If we parameterize C_2 by $\mathbf{r}(t) = \langle t, y \rangle,\ a \le t \le x,$ then

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} \int_{C^2} \mathbf{F} \bullet d\mathbf{r} \\ &= \frac{\partial}{\partial x} \int_a^x \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t) dt \\ &= \frac{\partial}{\partial x} \int_a^x \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d}{dt}(\langle t, y \rangle) dt \\ &= \frac{\partial}{\partial x} \int_a^x \mathbf{F}(r(t)) \bullet \langle 1, 0 \rangle\, dt \\ &= \frac{\partial}{\partial x} \int_a^x P(t, y) dt. \end{aligned}$$

By the Fundamental Theorem of Calculus (part 1),

$$f_x = \frac{\partial}{\partial x} \int_a^x P(t, y) dt = P(x, y).$$

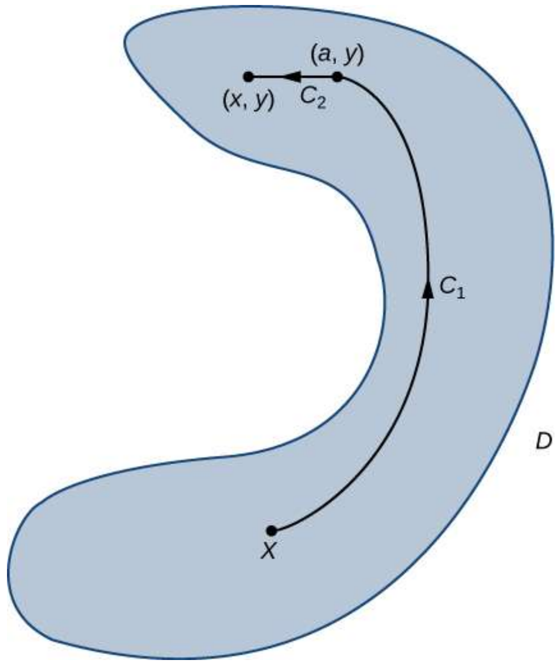


Figure 6.30 Here, C_1 is any path from C to $(a,\ y)$ that stays inside D, and C_2 is the horizontal line segment from $(a,\ y)$ to $(x,\ y)$.

A similar argument using a vertical line segment rather than a horizontal line segment shows that $f_y = Q(x,\ y)$.

Therefore $\nabla f = \mathbf{F}$ and **F** is conservative.

□

We have spent a lot of time discussing and proving **Path Independence of Conservative Fields** and **The Path Independence Test for Conservative Fields**, but we can summarize them simply: a vector field **F** on an open and connected domain is conservative if and only if it is independent of path. This is important to know because conservative vector fields are extremely important in applications, and these theorems give us a different way of viewing what it means to be conservative using path independence.

Example 6.30

Showing That a Vector Field Is Not Conservative

Use path independence to show that vector field $\mathbf{F}(x,\ y) = \langle x^2 y,\ y + 5 \rangle$ is not conservative.

Solution

We can indicate that **F** is not conservative by showing that **F** is not path independent. We do so by giving two different paths, C_1 and C_2, that both start at $(0,\ 0)$ and end at $(1,\ 1)$, and yet $\int_{C_1} \mathbf{F} \cdot dr \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

Let C_1 be the curve with parameterization $r_1(t) = \langle t,\ t \rangle,\ 0 \le t \le 1$ and let C_2 be the curve with parameterization $r_2(t) = \langle t,\ t^2 \rangle,\ 0 \le t \le 1$ (**Figure 6.31**). Then

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot dr &= \int_0^1 \mathbf{F}(r_1(t)) \cdot r_1{}'(t)dt \\ &= \int_0^1 \langle t^3, t+5 \rangle \cdot \langle 1, 1 \rangle\, dt = \int_0^1 \left(t^3 + t + 5\right)dt \\ &= \left[\frac{t^4}{4} + \frac{t^2}{2} + 5t\right]_0^1 = \frac{23}{4}\end{aligned}$$

and

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot dr &= \int_0^1 \mathbf{F}(r_2(t)) \cdot r_2{}'(t)dt \\ &= \int_0^1 \langle t^4, t^2+5 \rangle \cdot \langle 1, 2t \rangle\, dt = \int_0^1 \left(t^4 + 2t^3 + 10t\right)dt \\ &= \left[\frac{t^5}{5} + \frac{t^4}{2} + 5t^2\right]_0^1 = \frac{57}{10}.\end{aligned}$$

Since $\int_{C_1} \mathbf{F} \bullet d\mathbf{r} \neq \int_{C_2} \mathbf{F} \bullet d\mathbf{r}$, the value of a line integral of **F** depends on the path between two given points. Therefore, **F** is not independent of path, and **F** is not conservative.

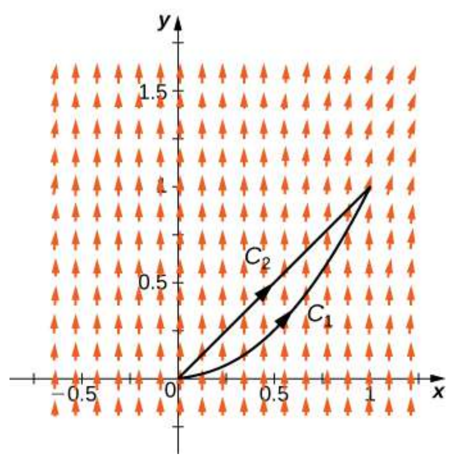


Figure 6.31 Curves C_1 and C_2 are both oriented from left to right.

6.27 Show that $\mathbf{F}(x, y) = \langle xy, x^2y^2 \rangle$ is not path independent by considering the line segment from $(0, 0)$ to $(0, 2)$ and the piece of the graph of $y = \frac{x^2}{2}$ that goes from $(0, 0)$ to $(0, 2)$.

Conservative Vector Fields and Potential Functions

As we have learned, the Fundamental Theorem for Line Integrals says that if **F** is conservative, then calculating $\int_C \mathbf{F} \cdot dr$ has two steps: first, find a potential function f for **F** and, second, calculate $f(P_1) - f(P_0)$, where P_1 is the endpoint of C and P_0 is the starting point. To use this theorem for a conservative field **F**, we must be able to find a potential function f for **F**. Therefore, we must answer the following question: Given a conservative vector field **F**, how do we find a function

f such that $\nabla f = \mathbf{F}$? Before giving a general method for finding a potential function, let's motivate the method with an example.

Example 6.31

Finding a Potential Function

Find a potential function for $\mathbf{F}(x, y) = \langle 2xy^3, 3x^2y^2 + \cos(y) \rangle$, thereby showing that $\mathbf{F}$ is conservative.

Solution

Suppose that $f(x, y)$ is a potential function for $\mathbf{F}$. Then, $\nabla f = \mathbf{F}$, and therefore

$$f_x = 2xy^3 \text{ and } f_y = 3x^2y^2 + \cos y.$$

Integrating the equation $f_x = 2xy^3$ with respect to x yields the equation

$$f(x, y) = x^2y^3 + h(y).$$

Notice that since we are integrating a two-variable function with respect to x, we must add a constant of integration that is a constant with respect to x, but may still be a function of y. The equation $f(x, y) = x^2y^3 + h(y)$ can be confirmed by taking the partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2y^3\right) + \frac{\partial}{\partial x}(h(y)) = 2xy^3 + 0 = 2xy^3.$$

Since f is a potential function for $\mathbf{F}$,

$$f_y = 3x^2y^2 + \cos(y),$$

and therefore

$$3x^2y^2 + g'(y) = 3x^2y^2 + \cos(y).$$

This implies that $h'(y) = \cos y$, so $h(y) = \sin y + C$. Therefore, *any* function of the form $f(x, y) = x^2y^3 + \sin(y) + C$ is a potential function. Taking, in particular, $C = 0$ gives the potential function $f(x, y) = x^2y^3 + \sin(y)$.

To verify that f is a potential function, note that $\nabla f = \langle 2xy^3, 3x^2y^2 + \cos y \rangle = \mathbf{F}$.

6.28 Find a potential function for $\mathbf{F}(x, y) = \langle e^x y^3 + y, 3e^x y^2 + x \rangle$.

The logic of the previous example extends to finding the potential function for any conservative vector field in $\mathbb{R}^2$. Thus, we have the following problem-solving strategy for finding potential functions:

Problem-Solving Stragegy: Finding a Potential Function for a Conservative Vector Field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$

1. Integrate P with respect to x. This results in a function of the form $g(x, y) + h(y)$, where $h(y)$ is unknown.

2. Take the partial derivative of $g(x, y) + h(y)$ with respect to y, which results in the function $g_y(x, y) + h'(y)$.
3. Use the equation $g_y(x, y) + h'(y) = Q(x, y)$ to find $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$.
5. Any function of the form $f(x, y) = g(x, y) + h(y) + C$, where C is a constant, is a potential function for $\mathbf{F}$.

We can adapt this strategy to find potential functions for vector fields in $\mathbb{R}^3$, as shown in the next example.

Example 6.32

Finding a Potential Function in $\mathbb{R}^3$

Find a potential function for $\mathbf{F}(x, y) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z \rangle$, thereby showing that $\mathbf{F}$ is conservative.

Solution

Suppose that f is a potential function. Then, $\nabla f = \mathbf{F}$ and therefore $f_x = 2xy$. Integrating this equation with respect to x yields the equation $f(x, y, z) = x^2y + g(y, z)$ for some function g. Notice that, in this case, the constant of integration with respect to x is a function of y and z.

Since f is a potential function,

$$x^2 + 2yz^3 = f_y = x^2 + g_y.$$

Therefore,

$$g_y = 2yz^3.$$

Integrating this function with respect to y yields

$$g(y, z) = y^2z^3 + h(z)$$

for some function $h(z)$ of z alone. (Notice that, because we know that g is a function of only y and z, we do not need to write $g(y, z) = y^2z^3 + h(x, z)$.) Therefore,

$$f(x, y, z) = x^2y + g(y, z) = x^2y + y^2z^3 + h(z).$$

To find f, we now must only find h. Since f is a potential function,

$$3y^2z^2 + 2z = g_z = 3y^2z^2 + h'(z).$$

This implies that $h'(z) = 2z$, so $h(z) = z^2 + C$. Letting $C = 0$ gives the potential function

$$f(x, y, z) = x^2y + y^2z^3 + z^2.$$

To verify that f is a potential function, note that $\nabla f = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z \rangle = \mathbf{F}$.

6.29 Find a potential function for $\mathbf{F}(x, y, z) = \langle 12x^2, \cos y \cos z, 1 - \sin y \sin z \rangle$.

We can apply the process of finding a potential function to a gravitational force. Recall that, if an object has unit mass and is located at the origin, then the gravitational force in $\mathbb{R}^2$ that the object exerts on another object of unit mass at the point (x, y) is given by vector field

$$\mathbf{F}(x, y) = -G \left\langle \frac{x}{\left(x^2 + y^2\right)^{3/2}}, \frac{y}{\left(x^2 + y^2\right)^{3/2}} \right\rangle,$$

where G is the universal gravitational constant. In the next example, we build a potential function for $\mathbf{F}$, thus confirming what we already know: that gravity is conservative.

Example 6.33

Finding a Potential Function

Find a potential function f for $\mathbf{F}(x, y) = -G \left\langle \frac{x}{\left(x^2 + y^2\right)^{3/2}}, \frac{y}{\left(x^2 + y^2\right)^{3/2}} \right\rangle$.

Solution

Suppose that f is a potential function. Then, $\nabla f = \mathbf{F}$ and therefore

$$f_x = \frac{-Gx}{\left(x^2 + y^2\right)^{3/2}}.$$

To integrate this function with respect to *x*, we can use *u*-substitution. If $u = x^2 + y^2$, then $\frac{du}{2} = x dx$, so

$$\begin{aligned} \int \frac{-Gx}{\left(x^2 + y^2\right)^{3/2}} dx &= \int \frac{-G}{2u^{3/2}} du \\ &= \frac{G}{\sqrt{u}} + h(y) \\ &= \frac{G}{\sqrt{x^2 + y^2}} + h(y) \end{aligned}$$

for some function $h(y)$. Therefore,

$$f(x, y) = \frac{G}{\sqrt{x^2 + y^2}} + h(y).$$

Since f is a potential function for $\mathbf{F}$,

$$f_y = \frac{-Gy}{\left(x^2 + y^2\right)^{3/2}}.$$

Since $f(x, y) = \frac{G}{\sqrt{x^2 + y^2}} + h(y)$, f_y also equals $\frac{-Gy}{\left(x^2 + y^2\right)^{3/2}} + h'(y)$.

Therefore,

$$\frac{-Gy}{\left(x^2 + y^2\right)^{3/2}} + h'(y) = \frac{-Gy}{\left(x^2 + y^2\right)^{3/2}},$$

which implies that $h'(y) = 0$. Thus, we can take $h(y)$ to be any constant; in particular, we can let $h(y) = 0$.

The function

$$f(x,\ y) = \frac{G}{\sqrt{x^2 + y^2}}$$

is a potential function for the gravitational field **F**. To confirm that f is a potential function, note that

$$\begin{aligned}\nabla f &= \left\langle -\frac{1}{2}\frac{G}{\left(x^2+y^2\right)^{3/2}}(2x),\ -\frac{1}{2}\frac{G}{\left(x^2+y^2\right)^{3/2}}(2y) \right\rangle \\ &= \left\langle \frac{-Gx}{\left(x^2+y^2\right)^{3/2}}, \frac{-Gy}{\left(x^2+y^2\right)^{3/2}} \right\rangle \\ &= \mathbf{F}.\end{aligned}$$

6.30 Find a potential function f for the three-dimensional gravitational force $\mathbf{F}(x,\ y,\ z) = \left\langle \frac{-Gx}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{-Gy}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{-Gz}{\left(x^2+y^2+z^2\right)^{3/2}} \right\rangle$.

Testing a Vector Field

Until now, we have worked with vector fields that we know are conservative, but if we are not told that a vector field is conservative, we need to be able to test whether it is conservative. Recall that, if **F** is conservative, then **F** has the cross-partial property (see **The Cross-Partial Property of Conservative Vector Fields**). That is, if $\mathbf{F} = \langle P, Q, R \rangle$ is conservative, then $P_y = Q_x, P_z = R_x,$ and $Q_z = R_y$. So, if **F** has the cross-partial property, then is **F** conservative? If the domain of **F** is open and simply connected, then the answer is yes.

Theorem 6.10: The Cross-Partial Test for Conservative Fields

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x, P_z = R_x,$ and $Q_z = R_y$ throughout D, then **F** is conservative.

Although a proof of this theorem is beyond the scope of the text, we can discover its power with some examples. Later, we see why it is necessary for the region to be simply connected.

Combining this theorem with the cross-partial property, we can determine whether a given vector field is conservative:

Theorem 6.11: Cross-Partial Property of Conservative Fields

Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field on an open, simply connected region D. Then $P_y = Q_x, P_z = R_x,$ and $Q_z = R_y$ throughout D if and only if **F** is conservative.

The version of this theorem in $\mathbb{R}^2$ is also true. If $\mathbf{F} = \langle P, Q \rangle$ is a vector field on an open, simply connected domain in $\mathbb{R}^2$, then **F** is conservative if and only if $P_y = Q_x$.

Example 6.34

Determining Whether a Vector Field Is Conservative

Determine whether vector field $\mathbf{F}(x, y, z) = \langle xy^2 z, x^2 yz, z^2 \rangle$ is conservative.

Solution

Note that the domain of **F** is all of $\mathbb{R}^2$ and $\mathbb{R}^3$ is simply connected. Therefore, we can use **Cross-Partial Property of Conservative Fields** to determine whether **F** is conservative. Let

$$P(x, y, z) = xy^2 z, \; Q(x, y, z) = x^2 yz, \text{ and } R(x, y, z) = z^2.$$

Since $Q_z = x^2 y$ and $R_y = 0,$ the vector field is not conservative.

Example 6.35

Determining Whether a Vector Field Is Conservative

Determine vector field $\mathbf{F}(x, y) = \langle x\ln(y), \frac{x^2}{2y} \rangle$ is conservative.

Solution

Note that the domain of **F** is the part of $\mathbb{R}^2$ in which $y > 0.$ Thus, the domain of **F** is part of a plane above the x-axis, and this domain is simply connected (there are no holes in this region and this region is connected). Therefore, we can use **Cross-Partial Property of Conservative Fields** to determine whether **F** is conservative. Let

$$P(x, y) = x\ln(y) \text{ and } Q(x, y) = \frac{x^2}{2y}.$$

Then $P_y = \frac{x}{y} = Q_x$ and thus **F** is conservative.

6.31 Determine whether $\mathbf{F}(x, y) = \langle \sin x \cos y, \cos x \sin y \rangle$ is conservative.

When using **Cross-Partial Property of Conservative Fields**, it is important to remember that a theorem is a tool, and like any tool, it can be applied only under the right conditions. In the case of **Cross-Partial Property of Conservative Fields**, the theorem can be applied only if the domain of the vector field is simply connected.

To see what can go wrong when misapplying the theorem, consider the vector field from **Example 6.30**:

$$\mathbf{F}(x, y) = \frac{y}{x^2 + y^2}\mathbf{i} + \frac{-x}{x^2 + y^2}\mathbf{j}.$$

This vector field satisfies the cross-partial property, since

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)=\frac{\left(x^2+y^2\right)-y(2y)}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

and

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)=\frac{-\left(x^2+y^2\right)+x(2x)}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}.$$

Since **F** satisfies the cross-partial property, we might be tempted to conclude that **F** is conservative. However, **F** is not conservative. To see this, let

$$\mathbf{r}(t)=\langle \cos t, \sin t\rangle,\ 0\le t\le\pi$$

be a parameterization of the upper half of a unit circle oriented counterclockwise (denote this C_1) and let

$$\mathrm{s}(t)=\langle \cos t, -\sin t\rangle,\ 0\le t\le\pi$$

be a parameterization of the lower half of a unit circle oriented clockwise (denote this C_2). Notice that C_1 and C_2 have the same starting point and endpoint. Since $\sin^2 t+\cos^2 t=1$,

$$\mathbf{F}(\mathbf{r}(t))\bullet\mathbf{r}'(t)=\langle \sin(t), -\cos(t)\rangle\bullet\langle -\sin(t), \cos(t)\rangle=-1$$

and

$$\begin{aligned}\mathbf{F}(s(t))\cdot\mathbf{s}'(t)&=\langle -\sin t, -\cos t\rangle\cdot\langle -\sin t, -\cos t\rangle\\&=\sin^2 t+\cos^2 t\\&=1.\end{aligned}$$

Therefore,

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r}=\int_0^{\pi}-1dt=-\pi \text{ and } \int_{C_2}\mathbf{F}\cdot dr=\int_0^{\pi}1dt=\pi.$$

Thus, C_1 and C_2 have the same starting point and endpoint, but $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}\neq\int_{C_2}\mathbf{F}\cdot d\mathbf{r}$. Therefore, **F** is not independent of path and **F** is not conservative.

To summarize: **F** satisfies the cross-partial property and yet **F** is not conservative. What went wrong? Does this contradict **Cross-Partial Property of Conservative Fields**? The issue is that the domain of **F** is all of $\mathbb{R}^2$ except for the origin. In other words, the domain of **F** has a hole at the origin, and therefore the domain is not simply connected. Since the domain is not simply connected, **Cross-Partial Property of Conservative Fields** does not apply to **F**.

We close this section by looking at an example of the usefulness of the Fundamental Theorem for Line Integrals. Now that we can test whether a vector field is conservative, we can always decide whether the Fundamental Theorem for Line Integrals can be used to calculate a vector line integral. If we are asked to calculate an integral of the form $\int_C\mathbf{F}\cdot d\mathbf{r}$, then our first question should be: Is **F** conservative? If the answer is yes, then we should find a potential function and use the Fundamental Theorem for Line Integrals to calculate the integral. If the answer is no, then the Fundamental Theorem for Line Integrals can't help us and we have to use other methods, such as using **Equation 6.9**.

Example 6.36

Using the Fundamental Theorem for Line Integrals

Calculate line integral $\int_C\mathbf{F}\cdot dr$, where $\mathbf{F}(x, y, z)=\langle 2xe^y z+e^x z, x^2 e^y z, x^2 e^y+e^x\rangle$ and C is any smooth curve that goes from the origin to $(1, 1, 1)$.

Solution

Before trying to compute the integral, we need to determine whether **F** is conservative and whether the domain of **F** is simply connected. The domain of **F** is all of $\mathbb{R}^3$, which is connected and has no holes. Therefore, the domain of **F** is simply connected. Let

$$P(x, y, z) = 2xe^y z + e^x z,\ Q(x, y, z) = x^2 e^y z,\ \text{and } R(x, y, z) = x^2 e^y + e^x$$

so that $\mathbf{F} = \langle P, Q, R \rangle$. Since the domain of **F** is simply connected, we can check the cross partials to determine whether **F** is conservative. Note that

$$\begin{aligned} P_y &= 2xe^y z = Q_x \\ P_z &= 2xe^y + e^x = R_x \\ Q_z &= x^2 e^y = R_y. \end{aligned}$$

Therefore, **F** is conservative.

To evaluate $\int_C \mathbf{F} \cdot dr$ using the Fundamental Theorem for Line Integrals, we need to find a potential function f for **F**. Let f be a potential function for **F**. Then, $\nabla f = \mathbf{F}$, and therefore $f_x = 2xe^y z + e^x z$. Integrating this equation with respect to x gives $f(x, y, z) = x^2 e^y z + e^x z + h(y, z)$ for some function h. Differentiating this equation with respect to y gives $x^2 e^y z + h_y = Q = x^2 e^y z$, which implies that $h_y = 0$. Therefore, h is a function of z only, and $f(x, y, z) = x^2 e^y z + e^x z + h(z)$. To find h, note that $f_z = x^2 e^y + e^x + h'(z) = R = x^2 e^y + e^x$. Therefore, $h'(z) = 0$ and we can take $h(z) = 0$. A potential function for **F** is $f(x, y, z) = x^2 e^y z + e^x z$.

Now that we have a potential function, we can use the Fundamental Theorem for Line Integrals to evaluate the integral. By the theorem,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} \\ &= f(1, 1, 1) - f(0, 0, 0) \\ &= 2e. \end{aligned}$$

Analysis

Notice that if we hadn't recognized that **F** is conservative, we would have had to parameterize C and use **Equation 6.9**. Since curve C is unknown, using the Fundamental Theorem for Line Integrals is much simpler.

6.32 Calculate integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle \sin x \sin y, 5 - \cos x \cos y \rangle$ and C is a semicircle with starting point $(0, \pi)$ and endpoint $(0, -\pi)$.

Example 6.37

Work Done on a Particle

Let $\mathbf{F}(x, y) = \langle 2xy^2, 2x^2 y \rangle$ be a force field. Suppose that a particle begins its motion at the origin and ends its movement at any point in a plane that is not on the x-axis or the y-axis. Furthermore, the particle's motion can be modeled with a smooth parameterization. Show that **F** does positive work on the particle.

Solution

We show that **F** does positive work on the particle by showing that **F** is conservative and then by using the Fundamental Theorem for Line Integrals.

To show that **F** is conservative, suppose $f(x, y)$ were a potential function for **F**. Then, $\nabla f = \mathbf{F} = \langle 2xy^2, 2x^2 y \rangle$ and therefore $f_x = 2xy^2$ and $f_y = 2x^2 y$. Equation $f_x = 2xy^2$ implies that $f(x, y) = x^2 y^2 + h(y)$. Deriving both sides with respect to y yields $f_y = 2x^2 y + h'(y)$. Therefore, $h'(y) = 0$ and we can take $h(y) = 0$.

If $f(x, y) = x^2 y^2$, then note that $\nabla f = \langle 2xy^2, 2x^2 y \rangle = \mathbf{F}$, and therefore f is a potential function for **F**.

Let (a, b) be the point at which the particle stops is motion, and let C denote the curve that models the particle's motion. The work done by **F** on the particle is $\int_C \mathbf{F} \cdot d\mathbf{r}$. By the Fundamental Theorem for Line Integrals,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} \\ &= f(a, b) - f(0, 0) \\ &= a^2 b^2.\end{aligned}$$

Since $a \neq 0$ and $b \neq 0$, by assumption, $a^2 b^2 > 0$. Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$, and **F** does positive work on the particle.

Analysis

Notice that this problem would be much more difficult without using the Fundamental Theorem for Line Integrals. To apply the tools we have learned, we would need to give a curve parameterization and use **Equation 6.9**. Since the path of motion C can be as exotic as we wish (as long as it is smooth), it can be very difficult to parameterize the motion of the particle.

6.33 Let $\mathbf{F}(x, y) = \langle 4x^3 y^4, 4x^4 y^3 \rangle$, and suppose that a particle moves from point $(4, 4)$ to $(1, 1)$ along any smooth curve. Is the work done by **F** on the particle positive, negative, or zero?

6.3 EXERCISES

99. *True* or *False?* If vector field $\mathbf{F}$ is conservative on the open and connected region D, then line integrals of $\mathbf{F}$ are path independent on D, regardless of the shape of D.

100. *True* or *False?* Function $\mathbf{r}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, where $0 \le t \le 1$, parameterizes the straight-line segment from $\mathbf{a}$ to $\mathbf{b}$.

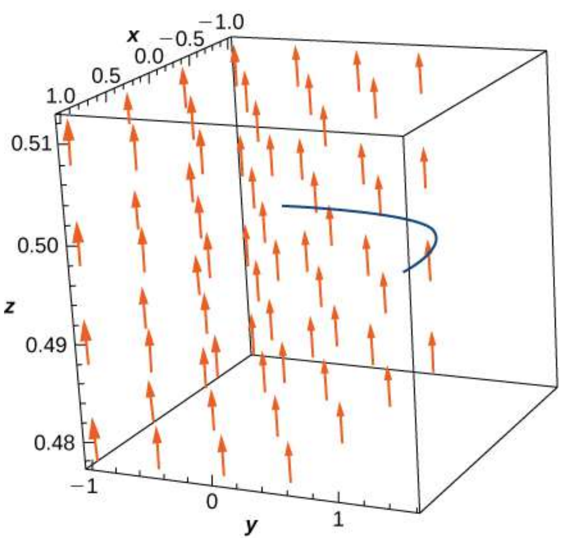
101. *True* or *False?* Vector field $\mathbf{F}(x, y, z) = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$ is conservative.

102. *True* or *False?* Vector field $\mathbf{F}(x, y, z) = y\mathbf{i} + (x + z)\mathbf{j} - y\mathbf{k}$ is conservative.

103. Justify the Fundamental Theorem of Line Integrals for $\int_C \mathbf{F} \cdot d\mathbf{r}$ in the case when $\mathbf{F}(x, y) = (2x + 2y)\mathbf{i} + (2x + 2y)\mathbf{j}$ and C is a portion of the positively oriented circle $x^2 + y^2 = 25$ from (5, 0) to (3, 4).

104. **[T]** Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, ,] where $\mathbf{F}(x, y) = \left(ye^{xy} + \cos x\right)\mathbf{i} + \left(xe^{xy} + \dfrac{1}{y^2 + 1}\right)\mathbf{j}$ and C is a portion of curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

105. **[T]** Evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (e^x \sin y - y)\mathbf{i} + (e^x \cos y - x - 2)\mathbf{j}$, and C is the path given by $r(t) = \left[t^3 \sin \frac{\pi t}{2}\right]\mathbf{i} - \left[\frac{\pi}{2}\cos\left(\frac{\pi t}{2} + \frac{\pi}{2}\right)\right]\mathbf{j}$ for $0 \le t \le 1$.



For the following exercises, determine whether the vector field is conservative and, if it is, find the potential function.

106. $\mathbf{F}(x, y) = 2xy^3\,\mathbf{i} + 3y^2x^2\,\mathbf{j}$

107. $\mathbf{F}(x, y) = (-y + e^x \sin y)\mathbf{i} + [(x + 2)e^x \cos y]\mathbf{j}$

108. $\mathbf{F}(x, y) = \left(e^{2x} \sin y\right)\mathbf{i} + \left[e^{2x} \cos y\right]\mathbf{j}$

109. $\mathbf{F}(x, y) = (6x + 5y)\mathbf{i} + (5x + 4y)\mathbf{j}$

110. $\mathbf{F}(x, y) = [2x\cos(y) - y\cos(x)]\mathbf{i} + \left[-x^2\sin(y) - \sin(x)\right]\mathbf{j}$

111. $\mathbf{F}(x, y) = [ye^x + \sin(y)]\mathbf{i} + [e^x + x\cos(y)]\mathbf{j}$

For the following exercises, evaluate the line integrals using the Fundamental Theorem of Line Integrals.

112. $\oint_C (y\mathbf{i} + x\mathbf{j}) \cdot d\mathbf{r}$, where C is any path from (0, 0) to (2, 4)

113. $\oint_C (2y\,dx + 2x\,dy)$, where C is the line segment from (0, 0) to (4, 4)

114. **[T]**
$$\oint_C \left[\arctan \frac{y}{x} - \frac{xy}{x^2 + y^2}\right]dx + \left[\frac{x^2}{x^2 + y^2} + e^{-y}(1 - y)\right]dy,$$
where C is any smooth curve from (1, 1) to $(-1, 2)$

115. Find the conservative vector field for the potential function
$$f(x, y) = 5x^2 + 3xy + 10y^2.$$

For the following exercises, determine whether the vector field is conservative and, if so, find a potential function.

116. $\mathbf{F}(x, y) = (12xy)\mathbf{i} + 6\left(x^2 + y^2\right)\mathbf{j}$

117. $\mathbf{F}(x, y) = (e^x \cos y)\mathbf{i} + 6(e^x \sin y)\mathbf{j}$

118. $\mathbf{F}(x, y) = \left(2xye^{x^2 y}\right)\mathbf{i} + 6\left(x^2 e^{x^2 y}\right)\mathbf{j}$

119. $\mathbf{F}(x, y, z) = (ye^z)\mathbf{i} + (xe^z)\mathbf{j} + (xye^z)\mathbf{k}$

120. $\mathbf{F}(x, y, z) = (\sin y)\mathbf{i} - (x \cos y)\mathbf{j} + \mathbf{k}$

121. $\mathbf{F}(x, y, z) = \left(\frac{1}{y}\right)\mathbf{i} + \left(\frac{x}{y^2}\right)\mathbf{j} + (2z - 1)\mathbf{k}$

122. $\mathbf{F}(x, y, z) = 3z^2\,\mathbf{i} - \cos y\mathbf{j} + 2xz\mathbf{k}$

123. $\mathbf{F}(x, y, z) = (2xy)\mathbf{i} + \left(x^2 + 2yz\right)\mathbf{j} + y^2\,\mathbf{k}$

For the following exercises, determine whether the given vector field is conservative and find a potential function.

124. $\mathbf{F}(x, y) = (e^x \cos y)\mathbf{i} + 6(e^x \sin y)\mathbf{j}$

125. $\mathbf{F}(x, y) = \left(2xye^{x^2 y}\right)\mathbf{i} + 6\left(x^2 e^{x^2 y}\right)\mathbf{j}$

For the following exercises, evaluate the integral using the Fundamental Theorem of Line Integrals.

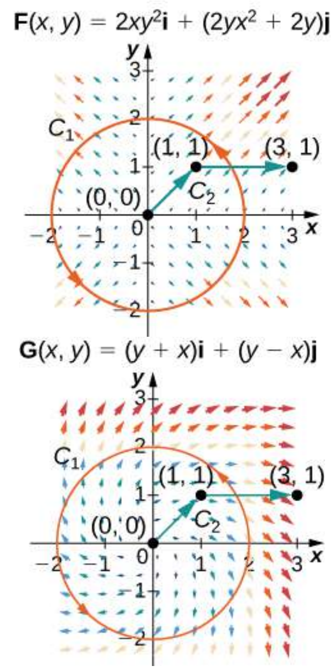
126. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y, z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $\left(1, \frac{1}{2}, 2\right)$ and ends at $(2, 1, -1)$.

127. **[T]** Evaluate $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y) = xy + e^x$ and C is a straight line from $(0, 0)$ to $(2, 1)$.

128. **[T]** Evaluate $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y) = x^2 y - x$ and C is any path in a plane from (1, 2) to (3, 2).

129. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y, z) = xyz^2 - yz$ and C has initial point (1, 2) and terminal point (3, 5).

For the following exercises, let $\mathbf{F}(x, y) = 2xy^2\,\mathbf{i} + \left(2yx^2 + 2y\right)\mathbf{j}$ and $G(x, y) = (y + x)\mathbf{i} + (y - x)\mathbf{j}$, and let C_1 be the curve consisting of the circle of radius 2, centered at the origin and oriented counterclockwise, and C_2 be the curve consisting of a line segment from (0, 0) to (1, 1) followed by a line segment from (1, 1) to (3, 1).



130. Calculate the line integral of **F** over C_1.

131. Calculate the line integral of **G** over C_1.

132. Calculate the line integral of **F** over C_2.

133. Calculate the line integral of **G** over C_2.

134. **[T]** Let $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + z\sin(yz)\mathbf{j} + y\sin(yz)\mathbf{k}$. Calculate $\oint_C \mathbf{F} \cdot dr$, where C is a path from $A = (0, 0, 1)$ to $B = (3, 1, 2)$.

135. **[T]** Find line integral $\oint_C \mathbf{F} \cdot dr$ of vector field $\mathbf{F}(x, y, z) = 3x^2 z\mathbf{i} + z^2\,\mathbf{j} + \left(x^3 + 2yz\right)\mathbf{k}$ along curve C parameterized by $r(t) = \left(\frac{\ln t}{\ln 2}\right)\mathbf{i} + t^{3/2}\,\mathbf{j} + t\cos(\pi t),\ 1 \le t \le 4$.

For the following exercises, show that the following vector fields are conservative by using a computer. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the given curve.

136. $\mathbf{F} = \left(xy^2 + 3x^2 y\right)\mathbf{i} + (x + y)x^2\,\mathbf{j}$; C is the curve consisting of line segments from $(1, 1)$ to $(0, 2)$ to $(3, 0)$.

137. $\mathbf{F} = \frac{2x}{y^2+1}\mathbf{i} - \frac{2y\left(x^2+1\right)}{\left(y^2+1\right)^2}\mathbf{j}$; C is parameterized by $x = t^3 - 1,\ y = t^6 - t,\ 0 \le t \le 1.$

138. **[T]** $\mathbf{F} = \left[\cos\left(xy^2\right) - xy^2\sin\left(xy^2\right)\right]\mathbf{i} - 2x^2y\sin\left(xy^2\right)\mathbf{j}$; C is curve $\left(e^t,\ e^{t+1}\right),\ -1 \le t \le 0.$

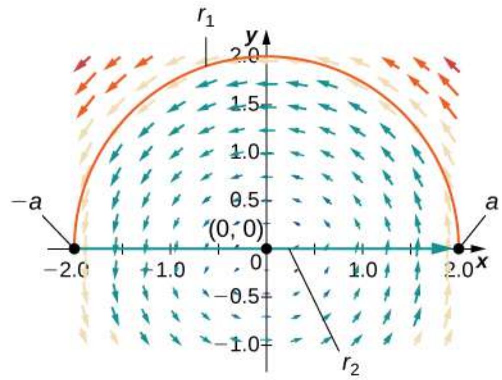
139. The mass of Earth is approximately 6×10^{27} g and that of the Sun is 330,000 times as much. The gravitational constant is $6.7 \times 10^{-8}\,\text{cm}^3/\text{s}^2 \cdot \text{g}$. The distance of Earth from the Sun is about 1.5×10^{12} cm. Compute, approximately, the work necessary to increase the distance of Earth from the Sun by 1 cm.

140. **[T]** Let $\mathbf{F} = (x,\ y,\ z) = \left(e^x \sin y\right)\mathbf{i} + \left(e^x \cos y\right)\mathbf{j} + z^2\mathbf{k}$. Evaluate the integral $\int_C \mathbf{F} \cdot ds$, where $\mathbf{c}(t) = \left(\sqrt{t},\ t^3,\ e^{\sqrt{t}}\right),\ 0 \le t \le 1.$

141. **[T]** Let $\mathbf{c} : [1,\ 2] \to \mathbb{R}^2$ be given by $x = e^{t-1},\ y = \sin\left(\frac{\pi}{t}\right)$. Use a computer to compute the integral $\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C 2x \cos y\,dx - x^2 \sin y\,dy$, where $\mathbf{F} = (2x \cos y)\mathbf{i} - \left(x^2 \sin y\right)\mathbf{j}.$

142. **[T]** Use a computer algebra system to find the mass of a wire that lies along curve $\mathbf{r}(t) = \left(t^2 - 1\right)\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 1$, if the density is $\frac{3}{2}t$.

143. Find the circulation and flux of field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ around and across the closed semicircular path that consists of semicircular arch $\mathbf{r}_1(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0 \le t \le \pi$, followed by line segment $\mathbf{r}_2(t) = t\mathbf{i},\ -a \le t \le a.$



144. Compute $\int_C \cos x \cos y\,dx - \sin x \sin y\,dy$, where $\mathbf{c}(t) = \left(t,\ t^2\right),\ 0 \le t \le 1.$

145. Complete the proof of **The Path Independence Test for Conservative Fields** by showing that $f_y = Q(x,\ y).$

6.4 | Green's Theorem

Learning Objectives

6.4.1 Apply the circulation form of Green's theorem.
6.4.2 Apply the flux form of Green's theorem.
6.4.3 Calculate circulation and flux on more general regions.

In this section, we examine Green's theorem, which is an extension of the Fundamental Theorem of Calculus to two dimensions. Green's theorem has two forms: a circulation form and a flux form, both of which require region *D* in the double integral to be simply connected. However, we will extend Green's theorem to regions that are not simply connected.

Put simply, Green's theorem relates a line integral around a simply closed plane curve *C* and a double integral over the region enclosed by *C*. The theorem is useful because it allows us to translate difficult line integrals into more simple double integrals, or difficult double integrals into more simple line integrals.

Extending the Fundamental Theorem of Calculus

Recall that the Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)dx = F(b) - F(a).$$

As a geometric statement, this equation says that the integral over the region below the graph of $F'(x)$ and above the line segment $[a, b]$ depends only on the value of *F* at the endpoints *a* and *b* of that segment. Since the numbers *a* and *b* are the boundary of the line segment $[a, b]$, the theorem says we can calculate integral $\int_a^b F'(x)dx$ based on information about the boundary of line segment $[a, b]$ (**Figure 6.32**). The same idea is true of the Fundamental Theorem for Line Integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

When we have a potential function (an "antiderivative"), we can calculate the line integral based solely on information about the boundary of curve *C*.

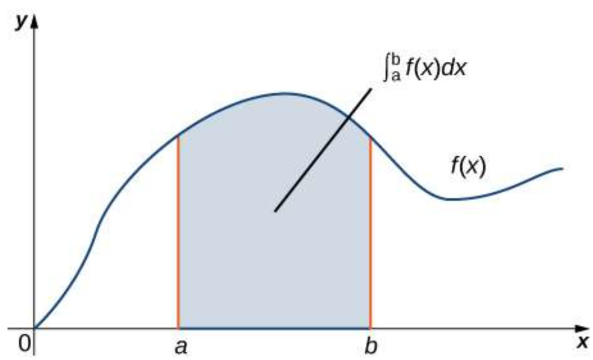


Figure 6.32 The Fundamental Theorem of Calculus says that the integral over line segment $[a, b]$ depends only on the values of the antiderivative at the endpoints of $[a, b]$.

Green's theorem takes this idea and extends it to calculating double integrals. Green's theorem says that we can calculate a double integral over region *D* based solely on information about the boundary of *D*. Green's theorem also says we can calculate a line integral over a simple closed curve *C* based solely on information about the region that *C* encloses. In particular, Green's theorem connects a double integral over region *D* to a line integral around the boundary of *D*.

Circulation Form of Green's Theorem

The first form of Green's theorem that we examine is the circulation form. This form of the theorem relates the vector line integral over a simple, closed plane curve C to a double integral over the region enclosed by C. Therefore, the circulation of a vector field along a simple closed curve can be transformed into a double integral and vice versa.

Theorem 6.12: Green's Theorem, Circulation Form

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise (**Figure 6.33**). Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA. \tag{6.13}$$

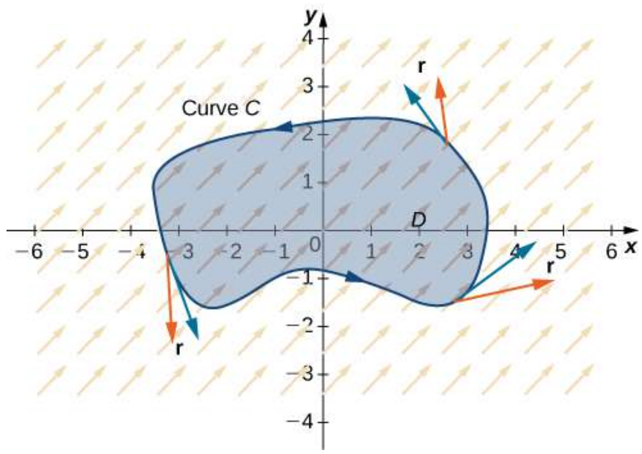


Figure 6.33 The circulation form of Green's theorem relates a line integral over curve C to a double integral over region D.

Notice that Green's theorem can be used only for a two-dimensional vector field $\mathbf{F}$. If $\mathbf{F}$ is a three-dimensional field, then Green's theorem does not apply. Since

$$\int_C P dx + Q dy = \int_C \mathbf{F} \cdot \mathbf{T} ds,$$

this version of Green's theorem is sometimes referred to as the *tangential form* of Green's theorem.

The proof of Green's theorem is rather technical, and beyond the scope of this text. Here we examine a proof of the theorem in the special case that D is a rectangle. For now, notice that we can quickly confirm that the theorem is true for the special case in which $\mathbf{F} = \langle P, Q \rangle$ is conservative. In this case,

$$\oint_C P dx + Q dy = 0$$

because the circulation is zero in conservative vector fields. By **Cross-Partial Property of Conservative Fields**, $\mathbf{F}$ satisfies the cross-partial condition, so $P_y = Q_x$. Therefore,

$$\iint_D (Q_x - P_y) dA = \iint_D 0 dA = 0 = \oint_C P dx + Q dy,$$

which confirms Green's theorem in the case of conservative vector fields.

Proof

Let's now prove that the circulation form of Green's theorem is true when the region D is a rectangle. Let D be the rectangle $[a, b] \times [c, d]$ oriented counterclockwise. Then, the boundary C of D consists of four piecewise smooth pieces C_1, C_2, C_3, and C_4 (**Figure 6.34**). We parameterize each side of D as follows:

$$\begin{aligned}
C_1 : \mathbf{r}_1(t) &= \langle t, c \rangle, a \le t \le b \\
C_2 : \mathbf{r}_2(t) &= \langle b, t \rangle, c \le t \le d \\
-C_3 : \mathbf{r}_3(t) &= \langle t, d \rangle, a \le t \le b \\
-C_4 : \mathbf{r}_4(t) &= \langle a, t \rangle, c \le t \le d.
\end{aligned}$$

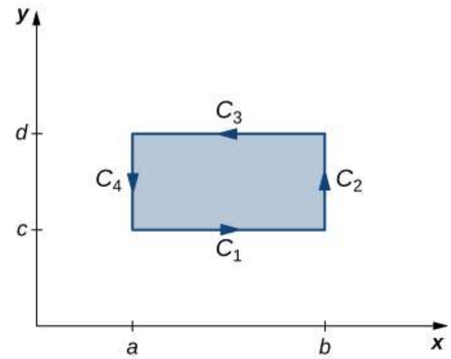


Figure 6.34 Rectangle D is oriented counterclockwise.

Then,

$$\begin{aligned}
\int_C \mathbf{F} \bullet d\mathbf{r} &= \int_{C_1} \mathbf{F} \bullet d\mathbf{r} + \int_{C_2} \mathbf{F} \bullet d\mathbf{r} + \int_{C_3} \mathbf{F} \bullet d\mathbf{r} + \int_{C_4} \mathbf{F} \bullet d\mathbf{r} \\
&= \int_{C_1} \mathbf{F} \bullet d\mathbf{r} + \int_{C_2} \mathbf{F} \bullet d\mathbf{r} - \int_{-C_3} \mathbf{F} \bullet d\mathbf{r} - \int_{-C_4} \mathbf{F} \bullet d\mathbf{r} \\
&= \int_a^b \mathbf{F}(\mathbf{r}_1(t)) \bullet \mathbf{r}_1(t)dt + \int_c^d \mathbf{F}(\mathbf{r}_2(t)) \bullet \mathbf{r}_2(t)dt \\
&\quad -\int_a^b \mathbf{F}(\mathbf{r}_3(t)) \bullet \mathbf{r}_3(t)dt - \int_c^d \mathbf{F}(\mathbf{r}_4(t)) \bullet \mathbf{r}_4(t)dt \\
&= \int_a^b P(t, c)dt + \int_c^d Q(b, t)dt - \int_a^b P(t, d)dt - \int_c^d Q(a, t)dt \\
&= \int_a^b (P(t, c) - P(t, d))dt + \int_c^d (Q(b, t) - Q(a, t))dt \\
&= -\int_a^b (P(t, d) - P(t, c))dt + \int_c^d (Q(b, t) - Q(a, t))dt.
\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$P(t, d) - P(t, c) = \int_c^d \frac{\partial}{\partial y} P(t, y)dy \text{ and } Q(b, t) - Q(a, t) = \int_a^b \frac{\partial}{\partial x} Q(x, t)dx.$$

Therefore,

$$\begin{aligned}
&-\int_a^b (P(t, d) - P(t, c))dt + \int_c^d (Q(b, t) - Q(a, t))dt \\
&= -\int_a^b \int_c^d \frac{\partial}{\partial y} P(t, y)dydt + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x, t)dxdt.
\end{aligned}$$

But,

$$-\int_a^b \int_c^d \frac{\partial}{\partial y} P(t, y) dy dt + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x, t) dx dt = -\int_a^b \int_c^d \frac{\partial}{\partial y} P(x, y) dy dx + \int_c^d \int_a^b \frac{\partial}{\partial x} Q(x, y) dx dy$$

$$= \int_a^b \int_c^d (Q_x - P_y) dy dx$$

$$= \iint_D (Q_x - P_y) dA.$$

Therefore, $\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_D (Q_x - P_y) dA$ and we have proved Green's theorem in the case of a rectangle.

To prove Green's theorem over a general region D, we can decompose D into many tiny rectangles and use the proof that the theorem works over rectangles. The details are technical, however, and beyond the scope of this text.

□

Example 6.38

Applying Green's Theorem over a Rectangle

Calculate the line integral

$$\oint_C x^2 y dx + (y - 3) dy,$$

where C is a rectangle with vertices $(1, 1)$, $(4, 1)$, $(4, 5)$, and $(1, 5)$ oriented counterclockwise.

Solution

Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle x^2 y, y - 3 \rangle$. Then, $Q_x = 0$ and $P_y = x^2$. Therefore, $Q_x - P_y = -x^2$.

Let D be the rectangular region enclosed by C (**Figure 6.35**). By Green's theorem,

$$\oint_C x^2 y dx + (y - 3) dy = \iint_D (Q_x - P_y) dA$$

$$= \iint_D -x^2 dA = \int_1^5 \int_1^4 -x^2 dx dy$$

$$= \int_1^5 -21 dy = -84.$$

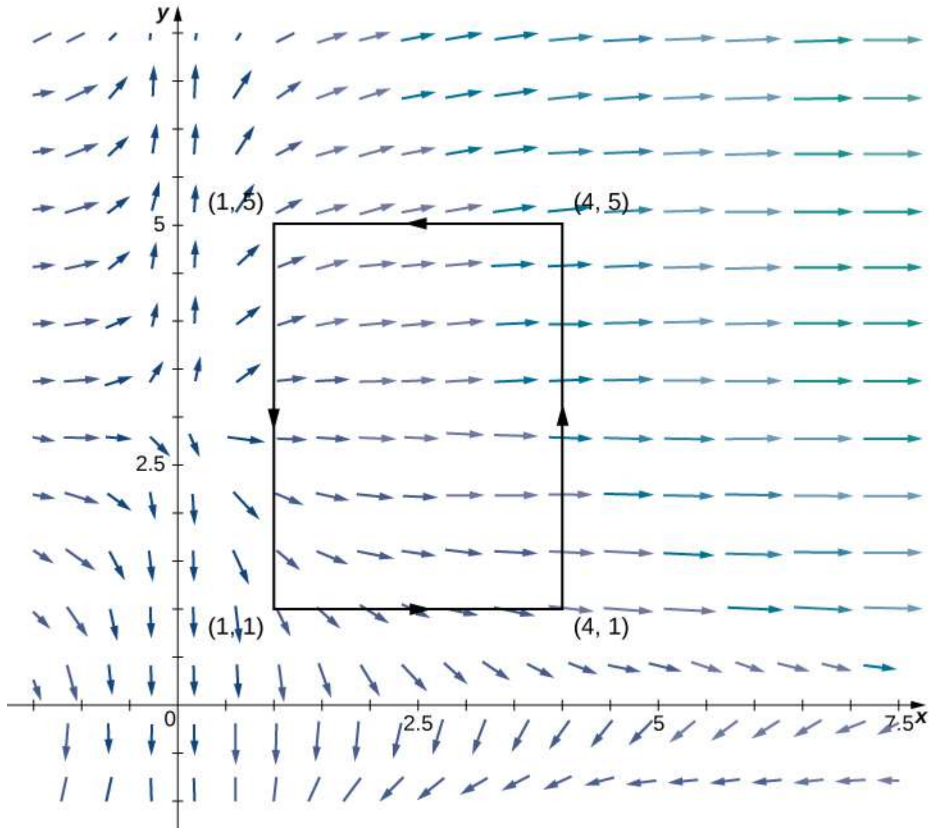


Figure 6.35 The line integral over the boundary of the rectangle can be transformed into a double integral over the rectangle.

Analysis

If we were to evaluate this line integral without using Green’s theorem, we would need to parameterize each side of the rectangle, break the line integral into four separate line integrals, and use the methods from **Line Integrals** to evaluate each integral. Furthermore, since the vector field here is not conservative, we cannot apply the Fundamental Theorem for Line Integrals. Green’s theorem makes the calculation much simpler.

Example 6.39

Applying Green’s Theorem to Calculate Work

Calculate the work done on a particle by force field

$$\mathbf{F}(x, y) = \langle y + \sin x, e^y - x \rangle$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2, 0)$.

Solution

Let C denote the circle and let D be the disk enclosed by C. The work done on the particle is

$$W = \oint_C (y + \sin x)dx + (e^y - x)dy.$$

As with **Example 6.38**, this integral can be calculated using tools we have learned, but it is easier to use the double integral given by Green's theorem (**Figure 6.36**).

Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle y + \sin x, e^y - x \rangle$. Then, $Q_x = -1$ and $P_y = 1$. Therefore, $Q_x - P_y = -2$.

By Green's theorem,

$$\begin{aligned} W &= \oint_C (y + \sin(x))dx + (e^y - x)dy \\ &= \iint_D (Q_x - P_y)dA = \iint_D -2dA \\ &= -2(\text{area}(D)) = -2\pi\left(2^2\right) = -8\pi. \end{aligned}$$

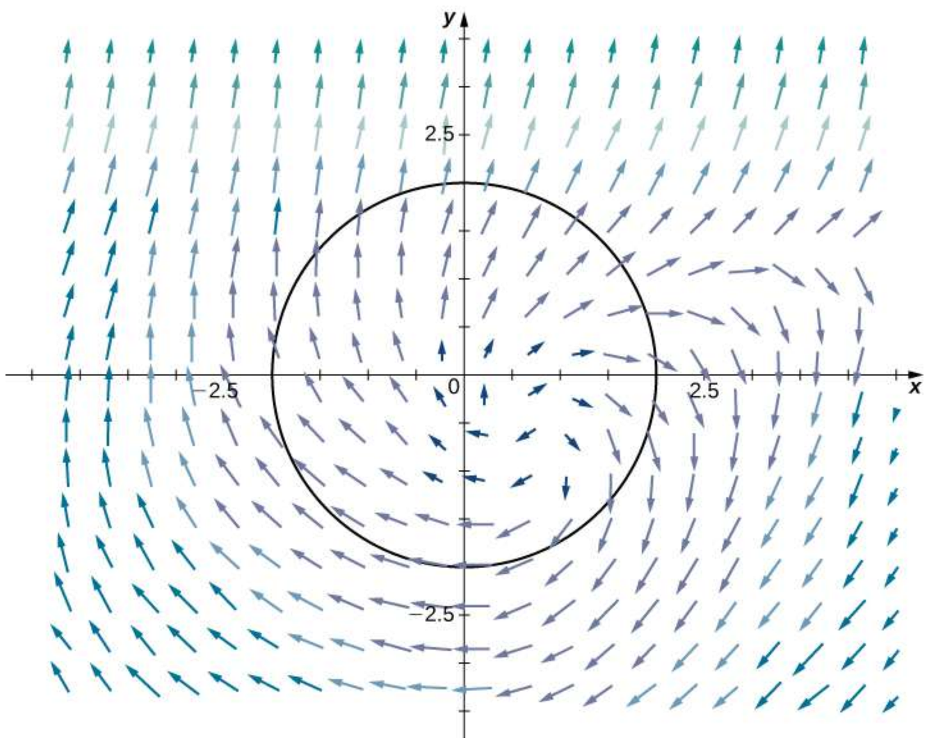


Figure 6.36 The line integral over the boundary circle can be transformed into a double integral over the disk enclosed by the circle.

6.34 Use Green's theorem to calculate line integral

$$\oint_C \sin(x^2)dx + (3x - y)dy,$$

where C is a right triangle with vertices $(-1, 2)$, $(4, 2)$, and $(4, 5)$ oriented counterclockwise.

In the preceding two examples, the double integral in Green's theorem was easier to calculate than the line integral, so we used the theorem to calculate the line integral. In the next example, the double integral is more difficult to calculate than the line integral, so we use Green's theorem to translate a double integral into a line integral.

Example 6.40

Applying Green's Theorem over an Ellipse

Calculate the area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (**Figure 6.37**).

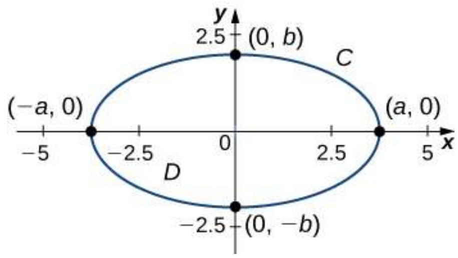


Figure 6.37 Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is denoted by C.

Solution

Let C denote the ellipse and let D be the region enclosed by C. Recall that ellipse C can be parameterized by

$$x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi.$$

Calculating the area of D is equivalent to computing double integral $\iint_D dA$. To calculate this integral without Green's theorem, we would need to divide D into two regions: the region above the x-axis and the region below. The area of the ellipse is

$$\int_{-a}^{a}\int_{0}^{\sqrt{b^2-(bx/a)^2}} dy\,dx + \int_{-a}^{a}\int_{-\sqrt{b^2-(bx/a)^2}}^{0} dy\,dx.$$

These two integrals are not straightforward to calculate (although when we know the value of the first integral, we know the value of the second by symmetry). Instead of trying to calculate them, we use Green's theorem to transform $\iint_D dA$ into a line integral around the boundary C.

Consider vector field

$$\mathbf{F}(x, y) = \langle P, Q \rangle = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle.$$

Then, $Q_x = \frac{1}{2}$ and $P_y = -\frac{1}{2}$, and therefore $Q_x - P_y = 1$. Notice that $\mathbf{F}$ was chosen to have the property that $Q_x - P_y = 1$. Since this is the case, Green's theorem transforms the line integral of $\mathbf{F}$ over C into the double integral of 1 over D.

By Green's theorem,

$$\begin{aligned}\iint_D dA &= \iint_D (Q_x - P_y)dA \\ &= \int_C \mathbf{F} \bullet d\mathbf{r} = \frac{1}{2}\int_C -ydx + xdy \\ &= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t) + a(\cos t)b\cos t dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t dt = \frac{1}{2}\int_0^{2\pi} ab dt = \pi ab.\end{aligned}$$

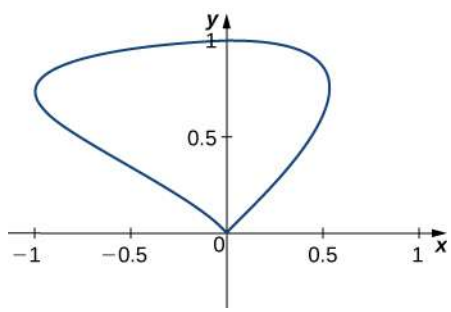
Therefore, the area of the ellipse is πab.

In **Example 6.40**, we used vector field $\mathbf{F}(x, y) = \langle P, Q \rangle = \langle -\frac{y}{2}, \frac{x}{2} \rangle$ to find the area of any ellipse. The logic of the previous example can be extended to derive a formula for the area of any region D. Let D be any region with a boundary that is a simple closed curve C oriented counterclockwise. If $\mathbf{F}(x, y) = \langle P, Q \rangle = \langle -\frac{y}{2}, \frac{x}{2} \rangle$, then $Q_x - P_y = 1$. Therefore, by the same logic as in **Example 6.40**,

$$\text{area of } D = \iint_D dA = \frac{1}{2}\oint_C -ydx + xdy. \tag{6.14}$$

It's worth noting that if $\mathbf{F} = \langle P, Q \rangle$ is any vector field with $Q_x - P_y = 1$, then the logic of the previous paragraph works. So. **Equation 6.14** is not the only equation that uses a vector field's mixed partials to get the area of a region.

6.35 Find the area of the region enclosed by the curve with parameterization $\mathbf{r}(t) = \langle \sin t \cos t, \sin t \rangle, 0 \le t \le \pi$.



Flux Form of Green's Theorem

The circulation form of Green's theorem relates a double integral over region D to line integral $\oint_C \mathbf{F} \cdot \mathbf{T} ds$, where C is the boundary of D. The flux form of Green's theorem relates a double integral over region D to the flux across boundary C. The flux of a fluid across a curve can be difficult to calculate using the flux line integral. This form of Green's theorem allows us to translate a difficult flux integral into a double integral that is often easier to calculate.

Theorem 6.13: Green's Theorem, Flux Form

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise (**Figure 6.38**). Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on an open region containing D. Then,

$$\oint_C \mathbf{F} \cdot \mathbf{N} ds = \iint_D P_x + Q_y \, dA. \tag{6.15}$$

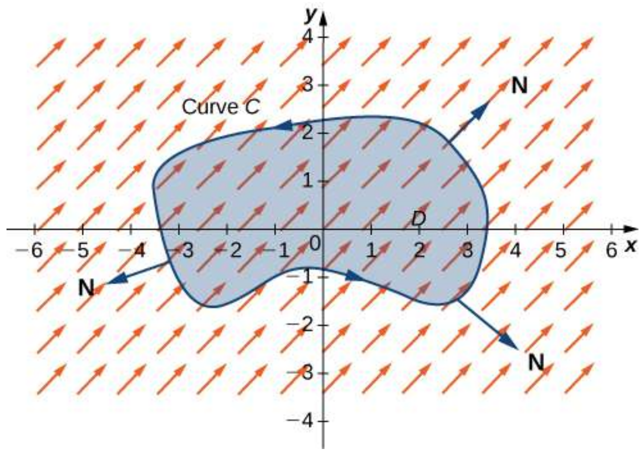


Figure 6.38 The flux form of Green's theorem relates a double integral over region D to the flux across curve C.

Because this form of Green's theorem contains unit normal vector **N**, it is sometimes referred to as the *normal form* of Green's theorem.

Proof

Recall that $\oint_C \mathbf{F} \cdot \mathbf{N} ds = \oint_C -Q dx + P dy$. Let $M = -Q$ and $N = P$. By the circulation form of Green's theorem,

$$\begin{aligned} \oint_C -Q dx + P dy &= \oint_C M dx + N dy \\ &= \iint_D N_x - M_y \, dA \\ &= \iint_D P_x - (-Q)_y \, dA \\ &= \iint_D P_x + Q_y \, dA. \end{aligned}$$

□

Example 6.41

Applying Green's Theorem for Flux across a Circle

Let C be a circle of radius r centered at the origin (**Figure 6.39**) and let $\mathbf{F}(x, y) = \langle x, y \rangle$. Calculate the flux across C.

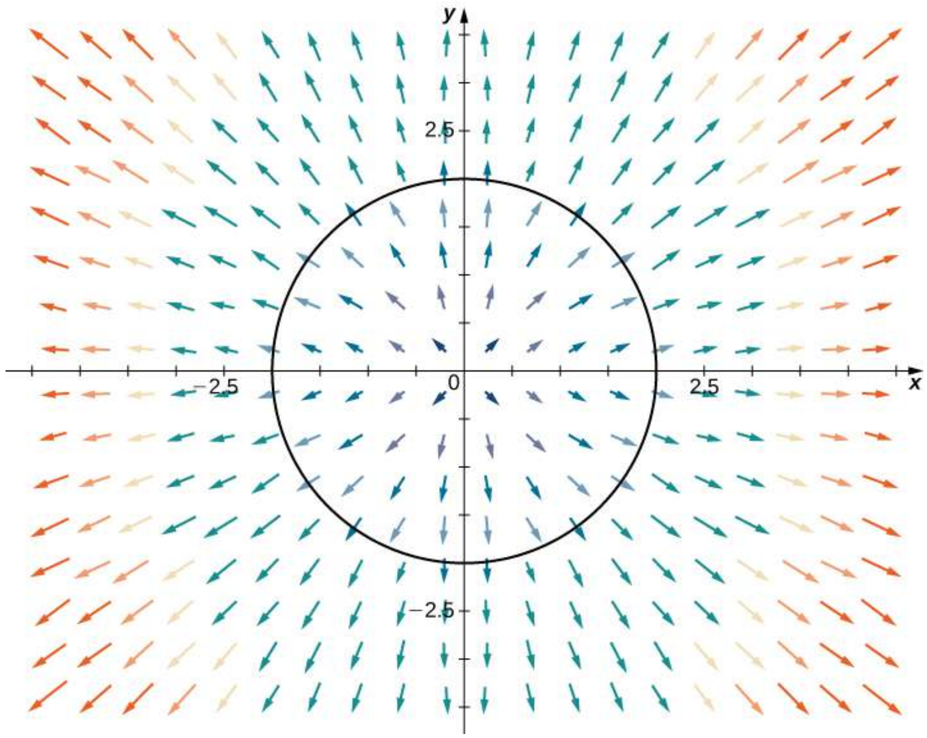


Figure 6.39 Curve *C* is a circle of radius *r* centered at the origin.

Solution

Let *D* be the disk enclosed by *C*. The flux across *C* is $\oint_C \mathbf{F} \cdot \mathbf{N} ds$. We could evaluate this integral using tools we have learned, but Green's theorem makes the calculation much more simple. Let $P(x, y) = x$ and $Q(x, y) = y$ so that $\mathbf{F} = \langle P, Q \rangle$. Note that $P_x = 1 = Q_y$, and therefore $P_x + Q_y = 2$. By Green's theorem,

$$\int_C \mathbf{F} \bullet \mathbf{N} ds = \iint_D 2 dA = 2 \iint_D dA.$$

Since $\iint_D dA$ is the area of the circle, $\iint_D dA = \pi r^2$. Therefore, the flux across *C* is $2\pi r^2$.

Example 6.42

Applying Green's Theorem for Flux across a Triangle

Let *S* be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 3)$ oriented clockwise (**Figure 6.40**). Calculate the flux of $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle x^2 + e^y, x + y \rangle$ across *S*.

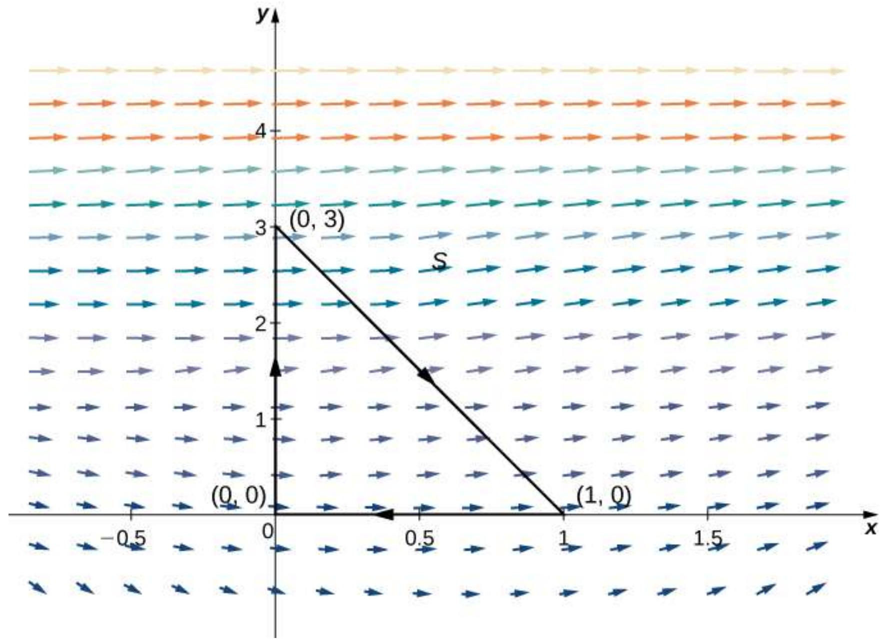


Figure 6.40 Curve S is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 3)$ oriented clockwise.

Solution

To calculate the flux without Green's theorem, we would need to break the flux integral into three line integrals, one integral for each side of the triangle. Using Green's theorem to translate the flux line integral into a single double integral is much more simple.

Let D be the region enclosed by S. Note that $P_x = 2x$ and $Q_y = 1$; therefore, $P_x + Q_y = 2x + 1$. Green's theorem applies only to simple closed curves oriented counterclockwise, but we can still apply the theorem because $\oint_C \mathbf{F} \cdot \mathbf{N} ds = -\oint_{-S} \mathbf{F} \cdot \mathbf{N} ds$ and $-S$ is oriented counterclockwise. By Green's theorem, the flux is

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{N} ds &= \oint_{-S} \mathbf{F} \cdot \mathbf{N} ds \\ &= -\iint_D (P_x + Q_y) dA \\ &= -\iint_D (2x + 1) dA. \end{aligned}$$

Notice that the top edge of the triangle is the line $y = -3x + 3$. Therefore, in the iterated double integral, the y-values run from $y = 0$ to $y = -3x + 3$, and we have

$$\begin{aligned} -\iint_D (2x + 1) dA &= -\int_0^1 \int_0^{-3x+3} (2x + 1) dy dx \\ &= -\int_0^1 (2x + 1)(-3x + 3) dx = -\int_0^1 \left(-6x^2 + 3x + 3\right) dx \\ &= -\left[-2x^3 + \frac{3x^2}{2} + 3x\right]_0^1 = -\frac{5}{2}. \end{aligned}$$

6.36 Calculate the flux of $\mathbf{F}(x, y) = \langle x^3, y^3 \rangle$ across a unit circle oriented counterclockwise.

Example 6.43

Applying Green's Theorem for Water Flow across a Rectangle

Water flows from a spring located at the origin. The velocity of the water is modeled by vector field $\mathbf{v}(x, y) = \langle 5x + y, x + 3y \rangle$ m/sec. Find the amount of water per second that flows across the rectangle with vertices $(-1, -2), (1, -2), (1, 3),$ and $(-1, 3),$ oriented counterclockwise (**Figure 6.41**).

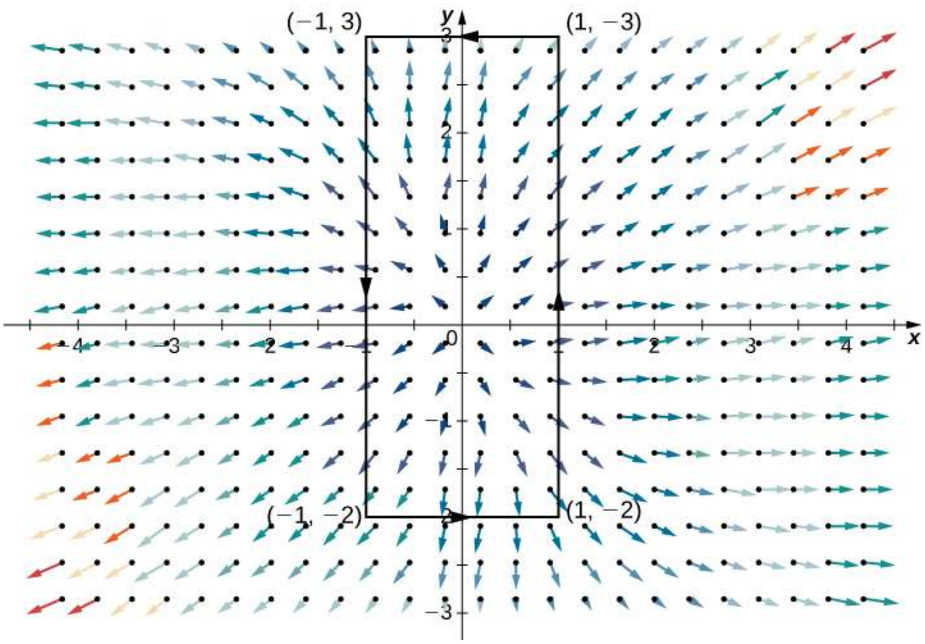


Figure 6.41 Water flows across the rectangle with vertices $(-1, -2), (1, -2), (1, 3),$ and $(-1, 3),$ oriented counterclockwise.

Solution

Let C represent the given rectangle and let D be the rectangular region enclosed by C. To find the amount of water flowing across C, we calculate flux $\int_C \mathbf{v} \cdot d\mathbf{r}$. Let $P(x, y) = 5x + y$ and $Q(x, y) = x + 3y$ so that $\mathbf{v} = (P, Q)$. Then, $P_x = 5$ and $Q_y = 3$. By Green's theorem,

$$\begin{aligned}\int_C \mathbf{v} \cdot d\mathbf{r} &= \iint_D (P_x + Q_y)\, dA \\ &= \iint_D 8\, dA \\ &= 8(\text{area of } D) = 80.\end{aligned}$$

Therefore, the water flux is 80 m^2/sec.

Recall that if vector field **F** is conservative, then **F** does no work around closed curves—that is, the circulation of **F** around a closed curve is zero. In fact, if the domain of **F** is simply connected, then **F** is conservative if and only if the circulation of **F** around any closed curve is zero. If we replace "circulation of **F**" with "flux of **F**," then we get a definition of a source-free vector field. The following statements are all equivalent ways of defining a source-free field $\mathbf{F} = \langle P, Q \rangle$ on a simply connected domain (note the similarities with properties of conservative vector fields):

1. The flux $\oint_C \mathbf{F} \cdot \mathbf{N} ds$ across any closed curve C is zero.
2. If C_1 and C_2 are curves in the domain of $\mathbf{F}$ with the same starting points and endpoints, then $\int_{C_1} \mathbf{F} \cdot \mathbf{N} ds = \int_{C_2} \mathbf{F} \cdot \mathbf{N} ds.$ In other words, flux is independent of path.
3. There is a **stream function** $g(x, y)$ for $\mathbf{F}$. A stream function for $\mathbf{F} = \langle P, Q \rangle$ is a function g such that $P = g_y$ and $Q = -g_x.$ Geometrically, $\mathbf{F} = (a, b)$ is tangential to the level curve of g at $(a, b).$ Since the gradient of g is perpendicular to the level curve of g at $(a, b),$ stream function g has the property $\mathbf{F}(a, b) \bullet \nabla g(a, b) = 0$ for any point (a, b) in the domain of g. (Stream functions play the same role for source-free fields that potential functions play for conservative fields.)
4. $P_x + Q_y = 0$

Example 6.44

Finding a Stream Function

Verify that rotation vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$ is source free, and find a stream function for $\mathbf{F}$.

Solution

Note that the domain of $\mathbf{F}$ is all of $\mathbb{R}^2,$ which is simply connected. Therefore, to show that $\mathbf{F}$ is source free, we can show any of items 1 through 4 from the previous list to be true. In this example, we show that item 4 is true. Let $P(x, y) = y$ and $Q(x, y) = -x.$ Then $P_x + 0 = Q_y,$ and therefore $P_x + Q_y = 0.$ Thus, $\mathbf{F}$ is source free.

To find a stream function for $\mathbf{F}$, proceed in the same manner as finding a potential function for a conservative field. Let g be a stream function for $\mathbf{F}$. Then $g_y = y,$ which implies that

$$g(x, y) = \frac{y^2}{2} + h(x).$$

Since $-g_x = Q = -x,$ we have $h'(x) = x.$ Therefore,

$$h(x) = \frac{x^2}{2} + C.$$

Letting $C = 0$ gives stream function

$$g(x, y) = \frac{x^2}{2} + \frac{y^2}{2}.$$

To confirm that g is a stream function for $\mathbf{F}$, note that $g_y = y = P$ and $-g_x = -x = Q.$

Notice that source-free rotation vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$ is perpendicular to conservative radial vector field $\nabla g = \langle x, y \rangle$ (Figure 6.42).

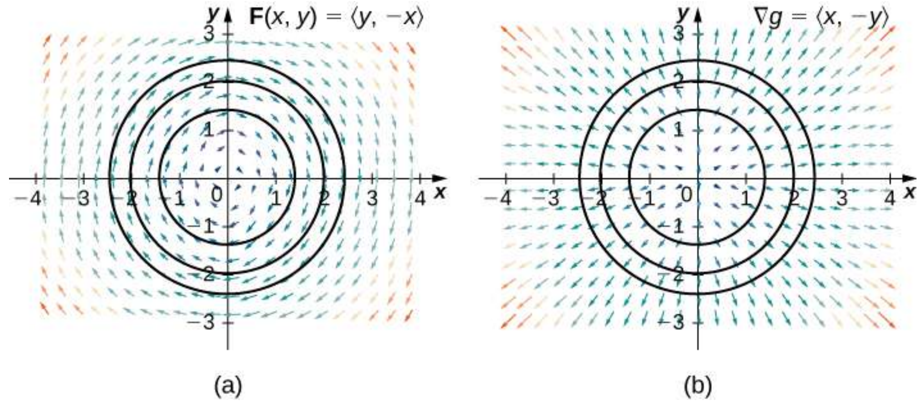


Figure 6.42 (a) In this image, we see the three-level curves of g and vector field **F**. Note that the **F** vectors on a given level curve are tangent to the level curve. (b) In this image, we see the three-level curves of g and vector field ∇g. The gradient vectors are perpendicular to the corresponding level curve. Therefore, $\mathbf{F}(a, b) \bullet \nabla g(a, b) = 0$ for any point in the domain of g.

6.37 Find a stream function for vector field $\mathbf{F}(x, y) = \langle x \sin y, \cos y \rangle$.

Vector fields that are both conservative and source free are important vector fields. One important feature of conservative and source-free vector fields on a simply connected domain is that any potential function f of such a field satisfies Laplace's equation $f_{xx} + f_{yy} = 0$. Laplace's equation is foundational in the field of partial differential equations because it models such phenomena as gravitational and magnetic potentials in space, and the velocity potential of an ideal fluid. A function that satisfies Laplace's equation is called a *harmonic* function. Therefore any potential function of a conservative and source-free vector field is harmonic.

To see that any potential function of a conservative and source-free vector field on a simply connected domain is harmonic, let f be such a potential function of vector field $\mathbf{F} = \langle P, Q \rangle$. Then, $f_x = P$ and $f_x = Q$ because $\nabla f = \mathbf{F}$. Therefore, $f_{xx} = P_x$ and $f_{yy} = Q_y$. Since **F** is source free, $f_{xx} + f_{yy} = P_x + Q_y = 0$, and we have that f is harmonic.

Example 6.45

Satisfying Laplace's Equation

For vector field $\mathbf{F}(x, y) = \langle e^x \sin y, e^x \cos y \rangle$, verify that the field is both conservative and source free, find a potential function for **F**, and verify that the potential function is harmonic.

Solution

Let $P(x, y) = e^x \sin y$ and $Q(x, y) = e^x \cos y$. Notice that the domain of **F** is all of two-space, which is simply connected. Therefore, we can check the cross-partials of **F** to determine whether **F** is conservative. Note that $P_y = e^x \cos y = Q_x$, so **F** is conservative. Since $P_x = e^x \sin y$ and $Q_y = e^x \sin y$, $P_x + Q_y = 0$ and the field is source free.

To find a potential function for **F**, let f be a potential function. Then, $\nabla f = \mathbf{F}$, so $f_x = e^x \sin y$. Integrating this equation with respect to x gives $f(x, y) = e^x \sin y + h(y)$. Since $f_y = e^x \cos y$, differentiating f with respect to y gives $e^x \cos y = e^x \cos y + h'(y)$. Therefore, we can take $h(y) = 0$, and $f(x, y) = e^x \sin y$ is a potential function for f.

To verify that f is a harmonic function, note that $f_{xx} = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$ and

$f_{yy} = \frac{\partial}{\partial x}(e^x \cos y) = -e^x \sin y$. Therefore, $f_{xx} + f_{yy} = 0$, and f satisfies Laplace's equation.

6.38 Is the function $f(x, y) = e^{x + 5y}$ harmonic?

Green's Theorem on General Regions

Green's theorem, as stated, applies only to regions that are simply connected—that is, Green's theorem as stated so far cannot handle regions with holes. Here, we extend Green's theorem so that it does work on regions with finitely many holes (**Figure 6.43**).

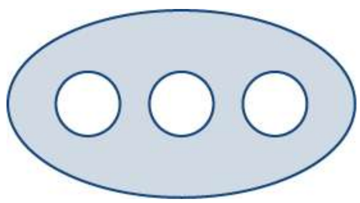


Figure 6.43 Green's theorem, as stated, does not apply to a nonsimply connected region with three holes like this one.

Before discussing extensions of Green's theorem, we need to go over some terminology regarding the boundary of a region. Let D be a region and let C be a component of the boundary of D. We say that C is *positively oriented* if, as we walk along C in the direction of orientation, region D is always on our left. Therefore, the counterclockwise orientation of the boundary of a disk is a positive orientation, for example. Curve C is *negatively oriented* if, as we walk along C in the direction of orientation, region D is always on our right. The clockwise orientation of the boundary of a disk is a negative orientation, for example.

Let D be a region with finitely many holes (so that D has finitely many boundary curves), and denote the boundary of D by ∂D (**Figure 6.44**). To extend Green's theorem so it can handle D, we divide region D into two regions, D_1 and D_2 (with respective boundaries ∂D_1 and ∂D_2), in such a way that $D = D_1 \cup D_2$ and neither D_1 nor D_2 has any holes (**Figure 6.44**).

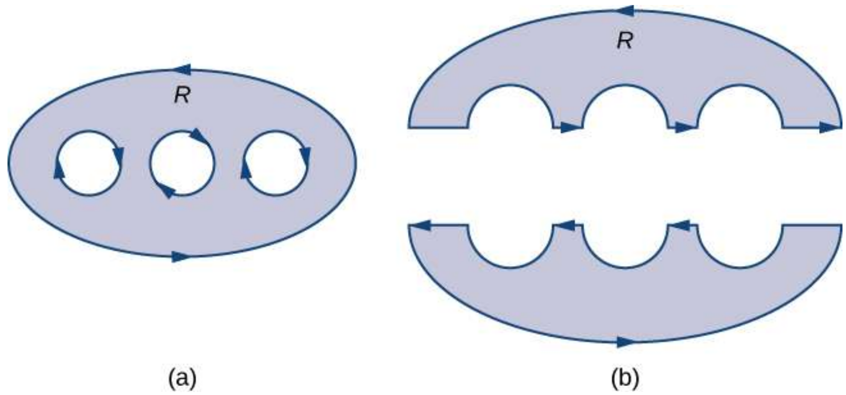


Figure 6.44 (a) Region D with an oriented boundary has three holes. (b) Region D split into two simply connected regions has no holes.

Assume the boundary of D is oriented as in the figure, with the inner holes given a negative orientation and the outer boundary given a positive orientation. The boundary of each simply connected region D_1 and D_2 is positively oriented. If $\mathbf{F}$ is a vector field defined on D, then Green's theorem says that

$$\begin{aligned}
\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} &= \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{\partial D_2} \mathbf{F} \cdot d\mathbf{r} \\
&= \iint_{D_1} Q_x - P_y \, dA + \iint_{D_2} Q_x - P_y \, dA \\
&= \iint_D (Q_x - P_y) dA.
\end{aligned}$$

Therefore, Green's theorem still works on a region with holes.

To see how this works in practice, consider annulus D in **Figure 6.45** and suppose that $\mathbf{F} = \langle P, Q \rangle$ is a vector field defined on this annulus. Region D has a hole, so it is not simply connected. Orient the outer circle of the annulus counterclockwise and the inner circle clockwise (**Figure 6.45**) so that, when we divide the region into D_1 and D_2, we are able to keep the region on our left as we walk along a path that traverses the boundary. Let D_1 be the upper half of the annulus and D_2 be the lower half. Neither of these regions has holes, so we have divided D into two simply connected regions.

We label each piece of these new boundaries as P_i for some i, as in **Figure 6.45**. If we begin at P and travel along the oriented boundary, the first segment is P_1, then P_2, P_3, and P_4. Now we have traversed D_1 and returned to P. Next, we start at P again and traverse D_2. Since the first piece of the boundary is the same as P_4 in D_1, but oriented in the opposite direction, the first piece of D_2 is $-P_4$. Next, we have P_5, then $-P_2$, and finally P_6.

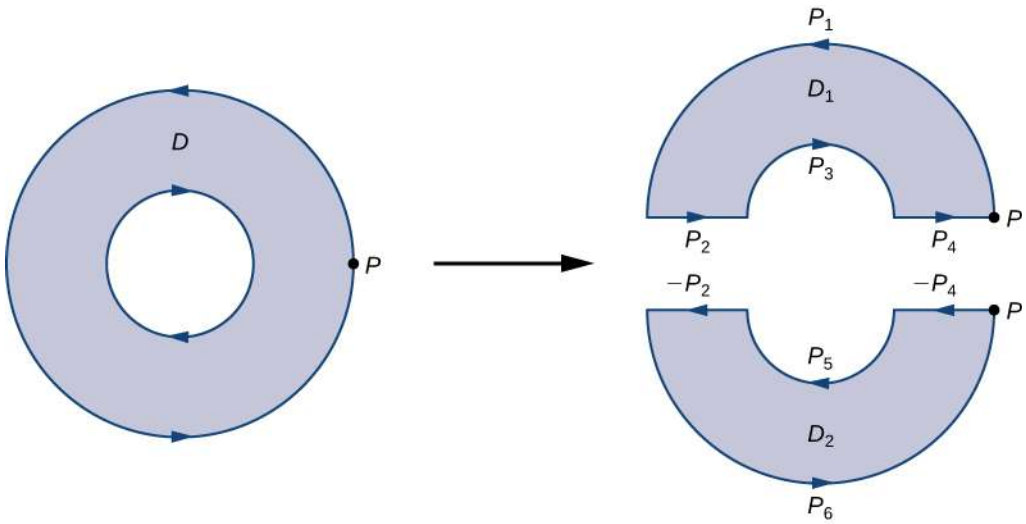


Figure 6.45 Breaking the annulus into two separate regions gives us two simply connected regions. The line integrals over the common boundaries cancel out.

Figure 6.45 shows a path that traverses the boundary of *D*. Notice that this path traverses the boundary of region D_1, returns to the starting point, and then traverses the boundary of region D_2. Furthermore, as we walk along the path, the region is always on our left. Notice that this traversal of the P_i paths covers the entire boundary of region *D*. If we had only traversed one portion of the boundary of *D*, then we cannot apply Green's theorem to *D*.

The boundary of the upper half of the annulus, therefore, is $P_1 \cup P_2 \cup P_3 \cup P_4$ and the boundary of the lower half of the annulus is $-P_4 \cup P_5 \cup -P_2 \cup P_6$. Then, Green's theorem implies

$$\begin{aligned}
\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} &= \int_{P_1} \mathbf{F} \cdot d\mathbf{r} + \int_{P_2} \mathbf{F} \cdot d\mathbf{r} + \int_{P_3} \mathbf{F} \cdot d\mathbf{r} + \int_{P_4} \mathbf{F} \cdot d\mathbf{r} + \int_{-P_4} \mathbf{F} \cdot d\mathbf{r} + \int_{P_5} \mathbf{F} \cdot d\mathbf{r} - \int_{P_2} \mathbf{F} \cdot d\mathbf{r} + \int_{P_6} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{P_1} \mathbf{F} \cdot d\mathbf{r} + \int_{P_2} \mathbf{F} \cdot d\mathbf{r} + \int_{P_3} \mathbf{F} \cdot d\mathbf{r} + \int_{P_4} \mathbf{F} \cdot d\mathbf{r} - \int_{P_4} \mathbf{F} \cdot d\mathbf{r} + \int_{P_5} \mathbf{F} \cdot d\mathbf{r} - \int_{P_2} \mathbf{F} \cdot d\mathbf{r} + \int_{P_6} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{P_1} \mathbf{F} \cdot d\mathbf{r} + \int_{P_3} \mathbf{F} \cdot d\mathbf{r} + \int_{P_5} \mathbf{F} \cdot d\mathbf{r} + \int_{P_6} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\partial D_2} \mathbf{F} \cdot d\mathbf{r} \\
&= \iint_{D_1} (Q_x - P_y) dA + \iint_{D_2} (Q_x - P_y) dA \\
&= \iint_{D} (Q_x - P_y) dA.
\end{aligned}$$

Therefore, we arrive at the equation found in Green's theorem—namely,

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y) dA.$$

The same logic implies that the flux form of Green's theorem can also be extended to a region with finitely many holes:

$$\oint_C \mathbf{F} \cdot \mathbf{N} ds = \iint_D (P_x + Q_y) dA.$$

Example 6.46

Using Green's Theorem on a Region with Holes

Calculate integral

$$\oint_{\partial D}\left(\sin x - \frac{y^3}{3}\right)dx + \left(\frac{y^3}{3} + \sin y\right)dy,$$

where D is the annulus given by the polar inequalities $1 \leq \mathbf{r} \leq 2, \quad 0 \leq \theta \leq 2\pi$.

Solution

Although D is not simply connected, we can use the extended form of Green's theorem to calculate the integral. Since the integration occurs over an annulus, we convert to polar coordinates:

$$\begin{aligned}
\oint_{\partial D}\left(\sin x - \frac{y^3}{3}\right)dx + \left(\frac{x^3}{3} + \sin y\right)dy &= \iint_D (Q_x - P_y)dA \\
&= \iint_D (x^2 + y^2)dA \\
&= \int_0^{2\pi}\int_1^2 r^3\,drd\theta = \int_0^{2\pi}\frac{15}{4}d\theta \\
&= \frac{15\pi}{2}.
\end{aligned}$$

Example 6.47

Using the Extended Form of Green's Theorem

Let $\mathbf{F} = \langle P, Q \rangle = \langle \frac{y}{x^2 + y^2}, -\frac{x}{x^2 + y^2} \rangle$ and let C be any simple closed curve in a plane oriented counterclockwise. What are the possible values of $\oint_C \mathbf{F} \cdot d\mathbf{r}$?

Solution

We use the extended form of Green's theorem to show that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is either 0 or -2π—that is, no matter how crazy curve C is, the line integral of **F** along C can have only one of two possible values. We consider two cases: the case when C encompasses the origin and the case when C does not encompass the origin.

Case 1: C Does Not Encompass the Origin

In this case, the region enclosed by C is simply connected because the only hole in the domain of **F** is at the origin. We showed in our discussion of cross-partials that **F** satisfies the cross-partial condition. If we restrict the domain of **F** just to C and the region it encloses, then **F** with this restricted domain is now defined on a simply connected domain. Since **F** satisfies the cross-partial property on its restricted domain, the field **F** is conservative on this simply connected region and hence the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is zero.

Case 2: C Does Encompass the Origin

In this case, the region enclosed by C is not simply connected because this region contains a hole at the origin. Let C_1 be a circle of radius a centered at the origin so that C_1 is entirely inside the region enclosed by C (**Figure 6.46**). Give C_1 a clockwise orientation.

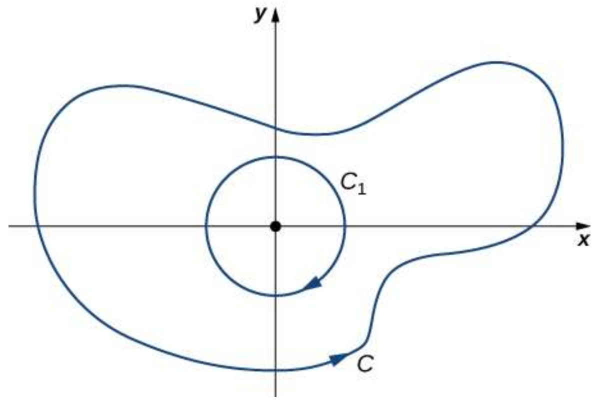


Figure 6.46 Choose circle C_1 centered at the origin that is contained entirely inside C.

Let D be the region between C_1 and C, and C is orientated counterclockwise. By the extended version of Green's theorem,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} + \int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \iint_D Q_x - P_y dA \\ &= \iint_D -\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{y^2-x^2}{(x^2+y^2)^2} dA \\ &= 0,\end{aligned}$$

and therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{C_1} \mathbf{F}\cdot d\mathbf{r}.$$

Since C_1 is a specific curve, we can evaluate $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$. Let

$$x = a\cos t,\; y = a\sin t,\; 0 \le t \le 2\pi$$

be a parameterization of C_1. Then,

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt \\ &= \int_0^{2\pi} \left\langle -\frac{\sin(t)}{a},\; -\frac{\cos(t)}{a} \right\rangle \cdot \langle -a\sin(t), -a\cos(t)\rangle\, dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t)dt = \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Therefore, $\int_C \mathbf{F}\cdot ds = -2\pi.$

6.39 Calculate integral $\oint_{\partial D} \mathbf{F}\cdot d\mathbf{r}$, where D is the annulus given by the polar inequalities $2 \le r \le 5,\; 0 \le \theta \le 2\pi$, and $\mathbf{F}(x, y) = \langle x^3, 5x + e^y \sin y \rangle$.

Student PROJECT

Measuring Area from a Boundary: The Planimeter

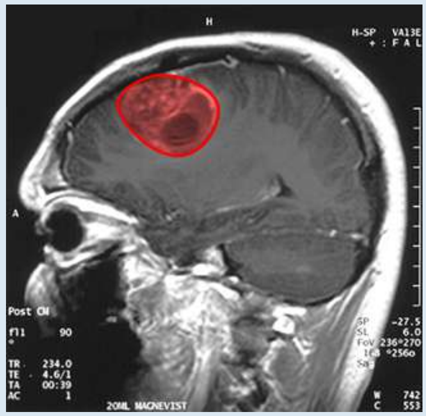


Figure 6.47 This magnetic resonance image of a patient's brain shows a tumor, which is highlighted in red. (credit: modification of work by Christaras A, Wikimedia Commons)

Imagine you are a doctor who has just received a magnetic resonance image of your patient's brain. The brain has a tumor (**Figure 6.47**). How large is the tumor? To be precise, what is the area of the red region? The red cross-section of the tumor has an irregular shape, and therefore it is unlikely that you would be able to find a set of equations or inequalities for the region and then be able to calculate its area by conventional means. You could approximate the area by chopping the region into tiny squares (a Riemann sum approach), but this method always gives an answer with some error.

Instead of trying to measure the area of the region directly, we can use a device called a *rolling planimeter* to calculate the area of the region exactly, simply by measuring its boundary. In this project you investigate how a planimeter works, and you use Green's theorem to show the device calculates area correctly.

A rolling planimeter is a device that measures the area of a planar region by tracing out the boundary of that region (**Figure 6.48**). To measure the area of a region, we simply run the tracer of the planimeter around the boundary of the region. The planimeter measures the number of turns through which the wheel rotates as we trace the boundary; the area of the shape is proportional to this number of wheel turns. We can derive the precise proportionality equation using Green's theorem. As the tracer moves around the boundary of the region, the tracer arm rotates and the roller moves back and forth (but does not rotate).

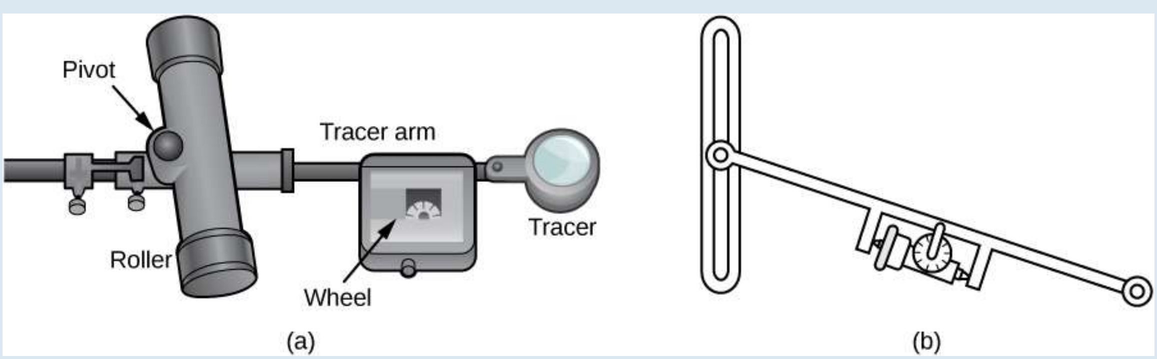


Figure 6.48 (a) A rolling planimeter. The pivot allows the tracer arm to rotate. The roller itself does not rotate; it only moves back and forth. (b) An interior view of a rolling planimeter. Notice that the wheel cannot turn if the planimeter is moving back and forth with the tracer arm perpendicular to the roller.

Let C denote the boundary of region D, the area to be calculated. As the tracer traverses curve C, assume the roller moves along the y-axis (since the roller does not rotate, one can assume it moves along a straight line). Use the coordinates (x, y) to represent points on boundary C, and coordinates $(0, Y)$ to represent the position of the pivot. As the planimeter traces C, the pivot moves along the y-axis while the tracer arm rotates on the pivot.

Watch a **short animation (http://www.openstaxcollege.org/l/20_planimeter)** of a planimeter in action.

Begin the analysis by considering the motion of the tracer as it moves from point (x, y) counterclockwise to point $(x + dx, y + dy)$ that is close to (x, y) (**Figure 6.49**). The pivot also moves, from point $(0, Y)$ to nearby point $(0, Y + dY)$. How much does the wheel turn as a result of this motion? To answer this question, break the motion into two parts. First, roll the pivot along the y-axis from $(0, Y)$ to $(0, Y + dY)$ without rotating the tracer arm. The tracer arm then ends up at point $(x, y + dY)$ while maintaining a constant angle ϕ with the x-axis. Second, rotate the tracer arm by an angle $d\theta$ without moving the roller. Now the tracer is at point $(x + dx, y + dy)$. Let l be the distance from the pivot to the wheel and let L be the distance from the pivot to the tracer (the length of the tracer arm).

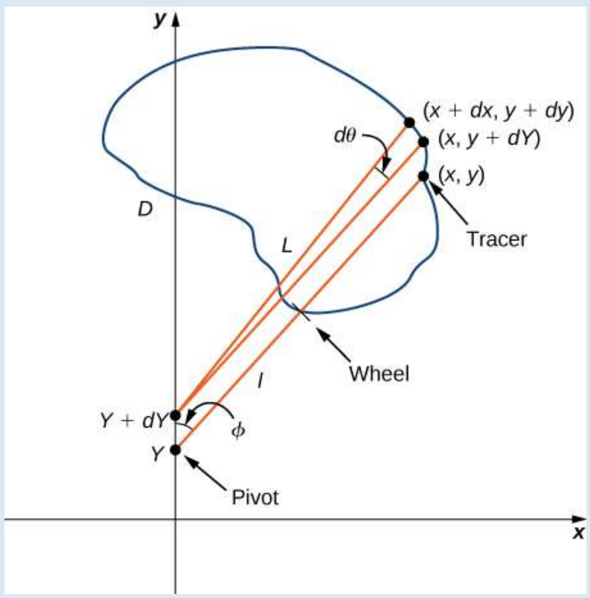


Figure 6.49 Mathematical analysis of the motion of the planimeter.

1. Explain why the total distance through which the wheel rolls the small motion just described is $\sin \phi dY + l d\theta = \frac{x}{L} dY + l d\theta$.

2. Show that $\oint_C d\theta = 0$.

3. Use step 2 to show that the total rolling distance of the wheel as the tracer traverses curve C is

   Total wheel roll $= \frac{1}{L} \oint_C x dY$.

   Now that you have an equation for the total rolling distance of the wheel, connect this equation to Green's theorem to calculate area D enclosed by C.

4. Show that $x^2 + (y - Y)^2 = L^2$.

5. Assume the orientation of the planimeter is as shown in **Figure 6.49**. Explain why $Y \le y$, and use this inequality to show there is a unique value of Y for each point (x, y): $Y = y = \sqrt{L^2 - x^2}$.

6. Use step 5 to show that $dY = dy + \frac{x}{\sqrt{L^2 - x^2}} dx$.

7. Use Green's theorem to show that $\oint_C \frac{x}{\sqrt{L^2 - x^2}} dx = 0$.

8. Use step 7 to show that the total wheel roll is
Total wheel roll $= \frac{1}{L} \oint_C x dy$.

   It took a bit of work, but this equation says that the variable of integration *Y* in step 3 can be replaced with *y*.

9. Use Green's theorem to show that the area of *D* is $\oint_C x dy$. The logic is similar to the logic used to show that the area of $D = \frac{1}{2} \oint_C -y dx + x dy$.

10. Conclude that the area of *D* equals the length of the tracer arm multiplied by the total rolling distance of the wheel.
You now know how a planimeter works and you have used Green's theorem to justify that it works. To calculate the area of a planar region *D*, use a planimeter to trace the boundary of the region. The area of the region is the length of the tracer arm multiplied by the distance the wheel rolled.

6.4 EXERCISES

For the following exercises, evaluate the line integrals by applying Green's theorem.

146. $\int_C 2xydx + (x + y)dy$, where C is the path from (0, 0) to (1, 1) along the graph of $y = x^3$ and from (1, 1) to (0, 0) along the graph of $y = x$ oriented in the counterclockwise direction

147. $\int_C 2xydx + (x + y)dy$, where C is the boundary of the region lying between the graphs of $y = 0$ and $y = 4 - x^2$ oriented in the counterclockwise direction

148. $\int_C 2\arctan\left(\frac{y}{x}\right)dx + \ln\left(x^2 + y^2\right)dy$, where C is defined by $x = 4 + 2\cos\theta,\ y = 4\sin\theta$ oriented in the counterclockwise direction

149. $\int_C \sin x \cos ydx + (xy + \cos x \sin y)dy$, where C is the boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$ oriented in the counterclockwise direction

150. $\int_C xydx + (x + y)dy$, where C is the boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ oriented in the counterclockwise direction

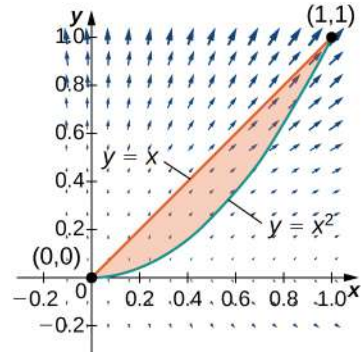
151. $\oint_C (-ydx + xdy)$, where C consists of line segment C_1 from $(-1, 0)$ to $(1, 0)$, followed by the semicircular arc C_2 from (1, 0) back to (1, 0)

For the following exercises, use Green's theorem.

152. Let C be the curve consisting of line segments from (0, 0) to (1, 1) to (0, 1) and back to (0, 0). Find the value of $\int_C xydx + \sqrt{y^2 + 1}dy$.

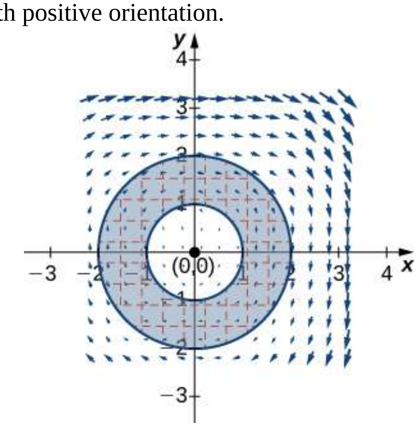
153. Evaluate line integral $\int_C xe^{-2x}dx + \left(x^4 + 2x^2y^2\right)dy$, where C is the boundary of the region between circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and is a positively oriented curve.

154. Find the counterclockwise circulation of field $\mathbf{F}(x,\ y) = xy\mathbf{i} + y^2\mathbf{j}$ around and over the boundary of the region enclosed by curves $y = x^2$ and $y = x$ in the first quadrant and oriented in the counterclockwise direction.



155. Evaluate $\oint_C y^3dx - x^3y^2dy$, where C is the positively oriented circle of radius 2 centered at the origin.

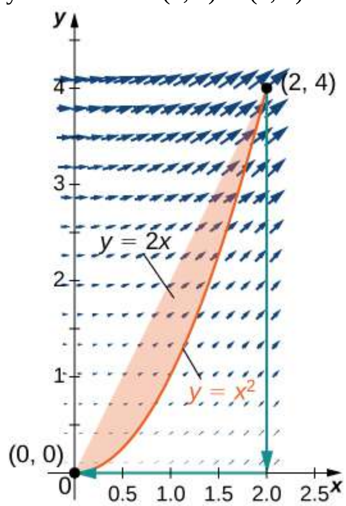
156. Evaluate $\oint_C y^3dx - x^3dy$, where C includes the two circles of radius 2 and radius 1 centered at the origin, both with positive orientation.



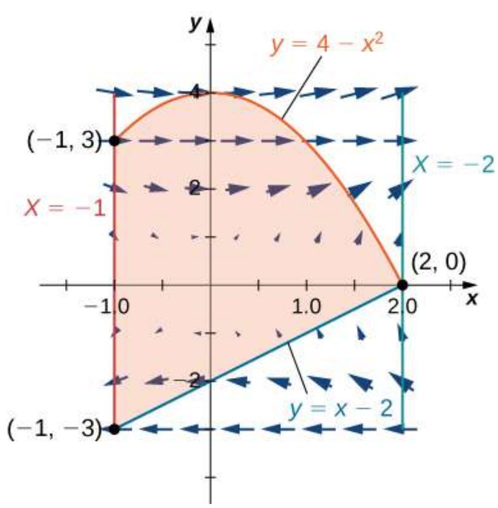
157. Calculate $\oint_C -x^2ydx + xy^2dy$, where C is a circle of radius 2 centered at the origin and oriented in the counterclockwise direction.

158. Calculate integral $\oint_C 2[y + x\sin(y)]dx + \left[x^2\cos(y) - 3y^2\right]dy$ along triangle C with vertices (0, 0), (1, 0) and (1, 1), oriented counterclockwise, using Green's theorem.

159. Evaluate integral $\oint_C \left(x^2 + y^2\right)dx + 2xydy,$ where C is the curve that follows parabola $y = x^2$ from $(0, 0)(2, 4),$ then the line from (2, 4) to (2, 0), and finally the line from (2, 0) to (0, 0).



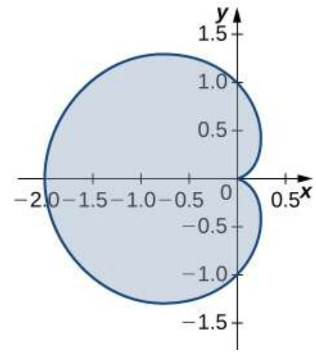
160. Evaluate line integral $\oint_C (y - \sin(y)\cos(y))dx + 2x\sin^2(y)dy,$ where C is oriented in a counterclockwise path around the region bounded by $x = -1,\ x = 2,\ y = 4 - x^2,$ and $y = x - 2.$



For the following exercises, use Green's theorem to find the area.

161. Find the area between ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and circle $x^2 + y^2 = 25.$

162. Find the area of the region enclosed by parametric equation $p(\theta) = \left(\cos(\theta) - \cos^2(\theta)\right)\mathbf{i} + (\sin(\theta) - \cos(\theta)\sin(\theta))\mathbf{j}$ for $0 \le \theta \le 2\pi.$



163. Find the area of the region bounded by hypocycloid $\mathbf{r}(t) = \cos^3(t)\mathbf{i} + \sin^3(t)\mathbf{j}.$ The curve is parameterized by $t \in [0, 2\pi].$

164. Find the area of a pentagon with vertices $(0, 4),\ (4, 1),\ (3, 0),\ (-1, -1),$ and $(-2, 2).$

165. Use Green's theorem to evaluate $\int_{C+} \left(y^2 + x^3\right)dx + x^4\,dy,$ where C^+ is the perimeter of square $[0, 1] \times [0, 1]$ oriented counterclockwise.

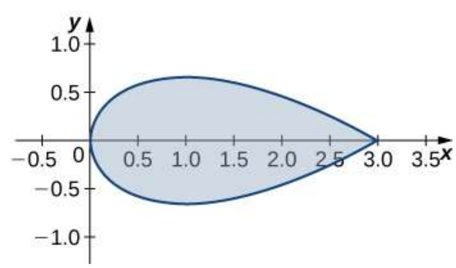
166. Use Green's theorem to prove the area of a disk with radius a is $A = \pi a^2.$

167. Use Green's theorem to find the area of one loop of a four-leaf rose $r = 3\sin 2\theta.$ (*Hint*: $xdy - ydx = \mathbf{r}^2\,d\theta$).

168. Use Green's theorem to find the area under one arch of the cycloid given by parametric plane $x = t - \sin t,\ y = 1 - \cos t,\ t \ge 0.$

169. Use Green's theorem to find the area of the region enclosed by curve

$$\mathbf{r}(t) = t^2\,\mathbf{i} + \left(\frac{t^3}{3} - t\right)\mathbf{j},\ -\sqrt{3} \le t\sqrt{3}.$$



170. **[T]** Evaluate Green's theorem using a computer algebra system to evaluate the integral $\int_C xe^y\,dx + e^x\,dy,$ where C is the circle given by $x^2 + y^2 = 4$ and is oriented in the counterclockwise direction.

171. Evaluate $\int_C \left(x^2 y - 2xy + y^2\right)ds,$ where C is the boundary of the unit square $0 \le x \le 1, 0 \le y \le 1,$ traversed counterclockwise.

172. Evaluate $\int_C \frac{-(y+2)dx + (x-1)dy}{(x-1)^2 + (y+2)^2},$ where C is any simple closed curve with an interior that does not contain point $(1, -2)$ traversed counterclockwise.

173. Evaluate $\int_C \frac{xdx + ydy}{x^2 + y^2},$ where C is any piecewise, smooth simple closed curve enclosing the origin, traversed counterclockwise.

For the following exercises, use Green's theorem to calculate the work done by force **F** on a particle that is moving counterclockwise around closed path C.

174. $\mathbf{F}(x, y) = xy\mathbf{i} + (x+y)\mathbf{j}, \quad C : x^2 + y^2 = 4$

175. $\mathbf{F}(x, y) = \left(x^{3/2} - 3y\right)\mathbf{i} + \left(6x + 5\sqrt{y}\right)\mathbf{j}, \quad C:$ boundary of a triangle with vertices (0, 0), (5, 0), and (0, 5)

176. Evaluate $\int_C \left(2x^3 - y^3\right)dx + \left(x^3 + y^3\right)dy,$ where C is a unit circle oriented in the counterclockwise direction.

177. A particle starts at point $(-2, 0),$ moves along the x-axis to (2, 0), and then travels along semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's theorem to find the work done on this particle by force field $\mathbf{F}(x, y) = x\mathbf{i} + \left(x^3 + 3xy^2\right)\mathbf{j}.$

178. David and Sandra are skating on a frictionless pond in the wind. David skates on the inside, going along a circle of radius 2 in a counterclockwise direction. Sandra skates once around a circle of radius 3, also in the counterclockwise direction. Suppose the force of the wind at point (x, y) (x, y) (x, y) is $\mathbf{F}(x, y) = \left(x^2 y + 10y\right)\mathbf{i} + \left(x^3 + 2xy^2\right)\mathbf{j}.$ Use Green's theorem to determine who does more work.

179. Use Green's theorem to find the work done by force field $\mathbf{F}(x, y) = (3y - 4x)\mathbf{i} + (4x - y)\mathbf{j}$ when an object moves once counterclockwise around ellipse $4x^2 + y^2 = 4.$

180. Use Green's theorem to evaluate line integral $\oint_C e^{2x}\sin 2y dx + e^{2x}\cos 2y dy,$ where C is ellipse $9(x-1)^2 + 4(y-3)^2 = 36$ oriented counterclockwise.

181. Evaluate line integral $\oint_C y^2\,dx + x^2\,dy,$ where C is the boundary of a triangle with vertices $(0, 0), (1, 1),$ and $(1, 0),$ with the counterclockwise orientation.

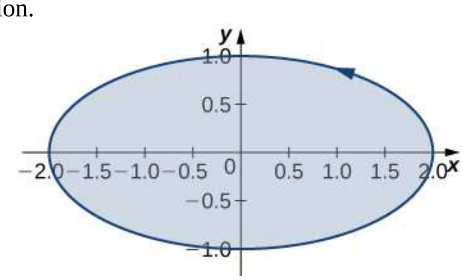
182. Use Green's theorem to evaluate line integral $\int_C \mathbf{h} \cdot d\mathbf{r}$ if $\mathbf{h}(x, y) = e^y\mathbf{i} - \sin \pi x\mathbf{j},$ where C is a triangle with vertices (1, 0), (0, 1), and $(-1, 0)$ $(-1, 0)$ traversed counterclockwise.

183. Use Green's theorem to evaluate line integral $\int_C \sqrt{1 + x^3}dx + 2xydy$ where C is a triangle with vertices (0, 0), (1, 0), and (1, 3) oriented clockwise.

184. Use Green's theorem to evaluate line integral $\int_C x^2 ydx - xy^2 dy$ where C is a circle $x^2 + y^2 = 4$ oriented counterclockwise.

185. Use Green's theorem to evaluate line integral $\int_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4 + 1}\right)dy$ where C is circle $x^2 + y^2 = 9$ oriented in the counterclockwise direction.

186. Use Green's theorem to evaluate line integral $\int_C (3x - 5y)dx + (x - 6y)dy,$ where C is ellipse $\frac{x^2}{4} + y^2 = 1$ and is oriented in the counterclockwise direction.



187. Let C be a triangular closed curve from (0, 0) to (1, 0) to (1, 1) and finally back to (0, 0). Let $\mathbf{F}(x, y) = 4y\mathbf{i} + 6x^2\mathbf{j}$. Use Green's theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{s}$.

188. Use Green's theorem to evaluate line integral $\oint_C ydx - xdy,$ where C is circle $x^2 + y^2 = a^2$ oriented in the clockwise direction.

189. Use Green's theorem to evaluate line integral $\oint_C (y + x)dx + (x + \sin y)dy,$ where C is any smooth simple closed curve joining the origin to itself oriented in the counterclockwise direction.

190. Use Green's theorem to evaluate line integral $\oint_C \left(y - \ln\left(x^2 + y^2\right)\right)dx + \left(2 \arctan \frac{y}{x}\right)dy,$ where C is the positively oriented circle $(x-2)^2 + (y-3)^2 = 1.$

191. Use Green's theorem to evaluate $\oint_C xydx + x^3y^3dy,$ where C is a triangle with vertices (0, 0), (1, 0), and (1, 2) with positive orientation.

192. Use Green's theorem to evaluate line integral $\int_C \sin ydx + x\cos ydy,$ where C is ellipse $x^2 + xy + y^2 = 1$ oriented in the counterclockwise direction.

193. Let $\mathbf{F}(x, y) = \left(\cos\left(x^5\right)\right) - \frac{1}{3}y^3\mathbf{i} + \frac{1}{3}x^3\mathbf{j}$. Find the counterclockwise circulation $\oint_C \mathbf{F} \cdot d\mathbf{r},$ where C is a curve consisting of the line segment joining $(-2, 0)$ and $(-1, 0),$ half circle $y = \sqrt{1 - x^2},$ the line segment joining (1, 0) and (2, 0), and half circle $y = \sqrt{4 - x^2}.$

194. Use Green's theorem to evaluate line integral $\int_C \sin\left(x^3\right)dx + 2ye^{x^2}dy,$ where C is a triangular closed curve that connects the points (0, 0), (2, 2), and (0, 2) counterclockwise.

195. Let C be the boundary of square $0 \le x \le \pi, 0 \le y \le \pi,$ traversed counterclockwise. Use Green's theorem to find $\int_C \sin(x + y)dx + \cos(x + y)dy.$

196. Use Green's theorem to evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r},$ where $\mathbf{F}(x, y) = \left(y^2 - x^2\right)\mathbf{i} + \left(x^2 + y^2\right)\mathbf{j},$ and C is a triangle bounded by $y = 0, x = 3,$ and $y = x,$ oriented counterclockwise.

197. Use Green's Theorem to evaluate integral $\int_C \mathbf{F} \cdot d\mathbf{r},$ where $\mathbf{F}(x, y) = \left(xy^2\right)\mathbf{i} + x\mathbf{j},$ and C is a unit circle oriented in the counterclockwise direction.

198. Use Green's theorem in a plane to evaluate line integral $\oint_C \left(xy + y^2\right)dx + x^2dy,$ where C is a closed curve of a region bounded by $y = x$ and $y = x^2$ oriented in the counterclockwise direction.

199. Calculate the outward flux of $\mathbf{F} = -x\mathbf{i} + 2y\mathbf{j}$ over a square with corners $(\pm 1, \pm 1),$ where the unit normal is outward pointing and oriented in the counterclockwise direction.

200. **[T]** Let C be circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction. Evaluate $\oint_C \left[\left(3y - e^{\tan - 1x}\right)dx + \left(7x + \sqrt{y^4 + 1}\right)dy\right]$ using a computer algebra system.

201. Find the flux of field $\mathbf{F} = -x\mathbf{i} + y\mathbf{j}$ across $x^2 + y^2 = 16$ oriented in the counterclockwise direction.

202. Let $\mathbf{F} = \left(y^2 - x^2\right)\mathbf{i} + \left(x^2 + y^2\right)\mathbf{j},$ and let C be a triangle bounded by $y = 0,\ x = 3,$ and $y = x$ oriented in the counterclockwise direction. Find the outward flux of **F** through C.

203. **[T]** Let C be unit circle $x^2 + y^2 = 1$ traversed once counterclockwise. Evaluate $\int_C \left[-y^3 + \sin(xy) + xy\cos(xy)\right]dx + \left[x^3 + x^2\cos(xy)\right]dy$ by using a computer algebra system.

204. **[T]** Find the outward flux of vector field $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j}$ across the boundary of annulus $R = \left\{(x, y) : 1 \le x^2 + y^2 \le 4\right\} = \{(r, \theta) : 1 \le r \le 2, 0 \le \theta \le 2\pi\}$ using a computer algebra system.

205. Consider region R bounded by parabolas $y = x^2$ and $x = y^2$. Let C be the boundary of R oriented counterclockwise. Use Green's theorem to evaluate $\oint_C \left(y + e^{\sqrt{x}}\right)dx + \left(2x + \cos\left(y^2\right)\right)dy.$

6.5 | Divergence and Curl

Learning Objectives

6.5.1 Determine divergence from the formula for a given vector field.
6.5.2 Determine curl from the formula for a given vector field.
6.5.3 Use the properties of curl and divergence to determine whether a vector field is conservative.

In this section, we examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

In addition to defining curl and divergence, we look at some physical interpretations of them, and show their relationship to conservative and source-free vector fields.

Divergence

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field **F** in $\mathbb{R}^2$ or $\mathbb{R}^3$ at a particular point P is a measure of the "outflowing-ness" of the vector field at P. If **F** represents the velocity of a fluid, then the divergence of **F** at P measures the net rate of change with respect to time of the amount of fluid flowing away from P (the tendency of the fluid to flow "out of" P). In particular, if the amount of fluid flowing into P is the same as the amount flowing out, then the divergence at P is zero.

Definition

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z all exist, then the **divergence** of **F** is defined by

$$\text{div}\,\mathbf{F} = P_x + Q_y + R_z = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{6.16}$$

Note the divergence of a vector field is not a vector field, but a scalar function. In terms of the gradient operator $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$, divergence can be written symbolically as the dot product

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}.$$

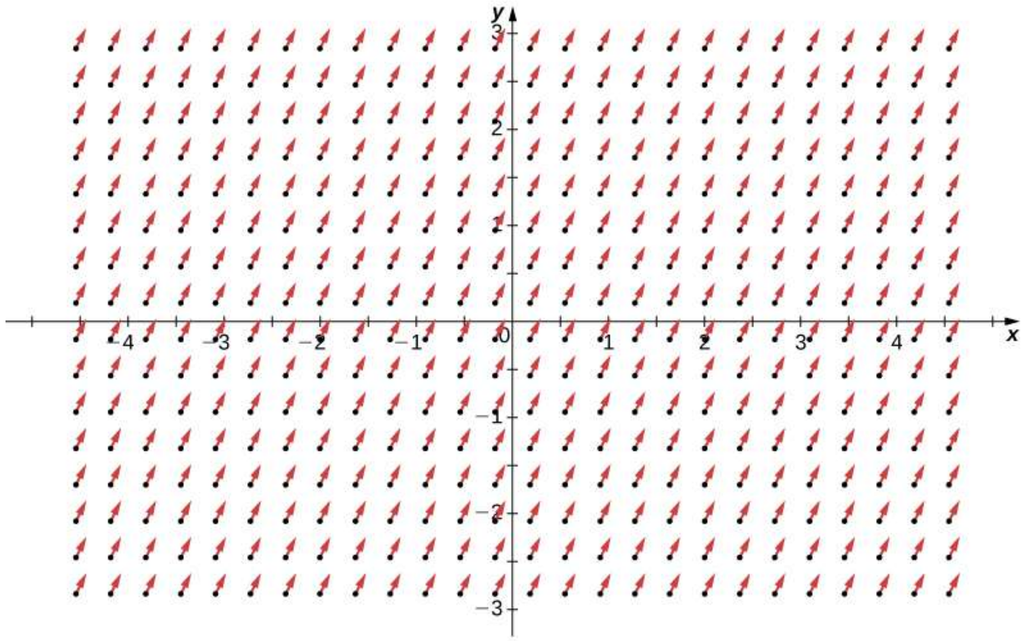
Note this is merely helpful notation, because the dot product of a vector of operators and a vector of functions is not meaningfully defined given our current definition of dot product.

If $\mathbf{F} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, and P_x and Q_y both exist, then the divergence of **F** is defined similarly as

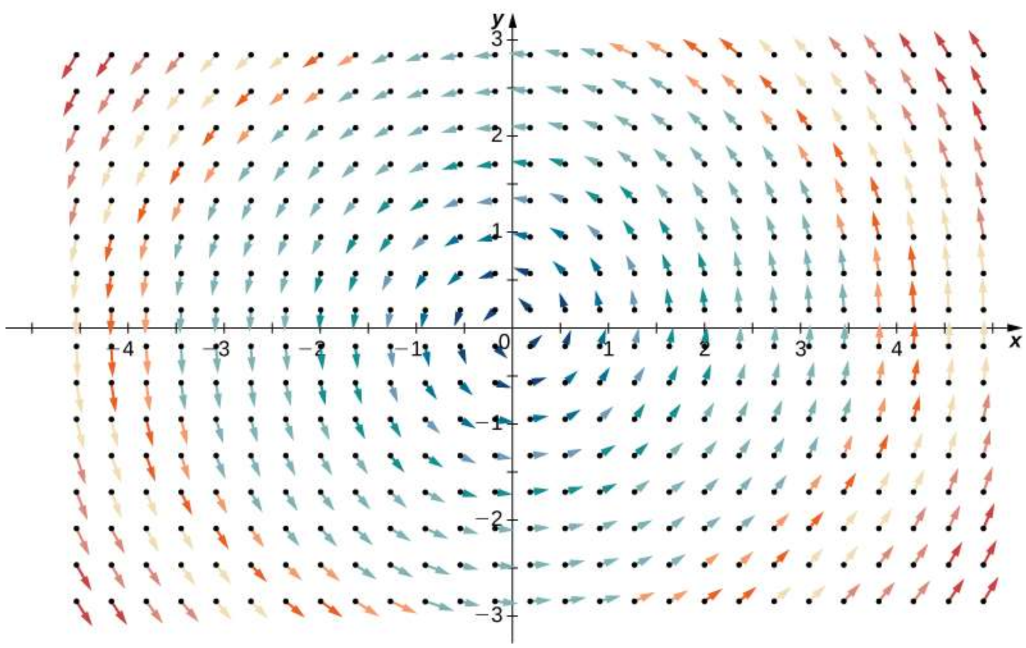
$$\text{div}\,\mathbf{F} = P_x + Q_y = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \nabla \cdot \mathbf{F}.$$

To illustrate this point, consider the two vector fields in **Figure 6.50**. At any particular point, the amount flowing in is the same as the amount flowing out, so at every point the "outflowing-ness" of the field is zero. Therefore, we expect the divergence of both fields to be zero, and this is indeed the case, as

$$\text{div}(\langle 1, 2 \rangle) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) = 0 \text{ and } \text{div}(\langle -y, x \rangle) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$



(a)



(b)

Figure 6.50 (a) Vector field $\langle 1, 2 \rangle$ has zero divergence. (b) Vector field $\langle -y, x \rangle$ also has zero divergence.

By contrast, consider radial vector field $\mathbf{R}(x, y) = \langle -x, -y \rangle$ in **Figure 6.51**. At any given point, more fluid is flowing in than is flowing out, and therefore the "outgoingness" of the field is negative. We expect the divergence of this field to be negative, and this is indeed the case, as $\text{div}(\mathbf{R}) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2$.

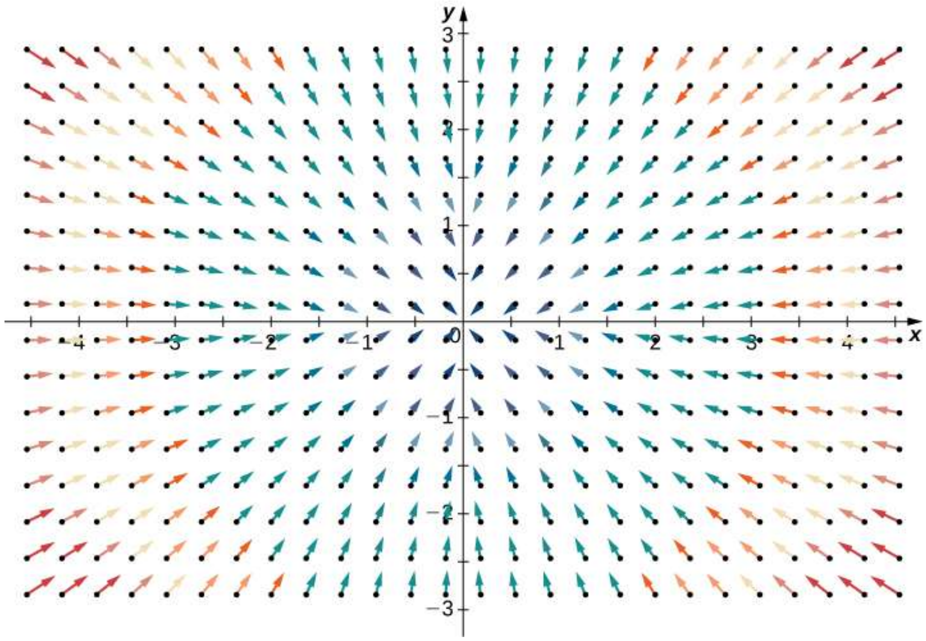


Figure 6.51 This vector field has negative divergence.

To get a global sense of what divergence is telling us, suppose that a vector field in $\mathbb{R}^2$ represents the velocity of a fluid. Imagine taking an elastic circle (a circle with a shape that can be changed by the vector field) and dropping it into a fluid. If the circle maintains its exact area as it flows through the fluid, then the divergence is zero. This would occur for both vector fields in **Figure 6.50**. On the other hand, if the circle's shape is distorted so that its area shrinks or expands, then the divergence is not zero. Imagine dropping such an elastic circle into the radial vector field in **Figure 6.51** so that the center of the circle lands at point (3, 3). The circle would flow toward the origin, and as it did so the front of the circle would travel more slowly than the back, causing the circle to "scrunch" and lose area. This is how you can see a negative divergence.

Example 6.48

Calculating Divergence at a Point

If $\mathbf{F}(x, y, z) = e^x \mathbf{i} + yz\mathbf{j} - y^2\mathbf{k},$ then find the divergence of $\mathbf{F}$ at $(0, 2, -1)$.

Solution

The divergence of $\mathbf{F}$ is

$$\frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}\left(yz^2\right) = e^x + z - 2yz.$$

Therefore, the divergence at $(0, 2, -1)$ is $e^0 - 1 + 4 = 4.$ If $\mathbf{F}$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at point $(0, 2, -1)$.

6.40 Find $\text{div}\,\mathbf{F}$ for $\mathbf{F}(x, y, z) = \langle xy, 5 - z^2 y, x^2 + y^2 \rangle$.

One application for divergence occurs in physics, when working with magnetic fields. A magnetic field is a vector field that models the influence of electric currents and magnetic materials. Physicists use divergence in Gauss's law for magnetism, which states that if $\mathbf{B}$ is a magnetic field, then $\nabla \cdot \mathbf{B} = 0$; in other words, the divergence of a magnetic field is zero.

Example 6.49

Determining Whether a Field Is Magnetic

Is it possible for $\mathbf{F}(x, y) = \langle x^2 y, y - xy^2 \rangle$ to be a magnetic field?

Solution

If $\mathbf{F}$ were magnetic, then its divergence would be zero. The divergence of $\mathbf{F}$ is

$$\frac{\partial}{\partial x}\left(x^2 y\right) + \frac{\partial}{\partial y}\left(y - xy^2\right) = 2xy + 1 - 2xy = 1$$

and therefore $\mathbf{F}$ cannot model a magnetic field (Figure 6.52).

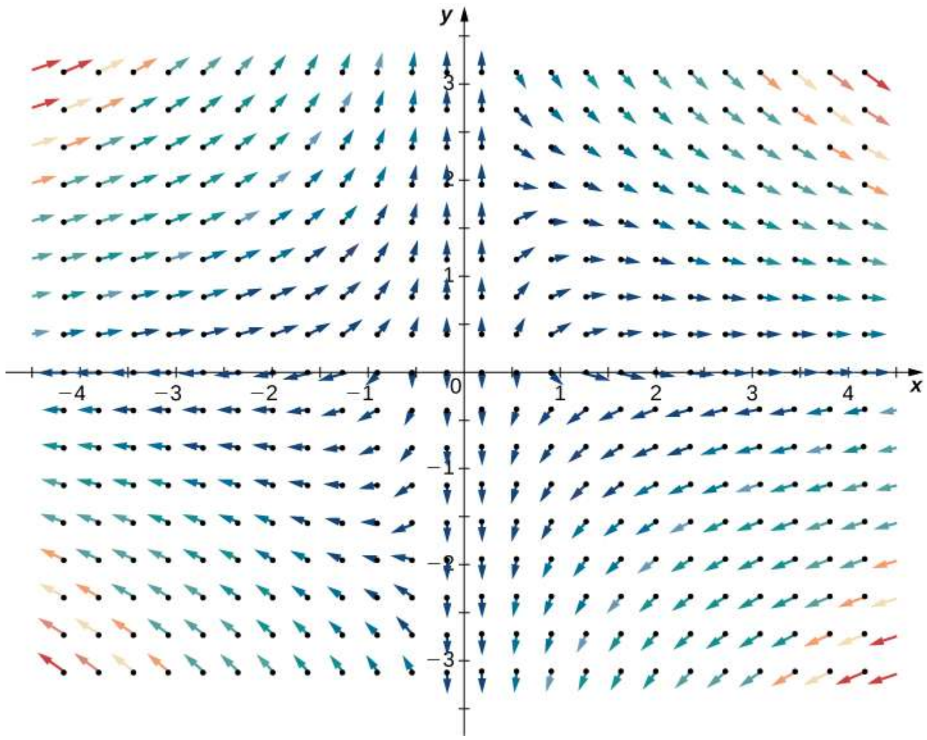


Figure 6.52 The divergence of vector field $\mathbf{F}(x, y) = \langle x^2 y, y - xy^2 \rangle$ is one, so it cannot model a magnetic field.

Another application for divergence is detecting whether a field is source free. Recall that a source-free field is a vector field that has a stream function; equivalently, a source-free field is a field with a flux that is zero along any closed curve. The next two theorems say that, under certain conditions, source-free vector fields are precisely the vector fields with zero divergence.

Theorem 6.14: Divergence of a Source-Free Vector Field

If $\mathbf{F} = \langle P, Q \rangle$ is a source-free continuous vector field with differentiable component functions, then $\text{div}\,\mathbf{F} = 0$.

Proof

Since $\mathbf{F}$ is source free, there is a function $g(x, y)$ with $g_y = P$ and $-g_x = Q$. Therefore, $\mathbf{F} = \langle g_y, -g_x \rangle$ and $\text{div}\,\mathbf{F} = g_{yx} - g_{xy} = 0$ by Clairaut's theorem.

□

The converse of **Divergence of a Source-Free Vector Field** is true on simply connected regions, but the proof is too technical to include here. Thus, we have the following theorem, which can test whether a vector field in $\mathbb{R}^2$ is source free.

Theorem 6.15: Divergence Test for Source-Free Vector Fields

Let $\mathbf{F} = \langle P, Q \rangle$ be a continuous vector field with differentiable component functions with a domain that is simply connected. Then, $\text{div}\,\mathbf{F} = 0$ if and only if $\mathbf{F}$ is source free.

Example 6.50

Determining Whether a Field Is Source Free

Is field $\mathbf{F}(x, y) = \langle x^2 y, 5 - xy^2 \rangle$ source free?

Solution

Note the domain of $\mathbf{F}$ is $\mathbb{R}^2$, which is simply connected. Furthermore, $\mathbf{F}$ is continuous with differentiable component functions. Therefore, we can use **Divergence Test for Source-Free Vector Fields** to analyze $\mathbf{F}$. The divergence of $\mathbf{F}$ is

$$\frac{\partial}{\partial x}\left(x^2 y\right) + \frac{\partial}{\partial y}\left(5 - xy^2\right) = 2xy - 2xy = 0.$$

Therefore, $\mathbf{F}$ is source free by **Divergence Test for Source-Free Vector Fields**.

6.41 Let $\mathbf{F}(x, y) = \langle -ay, bx \rangle$ be a rotational field where a and b are positive constants. Is $\mathbf{F}$ source free?

Recall that the flux form of Green's theorem says that

$$\oint_C \mathbf{F} \cdot \mathbf{N} ds = \iint_D P_x + Q_y \, dA,$$

where C is a simple closed curve and D is the region enclosed by C. Since $P_x + Q_y = \text{div}\,\mathbf{F}$, Green's theorem is sometimes written as

$$\oint_C \mathbf{F} \cdot \mathbf{N} ds = \iint_D \text{div}\,\mathbf{F} \, dA.$$

Therefore, Green's theorem can be written in terms of divergence. If we think of divergence as a derivative of sorts, then Green's theorem says the "derivative" of $\mathbf{F}$ on a region can be translated into a line integral of $\mathbf{F}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on a line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using divergence, we can see that Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can use all of what we have learned in the application of divergence. Let $\mathbf{v}$ be a vector field modeling the velocity of a fluid. Since the divergence of $\mathbf{v}$ at point P measures the “outflowing-ness” of the fluid at P, $\text{div}\,\mathbf{v}(P) > 0$ implies that more fluid is flowing out of P than flowing in. Similarly, $\text{div}\,\mathbf{v}(P) < 0$ implies the more fluid is flowing in to P than is flowing out, and $\text{div}\,\mathbf{v}(P) = 0$ implies the same amount of fluid is flowing in as flowing out.

Example 6.51

Determining Flow of a Fluid

Suppose $\mathbf{v}(x, y) = \langle -xy, y \rangle, y > 0$ models the flow of a fluid. Is more fluid flowing into point $(1, 4)$ than flowing out?

Solution

To determine whether more fluid is flowing into $(1, 4)$ than is flowing out, we calculate the divergence of $\mathbf{v}$ at $(1, 4)$:

$$\text{div}(\mathbf{v}) = \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(y) = -y + 1.$$

To find the divergence at $(1, 4)$, substitute the point into the divergence: $-4 + 1 = -3$. Since the divergence of $\mathbf{v}$ at $(1, 4)$ is negative, more fluid is flowing in than flowing out (**Figure 6.53**).

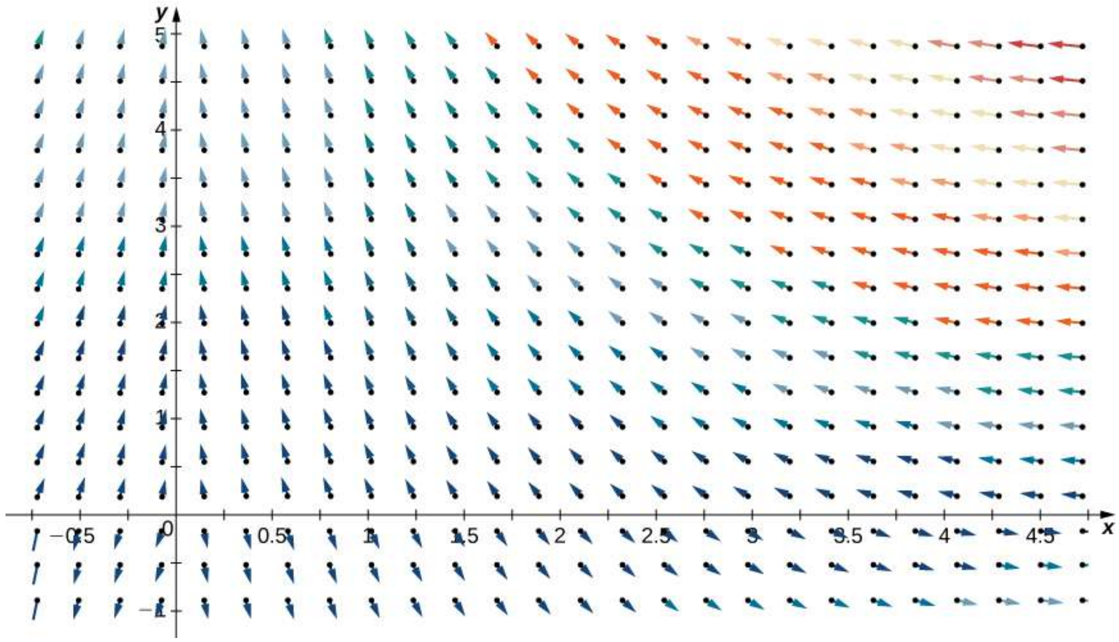


Figure 6.53 Vector field $\mathbf{v}(x, y) = \langle -xy, y \rangle$ has negative divergence at $(1, 4)$.

6.42 For vector field $\mathbf{v}(x, y) = \langle -xy, y \rangle, y > 0,$ find all points P such that the amount of fluid flowing in to P equals the amount of fluid flowing out of P.

Curl

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. Suppose that $\mathbf{F}$ represents the velocity field of a fluid. Then, the curl of $\mathbf{F}$ at point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector. The magnitude of the curl vector at P measures how quickly the particles rotate around this axis. In other words, the curl at a point is a measure of the vector field's "spin" at that point. Visually, imagine placing a paddlewheel into a fluid at P, with the axis of the paddlewheel aligned with the curl vector (**Figure 6.54**). The curl measures the tendency of the paddlewheel to rotate.

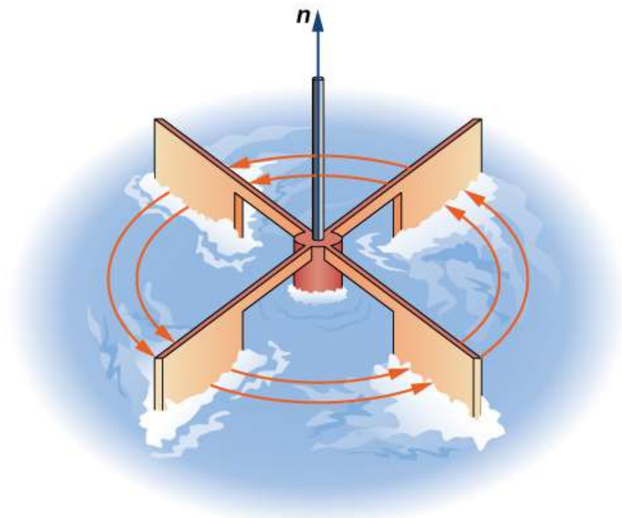


Figure 6.54 To visualize curl at a point, imagine placing a small paddlewheel into the vector field at a point.

Consider the vector fields in **Figure 6.50**. In part (a), the vector field is constant and there is no spin at any point. Therefore, we expect the curl of the field to be zero, and this is indeed the case. Part (b) shows a rotational field, so the field has spin. In particular, if you place a paddlewheel into a field at any point so that the axis of the wheel is perpendicular to a plane, the wheel rotates counterclockwise. Therefore, we expect the curl of the field to be nonzero, and this is indeed the case (the curl is $2\mathbf{k}$).

To see what curl is measuring globally, imagine dropping a leaf into the fluid. As the leaf moves along with the fluid flow, the curl measures the tendency of the leaf to rotate. If the curl is zero, then the leaf doesn't rotate as it moves through the fluid.

Definition

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$, and $P_x, Q_y,$ and R_z all exist, then the **curl** of **F** is defined by

$$\begin{aligned}\text{curl } \mathbf{F} &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.\end{aligned} \tag{6.17}$$

Note that the curl of a vector field is a vector field, in contrast to divergence.

The definition of curl can be difficult to remember. To help with remembering, we use the notation $\nabla \times \mathbf{F}$ to stand for a "determinant" that gives the curl formula:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

The determinant of this matrix is

$$(R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} = \text{curl } \mathbf{F}.$$

Thus, this matrix is a way to help remember the formula for curl. Keep in mind, though, that the word *determinant* is used very loosely. A determinant is not really defined on a matrix with entries that are three vectors, three operators, and three functions.

If $\mathbf{F} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$, then the curl of **F**, by definition, is

$$\text{curl } \mathbf{F} = (Q_x - P_y)\mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Example 6.52

Finding the Curl of a Three-Dimensional Vector Field

Find the curl of $\mathbf{F}(P, Q, R) = \langle x^2 z, e^y + xz, xyz \rangle$.

Solution

The curl is

$$\begin{aligned} \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ &= (xz - x)\mathbf{i} + (x^2 - yz)\mathbf{j} + z\mathbf{k}. \end{aligned}$$

6.43 Find the curl of $\mathbf{F} = \langle \sin x \cos z, \sin y \sin z, \cos x \cos y \rangle$ at point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

Example 6.53

Finding the Curl of a Two-Dimensional Vector Field

Find the curl of $\mathbf{F} = \langle P, Q \rangle = \langle y, 0 \rangle$.

Solution

Notice that this vector field consists of vectors that are all parallel. In fact, each vector in the field is parallel to the x-axis. This fact might lead us to the conclusion that the field has no spin and that the curl is zero. To test this theory, note that

$$\text{curl } \mathbf{F} = (Q_x - P_y)\mathbf{k} = -\mathbf{k} \neq 0.$$

Therefore, this vector field does have spin. To see why, imagine placing a paddlewheel at any point in the first quadrant (Figure 6.55). The larger magnitudes of the vectors at the top of the wheel cause the wheel to rotate. The wheel rotates in the clockwise (negative) direction, causing the coefficient of the curl to be negative.

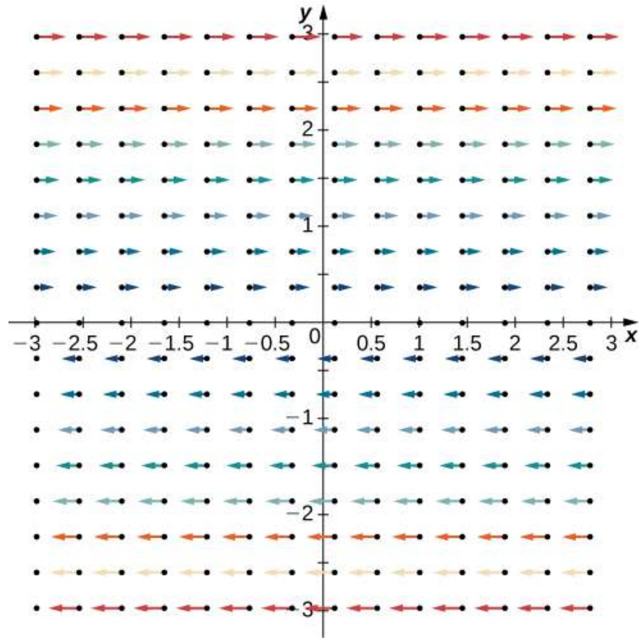


Figure 6.55 Vector field $\mathbf{F}(x, y) = \langle y, 0 \rangle$ consists of vectors that are all parallel.

Note that if $\mathbf{F} = \langle P, Q \rangle$ is a vector field in a plane, then $\text{curl}\,\mathbf{F} \cdot \mathbf{k} = (Q_x - P_y)\mathbf{k} \cdot \mathbf{k} = Q_x - P_y$. Therefore, the circulation form of Green's theorem is sometimes written as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl}\,\mathbf{F} \cdot \mathbf{k}\, dA,$$

where C is a simple closed curve and D is the region enclosed by C. Therefore, the circulation form of Green's theorem can be written in terms of the curl. If we think of curl as a derivative of sorts, then Green's theorem says that the "derivative" of **F** on a region can be translated into a line integral of **F** along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using curl, we can see the circulation form of Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can now use what we have learned about curl to show that gravitational fields have no "spin." Suppose there is an object at the origin with mass m_1 at the origin and an object with mass m_2. Recall that the gravitational force that object 1 exerts on object 2 is given by field

$$\mathbf{F}(x, y, z) = -Gm_2m_2 \left\langle \frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}}, \frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}}, \frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}} \right\rangle .$$

Example 6.54

Determining the Spin of a Gravitational Field

Show that a gravitational field has no spin.

Solution

To show that $\mathbf{F}$ has no spin, we calculate its curl. Let $P(x, y, z) = \frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}}$, $Q(x, y, z) = \frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}}$, and $R(x, y, z) = \frac{z}{\left(x^2 + y^2 + z^2\right)^{3/2}}$. Then,

$$\begin{aligned}
\operatorname{curl} \mathbf{F} &= -Gm_1 m_2\left[\left(R_y - Q_z\right)\mathbf{i} + \left(P_z - R_x\right)\mathbf{j} + \left(Q_x - P_y\right)\mathbf{k}\right] \\
&= -Gm_1 m_2 \begin{bmatrix} \left(\frac{-3yz}{\left(x^2 + y^2 + z^2\right)^{5/2}} - \left(\frac{-3yz}{\left(x^2 + y^2 + z^2\right)^{5/2}}\right)\right)\mathbf{i} \\ +\left(\frac{-3xz}{\left(x^2 + y^2 + z^2\right)^{5/2}} - \left(\frac{-3xz}{\left(x^2 + y^2 + z^2\right)^{5/2}}\right)\right)\mathbf{j} \\ +\left(\frac{-3xy}{\left(x^2 + y^2 + z^2\right)^{5/2}} - \left(\frac{-3xy}{\left(x^2 + y^2 + z^2\right)^{5/2}}\right)\right)\mathbf{k} \end{bmatrix} \\
&= 0.
\end{aligned}$$

Since the curl of the gravitational field is zero, the field has no spin.

6.44 Field $\mathbf{v}(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$ models the flow of a fluid. Show that if you drop a leaf into this fluid, as the leaf moves over time, the leaf does not rotate.

Using Divergence and Curl

Now that we understand the basic concepts of divergence and curl, we can discuss their properties and establish relationships between them and conservative vector fields.

If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$, then the curl of $\mathbf{F}$ is also a vector field in $\mathbb{R}^3$. Therefore, we can take the divergence of a curl. The next theorem says that the result is always zero. This result is useful because it gives us a way to show that some vector fields are not the curl of any other field. To give this result a physical interpretation, recall that divergence of a velocity field $\mathbf{v}$ at point P measures the tendency of the corresponding fluid to flow out of P. Since $\operatorname{div} \operatorname{curl}(\mathbf{v}) = 0$, the net rate of flow in vector field curl($\mathbf{v}$) at any point is zero. Taking the curl of vector field $\mathbf{F}$ eliminates whatever divergence was present in $\mathbf{F}$.

Theorem 6.16: Divergence of the Curl

Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field in $\mathbb{R}^3$ such that the component functions all have continuous second-order partial derivatives. Then, $\operatorname{div} \operatorname{curl}(\mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Proof

By the definitions of divergence and curl, and by Clairaut's theorem,

$$\begin{aligned}\text{div curl } \mathbf{F} &= \text{div}\left[(R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}\right] \\ &= R_{yx} - Q_{xz} + P_{yz} - R_{yx} + Q_{zx} - P_{zy} \\ &= 0.\end{aligned}$$

□

Example 6.55

Showing That a Vector Field Is Not the Curl of Another

Show that $\mathbf{F}(x, y, z) = e^x \mathbf{i} + yz\mathbf{j} + xz^2 \mathbf{k}$ is not the curl of another vector field. That is, show that there is no other vector **G** with $\text{curl } \mathbf{G} = \mathbf{F}$.

Solution

Notice that the domain of **F** is all of $\mathbb{R}^3$ and the second-order partials of **F** are all continuous. Therefore, we can apply the previous theorem to **F**.

The divergence of **F** is $e^x + z + 2xz$. If **F** were the curl of vector field **G**, then $\text{div } \mathbf{F} = \text{div curl } \mathbf{G} = 0$. But, the divergence of **F** is not zero, and therefore **F** is not the curl of any other vector field.

6.45 Is it possible for $\mathbf{G}(x, y, z) = \langle \sin x, \cos y, \sin(xyz) \rangle$ to be the curl of a vector field?

With the next two theorems, we show that if **F** is a conservative vector field then its curl is zero, and if the domain of **F** is simply connected then the converse is also true. This gives us another way to test whether a vector field is conservative.

Theorem 6.17: Curl of a Conservative Vector Field

If $\mathbf{F} = \langle P, Q, R \rangle$ is conservative, then $\text{curl } \mathbf{F} = 0$.

Proof

Since conservative vector fields satisfy the cross-partials property, all the cross-partials of **F** are equal. Therefore,

$$\begin{aligned}\text{curl } \mathbf{F} &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \\ &= 0.\end{aligned}$$

□

The same theorem is true for vector fields in a plane.

Since a conservative vector field is the gradient of a scalar function, the previous theorem says that $\text{curl}\,(\nabla f) = 0$ for any scalar function f. In terms of our curl notation, $\nabla \times \nabla(f) = 0$. This equation makes sense because the cross product of a vector with itself is always the zero vector. Sometimes equation $\nabla \times \nabla(f) = 0$ is simplified as $\nabla \times \nabla = 0$.

Theorem 6.18: Curl Test for a Conservative Field

Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field in space on a simply connected domain. If $\text{curl } \mathbf{F} = 0$, then **F** is conservative.

Proof

Since $\text{curl}\,\mathbf{F} = 0,$ we have that $R_y = Q_z, P_z = R_x,$ and $Q_x = P_y.$ Therefore, **F** satisfies the cross-partials property on a simply connected domain, and **Cross-Partial Property of Conservative Fields** implies that **F** is conservative.

□

The same theorem is also true in a plane. Therefore, if **F** is a vector field in a plane or in space and the domain is simply connected, then **F** is conservative if and only if $\text{curl}\,\mathbf{F} = 0.$

Example 6.56

Testing Whether a Vector Field Is Conservative

Use the curl to determine whether $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$ is conservative.

Solution

Note that the domain of **F** is all of $\mathbb{R}^3,$ which is simply connected (**Figure 6.56**). Therefore, we can test whether **F** is conservative by calculating its curl.

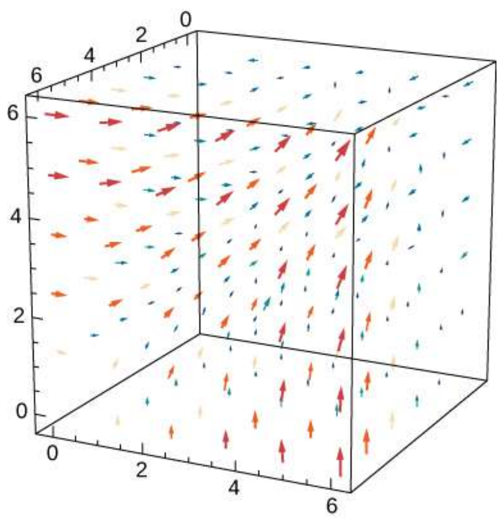


Figure 6.56 The curl of vector field $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$ is zero.

The curl of **F** is

$$\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\mathbf{i} + \left(\frac{\partial}{\partial y}yz - \frac{\partial}{\partial z}xy\right)\mathbf{j} + \left(\frac{\partial}{\partial y}xz - \frac{\partial}{\partial z}yz\right)\mathbf{k} = (x - x)\mathbf{i} + (y - y)\mathbf{j} + (z - z)\mathbf{k} = 0.$$

Thus, **F** is conservative.

We have seen that the curl of a gradient is zero. What is the divergence of a gradient? If f is a function of two variables, then $\text{div}(\nabla f) = \nabla \cdot (\nabla f) = f_{xx} + f_{yy}.$ We abbreviate this "double dot product" as $\nabla^2.$ This operator is called the *Laplace operator*, and in this notation Laplace's equation becomes $\nabla^2 f = 0.$ Therefore, a harmonic function is a function that becomes zero after taking the divergence of a gradient.

Similarly, if f is a function of three variables then

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz}.$$

Using this notation we get Laplace's equation for harmonic functions of three variables:

$$\nabla^2 f = 0.$$

Harmonic functions arise in many applications. For example, the potential function of an electrostatic field in a region of space that has no static charge is harmonic.

Example 6.57

Analyzing a Function

Is it possible for $f(x, y) = x^2 + x - y$ to be the potential function of an electrostatic field that is located in a region of $\mathbb{R}^2$ free of static charge?

Solution

If f were such a potential function, then f would be harmonic. Note that $f_{xx} = 2$ and $f_{yy} = 0,$ and so $f_{xx} + f_{yy} \neq 0.$ Therefore, f is not harmonic and f cannot represent an electrostatic potential.

6.46 Is it possible for function $f(x, y) = x^2 - y^2 + x$ to be the potential function of an electrostatic field located in a region of $\mathbb{R}^2$ free of static charge?

6.5 EXERCISES

For the following exercises, determine whether the statement is *true or false*.

206. If the coordinate functions of $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ have continuous second partial derivatives, then $\text{curl}\,(\text{div}(\mathbf{F}))$ equals zero.

207. $\nabla \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 1.$

208. All vector fields of the form $\mathbf{F}(x, y, z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$ are conservative.

209. If $\text{curl}\,\mathbf{F} = 0,$ then $\mathbf{F}$ is conservative.

210. If $\mathbf{F}$ is a constant vector field then $\text{div}\,\mathbf{F} = 0.$

211. If $\mathbf{F}$ is a constant vector field then $\text{curl}\,\mathbf{F} = 0.$

For the following exercises, find the curl of $\mathbf{F}$.

212. $\mathbf{F}(x, y, z) = xy^2z^4\mathbf{i} + \left(2x^2y + z\right)\mathbf{j} + y^3z^2\mathbf{k}$

213. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + y^2x\mathbf{j} + (y + 2z)\mathbf{k}$

214. $\mathbf{F}(x, y, z) = 3xyz^2\mathbf{i} + y^2\sin z\mathbf{j} + xe^{2z}\mathbf{k}$

215. $\mathbf{F}(x, y, z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$

216. $\mathbf{F}(x, y, z) = (x\cos y)\mathbf{i} + xy^2\mathbf{j}$

217. $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$

218. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + x^2y^2z^2\mathbf{j} + y^2z^3\mathbf{k}$

219. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

220. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

221. $\mathbf{F}(x, y, z) = ax\mathbf{i} + by\mathbf{j} + c\mathbf{k}$ for constants a, b, c

For the following exercises, find the divergence of $\mathbf{F}$.

222. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + y^2x\mathbf{j} + (y + 2z)\mathbf{k}$

223. $\mathbf{F}(x, y, z) = 3xyz^2\mathbf{i} + y^2\sin z\mathbf{j} + xe^{2z}\mathbf{k}$

224. $\mathbf{F}(x, y) = (\sin x)\mathbf{i} + (\cos y)\mathbf{j}$

225. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

226. $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$

227. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$

228. $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$

229. $\mathbf{F}(x, y, z) = ax\mathbf{i} + by\mathbf{j} + c\mathbf{k}$ for constants a, b, c

230. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + x^2y^2z^2\mathbf{j} + y^2z^3\mathbf{k}$

231. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

For the following exercises, determine whether each of the given scalar functions is harmonic.

232. $u(x, y, z) = e^{-x}(\cos y - \sin y)$

233. $w(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$

234. If $\mathbf{F}(x, y, z) = 2\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k},$ find $\text{curl}\,(\mathbf{F} \times \mathbf{G}).$

235. If $\mathbf{F}(x, y, z) = 2\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k},$ find $\text{div}\,(\mathbf{F} \times \mathbf{G}).$

236. Find $\text{div}\,\mathbf{F},$ given that $\mathbf{F} = \nabla f,$ where $f(x, y, z) = xy^3z^2.$

237. Find the divergence of $\mathbf{F}$ for vector field $\mathbf{F}(x, y, z) = \left(y^2 + z^2\right)(x + y)\mathbf{i} + \left(z^2 + x^2\right)(y + z)\mathbf{j} + \left(x^2 + y^2\right)(z + x)\mathbf{k}.$

238. Find the divergence of $\mathbf{F}$ for vector field $\mathbf{F}(x, y, z) = f_1(y, z)\mathbf{i} + f_2(x, z)\mathbf{j} + f_3(x, y)\mathbf{k}.$

For the following exercises, use $r = |\mathbf{r}|$ and $\mathbf{r} = (x, y, z).$

239. Find the $\text{curl}\,\mathbf{r}.$

240. Find the $\text{curl}\,\dfrac{\mathbf{r}}{r}.$

241. Find the $\text{curl}\,\dfrac{\mathbf{r}}{r^3}.$

242. Let $\mathbf{F}(x, y) = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2},$ where $\mathbf{F}$ is defined on $\{(x, y) \in \mathbb{R} \,|\, (x, y) \neq (0, 0)\}.$ Find $\text{curl}\,\mathbf{F}.$

For the following exercises, use a computer algebra system to find the curl of the given vector fields.

243. **[T]** $\mathbf{F}(x, y, z) = \arctan\left(\frac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\mathbf{j} + \mathbf{k}$

244. **[T]** $\mathbf{F}(x, y, z) = \sin(x - y)\mathbf{i} + \sin(y - z)\mathbf{j} + \sin(z - x)\mathbf{k}$

For the following exercises, find the divergence of **F** at the given point.

245. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ at $(2, -1, 3)$

246. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ at $(1, 2, 3)$

247. $\mathbf{F}(x, y, z) = e^{-xy}\mathbf{i} + e^{xz}\mathbf{j} + e^{yz}\mathbf{k}$ at $(3, 2, 0)$

248. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ at $(1, 2, 1)$

249. $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j}$ at $(0, 0, 3)$

For the following exercises, find the curl of **F** at the given point.

250. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ at $(2, -1, 3)$

251. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ at $(1, 2, 3)$

252. $\mathbf{F}(x, y, z) = e^{-xy}\mathbf{i} + e^{xz}\mathbf{j} + e^{yz}\mathbf{k}$ at $(3, 2, 0)$

253. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ at $(1, 2, 1)$

254. $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j}$ at $(0, 0, 3)$

255. Let $\mathbf{F}(x, y, z) = \left(3x^2 y + az\right)\mathbf{i} + x^3\mathbf{j} + \left(3x + 3z^2\right)\mathbf{k}$. For what value of a is **F** conservative?

256. Given vector field $\mathbf{F}(x, y) = \frac{1}{x^2 + y^2}(-y, x)$ on domain $D = \frac{\mathbb{R}^2}{\{(0, 0)\}} = \left\{(x, y) \in \mathbb{R}^2 \middle| (x, y) \neq (0, 0)\right\}$, is **F** conservative?

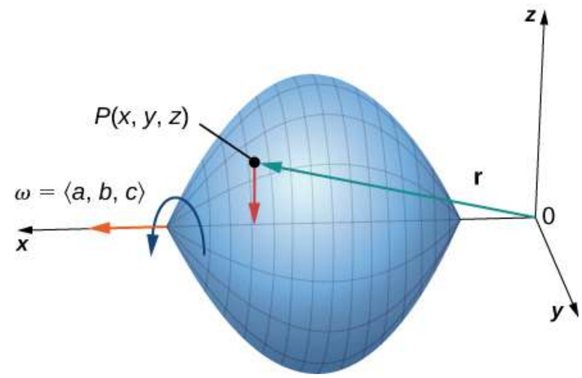
257. Given vector field $\mathbf{F}(x, y) = \frac{1}{x^2 + y^2}(x, y)$ on domain $D = \frac{\mathbb{R}^2}{\{(0, 0)\}}$, is **F** conservative?

258. Find the work done by force field $\mathbf{F}(x, y) = e^{-y}\mathbf{i} - xe^{-y}\mathbf{j}$ in moving an object from $P(0, 1)$ to $Q(2, 0)$. Is the force field conservative?

259. Compute divergence $\mathbf{F} = (\sinh x)\mathbf{i} + (\cosh y)\mathbf{j} - xyz\mathbf{k}$.

260. Compute curl $\mathbf{F} = (\sinh x)\mathbf{i} + (\cosh y)\mathbf{j} - xyz\mathbf{k}$.

For the following exercises, consider a rigid body that is rotating about the x-axis counterclockwise with constant angular velocity $\omega = \langle a, b, c \rangle$. If P is a point in the body located at $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the velocity at P is given by vector field $\mathbf{F} = \omega \times \mathbf{r}$.



261. Express **F** in terms of **i**, **j**, and **k** vectors.

262. Find $\text{div}\,\mathbf{F}$.

263. Find $\text{curl}\,\mathbf{F}$

In the following exercises, suppose that $\nabla \cdot \mathbf{F} = 0$ and $\nabla \cdot \mathbf{G} = 0$.

264. Does $\mathbf{F} + \mathbf{G}$ necessarily have zero divergence?

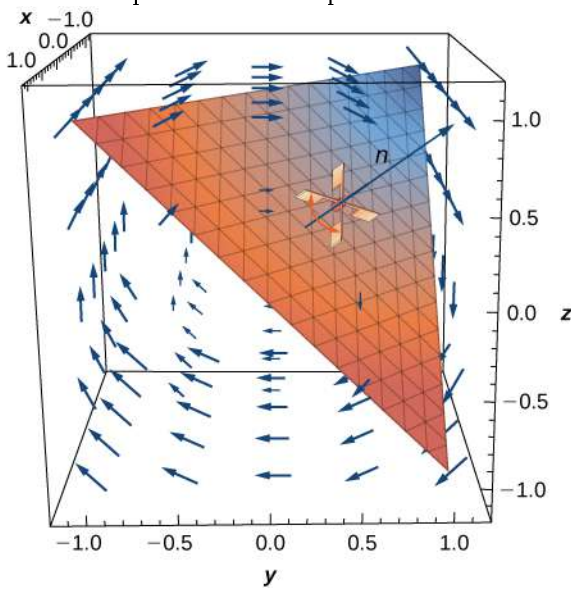
265. Does $\mathbf{F} \times \mathbf{G}$ necessarily have zero divergence?

In the following exercises, suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\mathbf{F} = -k\nabla T$, where $k > 0$ is a property of the material. The heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\nabla \cdot \mathbf{F} = -k\nabla \cdot \nabla T = -k\nabla^2 T$.

266. Compute the heat flow vector field.

267. Compute the divergence.

268. **[T]** Consider rotational velocity field $\mathbf{v} = \langle 0, 10z, -10y \rangle$. If a paddlewheel is placed in plane $x + y + z = 1$ with its axis normal to this plane, using a computer algebra system, calculate how fast the paddlewheel spins in revolutions per unit time.



6.6 | Surface Integrals

Learning Objectives

6.6.1 Find the parametric representations of a cylinder, a cone, and a sphere.
6.6.2 Describe the surface integral of a scalar-valued function over a parametric surface.
6.6.3 Use a surface integral to calculate the area of a given surface.
6.6.4 Explain the meaning of an oriented surface, giving an example.
6.6.5 Describe the surface integral of a vector field.
6.6.6 Use surface integrals to solve applied problems.

We have seen that a line integral is an integral over a path in a plane or in space. However, if we wish to integrate over a surface (a two-dimensional object) rather than a path (a one-dimensional object) in space, then we need a new kind of integral that can handle integration over objects in higher dimensions. We can extend the concept of a line integral to a surface integral to allow us to perform this integration.

Surface integrals are important for the same reasons that line integrals are important. They have many applications to physics and engineering, and they allow us to develop higher dimensional versions of the Fundamental Theorem of Calculus. In particular, surface integrals allow us to generalize Green's theorem to higher dimensions, and they appear in some important theorems we discuss in later sections.

Parametric Surfaces

A surface integral is similar to a line integral, except the integration is done over a surface rather than a path. In this sense, surface integrals expand on our study of line integrals. Just as with line integrals, there are two kinds of surface integrals: a surface integral of a scalar-valued function and a surface integral of a vector field.

However, before we can integrate over a surface, we need to consider the surface itself. Recall that to calculate a scalar or vector line integral over curve C, we first need to parameterize C. In a similar way, to calculate a surface integral over surface S, we need to parameterize S. That is, we need a working concept of a **parameterized surface** (or a **parametric surface**), in the same way that we already have a concept of a parameterized curve.

A parameterized surface is given by a description of the form

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle .$$

Notice that this parameterization involves two parameters, u and v, because a surface is two-dimensional, and therefore two variables are needed to trace out the surface. The parameters u and v vary over a region called the parameter domain, or **parameter space**—the set of points in the uv-plane that can be substituted into $\mathbf{r}$. Each choice of u and v in the parameter domain gives a point on the surface, just as each choice of a parameter t gives a point on a parameterized curve. The entire surface is created by making all possible choices of u and v over the parameter domain.

Definition

Given a parameterization of surface $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, the **parameter domain** of the parameterization is the set of points in the uv-plane that can be substituted into $\mathbf{r}$.

Example 6.58

Parameterizing a Cylinder

Describe surface S parameterized by

$$\mathbf{r}(u, v) = \langle \cos u, \sin u, v \rangle , -\infty < u < \infty, -\infty < v < \infty.$$

Solution

To get an idea of the shape of the surface, we first plot some points. Since the parameter domain is all of $\mathbb{R}^2$, we can choose any value for u and v and plot the corresponding point. If $u = v = 0$, then $\mathbf{r}(0, 0) = \langle 1, 0, 0 \rangle$, so point $(1, 0, 0)$ is on S. Similarly, points $\mathbf{r}(\pi, 2) = (-1, 0, 2)$ and $\mathbf{r}\left(\frac{\pi}{2}, 4\right) = (0, 1, 4)$ are on S.

Although plotting points may give us an idea of the shape of the surface, we usually need quite a few points to see the shape. Since it is time-consuming to plot dozens or hundreds of points, we use another strategy. To visualize S, we visualize two families of curves that lie on S. In the first family of curves we hold u constant; in the second family of curves we hold v constant. This allows us to build a “skeleton” of the surface, thereby getting an idea of its shape.

First, suppose that u is a constant K. Then the curve traced out by the parameterization is $\langle \cos K, \sin K, v \rangle$, which gives a vertical line that goes through point $(\cos K, \sin K, v)$ in the xy-plane.

Now suppose that v is a constant K. Then the curve traced out by the parameterization is $\langle \cos u, \sin u, K \rangle$, which gives a circle in plane $z = K$ with radius 1 and center $(0, 0, K)$.

If u is held constant, then we get vertical lines; if v is held constant, then we get circles of radius 1 centered around the vertical line that goes through the origin. Therefore the surface traced out by the parameterization is cylinder $x^2 + y^2 = 1$ (**Figure 6.57**).

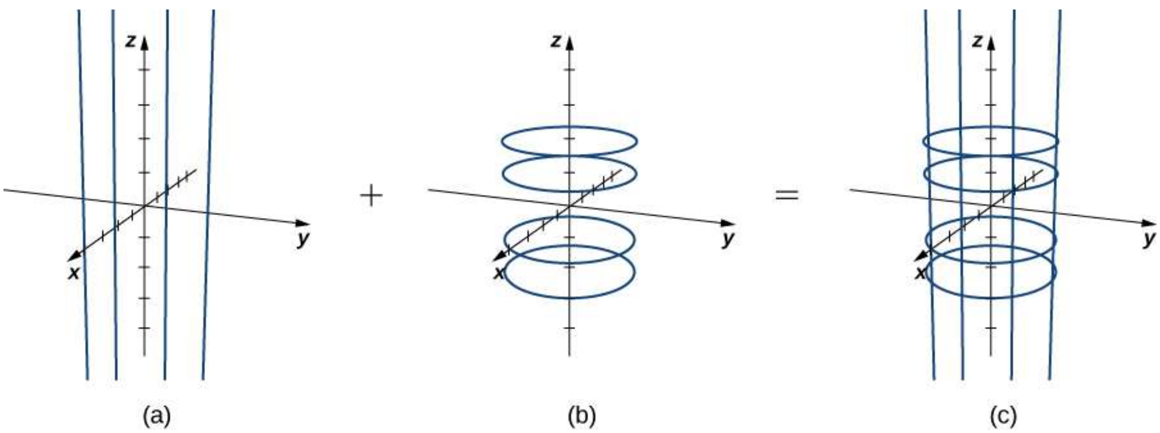


Figure 6.57 (a) Lines $\langle \cos K, \sin K, v \rangle$ for $K = 0, \frac{\pi}{2}, \pi,$ and $\frac{3\pi}{2}$. (b) Circles $\langle \cos u, \sin u, K \rangle$ for $K = -2, -1, 1,$ and 2. (c) The lines and circles together. As u and v vary, they describe a cylinder.

Notice that if $x = \cos u$ and $y = \sin u$, then $x^2 + y^2 = 1$, so points from S do indeed lie on the cylinder. Conversely, each point on the cylinder is contained in some circle $\langle \cos u, \sin u, k \rangle$ for some k, and therefore each point on the cylinder is contained in the parameterized surface (**Figure 6.58**).

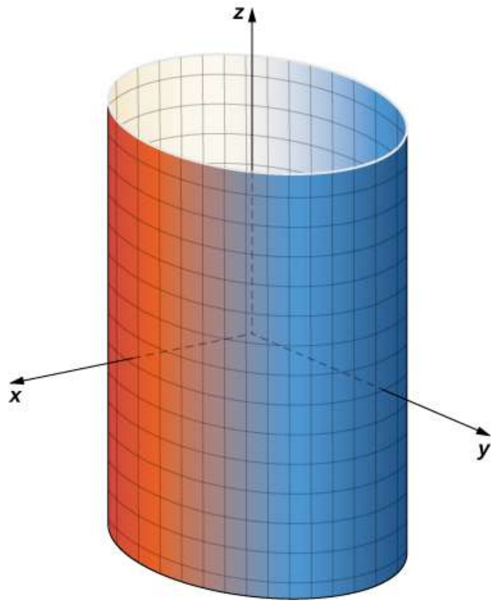


Figure 6.58 Cylinder $x^2 + y^2 = r^2$ has parameterization $\mathbf{r}(u, v) = \langle r\cos u, r\sin u, v \rangle$, $0 \le u \le 2\pi, -\infty < v < \infty.$

Analysis

Notice that if we change the parameter domain, we could get a different surface. For example, if we restricted the domain to $0 \le u \le \pi, 0 < v < 6,$ then the surface would be a half-cylinder of height 6.

6.47 Describe the surface with parameterization $\mathbf{r}(u, v) = \langle 2\cos u, 2\sin u, v \rangle$, $0 \le u < 2\pi, -\infty < v < \infty.$

It follows from **Example 6.58** that we can parameterize all cylinders of the form $x^2 + y^2 = R^2$. If S is a cylinder given by equation $x^2 + y^2 = R^2$, then a parameterization of S is

$$\mathbf{r}(u, v) = \langle R\cos u, R\sin u, v \rangle, \ 0 \le u < 2\pi, -\infty < v < \infty.$$

We can also find different types of surfaces given their parameterization, or we can find a parameterization when we are given a surface.

Example 6.59

Describing a Surface

Describe surface S parameterized by

$$\mathbf{r}(u, v) = \langle u\cos v, u\sin v, u^2 \rangle, \ 0 \le u < \infty, 0 \le v < 2\pi.$$

Solution

Notice that if u is held constant, then the resulting curve is a circle of radius u in plane $z = u$. Therefore, as u increases, the radius of the resulting circle increases. If v is held constant, then the resulting curve is a vertical parabola. Therefore, we expect the surface to be an elliptic paraboloid. To confirm this, notice that

$$\begin{aligned} x^2 + y^2 &= (u \cos v)^2 + (u \sin v)^2 \\ &= u^2 \cos^2 v + u^2 \sin^2 v \\ &= u^2 \\ &= z. \end{aligned}$$

Therefore, the surface is elliptic paraboloid $x^2 + y^2 = z$ (**Figure 6.59**).

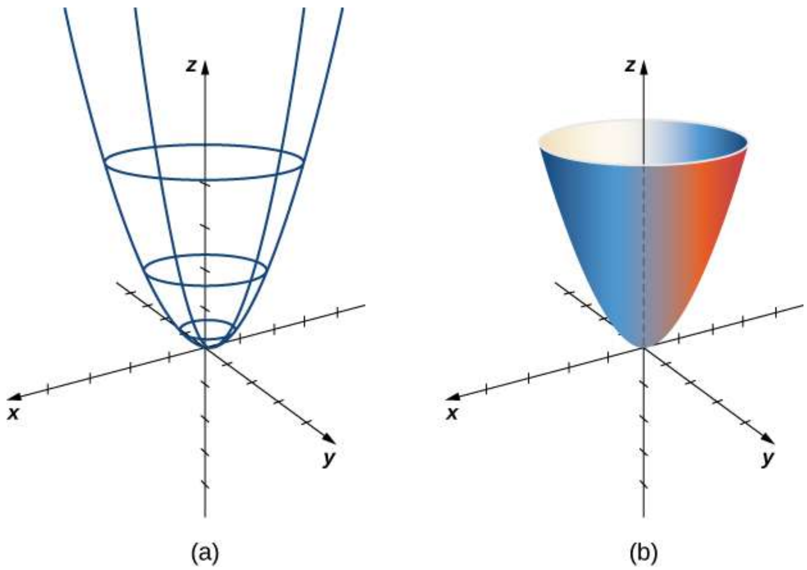


Figure 6.59 (a) Circles arise from holding u constant; the vertical parabolas arise from holding v constant. (b) An elliptic paraboloid results from all choices of u and v in the parameter domain.

6.48 Describe the surface parameterized by $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle$, $-\infty < u < \infty$, $0 \le v < 2\pi$.

Example 6.60

Finding a Parameterization

Give a parameterization of the cone $x^2 + y^2 = z^2$ lying on or above the plane $z = -2$.

Solution

The horizontal cross-section of the cone at height $z = u$ is circle $x^2 + y^2 = u^2$. Therefore, a point on the cone at height u has coordinates $(u \cos v, u \sin v, u)$ for angle v. Hence, a parameterization of the cone is $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle$. Since we are not interested in the entire cone, only the portion on or above plane $z = -2$, the parameter domain is given by $-2 \leq u < \infty, 0 \leq v < 2\pi$ (**Figure 6.60**).

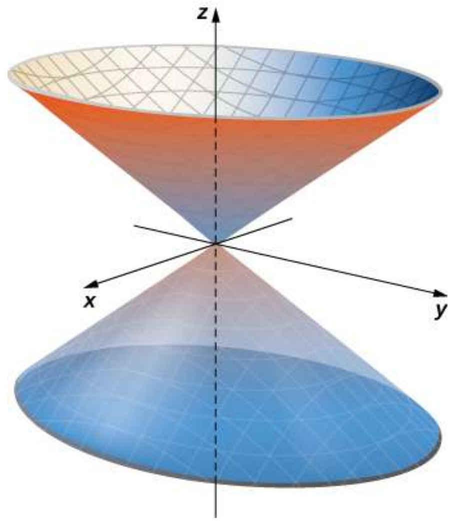


Figure 6.60 Cone $x^2 + y^2 = z^2$ has parameterization $r(u, v) = \langle u \cos v, u \sin v, u \rangle, -\infty < u < \infty, 0 \leq v \leq 2\pi.$

6.49 Give a parameterization for the portion of cone $x^2 + y^2 = z^2$ lying in the first octant.

We have discussed parameterizations of various surfaces, but two important types of surfaces need a separate discussion: spheres and graphs of two-variable functions. To parameterize a sphere, it is easiest to use spherical coordinates. The sphere of radius ρ centered at the origin is given by the parameterization

$$\mathbf{r}(\phi, \theta) = \langle \rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi \rangle, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi.$$

The idea of this parameterization is that as ϕ sweeps downward from the positive z-axis, a circle of radius $\rho \sin \phi$ is traced out by letting θ run from 0 to 2π. To see this, let ϕ be fixed. Then

$$\begin{aligned} x^2 + y^2 &= (\rho \cos \theta \sin \phi)^2 + (\rho \sin \theta \sin \phi)^2 \\ &= \rho^2 \sin^2 \phi \left(\cos^2 \theta + \sin^2 \theta\right) \\ &= \rho^2 \sin^2 \phi \\ &= (\rho \sin \phi)^2. \end{aligned}$$

This results in the desired circle (**Figure 6.61**).

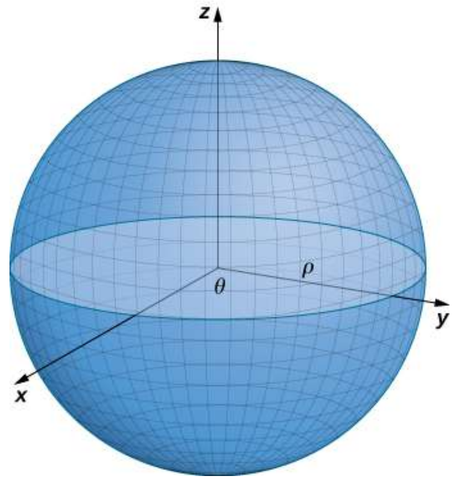


Figure 6.61 The sphere of radius ρ has parameterization $\mathbf{r}(\phi, \theta) = \langle \rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi \rangle,$ $0 \le \theta \le 2\pi, 0 \le \phi \le \pi.$

Finally, to parameterize the graph of a two-variable function, we first let $z = f(x, y)$ be a function of two variables. The simplest parameterization of the graph of f is $\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle,$ where x and y vary over the domain of f (**Figure 6.62**). For example, the graph of $f(x, y) = x^2 y$ can be parameterized by $\mathbf{r}(x, y) = \langle x, y, x^2 y \rangle,$ where the parameters x and y vary over the domain of f. If we only care about a piece of the graph of f—say, the piece of the graph over rectangle $[1, 3] \times [2, 5]$—then we can restrict the parameter domain to give this piece of the surface:

$$\mathbf{r}(x, y) = \langle x, y, x^2 y \rangle, 1 \le x \le 3, 2 \le y \le 5.$$

Similarly, if S is a surface given by equation $x = g(y, z)$ or equation $y = h(x, z),$ then a parameterization of S is $\mathbf{r}(y, z) = \langle g(y, z), y, z \rangle$ or $\mathbf{r}(x, z) = \langle x, h(x, z), z \rangle,$ respectively. For example, the graph of paraboloid $2y = x^2 + z^2$ can be parameterized by $\mathbf{r}(x, z) = \langle x, \frac{x^2 + z^2}{2}, z \rangle, 0 \le x < \infty, 0 \le z < \infty.$ Notice that we do not need to vary over the entire domain of y because x and z are squared.

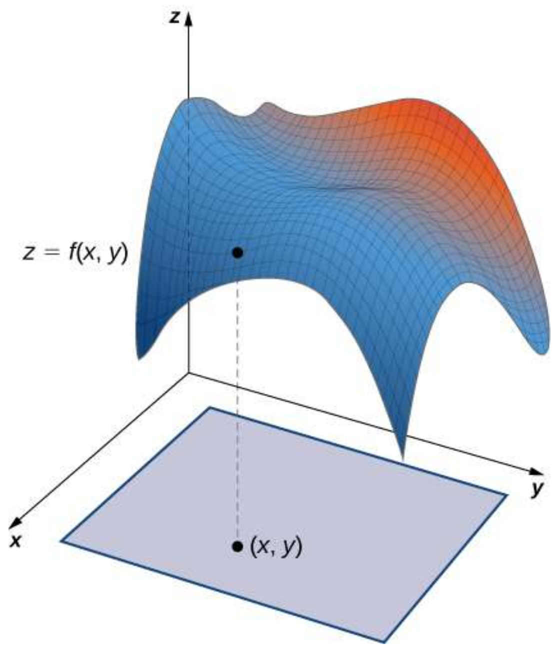


Figure 6.62 The simplest parameterization of the graph of a function is $\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle$.

Let's now generalize the notions of smoothness and regularity to a parametric surface. Recall that curve parameterization $\mathbf{r}(t), a \le t \le b$ is regular if $\mathbf{r}'(t) \ne 0$ for all t in $[a, b]$. For a curve, this condition ensures that the image of $\mathbf{r}$ really is a curve, and not just a point. For example, consider curve parameterization $\mathbf{r}(t) = \langle 1, 2 \rangle, 0 \le t \le 5$. The image of this parameterization is simply point $(1, 2)$, which is not a curve. Notice also that $\mathbf{r}'(t) = 0$. The fact that the derivative is zero indicates we are not actually looking at a curve.

Analogously, we would like a notion of regularity for surfaces so that a surface parameterization really does trace out a surface. To motivate the definition of regularity of a surface parameterization, consider parameterization

$$\mathbf{r}(u, v) = \langle 0, \cos v, 1 \rangle, 0 \le u \le 1, 0 \le v \le \pi.$$

Although this parameterization appears to be the parameterization of a surface, notice that the image is actually a line (**Figure 6.63**). How could we avoid parameterizations such as this? Parameterizations that do not give an actual surface? Notice that $\mathbf{r}_u = \langle 0, 0, 0 \rangle$ and $\mathbf{r}_v = \langle 0, -\sin v, 0 \rangle$, and the corresponding cross product is zero. The analog of the condition $\mathbf{r}'(t) = 0$ is that $\mathbf{r}_u \times \mathbf{r}_v$ is not zero for point (u, v) in the parameter domain, which is a regular parameterization.

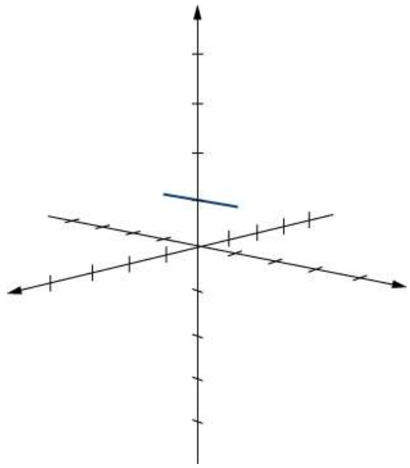


Figure 6.63 The image of parameterization $\mathbf{r}(u, v) = \langle 0, \cos v, 1 \rangle, 0 \le u \le 1, 0 \le v \le \pi$ is a line.

Definition

Parameterization $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ is a **regular parameterization** if $\mathbf{r}_u \times \mathbf{r}_v$ is not zero for point (u, v) in the parameter domain.

If parameterization $\mathbf{r}$ is regular, then the image of $\mathbf{r}$ is a two-dimensional object, as a surface should be. Throughout this chapter, parameterizations $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ are assumed to be regular.

Recall that curve parameterization $\mathbf{r}(t), a \le t \le b$ is smooth if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \ne 0$ for all t in $[a, b]$. Informally, a curve parameterization is smooth if the resulting curve has no sharp corners. The definition of a smooth surface parameterization is similar. Informally, a surface parameterization is *smooth* if the resulting surface has no sharp corners.

Definition

A surface parameterization $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ is *smooth* if vector $\mathbf{r}_u \times \mathbf{r}_v$ is not zero for any choice of u and v in the parameter domain.

A surface may also be *piecewise smooth* if it has smooth faces but also has locations where the directional derivatives do not exist.

Example 6.61

Identifying Smooth and Nonsmooth Surfaces

Which of the figures in Figure 6.64 is smooth?

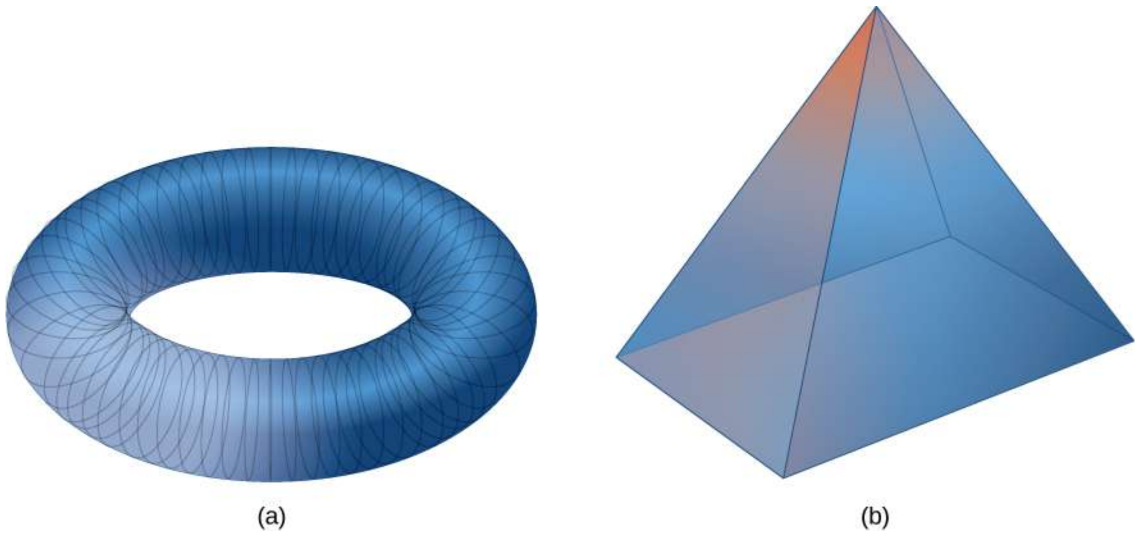


Figure 6.64 (a) This surface is smooth. (b) This surface is piecewise smooth.

Solution

The surface in **Figure 6.64**(a) can be parameterized by

$$\mathbf{r}(u, v) = \langle (2 + \cos v)\cos u, (2 + \cos v)\sin u, \sin v \rangle, 0 \le u < 2\pi, 0 \le v < 2\pi$$

(we can use technology to verify). Notice that vectors

$$\mathbf{r}_u = \langle -(2 + \cos v)\sin u, (2 + \cos v)\cos u, 0 \rangle \text{ and } \mathbf{r}_v = \langle -\sin v \cos u, -\sin v \sin u, \cos v \rangle$$

exist for any choice of u and v in the parameter domain, and

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(2 + \cos v)\sin u & (2 + \cos v)\cos u & 0 \\ -\sin v \cos u & -\sin v \sin u & \cos v \end{vmatrix} \\ &= [(2 + \cos v)\cos u \cos v]\mathbf{i} + [(2 + \cos v)\sin u \cos v]\mathbf{j} \\ &\quad + \left[(2 + \cos v)\sin v \sin^2 u + (2 + \cos v)\sin v \cos^2 u\right]\mathbf{k} \\ &= [(2 + \cos v)\cos u \cos v]\mathbf{i} + [(2 + \cos v)\sin u \cos v]\mathbf{j} + [(2 + \cos v)\sin v]\mathbf{k}.\end{aligned}$$

The $\mathbf{k}$ component of this vector is zero only if $v = 0$ or $v = \pi$. If $v = 0$ or $v = \pi$, then the only choices for u that make the $\mathbf{j}$ component zero are $u = 0$ or $u = \pi$. But, these choices of u do not make the $\mathbf{i}$ component zero. Therefore, $\mathbf{r}_u \times \mathbf{r}_v$ is not zero for any choice of u and v in the parameter domain, and the parameterization is smooth. Notice that the corresponding surface has no sharp corners.

In the pyramid in **Figure 6.64**(b), the sharpness of the corners ensures that directional derivatives do not exist at those locations. Therefore, the pyramid has no smooth parameterization. However, the pyramid consists of four smooth faces, and thus this surface is piecewise smooth.

6.50 Is the surface parameterization $\mathbf{r}(u, v) = \langle u^{2v}, v + 1, \sin u \rangle, 0 \le u \le 2, 0 \le v \le 3$ smooth?

Surface Area of a Parametric Surface

Our goal is to define a surface integral, and as a first step we have examined how to parameterize a surface. The second step is to define the surface area of a parametric surface. The notation needed to develop this definition is used throughout the rest of this chapter.

Let S be a surface with parameterization $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ over some parameter domain D. We assume here and throughout that the surface parameterization $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ is continuously differentiable—meaning, each component function has continuous partial derivatives. Assume for the sake of simplicity that D is a rectangle (although the following material can be extended to handle nonrectangular parameter domains). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of surface S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij}. Point P_{ij} corresponds to point (u_i, v_j) in the parameter domain.

Note that we can form a grid with lines that are parallel to the u-axis and the v-axis in the uv-plane. These grid lines correspond to a set of **grid curves** on surface S that is parameterized by $\mathbf{r}(u, v)$. Without loss of generality, we assume that P_{ij} is located at the corner of two grid curves, as in **Figure 6.65**. If we think of $\mathbf{r}$ as a mapping from the uv-plane to $\mathbb{R}^3$, the grid curves are the image of the grid lines under $\mathbf{r}$. To be precise, consider the grid lines that go through point (u_i, v_j). One line is given by $x = u_i, y = v$; the other is given by $x = u, y = v_j$. In the first grid line, the horizontal component is held constant, yielding a vertical line through (u_i, v_j). In the second grid line, the vertical component is held constant, yielding a horizontal line through (u_i, v_j). The corresponding grid curves are $\mathbf{r}(u_i, v)$ and $\mathbf{r}(u, v_j)$, and these curves intersect at point P_{ij}.

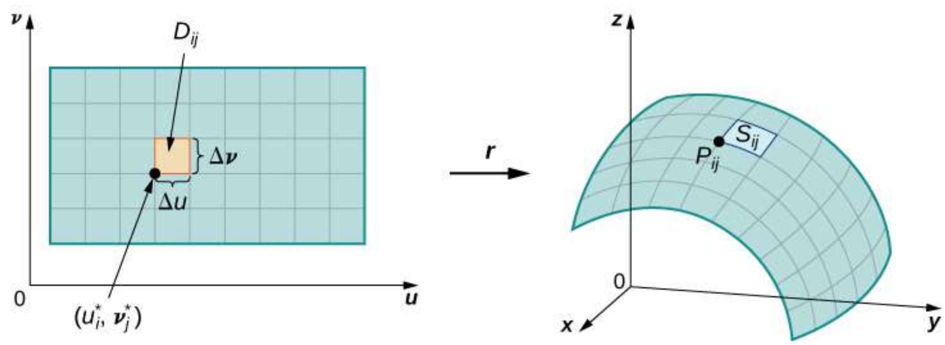


Figure 6.65 Grid lines on a parameter domain correspond to grid curves on a surface.

Now consider the vectors that are tangent to these grid curves. For grid curve $\mathbf{r}(u_i, v)$, the tangent vector at P_{ij} is

$$\mathbf{t}_v(P_{ij}) = \mathbf{r}_v(u_i, v_j) = \langle x_v(u_i, v_j), y_v(u_i, v_j), z_v(u_i, v_j) \rangle .$$

For grid curve $\mathbf{r}(u, v_j)$, the tangent vector at P_{ij} is

$$\mathbf{t}_u(P_{ij}) = \mathbf{r}_u(u_i, v_j) = \langle x_u(u_i, v_j), y_u(u_i, v_j), z_u(u_i, v_j) \rangle .$$

If vector $\mathbf{N} = \mathbf{t}_u(P_{ij}) \times \mathbf{t}_v(P_{ij})$ exists and is not zero, then the tangent plane at P_{ij} exists (**Figure 6.66**). If piece S_{ij} is small enough, then the tangent plane at point P_{ij} is a good approximation of piece S_{ij}.

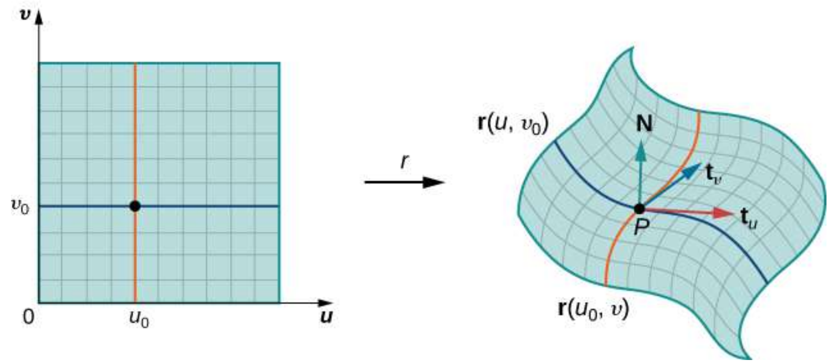


Figure 6.66 If the cross product of vectors $\mathbf{t}_u$ and $\mathbf{t}_v$ exists, then there is a tangent plane.

The tangent plane at P_{ij} contains vectors $\mathbf{t}_u(P_{ij})$ and $\mathbf{t}_v(P_{ij})$, and therefore the parallelogram spanned by $\mathbf{t}_u(P_{ij})$ and $\mathbf{t}_v(P_{ij})$ is in the tangent plane. Since the original rectangle in the uv-plane corresponding to S_{ij} has width Δu and length Δv, the parallelogram that we use to approximate S_{ij} is the parallelogram spanned by $\Delta u \mathbf{t}_u(P_{ij})$ and $\Delta v \mathbf{t}_v(P_{ij})$. In other words, we scale the tangent vectors by the constants Δu and Δv to match the scale of the original division of rectangles in the parameter domain. Therefore, the area of the parallelogram used to approximate the area of S_{ij} is

$$\Delta S_{ij} \approx \| (\Delta u \mathbf{t}_u(P_{ij})) \times (\Delta v \mathbf{t}_v(P_{ij})) \| = \| \mathbf{t}_u(P_{ij}) \times \mathbf{t}_v(P_{ij}) \| \Delta u \Delta v.$$

Varying point P_{ij} over all pieces S_{ij} and the previous approximation leads to the following definition of surface area of a parametric surface (**Figure 6.67**).

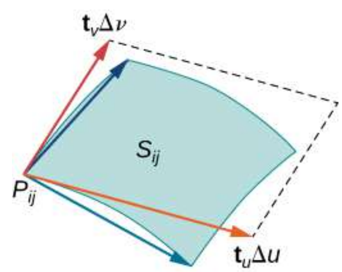


Figure 6.67 The parallelogram spanned by $\mathbf{t}_u$ and $\mathbf{t}_v$ approximates the piece of surface S_{ij}.

Definition

Let $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ with parameter domain D be a smooth parameterization of surface S. Furthermore, assume that S is traced out only once as (u, v) varies over D. The **surface area** of S is

$$\iint_D \| \mathbf{t}_u \times \mathbf{t}_v \| \, dA, \tag{6.18}$$

where $\mathbf{t}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$ and $\mathbf{t}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$.

Example 6.62

Calculating Surface Area

Calculate the lateral surface area (the area of the "side," not including the base) of the right circular cone with height h and radius r.

Solution

Before calculating the surface area of this cone using **Equation 6.18**, we need a parameterization. We assume this cone is in $\mathbb{R}^3$ with its vertex at the origin (**Figure 6.68**). To obtain a parameterization, let α be the angle that is swept out by starting at the positive z-axis and ending at the cone, and let $k = \tan\alpha$. For a height value v with $0 \leq v \leq h$, the radius of the circle formed by intersecting the cone with plane $z = v$ is kv. Therefore, a parameterization of this cone is

$$\mathbf{s}(u, v) = \langle kv\cos u, kv\sin u, v \rangle, 0 \leq u < 2\pi, 0 \leq v \leq h.$$

The idea behind this parameterization is that for a fixed v value, the circle swept out by letting u vary is the circle at height v and radius kv. As v increases, the parameterization sweeps out a "stack" of circles, resulting in the desired cone.

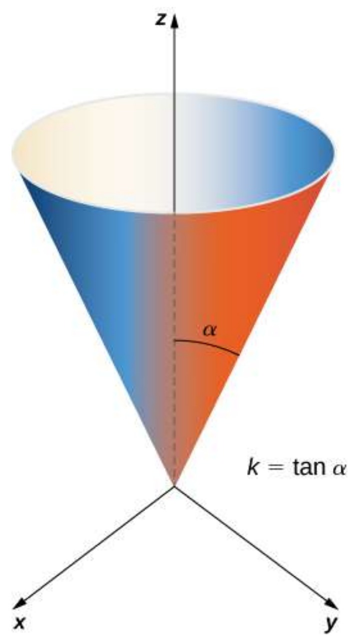


Figure 6.68 The right circular cone with radius $r = kh$ and height h has parameterization $\mathbf{s}(u, v) = \langle kv\cos u, kv\sin u, v \rangle, 0 \leq u < 2\pi, 0 \leq v \leq h.$

With a parameterization in hand, we can calculate the surface area of the cone using **Equation 6.18**. The tangent vectors are $\mathbf{t}_u = \langle -kv\sin u, kv\cos u, 0 \rangle$ and $\mathbf{t}_v = \langle k\cos u, k\sin u, 1 \rangle$. Therefore,

$$\begin{aligned}\mathbf{t}_u \times \mathbf{t}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -kv\sin u & kv\cos u & 0 \\ k\cos u & k\sin u & 1 \end{vmatrix} \\ &= \langle kv\cos u, kv\sin u, -k^2 v\sin^2 u - k^2 v\cos^2 u \rangle \\ &= \langle kv\cos u, kv\sin u, -k^2 v \rangle.\end{aligned}$$

The magnitude of this vector is

$$\begin{aligned} \| \langle kv\cos u, kv\sin u, -k^2 v \rangle \| &= \sqrt{k^2 v^2 \cos^2 u + k^2 v^2 \sin^2 u + k^4 v^2} \\ &= \sqrt{k^2 v^2 + k^4 v^2} \\ &= kv\sqrt{1+k^2}. \end{aligned}$$

By **Equation 6.18**, the surface area of the cone is

$$\begin{aligned} \iint_D \| \mathbf{t}_u \times \mathbf{t}_v \| \, dA &= \int_0^h \int_0^{2\pi} kv\sqrt{1+k^2} du dv \\ &= 2\pi k\sqrt{1+k^2} \int_0^h v dv \\ &= 2\pi k\sqrt{1+k^2} \left[\frac{v^2}{2}\right]_0^h \\ &= \pi k h^2 \sqrt{1+k^2}. \end{aligned}$$

Since $k = \tan\alpha = r/h$,

$$\begin{aligned} \pi k h^2 \sqrt{1+k^2} &= \pi \frac{r}{h} h^2 \sqrt{1+\frac{r^2}{h^2}} \\ &= \pi r h \sqrt{1+\frac{r^2}{h^2}} \\ &= \pi r \sqrt{h^2 + h^2\left(\frac{r^2}{h^2}\right)} \\ &= \pi r \sqrt{h^2 + r^2}. \end{aligned}$$

Therefore, the lateral surface area of the cone is $\pi r\sqrt{h^2+r^2}$.

Analysis

The surface area of a right circular cone with radius r and height h is usually given as $\pi r^2 + \pi r\sqrt{h^2+r^2}$. The reason for this is that the circular base is included as part of the cone, and therefore the area of the base πr^2 is added to the lateral surface area $\pi r\sqrt{h^2+r^2}$ that we found.

6.51 Find the surface area of the surface with parameterization $\mathbf{r}(u, v) = \langle u+v, u^2, 2v \rangle$, $0 \le u \le 3$, $0 \le v \le 2$.

Example 6.63

Calculating Surface Area

Show that the surface area of the sphere $x^2 + y^2 + z^2 = r^2$ is $4\pi r^2$.

Solution

The sphere has parameterization

$$\langle\, r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi\,\rangle,\ 0 \le \theta < 2\pi,\ 0 \le \phi \le \pi.$$

The tangent vectors are

$$\mathbf{t}_\theta = \langle\, -r\sin\theta\sin\phi,\ r\cos\theta\sin\phi,\ 0\,\rangle \text{ and } \mathbf{t}_\phi = \langle\, r\cos\theta\cos\phi,\ r\sin\theta\cos\phi,\ -r\sin\phi\,\rangle.$$

Therefore,

$$\begin{aligned}\mathbf{t}_\phi \times \mathbf{t}_\theta &= \langle\, r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin^2\theta\sin\phi\cos\phi + r^2\cos^2\theta\sin\phi\cos\phi\,\rangle \\ &= \langle\, r^2\cos\theta\sin^2\phi,\ r^2\sin\theta\sin^2\phi,\ r^2\sin\phi\cos\phi\,\rangle.\end{aligned}$$

Now,

$$\begin{aligned}\| \mathbf{t}_\phi \times \mathbf{t}_\theta &= \sqrt{r^4\sin^4\phi\cos^2\theta + r^4\sin^4\phi\sin^2\theta + r^4\sin^2\phi\cos^2\phi} \\ &= \sqrt{r^4\sin^4\phi + r^4\sin^2\phi\cos^2\phi} \\ &= r^2\sqrt{\sin^2\phi} \\ &= r\sin\phi.\end{aligned}$$

Notice that $\sin\phi \ge 0$ on the parameter domain because $0 \le \phi < \pi$, and this justifies equation $\sqrt{\sin^2\phi} = \sin\phi$. The surface area of the sphere is

$$\int_0^{2\pi}\int_0^{\pi} r^2\sin\phi\, d\phi\, d\theta = r^2\int_0^{2\pi} 2\, d\theta = 4\pi r^2.$$

We have derived the familiar formula for the surface area of a sphere using surface integrals.

6.52 Show that the surface area of cylinder $x^2 + y^2 = r^2,\ 0 \le z \le h$ is $2\pi rh$. Notice that this cylinder does not include the top and bottom circles.

In addition to parameterizing surfaces given by equations or standard geometric shapes such as cones and spheres, we can also parameterize surfaces of revolution. Therefore, we can calculate the surface area of a surface of revolution by using the same techniques. Let $y = f(x) \ge 0$ be a positive single-variable function on the domain $a \le x \le b$ and let S be the surface obtained by rotating f about the x-axis (**Figure 6.69**). Let θ be the angle of rotation. Then, S can be parameterized with parameters x and θ by

$$\mathbf{r}(x, \theta) = \langle\, x,\ f(x)\cos\theta,\ f(x)\sin\theta\,\rangle,\ a \le x \le b,\ 0 \le x < 2\pi.$$

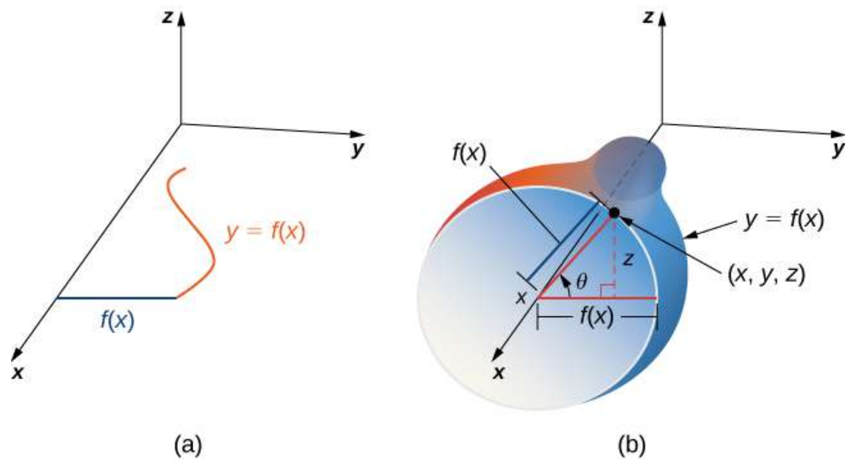


Figure 6.69 We can parameterize a surface of revolution by $\mathbf{r}(x, \theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle, \quad a \le x \le b, 0 \le x < 2\pi.$

Example 6.64

Calculating Surface Area

Find the area of the surface of revolution obtained by rotating $y = x^2, 0 \le x \le b$ about the x-axis (**Figure 6.70**).

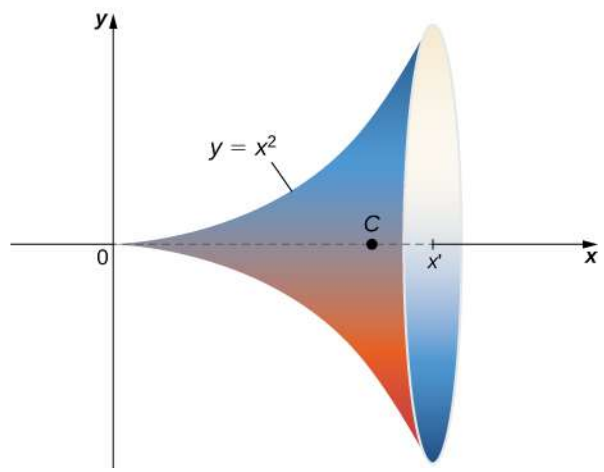


Figure 6.70 A surface integral can be used to calculate the surface area of this solid of revolution.

Solution

This surface has parameterization

$$\mathbf{r}(x, \theta) = \langle x, x^2\cos\theta, x^2\sin\theta \rangle, 0 \le x \le b, 0 \le x < 2\pi.$$

The tangent vectors are $\mathbf{t}_x = \langle 1, 2x\cos\theta, 2x\sin\theta \rangle$ and $\mathbf{t}_\theta = \langle 0, -x^2\sin\theta, -x^2\cos\theta \rangle$. Therefore,

$$\begin{aligned} \mathbf{t}_x \times \mathbf{t}_\theta &= \langle 2x^3\cos^2\theta + 2x^3\sin^2\theta, -x^2\cos\theta, -x^2\sin\theta \rangle \\ &= \langle 2x^3, -x^2\cos\theta, -x^2\sin\theta \rangle \end{aligned}$$

and

$$\begin{aligned} \mathbf{t}_x \times \mathbf{t}_\theta &= \sqrt{4x^6 + x^4 \cos^2\theta + x^4 \sin^2\theta} \\ &= \sqrt{4x^6 + x^4} \\ &= x^2\sqrt{4x^2 + 1}. \end{aligned}$$

The area of the surface of revolution is

$$\begin{aligned} \int_0^b \int_0^\pi x^2\sqrt{4x^2+1}\,d\theta dx &= 2\pi \int_0^b x^2\sqrt{4x^2+1}\,dx \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4x^2+1}\left(8x^3+x\right)\sinh^{-1}(2x)\right)\right]_0^b \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4b^2+1}\left(8b^3+b\right)\sinh^{-1}(2b)\right)\right]. \end{aligned}$$

6.53 Use **Equation 6.18** to find the area of the surface of revolution obtained by rotating curve $y = \sin x,\ 0 \le x \le \pi$ about the x-axis.

Surface Integral of a Scalar-Valued Function

Now that we can parameterize surfaces and we can calculate their surface areas, we are able to define surface integrals. First, let's look at the surface integral of a scalar-valued function. Informally, the surface integral of a scalar-valued function is an analog of a scalar line integral in one higher dimension. The domain of integration of a scalar line integral is a parameterized curve (a one-dimensional object); the domain of integration of a scalar surface integral is a parameterized surface (a two-dimensional object). Therefore, the definition of a surface integral follows the definition of a line integral quite closely. For scalar line integrals, we chopped the domain curve into tiny pieces, chose a point in each piece, computed the function at that point, and took a limit of the corresponding Riemann sum. For scalar surface integrals, we chop the domain *region* (no longer a curve) into tiny pieces and proceed in the same fashion.

Let S be a piecewise smooth surface with parameterization $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ with parameter domain D and let $f(x, y, z)$ be a function with a domain that contains S. For now, assume the parameter domain D is a rectangle, but we can extend the basic logic of how we proceed to any parameter domain (the choice of a rectangle is simply to make the notation more manageable). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij}, evaluate P_{ij} at f, and multiply by area ΔS_{ij} to form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f\left(P_{ij}\right)\Delta S_{ij}.$$

To define a surface integral of a scalar-valued function, we let the areas of the pieces of S shrink to zero by taking a limit.

Definition

The **surface integral of a scalar-valued function** of f over a piecewise smooth surface S is

$$\iint_S f(x, y, z)\,dS = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(P_{ij}\right)\Delta S_{ij}.$$

Again, notice the similarities between this definition and the definition of a scalar line integral. In the definition of a line integral we chop a curve into pieces, evaluate a function at a point in each piece, and let the length of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. In the definition of a surface integral, we chop a surface into pieces, evaluate a function at a point in each piece, and let the area of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. Thus, a surface integral is similar to a line integral but in one higher dimension.

The definition of a scalar line integral can be extended to parameter domains that are not rectangles by using the same logic used earlier. The basic idea is to chop the parameter domain into small pieces, choose a sample point in each piece, and so on. The exact shape of each piece in the sample domain becomes irrelevant as the areas of the pieces shrink to zero.

Scalar surface integrals are difficult to compute from the definition, just as scalar line integrals are. To develop a method that makes surface integrals easier to compute, we approximate surface areas ΔS_{ij} with small pieces of a tangent plane, just as we did in the previous subsection. Recall the definition of vectors $\mathbf{t}_u$ and $\mathbf{t}_v$:

$$\mathbf{t}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{t}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

From the material we have already studied, we know that

$$\Delta S_{ij} \approx \| \mathbf{t}_u(P_{ij}) \times \mathbf{t}_v(P_{ij}) \| \Delta u \Delta v.$$

Therefore,

$$\iint_S f(x, y, z)dS \approx \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) \| \mathbf{t}_u(P_{ij}) \times \mathbf{t}_v(P_{ij}) \| \Delta u \Delta v.$$

This approximation becomes arbitrarily close to $\lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}$ as we increase the number of pieces S_{ij} by letting *m* and *n* go to infinity. Therefore, we have the following equation to calculate scalar surface integrals:

$$\iint_S f(x, y, z)dS = \iint_D f(\mathbf{r}(u, v)) \| \mathbf{t}_u \times \mathbf{t}_v \| dA. \tag{6.19}$$

Equation 6.19 allows us to calculate a surface integral by transforming it into a double integral. This equation for surface integrals is analogous to **Equation 6.20** for line integrals:

$$\iint_C f(x, y, z)ds = \int_a^b f(\mathbf{r}(t)) \| \mathbf{r}'(t) \| dt.$$

In this case, vector $\mathbf{t}_u \times \mathbf{t}_v$ is perpendicular to the surface, whereas vector $\mathbf{r}'(t)$ is tangent to the curve.

Example 6.65

Calculating a Surface Integral

Calculate surface integral $\iint_S 5dS,$ where S is the surface with parameterization $\mathbf{r}(u, v) = \langle u, u^2, v \rangle$ for $0 \le u \le 2$ and $0 \le v \le u.$

Solution

Notice that this parameter domain *D* is a triangle, and therefore the parameter domain is not rectangular. This is not an issue though, because **Equation 6.19** does not place any restrictions on the shape of the parameter domain.

To use **Equation 6.19** to calculate the surface integral, we first find vector $\mathbf{t}_u$ and $\mathbf{t}_v$. Note that $\mathbf{t}_u = \langle 1, 2u, 0 \rangle$ and $\mathbf{t}_v = \langle 0, 0, 1 \rangle$. Therefore,

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle\, 2u,\ -1,\ 0 \,\rangle$$

and

$$\|\, \mathbf{t}_u \times \mathbf{t}_v \,\| \ = \sqrt{1+4u^2}.$$

By **Equation 6.19**,

$$\begin{aligned} \iint_S 5\,dS &= 5 \iint_D u\sqrt{1+4u^2}\,dA \\ &= 5\int_0^2 \int_0^u \sqrt{1+4u^2}\,dv\,du = 5\int_0^2 u\sqrt{1+4u^2}\,du \\ &= 5\left[\frac{\left(1+4u^2\right)^{3/2}}{3}\right]_0^2 = \frac{5\left(17^{3/2}-1\right)}{3} \approx 115.15. \end{aligned}$$

Example 6.66

Calculating the Surface Integral of a Cylinder

Calculate surface integral $\iint_S \left(x+y^2\right) dS,$ where S is cylinder $x^2+y^2=4,\ 0 \le z \le 3$ (**Figure 6.71**).

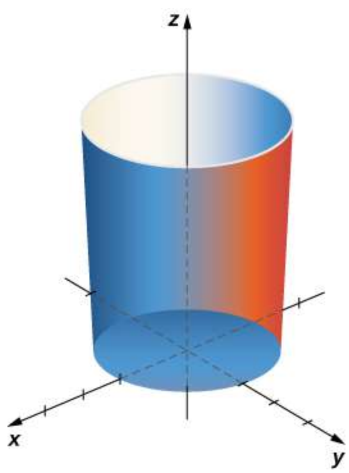


Figure 6.71 Integrating function $f(x,\ y,\ z) = x + y^2$ over a cylinder.

Solution

To calculate the surface integral, we first need a parameterization of the cylinder. Following **Example 6.58**, a parameterization is

$$\mathbf{r}(u,\ v) = \langle\, \cos u,\ \sin u,\ v \,\rangle,\ 0 \le u \le 2\pi,\ 0 \le v \le 3.$$

The tangent vectors are $\mathbf{t}_u = \langle \sin u, \cos u, 0 \rangle$ and $\mathbf{t}_v = \langle 0, 0, 1 \rangle$. Then,

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos u, \sin u, 0 \rangle$$

and $\| \mathbf{t}_u \times \mathbf{t}_v \| = \sqrt{\cos^2 u + \sin^2 u} = 1$. By **Equation 6.19**,

$$\begin{aligned} \iint_S f(x, y, z)dS &= \iint_D f(\mathbf{r}(u, v)) \| \mathbf{t}_u \times \mathbf{t}_v \| \, dA \\ &= \int_0^3 \int_0^{2\pi} \left(\cos u + \sin^2 u\right) du dv \\ &= \int_0^3 \left[\sin u + \frac{u}{2} - \frac{\sin(2u)}{4}\right]_0^{2\pi} dv = \int_0^3 \pi dv = 3\pi. \end{aligned}$$

6.54 Calculate $\iint_S \left(x^2 - z\right) dS,$ where S is the surface with parameterization $\mathbf{r}(u, v) = \langle v, u^2 + v^2, 1 \rangle, \ 0 \le u \le 2, \ 0 \le v \le 3.$

Example 6.67

Calculating the Surface Integral of a Piece of a Sphere

Calculate surface integral $\iint_S f(x, y, z)dS,$ where $f(x, y, z) = z^2$ and S is the surface that consists of the piece of sphere $x^2 + y^2 + z^2 = 4$ that lies on or above plane $z = 1$ and the disk that is enclosed by intersection plane $z = 1$ and the given sphere (**Figure 6.72**).

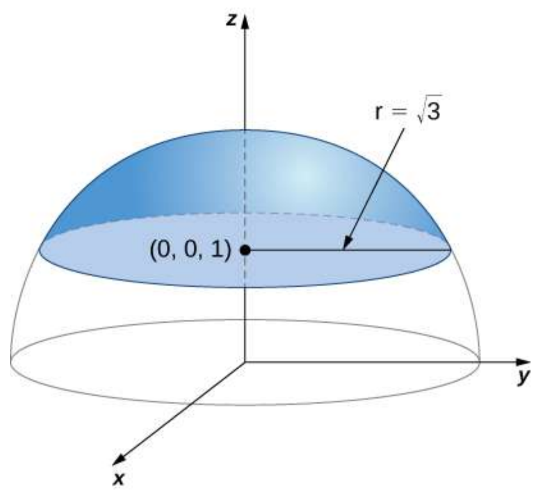


Figure 6.72 Calculating a surface integral over surface S.

Solution

Notice that S is not smooth but is piecewise smooth; S can be written as the union of its base S_1 and its spherical top S_2, and both S_1 and S_2 are smooth. Therefore, to calculate $\iint_S z^2\, dS$, we write this integral as $\iint_{S_1} z^2\, dS + \iint_{S_2} z^2\, dS$ and we calculate integrals $\iint_{S_1} z^2\, dS$ and $\iint_{S_2} z^2\, dS$.

First, we calculate $\iint_{S_1} z^2\, dS$. To calculate this integral we need a parameterization of S_1. This surface is a disk in plane $z = 1$ centered at $(0, 0, 1)$. To parameterize this disk, we need to know its radius. Since the disk is formed where plane $z = 1$ intersects sphere $x^2 + y^2 + z^2 = 4$, we can substitute $z = 1$ into equation $x^2 + y^2 + z^2 = 4$:

$$x^2 + y^2 + 1 = 4 \Rightarrow x^2 + y^2 = 3.$$

Therefore, the radius of the disk is $\sqrt{3}$ and a parameterization of S_1 is $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 1 \rangle, 0 \le u \le \sqrt{3}, 0 \le v \le 2\pi$. The tangent vectors are $\mathbf{t}_u = \langle \cos v, \sin v, 0 \rangle$ and $\mathbf{t}_v = \langle -u \sin v, u\cos v, 0 \rangle$, and thus

$$\mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = \langle 0, 0, u \cos^2 v + u \sin^2 v \rangle = \langle 0, 0, u \rangle.$$

The magnitude of this vector is u. Therefore,

$$\begin{aligned} \iint_{S_1} z^2\, dS &= \int_0^{\sqrt{3}} \int_0^{2\pi} f(\mathbf{r}(u, v)) \, \| \mathbf{t}_u \times \mathbf{t}_v \| \, dv\, du \\ &= \int_0^{\sqrt{3}} \int_0^{2\pi} u \, dv\, du \\ &= 2\pi \int_0^{\sqrt{3}} u du \\ &= 2\pi\sqrt{3}. \end{aligned}$$

Now we calculate $\iint_{S_2} dS$. To calculate this integral, we need a parameterization of S_2. The parameterization of full sphere $x^2 + y^2 + z^2 = 4$ is

$$\mathbf{r}(\phi, \theta) = \langle 2 \cos \theta \sin \phi, 2 \sin \theta \sin \phi, 2 \cos \phi \rangle, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi.$$

Since we are only taking the piece of the sphere on or above plane $z = 1$, we have to restrict the domain of ϕ. To see how far this angle sweeps, notice that the angle can be located in a right triangle, as shown in **Figure 6.73** (the $\sqrt{3}$ comes from the fact that the base of S is a disk with radius $\sqrt{3}$). Therefore, the tangent of ϕ is $\sqrt{3}$, which implies that ϕ is $\pi/6$. We now have a parameterization of S_2:

$$\mathbf{r}(\phi, \theta) = \langle 2 \cos \theta \sin \phi, 2 \sin \theta \sin \phi, 2 \cos \phi \rangle, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/3.$$

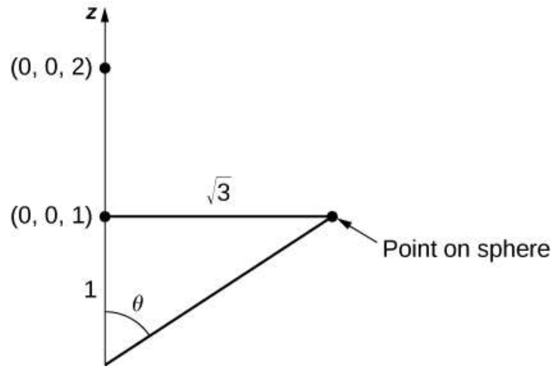


Figure 6.73 The maximum value of ϕ has a tangent value of $\sqrt{3}$.

The tangent vectors are

$$\mathbf{t}_\phi = \langle 2\cos\theta\cos\phi,\ 2\sin\theta\cos\phi,\ -2\sin\phi \rangle \text{ and } \mathbf{t}_\theta = \langle -2\sin\theta\sin\phi,\ u\cos\theta\sin\phi,\ 0 \rangle,$$

and thus

$$\begin{aligned}\mathbf{t}_\phi \times \mathbf{t}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \end{vmatrix} \\ &= \langle 4\cos\theta\sin^2\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos^2\theta\cos\phi\sin\phi + 4\sin^2\theta\cos\phi\sin\phi \rangle \\ &= \langle 4\cos\theta\sin^2\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos\phi\sin\phi \rangle.\end{aligned}$$

The magnitude of this vector is

$$\begin{aligned}\|\mathbf{t}_\phi \times \mathbf{t}_\theta\| &= \sqrt{16\cos^2\theta\sin^4\phi + 16\sin^2\theta\sin^4\phi + 16\cos^2\phi\sin^2\phi} \\ &= 4\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}.\end{aligned}$$

Therefore,

$$\begin{aligned}\iint_{S_2} z\,dS &= \int_0^{\pi/6}\int_0^{2\pi} f(\mathbf{r}(\phi,\theta))\,\|\mathbf{t}_\phi \times \mathbf{t}_\theta\|\,d\theta\,d\phi \\ &= \int_0^{\pi/6}\int_0^{2\pi} 16\cos^2\phi\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}\,d\theta\,d\phi \\ &= 32\pi\int_0^{\pi/6} \cos^2\phi\sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6} \cos^2\phi\sin\phi\sqrt{\sin^2\phi + \cos^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6} \cos^2\phi\sin\phi\,d\phi \\ &= 32\pi\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/6} = 32\pi\left[\frac{1}{3} - \frac{\sqrt{3}}{8}\right] = \frac{32\pi}{3} - 4\sqrt{3}.\end{aligned}$$

Since $\iint_S z^2\,dS = \iint_{S_1} z^2\,dS + \iint_{S_2} z^2\,dS$, we have $\iint_S z^2\,dS = (2\pi - 4)\sqrt{3} + \frac{32\pi}{3}$.

Analysis

In this example we broke a surface integral over a piecewise surface into the addition of surface integrals over smooth subsurfaces. There were only two smooth subsurfaces in this example, but this technique extends to finitely many smooth subsurfaces.

6.55 Calculate line integral $\iint_S (x-y)dS,$ where S is cylinder $x^2+y^2=1,\ 0 \le z \le 2,$ including the circular top and bottom.

Scalar surface integrals have several real-world applications. Recall that scalar line integrals can be used to compute the mass of a wire given its density function. In a similar fashion, we can use scalar surface integrals to compute the mass of a sheet given its density function. If a thin sheet of metal has the shape of surface S and the density of the sheet at point (x, y, z) is $\rho(x, y, z),$ then mass m of the sheet is $m = \iint_S \rho(x, y, z)dS.$

Example 6.68

Calculating the Mass of a Sheet

A flat sheet of metal has the shape of surface $z = 1 + x + 2y$ that lies above rectangle $0 \le x \le 4$ and $0 \le y \le 2.$ If the density of the sheet is given by $\rho(x, y, z) = x^2yz,$ what is the mass of the sheet?

Solution

Let S be the surface that describes the sheet. Then, the mass of the sheet is given by $m = \iint_S x^2yzdS.$

To compute this surface integral, we first need a parameterization of S. Since S is given by the function $f(x, y) = 1 + x + 2y,$ a parameterization of S is $\mathbf{r}(x, y) = \langle x, y, 1 + x + 2y \rangle,\ 0 \le x \le 4,\ 0 \le y \le 2.$

The tangent vectors are $\mathbf{t}_x = \langle 1, 0, 1 \rangle$ and $\mathbf{t}_y = \langle 1, 0, 2 \rangle$. Therefore, $\mathbf{t}_x \times \mathbf{t}_y = \langle -1, -2, 1 \rangle$ and $\| \mathbf{t}_x \times \mathbf{t}_y \| = \sqrt{6}.$ By **Equation 6.5**,

$$\begin{aligned} m &= \iint_S x^2yz\,dS \\ &= \sqrt{6}\int_0^4 \int_0^2 x^2y(1 + x + 2y)dydx \\ &= \sqrt{6}\int_0^4 \frac{22x^2}{3} + 2x^3\,dx \\ &= \frac{2560\sqrt{6}}{9} \\ &\approx 696.74. \end{aligned}$$

6.56 A piece of metal has a shape that is modeled by paraboloid $z = x^2 + y^2,\ 0 \le z \le 4,$ and the density of the metal is given by $\rho(x, y, z) = z + 1.$ Find the mass of the piece of metal.

Orientation of a Surface

Recall that when we defined a scalar line integral, we did not need to worry about an orientation of the curve of integration. The same was true for scalar surface integrals: we did not need to worry about an "orientation" of the surface of integration.

On the other hand, when we defined vector line integrals, the curve of integration needed an orientation. That is, we needed the notion of an oriented curve to define a vector line integral without ambiguity. Similarly, when we define a surface integral of a vector field, we need the notion of an oriented surface. An oriented surface is given an "upward" or "downward" orientation or, in the case of surfaces such as a sphere or cylinder, an "outward" or "inward" orientation.

Let S be a smooth surface. For any point (x, y, z) on S, we can identify two unit normal vectors $\mathbf{N}$ and $-\mathbf{N}$. If it is possible to choose a unit normal vector $\mathbf{N}$ at every point (x, y, z) on S so that $\mathbf{N}$ varies continuously over S, then S is "*orientable*." Such a choice of unit normal vector at each point gives the **orientation of a surface** S. If you think of the normal field as describing water flow, then the side of the surface that water flows toward is the "negative" side and the side of the surface at which the water flows away is the "positive" side. Informally, a choice of orientation gives S an "outer" side and an "inner" side (or an "upward" side and a "downward" side), just as a choice of orientation of a curve gives the curve "forward" and "backward" directions.

Closed surfaces such as spheres are orientable: if we choose the outward normal vector at each point on the surface of the sphere, then the unit normal vectors vary continuously. This is called the *positive orientation of the closed surface* (**Figure 6.74**). We also could choose the inward normal vector at each point to give an "inward" orientation, which is the negative orientation of the surface.

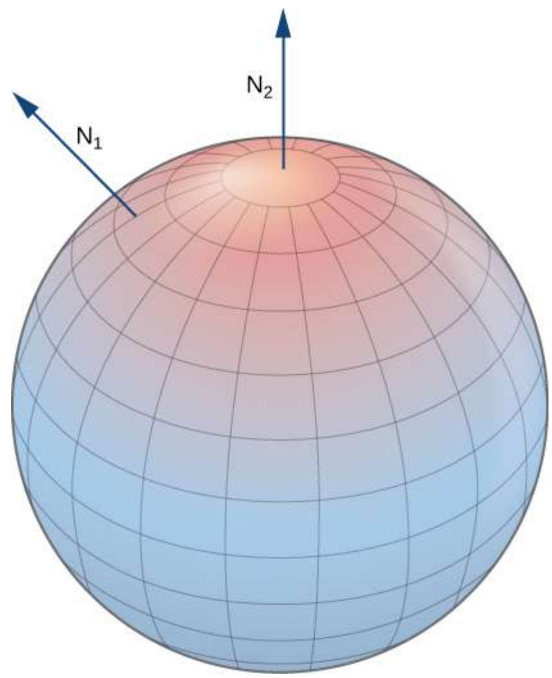


Figure 6.74 An oriented sphere with positive orientation.

A portion of the graph of any smooth function $z = f(x, y)$ is also orientable. If we choose the unit normal vector that points "above" the surface at each point, then the unit normal vectors vary continuously over the surface. We could also choose the unit normal vector that points "below" the surface at each point. To get such an orientation, we parameterize the graph of f in the standard way: $\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle$, where x and y vary over the domain of f. Then, $\mathbf{t}_x = \langle 1, 0, f_x \rangle$ and $\mathbf{t}_y = \langle 0, 1, f_y \rangle$, and therefore the cross product $\mathbf{t}_x \times \mathbf{t}_y$ (which is normal to the surface at any point on the surface) is $\langle -f_x, -f_y, 1 \rangle$. Since the z component of this vector is one, the corresponding unit normal vector points "upward," and the upward side of the surface is chosen to be the "positive" side.

Let S be a smooth orientable surface with parameterization $\mathbf{r}(u, v)$. For each point $\mathbf{r}(a, b)$ on the surface, vectors $\mathbf{t}_u$ and $\mathbf{t}_v$ lie in the tangent plane at that point. Vector $\mathbf{t}_u \times \mathbf{t}_v$ is normal to the tangent plane at $\mathbf{r}(a, b)$ and is therefore normal to S at that point. Therefore, the choice of unit normal vector

$$\mathbf{N} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{\| \mathbf{t}_u \times \mathbf{t}_v \|}$$

gives an orientation of surface S.

Example 6.69

Choosing an Orientation

Give an orientation of cylinder $x^2 + y^2 = r^2,\ 0 \le z \le h$.

Solution

This surface has parameterization

$$\mathbf{r}(u, v) = \langle r\cos u, r\sin u, v \rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h.$$

The tangent vectors are $\mathbf{t}_u = \langle -r\sin u, r\cos u, 0 \rangle$ and $\mathbf{t}_v = \langle 0, 0, 1 \rangle$. To get an orientation of the surface, we compute the unit normal vector

$$\mathbf{N} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{\| \mathbf{t}_u \times \mathbf{t}_v \|}.$$

In this case, $\mathbf{t}_u \times \mathbf{t}_v = \langle r\cos u, r\sin u, 0 \rangle$ and therefore

$$\| \mathbf{t}_u \times \mathbf{t}_v \| = \sqrt{r^2\cos^2 u + r^2\sin^2 u} = r.$$

An orientation of the cylinder is

$$\mathbf{N}(u, v) = \frac{\langle r\cos u, r\sin u, 0 \rangle}{r} = \langle \cos u, \sin u, 0 \rangle.$$

Notice that all vectors are parallel to the *xy*-plane, which should be the case with vectors that are normal to the cylinder. Furthermore, all the vectors point outward, and therefore this is an outward orientation of the cylinder (**Figure 6.75**).

Figure 6.75 If all the vectors normal to a cylinder point outward, then this is an outward orientation of the cylinder.

6.57 Give the "upward" orientation of the graph of $f(x, y) = xy$.

Since every curve has a "forward" and "backward" direction (or, in the case of a closed curve, a clockwise and counterclockwise direction), it is possible to give an orientation to any curve. Hence, it is possible to think of every curve as an oriented curve. This is not the case with surfaces, however. Some surfaces cannot be oriented; such surfaces are called *nonorientable*. Essentially, a surface can be oriented if the surface has an "inner" side and an "outer" side, or an "upward" side and a "downward" side. Some surfaces are twisted in such a fashion that there is no well-defined notion of an "inner" or "outer" side.

The classic example of a nonorientable surface is the Möbius strip. To create a Möbius strip, take a rectangular strip of paper, give the piece of paper a half-twist, and the glue the ends together (**Figure 6.76**). Because of the half-twist in the strip, the surface has no "outer" side or "inner" side. If you imagine placing a normal vector at a point on the strip and having the vector travel all the way around the band, then (because of the half-twist) the vector points in the opposite direction when it gets back to its original position. Therefore, the strip really only has one side.

Figure 6.76 The construction of a Möbius strip.

Since some surfaces are nonorientable, it is not possible to define a vector surface integral on all piecewise smooth surfaces. This is in contrast to vector line integrals, which can be defined on any piecewise smooth curve.

Surface Integral of a Vector Field

With the idea of orientable surfaces in place, we are now ready to define a **surface integral of a vector field**. The definition is analogous to the definition of the flux of a vector field along a plane curve. Recall that if **F** is a two-dimensional vector field and *C* is a plane curve, then the definition of the flux of **F** along *C* involved chopping *C* into small pieces, choosing a point inside each piece, and calculating $\mathbf{F}\cdot\mathbf{N}$ at the point (where **N** is the unit normal vector at the point). The definition of a surface integral of a vector field proceeds in the same fashion, except now we chop surface *S* into small pieces, choose a point in the small (two-dimensional) piece, and calculate $\mathbf{F}\cdot\mathbf{N}$ at the point.

To place this definition in a real-world setting, let *S* be an oriented surface with unit normal vector **N**. Let **v** be a velocity field of a fluid flowing through *S*, and suppose the fluid has density $\rho(x, y, z)$. Imagine the fluid flows through *S*, but *S* is completely permeable so that it does not impede the fluid flow (**Figure 6.77**). The **mass flux** of the fluid is the rate of mass flow per unit area. The mass flux is measured in mass per unit time per unit area. How could we calculate the mass flux of the fluid across *S*?

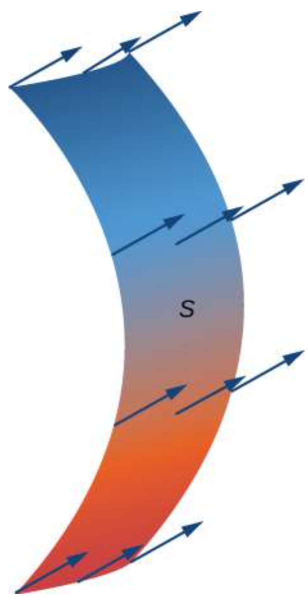


Figure 6.77 Fluid flows across a completely permeable surface *S*.

The rate of flow, measured in mass per unit time per unit area, is $\rho\mathbf{N}$. To calculate the mass flux across *S*, chop *S* into small pieces S_{ij}. If S_{ij} is small enough, then it can be approximated by a tangent plane at some point *P* in S_{ij}. Therefore, the unit normal vector at *P* can be used to approximate $\mathbf{N}(x, y, z)$ across the entire piece S_{ij}, because the normal vector to a plane does not change as we move across the plane. The component of the vector $\rho\mathbf{v}$ at *P* in the direction of **N** is $\rho\mathbf{v}\cdot\mathbf{N}$ at *P*. Since S_{ij} is small, the dot product $\rho\mathbf{v}\cdot\mathbf{N}$ changes very little as we vary across S_{ij}, and therefore $\rho\mathbf{v}\cdot\mathbf{N}$ can be taken as approximately constant across S_{ij}. To approximate the mass of fluid per unit time flowing across S_{ij} (and not just locally at point *P*), we need to multiply $(\rho\mathbf{v}\cdot\mathbf{N})(P)$ by the area of S_{ij}. Therefore, the mass of fluid per unit time flowing across S_{ij} in the direction of **N** can be approximated by $(\rho\mathbf{v} \cdot \mathbf{N})\Delta S_{ij}$, where **N**, ρ, and **v** are all evaluated at P (**Figure 6.78**). This is analogous to the flux of two-dimensional vector field **F** across plane curve C, in which we approximated flux across a small piece of C with the expression $(\mathbf{F} \cdot \mathbf{N})\Delta s$. To approximate the mass flux across S, form the sum $\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\mathbf{v} \cdot \mathbf{N})\Delta \mathbf{S}_{ij}$. As pieces S_{ij} get smaller, the sum $\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\mathbf{v} \cdot \mathbf{N})\Delta \mathbf{S}_{ij}$ gets arbitrarily close to the mass flux. Therefore, the mass flux is

$$\iint_S \rho\mathbf{v} \cdot \mathbf{N}dS = \lim_{m,\, n \to \infty}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\mathbf{v} \cdot \mathbf{N})\Delta \mathbf{S}_{ij}.$$

This is a surface integral of a vector field. Letting the vector field $\rho\mathbf{v}$ be an arbitrary vector field **F** leads to the following definition.

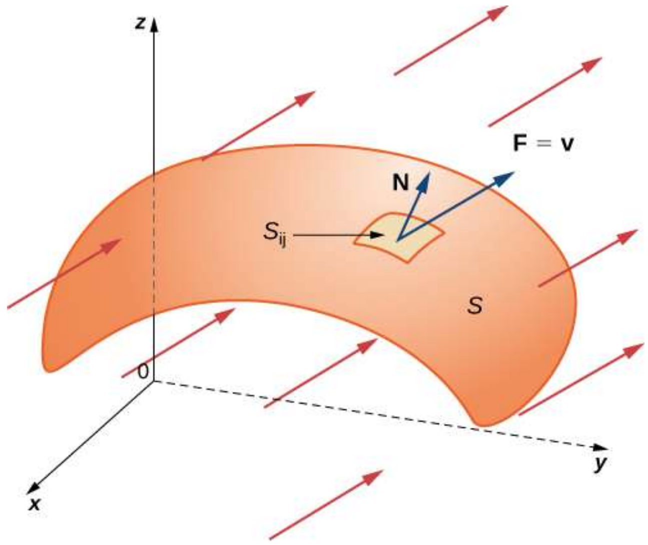


Figure 6.78 The mass of fluid per unit time flowing across S_{ij} in the direction of **N** can be approximated by $(\rho\mathbf{v} \cdot \mathbf{N})\Delta S_{ij}$.

Definition

Let **F** be a continuous vector field with a domain that contains oriented surface S with unit normal vector **N**. The **surface integral** of **F** over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{N}dS. \tag{6.20}$$

Notice the parallel between this definition and the definition of vector line integral $\int_C \mathbf{F} \cdot \mathbf{N}ds$. A surface integral of a vector field is defined in a similar way to a flux line integral across a curve, except the domain of integration is a surface (a two-dimensional object) rather than a curve (a one-dimensional object). Integral $\iint_S \mathbf{F} \cdot \mathbf{N}dS$ is called the *flux of **F** across S*, just as integral $\int_C \mathbf{F} \cdot \mathbf{N}ds$ is the flux of **F** across curve C. A surface integral over a vector field is also called a **flux integral**.

Just as with vector line integrals, surface integral $\iint_S \mathbf{F} \cdot \mathbf{N}d\mathbf{S}$ is easier to compute after surface S has been parameterized. Let $\mathbf{r}(u, v)$ be a parameterization of S with parameter domain D. Then, the unit normal vector is given by

$\mathbf{N} = \dfrac{\mathbf{t}_u \times \mathbf{t}_v}{\| \mathbf{t}_u \times \mathbf{t}_v \|}$ and, from **Equation 6.20**, we have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{N} d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{N} dS \\
&= \iint_S \mathbf{F} \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{\| \mathbf{t}_u \times \mathbf{t}_v \|} dS \\
&= \iint_D \left(\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{\| \mathbf{t}_u \times \mathbf{t}_v \|} \right) \mathbf{t}_u \times \mathbf{t}_v \| \, dA \\
&= \iint_D (\mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v)) dA.
\end{aligned}$$

Therefore, to compute a surface integral over a vector field we can use the equation

$$\iint_S \mathbf{F} \cdot \mathbf{N} d\mathbf{S} = \iint_D (\mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v)) dA. \tag{6.21}$$

Example 6.70

Calculating a Surface Integral

Calculate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{N} d\mathbf{S}$, where $\mathbf{F} = \langle -y, x, 0 \rangle$ and S is the surface with parameterization $\mathbf{r}(u, v) = \langle u, v^2 - u, u + v \rangle$, $0 \le u < 3$, $0 \le v \le 4$.

Solution

The tangent vectors are $\mathbf{t}_u = \langle 1, -1, 1 \rangle$ and $\mathbf{t}_v = \langle 0, 2v, 1 \rangle$. Therefore,

$$\mathbf{t}_u \times \mathbf{t}_v = \langle -1 - 2v, -1, 2v \rangle .$$

By **Equation 6.21**,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^4 \int_0^3 \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v) du dv \\
&= \int_0^4 \int_0^3 \langle u - v^2, u, 0 \rangle \cdot \langle -1 - 2v, -1, 2v \rangle \, du dv \\
&= \int_0^4 \int_0^3 \left[\left(u - v^2 \right) (-1 - 2v) - u \right] du dv \\
&= \int_0^4 \int_0^3 \left(2v^3 + v^2 - 2uv - 2u \right) du dv \\
&= \int_0^4 \left[2v^3 u + v^2 u - vu^2 - u^2 \right]_0^3 dv \\
&= \int_0^4 \left(6v^3 + 3v^2 - 9v - 9 \right) dv \\
&= \left[\frac{3v^4}{2} + v^3 - \frac{9v^2}{2} - 9v \right]_0^4 \\
&= 340.
\end{aligned}$$

Therefore, the flux of **F** across S is 340.

6.58 Calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle 0, -z, y \rangle$ and S is the portion of the unit sphere in the first octant with outward orientation.

Example 6.71

Calculating Mass Flow Rate

Let $\mathbf{v}(x, y, z) = \langle 2x, 2y, z \rangle$ represent a velocity field (with units of meters per second) of a fluid with constant density 80 kg/m^3. Let S be hemisphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$ such that S is oriented outward. Find the mass flow rate of the fluid across S.

Solution

A parameterization of the surface is

$$\mathbf{r}(\phi, \theta) = \langle 3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi \rangle, \ 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi/2.$$

As in **Example 6.64**, the tangent vectors are

$$\mathbf{t}_\theta \ \langle -3\sin\theta\sin\phi, 3\cos\theta\sin\phi, 0 \rangle \ \text{and} \ \mathbf{t}_\phi \ \langle 3\cos\theta\cos\phi, 3\sin\theta\cos\phi, -3\sin\phi \rangle,$$

and their cross product is

$$\mathbf{t}_\phi \times \mathbf{t}_\theta = \langle 9\cos\theta\sin^2\phi, 9\sin\theta\sin^2\phi, 9\sin\phi\cos\phi \rangle.$$

Notice that each component of the cross product is positive, and therefore this vector gives the outward orientation. Therefore we use the orientation $\mathbf{N} = \langle 9\cos\theta\sin^2\phi, 9\sin\theta\sin^2\phi, 9\sin\phi\cos\phi \rangle$ for the sphere.

By **Equation 6.20**,

$$\begin{aligned}
\iint_S \rho\mathbf{v} \cdot d\mathbf{S} &= 80\int_0^{2\pi}\int_0^{\pi/2} \mathbf{v}(\mathbf{r}(\phi, \theta)) \cdot \left(\mathbf{t}_\phi \times \mathbf{t}_\theta\right) d\phi d\theta \\
&= 80\int_0^{2\pi}\int_0^{\pi/2} \\
&= 80\int_0^{2\pi}\int_o^{\pi/2} \langle 6\cos\theta\sin\phi, 6\sin\theta\sin\phi, 3\cos\phi \rangle \\
&\qquad \cdot \langle 9\cos\theta\sin^2\phi, 9\sin\theta\sin^2\phi, 9\sin\phi\cos\phi \rangle \, d\phi d\theta \\
&= 80\int_0^{2\pi}\int_0^{\pi/2} 54\sin^3\phi + 27\cos^2\phi\sin\phi d\phi d\theta \\
&= 80\int_0^{2\pi}\int_0^{\pi/2} 54\left(1 - \cos^2\phi\right)\sin\phi + 27\cos^2\phi\sin\phi d\phi d\theta \\
&= 80\int_0^{2\pi}\int_0^{\pi/2} 54\sin\phi - 27\cos^2\phi\sin\phi d\phi d\theta \\
&= 80\int_0^{2\pi}\left[-54\cos\phi + 9\cos^3\phi\right]_{\phi=0}^{\phi=2\pi} d\theta \\
&= 80\int_0^{2\pi} 45 d\theta = 7200\pi.
\end{aligned}$$

Therefore, the mass flow rate is 7200π kg/sec/m^2.

6.59 Let $\mathbf{v}(x, y, z) = \langle x^2 + y^2, z, 4y \rangle$ m/sec represent a velocity field of a fluid with constant density 100 kg/m^3. Let S be the half-cylinder $\mathbf{r}(u, v) = \langle \cos u, \sin u, v \rangle, \ 0 \leq u \leq \pi, \ 0 \leq v \leq 2$ oriented outward. Calculate the mass flux of the fluid across S.

In **Example 6.70**, we computed the mass flux, which is the rate of mass flow per unit area. If we want to find the flow rate (measured in volume per time) instead, we can use flux integral $\int\int_S \mathbf{v} \bullet \mathbf{N} dS,$ which leaves out the density. Since the flow rate of a fluid is measured in volume per unit time, flow rate does not take mass into account. Therefore, we have the following characterization of the flow rate of a fluid with velocity **v** across a surface *S*:

$$\text{Flow rate of fluid ac oss } S = \int\int_S \mathbf{v} \bullet d\mathbf{S}.$$

To compute the flow rate of the fluid in **Example 6.68**, we simply remove the density constant, which gives a flow rate of $90\pi \text{ m}^3/\text{sec}$.

Both mass flux and flow rate are important in physics and engineering. Mass flux measures how much mass is flowing across a surface; flow rate measures how much volume of fluid is flowing across a surface.

In addition to modeling fluid flow, surface integrals can be used to model heat flow. Suppose that the temperature at point (x, y, z) in an object is $T(x, y, z)$. Then the **heat flow** is a vector field proportional to the negative temperature gradient in the object. To be precise, the heat flow is defined as vector field $\mathbf{F} = -k\nabla T,$ where the constant *k* is the *thermal conductivity* of the substance from which the object is made (this constant is determined experimentally). The rate of heat flow across surface *S* in the object is given by the flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S -k\nabla T \cdot d\mathbf{S}.$$

Example 6.72

Calculating Heat Flow

A cast-iron solid cylinder is given by inequalities $x^2 + y^2 \le 1, \quad 1 \le z \le 4.$ The temperature at point (x, y, z) in a region containing the cylinder is $T(x, y, z) = (x^2 + y^2)z.$ Given that the thermal conductivity of cast iron is 55, find the heat flow across the boundary of the solid if this boundary is oriented outward.

Solution

Let *S* denote the boundary of the object. To find the heat flow, we need to calculate flux integral $\iint_S -k\nabla T \cdot d\mathbf{S}.$

Notice that *S* is not a smooth surface but is piecewise smooth, since *S* is the union of three smooth surfaces (the circular top and bottom, and the cylindrical side). Therefore, we calculate three separate integrals, one for each smooth piece of *S*. Before calculating any integrals, note that the gradient of the temperature is $\nabla T = \langle 2xz, 2yz, x^2 + y^2 \rangle.$

First we consider the circular bottom of the object, which we denote $S_1.$ We can see that S_1 is a circle of radius 1 centered at point $(0, 0, 1),$ sitting in plane $z = 1.$ This surface has parameterization $\mathbf{r}(u, v) = \langle v\cos u, v\sin u, 1 \rangle, 0 \le u < 2\pi, 0 \le v \le 1.$ Therefore,

$$\mathbf{t}_u = \langle -v\sin u, v\cos u, 0 \rangle \text{ and } \mathbf{t}_v = \langle \cos u, v\sin u, 0 \rangle,$$

and

$$\mathbf{t}_u \times \mathbf{t}_v = \langle 0, 0, -v\sin^2 u - v\cos^2 u \rangle = \langle 0, 0, -v \rangle.$$

Since the surface is oriented outward and S_1 is the bottom of the object, it makes sense that this vector points downward. By **Equation 6.21**, the heat flow across S_1 is

$$\begin{aligned}\iint_{S_1} -k\nabla T\cdot d\mathbf{S} &= -55\int_0^{2\pi}\int_0^1 \nabla T(u,\, v)\cdot(\mathbf{t}_u\times\mathbf{t}_v)dvdu\\ &= -55\int_0^{2\pi}\int_0^1 \langle\, 2v\cos u,\, 2v\sin u,\, v^2\cos^2 u + v^2\sin^2 u\,\rangle\cdot\langle\, 0,\, 0,\, -v\,\rangle\, dvdu\\ &= -55\int_0^{2\pi}\int_0^1 \langle\, 2v\cos u,\, 2v\sin u,\, v^2\,\rangle\cdot\langle\, 0,\, 0,\, -v\,\rangle\, dvdu\\ &= -55\int_0^{2\pi}\int_0^1 -v^3 dvdu = -55\int_0^{2\pi} -\frac{1}{4}du = \frac{55\pi}{2}.\end{aligned}$$

Now let's consider the circular top of the object, which we denote S_2. We see that S_2 is a circle of radius 1 centered at point $(0, 0, 4)$, sitting in plane $z = 4$. This surface has parameterization $\mathbf{r}(u,\, v) = \langle\, v\cos u,\, v\sin u,\, 4\,\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le 1$. Therefore,

$$\mathbf{t}_u = \langle\, -v\sin u,\, v\cos u,\, 0\,\rangle \text{ and } \mathbf{t}_v = \langle\, \cos u,\, v\sin u,\, 0\,\rangle,$$

and

$$\mathbf{t}_u\times\mathbf{t}_v = \langle\, 0,\, 0,\, -v\sin^2 u - v\cos^2 u\,\rangle = \langle\, 0,\, 0,\, -v\,\rangle.$$

Since the surface is oriented outward and S_1 is the top of the object, we instead take vector $\mathbf{t}_v\times\mathbf{t}_u = \langle\, 0,\, 0,\, v\,\rangle$. By **Equation 6.21**, the heat flow across S_1 is

$$\begin{aligned}\int\int_{S_2} -k\nabla T \bullet d\mathbf{S} &= -55\int_0^{2\pi}\int_0^1 \nabla T(u,\, v)\bullet(\mathbf{t}_v\times\mathbf{t}_u)dvdu\\ &= -55\int_0^{2\pi}\int_0^1 \langle\, 8v\cos u,\, 8v\sin u,\, v^2\cos^2 u + v^2\sin^2 u\,\rangle \bullet \langle\, 0,\, 0,\, v\,\rangle\, dvdu\\ &= -55\int_0^{2\pi}\int_0^1 \langle\, 8v\cos u,\, 8v\sin u,\, v^2\,\rangle \bullet \langle\, 0,\, 0,\, v\,\rangle\, dvdu\\ &= -55\int_0^{2\pi}\int_0^1 v^3\, dvdu = -\frac{55\pi}{2}.\end{aligned}$$

Last, let's consider the cylindrical side of the object. This surface has parameterization $\mathbf{r}(u,\, v) = \langle\, \cos u,\, \sin u,\, v\,\rangle,\ 0 \le u < 2\pi,\ 1 \le v \le 4$. By **Example 6.66**, we know that $\mathbf{t}_u\times\mathbf{t}_v = \langle\, \cos u,\, \sin u,\, 0\,\rangle$. By **Equation 6.21**,

$$\begin{aligned}\iint_{S_3} -k\nabla T \bullet d\mathbf{S} &= -55\int_0^{2\pi}\int_1^4 \nabla T(u,\, v)\bullet(\mathbf{t}_v\times\mathbf{t}_u)dvdu\\ &= -55\int_0^{2\pi}\int_1^4 \langle\, 2v\cos u,\, 2v\sin u,\, \cos^2 u + \sin^2 u\,\rangle \bullet \langle\, \cos u,\, \sin u,\, 0\,\rangle\, dvdu\\ &= -55\int_0^{2\pi}\int_0^1 \langle\, 2v\cos u,\, 2v\sin u,\, 1\,\rangle \bullet \langle\, \cos u,\, \sin u,\, 0\,\rangle\, dvdu\\ &= -55\int_0^{2\pi}\int_0^1 \left(2v\cos^2 u + 2v\sin^2 u\right)dvdu\\ &= -55\int_0^{2\pi}\int_0^1 2v dvdu = -55\int_0^{2\pi} du = -110\pi.\end{aligned}$$

Therefore, the rate of heat flow across S is $\frac{55\pi}{2} - \frac{55\pi}{2} - 110\pi = -110\pi.$

6.60 A cast-iron solid ball is given by inequality $x^2 + y^2 + z^2 \leq 1$. The temperature at a point in a region containing the ball is $T(x, y, z) = \frac{1}{3}\left(x^2 + y^2 + z^2\right)$. Find the heat flow across the boundary of the solid if this boundary is oriented outward.

6.6 EXERCISES

For the following exercises, determine whether the statements are *true or false*.

269. If surface S is given by $\{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, z = 10\}$, then $\iint_S f(x, y, z)dS = \int_0^1 \int_0^1 f(x, y, 10)dxdy.$

270. If surface S is given by $\{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, z = x\}$, then $\iint_S f(x, y, z)dS = \int_0^1 \int_0^1 f(x, y, x)dxdy.$

271. Surface $\mathbf{r} = \langle v \cos u, v \sin u, v^2 \rangle$, for $0 \leq u \leq \pi, 0 \leq v \leq 2$, is the same as surface $\mathbf{r} = \langle \sqrt{v} \cos 2u, \sqrt{v} \sin 2u, v \rangle$, for $0 \leq u \leq \frac{\pi}{2}, 0 \leq v \leq 4$.

272. Given the standard parameterization of a sphere, normal vectors $\mathrm{t}_u \times \mathrm{t}_v$ are outward normal vectors.

For the following exercises, find parametric descriptions for the following surfaces.

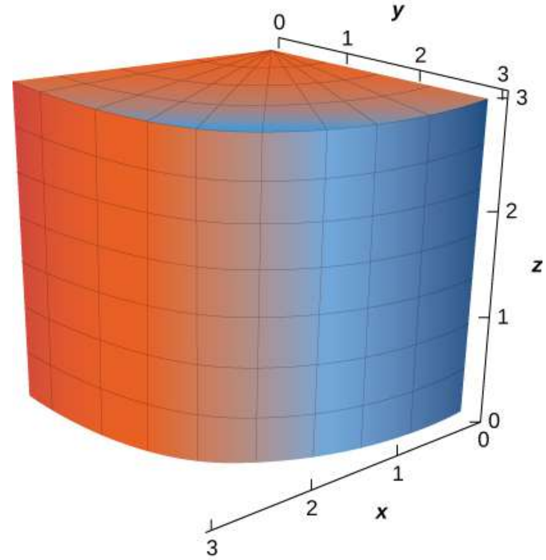
273. Plane $3x - 2y + z = 2$

274. Paraboloid $z = x^2 + y^2$, for $0 \leq z \leq 9$.

275. Plane $2x - 4y + 3z = 16$

276. The frustum of cone $z^2 = x^2 + y^2$, for $2 \leq z \leq 8$

277. The portion of cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \leq z \leq 3$



278. A cone with base radius r and height h, where r and h are positive constants

For the following exercises, use a computer algebra system to approximate the area of the following surfaces using a parametric description of the surface.

279. **[T]** Half cylinder $\{(r, \theta, z) : r = 4, 0 \leq \theta \leq \pi, 0 \leq z \leq 7\}$

280. **[T]** Plane $z = 10 - x - y$ above square $|x| \leq 2, |y| \leq 2$

For the following exercises, let S be the hemisphere $x^2 + y^2 + z^2 = 4$, with $z \geq 0$, and evaluate each surface integral, in the counterclockwise direction.

281. $\iint_S z dS$

282. $\iint_S (x - 2y)dS$

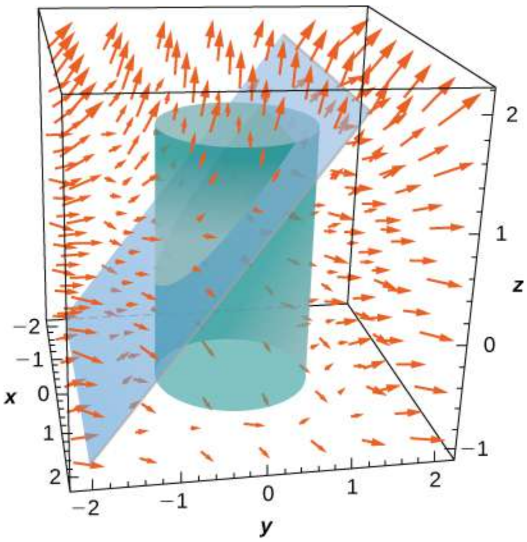
283. $\iint_S (x^2 + y^2)z dS$

For the following exercises, evaluate $\int\int_S \mathbf{F} \cdot \mathbf{N} ds$ for vector field $\mathbf{F}$, where $\mathbf{N}$ is an outward normal vector to surface *S*.

284. $\mathbf{F}(x,\ y,\ z) = x\mathbf{i} + 2y\mathbf{j} - 3z\mathbf{k},$ and S is that part of plane $15x - 12y + 3z = 6$ that lies above unit square $0 \le x \le 1,\ 0 \le y \le 1.$

285. $\mathbf{F}(x,\ y,\ z) = x\mathbf{i} + y\mathbf{j},$ and S is hemisphere $z = \sqrt{1 - x^2 - y^2}.$

286. $\mathbf{F}(x,\ y,\ z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k},$ and S is the portion of plane $z = y + 1$ that lies inside cylinder $x^2 + y^2 = 1.$



For the following exercises, approximate the mass of the homogeneous lamina that has the shape of given surface S. Round to four decimal places.

287. **[T]** S is surface $z = 4 - x - 2y,$ with $z \ge 0, x \ge 0, y \ge 0; \xi = x.$

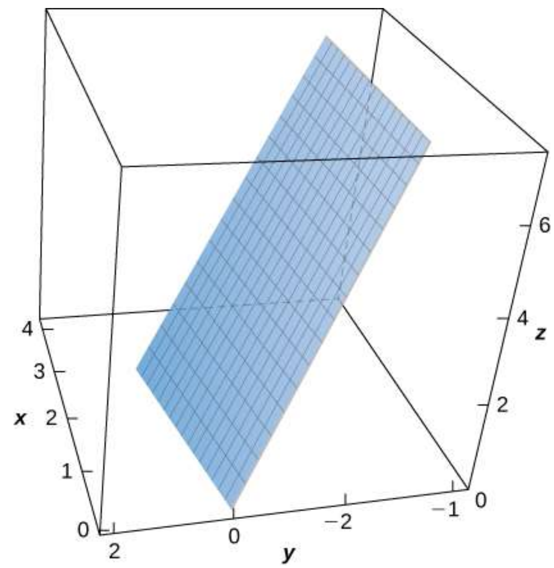
288. **[T]** S is surface $z = x^2 + y^2,$ with $z \le 1; \xi = z.$

289. **[T]** S is surface $x^2 + y^2 + x^2 = 5,$ with $z \ge 1; \xi = \theta^2.$

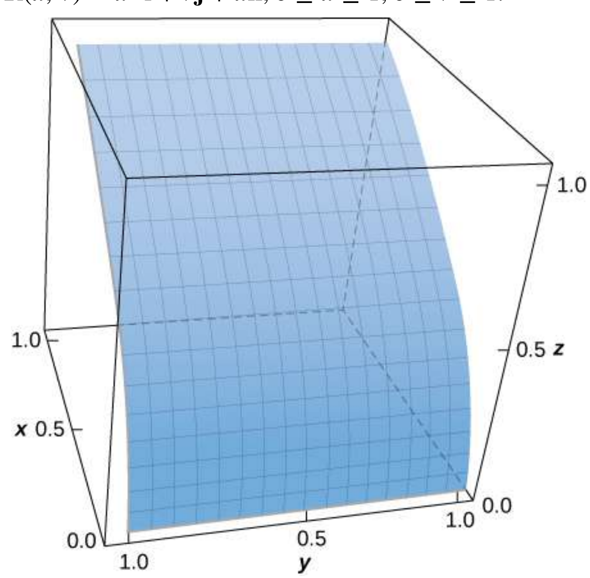
290. Evaluate $\iint_S \left(y^2 z\mathbf{i} + y^3\mathbf{j} + xz\mathbf{k}\right) \cdot d\mathbf{S},$ where S is the surface of cube $-1 \le x \le 1,\ -1 \le y \le 1,$ and $0 \le z \le 2.$ in a counterclockwise direction.

291. Evaluate surface integral $\iint_S g dS,$ where $g(x,\ y,\ z) = xz + 2x^2 - 3xy$ and S is the portion of plane $2x - 3y + z = 6$ that lies over unit square R: $0 \le x \le 1, 0 \le y \le 1.$

292. Evaluate $\iint_S (x + y + z) d\mathbf{S},$ where S is the surface defined parametrically by $\mathbf{R}(u,\ v) = (2u + v)\mathbf{i} + (u - 2v)\mathbf{j} + (u + 3v)\mathbf{k}$ for $0 \le u \le 1,$ and $0 \le v \le 2.$



293. **[T]** Evaluate $\iint_S (x - y^2 + z) d\mathbf{S},$ where S is the surface defined by $\mathbf{R}(u,\ v) = u^2\mathbf{i} + v\mathbf{j} + u\mathbf{k}, 0 \le u \le 1, 0 \le v \le 1.$

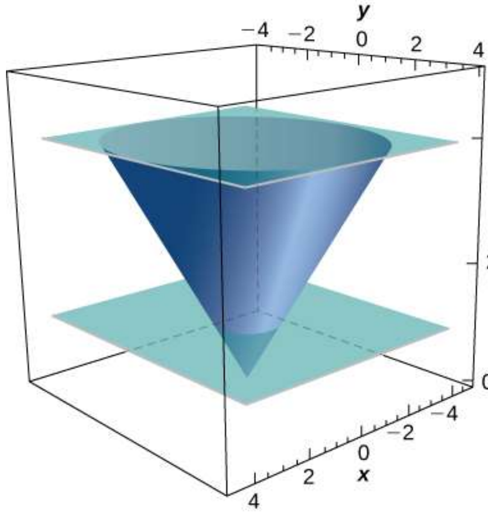


294. **[T]** Evaluate where S is the surface defined by $\mathbf{R}(u,\ v) = u\mathbf{i} - u^2\mathbf{j} + v\mathbf{k}, 0 \le u \le 2, 0 \le v \le 1.$ for $0 \le u \le 1, 0 \le v \le 2.$

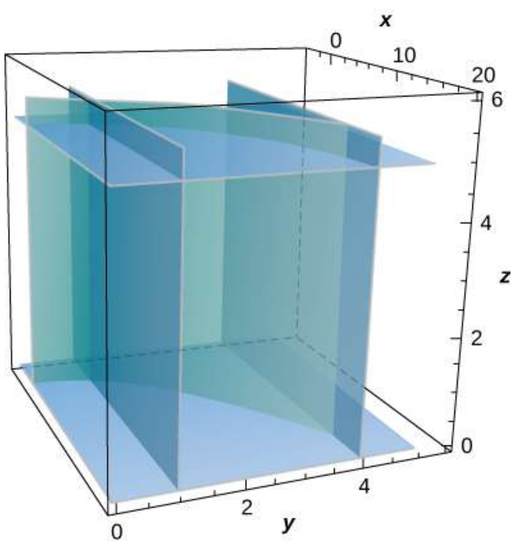
295. Evaluate $\iint_S \left(x^2 + y^2\right) d\mathbf{S},$ where S is the surface bounded above hemisphere $z = \sqrt{1 - x^2 - y^2},$ and below by plane $z = 0.$

296. Evaluate $\iint_S \left(x^2 + y^2 + z^2\right) d\mathbf{S}$, where S is the portion of plane $z = x + 1$ that lies inside cylinder $x^2 + y^2 = 1$.

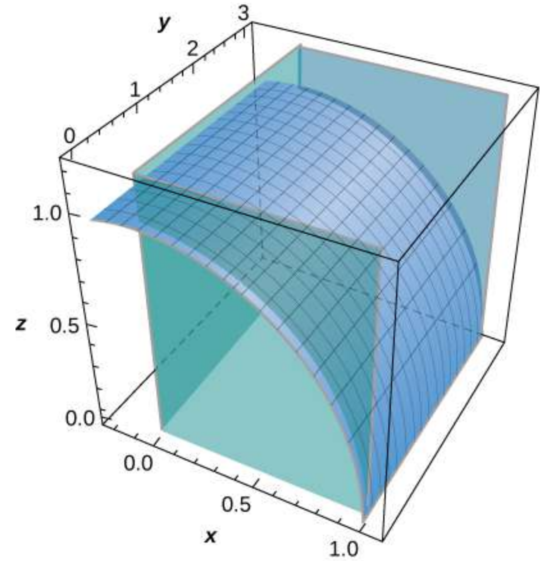
297. **[T]** Evaluate $\iint_S x^2 z dS$, where S is the portion of cone $z^2 = x^2 + y^2$ that lies between planes $z = 1$ and $z = 4$.



298. **[T]** Evaluate $\iint_S (xz/y) dS$, where S is the portion of cylinder $x = y^2$ that lies in the first octant between planes $z = 0$, $z = 5$, $y = 1$, and $y = 4$.



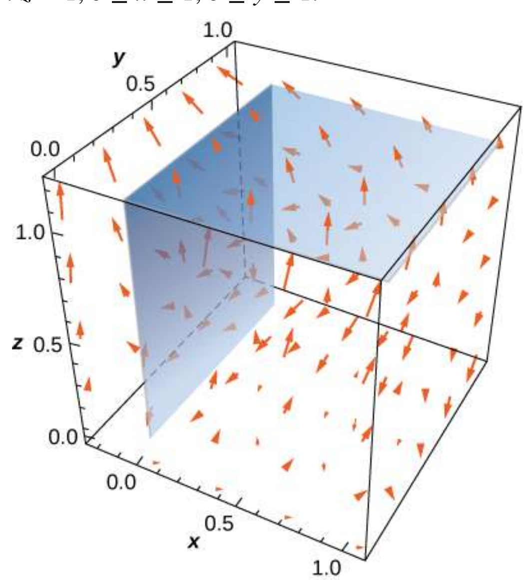
299. **[T]** Evaluate $\iint_S (z + y) dS$, where S is the part of the graph of $z = \sqrt{1 - x^2}$ in the first octant between the xz-plane and plane $y = 3$.



300. Evaluate $\iint_S xyz d\mathrm{S}$ if S is the part of plane $z = x + y$ that lies over the triangular region in the xy-plane with vertices (0, 0, 0), (1, 0, 0), and (0, 2, 0).

301. Find the mass of a lamina of density $\xi(x, y, z) = z$ in the shape of hemisphere $z = \left(a^2 - x^2 - y^2\right)^{1/2}$.

302. Compute $\int\int_S \mathbf{F} \cdot \mathbf{N} dS$, where $\mathbf{F}(x, y, z) = x\mathbf{i} - 5y\mathbf{j} + 4z\mathbf{k}$ and $\mathbf{N}$ is an outward normal vector S, where S is the union of two squares $S_1 : x = 0, 0 \le y \le 1, 0 \le z \le 1$ and $S_2 : z = 1, 0 \le x \le 1, 0 \le y \le 1$.



303. Compute $\int\int_S \mathbf{F} \cdot \mathbf{N} dS,$ where $\mathbf{F}(x, y, z) = xy\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k}$ and $\mathbf{N}$ is an outward normal vector S, where S is the triangular region cut off from plane $x + y + z = 1$ by the positive coordinate axes.

304. Compute $\int\int_S \mathbf{F} \cdot \mathbf{N} dS,$ where $\mathbf{F}(x, y, z) = 2yz\mathbf{i} + \left(\tan^{-1} xz\right)\mathbf{j} + e^{xy}\mathbf{k}$ and $\mathbf{N}$ is an outward normal vector S, where S is the surface of sphere $x^2 + y^2 + z^2 = 1.$

305. Compute $\int\int_S \mathbf{F} \cdot \mathbf{N} dS,$ where $\mathbf{F}(x, y, z) = xyz\mathbf{i} + xyz\mathbf{j} + xyz\mathbf{k}$ and $\mathbf{N}$ is an outward normal vector S, where S is the surface of the five faces of the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ missing $z = 0.$

For the following exercises, express the surface integral as an iterated double integral by using a projection on S on the yz-plane.

306. $\iint_S xy^2 z^3 \, d\mathrm{S};$ S is the first-octant portion of plane $2x + 3y + 4z = 12.$

307. $\iint_S \left(x^2 - 2y + z\right) d\mathrm{S};$ S is the portion of the graph of $4x + y = 8$ bounded by the coordinate planes and plane $z = 6.$

For the following exercises, express the surface integral as an iterated double integral by using a projection on S on the xz-plane

308. $\iint_S xy^2 z^3 \, d\mathrm{S};$ S is the first-octant portion of plane $2x + 3y + 4z = 12.$

309. $\iint_S \left(x^2 - 2y + z\right) d\mathrm{S};$ S is the portion of the graph of $4x + y = 8$ bounded by the coordinate planes and plane $z = 6.$

310. Evaluate surface integral $\iint_S yz d\mathrm{S},$ where S is the first-octant part of plane $x + y + z = \lambda,$ where λ is a positive constant.

311. Evaluate surface integral $\iint_S \left(x^2 z + y^2 z\right) d\mathrm{S},$ where S is hemisphere $x^2 + y^2 + z^2 = a^2,\ z \ge 0.$

312. Evaluate surface integral $\iint_S z d\mathrm{A},$ where S is surface $z = \sqrt{x^2 + y^2},\ 0 \le z \le 2.$

313. Evaluate surface integral $\iint_S x^2 yz dS,$ where S is the part of plane $z = 1 + 2x + 3y$ that lies above rectangle $0 \le x \le 3$ and $0 \le y \le 2.$

314. Evaluate surface integral $\iint_S yz dS,$ where S is plane $x + y + z = 1$ that lies in the first octant.

315. Evaluate surface integral $\iint_S yz d\mathrm{S},$ where S is the part of plane $z = y + 3$ that lies inside cylinder $x^2 + y^2 = 1.$

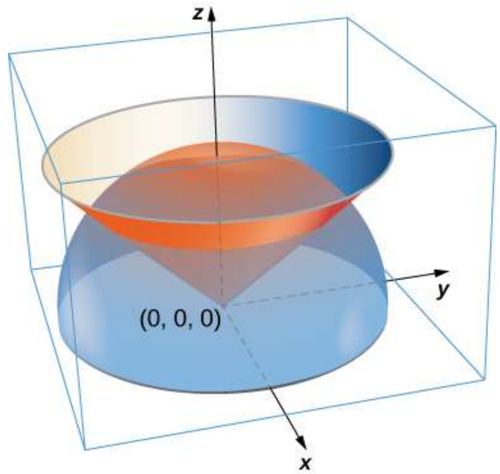
For the following exercises, use geometric reasoning to evaluate the given surface integrals.

316. $\iint_S \sqrt{x^2 + y^2 + z^2} dS,$ where S is surface $x^2 + y^2 + z^2 = 4,\ z \ge 0$

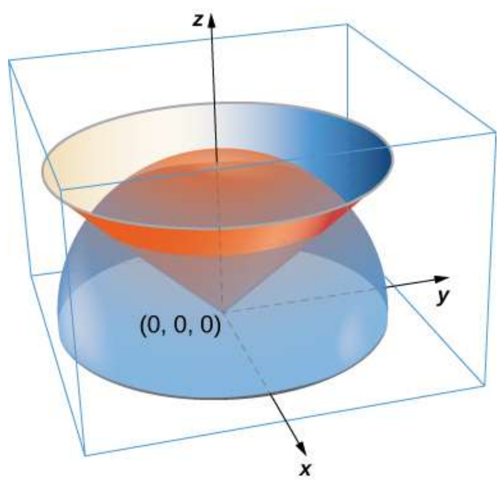
317. $\iint_S (x\mathbf{i} + y\mathbf{j}) \cdot d\mathbf{S},$ where S is surface $x^2 + y^2 = 4,\ 1 \le z \le 3,$ oriented with unit normal vectors pointing outward

318. $\iint_S (z\mathbf{k}) \cdot d\mathbf{S},$ where S is disc $x^2 + y^2 \le 9$ on plane $z = 4,$ oriented with unit normal vectors pointing upward

319. A lamina has the shape of a portion of sphere $x^2 + y^2 + z^2 = a^2$ that lies within cone $z = \sqrt{x^2 + y^2}$. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Determine the mass of the lamina if $\delta(x, y, z) = x^2 y^2 z$.



320. A lamina has the shape of a portion of sphere $x^2 + y^2 + z^2 = a^2$ that lies within cone $z = \sqrt{x^2 + y^2}$. Let S be the spherical shell centered at the origin with radius a, and let C be the right circular cone with a vertex at the origin and an axis of symmetry that coincides with the z-axis. Suppose the vertex angle of the cone is ϕ_0, with $0 \leq \phi_0 < \frac{\pi}{2}$. Determine the mass of that portion of the shape enclosed in the intersection of S and C. Assume $\delta(x, y, z) = x^2 y^2 z$.



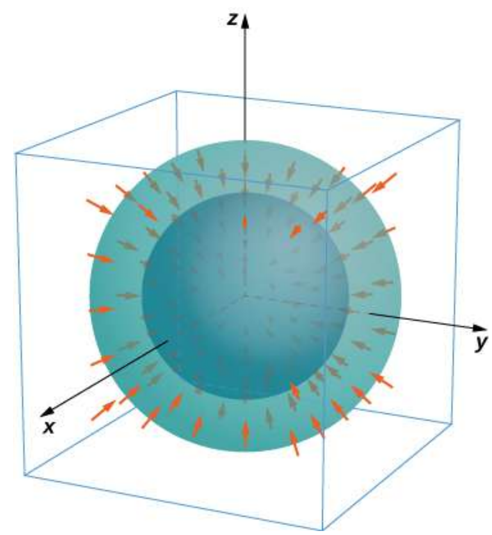
321. A paper cup has the shape of an inverted right circular cone of height 6 in. and radius of top 3 in. If the cup is full of water weighing 62.5 lb/ft^3, find the total force exerted by the water on the inside surface of the cup.

For the following exercises, the heat flow vector field for conducting objects i $\mathbf{F} = -k\nabla T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Find the outward flux of **F** across the following surfaces S for the given temperature distributions and assume $k = 1$.

322. $T(x, y, z) = 100e^{-x-y}$; S consists of the faces of cube $|x| \leq 1, |y| \leq 1, |z| \leq 1$.

323. $T(x, y, z) = -\ln\left(x^2 + y^2 + z^2\right)$; S is sphere $x^2 + y^2 + z^2 = a^2$.

For the following exercises, consider the radial fields $\mathbf{F} = \dfrac{\langle x, y, z \rangle}{\left(x^2 + y^2 + z^2\right)^{\frac{p}{2}}} = \dfrac{\mathbf{r}}{|\mathbf{r}|^p}$, where p is a real number. Let S consist of spheres A and B centered at the origin with radii $0 < a < b$. The total outward flux across S consists of the outward flux across the outer sphere B less the flux into S across inner sphere A.



324. Find the total flux across S with $p = 0$.

325. Show that for $p = 3$ the flux across S is independent of a and b.

6.7 | Stokes' Theorem

Learning Objectives

6.7.1 Explain the meaning of Stokes' theorem.
6.7.2 Use Stokes' theorem to evaluate a line integral.
6.7.3 Use Stokes' theorem to calculate a surface integral.
6.7.4 Use Stokes' theorem to calculate a curl.

In this section, we study Stokes' theorem, a higher-dimensional generalization of Green's theorem. This theorem, like the Fundamental Theorem for Line Integrals and Green's theorem, is a generalization of the Fundamental Theorem of Calculus to higher dimensions. Stokes' theorem relates a vector surface integral over surface S in space to a line integral around the boundary of S. Therefore, just as the theorems before it, Stokes' theorem can be used to reduce an integral over a geometric object S to an integral over the boundary of S.

In addition to allowing us to translate between line integrals and surface integrals, Stokes' theorem connects the concepts of curl and circulation. Furthermore, the theorem has applications in fluid mechanics and electromagnetism. We use Stokes' theorem to derive Faraday's law, an important result involving electric fields.

Stokes' Theorem

Stokes' theorem says we can calculate the flux of curl $\mathbf{F}$ across surface S by knowing information only about the values of $\mathbf{F}$ along the boundary of S. Conversely, we can calculate the line integral of vector field $\mathbf{F}$ along the boundary of surface S by translating to a double integral of the curl of $\mathbf{F}$ over S.

Let S be an oriented smooth surface with unit normal vector $\mathbf{N}$. Furthermore, suppose the boundary of S is a simple closed curve C. The orientation of S induces the positive orientation of C if, as you walk in the positive direction around C with your head pointing in the direction of $\mathbf{N}$, the surface is always on your left. With this definition in place, we can state Stokes' theorem.

Theorem 6.19: Stokes' Theorem

Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation (**Figure 6.79**). If $\mathbf{F}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}.$$

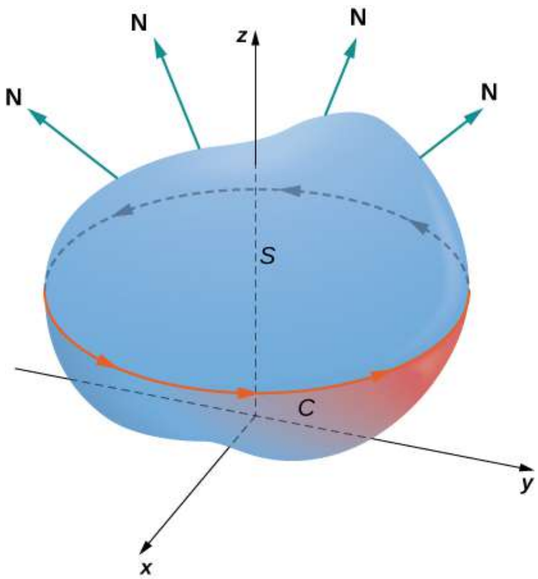


Figure 6.79 Stokes' theorem relates the flux integral over the surface to a line integral around the boundary of the surface. Note that the orientation of the curve is positive.

Suppose surface S is a flat region in the xy-plane with upward orientation. Then the unit normal vector is $\mathbf{k}$ and surface integral $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ is actually the double integral $\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{k}dA$. In this special case, Stokes' theorem gives $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{k}dA$. However, this is the flux form of Green's theorem, which shows us that Green's theorem is a special case of Stokes' theorem. Green's theorem can only handle surfaces in a plane, but Stokes' theorem can handle surfaces in a plane or in space.

The complete proof of Stokes' theorem is beyond the scope of this text. We look at an intuitive explanation for the truth of the theorem and then see proof of the theorem in the special case that surface S is a portion of a graph of a function, and S, the boundary of S, and $\mathbf{F}$ are all fairly tame.

Proof

First, we look at an informal proof of the theorem. This proof is not rigorous, but it is meant to give a general feeling for why the theorem is true. Let S be a surface and let D be a small piece of the surface so that D does not share any points with the boundary of S. We choose D to be small enough so that it can be approximated by an oriented square E. Let D inherit its orientation from S, and give E the same orientation. This square has four sides; denote them E_l, E_r, E_u, and E_d for the left, right, up, and down sides, respectively. On the square, we can use the flux form of Green's theorem:

$$\int_{E_l+E_d+E_r+E_u} \mathbf{F}\cdot d\mathbf{r} = \iint_E \text{curl}\,\mathbf{F}\cdot\mathbf{N}dS = \iint_E \text{curl}\,\mathbf{F}\cdot d\mathbf{S}.$$

To approximate the flux over the entire surface, we add the values of the flux on the small squares approximating small pieces of the surface (**Figure 6.80**). By Green's theorem, the flux across each approximating square is a line integral over its boundary. Let F be an approximating square with an orientation inherited from S and with a right side E_l (so F is to the left of E). Let F_r denote the right side of F; then, $E_l = -F_r$. In other words, the right side of F is the same curve as the left side of E, just oriented in the opposite direction. Therefore,

$$\int_{E_l} \mathbf{F}\cdot d\mathbf{r} = -\int_{F_r} \mathbf{F}\cdot d\mathbf{r}.$$

As we add up all the fluxes over all the squares approximating surface S, line integrals $\int_{E_l} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{F_r} \mathbf{F}\cdot d\mathbf{r}$ cancel each other out. The same goes for the line integrals over the other three sides of E. These three line integrals cancel out with the line integral of the lower side of the square above E, the line integral over the left side of the square to the right of E, and the line integral over the upper side of the square below E (**Figure 6.81**). After all this cancelation occurs over all the approximating squares, the only line integrals that survive are the line integrals over sides approximating the boundary of S. Therefore, the sum of all the fluxes (which, by Green's theorem, is the sum of all the line integrals around the boundaries of approximating squares) can be approximated by a line integral over the boundary of S. In the limit, as the areas of the approximating squares go to zero, this approximation gets arbitrarily close to the flux.

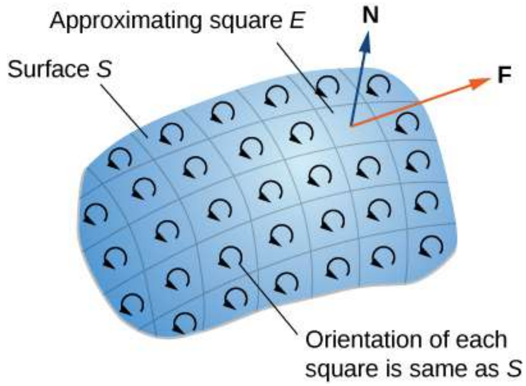


Figure 6.80 Chop the surface into small pieces. The pieces should be small enough that they can be approximated by a square.

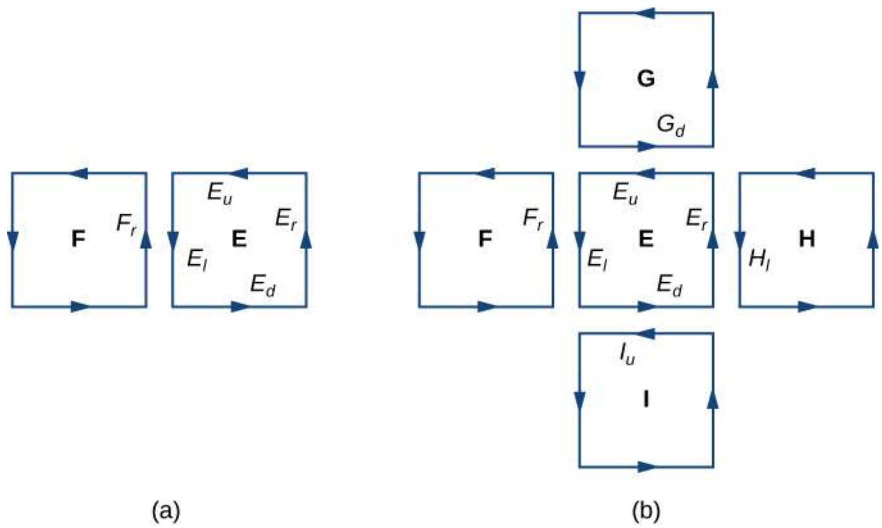


Figure 6.81 (a) The line integral along E_l cancels out the line integral along F_r because $E_l = -F_r$. (b) The line integral along any of the sides of E cancels out with the line integral along a side of an adjacent approximating square.

Let's now look at a rigorous proof of the theorem in the special case that S is the graph of function $z = f(x, y)$, where x and y vary over a bounded, simply connected region D of finite area (**Figure 6.82**). Furthermore, assume that f has continuous second-order partial derivatives. Let C denote the boundary of S and let C' denote the boundary of D. Then, D is the "shadow" of S in the plane and C' is the "shadow" of C. Suppose that S is oriented upward. The counterclockwise orientation of C is positive, as is the counterclockwise orientation of C'. Let $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ be a vector field with component functions that have continuous partial derivatives.

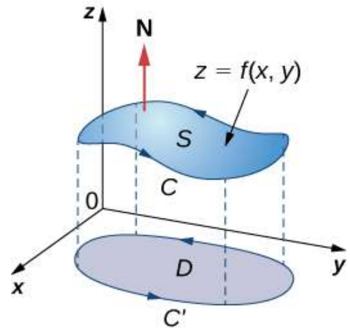


Figure 6.82 D is the "shadow," or projection, of S in the plane and C' is the projection of C.

We take the standard parameterization of $S : x = x,\ y = y,\ z = g(x, y)$. The tangent vectors are $\mathbf{t}_x = \langle 1, 0, g_x \rangle$ and $\mathbf{t}_y = \langle 0, 1, g_y \rangle$, and therefore, $\mathbf{t}_x \cdot \mathbf{t}_y = \langle -g_x, -g_y, 1 \rangle$. By **Equation 6.19**,

$$\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \iint_D \left[-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y)\right]dA,$$

where the partial derivatives are all evaluated at $(x, y, g(x, y))$, making the integrand depend on x and y only. Suppose $\langle x(t), y(t) \rangle, a \le t \le b$ is a parameterization of C'. Then, a parameterization of C is $\langle x(t), y(t), g(x(t), y(t)) \rangle, a \le t \le b$. Armed with these parameterizations, the Chain rule, and Green's theorem, and keeping in mind that P, Q, and R are all functions of x and y, we can evaluate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b (Px'(t) + Qy'(t) + Rz'(t))dt \\
&= \int_a^b \left[Px'(t) + Qy'(t) + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right]dt \\
&= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)x'(t) + \left(Q + R\frac{\partial z}{\partial y}\right)y'(t)\right]dt \\
&= \int_{C'} \left(P + R\frac{\partial z}{\partial x}\right)dx + \left(Q + R\frac{\partial z}{\partial y}\right)dy \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right]dA \\
&= \iint_D \begin{pmatrix} \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x \partial y}\right) \\ -\left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y \partial x}\right) \end{pmatrix} dA.
\end{aligned}$$

By Clairaut's theorem, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$. Therefore, four of the terms disappear from this double integral, and we are left with

$$\iint_D \left[-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y)\right]dA,$$

which equals $\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$.

□

We have shown that Stokes' theorem is true in the case of a function with a domain that is a simply connected region of finite area. We can quickly confirm this theorem for another important case: when vector field $\mathbf{F}$ is conservative. If $\mathbf{F}$ is conservative, the curl of **F** is zero, so $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = 0.$ Since the boundary of S is a closed curve, $\int_C \mathbf{F} \cdot d\mathbf{r}$ is also zero.

Example 6.73

Verifying Stokes' Theorem for a Specific Case

Verify that Stokes' theorem is true for vector field $\mathbf{F}(x, y, z) = \langle y, 2z, x^2 \rangle$ and surface S, where S is the paravbolid $z = 4 - x^2 - y^2$.

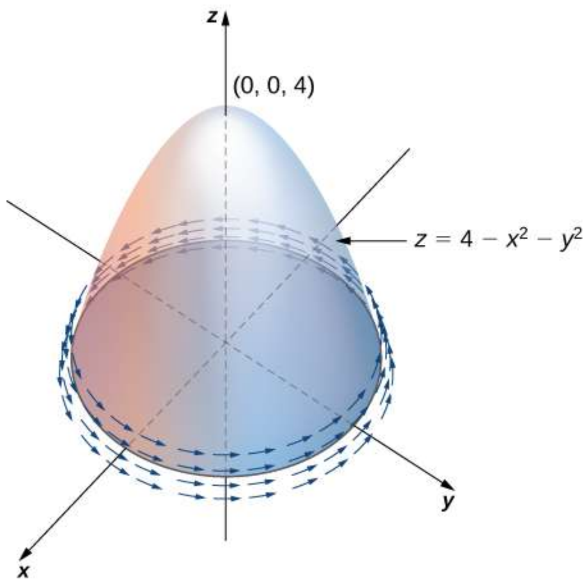


Figure 6.83 Verifying Stokes' theorem for a hemisphere in a vector field.

Solution

As a surface integral, you have $g(x, y) = 4 - x^2 - y^2,\ g_x = -2y$ and

$$\text{curl}\, \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2z & x^2 \end{vmatrix} = \langle -2, -2x, -1 \rangle.$$

By **Equation 6.19**,

$$\begin{aligned} \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} &= \iint_D \text{curl}\, \mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{t}_\phi \times \mathbf{t}_\theta) dA \\ &= \iint_D \langle -2, -2x, -1 \rangle \cdot \langle 2x, 2y, 1 \rangle\, dA \\ &= \int_{-2}^{2} \int_{\sqrt{4 - x^2}}^{\sqrt{4 - x^2}} (-4x - 4xy - 1) dy dx \\ &= \int_{-2}^{2} \left(-8x\sqrt{4 - x^2} - 2\sqrt{4 - x^2} \right) dx \\ &= -4\pi \end{aligned}$$

As a line integral, you can parameterize C by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 0 \rangle \; 0 \le t \le 2\pi$. By **Equation 6.19**,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle 2\sin t, 0, 4\cos^2 t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt \\ &= \int_0^{2\pi} -4\sin^2 t dt = -4\pi\end{aligned} \tag{6.22}$$

Therefore, we have verified Stokes' theorem for this example.

6.61 Verify that Stokes' theorem is true for vector field $\mathbf{F}(x, y, z) = \langle y, x, -z \rangle$ and surface S, where S is the upwardly oriented portion of the graph of $f(x, y) = x^2 y$ over a triangle in the xy-plane with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$.

Applying Stokes' Theorem

Stokes' theorem translates between the flux integral of surface S to a line integral around the boundary of S. Therefore, the theorem allows us to compute surface integrals or line integrals that would ordinarily be quite difficult by translating the line integral into a surface integral or vice versa. We now study some examples of each kind of translation.

Example 6.74

Calculating a Surface Integral

Calculate surface integral $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$, where S is the surface, oriented outward, in **Figure 6.84** and $\mathbf{F} = \langle z, 2xy, x + y \rangle$.

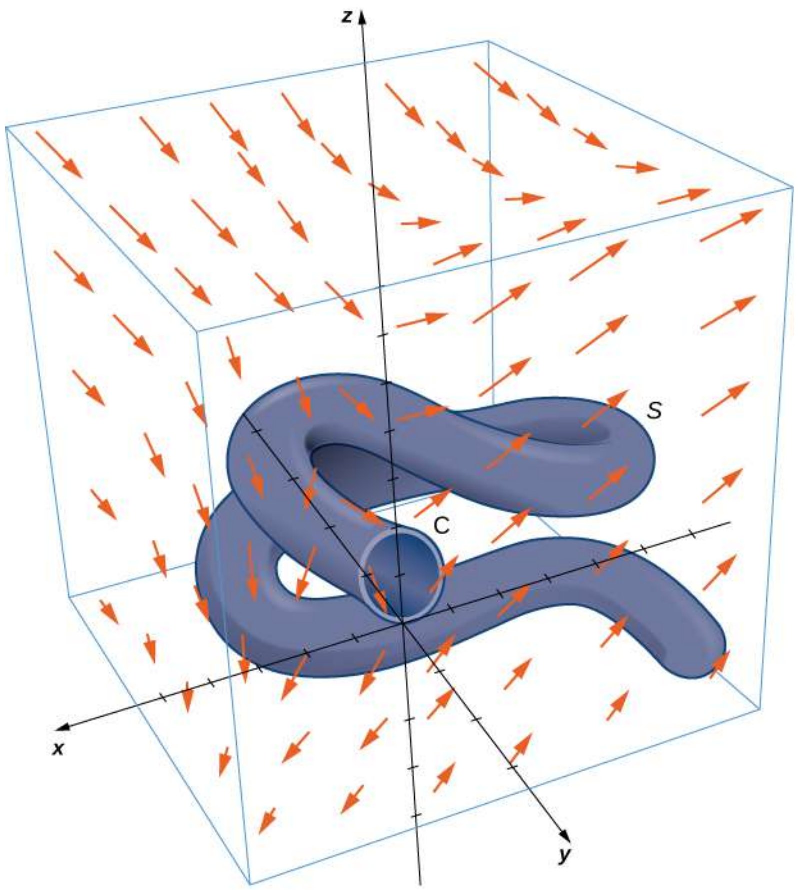


Figure 6.84 A complicated surface in a vector field.

Solution

Note that to calculate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ without using Stokes' theorem, we would need to use **Equation 6.19**. Use of this equation requires a parameterization of S. Surface S is complicated enough that it would be extremely difficult to find a parameterization. Therefore, the methods we have learned in previous sections are not useful for this problem. Instead, we use Stokes' theorem, noting that the boundary C of the surface is merely a single circle with radius 1.

The curl of $\mathbf{F}$ is $\langle 1, 1, 2y \rangle$. By Stokes' theorem,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r},$$

where C has parameterization $\mathbf{r}(t) = \langle -\sin t, 0, 1 - \cos t \rangle$, $0 \le t < 2\pi$. By **Equation 6.9**,

$$
\begin{aligned}
\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} &= \int_C \mathbf{F}\cdot d\mathbf{r}\\
&= \int_0^{2\pi} \langle 1-\cos t,\, 0,\, -\sin t\rangle \cdot \langle -\cos t,\, 0,\, \sin t\rangle\, dt\\
&= \int_0^{2\pi} \left(-\cos t + \cos^2 t - \sin^2 t\right) dt\\
&= \left[-\sin t + \tfrac{1}{2}\sin(2t)\right]_0^{2\pi}\\
&= (-\sin(2\pi) + \tfrac{1}{2}\sin(4\pi)) - (-\sin 0 + \tfrac{1}{2}\sin 0)\\
&= 0.
\end{aligned}
$$

An amazing consequence of Stokes' theorem is that if S' is any other smooth surface with boundary C and the same orientation as S, then $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = 0$ because Stokes' theorem says the surface integral depends on the line integral around the boundary only.

In **Example 6.74**, we calculated a surface integral simply by using information about the boundary of the surface. In general, let S_1 and S_2 be smooth surfaces with the same boundary C and the same orientation. By Stokes' theorem,

$$\iint_{S_1} \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2} \text{curl}\,\mathbf{F}\cdot d\mathbf{S}. \tag{6.23}$$

Therefore, if $\iint_{S_1} \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ is difficult to calculate but $\iint_{S_2} \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ is easy to calculate, Stokes' theorem allows us to calculate the easier surface integral. In **Example 6.74**, we could have calculated $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ by calculating $\iint_{S'} \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$, where S' is the disk enclosed by boundary curve C (a much more simple surface with which to work).

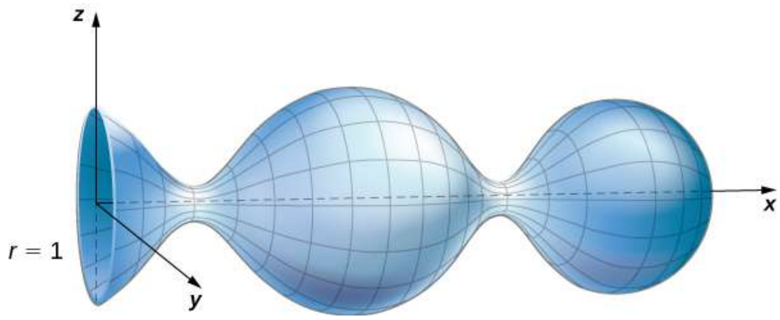
Equation 6.23 shows that flux integrals of curl vector fields are **surface independent** in the same way that line integrals of gradient fields are path independent. Recall that if $\mathbf{F}$ is a two-dimensional conservative vector field defined on a simply connected domain, f is a potential function for $\mathbf{F}$, and C is a curve in the domain of $\mathbf{F}$, then $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the endpoints of C. Therefore if C' is any other curve with the same starting point and endpoint as C (that is, C' has the same orientation as C), then $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'} \mathbf{F}\cdot d\mathbf{r}$. In other words, the value of the integral depends on the boundary of the path only; it does not really depend on the path itself.

Analogously, suppose that S and S' are surfaces with the same boundary and same orientation, and suppose that $\mathbf{G}$ is a three-dimensional vector field that can be written as the curl of another vector field $\mathbf{F}$ (so that $\mathbf{F}$ is like a "potential field" of $\mathbf{G}$). By **Equation 6.23**,

$$\iint_S \mathbf{G}\cdot d\mathbf{S} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S'} \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_{S'} \mathbf{G}\cdot d\mathbf{S}.$$

Therefore, the flux integral of $\mathbf{G}$ does not depend on the surface, only on the boundary of the surface. Flux integrals of vector fields that can be written as the curl of a vector field are surface independent in the same way that line integrals of vector fields that can be written as the gradient of a scalar function are path independent.

6.62 Use Stokes' theorem to calculate surface integral $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle z, x, y \rangle$ and S is the surface as shown in the following figure. The boundary curve, C, is oriented clockwise.



Example 6.75

Calculating a Line Integral

Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle xy, x^2 + y^2 + z^2, yz \rangle$ and C is the boundary of the parallelogram with vertices $(0, 0, 1), (0, 1, 0), (2, 0, -1),$ and $(2, 1, -2)$.

Solution

To calculate the line integral directly, we need to parameterize each side of the parallelogram separately, calculate four separate line integrals, and add the result. This is not overly complicated, but it is time-consuming.

By contrast, let's calculate the line integral using Stokes' theorem. Let S denote the surface of the parallelogram. Note that S is the portion of the graph of $z = 1 - x - y$ for (x, y) varying over the rectangular region with vertices $(0, 0),$ $(0, 1),$ $(2, 0),$ and $(2, 1)$ in the xy-plane. Therefore, a parameterization of S is $\langle x, y, 1 - x - y \rangle, 0 \le x \le 2, 0 \le y \le 1$. The curl of $\mathbf{F}$ is $-\langle z, 0, x \rangle$, and Stokes' theorem and **Equation 6.19** give

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} \\ &= \int_0^2 \int_0^1 \text{curl}\, \mathbf{F}(x, y) \cdot (\mathbf{t}_x \times \mathbf{t}_y) dy dx \\ &= \int_0^2 \int_0^1 \langle -(1 - x - y), 0, x \rangle \cdot (\langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle) dy dx \\ &= \int_0^2 \int_0^1 \langle x + y - 1, 0, x \rangle \cdot \langle 1, 1, 1 \rangle\, dy dx \\ &\int_0^2 \int_0^1 2x + y - 1 dy dx \\ &= 3.\end{aligned}$$

6.63 Use Stokes' theorem to calculate line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle z, x, y \rangle$ and C is oriented clockwise and is the boundary of a triangle with vertices $(0, 0, 1), (3, 0, -2),$ and $(0, 1, 2)$.

Interpretation of Curl

In addition to translating between line integrals and flux integrals, Stokes' theorem can be used to justify the physical interpretation of curl that we have learned. Here we investigate the relationship between curl and circulation, and we use Stokes' theorem to state Faraday's law—an important law in electricity and magnetism that relates the curl of an electric field to the rate of change of a magnetic field.

Recall that if C is a closed curve and $\mathbf{F}$ is a vector field defined on C, then the circulation of $\mathbf{F}$ around C is line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$. If $\mathbf{F}$ represents the velocity field of a fluid in space, then the circulation measures the tendency of the fluid to move in the direction of C.

Let $\mathbf{F}$ be a continuous vector field and let D_r be a small disk of radius r with center P_0 (**Figure 6.85**). If D_r is small enough, then $(\text{curl}\,\mathbf{F})(P) \approx (\text{curl}\,\mathbf{F})(P_0)$ for all points P in D_r because the curl is continuous. Let C_r be the boundary circle of D_r. By Stokes' theorem,

$$\int_{C_r} \mathbf{F} \cdot d\mathbf{r} = \iint_{D_r} \text{curl}\,\mathbf{F} \cdot \mathbf{N}\,dS \approx \iint_{D_r} (\text{curl}\,\mathbf{F})(P_0) \cdot \mathbf{N}(P_0)\,dS.$$

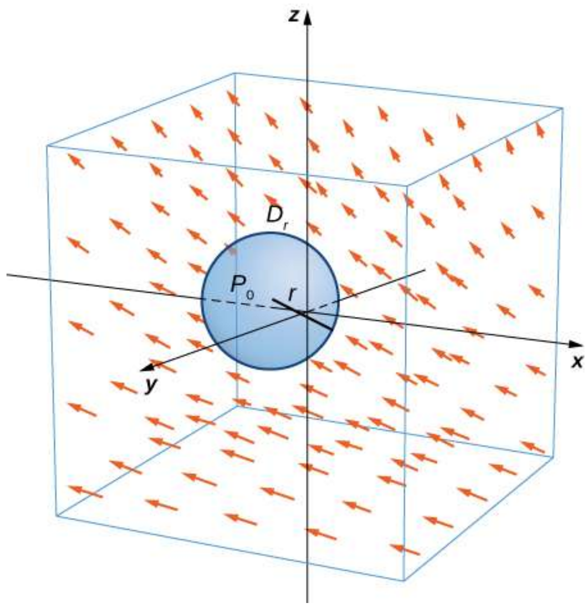


Figure 6.85 Disk D_r is a small disk in a continuous vector field.

The quantity $(\text{curl}\,\mathbf{F})(P_0) \cdot \mathbf{N}(P_0)$ is constant, and therefore

$$\iint_{D_r} (\text{curl}\,\mathbf{F})(P_0) \cdot \mathbf{N}(P_0)\,dS = \pi r^2 \left[(\text{curl}\,\mathbf{F})(P_0) \cdot \mathbf{N}(P_0)\right].$$

Thus

$$\int_{C_r} \mathbf{F} \cdot d\mathbf{r} \approx \pi r^2 \left[(\text{curl}\,\mathbf{F})(P_0) \cdot \mathbf{N}(P_0)\right],$$

and the approximation gets arbitrarily close as the radius shrinks to zero. Therefore Stokes' theorem implies that

$$(\text{curl}\,\mathbf{F})(P_0) \cdot \mathbf{N}(P_0) = \lim_{r \to 0^+} \frac{1}{\pi r^2} \int_{C_r} \mathbf{F} \cdot d\mathbf{r}.$$

This equation relates the curl of a vector field to the circulation. Since the area of the disk is πr^2, this equation says we can view the curl (in the limit) as the circulation per unit area. Recall that if $\mathbf{F}$ is the velocity field of a fluid, then circulation

$\oint_{C_r} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_r} \mathbf{F} \cdot \mathbf{T} ds$ is a measure of the tendency of the fluid to move around C_r. The reason for this is that $\mathbf{F} \cdot \mathbf{T}$ is a component of **F** in the direction of **T**, and the closer the direction of **F** is to **T**, the larger the value of $\mathbf{F} \cdot \mathbf{T}$ (remember that if **a** and **b** are vectors and **b** is fixed, then the dot product $\mathbf{a} \cdot \mathbf{b}$ is maximal when **a** points in the same direction as **b**). Therefore, if **F** is the velocity field of a fluid, then $\text{curl}\,\mathbf{F} \cdot \mathbf{N}$ is a measure of how the fluid rotates about axis **N**. The effect of the curl is largest about the axis that points in the direction of **N**, because in this case $\text{curl}\,\mathbf{F} \cdot \mathbf{N}$ is as large as possible.

To see this effect in a more concrete fashion, imagine placing a tiny paddlewheel at point P_0 (**Figure 6.86**). The paddlewheel achieves its maximum speed when the axis of the wheel points in the direction of curl**F**. This justifies the interpretation of the curl we have learned: curl is a measure of the rotation in the vector field about the axis that points in the direction of the normal vector **N**, and Stokes' theorem justifies this interpretation.

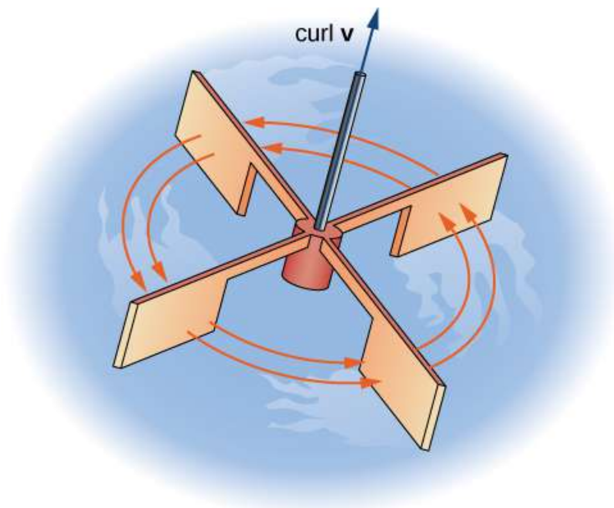


Figure 6.86 To visualize curl at a point, imagine placing a tiny paddlewheel at that point in the vector field.

Now that we have learned about Stokes' theorem, we can discuss applications in the area of electromagnetism. In particular, we examine how we can use Stokes' theorem to translate between two equivalent forms of Faraday's law. Before stating the two forms of Faraday's law, we need some background terminology.

Let C be a closed curve that models a thin wire. In the context of electric fields, the wire may be moving over time, so we write $C(t)$ to represent the wire. At a given time t, curve $C(t)$ may be different from original curve C because of the movement of the wire, but we assume that $C(t)$ is a closed curve for all times t. Let $D(t)$ be a surface with $C(t)$ as its boundary, and orient $C(t)$ so that $D(t)$ has positive orientation. Suppose that $C(t)$ is in a magnetic field $\mathbf{B}(t)$ that can also change over time. In other words, **B** has the form

$$\mathbf{B}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle,$$

where P, Q, and R can all vary continuously over time. We can produce current along the wire by changing field $\mathbf{B}(t)$ (this is a consequence of Ampere's law). Flux $\phi(t) = \iint_{D(t)} \mathbf{B}(t) \cdot d\mathbf{S}$ creates electric field $\mathbf{E}(t)$ that does work. The integral form of Faraday's law states that

$$\text{Work} = \int_{C(t)} \mathbf{E}(t) \cdot d\mathbf{r} = -\frac{\partial \phi}{\partial t}.$$

In other words, the work done by **E** is the line integral around the boundary, which is also equal to the rate of change of the flux with respect to time. The differential form of Faraday's law states that

$$\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Using Stokes' theorem, we can show that the differential form of Faraday's law is a consequence of the integral form. By Stokes' theorem, we can convert the line integral in the integral form into surface integral

$$-\frac{\partial \phi}{\partial t} = \int_{C(t)} \mathbf{E}(t) \cdot d\mathbf{r} = \iint_{D(t)} \text{curl } \mathbf{E}(t) \cdot d\mathbf{S}.$$

Since $\phi(t) = \iint_{D(t)} \mathbf{B}(t) \cdot d\mathbf{S},$ then as long as the integration of the surface does not vary with time we also have

$$-\frac{\partial \phi}{\partial t} = \iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}.$$

Therefore,

$$\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \iint_{D(t)} \text{curl } \mathbf{E} \cdot d\mathbf{S}.$$

To derive the differential form of Faraday's law, we would like to conclude that $\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$ In general, the equation

$$\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \iint_{D(t)} \text{curl } \mathbf{E} \cdot d\mathbf{S}$$

is not enough to conclude that $\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$ The integral symbols do not simply "cancel out," leaving equality of the integrands. To see why the integral symbol does not just cancel out in general, consider the two single-variable integrals $\int_0^1 x dx$ and $\int_0^1 f(x) dx,$ where

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1/2 \\ 0, & 1/2 \leq x \leq 1. \end{cases}$$

Both of these integrals equal $\frac{1}{2},$ so $\int_0^1 x dx = \int_0^1 f(x) dx.$ However, $x \neq f(x).$ Analogously, with our equation $\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \iint_{D(t)} \text{curl } \mathbf{E} \cdot d\mathbf{S},$ we cannot simply conclude that $\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ just because their integrals are equal. However, in our context, equation $\iint_{D(t)} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \iint_{D(t)} \text{curl } \mathbf{E} \cdot d\mathbf{S}$ is true for *any* region, however small (this is in contrast to the single-variable integrals just discussed). If $\mathbf{F}$ and $\mathbf{G}$ are three-dimensional vector fields such that $\iint_s \mathbf{F} \cdot d\mathbf{S} = \iint_s \mathbf{G} \cdot d\mathbf{S}$ for any surface S, then it is possible to show that $\mathbf{F} = \mathbf{G}$ by shrinking the area of S to zero by taking a limit (the smaller the area of S, the closer the value of $\iint_s \mathbf{F} \cdot d\mathbf{S}$ to the value of $\mathbf{F}$ at a point inside S). Therefore, we can let area $D(t)$ shrink to zero by taking a limit and obtain the differential form of Faraday's law:

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

In the context of electric fields, the curl of the electric field can be interpreted as the negative of the rate of change of the corresponding magnetic field with respect to time.

Example 6.76

Using Faraday's Law

Calculate the curl of electric field $\mathbf{E}$ if the corresponding magnetic field is constant field $\mathbf{B}(t) = \langle 1, -4, 2 \rangle$.

Solution

Since the magnetic field does not change with respect to time, $-\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}.$ By Faraday's law, the curl of the electric field is therefore also zero.

Analysis

A consequence of Faraday's law is that the curl of the electric field corresponding to a constant magnetic field is always zero.

6.64 Calculate the curl of electric field **E** if the corresponding magnetic field is $\mathbf{B}(t) = \langle tx, ty, -2tz \rangle$, $0 \le t < \infty$.

Notice that the curl of the electric field does not change over time, although the magnetic field does change over time.

6.7 EXERCISES

For the following exercises, without using Stokes' theorem, calculate directly both the flux of $\text{curl}\,\mathbf{F}\cdot\mathbf{N}$ over the given surface and the circulation integral around its boundary, assuming all boundaries are oriented clockwise as viewed from above.

326. $\mathbf{F}(x, y, z) = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k};$ S is the first-octant portion of plane $x + y + z = 1.$

327. $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k};$ S is hemisphere $z = \left(a^2 - x^2 - y^2\right)^{1/2}.$

328. $\mathbf{F}(x, y, z) = y^2\mathbf{i} + 2x\mathbf{j} + 5\mathbf{k};$ S is hemisphere $z = \left(4 - x^2 - y^2\right)^{1/2}.$

329. $\mathbf{F}(x, y, z) = z\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k};$ S is upper hemisphere $z = \sqrt{9 - x^2 - y^2}.$

330. $\mathbf{F}(x, y, z) = (x + 2z)\mathbf{i} + (y - x)\mathbf{j} + (z - y)\mathbf{k};$ S is a triangular region with vertices (3, 0, 0), (0, 3/2, 0), and (0, 0, 3).

331. $\mathbf{F}(x, y, z) = 2y\mathbf{i} - 6z\mathbf{j} + 3x\mathbf{k};$ S is a portion of paraboloid $z = 4 - x^2 - y^2$ and is above the xy-plane.

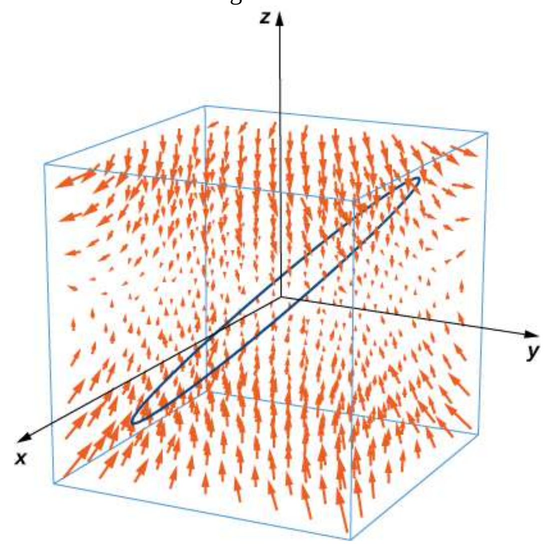
For the following exercises, use Stokes' theorem to evaluate $\iint_S(\text{curl}\,\mathbf{F}\cdot\mathbf{N})dS$ for the vector fields and surface.

332. $\mathbf{F}(x, y, z) = xy\mathbf{i} - z\mathbf{j}$ and S is the surface of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1,$ except for the face where $z = 0,$ and using the outward unit normal vector.

333. $\mathbf{F}(x, y, z) = xy\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k};$ and C is the intersection of paraboloid $z = x^2 + y^2$ and plane $z = y,$ and using the outward normal vector.

334. $\mathbf{F}(x, y, z) = 4y\mathbf{i} + z\mathbf{j} + 2y\mathbf{k}$ and C is the intersection of sphere $x^2 + y^2 + z^2 = 4$ with plane $z = 0,$ and using the outward normal vector

335. Use Stokes' theorem to evaluate $\int_C\left[2xy^2zdx + 2x^2yzdy + \left(x^2y^2 - 2z\right)dz\right],$ where C is the curve given by $x = \cos t,\ y = \sin t,\ z = \sin t,\ 0 \le t \le 2\pi,$ traversed in the direction of increasing t.



336. **[T]** Use a computer algebraic system (CAS) and Stokes' theorem to approximate line integral $\int_C(ydx + zdy + xdz),$ where C is the intersection of plane $x + y = 2$ and surface $x^2 + y^2 + z^2 = 2(x + y),$ traversed counterclockwise viewed from the origin.

337. **[T]** Use a CAS and Stokes' theorem to approximate line integral $\int_C(3ydx + 2zdy - 5xdz),$ where C is the intersection of the xy-plane and hemisphere $z = \sqrt{1 - x^2 - y^2},$ traversed counterclockwise viewed from the top—that is, from the positive z-axis toward the xy-plane.

338. **[T]** Use a CAS and Stokes' theorem to approximate line integral $\int_C[(1 + y)zdx + (1 + z)xdy + (1 + x)ydz],$ where C is a triangle with vertices $(1, 0, 0),$ $(0, 1, 0),$ and $(0, 0, 1)$ oriented counterclockwise.

339. Use Stokes' theorem to evaluate $\iint_S\text{curl}\,\mathbf{F}\cdot d\mathbf{S},$ where $\mathbf{F}(x, y, z) = e^{xy}\cos z\mathbf{i} + x^2z\mathbf{j} + xy\mathbf{k},$ and S is half of sphere $x = \sqrt{1 - y^2 - z^2},$ oriented out toward the positive x-axis.

340. **[T]** Use a CAS and Stokes' theorem to evaluate $\iint_S (\text{curl }\mathbf{F}\cdot\mathbf{N})dS,$ where $\mathbf{F}(x, y, z) = x^2 y\mathbf{i} + xy^2\mathbf{j} + z^3\mathbf{k}$ and C is the curve of the intersection of plane $3x + 2y + z = 6$ and cylinder $x^2 + y^2 = 4,$ oriented clockwise when viewed from above.

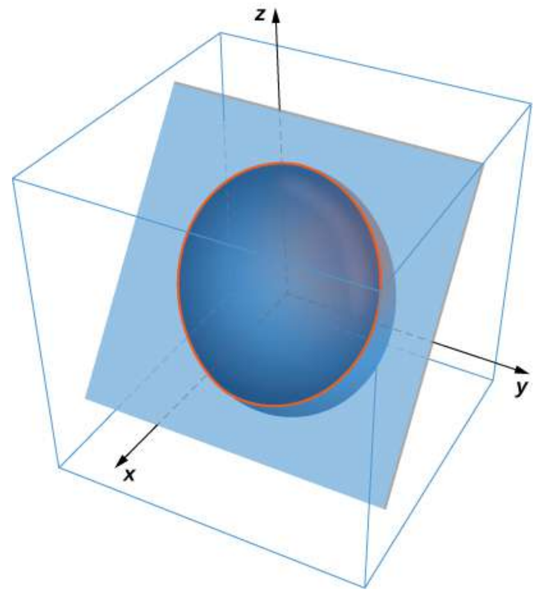
341. **[T]** Use a CAS and Stokes' theorem to evaluate $\iint\limits_S \text{curl }\mathbf{F}\cdot d\mathbf{S},$ where $\mathbf{F}(x, y, z) = \left(\sin(y+z) - yx^2 - \frac{y^3}{3}\right)\mathbf{i} + x\cos(y+z)\mathbf{j} + \cos(2y)\mathbf{k}$ and S consists of the top and the four sides but not the bottom of the cube with vertices $(\pm1, \pm1, \pm1),$ oriented outward.

342. **[T]** Use a CAS and Stokes' theorem to evaluate $\iint\limits_S \text{curl }\mathbf{F}\cdot d\mathbf{S},$ where $\mathbf{F}(x, y, z) = z^2\mathbf{i} - 3xy\mathbf{j} + x^3y^3\mathbf{k}$ and S is the top part of $z = 5 - x^2 - y^2$ above plane $z = 1,$ and S is oriented upward.

343. Use Stokes' theorem to evaluate $\iint_S (\text{curl }\mathbf{F}\cdot\mathbf{N})dS,$ where $\mathbf{F}(x, y, z) = z^2\mathbf{i} + y^2\mathbf{j} + x\mathbf{k}$ and S is a triangle with vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1) with counterclockwise orientation.

344. Use Stokes' theorem to evaluate line integral $\int\limits_C (z\,dx + x\,dy + y\,dz),$ where C is a triangle with vertices (3, 0, 0), (0, 0, 2), and (0, 6, 0) traversed in the given order.

345. Use Stokes' theorem to evaluate $\int\limits_C \left(\frac{1}{2}y^2\,dx + z\,dy + x\,dz\right),$ where C is the curve of intersection of plane $x + z = 1$ and ellipsoid $x^2 + 2y^2 + z^2 = 1,$ oriented clockwise from the origin.



346. Use Stokes' theorem to evaluate $\iint_S (\text{curl }\mathbf{F}\cdot\mathbf{N})dS,$ where $\mathbf{F}(x, y, z) = x\mathbf{i} + y^2\mathbf{j} + ze^{xy}\mathbf{k}$ and S is the part of surface $z = 1 - x^2 - 2y^2$ with $z \ge 0,$ oriented counterclockwise.

347. Use Stokes' theorem for vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + 3x\mathbf{j} + 2z\mathbf{k}$ where S is surface $z = 1 - x^2 - 2y^2, z \ge 0,$ C is boundary circle $x^2 + y^2 = 1,$ and S is oriented in the positive z-direction.

348. Use Stokes' theorem for vector field $\mathbf{F}(x, y, z) = -\frac{3}{2}y^2\mathbf{i} - 2xy\mathbf{j} + yz\mathbf{k},$ where S is that part of the surface of plane $x + y + z = 1$ contained within triangle C with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1), traversed counterclockwise as viewed from above.

349. A certain closed path C in plane $2x + 2y + z = 1$ is known to project onto unit circle $x^2 + y^2 = 1$ in the xy-plane. Let c be a constant and let $\mathbf{R}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$ Use Stokes' theorem to evaluate $\int_C (c\mathbf{k}\times\mathbf{R})\cdot d\mathbf{S}.$

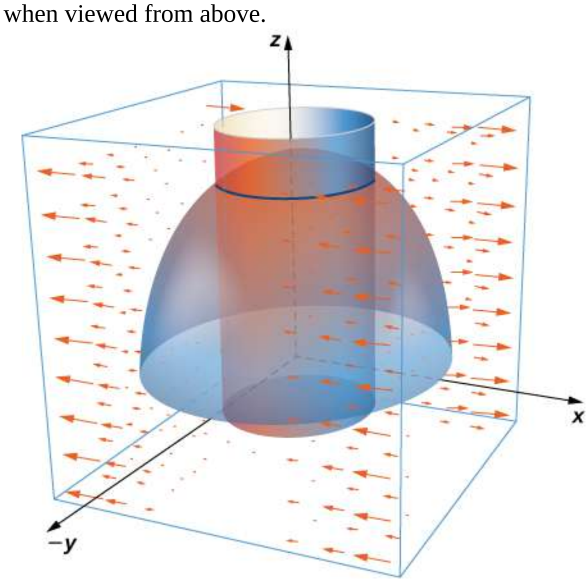
350. Use Stokes' theorem and let C be the boundary of surface $z = x^2 + y^2$ with $0 \le x \le 2$ and $0 \le y \le 1$, oriented with upward facing normal. Define $\mathbf{F}(x, y, z) = \left[\sin(x^3) + xz\right]\mathbf{i} + (x - yz)\mathbf{j} + \cos(z^4)\mathbf{k}$ and evaluate $\int_C \mathbf{F} \cdot d\mathbf{S}$.

351. Let S be hemisphere $x^2 + y^2 + z^2 = 4$ with $z \ge 0$, oriented upward. Let $\mathbf{F}(x, y, z) = x^2 e^{yz}\mathbf{i} + y^2 e^{xz}\mathbf{j} + z^2 e^{xy}\mathbf{k}$ be a vector field. Use Stokes' theorem to evaluate $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$.

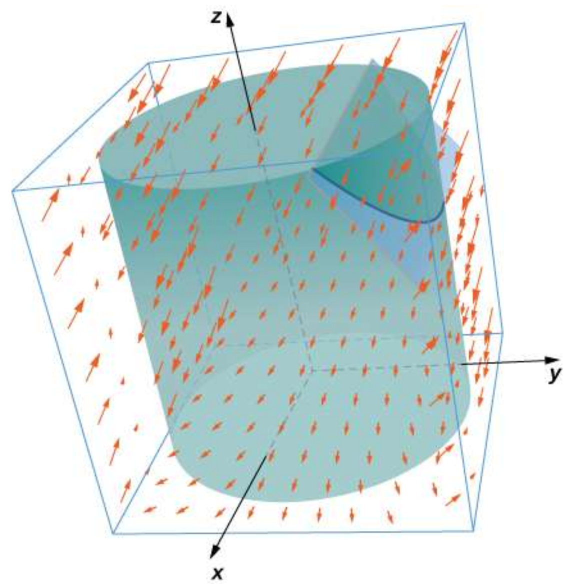
352. Let $\mathbf{F}(x, y, z) = xy\mathbf{i} + \left(e^{z^2} + y\right)\mathbf{j} + (x + y)\mathbf{k}$ and let S be the graph of function $y = \frac{x^2}{9} + \frac{z^2}{9} - 1$ with $z \le 0$ oriented so that the normal vector S has a positive y component. Use Stokes' theorem to compute integral $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$.

353. Use Stokes' theorem to evaluate $\oint \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and C is a triangle with vertices $(0, 0, 0)$, $(2, 0, 0)$ and $(0, -2, 2)$ oriented counterclockwise when viewed from above.

354. Use the surface integral in Stokes' theorem to calculate the circulation of field $\mathbf{F}$, $\mathbf{F}(x, y, z) = x^2 y^3 \mathbf{i} + \mathbf{j} + z\mathbf{k}$ around C, which is the intersection of cylinder $x^2 + y^2 = 4$ and hemisphere $x^2 + y^2 + z^2 = 16,\ z \ge 0$, oriented counterclockwise when viewed from above.



355. Use Stokes' theorem to compute $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \mathbf{i} + xy^2\mathbf{j} + xy^2\mathbf{k}$ and S is a part of plane $y + z = 2$ inside cylinder $x^2 + y^2 = 1$ and oriented counterclockwise.



356. Use Stokes' theorem to evaluate $\iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and S is the part of plane $x + y + z = 1$ in the positive octant and oriented counterclockwise $x \ge 0, y \ge 0, z \ge 0$.

357. Let $\mathbf{F}(x, y, z) = xy\mathbf{i} + 2z\mathbf{j} - 2y\mathbf{k}$ and let C be the intersection of plane $x + z = 5$ and cylinder $x^2 + y^2 = 9$, which is oriented counterclockwise when viewed from the top. Compute the line integral of $\mathbf{F}$ over C using Stokes' theorem.

358. **[T]** Use a CAS and let $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + (yz - x)\mathbf{j} + e^{yxz}\mathbf{k}$. Use Stokes' theorem to compute the surface integral of curl $\mathbf{F}$ over surface S with inward orientation consisting of cube $[0, 1] \times [0, 1] \times [0, 1]$ with the right side missing.

359. Let S be ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ oriented counterclockwise and let $\mathbf{F}$ be a vector field with component functions that have continuous partial derivatives.

360. Let S be the part of paraboloid $z = 9 - x^2 - y^2$ with $z \ge 0$. Verify Stokes' theorem for vector field $\mathbf{F}(x, y, z) = 3z\mathbf{i} + 4x\mathbf{j} + 2y\mathbf{k}$.

361. **[T]** Use a CAS and Stokes' theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F}(x, y, z) = (3z - \sin x)\mathbf{i} + \left(x^2 + e^y\right)\mathbf{j} + \left(y^3 - \cos z\right)\mathbf{k}$, where C is the curve given by $x = \cos t,\ y = \sin t,\ z = 1;\ 0 \le t \le 2\pi$.

362. **[T]** Use a CAS and Stokes' theorem to evaluate $\mathbf{F}(x, y, z) = 2y\mathbf{i} + e^z\mathbf{j} - \arctan x\mathbf{k}$ with S as a portion of paraboloid $z = 4 - x^2 - y^2$ cut off by the xy-plane oriented counterclockwise.

363. **[T]** Use a CAS to evaluate $\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$ and S is the surface parametrically by $\mathbf{r}(r, \theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + \left(4 - r^2\right)\mathbf{k}$ $(0 \le \theta \le 2\pi, 0 \le r \le 3)$.

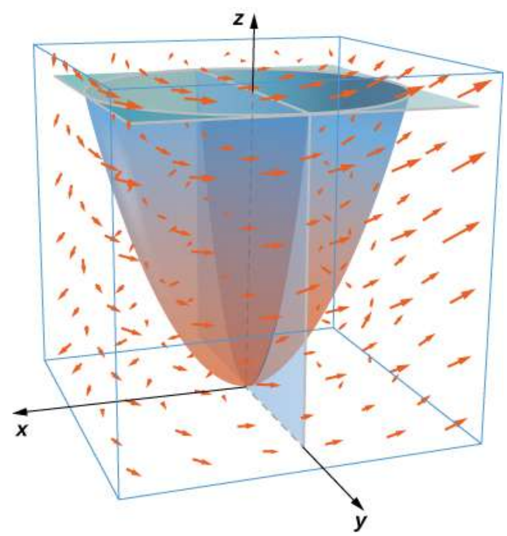
364. Let S be paraboloid $z = a\left(1 - x^2 - y^2\right)$, for $z \ge 0$, where $a > 0$ is a real number. Let $\mathbf{F} = \langle x - y, y + z, z - x \rangle$. For what value(s) of a (if any) does $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS$ have its maximum value?

For the following application exercises, the goal is to evaluate $A = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS$, where $\mathbf{F} = \langle xz, -xz, xy \rangle$ and S is the upper half of ellipsoid $x^2 + y^2 + 8z^2 = 1$, where $z \ge 0$.

365. Evaluate a surface integral over a more convenient surface to find the value of A.

366. Evaluate A using a line integral.

367. Take paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$, and slice it with plane $y = 0$. Let S be the surface that remains for $y \ge 0$, including the planar surface in the xz-plane. Let C be the semicircle and line segment that bounded the cap of S in plane $z = 4$ with counterclockwise orientation. Let $\mathbf{F} = \langle 2z + y, 2x + z, 2y + x \rangle$. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS$.



For the following exercises, let S be the disk enclosed by curve $C : \mathbf{r}(t) = \langle \cos\varphi\cos t, \sin t, \sin\varphi\cos t \rangle$, for $0 \le t \le 2\pi$, where $0 \le \varphi \le \frac{\pi}{2}$ is a fixed angle.

368. What is the length of C in terms of φ?

369. What is the circulation of C of vector field $\mathbf{F} = \langle -y, -z, x \rangle$ as a function of φ?

370. For what value of φ is the circulation a maximum?

371. Circle C in plane $x + y + z = 8$ has radius 4 and center (2, 3, 3). Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $F = \langle 0, -z, 2y \rangle$, where C has a counterclockwise orientation when viewed from above.

372. Velocity field $\mathbf{v} = \langle 0, 1 - x^2, 0 \rangle$, for $|x| \le 1$ and $|z| \le 1$, represents a horizontal flow in the y-direction. Compute the curl of $\mathbf{v}$ in a clockwise rotation.

373. Evaluate integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS$, where $\mathbf{F} = -xz\mathbf{i} + yz\mathbf{j} + xye^z\mathbf{k}$ and S is the cap of paraboloid $z = 5 - x^2 - y^2$ above plane $z = 3$, and $\mathbf{n}$ points in the positive z-direction on S.

For the following exercises, use Stokes' theorem to find the circulation of the following vector fields around any smooth, simple closed curve C.

374. $\mathbf{F} = \nabla(x \sin ye^z)$

375. $\mathbf{F} = \langle y^2 z^3, z2xyz^3, 3xy^2 z^2 \rangle$

6.8 | The Divergence Theorem

Learning Objectives

6.8.1 Explain the meaning of the divergence theorem.
6.8.2 Use the divergence theorem to calculate the flux of a vector field.
6.8.3 Apply the divergence theorem to an electrostatic field.

We have examined several versions of the Fundamental Theorem of Calculus in higher dimensions that relate the integral around an oriented boundary of a domain to a “derivative” of that entity on the oriented domain. In this section, we state the divergence theorem, which is the final theorem of this type that we will study. The divergence theorem has many uses in physics; in particular, the divergence theorem is used in the field of partial differential equations to derive equations modeling heat flow and conservation of mass. We use the theorem to calculate flux integrals and apply it to electrostatic fields.

Overview of Theorems

Before examining the divergence theorem, it is helpful to begin with an overview of the versions of the Fundamental Theorem of Calculus we have discussed:

1. **The Fundamental Theorem of Calculus**:

$$\int_a^b f'(x)dx = f(b) - f(a).$$

This theorem relates the integral of derivative f' over line segment $[a, b]$ along the x-axis to a difference of f evaluated on the boundary.

2. **The Fundamental Theorem for Line Integrals**:

$$\int_C \nabla f \cdot d\mathbf{r} = f(P_1) - f(P_0),$$

where P_0 is the initial point of C and P_1 is the terminal point of C. The Fundamental Theorem for Line Integrals allows path C to be a path in a plane or in space, not just a line segment on the x-axis. If we think of the gradient as a derivative, then this theorem relates an integral of derivative ∇f over path C to a difference of f evaluated on the boundary of C.

3. **Green’s theorem, circulation form**:

$$\iint_D (Q_x - P_y)dA = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Since $Q_x - P_y = \text{curl } \mathbf{F} \cdot \mathbf{k}$ and curl is a derivative of sorts, Green’s theorem relates the integral of derivative curl**F** over planar region D to an integral of **F** over the boundary of D.

4. **Green’s theorem, flux form**:

$$\iint_D (P_x + Q_y)dA = \int_C \mathbf{F} \cdot \mathbf{N}ds.$$

Since $P_x + Q_y = \text{div } \mathbf{F}$ and divergence is a derivative of sorts, the flux form of Green’s theorem relates the integral of derivative div**F** over planar region D to an integral of **F** over the boundary of D.

5. **Stokes’ theorem**:

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

If we think of the curl as a derivative of sorts, then Stokes’ theorem relates the integral of derivative curl**F** over surface S (not necessarily planar) to an integral of $\mathbf{F}$ over the boundary of S.

Stating the Divergence Theorem

The divergence theorem follows the general pattern of these other theorems. If we think of divergence as a derivative of sorts, then the **divergence theorem** relates a triple integral of derivative div$\mathbf{F}$ over a solid to a flux integral of $\mathbf{F}$ over the boundary of the solid. More specifically, the divergence theorem relates a flux integral of vector field $\mathbf{F}$ over a closed surface S to a triple integral of the divergence of $\mathbf{F}$ over the solid enclosed by S.

Theorem 6.20: The Divergence Theorem

Let S be a piecewise, smooth closed surface that encloses solid E in space. Assume that S is oriented outward, and let $\mathbf{F}$ be a vector field with continuous partial derivatives on an open region containing E (**Figure 6.87**). Then

$$\iiint_E \text{div}\,\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot d\mathbf{S}. \tag{6.24}$$

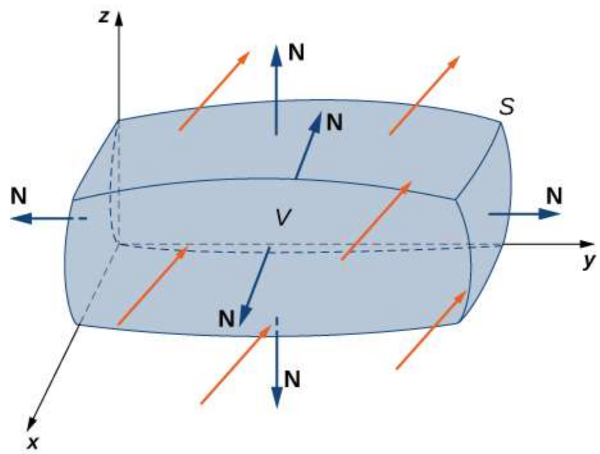


Figure 6.87 The divergence theorem relates a flux integral across a closed surface S to a triple integral over solid E enclosed by the surface.

Recall that the flux form of Green's theorem states that $\iint_D \text{div}\,\mathbf{F}\,dA = \int_C \mathbf{F}\cdot\mathbf{N}\,ds$. Therefore, the divergence theorem is a version of Green's theorem in one higher dimension.

The proof of the divergence theorem is beyond the scope of this text. However, we look at an informal proof that gives a general feel for why the theorem is true, but does not prove the theorem with full rigor. This explanation follows the informal explanation given for why Stokes' theorem is true.

Proof

Let B be a small box with sides parallel to the coordinate planes inside E (**Figure 6.88**). Let the center of B have coordinates (x, y, z) and suppose the edge lengths are $\Delta x, \Delta y,$ and Δz (**Figure 6.88**(b)). The normal vector out of the top of the box is $\mathbf{k}$ and the normal vector out of the bottom of the box is $-\mathbf{k}$. The dot product of $\mathbf{F} = \langle P, Q, R \rangle$ with $\mathbf{k}$ is R and the dot product with $-\mathbf{k}$ is $-R$. The area of the top of the box (and the bottom of the box) ΔS is $\Delta x \Delta y$.

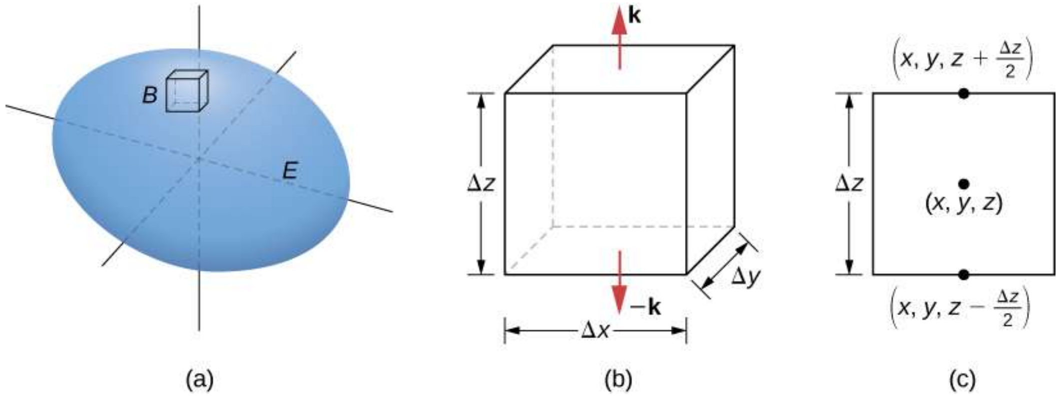


Figure 6.88 (a) A small box *B* inside surface *E* has sides parallel to the coordinate planes. (b) Box *B* has side lengths $\Delta x, \ \Delta y,$ and Δz (c) If we look at the side view of *B*, we see that, since $(x, \ y, \ z)$ is the center of the box, to get to the top of the box we must travel a vertical distance of $\Delta z/2$ up from $(x, \ y, \ z).$ Similarly, to get to the bottom of the box we must travel a distance $\Delta z/2$ down from $(x, \ y, \ z).$

The flux out of the top of the box can be approximated by $R\left(x, \ y, \ z + \frac{\Delta z}{2}\right)\Delta x\Delta y$ (**Figure 6.88**(c)) and the flux out of the bottom of the box is $-R\left(x, \ y, \ z - \frac{\Delta z}{2}\right)\Delta x\Delta y.$ If we denote the difference between these values as $\Delta R,$ then the net flux in the vertical direction can be approximated by $\Delta R\Delta x\Delta y.$ However,

$$\Delta R\Delta x\Delta y = \left(\frac{\Delta R}{\Delta z}\right)\Delta x\Delta y\Delta z \approx \left(\frac{\partial R}{\partial z}\right)\Delta V.$$

Therefore, the net flux in the vertical direction can be approximated by $\left(\frac{\partial R}{\partial z}\right)\Delta V.$ Similarly, the net flux in the *x*-direction can be approximated by $\left(\frac{\partial P}{\partial x}\right)\Delta V$ and the net flux in the *y*-direction can be approximated by $\left(\frac{\partial Q}{\partial y}\right)\Delta V.$ Adding the fluxes in all three directions gives an approximation of the total flux out of the box:

$$\text{Total flu} \quad \approx \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\Delta V = \text{div}\,\mathbf{F}\Delta V.$$

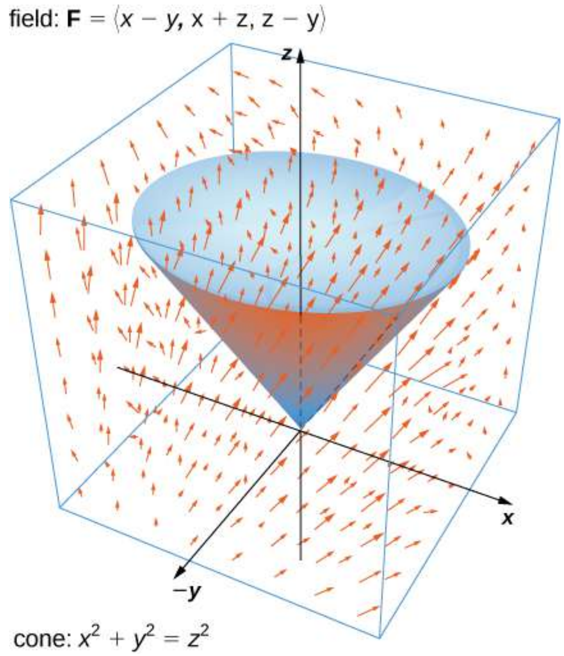
This approximation becomes arbitrarily close to the value of the total flux as the volume of the box shrinks to zero. The sum of $\text{div}\,\mathbf{F}\Delta V$ over all the small boxes approximating *E* is approximately $\iiint_E \text{div}\,\mathbf{F}dV.$ On the other hand, the sum of $\text{div}\,\mathbf{F}\Delta V$ over all the small boxes approximating *E* is the sum of the fluxes over all these boxes. Just as in the informal proof of Stokes' theorem, adding these fluxes over all the boxes results in the cancelation of a lot of the terms. If an approximating box shares a face with another approximating box, then the flux over one face is the negative of the flux over the shared face of the adjacent box. These two integrals cancel out. When adding up all the fluxes, the only flux integrals that survive are the integrals over the faces approximating the boundary of *E*. As the volumes of the approximating boxes shrink to zero, this approximation becomes arbitrarily close to the flux over *S*.

□

Example 6.77

Verifying the Divergence Theorem

Verify the divergence theorem for vector field $\mathbf{F} = \langle x - y, x + z, z - y \rangle$ and surface *S* that consists of cone $x^2 + y^2 = z^2, 0 \le z \le 1,$ and the circular top of the cone (see the following figure). Assume this surface is positively oriented.



Solution

Let E be the solid cone enclosed by S. To verify the theorem for this example, we show that $\iiint_E \text{div } \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$ by calculating each integral separately.

To compute the triple integral, note that $\text{div } \mathbf{F} = P_x + Q_y + R_z = 2,$ and therefore the triple integral is

$$\begin{aligned} \iiint_E \text{div } \mathbf{F} dV &= 2 \iiint_E dV \\ &= 2(\text{volume of } E). \end{aligned}$$

The volume of a right circular cone is given by $\pi r^2 \frac{h}{3}$. In this case, $h = r = 1$. Therefore,

$$\iiint_E \text{div } \mathbf{F} dV = 2(\text{volume of } E) = \frac{2\pi}{3}.$$

To compute the flux integral, first note that S is piecewise smooth; S can be written as a union of smooth surfaces. Therefore, we break the flux integral into two pieces: one flux integral across the circular top of the cone and one flux integral across the remaining portion of the cone. Call the circular top S_1 and the portion under the top S_2. We start by calculating the flux across the circular top of the cone. Notice that S_1 has parameterization

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 1 \rangle, 0 \le u \le 1, 0 \le v \le 2\pi.$$

Then, the tangent vectors are $\mathbf{t}_u = \langle \cos v, \sin v, 0 \rangle$ and $\mathbf{t}_v = \langle -u \cos v, u \sin v, 0 \rangle$. Therefore, the flux across S_1 is

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v) dA \\ &= \int_0^1 \int_0^{2\pi} \langle u \cos v - u \sin v,\ u \cos v + 1,\ 1 - u \sin v \rangle \cdot \langle 0, 0, u \rangle\, dvdu \\ &= \int_0^1 \int_0^{2\pi} u - u^2 \sin v\, dvdu = \pi.\end{aligned}$$

We now calculate the flux over S_2. A parameterization of this surface is

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \ 0 \le u \le 1, \ 0 \le v \le 2\pi.$$

The tangent vectors are $\mathbf{t}_u = \langle \cos v, \sin v, 1 \rangle$ and $\mathbf{t}_v = \langle -u \sin v, u \cos v, 0 \rangle$, so the cross product is

$$\mathbf{t}_u \times \mathbf{t}_v = \langle -u \cos v, -u \sin v, u \rangle.$$

Notice that the negative signs on the x and y components induce the negative (or inward) orientation of the cone. Since the surface is positively oriented, we use vector $\mathbf{t}_v \times \mathbf{t}_u = \langle u \cos v, u \sin v, -u \rangle$ in the flux integral. The flux across S_2 is then

$$\begin{aligned}\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{2\pi} \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{t}_v \times \mathbf{t}_u) dA \\ &= \int_0^1 \int_0^{2\pi} \langle u \cos v - u \sin v,\ u \cos v + u,\ u - \sin v \rangle \cdot \langle u \cos v, u \sin v, -u \rangle \\ &= \int_0^1 \int_0^{2\pi} u^2 \cos^2 v + 2u^2 \sin v - u^2\, dvdu = -\frac{\pi}{3}.\end{aligned}$$

The total flux across S is

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{2\pi}{3} = \iiint_E \operatorname{div} \mathbf{F} dV,$$

and we have verified the divergence theorem for this example.

6.65 Verify the divergence theorem for vector field $\mathbf{F}(x, y, z) = \langle x + y + z, y, 2x - y \rangle$ and surface S given by the cylinder $x^2 + y^2 = 1, \ 0 \le z \le 3$ plus the circular top and bottom of the cylinder. Assume that S is positively oriented.

Recall that the divergence of continuous field $\mathbf{F}$ at point P is a measure of the "outflowing-ness" of the field at P. If $\mathbf{F}$ represents the velocity field of a fluid, then the divergence can be thought of as the rate per unit volume of the fluid flowing out less the rate per unit volume flowing in. The divergence theorem confirms this interpretation. To see this, let P be a point and let B_r be a ball of small radius r centered at P (**Figure 6.89**). Let S_r be the boundary sphere of B_r. Since the radius is small and $\mathbf{F}$ is continuous, $\operatorname{div} \mathbf{F}(Q) \approx \operatorname{div} \mathbf{F}(P)$ for all other points Q in the ball. Therefore, the flux across S_r can be approximated using the divergence theorem:

$$\iint_{S_r} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_r} \operatorname{div} \mathbf{F} dV \approx \iiint_{B_r} \operatorname{div} \mathbf{F}(P) dV.$$

Since $\operatorname{div} \mathbf{F}(P)$ is a constant,

$$\iiint_{B_r} \operatorname{div} \mathbf{F}(P) dV = \operatorname{div} \mathbf{F}(P) V(B_r).$$

Therefore, flux $\iint_{S_r} \mathbf{F} \cdot d\mathbf{S}$ can be approximated by $\text{div}\,\mathbf{F}(P)V(B_r)$. This approximation gets better as the radius shrinks to zero, and therefore

$$\text{div}\,\mathbf{F}(P) = \lim_{r \to 0} \frac{1}{V(B_r)} \iint_{S_r} \mathbf{F} \cdot d\mathbf{S}.$$

This equation says that the divergence at *P* is the net rate of outward flux of the fluid per unit volume.

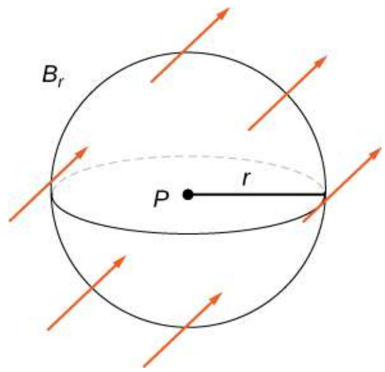


Figure 6.89 Ball B_r of small radius *r* centered at *P*.

Using the Divergence Theorem

The divergence theorem translates between the flux integral of closed surface *S* and a triple integral over the solid enclosed by *S*. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

Example 6.78

Applying the Divergence Theorem

Calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where *S* is cylinder $x^2 + y^2 = 1,\ 0 \le z \le 2$, including the circular top and bottom, and $\mathbf{F} = \left\langle \frac{x^3}{3} + yz,\ \frac{y^3}{3} - \sin(xz),\ z - x - y \right\rangle$.

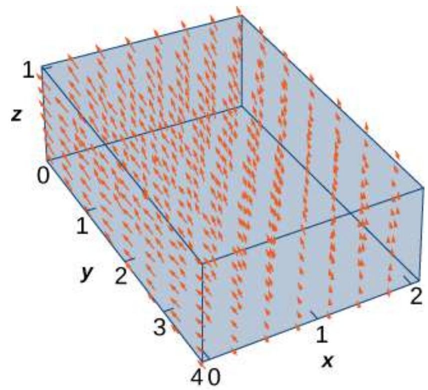
Solution

We could calculate this integral without the divergence theorem, but the calculation is not straightforward because we would have to break the flux integral into three separate integrals: one for the top of the cylinder, one for the bottom, and one for the side. Furthermore, each integral would require parameterizing the corresponding surface, calculating tangent vectors and their cross product, and using **Equation 6.19**.

By contrast, the divergence theorem allows us to calculate the single triple integral $\iiint_E \text{div}\,\mathbf{F}\,dV$, where *E* is the solid enclosed by the cylinder. Using the divergence theorem and converting to cylindrical coordinates, we have

$$\begin{aligned} \iint_s \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \text{div}\,\mathbf{F}\,dV \\ &= \iiint_E \left(x^2 + y^2 + 1\right) dV \\ &= \int_0^{2\pi} \int_0^1 \int_0^2 \left(r^2 + 1\right) r\,dz\,dr\,d\theta \\ &= \frac{3}{2} \int_0^{2\pi} d\theta = 3\pi. \end{aligned}$$

6.66 Use the divergence theorem to calculate flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the boundary of the box given by $0 \le x \le 2,\ 1 \le y \le 4,\ 0 \le z \le 1$, and $\mathbf{F} = \langle x^2 + yz,\ y - z,\ 2x + 2y + 2z \rangle$ (see the following figure).



Example 6.79

Applying the Divergence Theorem

Let $\mathbf{v} = \langle -\frac{y}{z}, \frac{x}{z}, 0 \rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4,\ 2 \le y \le 5,\ 1 \le z \le 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.

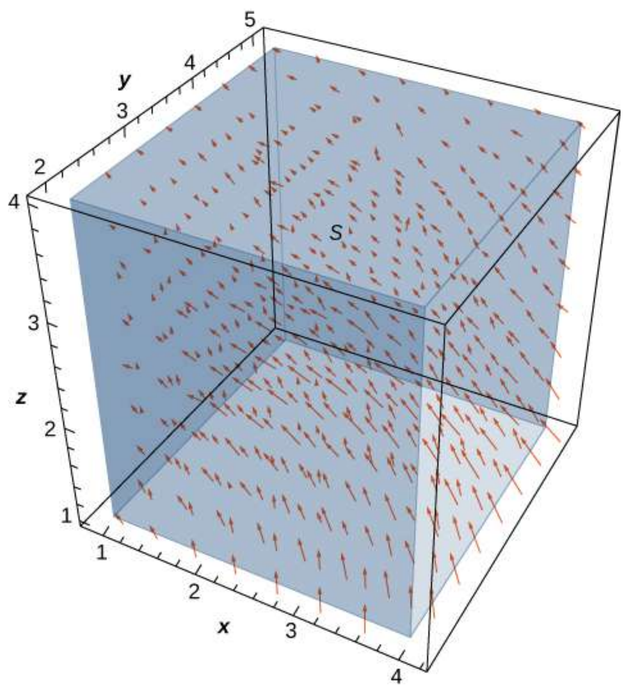


Figure 6.90 Vector field $\mathbf{v} = \langle -\frac{y}{z}, \frac{x}{z}, 0 \rangle$.

Solution

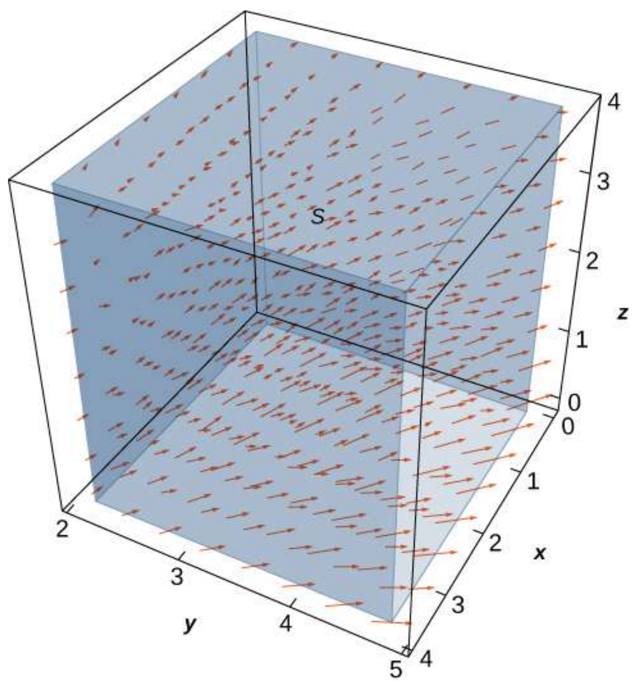
The flow rate of the fluid across S is $\iint_S \mathbf{v} \cdot d\mathbf{S}$. Before calculating this flux integral, let's discuss what the value of the integral should be. Based on **Figure 6.90**, we see that if we place this cube in the fluid (as long as the cube doesn't encompass the origin), then the rate of fluid entering the cube is the same as the rate of fluid exiting the cube. The field is rotational in nature and, for a given circle parallel to the *xy*-plane that has a center on the *z*-axis, the vectors along that circle are all the same magnitude. That is how we can see that the flow rate is the same entering and exiting the cube. The flow into the cube cancels with the flow out of the cube, and therefore the flow rate of the fluid across the cube should be zero.

To verify this intuition, we need to calculate the flux integral. Calculating the flux integral directly requires breaking the flux integral into six separate flux integrals, one for each face of the cube. We also need to find tangent vectors, compute their cross product, and use **Equation 6.19**. However, using the divergence theorem makes this calculation go much more quickly:

$$\begin{aligned}\iint_S \mathbf{v} \cdot d\mathbf{S} &= \iiint_C \operatorname{div}(\mathbf{v}) dV \\ &= \iiint_C 0 \, dV = 0.\end{aligned}$$

Therefore the flux is zero, as expected.

6.67 Let $\mathbf{v} = \left\langle \frac{x}{z}, \frac{y}{z}, 0 \right\rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \leq x \leq 4,\ 2 \leq y \leq 5,\ 1 \leq z \leq 4,$ and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.



Example 6.79 illustrates a remarkable consequence of the divergence theorem. Let S be a piecewise, smooth closed surface and let $\mathbf{F}$ be a vector field defined on an open region containing the surface enclosed by S. If $\mathbf{F}$ has the form $\mathbf{F} = \langle f(y, z), g(x, z), h(x, y) \rangle$, then the divergence of $\mathbf{F}$ is zero. By the divergence theorem, the flux of $\mathbf{F}$ across S is also zero. This makes certain flux integrals incredibly easy to calculate. For example, suppose we wanted to calculate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is a cube and

$$\mathbf{F} = \langle \sin(y)e^{yz}, x^2z^2, \cos(xy)e^{\sin x} \rangle .$$

Calculating the flux integral directly would be difficult, if not impossible, using techniques we studied previously. At the very least, we would have to break the flux integral into six integrals, one for each face of the cube. But, because the divergence of this field is zero, the divergence theorem immediately shows that the flux integral is zero.

We can now use the divergence theorem to justify the physical interpretation of divergence that we discussed earlier. Recall that if **F** is a continuous three-dimensional vector field and P is a point in the domain of **F**, then the divergence of **F** at P is a measure of the “outflowing-ness” of **F** at P. If **F** represents the velocity field of a fluid, then the divergence of **F** at P is a measure of the net flow rate out of point P (the flow of fluid out of P less the flow of fluid in to P). To see how the divergence theorem justifies this interpretation, let B_r be a ball of very small radius r with center P, and assume that B_r is in the domain of **F**. Furthermore, assume that B_r has a positive, outward orientation. Since the radius of B_r is small and **F** is continuous, the divergence of **F** is approximately constant on B_r. That is, if P' is any point in B_r, then $\operatorname{div} \mathbf{F}(P) \approx \operatorname{div} \mathbf{F}(P')$. Let S_r denote the boundary sphere of B_r. We can approximate the flux across S_r using the divergence theorem as follows:

$$\begin{aligned} \iint_{S_r} \mathbf{F} \cdot d\mathbf{S} &= \iiint_{B_r} \operatorname{div} \mathbf{F}\, dV \\ &\approx \iiint_{B_r} \operatorname{div} \mathbf{F}(P) dV \\ &= \operatorname{div} \mathbf{F}(P) V(B_r). \end{aligned}$$

As we shrink the radius r to zero via a limit, the quantity $\operatorname{div} \mathbf{F}(P)V(B_r)$ gets arbitrarily close to the flux. Therefore,

$$\operatorname{div} \mathbf{F}(P) = \lim_{r \to 0} \frac{1}{V(B_r)} \iint_{S_r} \mathbf{F} \cdot d\mathbf{S}$$

and we can consider the divergence at P as measuring the net rate of outward flux per unit volume at P. Since “outflowing-ness” is an informal term for the net rate of outward flux per unit volume, we have justified the physical interpretation of divergence we discussed earlier, and we have used the divergence theorem to give this justification.

Application to Electrostatic Fields

The divergence theorem has many applications in physics and engineering. It allows us to write many physical laws in both an integral form and a differential form (in much the same way that Stokes’ theorem allowed us to translate between an integral and differential form of Faraday’s law). Areas of study such as fluid dynamics, electromagnetism, and quantum mechanics have equations that describe the conservation of mass, momentum, or energy, and the divergence theorem allows us to give these equations in both integral and differential forms.

One of the most common applications of the divergence theorem is to electrostatic fields. An important result in this subject is **Gauss’ law**. This law states that if S is a closed surface in electrostatic field **E**, then the flux of **E** across S is the total charge enclosed by S (divided by an electric constant). We now use the divergence theorem to justify the special case of this law in which the electrostatic field is generated by a stationary point charge at the origin.

If (x, y, z) is a point in space, then the distance from the point to the origin is $r = \sqrt{x^2 + y^2 + z^2}$. Let $\mathbf{F}_r$ denote radial vector field $\mathbf{F}_r = \frac{1}{r^2} \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$. The vector at a given position in space points in the direction of unit radial vector $\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$ and is scaled by the quantity $1/r^2$. Therefore, the magnitude of a vector at a given point is inversely proportional to the square of the vector’s distance from the origin. Suppose we have a stationary charge of q Coulombs at the origin, existing in a vacuum. The charge generates electrostatic field **E** given by

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\mathbf{F}_r,$$

where the approximation $\varepsilon_0 = 8.854 \times 10^{-12}$ farad (F)/m is an electric constant. (The constant ε_0 is a measure of the resistance encountered when forming an electric field in a vacuum.) Notice that **E** is a radial vector field similar to the gravitational field described in **Example 6.6**. The difference is that this field points outward whereas the gravitational field points inward. Because

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\mathbf{F}_r = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r^2} \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle\right),$$

we say that electrostatic fields obey an **inverse-square law**. That is, the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge (which in this case is at the origin). Given this vector field, we show that the flux across closed surface S is zero if the charge is outside of S, and that the flux is q/ε_0 if the charge is inside of S. In other words, the flux across S is the charge inside the surface divided by constant ε_0. This is a special case of Gauss' law, and here we use the divergence theorem to justify this special case.

To show that the flux across S is the charge inside the surface divided by constant ε_0, we need two intermediate steps. First we show that the divergence of $\mathbf{F}_r$ is zero and then we show that the flux of $\mathbf{F}_r$ across any smooth surface S is either zero or 4π. We can then justify this special case of Gauss' law.

Example 6.80

The Divergence of $\mathbf{F}_r$ Is Zero

Verify that the divergence of $\mathbf{F}_r$ is zero where $\mathbf{F}_r$ is defined (away from the origin).

Solution

Since $r = \sqrt{x^2 + y^2 + z^2}$, the quotient rule gives us

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) &= \frac{\partial}{\partial x}\left(\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right) \\ &= \frac{\left(x^2+y^2+z^2\right)^{3/2} - x\left[\frac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}2x\right]}{\left(x^2+y^2+z^2\right)^3} \\ &= \frac{r^3 - 3x^2 r}{r^6} = \frac{r^2 - 3x^2}{r^5}.\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{r^2 - 3y^2}{r^5} \text{ and } \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{r^2 - 3z^2}{r^5}.$$

Therefore,

$$\begin{aligned}\operatorname{div}\mathbf{F}_r &= \frac{r^2-3x^2}{r^5} + \frac{r^2-3y^2}{r^5} + \frac{r^2-3z^2}{r^5} \\ &= \frac{3r^2 - 3\left(x^2+y^2+z^2\right)}{r^5} \\ &= \frac{3r^2-3r^2}{r^5} = 0.\end{aligned}$$

Notice that since the divergence of $\mathbf{F}_r$ is zero and $\mathbf{E}$ is $\mathbf{F}_r$ scaled by a constant, the divergence of electrostatic field $\mathbf{E}$ is also zero (except at the origin).

Theorem 6.21: Flux across a Smooth Surface

Let S be a connected, piecewise smooth closed surface and let $\mathbf{F}_r = \frac{1}{r^2}\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$. Then,

$$\iint_S \mathbf{F}_r \cdot d\mathbf{S} = \begin{cases} 0 & \text{if } S \text{ does not encompass the origin} \\ 4\pi & \text{if } S \text{ encompasses the origin.} \end{cases}$$

In other words, this theorem says that the flux of $\mathbf{F}_r$ across any piecewise smooth closed surface *S* depends only on whether the origin is inside of *S*.

Proof

The logic of this proof follows the logic of **Example 6.46**, only we use the divergence theorem rather than Green's theorem.

First, suppose that *S* does not encompass the origin. In this case, the solid enclosed by *S* is in the domain of $\mathbf{F}_r,$ and since the divergence of $\mathbf{F}_r$ is zero, we can immediately apply the divergence theorem and find that $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is zero.

Now suppose that *S* does encompass the origin. We cannot just use the divergence theorem to calculate the flux, because the field is not defined at the origin. Let S_a be a sphere of radius *a* inside of *S* centered at the origin. The outward normal vector field on the sphere, in spherical coordinates, is

$$\mathbf{t}_\phi \times \mathbf{t}_\theta = \langle a^2 \cos\theta \sin^2\phi, a^2 \sin\theta \sin^2\phi, a^2 \sin\phi \cos\phi \rangle$$

(see **Example 6.64**). Therefore, on the surface of the sphere, the dot product $\mathbf{F}_r \cdot \mathbf{N}$ (in spherical coordinates) is

$$\begin{aligned} \mathbf{F}_r \cdot \mathbf{N} &= \left\langle \frac{\sin\phi\cos\theta}{a^2}, \frac{\sin\phi\sin\theta}{a^2}, \frac{\cos\phi}{a^2} \right\rangle \cdot \langle a^2 \cos\theta \sin^2\phi, a^2 \sin\theta \sin^2\phi, a^2 \sin\phi\cos\phi \rangle \\ &= \sin\phi(\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle \cdot \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle) \\ &= \sin\phi. \end{aligned}$$

The flux of $\mathbf{F}_r$ across S_a is

$$\iint_{S_a} \mathbf{F}_r \cdot \mathbf{N} dS = \int_0^{2\pi} \int_0^{\pi} \sin\phi d\phi d\theta = 4\pi.$$

Now, remember that we are interested in the flux across *S*, not necessarily the flux across $S_a.$ To calculate the flux across *S*, let *E* be the solid between surfaces S_a and *S*. Then, the boundary of *E* consists of S_a and *S*. Denote this boundary by $S - S_a$ to indicate that *S* is oriented outward but now S_a is oriented inward. We would like to apply the divergence theorem to solid *E*. Notice that the divergence theorem, as stated, can't handle a solid such as *E* because *E* has a hole. However, the divergence theorem can be extended to handle solids with holes, just as Green's theorem can be extended to handle regions with holes. This allows us to use the divergence theorem in the following way. By the divergence theorem,

$$\begin{aligned} \iint_{S - S_a} \mathbf{F}_r \cdot d\mathbf{S} &= \iint_S \mathbf{F}_r \cdot d\mathbf{S} - \iint_{S_a} \mathbf{F}_r \cdot d\mathbf{S} \\ &= \iiint_E \operatorname{div} \mathbf{F}_r \, dV \\ &= \iiint_E 0 dV = 0. \end{aligned}$$

Therefore,

$$\iint_S \mathbf{F}_r \cdot d\mathbf{S} = \iint_{S_a} \mathbf{F}_r \cdot d\mathbf{S} = 4\pi,$$

and we have our desired result.

□

Now we return to calculating the flux across a smooth surface in the context of electrostatic field $\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\mathbf{F}_r$ of a point charge at the origin. Let *S* be a piecewise smooth closed surface that encompasses the origin. Then

$$\begin{aligned}\iint_S \mathbf{E} \cdot d\mathbf{S} &= \iint_S \frac{q}{4\pi\varepsilon_0} \mathbf{F}_r \cdot d\mathbf{S} \\ &= \frac{q}{4\pi\varepsilon_0} \iint_S \mathbf{F}_r \cdot d\mathbf{S} \\ &= \frac{q}{\varepsilon_0}.\end{aligned}$$

If S does not encompass the origin, then

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{4\pi\varepsilon_0} \iint_S \mathbf{F}_r \cdot d\mathbf{S} = 0.$$

Therefore, we have justified the claim that we set out to justify: the flux across closed surface S is zero if the charge is outside of S, and the flux is q/ε_0 if the charge is inside of S.

This analysis works only if there is a single point charge at the origin. In this case, Gauss' law says that the flux of **E** across S is the total charge enclosed by S. Gauss' law can be extended to handle multiple charged solids in space, not just a single point charge at the origin. The logic is similar to the previous analysis, but beyond the scope of this text. In full generality, Gauss' law states that if S is a piecewise smooth closed surface and Q is the total amount of charge inside of S, then the flux of **E** across S is Q/ε_0.

Example 6.81

Using Gauss' law

Suppose we have four stationary point charges in space, all with a charge of 0.002 Coulombs (C). The charges are located at $(0, 1, 1)$, $(1, 1, 4)$, $(-1, 0, 0)$, and $(-2, -2, 2)$. Let **E** denote the electrostatic field generated by these point charges. If S is the sphere of radius 2 oriented outward and centered at the origin, then find $\iint_S \mathbf{E} \cdot d\mathbf{S}$.

Solution

According to Gauss' law, the flux of **E** across S is the total charge inside of S divided by the electric constant. Since S has radius 2, notice that only two of the charges are inside of S: the charge at $(0, 1, 1)$ and the charge at $(-1, 0, 0)$. Therefore, the total charge encompassed by S is 0.004 and, by Gauss' law,

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \frac{0.004}{8.854 \times 10^{-12}} \approx 4.518 \times 10^9 \text{ V-m}.$$

6.68 Work the previous example for surface S that is a sphere of radius 4 centered at the origin, oriented outward.

6.8 EXERCISES

For the following exercises, use a computer algebraic system (CAS) and the divergence theorem to evaluate surface integral $\int_S \mathbf{F} \cdot \mathbf{n} ds$ for the given choice of **F** and the boundary surface *S*. For each closed surface, assume **N** is the outward unit normal vector.

376. **[T]** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the surface of cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 < z \le 1$.

377. **[T]** $\mathbf{F}(x, y, z) = (\cos yz)\mathbf{i} + e^{xz}\mathbf{j} + 3z^2\mathbf{k}$; S is the surface of hemisphere $z = \sqrt{4 - x^2 - y^2}$ together with disk $x^2 + y^2 \le 4$ in the *xy*-plane.

378. **[T]** $\mathbf{F}(x, y, z) = \left(x^2 + y^2 - x^2\right)\mathbf{i} + x^2 y\mathbf{j} + 3z\mathbf{k}$; S is the surface of the five faces of unit cube $0 \le x \le 1,\ 0 \le y \le 1,\ 0 < z \le 1$.

379. **[T]** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the surface of paraboloid $z = x^2 + y^2$ for $0 \le z \le 9$.

380. **[T]** $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$; S is the surface of sphere $x^2 + y^2 + z^2 = 4$.

381. **[T]** $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \left(z^2 - 1\right)\mathbf{k}$; S is the surface of the solid bounded by cylinder $x^2 + y^2 = 4$ and planes $z = 0$ and $z = 1$.

382. **[T]** $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + yz^2\mathbf{j} + x^2 z\mathbf{k}$; S is the surface bounded above by sphere $\rho = 2$ and below by cone $\varphi = \frac{\pi}{4}$ in spherical coordinates. (Think of S as the surface of an "ice cream cone.")

383. **[T]** $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + 3a^2 z\mathbf{k}$ (constant $a > 0$); S is the surface bounded by cylinder $x^2 + y^2 = a^2$ and planes $z = 0$ and $z = 1$.

384. **[T]** Surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the solid bounded by paraboloid $z = x^2 + y^2$ and plane $z = 4$, and $\mathbf{F}(x, y, z) = \left(x + y^2 z^2\right)\mathbf{i} + \left(y + z^2 x^2\right)\mathbf{j} + \left(z + x^2 y^2\right)\mathbf{k}$

385. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \left(e^{y^2}\right)\mathbf{i} + \left(y + \sin\left(z^2\right)\right)\mathbf{j} + (z - 1)\mathbf{k}$ and S is upper hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$, oriented upward.

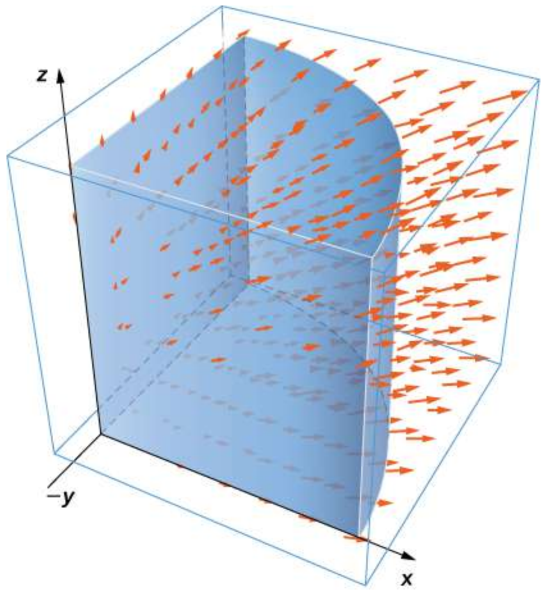
386. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^4\mathbf{i} - x^3 z^2\mathbf{j} + 4xy^2 z\mathbf{k}$ and S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

387. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ when $\mathbf{F}(x, y, z) = x^2 z^3\mathbf{i} + 2xyz^3\mathbf{j} + xz^4\mathbf{k}$ and S is the surface of the box with vertices $(\pm 1, \pm 2, \pm 3)$.

388. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ when $\mathbf{F}(x, y, z) = z\tan^{-1}\left(y^2\right)\mathbf{i} + z^3\ln\left(x^2 + 1\right)\mathbf{j} + z\mathbf{k}$ and S is a part of paraboloid $x^2 + y^2 + z = 2$ that lies above plane $z = 1$ and is oriented upward.

389. **[T]** Use a CAS and the divergence theorem to calculate flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \left(x^3 + y^3\right)\mathbf{i} + \left(y^3 + z^3\right)\mathbf{j} + \left(z^3 + x^3\right)\mathbf{k}$ and S is a sphere with center (0, 0) and radius 2.

390. Use the divergence theorem to compute the value of flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = (y^3 + 3x)\mathbf{i} + (xz + y)\mathbf{j} + [z + x^4 \cos(x^2 y)]\mathbf{k}$ and S is the area of the region bounded by $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$, and $0 \leq z \leq 1$.



391. Use the divergence theorem to compute flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\mathbf{j} - z\mathbf{k}$ and S consists of the union of paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$, and disk $x^2 + z^2 \leq 1$, $y = 1$, oriented outward. What is the flux through just the paraboloid?

392. Use the divergence theorem to compute flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x + y\mathbf{j} + z^4\mathbf{k}$ and S is a part of cone $z = \sqrt{x^2 + y^2}$ beneath top plane $z = 1$, oriented downward.

393. Use the divergence theorem to calculate surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F}(x, y, z) = x^4\mathbf{i} - x^3 z^2\mathbf{j} + 4xy^2 z\mathbf{k}$, where S is the surface bounded by cylinder $x^2 + y^2 = 1$ and planes $z = x + 2$ and $z = 0$.

394. Consider $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + (z + 1)\mathbf{k}$. Let E be the solid enclosed by paraboloid $z = 4 - x^2 - y^2$ and plane $z = 0$ with normal vectors pointing outside E. Compute flux $\boldsymbol{F}$ across the boundary of E using the divergence theorem.

For the following exercises, use a CAS along with the divergence theorem to compute the net outward flux for the fields across the given surfaces S.

395. **[T]** $\mathbf{F} = \langle x, -2y, 3z \rangle$; S is sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 6\}$.

396. **[T]** $\mathbf{F} = \langle x, 2y, z \rangle$; S is the boundary of the tetrahedron in the first octant formed by plane $x + y + z = 1$.

397. **[T]** $\mathbf{F} = \langle y - 2x, x^3 - y, y^2 - z \rangle$; S is sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 4\}$.

398. **[T]** $\mathbf{F} = \langle x, y, z \rangle$; S is the surface of paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, plus its base in the xy-plane.

For the following exercises, use a CAS and the divergence theorem to compute the net outward flux for the vector fields across the boundary of the given regions D.

399. **[T]** $\mathbf{F} = \langle z - x, x - y, 2y - z \rangle$; D is the region between spheres of radius 2 and 4 centered at the origin.

400. **[T]** $\mathbf{F} = \dfrac{\mathbf{r}}{|\mathbf{r}|} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$; D is the region between spheres of radius 1 and 2 centered at the origin.

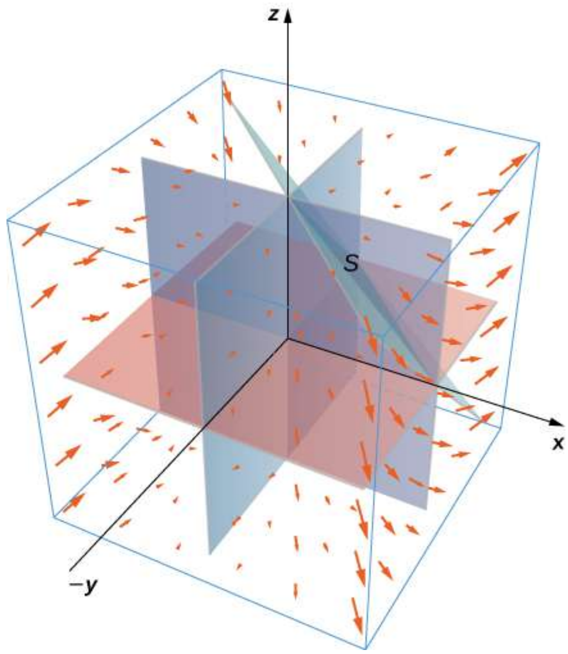
401. **[T]** $\mathbf{F} = \langle x^2, -y^2, z^2 \rangle$; D is the region in the first octant between planes $z = 4 - x - y$ and $z = 2 - x - y$.

402. Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 3xy\mathbf{j} + xz^2\mathbf{k}$. Use the divergence theorem to calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the surface of the cube with corners at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$, oriented outward.

403. Use the divergence theorem to find the outward flux of field $\mathbf{F}(x, y, z) = (x^3 - 3y)\mathbf{i} + (2yz + 1)\mathbf{j} + xyz\mathbf{k}$ through the cube bounded by planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$.

404. Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 3y\mathbf{j} + 5z\mathbf{k}$ and let S be hemisphere $z = \sqrt{9 - x^2 - y^2}$ together with disk $x^2 + y^2 \leq 9$ in the xy-plane. Use the divergence theorem.

405. Evaluate $\iint_S \mathbf{F} \cdot \mathbf{N} dS,$ where $\mathbf{F}(x,\ y,\ z) = x^2\mathbf{i} + xy\mathbf{j} + x^3y^3\mathbf{k}$ and S is the surface consisting of all faces except the tetrahedron bounded by plane $x + y + z = 1$ and the coordinate planes, with outward unit normal vector $\mathbf{N}$.



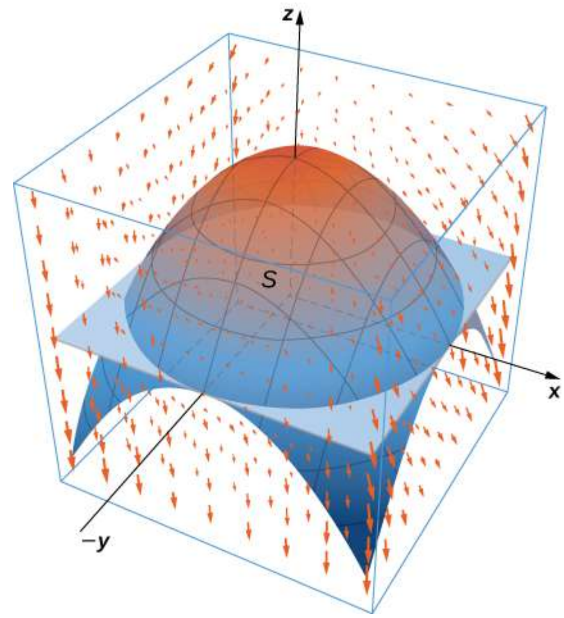
406. Find the net outward flux of field $\mathbf{F} = \langle\ bz - cy,\ cx - az,\ ay - bx\ \rangle$ across any smooth closed surface in $\mathbf{R}^3,$ where a, b, and c are constants.

407. Use the divergence theorem to evaluate $\iint_S \| \mathbf{R} \| \mathbf{R} \cdot n ds,$ where $\mathbf{R}(x,\ y,\ z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is sphere $x^2 + y^2 + z^2 = a^2,$ with constant $a > 0.$

408. Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S},$ where $\mathbf{F}(x,\ y,\ z) = y^2 z\mathbf{i} + y^3\mathbf{j} + xz\mathbf{k}$ and S is the boundary of the cube defined by $-1 \le x \le 1,\ -1 \le y \le 1,$ and $0 \le z \le 2.$

409. Let R be the region defined by $x^2 + y^2 + z^2 \le 1.$ Use the divergence theorem to find $\iiint_R z^2\, dV.$

410. Let E be the solid bounded by the xy-plane and paraboloid $z = 4 - x^2 - y^2$ so that S is the surface of the paraboloid piece together with the disk in the xy-plane that forms its bottom. If $\mathbf{F}(x,\ y,\ z) = \left(xz \sin(yz) + x^3\right)\mathbf{i} + \cos(yz)\mathbf{j} + \left(3zy^2 - e^{x^2 + y^2}\right)\mathbf{k},$ find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem.



411. Let E be the solid unit cube with diagonally opposite corners at the origin and (1, 1, 1), and faces parallel to the coordinate planes. Let S be the surface of E, oriented with the outward-pointing normal. Use a CAS to find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem if $\mathbf{F}(x,\ y,\ z) = 2xy\mathbf{i} + 3ye^z\mathbf{j} + x \sin z\mathbf{k}.$

412. Use the divergence theorem to calculate the flux of $\mathbf{F}(x,\ y,\ z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through sphere $x^2 + y^2 + z^2 = 1.$

413. Find $\iint_S \mathbf{F} \cdot d\mathbf{S},$ where $\mathbf{F}(x,\ y,\ z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the outwardly oriented surface obtained by removing cube $[1,\ 2] \times [1,\ 2] \times [1,\ 2]$ from cube $[0,\ 2] \times [0,\ 2] \times [0,\ 2].$

414. Consider radial vector field $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle\ x,\ y,\ z\ \rangle}{\left(x^2 + y^2 + z^2\right)^{1/2}}.$ Compute the surface integral, where S is the surface of a sphere of radius a centered at the origin.

415. Compute the flux of water through parabolic cylinder $S : y = x^2,$ from $0 \leq x \leq 2,\ 0 \leq z \leq 3,$ if the velocity vector is $\mathbf{F}(x, y, z) = 3z^2\,\mathbf{i} + 6\mathbf{j} + 6xz\mathbf{k}.$

416. **[T]** Use a CAS to find the flux of vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + z\mathbf{j} + \sqrt{x^2 + y^2}\mathbf{k}$ across the portion of hyperboloid $x^2 + y^2 = z^2 + 1$ between planes $z = 0$ and $z = \frac{\sqrt{3}}{3},$ oriented so the unit normal vector points away from the *z*-axis.

417. **[T]** Use a CAS to find the flux of vector field $\mathbf{F}(x, y, z) = (e^y + x)\mathbf{i} + (3\cos(xz) - y)\mathbf{j} + z\mathbf{k}$ through surface *S*, where *S* is given by $z^2 = 4x^2 + 4y^2$ from $0 \leq z \leq 4,$ oriented so the unit normal vector points downward.

418. **[T]** Use a CAS to compute $\iint_S \mathbf{F} \cdot d\mathbf{S},$ where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$ and *S* is a part of sphere $x^2 + y^2 + z^2 = 2$ with $0 \leq z \leq 1.$

419. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S},$ where $\mathbf{F}(x, y, z) = bxy^2\,\mathbf{i} + bx^2y\mathbf{j} + \left(x^2 + y^2\right)z^2\,\mathbf{k}$ and *S* is a closed surface bounding the region and consisting of solid cylinder $x^2 + y^2 \leq a^2$ and $0 \leq z \leq b.$

420. **[T]** Use a CAS to calculate the flux of $\mathbf{F}(x, y, z) = \left(x^3 + y\sin z\right)\mathbf{i} + \left(y^3 + z\sin x\right)\mathbf{j} + 3z\mathbf{k}$ across surface *S*, where *S* is the boundary of the solid bounded by hemispheres $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2},$ and plane $z = 0.$

421. Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S},$ where $\mathbf{F}(x, y, z) = xy\mathbf{i} - \frac{1}{2}y^2\,\mathbf{j} + z\mathbf{k}$ and *S* is the surface consisting of three pieces: $z = 4 - 3x^2 - 3y^2,\ 1 \leq z \leq 4$ on the top; $x^2 + y^2 = 1,\ 0 \leq z \leq 1$ on the sides; and $z = 0$ on the bottom.

422. **[T]** Use a CAS and the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S},$ where $\mathbf{F}(x, y, z) = (2x + y\cos z)\mathbf{i} + \left(x^2 - y\right)\mathbf{j} + y^2z\mathbf{k}$ and *S* is sphere $x^2 + y^2 + z^2 = 4$ orientated outward.

423. Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S},$ where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and *S* is the boundary of the solid enclosed by paraboloid $y = x^2 + z^2 - 2,$ cylinder $x^2 + z^2 = 1,$ and plane $x + y = 2,$ and *S* is oriented outward.

For the following exercises, Fourier's law of heat transfer states that the heat flow vector **F** at a point is proportional to the negative gradient of the temperature; that is, $\mathbf{F} = -k\nabla T,$ which means that heat energy flows hot regions to cold regions. The constant $k > 0$ is called the *conductivity*, which has metric units of joules per meter per second-kelvin or watts per meter-kelvin. A temperature function for region *D* is given. Use the divergence theorem to find net outward heat flux $\iint_S \mathbf{F} \cdot \mathbf{N}dS = -k\iint_S \nabla T \cdot \mathbf{N}dS$ across the boundary *S* of *D*, where $k = 1.$

424. $T(x, y, z) = 100 + x + 2y + z;$ $D = \{(x, y, z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$

425. $T(x, y, z) = 100 + e^{-z};$ $D = \{(x, y, z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$

426. $T(x, y, z) = 100e^{-x^2 - y^2 - z^2};$ *D* is the sphere of radius *a* centered at the origin.

CHAPTER 6 REVIEW

KEY TERMS

circulation the tendency of a fluid to move in the direction of curve C. If C is a closed curve, then the circulation of **F** along C is line integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$, which we also denote $\oint_C \mathbf{F} \cdot \mathbf{T} ds$

closed curve a curve for which there exists a parameterization $\mathbf{r}(t)$, $a \le t \le b$, such that $\mathbf{r}(a) = \mathbf{r}(b)$, and the curve is traversed exactly once

closed curve a curve that begins and ends at the same point

connected region a region in which any two points can be connected by a path with a trace contained entirely inside the region

conservative field a vector field for which there exists a scalar function f such that $\nabla f = \mathbf{F}$

curl the curl of vector field $\mathbf{F} = \langle P, Q, R \rangle$, denoted $\nabla \times \mathbf{F}$, is the "determinant" of the matrix $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ and is given by the expression $(R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

divergence the divergence of a vector field $\mathbf{F} = \langle P, Q, R \rangle$, denoted $\nabla \times \mathbf{F}$, is $P_x + Q_y + R_z$; it measures the "outflowing-ness" of a vector field

divergence theorem a theorem used to transform a difficult flux integral into an easier triple integral and vice versa

flux the rate of a fluid flowing across a curve in a vector field; the flux of vector field **F** across plane curve C is line integral $\int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\|\mathbf{n}(t)\|} ds$

flux integral another name for a surface integral of a vector field; the preferred term in physics and engineering

Fundamental Theorem for Line Integrals the value of line integral $\int_C \nabla f \cdot d\mathbf{r}$ depends only on the value of f at the endpoints of C: $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$

Gauss' law if S is a piecewise, smooth closed surface in a vacuum and Q is the total stationary charge inside of S, then the flux of electrostatic field **E** across S is Q/ε_0

gradient field a vector field **F** for which there exists a scalar function f such that $\nabla f = \mathbf{F}$; in other words, a vector field that is the gradient of a function; such vector fields are also called *conservative*

Green's theorem relates the integral over a connected region to an integral over the boundary of the region

grid curves curves on a surface that are parallel to grid lines in a coordinate plane

heat flow a vector field proportional to the negative temperature gradient in an object

independence of path a vector field **F** has path independence if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any curves C_1 and C_2 in the domain of **F** with the same initial points and terminal points

inverse-square law the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge

line integral the integral of a function along a curve in a plane or in space

mass flux the rate of mass flow of a fluid per unit area, measured in mass per unit time per unit area

orientation of a curve the orientation of a curve *C* is a specified direction of *C*

orientation of a surface if a surface has an "inner" side and an "outer" side, then an orientation is a choice of the inner or the outer side; the surface could also have "upward" and "downward" orientations

parameter domain (parameter space) the region of the *uv* plane over which the parameters *u* and *v* vary for parameterization $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$

parameterized surface (parametric surface) a surface given by a description of the form $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, where the parameters *u* and *v* vary over a parameter domain in the *uv*-plane

piecewise smooth curve an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

potential function a scalar function f such that $\nabla f = \mathbf{F}$

radial field a vector field in which all vectors either point directly toward or directly away from the origin; the magnitude of any vector depends only on its distance from the origin

regular parameterization parameterization $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ such that $\mathbf{r}_u \times \mathbf{r}_v$ is not zero for point (u, v) in the parameter domain

rotational field a vector field in which the vector at point (x, y) is tangent to a circle with radius $r = \sqrt{x^2 + y^2}$; in a rotational field, all vectors flow either clockwise or counterclockwise, and the magnitude of a vector depends only on its distance from the origin

scalar line integral the scalar line integral of a function f along a curve *C* with respect to arc length is the integral $\int_C f\, ds$, it is the integral of a scalar function f along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

simple curve a curve that does not cross itself

simply connected region a region that is connected and has the property that any closed curve that lies entirely inside the region encompasses points that are entirely inside the region

Stokes' theorem relates the flux integral over a surface *S* to a line integral around the boundary *C* of the surface *S*

stream function if $\mathbf{F} = \langle P, Q \rangle$ is a source-free vector field, then stream function *g* is a function such that $P = g_y$ and $Q = -g_x$

surface area the area of surface *S* given by the surface integral $\iint_S dS$

surface independent flux integrals of curl vector fields are surface independent if their evaluation does not depend on the surface but only on the boundary of the surface

surface integral an integral of a function over a surface

surface integral of a scalar-valued function a surface integral in which the integrand is a scalar function

surface integral of a vector field a surface integral in which the integrand is a vector field

unit vector field a vector field in which the magnitude of every vector is 1

vector field measured in $\mathbb{R}^2$, an assignment of a vector $\mathbf{F}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$; in $\mathbb{R}^3$, an assignment of a vector $\mathbf{F}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$

vector line integral the vector line integral of vector field **F** along curve *C* is the integral of the dot product of **F** with unit tangent vector **T** of *C* with respect to arc length, $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

KEY EQUATIONS

- **Vector field in $\mathbb{R}^2$**
  $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$
  or
  $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$
- **Vector field in $\mathbb{R}^3$**
  $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$
  or
  $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$
- **Calculating a scalar line integral**
  $\int_C f(x, y, z)ds = \int_a^b f(\mathbf{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$
- **Calculating a vector line integral**
  $\int_C \mathbf{F} \cdot ds = \int_C \mathbf{F} \cdot \mathbf{T}ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$
  or
  $\int_C Pdx + Qdy + Rdz = \int_a^b \left(P(\mathbf{r}(t))\frac{dx}{dt} + Q(\mathbf{r}(t))\frac{dy}{dt} + R(\mathbf{r}(t))\frac{dz}{dt}\right)dt$
- **Calculating flux**
  $\int_C \mathbf{F} \cdot \frac{\mathbf{n}(t)}{\| \mathbf{n}(t) \|} ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t)dt$
- **Fundamental Theorem for Line Integrals**
  $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
- **Circulation of a conservative field over curve *C* that encloses a simply connected region**
  $\oint_C \nabla f \cdot d\mathbf{r} = 0$
- **Green's theorem, circulation form**
  $\oint_C Pdx + Qdy = \iint_D Q_x - P_y\, dA,$ where C is the boundary of D
- **Green's theorem, flux form**
  $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y\, dA,$ where C is the boundary of D
- **Green's theorem, extended version**
  $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y\, dA$
- **Curl**
  $\nabla \times \mathbf{F} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}$
- **Divergence**
  $\nabla \cdot \mathbf{F} = P_x + Q_y + R_z$
- **Divergence of curl is zero**
  $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
- **Curl of a gradient is the zero vector**

$\nabla \times (\nabla f) = 0$

- **Scalar surface integral**
  $\int\int_S f(x, y, z)dS = \int\int_D f(\mathbf{r}(u, v))\|\mathbf{t}_u \times \mathbf{t}_v\|dA$
- **Flux integral**
  $\iint_S \mathbf{F} \cdot \mathbf{N}dS = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{t}_u \times \mathbf{t}_v)dA$
- **Stokes' theorem**
  $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$
- **Divergence theorem**
  $\iiint_E \text{div } \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$

KEY CONCEPTS

6.1 Vector Fields

- A vector field assigns a vector $\mathbf{F}(x, y)$ to each point (x, y) in a subset D of $\mathbb{R}^2$ or $\mathbb{R}^3$. $\mathbf{F}(x, y, z)$ to each point (x, y, z) in a subset D of $\mathbb{R}^3$.
- Vector fields can describe the distribution of vector quantities such as forces or velocities over a region of the plane or of space. They are in common use in such areas as physics, engineering, meteorology, oceanography.
- We can sketch a vector field by examining its defining equation to determine relative magnitudes in various locations and then drawing enough vectors to determine a pattern.
- A vector field $\mathbf{F}$ is called conservative if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

6.2 Line Integrals

- Line integrals generalize the notion of a single-variable integral to higher dimensions. The domain of integration in a single-variable integral is a line segment along the x-axis, but the domain of integration in a line integral is a curve in a plane or in space.
- If C is a curve, then the length of C is $\int_C ds$.
- There are two kinds of line integral: scalar line integrals and vector line integrals. Scalar line integrals can be used to calculate the mass of a wire; vector line integrals can be used to calculate the work done on a particle traveling through a field.
- Scalar line integrals can be calculated using **Equation 6.8**; vector line integrals can be calculated using **Equation 6.9**.
- Two key concepts expressed in terms of line integrals are flux and circulation. Flux measures the rate that a field crosses a given line; circulation measures the tendency of a field to move in the same direction as a given closed curve.

6.3 Conservative Vector Fields

- The theorems in this section require curves that are closed, simple, or both, and regions that are connected or simply connected.
- The line integral of a conservative vector field can be calculated using the Fundamental Theorem for Line Integrals. This theorem is a generalization of the Fundamental Theorem of Calculus in higher dimensions. Using this theorem usually makes the calculation of the line integral easier.
- Conservative fields are independent of path. The line integral of a conservative field depends only on the value of

the potential function at the endpoints of the domain curve.

- Given vector field **F**, we can test whether **F** is conservative by using the cross-partial property. If **F** has the cross-partial property and the domain is simply connected, then **F** is conservative (and thus has a potential function). If **F** is conservative, we can find a potential function by using the Problem-Solving Strategy.
- The circulation of a conservative vector field on a simply connected domain over a closed curve is zero.

6.4 Green's Theorem

- Green's theorem relates the integral over a connected region to an integral over the boundary of the region. Green's theorem is a version of the Fundamental Theorem of Calculus in one higher dimension.
- Green's Theorem comes in two forms: a circulation form and a flux form. In the circulation form, the integrand is $\mathbf{F} \cdot \mathbf{T}$. In the flux form, the integrand is $\mathbf{F} \cdot \mathbf{N}$.
- Green's theorem can be used to transform a difficult line integral into an easier double integral, or to transform a difficult double integral into an easier line integral.
- A vector field is source free if it has a stream function. The flux of a source-free vector field across a closed curve is zero, just as the circulation of a conservative vector field across a closed curve is zero.

6.5 Divergence and Curl

- The divergence of a vector field is a scalar function. Divergence measures the "outflowing-ness" of a vector field. If **v** is the velocity field of a fluid, then the divergence of **v** at a point is the outflow of the fluid less the inflow at the point.
- The curl of a vector field is a vector field. The curl of a vector field at point P measures the tendency of particles at P to rotate about the axis that points in the direction of the curl at P.
- A vector field with a simply connected domain is conservative if and only if its curl is zero.

6.6 Surface Integrals

- Surfaces can be parameterized, just as curves can be parameterized. In general, surfaces must be parameterized with two parameters.
- Surfaces can sometimes be oriented, just as curves can be oriented. Some surfaces, such as a Möbius strip, cannot be oriented.
- A surface integral is like a line integral in one higher dimension. The domain of integration of a surface integral is a surface in a plane or space, rather than a curve in a plane or space.
- The integrand of a surface integral can be a scalar function or a vector field. To calculate a surface integral with an integrand that is a function, use **Equation 6.19**. To calculate a surface integral with an integrand that is a vector field, use **Equation 6.20**.
- If S is a surface, then the area of S is $\iint_S dS$.

6.7 Stokes' Theorem

- Stokes' theorem relates a flux integral over a surface to a line integral around the boundary of the surface. Stokes' theorem is a higher dimensional version of Green's theorem, and therefore is another version of the Fundamental Theorem of Calculus in higher dimensions.
- Stokes' theorem can be used to transform a difficult surface integral into an easier line integral, or a difficult line integral into an easier surface integral.
- Through Stokes' theorem, line integrals can be evaluated using the simplest surface with boundary C.
- Faraday's law relates the curl of an electric field to the rate of change of the corresponding magnetic field. Stokes' theorem can be used to derive Faraday's law.

6.8 The Divergence Theorem

- The divergence theorem relates a surface integral across closed surface S to a triple integral over the solid enclosed by S. The divergence theorem is a higher dimensional version of the flux form of Green's theorem, and is therefore a higher dimensional version of the Fundamental Theorem of Calculus.
- The divergence theorem can be used to transform a difficult flux integral into an easier triple integral and vice versa.
- The divergence theorem can be used to derive Gauss' law, a fundamental law in electrostatics.

CHAPTER 6 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample.

427. Vector field $\mathbf{F}(x, y) = x^2 y\mathbf{i} + y^2 x\mathbf{j}$ is conservative.

428. For vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, if $P_y(x, y) = Q_x(x, y)$ in open region D, then $\int_{\partial D} Pdx + Qdy = 0$.

429. The divergence of a vector field is a vector field.

430. If $\text{curl}\,\mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

Draw the following vector fields.

431. $\mathbf{F}(x, y) = \frac{1}{2}\mathbf{i} + 2x\mathbf{j}$

432. $\mathbf{F}(x, y) = \sqrt{\dfrac{y\mathbf{i} + 3x\mathbf{j}}{x^2 + y^2}}$

Are the following the vector fields conservative? If so, find the potential function f such that $\mathbf{F} = \nabla f$.

433. $\mathbf{F}(x, y) = y\mathbf{i} + (x - 2e^y)\mathbf{j}$

434. $\mathbf{F}(x, y) = (6xy)\mathbf{i} + \left(3x^2 - ye^y\right)\mathbf{j}$

435. $\mathbf{F}(x, y, z) = \left(2xy + z^2\right)\mathbf{i} + \left(x^2 + 2yz\right)\mathbf{j} + \left(2xz + y^2\right)\mathbf{k}$

436. $\mathbf{F}(x, y, z) = (e^x y)\mathbf{i} + (e^x + z)\mathbf{j} + \left(e^x + y^2\right)\mathbf{k}$

Evaluate the following integrals.

437. $\int_C x^2 dy + (2x - 3xy)dx$, along $C : y = \frac{1}{2}x$ from (0, 0) to (4, 2)

438. $\int_C ydx + xy^2 dy$, where $C : x = \sqrt{t},\ y = t - 1,\ 0 \le t \le 1$

439. $\iint_S xy^2 dS$, where S is surface $z = x^2 - y,\ 0 \le x \le 1,\ 0 \le y \le 4$

Find the divergence and curl for the following vector fields.

440. $\mathbf{F}(x, y, z) = 3xyz\mathbf{i} + xye^z\mathbf{j} - 3xy\mathbf{k}$

441. $\mathbf{F}(x, y, z) = e^x\mathbf{i} + e^{xy}\mathbf{j} + e^{xyz}\mathbf{k}$

Use Green's theorem to evaluate the following integrals.

442. $\int_C 3xydx + 2xy^2 dy$, where C is a square with vertices (0, 0), (0, 2), (2, 2) and (2, 0)

443. $\oint_C 3ydx + (x + e^y)dy$, where C is a circle centered at the origin with radius 3

Use Stokes' theorem to evaluate $\int\int_S \text{curl}\,\mathbf{F} \cdot dS$.

444. $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$, where S is the upper half of the unit sphere

445. $\mathbf{F}(x, y, z) = y\mathbf{i} + xyz\mathbf{j} - 2zx\mathbf{k}$, where S is the upward-facing paraboloid $z = x^2 + y^2$ lying in cylinder $x^2 + y^2 = 1$

Use the divergence theorem to evaluate $\int\int_S \mathbf{F} \cdot dS$.

446. $\mathbf{F}(x, y, z) = \left(x^3 y\right)\mathbf{i} + \left(3y - e^x\right)\mathbf{j} + (z + x)\mathbf{k},$ over cube S defined by $-1 \le x \le 1,$ $0 \le y \le 2,$ $0 \le z \le 2$

447. $\mathbf{F}(x, y, z) = (2xy)\mathbf{i} + \left(-y^2\right)\mathbf{j} + \left(2z^3\right)\mathbf{k},$ where S is bounded by paraboloid $z = x^2 + y^2$ and plane $z = 2$

448. Find the amount of work performed by a 50-kg woman ascending a helical staircase with radius 2 m and height 100 m. The woman completes five revolutions during the climb.

449. Find the total mass of a thin wire in the shape of a semicircle with radius $\sqrt{2},$ and a density function of $\rho(x, y) = y + x^2.$

450. Find the total mass of a thin sheet in the shape of a hemisphere with radius 2 for $z \ge 0$ with a density function $\rho(x, y, z) = x + y + z.$

451. Use the divergence theorem to compute the value of the flux integral over the unit sphere with $\mathbf{F}(x, y, z) = 3z\mathbf{i} + 2y\mathbf{j} + 2x\mathbf{k}.$

7 | SECOND-ORDER DIFFERENTIAL EQUATIONS

Figure 7.1 A motorcycle suspension system is an example of a damped spring-mass system. The spring absorbs bumps and keeps the tire in contact with the road. The shock absorber damps the motion so the motorcycle does not continue to bounce after going over each bump. (credit: nSeika, Flickr)

Chapter Outline

Introduction

We have already studied the basics of differential equations, including separable first-order equations. In this chapter, we go a little further and look at second-order equations, which are equations containing second derivatives of the dependent variable. The solution methods we examine are different from those discussed earlier, and the solutions tend to involve trigonometric functions as well as exponential functions. Here we concentrate primarily on second-order equations with constant coefficients.

Such equations have many practical applications. The operation of certain electrical circuits, known as resistor–inductor–capacitor (*RLC*) circuits, can be described by second-order differential equations with constant coefficients. These circuits are found in all kinds of modern electronic devices—from computers to smartphones to televisions. Such circuits can be used to select a range of frequencies from the entire radio wave spectrum, and are they commonly used for tuning AM/FM radios. We look at these circuits more closely in **Applications**.

Spring-mass systems, such as motorcycle shock absorbers, are a second common application of second-order differential equations. For motocross riders, the suspension systems on their motorcycles are very important. The off-road courses on which they ride often include jumps, and losing control of the motorcycle when landing could cost them the race. The movement of the shock absorber depends on the amount of damping in the system. In this chapter, we model forced and unforced spring-mass systems with varying amounts of damping.

7.1 | Second-Order Linear Equations

Learning Objectives

- **7.1.1** Recognize homogeneous and nonhomogeneous linear differential equations.
- **7.1.2** Determine the characteristic equation of a homogeneous linear equation.
- **7.1.3** Use the roots of the characteristic equation to find the solution to a homogeneous linear equation.
- **7.1.4** Solve initial-value and boundary-value problems involving linear differential equations.

When working with differential equations, usually the goal is to find a solution. In other words, we want to find a function (or functions) that satisfies the differential equation. The technique we use to find these solutions varies, depending on the form of the differential equation with which we are working. Second-order differential equations have several important characteristics that can help us determine which solution method to use. In this section, we examine some of these characteristics and the associated terminology.

Homogeneous Linear Equations

Consider the second-order differential equation

$$xy'' + 2x^2 y' + 5x^3 y = 0.$$

Notice that y and its derivatives appear in a relatively simple form. They are multiplied by functions of x, but are not raised to any powers themselves, nor are they multiplied together. As discussed in **Introduction to Differential Equations (http://cnx.org/content/m53696/latest/)** , first-order equations with similar characteristics are said to be linear. The same is true of second-order equations. Also note that all the terms in this differential equation involve either y or one of its derivatives. There are no terms involving only functions of x. Equations like this, in which every term contains y or one of its derivatives, are called homogeneous.

Not all differential equations are homogeneous. Consider the differential equation

$$xy'' + 2x^2 y' + 5x^3 y = x^2.$$

The x^2 term on the right side of the equal sign does not contain y or any of its derivatives. Therefore, this differential equation is nonhomogeneous.

Definition

A second-order differential equation is linear if it can be written in the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x), \tag{7.1}$$

where $a_2(x)$, $a_1(x)$, $a_0(x)$, and $r(x)$ are real-valued functions and $a_2(x)$ is not identically zero. If $r(x) \equiv 0$—in other words, if $r(x) = 0$ for every value of x—the equation is said to be a **homogeneous linear equation**. If $r(x) \neq 0$ for some value of x, the equation is said to be a **nonhomogeneous linear equation**.

Visit this **website (http://www.openstaxcollege.org/l/20_Secondord)** to study more about second-order linear differential equations.

In linear differential equations, y and its derivatives can be raised only to the first power and they may not be multiplied by one another. Terms involving y^2 or $\sqrt{y'}$ make the equation nonlinear. Functions of y and its derivatives, such as $\sin y$ or $e^{y'}$, are similarly prohibited in linear differential equations.

Note that equations may not always be given in standard form (the form shown in the definition). It can be helpful to rewrite them in that form to decide whether they are linear, or whether a linear equation is homogeneous.

Example 7.1

Classifying Second-Order Equations

Classify each of the following equations as linear or nonlinear. If the equation is linear, determine further whether it is homogeneous or nonhomogeneous.

a. $y'' + 3x^4 y' + x^2 y^2 = x^3$
b. $(\sin x)y'' + (\cos x)y' + 3y = 0$
c. $4t^2 x'' + 3txx' + 4x = 0$
d. $5y'' + y = 4x^5$
e. $(\cos x)y'' - \sin y' + (\sin x)y - \cos x = 0$
f. $8ty'' - 6t^2 y' + 4ty - 3t^2 = 0$
g. $\sin(x^2)y'' - (\cos x)y' + x^2 y = y' - 3$
h. $y'' + 5xy' - 3y = \cos y$

Solution

a. This equation is nonlinear because of the y^2 term.
b. This equation is linear. There is no term involving a power or function of y, and the coefficients are all functions of x. The equation is already written in standard form, and $r(x)$ is identically zero, so the equation is homogeneous.
c. This equation is nonlinear. Note that, in this case, x is the dependent variable and t is the independent variable. The second term involves the product of x and x', so the equation is nonlinear.
d. This equation is linear. Since $r(x) = 4x^5$, the equation is nonhomogeneous.
e. This equation is nonlinear, because of the $\sin y'$ term.
f. This equation is linear. Rewriting it in standard form gives

$$8t^2 y'' - 6t^2 y' + 4ty = 3t^2.$$

With the equation in standard form, we can see that $r(t) = 3t^2$, so the equation is nonhomogeneous.

g. This equation looks like it's linear, but we should rewrite it in standard form to be sure. We get

$$\sin(x^2)y'' - (\cos x + 1)y' + x^2 y = -3.$$

This equation is, indeed, linear. With $r(x) = -3,$ it is nonhomogeneous.

h. This equation is nonlinear because of the $\cos y$ term.

Visit this **website (http://www.openstaxcollege.org/l/20_Secondord2)** that discusses second-order differential equations.

7.1 Classify each of the following equations as linear or nonlinear. If the equation is linear, determine further whether it is homogeneous or nonhomogeneous.

a. $(y'')^2 - y' + 8x^3 y = 0$

b. $(\sin t)y'' + \cos t - 3ty' = 0$

Later in this section, we will see some techniques for solving specific types of differential equations. Before we get to that, however, let's get a feel for how solutions to linear differential equations behave. In many cases, solving differential equations depends on making educated guesses about what the solution might look like. Knowing how various types of solutions behave will be helpful.

Example 7.2

Verifying a Solution

Consider the linear, homogeneous differential equation

$$x^2 y'' - xy' - 3y = 0.$$

Looking at this equation, notice that the coefficient functions are polynomials, with higher powers of x associated with higher-order derivatives of y. Show that $y = x^3$ is a solution to this differential equation.

Solution

Let $y = x^3$. Then $y' = 3x^2$ and $y'' = 6x$. Substituting into the differential equation, we see that

$$\begin{aligned} x^2 y'' - xy' - 3y &= x^2(6x) - x\left(3x^2\right) - 3\left(x^3\right) \\ &= 6x^3 - 3x^3 - 3x^3 \\ &= 0. \end{aligned}$$

7.2 Show that $y = 2x^2$ is a solution to the differential equation

$$\frac{1}{2}x^2 y'' - xy' + y = 0.$$

Although simply finding any solution to a differential equation is important, mathematicians and engineers often want to go beyond finding *one* solution to a differential equation to finding *all* solutions to a differential equation. In other words, we want to find a general solution**.** Just as with first-order differential equations, a general solution (or family of solutions) gives the entire set of solutions to a differential equation. An important difference between first-order and second-order equations is that, with second-order equations, we typically need to find two different solutions to the equation to find the general solution. If we find two solutions, then any linear combination of these solutions is also a solution. We state this fact as the following theorem.

Theorem 7.1: Superposition Principle

If $y_1(x)$ and $y_2(x)$ are solutions to a linear homogeneous differential equation, then the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are constants, is also a solution.

The proof of this superposition principle theorem is left as an exception.

Example 7.3

Verifying the Superposition Principle

Consider the differential equation

$$y'' - 4y' - 5y = 0.$$

Given that e^{-x} and e^{5x} are solutions to this differential equation, show that $4e^{-x} + e^{5x}$ is a solution.

Solution

We have

$$y(x) = 4e^{-x} + e^{5x}, \text{ so } y'(x) = -4e^{-x} + 5e^{5x} \text{ and } y''(x) = 4e^{-x} + 25e^{5x}.$$

Then

$$\begin{aligned} y'' - 4y' - 5y &= \left(4e^{-x} + 25e^{5x}\right) - 4\left(-4e^{-x} + 5e^{5x}\right) - 5\left(4e^{-x} + e^{5x}\right) \\ &= 4e^{-x} + 25e^{5x} + 16e^{-x} - 20e^{5x} - 20e^{-x} - 5e^{5x} \\ &= 0. \end{aligned}$$

Thus, $y(x) = 4e^{-x} + e^{5x}$ is a solution.

7.3 Consider the differential equation

$$y'' + 5y' + 6y = 0.$$

Given that e^{-2x} and e^{-3x} are solutions to this differential equation, show that $3e^{-2x} + 6e^{-3x}$ is a solution.

Unfortunately, to find the general solution to a second-order differential equation, it is not enough to find any two solutions and then combine them. Consider the differential equation

$$x'' + 7x' + 12x = 0.$$

Both e^{-3t} and $2e^{-3t}$ are solutions (check this). However, $x(t) = c_1 e^{-3t} + c_2\left(2e^{-3t}\right)$ is *not* the general solution. This expression does not account for all solutions to the differential equation. In particular, it fails to account for the function e^{-4t}, which is also a solution to the differential equation.

It turns out that to find the general solution to a second-order differential equation, we must find two linearly independent solutions. We define that terminology here.

Definition

A set of functions $f_1(x), f_2(x),\ldots,f_n(x)$ is said to be **linearly dependent** if there are constants $c_1, c_2,\ldots c_n,$ not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for all *x* over the interval of interest. A set of functions that is not linearly dependent is said to be **linearly independent**.

In this chapter, we usually test sets of only two functions for linear independence, which allows us to simplify this definition. From a practical perspective, we see that two functions are linearly dependent if either one of them is identically zero or if they are constant multiples of each other.

First we show that if the functions meet the conditions given previously, then they are linearly dependent. If one of the functions is identically zero—say, $f_2(x) \equiv 0$—then choose $c_1 = 0$ and $c_2 = 1,$ and the condition for linear dependence is satisfied. If, on the other hand, neither $f_1(x)$ nor $f_2(x)$ is identically zero, but $f_1(x) = Cf_2(x)$ for some constant $C,$ then choose $c_1 = \frac{1}{C}$ and $c_2 = -1,$ and again, the condition is satisfied.

Next, we show that if two functions are linearly dependent, then either one is identically zero or they are constant multiples of one another. Assume $f_1(x)$ and $f_2(x)$ are linearly independent. Then, there are constants, c_1 and $c_2,$ not both zero, such that

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all *x* over the interval of interest. Then,

$$c_1 f_1(x) = -c_2 f_2(x).$$

Now, since we stated that c_1 and c_2 can't both be zero, assume $c_2 \neq 0.$ Then, there are two cases: either $c_1 = 0$ or $c_1 \neq 0.$ If $c_1 = 0,$ then

$$\begin{aligned} 0 &= -c_2 f_2(x) \\ 0 &= f_2(x), \end{aligned}$$

so one of the functions is identically zero. Now suppose $c_1 \neq 0.$ Then,

$$f_1(x) = \left(-\frac{c_2}{c_1}\right) f_2(x)$$

and we see that the functions are constant multiples of one another.

Theorem 7.2: Linear Dependence of Two Functions

Two functions, $f_1(x)$ and $f_2(x),$ are said to be linearly dependent if either one of them is identically zero or if $f_1(x) = Cf_2(x)$ for some constant *C* and for all *x* over the interval of interest. Functions that are not linearly dependent are said to be *linearly independent*.

Example 7.4

Testing for Linear Dependence

Determine whether the following pairs of functions are linearly dependent or linearly independent.

a. $f_1(x) = x^2, \quad f_2(x) = 5x^2$

b. $f_1(x) = \sin x, \quad f_2(x) = \cos x$

c. $f_1(x) = e^{3x}, \quad f_2(x) = e^{-3x}$

d. $f_1(x) = 3x, \quad f_2(x) = 3x + 1$

Solution

a. $f_2(x) = 5f_1(x),$ so the functions are linearly dependent.

b. There is no constant C such that $f_1(x) = Cf_2(x),$ so the functions are linearly independent.

c. There is no constant C such that $f_1(x) = Cf_2(x),$ so the functions are linearly independent. Don't get confused by the fact that the exponents are constant multiples of each other. With two exponential functions, unless the exponents are equal, the functions are linearly independent.

d. There is no constant C such that $f_1(x) = Cf_2(x),$ so the functions are linearly independent.

7.4 Determine whether the following pairs of functions are linearly dependent or linearly independent: $f_1(x) = e^x, \quad f_2(x) = 3e^{3x}.$

If we are able to find two linearly independent solutions to a second-order differential equation, then we can combine them to find the general solution. This result is formally stated in the following theorem.

Theorem 7.3: General Solution to a Homogeneous Equation

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions to a second-order, linear, homogeneous differential equation, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are constants.

When we say a family of functions is the general solution to a differential equation, we mean that (1) every expression of that form is a solution and (2) every solution to the differential equation can be written in that form, which makes this theorem extremely powerful. If we can find two linearly independent solutions to a differential equation, we have, effectively, found *all* solutions to the differential equation—quite a remarkable statement. The proof of this theorem is beyond the scope of this text.

Example 7.5

Writing the General Solution

If $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$ are solutions to $y'' - 9y = 0,$ what is the general solution?

Solution

Note that y_1 and y_2 are not constant multiples of one another, so they are linearly independent. Then, the general solution to the differential equation is $y(t) = c_1 e^{3t} + c_2 e^{-3t}$.

7.5 If $y_1(x) = e^{3x}$ and $y_2(x) = xe^{3x}$ are solutions to $y'' - 6y' + 9y = 0$, what is the general solution?

Second-Order Equations with Constant Coefficients

Now that we have a better feel for linear differential equations, we are going to concentrate on solving second-order equations of the form

$$ay'' + by' + cy = 0, \tag{7.2}$$

where a, b, and c are constants.

Since all the coefficients are constants, the solutions are probably going to be functions with derivatives that are constant multiples of themselves. We need all the terms to cancel out, and if taking a derivative introduces a term that is not a constant multiple of the original function, it is difficult to see how that term cancels out. Exponential functions have derivatives that are constant multiples of the original function, so let's see what happens when we try a solution of the form $y(x) = e^{\lambda x}$, where λ (the lowercase Greek letter lambda) is some constant.

If $y(x) = e^{\lambda x}$, then $y'(x) = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$. Substituting these expressions into **Equation 7.1**, we get

$$\begin{aligned} ay'' + by' + cy &= a(\lambda^2 e^{\lambda x}) + b(\lambda e^{\lambda x}) + ce^{\lambda x} \\ &= e^{\lambda x}(a\lambda^2 + b\lambda + c). \end{aligned}$$

Since $e^{\lambda x}$ is never zero, this expression can be equal to zero for all x only if

$$a\lambda^2 + b\lambda + c = 0.$$

We call this the characteristic equation of the differential equation.

Definition

The **characteristic equation** of the differential equation $ay'' + by' + cy = 0$ is $a\lambda^2 + b\lambda + c = 0$.

The characteristic equation is very important in finding solutions to differential equations of this form. We can solve the characteristic equation either by factoring or by using the quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This gives three cases. The characteristic equation has (1) distinct real roots; (2) a single, repeated real root; or (3) complex conjugate roots. We consider each of these cases separately.

Distinct Real Roots

If the characteristic equation has distinct real roots λ_1 and λ_2, then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are linearly independent solutions to **Example 7.1**, and the general solution is given by

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x},$$

where c_1 and c_2 are constants.

For example, the differential equation $y'' + 9y' + 14y = 0$ has the associated characteristic equation $\lambda^2 + 9\lambda + 14 = 0$. This factors into $(\lambda + 2)(\lambda + 7) = 0$, which has roots $\lambda_1 = -2$ and $\lambda_2 = -7$. Therefore, the general solution to this differential equation is

$$y(x) = c_1 e^{-2x} + c_2 e^{-7x}.$$

Single Repeated Real Root

Things are a little more complicated if the characteristic equation has a repeated real root, λ. In this case, we know $e^{\lambda x}$ is a solution to **Equation 7.1**, but it is only one solution and we need two linearly independent solutions to determine the general solution. We might be tempted to try a function of the form $ke^{\lambda x}$, where *k* is some constant, but it would not be linearly independent of $e^{\lambda x}$. Therefore, let's try $xe^{\lambda x}$ as the second solution. First, note that by the quadratic formula,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

But, λ is a repeated root, so $b^2 - 4ac = 0$ and $\lambda = \frac{-b}{2a}$. Thus, if $y = xe^{\lambda x}$, we have

$$y' = e^{\lambda x} + \lambda x e^{\lambda x} \text{ and } y'' = 2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}.$$

Substituting these expressions into **Equation 7.1**, we see that

$$\begin{aligned} ay'' + by' + cy &= a(2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}) + b(e^{\lambda x} + \lambda x e^{\lambda x}) + cxe^{\lambda x} \\ &= xe^{\lambda x}(a\lambda^2 + b\lambda + c) + e^{\lambda x}(2a\lambda + b) \\ &= xe^{\lambda x}(0) + e^{\lambda x}\left(2a\left(\frac{-b}{2a}\right) + b\right) \\ &= 0 + e^{\lambda x}(0) \\ &= 0. \end{aligned}$$

This shows that $xe^{\lambda x}$ is a solution to **Equation 7.1**. Since $e^{\lambda x}$ and $xe^{\lambda x}$ are linearly independent, when the characteristic equation has a repeated root λ, the general solution to **Equation 7.1** is given by

$$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x},$$

where c_1 and c_2 are constants.

For example, the differential equation $y'' + 12y' + 36y = 0$ has the associated characteristic equation $\lambda^2 + 12\lambda + 36 = 0$. This factors into $(\lambda + 6)^2 = 0$, which has a repeated root $\lambda = -6$. Therefore, the general solution to this differential equation is

$$y(x) = c_1 e^{-6x} + c_2 x e^{-6x}.$$

Complex Conjugate Roots

The third case we must consider is when $b^2 - 4ac < 0$. In this case, when we apply the quadratic formula, we are taking the square root of a negative number. We must use the imaginary number $i = \sqrt{-1}$ to find the roots, which take the form $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$. The complex number $\alpha + \beta i$ is called the *conjugate* of $\alpha - \beta i$. Thus, we see that when $b^2 - 4ac < 0$, the roots of our characteristic equation are always complex conjugates.

This creates a little bit of a problem for us. If we follow the same process we used for distinct real roots—using the roots of the characteristic equation as the coefficients in the exponents of exponential functions—we get the functions $e^{(\alpha + \beta i)x}$ and $e^{(\alpha - \beta i)x}$ as our solutions. However, there are problems with this approach. First, these functions take on complex (imaginary) values, and a complete discussion of such functions is beyond the scope of this text. Second, even if we were comfortable with complex-value functions, in this course we do not address the idea of a derivative for such functions. So, if possible, we'd like to find two linearly independent *real-value* solutions to the differential equation. For purposes of this development, we are going to manipulate and differentiate the functions $e^{(\alpha+\beta i)x}$ and $e^{(\alpha-\beta i)x}$ as if they were real-value functions. For these particular functions, this approach is valid mathematically, but be aware that there are other instances when complex-value functions do not follow the same rules as real-value functions. Those of you interested in a more in-depth discussion of complex-value functions should consult a complex analysis text.

Based on the roots $\alpha \pm \beta i$ of the characteristic equation, the functions $e^{(\alpha+\beta i)x}$ and $e^{(\alpha-\beta i)x}$ are linearly independent solutions to the differential equation. and the general solution is given by

$$y(x) = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}.$$

Using some smart choices for c_1 and $c_2,$ and a little bit of algebraic manipulation, we can find two linearly independent, real-value solutions to **Equation 7.1** and express our general solution in those terms.

We encountered exponential functions with complex exponents earlier. One of the key tools we used to express these exponential functions in terms of sines and cosines was Euler's formula, which tells us that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

for all real numbers θ.

Going back to the general solution, we have

$$\begin{aligned} y(x) &= c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x} \\ &= c_1 e^{\alpha x} e^{\beta ix} + c_2 e^{\alpha x} e^{-\beta ix} \\ &= e^{\alpha x}\left(c_1 e^{\beta ix} + c_2 e^{-\beta ix}\right). \end{aligned}$$

Applying Euler's formula together with the identities $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x,$ we get

$$\begin{aligned} y(x) &= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos(-\beta x) + i\sin(-\beta x))\right] \\ &= e^{\alpha x}\left[(c_1 + c_2)\cos\beta x + (c_1 - c_2)i\sin\beta x\right]. \end{aligned}$$

Now, if we choose $c_1 = c_2 = \frac{1}{2},$ the second term is zero and we get

$$y(x) = e^{\alpha x}\cos\beta x$$

as a real-value solution to **Equation 7.1**. Similarly, if we choose $c_1 = -\frac{i}{2}$ and $c_2 = \frac{i}{2},$ the first term is zero and we get

$$y(x) = e^{\alpha x}\sin\beta x$$

as a second, linearly independent, real-value solution to **Equation 7.1**.

Based on this, we see that if the characteristic equation has complex conjugate roots $\alpha \pm \beta i,$ then the general solution to **Equation 7.1** is given by

$$\begin{aligned} y(x) &= c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x), \end{aligned}$$

where c_1 and c_2 are constants.

For example, the differential equation $y'' - 2y' + 5y = 0$ has the associated characteristic equation $\lambda^2 - 2\lambda + 5 = 0.$ By the quadratic formula, the roots of the characteristic equation are $1 \pm 2i.$ Therefore, the general solution to this differential equation is

$$y(x) = e^{x}(c_1\cos 2x + c_2\sin 2x).$$

Summary of Results

We can solve second-order, linear, homogeneous differential equations with constant coefficients by finding the roots of the associated characteristic equation. The form of the general solution varies, depending on whether the characteristic equation has distinct, real roots; a single, repeated real root; or complex conjugate roots. The three cases are summarized in **Table 7.1**.

Characteristic Equation Roots	General Solution to the Differential Equation
Distinct real roots, λ_1 and λ_2	$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
A repeated real root, λ	$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$
Complex conjugate roots $\alpha \pm \beta i$	$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Table 7.1 Summary of Characteristic Equation Cases

Problem-Solving Strategy: Using the Characteristic Equation to Solve Second-Order Differential Equations with Constant Coefficients

1. Write the differential equation in the form $ay'' + by' + cy = 0.$
2. Find the corresponding characteristic equation $a\lambda^2 + b\lambda + c = 0.$
3. Either factor the characteristic equation or use the quadratic formula to find the roots.
4. Determine the form of the general solution based on whether the characteristic equation has distinct, real roots; a single, repeated real root; or complex conjugate roots.

Example 7.6

Solving Second-Order Equations with Constant Coefficients

Find the general solution to the following differential equations. Give your answers as functions of x.

a. $y'' + 3y' - 4y = 0$

b. $y'' + 6y' + 13y = 0$

c. $y'' + 2y' + y = 0$

d. $y'' - 5y' = 0$

e. $y'' - 16y = 0$

f. $y'' + 16y = 0$

Solution

Note that all these equations are already given in standard form (step 1).

a. The characteristic equation is $\lambda^2 + 3\lambda - 4 = 0$ (step 2). This factors into $(\lambda + 4)(\lambda - 1) = 0,$ so the roots of the characteristic equation are $\lambda_1 = -4$ and $\lambda_2 = 1$ (step 3). Then the general solution to the differential equation is

$$y(x) = c_1 e^{-4x} + c_2 e^{x} \text{ (step 4)}.$$

b. The characteristic equation is $\lambda^2 + 6\lambda + 13 = 0$ (step 2). Applying the quadratic formula, we see this equation has complex conjugate roots $-3 \pm 2i$ (step 3). Then the general solution to the differential equation is

$$y(t) = e^{-3t}(c_1 \cos 2t + c_2 \sin 2t) \text{ (step 4)}.$$

c. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$ (step 2). This factors into $(\lambda + 1)^2 = 0,$ so the characteristic equation has a repeated real root $\lambda = -1$ (step 3). Then the general solution to the differential equation is

$$y(t) = c_1 e^{-t} + c_2 te^{-t} \text{ (step 4)}.$$

d. The characteristic equation is $\lambda^2 - 5\lambda$ (step 2). This factors into $\lambda(\lambda - 5) = 0,$ so the roots of the characteristic equation are $\lambda_1 = 0$ and $\lambda_2 = 5$ (step 3). Note that $e^{0x} = e^0 = 1,$ so our first solution is just a constant. Then the general solution to the differential equation is

$$y(x) = c_1 + c_2 e^{5x} \text{ (step 4)}.$$

e. The characteristic equation is $\lambda^2 - 16 = 0$ (step 2). This factors into $(\lambda + 4)(\lambda - 4) = 0,$ so the roots of the characteristic equation are $\lambda_1 = 4$ and $\lambda_2 = -4$ (step 3). Then the general solution to the differential equation is

$$y(x) = c_1 e^{4x} + c_2 e^{-4x} \text{ (step 4)}.$$

f. The characteristic equation is $\lambda^2 + 16 = 0$ (step 2). This has complex conjugate roots $\pm 4i$ (step 3). Note that $e^{0x} = e^0 = 1,$ so the exponential term in our solution is just a constant. Then the general solution to the differential equation is

$$y(t) = c_1 \cos 4t + c_2 \sin 4t \text{ (step 4)}.$$

7.6 Find the general solution to the following differential equations:

a. $y'' - 2y' + 10y = 0$

b. $y'' + 14y' + 49y = 0$

Initial-Value Problems and Boundary-Value Problems

So far, we have been finding general solutions to differential equations. However, differential equations are often used to describe physical systems, and the person studying that physical system usually knows something about the state of that system at one or more points in time. For example, if a constant-coefficient differential equation is representing how far a motorcycle shock absorber is compressed, we might know that the rider is sitting still on his motorcycle at the start of a race, time $t = t_0$. This means the system is at equilibrium, so $y(t_0) = 0,$ and the compression of the shock absorber is not changing, so $y'(t_0) = 0.$ With these two initial conditions and the general solution to the differential equation, we can find the *specific* solution to the differential equation that satisfies both initial conditions. This process is known as *solving an initial-value problem*. (Recall that we discussed initial-value problems in **Introduction to Differential Equations (http://cnx.org/content/m53696/latest/)** .) Note that second-order equations have two arbitrary constants in the general solution, and therefore we require two initial conditions to find the solution to the initial-value problem.

Sometimes we know the condition of the system at two different times. For example, we might know $y(t_0) = y_0$ and

$y(t_1) = y_1$. These conditions are called **boundary conditions**, and finding the solution to the differential equation that satisfies the boundary conditions is called solving a **boundary-value problem**.

Mathematicians, scientists, and engineers are interested in understanding the conditions under which an initial-value problem or a boundary-value problem has a unique solution. Although a complete treatment of this topic is beyond the scope of this text, it is useful to know that, within the context of constant-coefficient, second-order equations, initial-value problems are guaranteed to have a unique solution as long as two initial conditions are provided. Boundary-value problems, however, are not as well behaved. Even when two boundary conditions are known, we may encounter boundary-value problems with unique solutions, many solutions, or no solution at all.

Example 7.7

Solving an Initial-Value Problem

Solve the following initial-value problem: $y'' + 3y' - 4y = 0, \quad y(0) = 1, \quad y'(0) = -9.$

Solution

We already solved this differential equation in Example 7.6a. and found the general solution to be

$$y(x) = c_1 e^{-4x} + c_2 e^x.$$

Then

$$y'(x) = -4c_1 e^{-4x} + c_2 e^x.$$

When $x = 0$, we have $y(0) = c_1 + c_2$ and $y'(0) = -4c_1 + c_2$. Applying the initial conditions, we have

$$\begin{aligned} c_1 + c_2 &= 1 \\ -4c_1 + c_2 &= -9. \end{aligned}$$

Then $c_1 = 1 - c_2$. Substituting this expression into the second equation, we see that

$$\begin{aligned} -4(1 - c_2) + c_2 &= -9 \\ -4 + 4c_2 + c_2 &= -9 \\ 5c_2 &= -5 \\ c_2 &= -1. \end{aligned}$$

So, $c_1 = 2$ and the solution to the initial-value problem is

$$y(x) = 2e^{-4x} - e^x.$$

7.7 Solve the initial-value problem $y'' - 3y' - 10y = 0, \quad y(0) = 0, \quad y'(0) = 7.$

Example 7.8

Solving an Initial-Value Problem and Graphing the Solution

Solve the following initial-value problem and graph the solution:

$$y'' + 6y' + 13y = 0, \;\; y(0) = 0, \;\; y'(0) = 2$$

Solution

We already solved this differential equation in **Example 7.6**b. and found the general solution to be

$$y(x) = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x).$$

Then

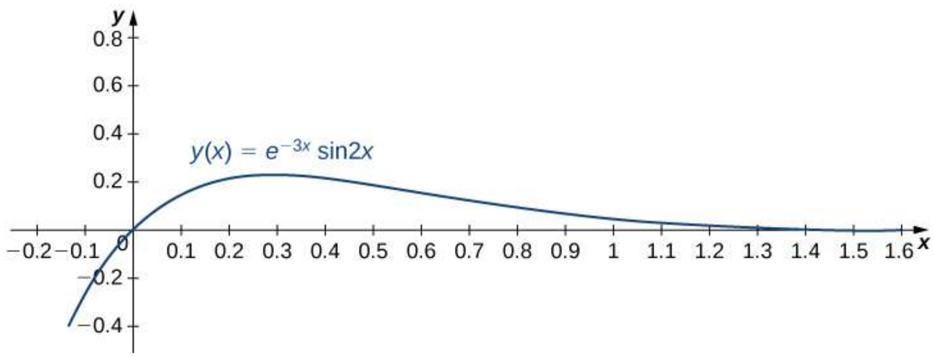
$$y'(x) = e^{-3x}(-2c_1 \sin 2x + 2c_2 \cos 2x) - 3e^{-3x}(c_1 \cos 2x + c_2 \sin 2x).$$

When $x = 0,$ we have $y(0) = c_1$ and $y'(0) = 2c_2 - 3c_1.$ Applying the initial conditions, we obtain

$$\begin{aligned} c_1 &= 0 \\ -3c_1 + 2c_2 &= 2. \end{aligned}$$

Therefore, $c_1 = 0,\quad c_2 = 1,$ and the solution to the initial value problem is shown in the following graph.

$$y = e^{-3x} \sin 2x.$$



7.8 Solve the following initial-value problem and graph the solution: $y'' - 2y' + 10y = 0,\ y(0) = 2,\ y'(0) = -1$

Example 7.9

Initial-Value Problem Representing a Spring-Mass System

The following initial-value problem models the position of an object with mass attached to a spring. Spring-mass systems are examined in detail in **Applications**. The solution to the differential equation gives the position of the mass with respect to a neutral (equilibrium) position (in meters) at any given time. (Note that for spring-mass systems of this type, it is customary to define the downward direction as positive.)

$$y'' + 2y' + y = 0,\ y(0) = 1,\ y'(0) = 0$$

Solve the initial-value problem and graph the solution. What is the position of the mass at time $t = 2$ sec? How fast is the mass moving at time $t = 1$ sec? In what direction?

Solution

In **Example 7.6**c. we found the general solution to this differential equation to be

$$y(t) = c_1 e^{-t} + c_2 te^{-t}.$$

Then

$$y'(t) = -c_1 e^{-t} + c_2\left(-te^{-t} + e^{-t}\right).$$

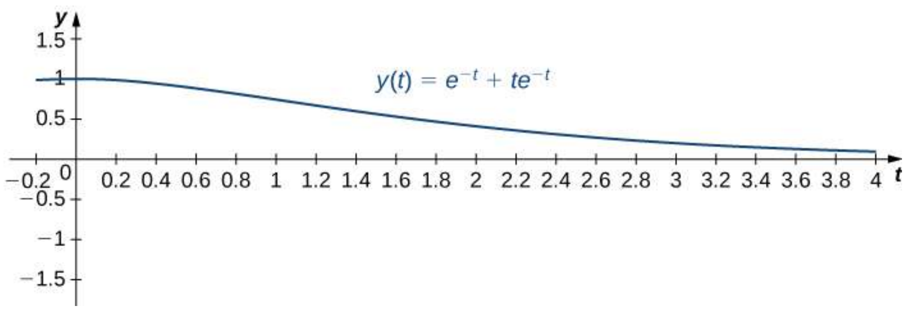
When $t = 0,$ we have $y(0) = c_1$ and $y'(0) = -c_1 + c_2$. Applying the initial conditions, we obtain

$$\begin{aligned} c_1 &= 1 \\ -c_1 + c_2 &= 0. \end{aligned}$$

Thus, $c_1 = 1,$ $c_2 = 1,$ and the solution to the initial value problem is

$$y(t) = e^{-t} + te^{-t}.$$

This solution is represented in the following graph. At time $t = 2,$ the mass is at position $y(2) = e^{-2} + 2e^{-2} = 3e^{-2} \approx 0.406$ m below equilibrium.



To calculate the velocity at time $t = 1,$ we need to find the derivative. We have $y(t) = e^{-t} + te^{-t},$ so

$$y'(t) = -e^{-t} + e^{-t} - te^{-t} = -te^{-t}.$$

Then $y'(1) = -e^{-1} \approx -0.3679$. At time $t = 1,$ the mass is moving upward at 0.3679 m/sec.

7.9 Suppose the following initial-value problem models the position (in feet) of a mass in a spring-mass system at any given time. Solve the initial-value problem and graph the solution. What is the position of the mass at time $t = 0.3$ sec? How fast is it moving at time $t = 0.1$ sec? In what direction?

$$y'' + 14y' + 49y = 0,\ y(0) = 0,\ y'(0) = 1$$

Example 7.10

Solving a Boundary-Value Problem

In **Example 7.6**f. we solved the differential equation $y'' + 16y = 0$ and found the general solution to be $y(t) = c_1 \cos 4t + c_2 \sin 4t$. If possible, solve the boundary-value problem if the boundary conditions are the following:

a. $y(0) = 0,\quad y\left(\frac{\pi}{4}\right) = 0$

b. $y(0) = 1,\quad y\left(\frac{\pi}{8}\right) = 0$

c. $y\left(\frac{\pi}{8}\right) = 0, \quad y\left(\frac{3\pi}{8}\right) = 2$

Solution

We have

$$y(x) = c_1 \cos 4t + c_2 \sin 4t.$$

a. Applying the first boundary condition given here, we get $y(0) = c_1 = 0$. So the solution is of the form $y(t) = c_2 \sin 4t$. When we apply the second boundary condition, though, we get $y\left(\frac{\pi}{4}\right) = c_2 \sin\left(4\left(\frac{\pi}{4}\right)\right) = c_2 \sin \pi = 0$ for all values of c_2. The boundary conditions are not sufficient to determine a value for c_2, so this boundary-value problem has infinitely many solutions. Thus, $y(t) = c_2 \sin 4t$ is a solution for any value of c_2.

b. Applying the first boundary condition given here, we get $y(0) = c_1 = 1$. Applying the second boundary condition gives $y\left(\frac{\pi}{8}\right) = c_2 = 0$, so $c_2 = 0$. In this case, we have a unique solution: $y(t) = \cos 4t$.

c. Applying the first boundary condition given here, we get $y\left(\frac{\pi}{8}\right) = c_2 = 0$. However, applying the second boundary condition gives $y\left(\frac{3\pi}{8}\right) = -c_2 = 2$, so $c_2 = -2$. We cannot have $c_2 = 0 = -2$, so this boundary value problem has no solution.

7.1 EXERCISES

Classify each of the following equations as linear or nonlinear. If the equation is linear, determine whether it is homogeneous or nonhomogeneous.

1. $x^3 y'' + (x-1)y' - 8y = 0$

2. $\left(1 + y^2\right)y'' + xy' - 3y = \cos x$

3. $xy'' + e^y y' = x$

4. $y'' + \frac{4}{x}y' - 8xy = 5x^2 + 1$

5. $y'' + (\sin x)y' - xy = 4y$

6. $y'' + \left(\frac{x+3}{y}\right)y' = 0$

For each of the following problems, verify that the given function is a solution to the differential equation. Use a graphing utility to graph the particular solutions for several values of c_1 and c_2. What do the solutions have in common?

7. **[T]** $y'' + 2y' - 3y = 0; \quad y(x) = c_1 e^x + c_2 e^{-3x}$

8. **[T]** $x^2 y'' - 2y - 3x^2 + 1 = 0; \quad y(x) = c_1 x^2 + c_2 x^{-1} + x^2 \ln(x) + \frac{1}{2}$

9. **[T]** $y'' + 14y' + 49y = 0; \quad y(x) = c_1 e^{-7x} + c_2 xe^{-7x}$

10. **[T]** $6y'' - 49y' + 8y = 0; \quad y(x) = c_1 e^{x/6} + c_2 e^{8x}$

Find the general solution to the linear differential equation.

11. $y'' - 3y' - 10y = 0$

12. $y'' - 7y' + 12y = 0$

13. $y'' + 4y' + 4y = 0$

14. $4y'' - 12y' + 9y = 0$

15. $2y'' - 3y' - 5y = 0$

16. $3y'' - 14y' + 8y = 0$

17. $y'' + y' + y = 0$

18. $5y'' + 2y' + 4y = 0$

19. $y'' - 121y = 0$

20. $8y'' + 14y' - 15y = 0$

21. $y'' + 81y = 0$

22. $y'' - y' + 11y = 0$

23. $2y'' = 0$

24. $y'' - 6y' + 9y = 0$

25. $3y'' - 2y' - 7y = 0$

26. $4y'' - 10y' = 0$

27. $36\frac{d^2 y}{dx^2} + 12\frac{dy}{dx} + y = 0$

28. $25\frac{d^2 y}{dx^2} - 80\frac{dy}{dx} + 64y = 0$

29. $\frac{d^2 y}{dx^2} - 9\frac{dy}{dx} = 0$

30. $4\frac{d^2 y}{dx^2} + 8y = 0$

Solve the initial-value problem.

31. $y'' + 5y' + 6y = 0, \quad y(0) = 0, \quad y'(0) = -2$

32. $y'' + 2y' - 8y = 0, \quad y(0) = 5, \quad y'(0) = 4$

33. $y'' + 4y = 0, \quad y(0) = 3, \quad y'(0) = 10$

34. $y'' - 18y' + 81y = 0, \quad y(0) = 1, \quad y'(0) = 5$

35. $y'' - y' - 30y = 0, \quad y(0) = 1, \quad y'(0) = -16$

36. $4y'' + 4y' - 8y = 0, \quad y(0) = 2, \quad y'(0) = 1$

37. $25y'' + 10y' + y = 0, \quad y(0) = 2, \quad y'(0) = 1$

38. $y'' + y = 0, \quad y(\pi) = 1, \quad y'(\pi) = -5$

Solve the boundary-value problem, if possible.

39. $y'' + y' - 42y = 0, \quad y(0) = 0, \quad y(1) = 2$

40. $9y'' + y = 0, \quad y\left(\frac{3\pi}{2}\right) = 6, \quad y(0) = -8$

41. $y'' + 10y' + 34y = 0, \quad y(0) = 6, \quad y(\pi) = 2$

42. $y'' + 7y' - 60y = 0, \quad y(0) = 4, \ \ y(2) = 0$

43. $y'' - 4y' + 4y = 0, \quad y(0) = 2, \ \ y(1) = -1$

44. $y'' - 5y' = 0, \quad y(0) = 3, \ \ y(-1) = 2$

45. $y'' + 9y = 0, \quad y(0) = 4, \ \ y\left(\frac{\pi}{3}\right) = -4$

46. $4y'' + 25y = 0, \quad y(0) = 2, \ \ y(2\pi) = -2$

47. Find a differential equation with a general solution that is $y = c_1 e^{x/5} + c_2 e^{-4x}$.

48. Find a differential equation with a general solution that is $y = c_1 e^{x} + c_2 e^{-4x/3}$.

For each of the following differential equations:

a. Solve the initial value problem.

b. **[T]** Use a graphing utility to graph the particular solution.

49. $y'' + 64y = 0; \quad y(0) = 3, \quad y'(0) = 16$

50. $y'' - 2y' + 10y = 0 \quad y(0) = 1, \quad y'(0) = 13$

51. $y'' + 5y' + 15y = 0 \quad y(0) = -2, \quad y'(0) = 7$

52. (Principle of superposition) Prove that if $y_1(x)$ and $y_2(x)$ are solutions to a linear homogeneous differential equation, $y'' + p(x)y' + q(x)y = 0,$ then the function $y(x) = c_1 y_1(x) + c_2 y_2(x),$ where c_1 and c_2 are constants, is also a solution.

53. Prove that if *a*, *b*, and *c* are positive constants, then all solutions to the second-order linear differential equation $ay'' + by' + cy = 0$ approach zero as $x \to \infty$. (*Hint:* Consider three cases: two distinct roots, repeated real roots, and complex conjugate roots.)

7.2 | Nonhomogeneous Linear Equations

Learning Objectives

7.2.1 Write the general solution to a nonhomogeneous differential equation.
7.2.2 Solve a nonhomogeneous differential equation by the method of undetermined coefficients.
7.2.3 Solve a nonhomogeneous differential equation by the method of variation of parameters.

In this section, we examine how to solve nonhomogeneous differential equations. The terminology and methods are different from those we used for homogeneous equations, so let's start by defining some new terms.

General Solution to a Nonhomogeneous Linear Equation

Consider the nonhomogeneous linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x).$$

The associated homogeneous equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{7.3}$$

is called the **complementary equation**. We will see that solving the complementary equation is an important step in solving a nonhomogeneous differential equation.

Definition

A solution $y_p(x)$ of a differential equation that contains no arbitrary constants is called a **particular solution** to the equation.

Theorem 7.4: General Solution to a Nonhomogeneous Equation

Let $y_p(x)$ be any particular solution to the nonhomogeneous linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x).$$

Also, let $c_1y_1(x) + c_2y_2(x)$ denote the general solution to the complementary equation. Then, the general solution to the nonhomogeneous equation is given by

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x). \tag{7.4}$$

Proof

To prove $y(x)$ is the general solution, we must first show that it solves the differential equation and, second, that any solution to the differential equation can be written in that form. Substituting $y(x)$ into the differential equation, we have

$$\begin{aligned} a_2(x)y'' + a_1(x)y' + a_0(x)y &= a_2(x)\left(c_1y_1 + c_2y_2 + y_p\right)'' + a_1(x)\left(c_1y_1 + c_2y_2 + y_p\right)' \\ &\quad + a_0(x)\left(c_1y_1 + c_2y_2 + y_p\right) \\ &= \left[a_2(x)(c_1y_1 + c_2y_2)'' + a_1(x)(c_1y_1 + c_2y_2)' + a_0(x)(c_1y_1 + c_2y_2)\right] \\ &\quad + a_2(x)y_p'' + a_1(x)y_p' + a_0(x)y_p \\ &= 0 + r(x) \\ &= r(x). \end{aligned}$$

So $y(x)$ is a solution.

Now, let $z(x)$ be any solution to $a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$. Then

$$\begin{aligned} a_2(x)(z - y_p)'' + a_1(x)(z - y_p)' + a_0(x)(z - y_p) &= \left(a_2(x)z'' + a_1(x)z' + a_0(x)z\right) \\ &\quad -\left(a_2(x)y_p{}'' + a_1(x)y_p{}' + a_0(x)y_p\right) \\ &= r(x) - r(x) \\ &= 0, \end{aligned}$$

so $z(x) - y_p(x)$ is a solution to the complementary equation. But, $c_1 y_1(x) + c_2 y_2(x)$ is the general solution to the complementary equation, so there are constants c_1 and c_2 such that

$$z(x) - y_p(x) = c_1 y_1(x) + c_2 y_2(x).$$

Hence, we see that $z(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$.

□

Example 7.11

Verifying the General Solution

Given that $y_p(x) = x$ is a particular solution to the differential equation $y'' + y = x,$ write the general solution and check by verifying that the solution satisfies the equation.

Solution

The complementary equation is $y'' + y = 0,$ which has the general solution $c_1 \cos x + c_2 \sin x$. So, the general solution to the nonhomogeneous equation is

$$y(x) = c_1 \cos x + c_2 \sin x + x.$$

To verify that this is a solution, substitute it into the differential equation. We have

$$y'(x) = -c_1 \sin x + c_2 \cos x + 1 \text{ and } y''(x) = -c_1 \cos x - c_2 \sin x.$$

Then

$$\begin{aligned} y''(x) + y(x) &= -c_1 \cos x - c_2 \sin x + c_1 \cos x + c_2 \sin x + x \\ &= x. \end{aligned}$$

So, $y(x)$ is a solution to $y'' + y = x$.

7.10 Given that $y_p(x) = -2$ is a particular solution to $y'' - 3y' - 4y = 8,$ write the general solution and verify that the general solution satisfies the equation.

In the preceding section, we learned how to solve homogeneous equations with constant coefficients. Therefore, for nonhomogeneous equations of the form $ay'' + by' + cy = r(x),$ we already know how to solve the complementary equation, and the problem boils down to finding a particular solution for the nonhomogeneous equation. We now examine two techniques for this: the method of undetermined coefficients and the method of variation of parameters.

Undetermined Coefficients

The **method of undetermined coefficients** involves making educated guesses about the form of the particular solution based on the form of $r(x)$. When we take derivatives of polynomials, exponential functions, sines, and cosines, we get polynomials, exponential functions, sines, and cosines. So when $r(x)$ has one of these forms, it is possible that the solution to the nonhomogeneous differential equation might take that same form. Let's look at some examples to see how this works.

Example 7.12

Undetermined Coefficients When $r(x)$ Is a Polynomial

Find the general solution to $y'' + 4y' + 3y = 3x$.

Solution

The complementary equation is $y'' + 4y' + 3y = 0$, with general solution $c_1 e^{-x} + c_2 e^{-3x}$. Since $r(x) = 3x$, the particular solution might have the form $y_p(x) = Ax + B$. If this is the case, then we have $y_p{}'(x) = A$ and $y_p{}''(x) = 0$. For y_p to be a solution to the differential equation, we must find values for A and B such that

$$\begin{aligned} y'' + 4y' + 3y &= 3x \\ 0 + 4(A) + 3(Ax + B) &= 3x \\ 3Ax + (4A + 3B) &= 3x. \end{aligned}$$

Setting coefficients of like terms equal, we have

$$\begin{aligned} 3A &= 3 \\ 4A + 3B &= 0. \end{aligned}$$

Then, $A = 1$ and $B = -\frac{4}{3}$, so $y_p(x) = x - \frac{4}{3}$ and the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-3x} + x - \frac{4}{3}.$$

In **Example 7.12**, notice that even though $r(x)$ did not include a constant term, it was necessary for us to include the constant term in our guess. If we had assumed a solution of the form $y_p = Ax$ (with no constant term), we would not have been able to find a solution. (Verify this!) If the function $r(x)$ is a polynomial, our guess for the particular solution should be a polynomial of the same degree, and it must include all lower-order terms, regardless of whether they are present in $r(x)$.

Example 7.13

Undetermined Coefficients When $r(x)$ Is an Exponential

Find the general solution to $y'' - y' - 2y = 2e^{3x}$.

Solution

The complementary equation is $y'' - y' - 2y = 0$, with the general solution $c_1 e^{-x} + c_2 e^{2x}$. Since $r(x) = 2e^{3x}$, the particular solution might have the form $y_p(x) = Ae^{3x}$. Then, we have $y_p{}'(x) = 3Ae^{3x}$ and $y_p{}''(x) = 9Ae^{3x}$. For y_p to be a solution to the differential equation, we must find a value for A such that

$$\begin{aligned} y'' - y' - 2y &= 2e^{3x} \\ 9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} &= 2e^{3x} \\ 4Ae^{3x} &= 2e^{3x}. \end{aligned}$$

So, $4A = 2$ and $A = 1/2$. Then, $y_p(x) = \left(\frac{1}{2}\right)e^{3x}$, and the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{2}e^{3x}.$$

7.11 Find the general solution to $y'' - 4y' + 4y = 7\sin t - \cos t$.

In the previous checkpoint, $r(x)$ included both sine and cosine terms. However, even if $r(x)$ included a sine term only or a cosine term only, both terms must be present in the guess. The method of undetermined coefficients also works with products of polynomials, exponentials, sines, and cosines. Some of the key forms of $r(x)$ and the associated guesses for $y_p(x)$ are summarized in **Table 7.2**.

$r(x)$	Initial guess for $y_p(x)$
k (a constant)	A (a constant)
$ax+b$	$Ax+B$ (*Note*: The guess must include both terms even if $b=0.$)
ax^2+bx+c	Ax^2+Bx+C (*Note*: The guess must include all three terms even if b or c are zero.)
Higher-order polynomials	Polynomial of the same order as $r(x)$
$ae^{\lambda x}$	$Ae^{\lambda x}$
$a\cos\beta x+b\sin\beta x$	$A\cos\beta x+B\sin\beta x$ (*Note*: The guess must include both terms even if either $a=0$ or $b=0.$)
$ae^{\alpha x}\cos\beta x+be^{\alpha x}\sin\beta x$	$Ae^{\alpha x}\cos\beta x+Be^{\alpha x}\sin\beta x$
$(ax^2+bx+c)e^{\lambda x}$	$(Ax^2+Bx+C)e^{\lambda x}$
$(a_2x^2+a_1x+a_0)\cos\beta x$ $+(b_2x^2+b_1x+b_0)\sin\beta x$	$(A_2x^2+A_1x+A_0)\cos\beta x$ $+(B_2x^2+B_1x+B_0)\sin\beta x$
$(a_2x^2+a_1x+a_0)e^{\alpha x}\cos\beta x$ $+(b_2x^2+b_1x+b_0)e^{\alpha x}\sin\beta x$	$(A_2x^2+A_1x+A_0)e^{\alpha x}\cos\beta x$ $+(B_2x^2+B_1x+B_0)e^{\alpha x}\sin\beta x$

Table 7.2 Key Forms for the Method of Undetermined Coefficients

Keep in mind that there is a key pitfall to this method. Consider the differential equation $y''+5y'+6y=3e^{-2x}.$ Based on the form of $r(x),$ we guess a particular solution of the form $y_p(x)=Ae^{-2x}.$ But when we substitute this expression into the differential equation to find a value for $A,$ we run into a problem. We have

$$y_p{}'(x)=-2Ae^{-2x}$$

and

$$y_p{}''=4Ae^{-2x},$$

so we want

$$\begin{aligned} y'' + 5y' + 6y &= 3e^{-2x} \\ 4Ae^{-2x} + 5\left(-2Ae^{-2x}\right) + 6Ae^{-2x} &= 3e^{-2x} \\ 4Ae^{-2x} - 10Ae^{-2x} + 6Ae^{-2x} &= 3e^{-2x} \\ 0 &= 3e^{-2x}, \end{aligned}$$

which is not possible.

Looking closely, we see that, in this case, the general solution to the complementary equation is $c_1 e^{-2x} + c_2 e^{-3x}$. The exponential function in $r(x)$ is actually a solution to the complementary equation, so, as we just saw, all the terms on the left side of the equation cancel out. We can still use the method of undetermined coefficients in this case, but we have to alter our guess by multiplying it by x. Using the new guess, $y_p(x) = Axe^{-2x}$, we have

$$y_p{}'(x) = A\left(e^{-2x} - 2xe^{-2x}\right)$$

and

$$y_p{}''(x) = -4Ae^{-2x} + 4Axe^{-2x}.$$

Substitution gives

$$\begin{aligned} y'' + 5y' + 6y &= 3e^{-2x} \\ \left(-4Ae^{-2x} + 4Axe^{-2x}\right) + 5\left(Ae^{-2x} - 2Axe^{-2x}\right) + 6Axe^{-2x} &= 3e^{-2x} \\ -4Ae^{-2x} + 4Axe^{-2x} + 5Ae^{-2x} - 10Axe^{-2x} + 6Axe^{-2x} &= 3e^{-2x} \\ Ae^{-2x} &= 3e^{-2x}. \end{aligned}$$

So, $A = 3$ and $y_p(x) = 3xe^{-2x}$. This gives us the following general solution

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x} + 3xe^{-2x}.$$

Note that if xe^{-2x} were also a solution to the complementary equation, we would have to multiply by x again, and we would try $y_p(x) = Ax^2 e^{-2x}$.

Problem-Solving Strategy: Method of Undetermined Coefficients

1. Solve the complementary equation and write down the general solution.
2. Based on the form of $r(x)$, make an initial guess for $y_p(x)$.
3. Check whether any term in the guess for $y_p(x)$ is a solution to the complementary equation. If so, multiply the guess by x. Repeat this step until there are no terms in $y_p(x)$ that solve the complementary equation.
4. Substitute $y_p(x)$ into the differential equation and equate like terms to find values for the unknown coefficients in $y_p(x)$.
5. Add the general solution to the complementary equation and the particular solution you just found to obtain the general solution to the nonhomogeneous equation.

Example 7.14

Solving Nonhomogeneous Equations

Find the general solutions to the following differential equations.

a. $y'' - 9y = -6\cos 3x$

b. $x'' + 2x' + x = 4e^{-t}$

c. $y'' - 2y' + 5y = 10x^2 - 3x - 3$

d. $y'' - 3y' = -12t$

Solution

a. The complementary equation is $y'' - 9y = 0,$ which has the general solution $c_1 e^{3x} + c_2 e^{-3x}$ (step 1). Based on the form of $r(x) = -6\cos 3x,$ our initial guess for the particular solution is $y_p(x) = A\cos 3x + B\sin 3x$ (step 2). None of the terms in $y_p(x)$ solve the complementary equation, so this is a valid guess (step 3).
Now we want to find values for A and $B,$ so substitute y_p into the differential equation. We have

$$y_p{}'(x) = -3A\sin 3x + 3B\cos 3x \text{ and } y_p{}''(x) = -9A\cos 3x - 9B\sin 3x,$$

so we want to find values of A and B such that

$$\begin{aligned} y'' - 9y &= -6\cos 3x \\ -9A\cos 3x - 9B\sin 3x - 9(A\cos 3x + B\sin 3x) &= -6\cos 3x \\ -18A\cos 3x - 18B\sin 3x &= -6\cos 3x. \end{aligned}$$

Therefore,

$$\begin{aligned} -18A &= -6 \\ -18B &= 0. \end{aligned}$$

This gives $A = \frac{1}{3}$ and $B = 0,$ so $y_p(x) = \left(\frac{1}{3}\right)\cos 3x$ (step 4).
Putting everything together, we have the general solution

$$y(x) = c_1 e^{3x} + c_2 e^{-3x} + \frac{1}{3}\cos 3x.$$

b. The complementary equation is $x'' + 2x' + x = 0,$ which has the general solution $c_1 e^{-t} + c_2 te^{-t}$ (step 1). Based on the form $r(t) = 4e^{-t},$ our initial guess for the particular solution is $x_p(t) = Ae^{-t}$ (step 2). However, we see that this guess solves the complementary equation, so we must multiply by t, which gives a new guess: $x_p(t) = Ate^{-t}$ (step 3). Checking this new guess, we see that it, too, solves the complementary equation, so we must multiply by t again, which gives $x_p(t) = At^2 e^{-t}$ (step 3 again). Now, checking this guess, we see that $x_p(t)$ does not solve the complementary equation, so this is a valid guess (step 3 yet again).
We now want to find a value for $A,$ so we substitute x_p into the differential equation. We have

$$\begin{aligned} x_p(t) &= At^2 e^{-t}, \text{ so} \\ x_p{}'(t) &= 2Ate^{-t} - At^2 e^{-t} \end{aligned}$$

and $x_p''(t) = 2Ae^{-t} - 2Ate^{-t} - \left(2Ate^{-t} - At^2e^{-t}\right) = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$.

Substituting into the differential equation, we want to find a value of A so that

$$\begin{aligned} x'' + 2x' + x &= 4e^{-t} \\ 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t} + 2\left(2Ate^{-t} - At^2e^{-t}\right) + At^2e^{-t} &= 4e^{-t} \\ 2Ae^{-t} &= 4e^{-t}. \end{aligned}$$

This gives $A = 2$, so $x_p(t) = 2t^2e^{-t}$ (step 4). Putting everything together, we have the general solution

$$x(t) = c_1e^{-t} + c_2te^{-t} + 2t^2e^{-t}.$$

c. The complementary equation is $y'' - 2y' + 5y = 0$, which has the general solution $c_1e^x\cos 2x + c_2e^x\sin 2x$ (step 1). Based on the form $r(x) = 10x^2 - 3x - 3$, our initial guess for the particular solution is $y_p(x) = Ax^2 + Bx + C$ (step 2). None of the terms in $y_p(x)$ solve the complementary equation, so this is a valid guess (step 3). We now want to find values for A, B, and C, so we substitute y_p into the differential equation. We have $y_p'(x) = 2Ax + B$ and $y_p''(x) = 2A$, so we want to find values of A, B, and C such that

$$\begin{aligned} y'' - 2y' + 5y &= 10x^2 - 3x - 3 \\ 2A - 2(2Ax + B) + 5\left(Ax^2 + Bx + C\right) &= 10x^2 - 3x - 3 \\ 5Ax^2 + (5B - 4A)x + (5C - 2B + 2A) &= 10x^2 - 3x - 3. \end{aligned}$$

Therefore,

$$\begin{aligned} 5A &= 10 \\ 5B - 4A &= -3 \\ 5C - 2B + 2A &= -3. \end{aligned}$$

This gives $A = 2$, $B = 1$, and $C = -1$, so $y_p(x) = 2x^2 + x - 1$ (step 4). Putting everything together, we have the general solution

$$y(x) = c_1e^x\cos 2x + c_2e^x\sin 2x + 2x^2 + x - 1.$$

d. The complementary equation is $y'' - 3y' = 0$, which has the general solution $c_1e^{3t} + c_2$ (step 1). Based on the form $r(t) = -12t$, our initial guess for the particular solution is $y_p(t) = At + B$ (step 2). However, we see that the constant term in this guess solves the complementary equation, so we must multiply by t, which gives a new guess: $y_p(t) = At^2 + Bt$ (step 3). Checking this new guess, we see that none of the terms in $y_p(t)$ solve the complementary equation, so this is a valid guess (step 3 again). We now want to find values for A and B, so we substitute y_p into the differential equation. We have $y_p'(t) = 2At + B$ and $y_p''(t) = 2A$, so we want to find values of A and B such that

$$\begin{aligned} y'' - 3y' &= -12t \\ 2A - 3(2At + B) &= -12t \\ -6At + (2A - 3B) &= -12t. \end{aligned}$$

Therefore,

$$\begin{aligned} -6A &= -12 \\ 2A - 3B &= 0. \end{aligned}$$

This gives $A = 2$ and $B = 4/3$, so $y_p(t) = 2t^2 + (4/3)t$ (step 4). Putting everything together, we have the general solution

$$y(t) = c_1 e^{3t} + c_2 + 2t^2 + \frac{4}{3}t.$$

7.12 Find the general solution to the following differential equations.

a. $y'' - 5y' + 4y = 3e^x$

b. $y'' + y' - 6y = 52\cos 2t$

Variation of Parameters

Sometimes, $r(x)$ is not a combination of polynomials, exponentials, or sines and cosines. When this is the case, the method of undetermined coefficients does not work, and we have to use another approach to find a particular solution to the differential equation. We use an approach called the **method of variation of parameters**.

To simplify our calculations a little, we are going to divide the differential equation through by a, so we have a leading coefficient of 1. Then the differential equation has the form

$$y'' + py' + qy = r(x),$$

where p and q are constants.

If the general solution to the complementary equation is given by $c_1 y_1(x) + c_2 y_2(x)$, we are going to look for a particular solution of the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$. In this case, we use the two linearly independent solutions to the complementary equation to form our particular solution. However, we are assuming the coefficients are functions of x, rather than constants. We want to find functions $u(x)$ and $v(x)$ such that $y_p(x)$ satisfies the differential equation. We have

$$\begin{aligned} y_p &= uy_1 + vy_2 \\ y_p{}' &= u'y_1 + uy_1{}' + v'y_2 + vy_2{}' \\ y_p{}'' &= (u'y_1 + v'y_2)' + u'y_1{}' + uy_1{}'' + v'y_2{}' + vy_2{}''. \end{aligned}$$

Substituting into the differential equation, we obtain

$$\begin{aligned} y_p{}'' + py_p{}' + qy_p &= \left[(u'y_1 + v'y_2)' + u'y_1{}' + uy_1{}'' + v'y_2{}' + vy_2{}''\right] \\ &\quad + p\left[u'y_1 + uy_1{}' + v'y_2 + vy_2{}'\right] + q\left[uy_1 + vy_2\right] \\ &= u\left[y_1{}'' + py_1{}' + qy_1\right] + v\left[y_2{}'' + py_2{}' + qy_2\right] \\ &\quad + (u'y_1 + v'y_2)' + p(u'y_1 + v'y_2) + (u'y_1{}' + v'y_2{}'). \end{aligned}$$

Note that y_1 and y_2 are solutions to the complementary equation, so the first two terms are zero. Thus, we have

$$\left(u' y_1 + v' y_2\right)' + p\left(u' y_1 + v' y_2\right) + \left(u' y_1{}' + v' y_2{}'\right) = r(x).$$

If we simplify this equation by imposing the additional condition $u' y_1 + v' y_2 = 0,$ the first two terms are zero, and this reduces to $u' y_1{}' + v' y_2{}' = r(x).$ So, with this additional condition, we have a system of two equations in two unknowns:

$$\begin{aligned} u' y_1 + v' y_2 &= 0 \\ u' y_1{}' + v' y_2{}' &= r(x). \end{aligned}$$

Solving this system gives us u' and $v',$ which we can integrate to find u and v.

Then, $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ is a particular solution to the differential equation. Solving this system of equations is sometimes challenging, so let's take this opportunity to review Cramer's rule, which allows us to solve the system of equations using determinants.

Rule: Cramer's Rule

The system of equations

$$\begin{aligned} a_1 z_1 + b_1 z_2 &= r_1 \\ a_2 z_1 + b_2 z_2 &= r_2 \end{aligned}$$

has a unique solution if and only if the determinant of the coefficients is not zero. In this case, the solution is given by

$$z_1 = \frac{\begin{vmatrix} r_1 & b_1 \\ r_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad z_2 = \frac{\begin{vmatrix} a_1 & r_1 \\ a_2 & r_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Example 7.15

Using Cramer's Rule

Use Cramer's rule to solve the following system of equations.

$$\begin{aligned} x^2 z_1 + 2xz_2 &= 0 \\ z_1 - 3x^2 z_2 &= 2x \end{aligned}$$

Solution

We have

$$\begin{aligned} a_1(x) &= x^2 \\ a_2(x) &= 1 \\ b_1(x) &= 2x \\ b_2(x) &= -3x^2 \\ r_1(x) &= 0 \\ r_2(x) &= 2x. \end{aligned}$$

Then,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} x^2 & 2x \\ 1 & -3x^2 \end{vmatrix} = -3x^4 - 2x$$

and

$$\begin{vmatrix} r_1 & b_1 \\ r_2 & b_2 \end{vmatrix} = \begin{vmatrix} 0 & 2x \\ 2x & -3x^2 \end{vmatrix} = 0 - 4x^2 = -4x^2.$$

Thus,

$$z_1 = \frac{\begin{vmatrix} r_1 & b_1 \\ r_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{-4x^2}{-3x^4 - 2x} = \frac{4x}{3x^3 + 2}.$$

In addition,

$$\begin{vmatrix} a_1 & r_1 \\ a_2 & r_2 \end{vmatrix} = \begin{vmatrix} x^2 & 0 \\ 1 & 2x \end{vmatrix} = 2x^3 - 0 = 2x^3.$$

Thus,

$$z_2 = \frac{\begin{vmatrix} a_1 & r_1 \\ a_2 & r_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{2x^3}{-3x^4 - 2x} = \frac{-2x^2}{3x^3 + 2}.$$

7.13 Use Cramer’s rule to solve the following system of equations.

$$\begin{aligned} 2xz_1 - 3z_2 &= 0 \\ x^2 z_1 + 4xz_2 &= x + 1 \end{aligned}$$

Problem-Solving Strategy: Method of Variation of Parameters

1. Solve the complementary equation and write down the general solution
$$c_1 y_1(x) + c_2 y_2(x).$$
2. Use Cramer’s rule or another suitable technique to find functions $u'(x)$ and $v'(x)$ satisfying
$$\begin{aligned} u' y_1 + v' y_2 &= 0 \\ u' y_1{}' + v' y_2{}' &= r(x). \end{aligned}$$
3. Integrate u' and v' to find $u(x)$ and $v(x)$. Then, $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ is a particular solution to the equation.
4. Add the general solution to the complementary equation and the particular solution found in step 3 to obtain the general solution to the nonhomogeneous equation.

Example 7.16

Using the Method of Variation of Parameters

Find the general solution to the following differential equations.

a. $y'' - 2y' + y = \frac{e^t}{t^2}$

b. $y'' + y = 3\sin^2 x$

Solution

a. The complementary equation is $y'' - 2y' + y = 0$ with associated general solution $c_1 e^t + c_2 te^t$. Therefore, $y_1(t) = e^t$ and $y_2(t) = te^t$. Calculating the derivatives, we get $y_1{}'(t) = e^t$ and $y_2{}'(t) = e^t + te^t$ (step 1). Then, we want to find functions $u'(t)$ and $v'(t)$ so that

$$\begin{aligned} u' e^t + v' te^t &= 0 \\ u' e^t + v'\left(e^t + te^t\right) &= \frac{e^t}{t^2}. \end{aligned}$$

Applying Cramer's rule, we have

$$u' = \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{t^2} & e^t + te^t \end{vmatrix}}{\begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}} = \frac{0 - te^t\left(\frac{e^t}{t^2}\right)}{e^t\left(e^t + te^t\right) - e^t te^t} = \frac{-\frac{e^{2t}}{t}}{e^{2t}} = -\frac{1}{t}$$

and

$$v' = \frac{\begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{t^2} \end{vmatrix}}{\begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}} = \frac{e^t\left(\frac{e^t}{t^2}\right)}{e^{2t}} = \frac{1}{t^2} \text{ (step 2).}$$

Integrating, we get

$$\begin{aligned} u &= -\int \frac{1}{t} dt = -\ln|t| \\ v &= \int \frac{1}{t^2} dt = -\frac{1}{t} \text{ (step 3).} \end{aligned}$$

Then we have

$$\begin{aligned} y_p &= -e^t \ln|t| - \frac{1}{t} te^t \\ &= -e^t \ln|t| - e^t \text{ (step 4).} \end{aligned}$$

The e^t term is a solution to the complementary equation, so we don't need to carry that term into our general solution explicitly. The general solution is

$$y(t) = c_1 e^t + c_2 te^t - e^t \ln|t| \text{ (step 5).}$$

b. The complementary equation is $y'' + y = 0$ with associated general solution $c_1 \cos x + c_2 \sin x$. So, $y_1(x) = \cos x$ and $y_2(x) = \sin x$ (step 1). Then, we want to find functions $u'(x)$ and $v'(x)$ such that

$$\begin{aligned} u' \cos x + v' \sin x &= 0 \\ -u' \sin x + v' \cos x &= 3 \sin x. \end{aligned}$$

Applying Cramer's rule, we have

$$u' = \frac{\begin{vmatrix} 0 & \sin x \\ 3\sin^2 x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{0 - 3\sin^3 x}{\cos^2 x + \sin^2 x} = -3\sin^3 x$$

and

$$v' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & 3\sin^2 x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{3\sin^2 x \cos x}{1} = 3\sin^2 x \cos x \text{ (step 2).}$$

Integrating first to find u, we get

$$u = \int -3\sin^3 x dx = -3\left[-\tfrac{1}{3}\sin^2 x \cos x + \tfrac{2}{3}\int \sin x dx\right] = \sin^2 x \cos x + 2\cos x.$$

Now, we integrate to find v. Using substitution (with $w = \sin x$), we get

$$v = \int 3\sin^2 x \cos x dx = \int 3w^2 dw = w^3 = \sin^3 x.$$

Then,

$$\begin{aligned} y_p &= \left(\sin^2 x \cos x + 2\cos x\right)\cos x + \left(\sin^3 x\right)\sin x \\ &= \sin x \cos x + 2\cos x + \sin x \\ &= 2\cos x + \sin x\left(\cos^2 x + \sin^2 x\right) \qquad \text{(step 4).} \\ &= 2\cos x + \sin x \\ &= \cos x + 1 \end{aligned}$$

The general solution is

$$y(x) = c_1 \cos x + c_2 \sin x + 1 + \cos^2 x \text{ (step 5).}$$

7.14 Find the general solution to the following differential equations.

a. $y'' + y = \sec x$

b. $x'' - 2x' + x = \frac{e^t}{t}$

7.2 EXERCISES

Solve the following equations using the method of undetermined coefficients.

54. $2y'' - 5y' - 12y = 6$

55. $3y'' + y' - 4y = 8$

56. $y'' - 6y' + 5y = e^{-x}$

57. $y'' + 16y = e^{-2x}$

58. $y'' - 4y = x^2 + 1$

59. $y'' - 4y' + 4y = 8x^2 + 4x$

60. $y'' - 2y' - 3y = \sin 2x$

61. $y'' + 2y' + y = \sin x + \cos x$

62. $y'' + 9y = e^x \cos x$

63. $y'' + y = 3\sin 2x + x\cos 2x$

64. $y'' + 3y' - 28y = 10e^{4x}$

65. $y'' + 10y' + 25y = xe^{-5x} + 4$

In each of the following problems,

a. Write the form for the particular solution $y_p(x)$ for the method of undetermined coefficients.
b. **[T]** Use a computer algebra system to find a particular solution to the given equation.

66. $y'' - y' - y = x + e^{-x}$

67. $y'' - 3y = x^2 - 4x + 11$

68. $y'' - y' - 4y = e^x \cos 3x$

69. $2y'' - y' + y = \left(x^2 - 5x\right)e^{-x}$

70. $4y'' + 5y' - 2y = e^{2x} + x\sin x$

71. $y'' - y' - 2y = x^2 e^x \sin x$

Solve the differential equation using either the method of undetermined coefficients or the variation of parameters.

72. $y'' + 3y' - 4y = 2e^x$

73. $y'' + 2y' = e^{3x}$

74. $y'' + 6y' + 9y = e^{-x}$

75. $y'' + 2y' - 8y = 6e^{2x}$

Solve the differential equation using the method of variation of parameters.

76. $4y'' + y = 2\sin x$

77. $y'' - 9y = 8x$

78. $y'' + y = \sec x, \quad 0 < x < \pi/2$

79. $y'' + 4y = 3\csc 2x, \quad 0 < x < \pi/2$

Find the unique solution satisfying the differential equation and the initial conditions given, where $y_p(x)$ is the particular solution.

80. $y'' - 2y' + y = 12e^x, \quad y_p(x) = 6x^2 e^x, \quad y(0) = 6, \quad y'(0) = 0$

81. $y'' - 7y' = 4xe^{7x}, \quad y_p(x) = \frac{2}{7}x^2 e^{7x} - \frac{4}{49}xe^{7x}, \quad y(0) = -1, \quad y'(0) = 0$

82. $y'' + y = \cos x - 4\sin x, \quad y_p(x) = 2x\cos x + \frac{1}{2}x\sin x, \quad y(0) = 8, \quad y'(0) = -4$

83. $y'' - 5y' = e^{5x} + 8e^{-5x}, \quad y_p(x) = \frac{1}{5}xe^{5x} + \frac{4}{25}e^{-5x}, \quad y(0) = -2, \quad y'(0) = 0$

In each of the following problems, two linearly independent solutions—y_1 and y_2—are given that satisfy the corresponding homogeneous equation. Use the method of variation of parameters to find a particular solution to the given nonhomogeneous equation. Assume $x > 0$ in each exercise.

84. $x^2 y'' + 2xy' - 2y = 3x, \quad y_1(x) = x, \quad y_2(x) = x^{-2}$

85. $x^2 y'' - 2y = 10x^2 - 1, \quad y_1(x) = x^2, \quad y_2(x) = x^{-1}$

7.3 | Applications

Learning Objectives

7.3.1 Solve a second-order differential equation representing simple harmonic motion.
7.3.2 Solve a second-order differential equation representing damped simple harmonic motion.
7.3.3 Solve a second-order differential equation representing forced simple harmonic motion.
7.3.4 Solve a second-order differential equation representing charge and current in an RLC series circuit.

We saw in the chapter introduction that second-order linear differential equations are used to model many situations in physics and engineering. In this section, we look at how this works for systems of an object with mass attached to a vertical spring and an electric circuit containing a resistor, an inductor, and a capacitor connected in series. Models such as these can be used to approximate other more complicated situations; for example, bonds between atoms or molecules are often modeled as springs that vibrate, as described by these same differential equations.

Simple Harmonic Motion

Consider a mass suspended from a spring attached to a rigid support. (This is commonly called a spring-mass system.) Gravity is pulling the mass downward and the restoring force of the spring is pulling the mass upward. As shown in **Figure 7.2**, when these two forces are equal, the mass is said to be at the equilibrium position. If the mass is displaced from equilibrium, it oscillates up and down. This behavior can be modeled by a second-order constant-coefficient differential equation.

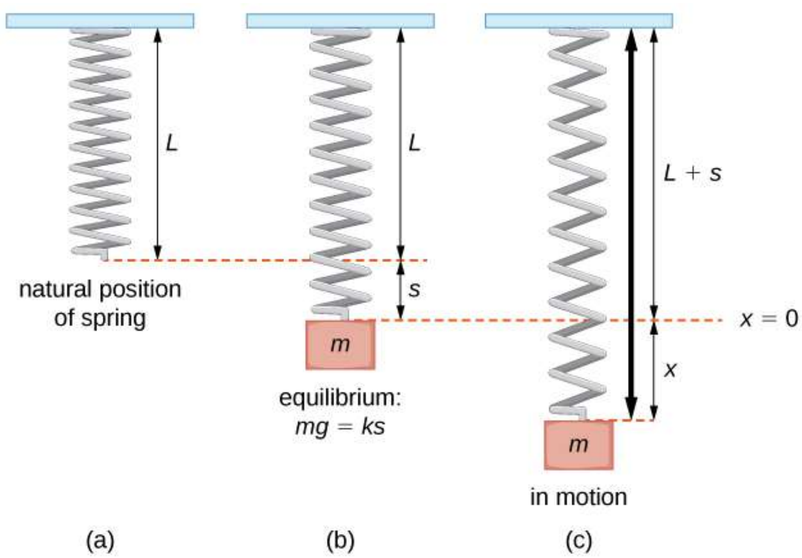


Figure 7.2 A spring in its natural position (a), at equilibrium with a mass *m* attached (b), and in oscillatory motion (c).

Let $x(t)$ denote the displacement of the mass from equilibrium. Note that for spring-mass systems of this type, it is customary to adopt the convention that down is positive. Thus, a positive displacement indicates the mass is *below* the equilibrium point, whereas a negative displacement indicates the mass is *above* equilibrium. Displacement is usually given in feet in the English system or meters in the metric system.

Consider the forces acting on the mass. The force of gravity is given by mg. In the English system, mass is in slugs and the acceleration resulting from gravity is in feet per second squared. The acceleration resulting from gravity is constant, so in the English system, $g = 32 \text{ ft/sec}^2$. Recall that 1 slug-foot/sec^2 is a pound, so the expression mg can be expressed in pounds. Metric system units are kilograms for mass and m/sec^2 for gravitational acceleration. In the metric system, we have $g = 9.8 \text{ m/sec}^2$.

According to Hooke's law, the restoring force of the spring is proportional to the displacement and acts in the opposite direction from the displacement, so the restoring force is given by $-k(s+x)$. The spring constant is given in pounds per foot in the English system and in newtons per meter in the metric system.

Now, by Newton's second law, the sum of the forces on the system (gravity plus the restoring force) is equal to mass times acceleration, so we have

$$\begin{aligned} mx'' &= -k(s+x)+mg \\ &= -ks-kx+mg. \end{aligned}$$

However, by the way we have defined our equilibrium position, $mg = ks$, the differential equation becomes

$$mx'' + kx = 0.$$

It is convenient to rearrange this equation and introduce a new variable, called the angular frequency, ω. Letting $\omega = \sqrt{k/m}$, we can write the equation as

$$x'' + \omega^2 x = 0. \tag{7.5}$$

This differential equation has the general solution

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t, \tag{7.6}$$

which gives the position of the mass at any point in time. The motion of the mass is called **simple harmonic motion**. The period of this motion (the time it takes to complete one oscillation) is $T = \frac{2\pi}{\omega}$ and the frequency is $f = \frac{1}{T} = \frac{\omega}{2\pi}$ (**Figure 7.3**).

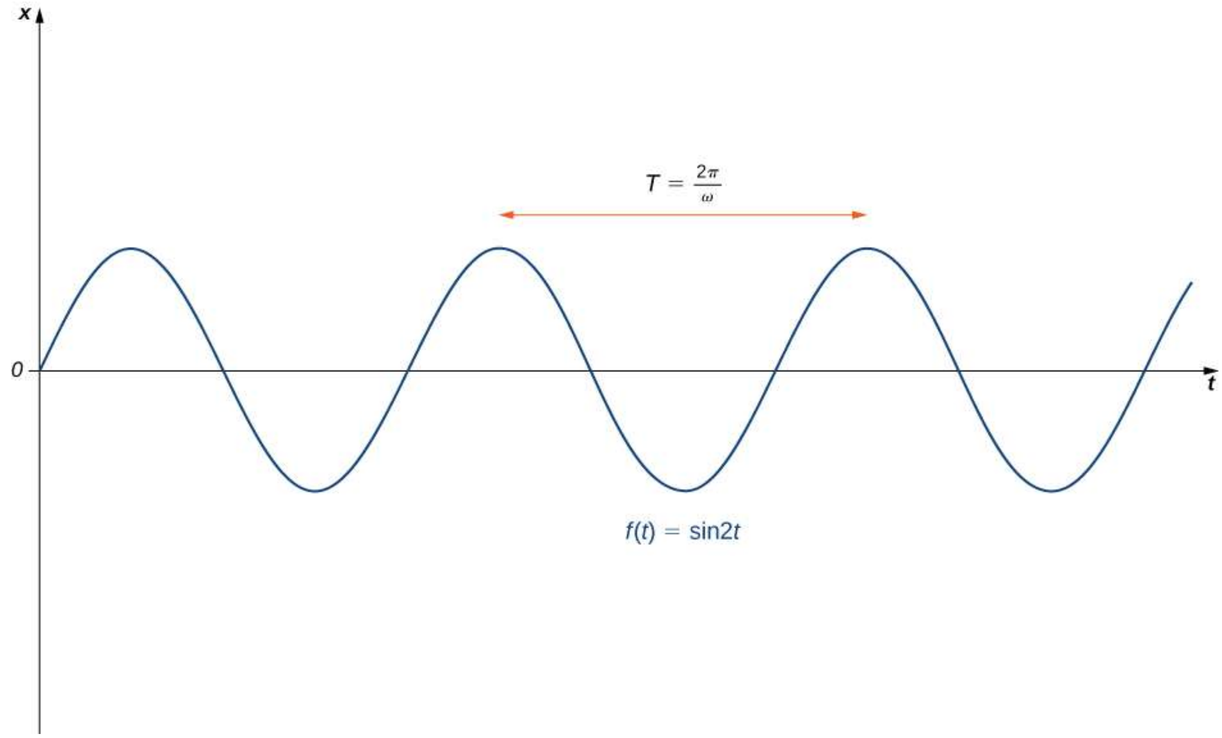


Figure 7.3 A graph of vertical displacement versus time for simple harmonic motion.

Example 7.17

Simple Harmonic Motion

Assume an object weighing 2 lb stretches a spring 6 in. Find the equation of motion if the spring is released from the equilibrium position with an upward velocity of 16 ft/sec. What is the period of the motion?

Solution

We first need to find the spring constant. We have

$$\begin{aligned} mg &= ks \\ 2 &= k\left(\frac{1}{2}\right) \\ k &= 4. \end{aligned}$$

We also know that weight W equals the product of mass m and the acceleration due to gravity g. In English units, the acceleration due to gravity is 32 ft/sec^2.

$$\begin{aligned} W &= mg \\ 2 &= m(32) \\ m &= \frac{1}{16} \end{aligned}$$

Thus, the differential equation representing this system is

$$\frac{1}{16}x'' + 4x = 0.$$

Multiplying through by 16, we get $x'' + 64x = 0,$ which can also be written in the form $x'' + (8^2)x = 0.$ This equation has the general solution

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t).$$

The mass was released from the equilibrium position, so $x(0) = 0,$ and it had an initial upward velocity of 16 ft/sec, so $x'(0) = -16.$ Applying these initial conditions to solve for c_1 and $c_2.$ gives

$$x(t) = -2 \sin 8t.$$

The period of this motion is $\frac{2\pi}{8} = \frac{\pi}{4}$ sec.

7.15 A 200-g mass stretches a spring 5 cm. Find the equation of motion of the mass if it is released from rest from a position 10 cm below the equilibrium position. What is the frequency of this motion?

Writing the general solution in the form $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ has some advantages. It is easy to see the link between the differential equation and the solution, and the period and frequency of motion are evident. This form of the function tells us very little about the amplitude of the motion, however. In some situations, we may prefer to write the solution in the form

$$x(t) = A \sin(\omega t + \phi). \tag{7.7}$$

Although the link to the differential equation is not as explicit in this case, the period and frequency of motion are still evident. Furthermore, the amplitude of the motion, A, is obvious in this form of the function. The constant ϕ is called a *phase shift* and has the effect of shifting the graph of the function to the left or right.

To convert the solution to this form, we want to find the values of A and ϕ such that

$$c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \sin(\omega t + \phi).$$

We first apply the trigonometric identity

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

to get

$$\begin{aligned} c_1 \cos(\omega t) + c_2 \sin(\omega t) &= A(\sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi) \\ &= A\sin\phi(\cos(\omega t)) + A\cos\phi(\sin(\omega t)). \end{aligned}$$

Thus,

$$c_1 = A\sin\phi \text{ and } c_2 = A\cos\phi.$$

If we square both of these equations and add them together, we get

$$\begin{aligned} c_1^2 + c_2^2 &= A^2 \sin\,\phi + A^2 \cos\,\phi \\ &= A^2\left(\sin^2\phi + \cos^2\phi\right) \\ &= A^2. \end{aligned}$$

Thus,

$$A = \sqrt{c_1^2 + c_2^2}.$$

Now, to find $\phi,$ go back to the equations for c_1 and $c_2,$ but this time, divide the first equation by the second equation to get

$$\begin{aligned} \frac{c_1}{c_2} &= \frac{A\sin\phi}{A\cos\phi} \\ &= \tan\phi. \end{aligned}$$

Then,

$$\tan\phi = \frac{c_1}{c_2}.$$

We summarize this finding in the following theorem.

Theorem 7.5: Solution to the Equation for Simple Harmonic Motion

The function $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ can be written in the form $x(t) = A\sin(\omega t + \phi),$ where $A = \sqrt{c_1^2 + c_2^2}$ and $\tan\phi = \frac{c_1}{c_2}.$

Note that when using the formula $\tan\phi = \frac{c_1}{c_2}$ to find $\phi,$ we must take care to ensure ϕ is in the right quadrant (**Figure 7.4**).

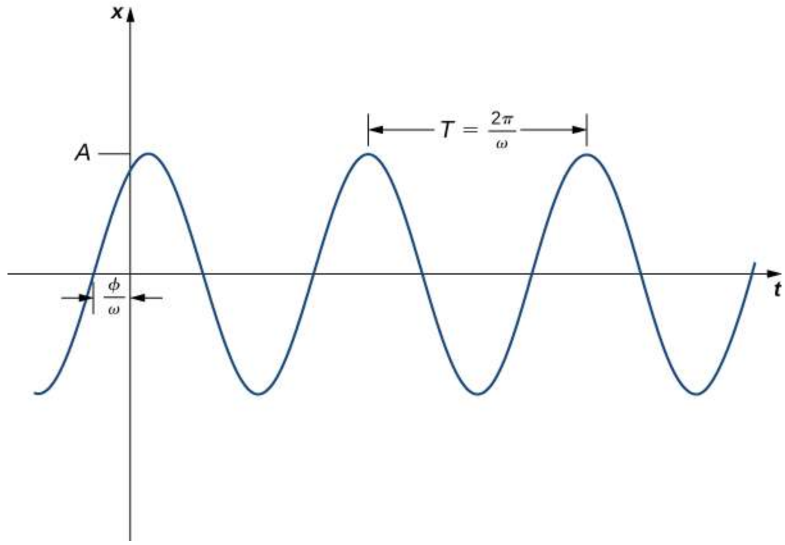


Figure 7.4 A graph of vertical displacement versus time for simple harmonic motion with a phase change.

Example 7.18

Expressing the Solution with a Phase Shift

Express the following functions in the form $A\sin(\omega t + \phi)$. What is the frequency of motion? The amplitude?

a. $x(t) = 2\cos(3t) + \sin(3t)$

b. $x(t) = 3\cos(2t) - 2\sin(2t)$

Solution

a. We have

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$$

and

$$\tan\phi = \frac{c_1}{c_2} = \frac{2}{1} = 2.$$

Note that both c_1 and c_2 are positive, so ϕ is in the first quadrant. Thus,

$$\phi \approx 1.107 \text{ rad},$$

so we have

$$x(t) = 2\cos(3t) + \sin(3t) = \sqrt{5}\sin(3t + 1.107).$$

The frequency is $\frac{\omega}{2\pi} = \frac{3}{2\pi} \approx 0.477$. The amplitude is $\sqrt{5}$.

b. We have

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

and

$$\tan\phi = \frac{c_1}{c_2} = \frac{3}{-2} = -\frac{3}{2}.$$

Note that c_1 is positive but c_2 is negative, so ϕ is in the fourth quadrant. Thus,

$$\phi \approx -0.983 \text{ rad},$$

so we have

$$\begin{aligned} x(t) &= 3\cos(2t) - 2\sin(2t) \\ &= \sqrt{13}\sin(2t - 0.983). \end{aligned}$$

The frequency is $\frac{\omega}{2\pi} = \frac{2}{2\pi} \approx 0.318.$ The amplitude is $\sqrt{13}.$

7.16 Express the function $x(t) = \cos(4t) + 4\sin(4t)$ in the form $A\sin(\omega t + \phi).$ What is the frequency of motion? The amplitude?

Damped Vibrations

With the model just described, the motion of the mass continues indefinitely. Clearly, this doesn't happen in the real world. In the real world, there is almost always some friction in the system, which causes the oscillations to die off slowly—an effect called *damping*. So now let's look at how to incorporate that damping force into our differential equation.

Physical spring-mass systems almost always have some damping as a result of friction, air resistance, or a physical damper, called a *dashpot* (a pneumatic cylinder; see **Figure 7.5**).

Figure 7.5 A dashpot is a pneumatic cylinder that dampens the motion of an oscillating system.

Because damping is primarily a friction force, we assume it is proportional to the velocity of the mass and acts in the opposite direction. So the damping force is given by $-bx'$ for some constant $b > 0.$ Again applying Newton's second law, the differential equation becomes

$$mx'' + bx' + kx = 0.$$

Then the associated characteristic equation is

$$m\lambda^2 + b\lambda + k = 0.$$

Applying the quadratic formula, we have

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

Just as in **Second-Order Linear Equations** we consider three cases, based on whether the characteristic equation has distinct real roots, a repeated real root, or complex conjugate roots.

Case 1: $b^2 > 4mk$

In this case, we say the system is *overdamped*. The general solution has the form

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

where both λ_1 and λ_2 are less than zero. Because the exponents are negative, the displacement decays to zero over time, usually quite quickly. Overdamped systems do not oscillate (no more than one change of direction), but simply move back toward the equilibrium position. **Figure 7.6** shows what typical critically damped behavior looks like.

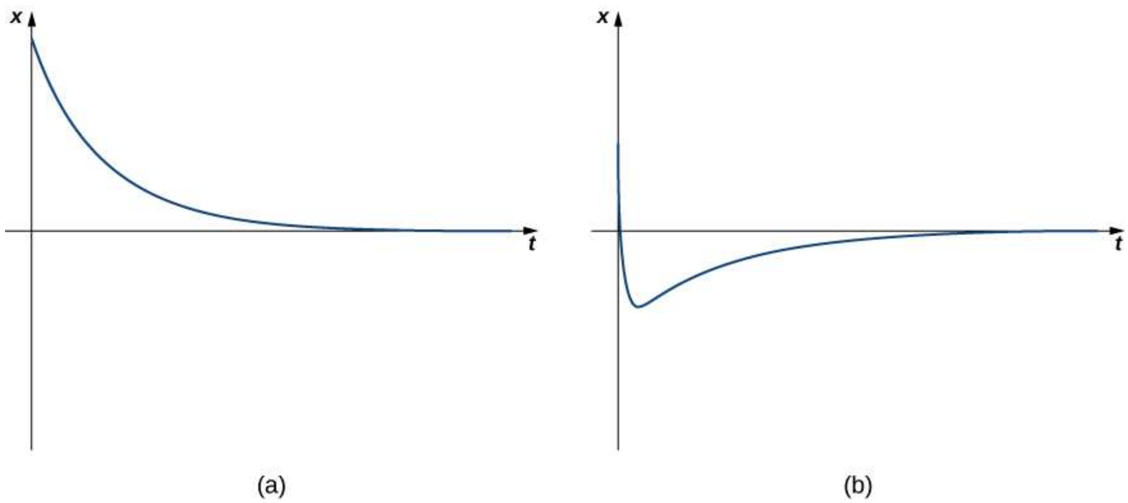


Figure 7.6 Behavior of an overdamped spring-mass system, with no change in direction (a) and only one change in direction (b).

Example 7.19

Overdamped Spring-Mass System

A 16-lb mass is attached to a 10-ft spring. When the mass comes to rest in the equilibrium position, the spring measures 15 ft 4 in. The system is immersed in a medium that imparts a damping force equal to $\frac{5}{2}$ times the instantaneous velocity of the mass. Find the equation of motion if the mass is pushed upward from the equilibrium position with an initial upward velocity of 5 ft/sec. What is the position of the mass after 10 sec? Its velocity?

Solution

The mass stretches the spring 5 ft 4 in., or $\frac{16}{3}$ ft. Thus, $16 = \left(\frac{16}{3}\right)k$, so $k = 3$. We also have $m = \frac{16}{32} = \frac{1}{2}$, so the differential equation is

$$\frac{1}{2}x'' + \frac{5}{2}x' + 3x = 0.$$

Multiplying through by 2 gives $x'' + 5x' + 6x = 0,$ which has the general solution

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

Applying the initial conditions, $x(0) = 0$ and $x'(0) = -5,$ we get

$$x(t) = -5e^{-2t} + 5e^{-3t}.$$

After 10 sec the mass is at position

$$x(10) = -5e^{-20} + 5e^{-30} \approx -1.0305 \times 10^{-8} \approx 0,$$

so it is, effectively, at the equilibrium position. We have $x'(t) = 10e^{-2t} - 15e^{-3t},$ so after 10 sec the mass is moving at a velocity of

$$x'(10) = 10e^{-20} - 15e^{-30} \approx 2.061 \times 10^{-8} \approx 0.$$

After only 10 sec, the mass is barely moving.

7.17 A 2-kg mass is attached to a spring with spring constant 24 N/m. The system is then immersed in a medium imparting a damping force equal to 16 times the instantaneous velocity of the mass. Find the equation of motion if it is released from rest at a point 40 cm below equilibrium.

Case 2: $b^2 = 4mk$

In this case, we say the system is *critically damped*. The general solution has the form

$$x(t) = c_1 e^{\lambda_1 t} + c_2 te^{\lambda_1 t},$$

where λ_1 is less than zero. The motion of a critically damped system is very similar to that of an overdamped system. It does not oscillate. However, with a critically damped system, if the damping is reduced even a little, oscillatory behavior results. From a practical perspective, physical systems are almost always either overdamped or underdamped (case 3, which we consider next). It is impossible to fine-tune the characteristics of a physical system so that b^2 and $4mk$ are exactly equal. **Figure 7.7** shows what typical critically damped behavior looks like.

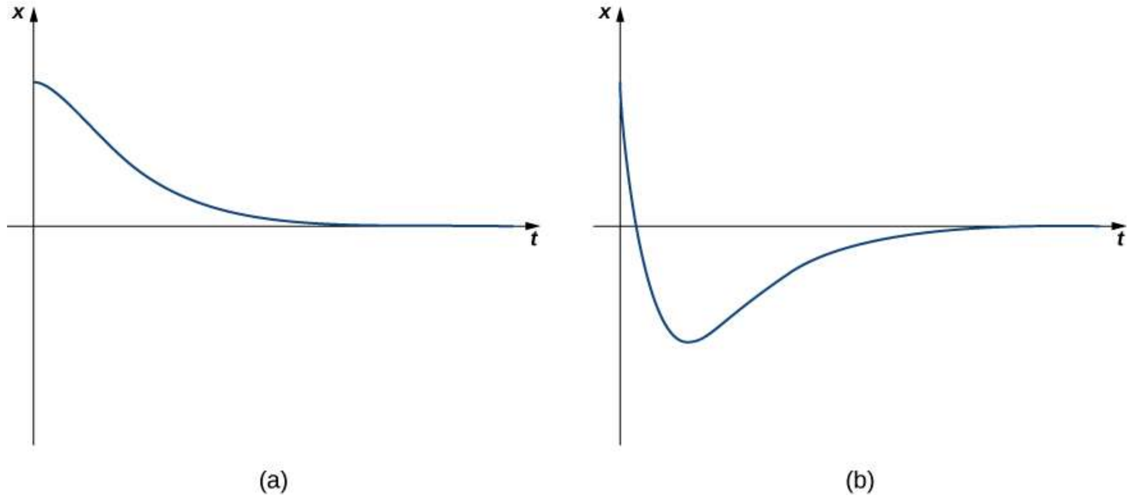


Figure 7.7 Behavior of a critically damped spring-mass system. The system graphed in part (a) has more damping than the system graphed in part (b).

Example 7.20

Critically Damped Spring-Mass System

A 1-kg mass stretches a spring 20 cm. The system is attached to a dashpot that imparts a damping force equal to 14 times the instantaneous velocity of the mass. Find the equation of motion if the mass is released from equilibrium with an upward velocity of 3 m/sec.

Solution

We have $mg = 1(9.8) = 0.2k,$ so $k = 49.$ Then, the differential equation is

$$x'' + 14x' + 49x = 0,$$

which has general solution

$$x(t) = c_1 e^{-7t} + c_2 te^{-7t}.$$

Applying the initial conditions $x(0) = 0$ and $x'(0) = -3$ gives

$$x(t) = -3te^{-7t}.$$

7.18 A 1-lb weight stretches a spring 6 in., and the system is attached to a dashpot that imparts a damping force equal to half the instantaneous velocity of the mass. Find the equation of motion if the mass is released from rest at a point 6 in. below equilibrium.

Case 3: $b^2 < 4mk$

In this case, we say the system is *underdamped*. The general solution has the form

$$x(t) = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right),$$

where α is less than zero. Underdamped systems do oscillate because of the sine and cosine terms in the solution. However, the exponential term dominates eventually, so the amplitude of the oscillations decreases over time. Figure 7.8 shows what typical underdamped behavior looks like.

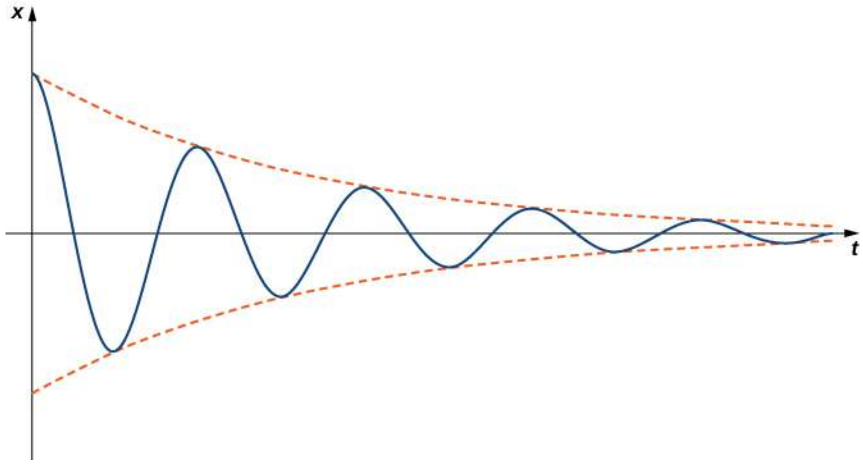


Figure 7.8 Behavior of an underdamped spring-mass system.

Note that for all damped systems, $\lim_{t \to \infty} x(t) = 0.$ The system always approaches the equilibrium position over time.

Example 7.21

Underdamped Spring-Mass System

A 16-lb weight stretches a spring 3.2 ft. Assume the damping force on the system is equal to the instantaneous velocity of the mass. Find the equation of motion if the mass is released from rest at a point 9 in. below equilibrium.

Solution

We have $k = \frac{16}{3.2} = 5$ and $m = \frac{16}{32} = \frac{1}{2},$ so the differential equation is

$$\frac{1}{2}x'' + x' + 5x = 0, \text{ or } x'' + 2x' + 10x = 0.$$

This equation has the general solution

$$x(t) = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t)).$$

Applying the initial conditions, $x(0) = \frac{3}{4}$ and $x'(0) = 0,$ we get

$$x(t) = e^{-t}\left(\frac{3}{4}\cos(3t) + \frac{1}{4}\sin(3t)\right).$$

7.19 A 1-kg mass stretches a spring 49 cm. The system is immersed in a medium that imparts a damping force equal to four times the instantaneous velocity of the mass. Find the equation of motion if the mass is released from rest at a point 24 cm above equilibrium.

Example 7.22

Chapter Opener: Modeling a Motorcycle Suspension System

Figure 7.9 (credit: modification of work by nSeika, Flickr)

For motocross riders, the suspension systems on their motorcycles are very important. The off-road courses on which they ride often include jumps, and losing control of the motorcycle when they land could cost them the race.

This suspension system can be modeled as a damped spring-mass system. We define our frame of reference with respect to the frame of the motorcycle. Assume the end of the shock absorber attached to the motorcycle frame is fixed. Then, the "mass" in our spring-mass system is the motorcycle wheel. We measure the position of the wheel with respect to the motorcycle frame. This may seem counterintuitive, since, in many cases, it is actually the motorcycle frame that moves, but this frame of reference preserves the development of the differential equation that was done earlier. As with earlier development, we define the downward direction to be positive.

When the motorcycle is lifted by its frame, the wheel hangs freely and the spring is uncompressed. This is the spring's natural position. When the motorcycle is placed on the ground and the rider mounts the motorcycle, the spring compresses and the system is in the equilibrium position (Figure 7.10).

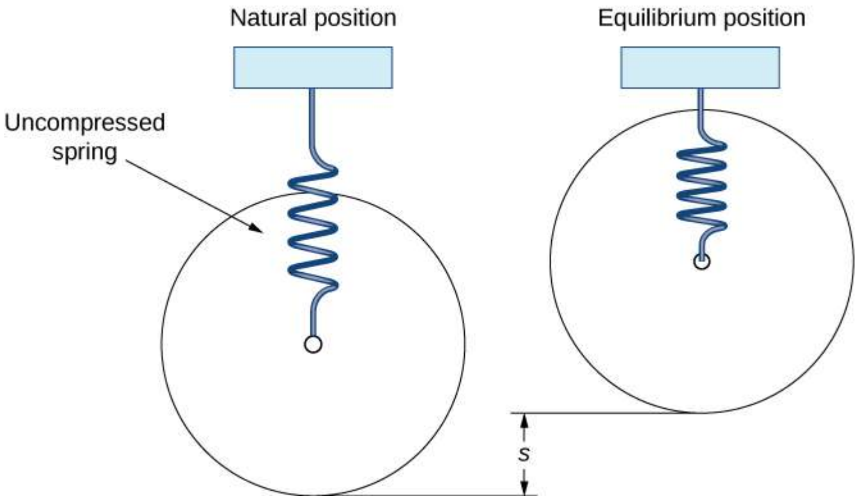


Figure 7.10 We can use a spring-mass system to model a motorcycle suspension.

This system can be modeled using the same differential equation we used before:

$$mx'' + bx' + kx = 0.$$

A motocross motorcycle weighs 204 lb, and we assume a rider weight of 180 lb. When the rider mounts the motorcycle, the suspension compresses 4 in., then comes to rest at equilibrium. The suspension system provides damping equal to 240 times the instantaneous vertical velocity of the motorcycle (and rider).

a. Set up the differential equation that models the behavior of the motorcycle suspension system.
b. We are interested in what happens when the motorcycle lands after taking a jump. Let time $t = 0$ denote the time when the motorcycle first contacts the ground. If the motorcycle hits the ground with a velocity of 10 ft/sec downward, find the equation of motion of the motorcycle after the jump.
c. Graph the equation of motion over the first second after the motorcycle hits the ground.

Solution

a. We have defined equilibrium to be the point where $mg = ks$, so we have

$$\begin{aligned} mg &= ks \\ 384 &= k\left(\frac{1}{3}\right) \\ k &= 1152. \end{aligned}$$

We also have

$$\begin{aligned} W &= mg \\ 384 &= m(32) \\ m &= 12. \end{aligned}$$

Therefore, the differential equation that models the behavior of the motorcycle suspension is

$$12x'' + 240x' + 1152x = 0.$$

Dividing through by 12, we get

$$x'' + 20x' + 96x = 0.$$

b. The differential equation found in part a. has the general solution

$$x(t) = c_1 e^{-8t} + c_2 e^{-12t}.$$

Now, to determine our initial conditions, we consider the position and velocity of the motorcycle wheel when the wheel first contacts the ground. Since the motorcycle was in the air prior to contacting the ground, the wheel was hanging freely and the spring was uncompressed. Therefore the wheel is 4 in. $\left(\frac{1}{3}\text{ ft}\right)$ below the equilibrium position (with respect to the motorcycle frame), and we have $x(0) = \frac{1}{3}$. According to the problem statement, the motorcycle has a velocity of 10 ft/sec downward when the motorcycle contacts the ground, so $x'(0) = 10$. Applying these initial conditions, we get $c_1 = \frac{7}{2}$ and $c_2 = -\left(\frac{19}{6}\right)$, so the equation of motion is

$$x(t) = \frac{7}{2}e^{-8t} - \frac{19}{6}e^{-12t}.$$

c. The graph is shown in **Figure 7.11**.

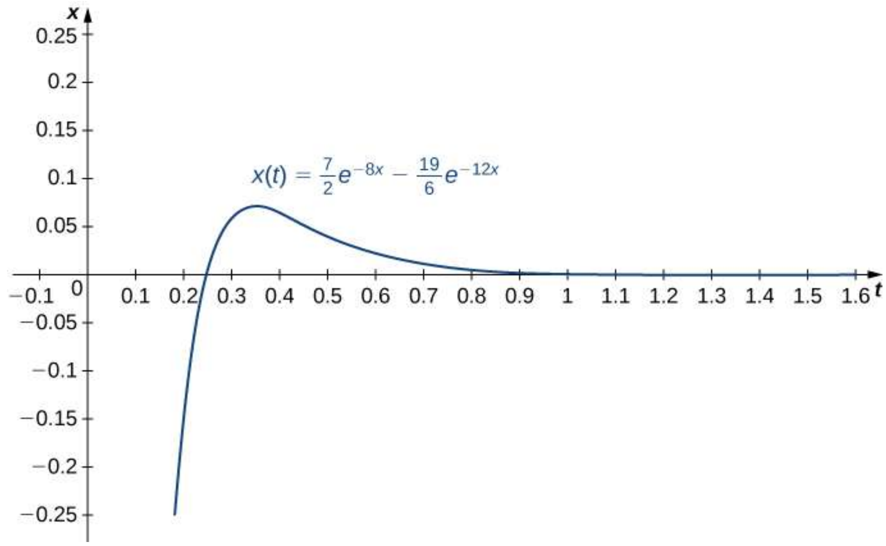


Figure 7.11 Graph of the equation of motion over a time of one second.

Student PROJECT

Landing Vehicle

NASA is planning a mission to Mars. To save money, engineers have decided to adapt one of the moon landing vehicles for the new mission. However, they are concerned about how the different gravitational forces will affect the suspension system that cushions the craft when it touches down. The acceleration resulting from gravity on the moon is 1.6 m/sec^2, whereas on Mars it is 3.7 m/sec^2.

The suspension system on the craft can be modeled as a damped spring-mass system. In this case, the spring is below the moon lander, so the spring is slightly compressed at equilibrium, as shown in **Figure 7.12**.

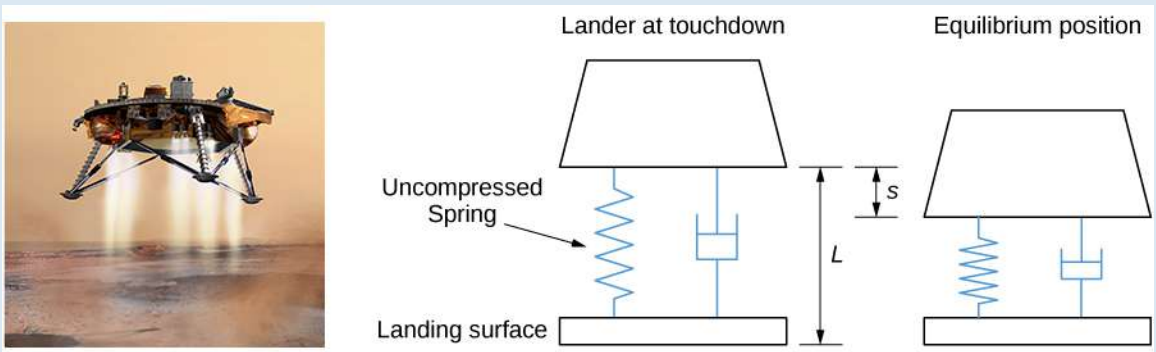


Figure 7.12 The landing craft suspension can be represented as a damped spring-mass system. (credit “lander”: NASA)

We retain the convention that down is positive. Despite the new orientation, an examination of the forces affecting the lander shows that the same differential equation can be used to model the position of the landing craft relative to equilibrium:

$$mx'' + bx' + kx = 0,$$

where m is the mass of the lander, b is the damping coefficient, and k is the spring constant.

1. The lander has a mass of 15,000 kg and the spring is 2 m long when uncompressed. The lander is designed to compress the spring 0.5 m to reach the equilibrium position under lunar gravity. The dashpot imparts a damping force equal to 48,000 times the instantaneous velocity of the lander. Set up the differential equation that models the motion of the lander when the craft lands on the moon.
2. Let time $t = 0$ denote the instant the lander touches down. The rate of descent of the lander can be controlled by the crew, so that it is descending at a rate of 2 m/sec when it touches down. Find the equation of motion of the lander on the moon.
3. If the lander is traveling too fast when it touches down, it could fully compress the spring and “bottom out.” Bottoming out could damage the landing craft and must be avoided at all costs. Graph the equation of motion found in part 2. If the spring is 0.5 m long when fully compressed, will the lander be in danger of bottoming out?
4. Assuming NASA engineers make no adjustments to the spring or the damper, how far does the lander compress the spring to reach the equilibrium position under Martian gravity?
5. If the lander crew uses the same procedures on Mars as on the moon, and keeps the rate of descent to 2 m/sec, will the lander bottom out when it lands on Mars?
6. What adjustments, if any, should the NASA engineers make to use the lander safely on Mars?

Forced Vibrations

The last case we consider is when an external force acts on the system. In the case of the motorcycle suspension system, for example, the bumps in the road act as an external force acting on the system. Another example is a spring hanging from a support; if the support is set in motion, that motion would be considered an external force on the system. We model these forced systems with the nonhomogeneous differential equation

$$mx'' + bx' + kx = f(t), \tag{7.8}$$

where the external force is represented by the $f(t)$ term. As we saw in **Nonhomogenous Linear Equations**, differential equations such as this have solutions of the form

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + x_p(t),$$

where $c_1 x_1(t) + c_2 x_2(t)$ is the general solution to the complementary equation and $x_p(t)$ is a particular solution to the nonhomogeneous equation. If the system is damped, $\lim\limits_{t \to \infty} c_1 x_1(t) + c_2 x_2(t) = 0.$ Since these terms do not affect the long-term behavior of the system, we call this part of the solution the *transient solution***.** The long-term behavior of the system is determined by $x_p(t),$ so we call this part of the solution the **steady-state solution**.

This **website (http://www.openstaxcollege.org/l/20_Oscillations)** shows a simulation of forced vibrations.

Example 7.23

Forced Vibrations

A mass of 1 slug stretches a spring 2 ft and comes to rest at equilibrium. The system is attached to a dashpot that imparts a damping force equal to eight times the instantaneous velocity of the mass. Find the equation of motion if an external force equal to $f(t) = 8\sin(4t)$ is applied to the system beginning at time $t = 0.$ What is the transient solution? What is the steady-state solution?

Solution

We have $mg = 1(32) = 2k,$ so $k = 16$ and the differential equation is

$$x'' + 8x' + 16x = 8\sin(4t).$$

The general solution to the complementary equation is

$$c_1 e^{-4t} + c_2 te^{-4t}.$$

Assuming a particular solution of the form $x_p(t) = A\cos(4t) + B\sin(4t)$ and using the method of undetermined coefficients, we find $x_p(t) = -\frac{1}{4}\cos(4t),$ so

$$x(t) = c_1 e^{-4t} + c_2 te^{-4t} - \frac{1}{4}\cos(4t).$$

At $t = 0,$ the mass is at rest in the equilibrium position, so $x(0) = x'(0) = 0.$ Applying these initial conditions to solve for c_1 and $c_2,$ we get

$$x(t) = \frac{1}{4}e^{-4t} + te^{-4t} - \frac{1}{4}\cos(4t).$$

The transient solution is $\frac{1}{4}e^{-4t} + te^{-4t}.$ The steady-state solution is $-\frac{1}{4}\cos(4t).$

7.20 A mass of 2 kg is attached to a spring with constant 32 N/m and comes to rest in the equilibrium position. Beginning at time $t = 0,$ an external force equal to $f(t) = 68e^{-2t}\cos(4t)$ is applied to the system. Find the equation of motion if there is no damping. What is the transient solution? What is the steady-state solution?

Student PROJECT

Resonance

Consider an undamped system exhibiting simple harmonic motion. In the real world, we never truly have an undamped system; –some damping always occurs. For theoretical purposes, however, we could imagine a spring-mass system contained in a vacuum chamber. With no air resistance, the mass would continue to move up and down indefinitely.

The frequency of the resulting motion, given by $f = \frac{1}{T} = \frac{\omega}{2\pi},$ is called the *natural frequency of the system*. If an external force acting on the system has a frequency close to the natural frequency of the system, a phenomenon called *resonance* results. The external force reinforces and amplifies the natural motion of the system.

1. Consider the differential equation $x'' + x = 0.$ Find the general solution. What is the natural frequency of the system?
2. Now suppose this system is subjected to an external force given by $f(t) = 5\cos t.$ Solve the initial-value problem $x'' + x = 5\cos t, \quad x(0) = 0, \quad x'(0) = 1.$
3. Graph the solution. What happens to the behavior of the system over time?
4. In the real world, there is always some damping. However, if the damping force is weak, and the external force is strong enough, real-world systems can still exhibit resonance. One of the most famous examples of resonance is the collapse of the Tacoma Narrows Bridge on November 7, 1940. The bridge had exhibited strange behavior ever since it was built. The roadway had a strange "bounce" to it. On the day it collapsed, a strong windstorm caused the roadway to twist and ripple violently. The bridge was unable to withstand these forces and it ultimately collapsed. Experts believe the windstorm exerted forces on the bridge that were very close to its natural frequency, and the resulting resonance ultimately shook the bridge apart.

   This **website (http://www.openstaxcollege.org/l/20_TacomaNarrow)** contains more information about the collapse of the Tacoma Narrows Bridge.

   During the short time the Tacoma Narrows Bridge stood, it became quite a tourist attraction. Several people were on site the day the bridge collapsed, and one of them caught the collapse on film. Watch the **video (http://www.openstaxcollege.org/l/20_TacomaNarro2)** to see the collapse.

5. Another real-world example of resonance is a singer shattering a crystal wineglass when she sings just the right note. When someone taps a crystal wineglass or wets a finger and runs it around the rim, a tone can be heard. That note is created by the wineglass vibrating at its natural frequency. If a singer then sings that same note at a high enough volume, the glass shatters as a result of resonance.

   The TV show *Mythbusters* aired an episode on this phenomenon. Visit this **website (http://www.openstaxcollege.org/l/20_glass)** to learn more about it. Adam Savage also described the experience. Watch this **video (http://www.openstaxcollege.org/l/20_glass2)** for his account.

The *RLC* Series Circuit

Consider an electrical circuit containing a resistor, an inductor, and a capacitor, as shown in **Figure 7.10**. Such a circuit is called an ***RLC* series circuit**. *RLC* circuits are used in many electronic systems, most notably as tuners in AM/FM radios. The tuning knob varies the capacitance of the capacitor, which in turn tunes the radio. Such circuits can be modeled by second-order, constant-coefficient differential equations.

Let $I(t)$ denote the current in the *RLC* circuit and $q(t)$ denote the charge on the capacitor. Furthermore, let *L* denote inductance in henrys (H), *R* denote resistance in ohms (Ω), and *C* denote capacitance in farads (F). Last, let $E(t)$ denote electric potential in volts (V).

Kirchhoff's voltage rule states that the sum of the voltage drops around any closed loop must be zero. So, we need to consider the voltage drops across the inductor (denoted E_L), the resistor (denoted E_R), and the capacitor (denoted E_C). Because the *RLC* circuit shown in **Figure 7.10** includes a voltage source, $E(t)$, which adds voltage to the circuit, we have $E_L + E_R + E_C = E(t)$.

We present the formulas below without further development. Those of you interested in the derivation of these formulas should consult a physics text. Using Faraday's law and Lenz's law, the voltage drop across an inductor can be shown to be proportional to the instantaneous rate of change of current, with proportionality constant *L*. Thus,

$$E_L = L\frac{dI}{dt}.$$

Next, according to Ohm's law, the voltage drop across a resistor is proportional to the current passing through the resistor, with proportionality constant *R*. Therefore,

$$E_R = RI.$$

Last, the voltage drop across a capacitor is proportional to the charge, *q*, on the capacitor, with proportionality constant $1/C$. Thus,

$$E_C = \frac{1}{C}q.$$

Adding these terms together, we get

$$L\frac{dI}{dt} + RI + \frac{1}{C}q = E(t).$$

Noting that $I = (dq)/(dt)$, this becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{7.9}$$

Mathematically, this system is analogous to the spring-mass systems we have been examining in this section.

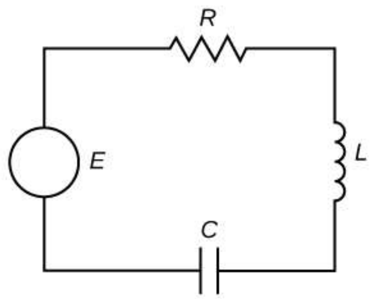


Figure 7.13 An *RLC* series circuit can be modeled by the same differential equation as a mass-spring system.

Example 7.24

The *RLC* Series Circuit

Find the charge on the capacitor in an *RLC* series circuit where $L = 5/3$ H, $R = 10\Omega$, $C = 1/30$ F, and $E(t) = 300$ V. Assume the initial charge on the capacitor is 0 C and the initial current is 9 A. What happens to the charge on the capacitor over time?

Solution

We have

$$\begin{aligned} L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q &= E(t) \\ \frac{5}{3}\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 30q &= 300 \\ \frac{d^2q}{dt^2} + 6\frac{dq}{dt} + 18q &= 180. \end{aligned}$$

The general solution to the complementary equation is

$$e^{-3t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right).$$

Assume a particular solution of the form $q_p = A,$ where A is a constant. Using the method of undetermined coefficients, we find $A = 10.$ So,

$$q(t) = e^{-3t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right) + 10.$$

Applying the initial conditions $q(0) = 0$ and $i(0) = \left((dq)/(dt)\right)(0) = 9,$ we find $c_1 = -10$ and $c_2 = -7.$ So the charge on the capacitor is

$$q(t) = -10e^{-3t}\cos(3t) - 7e^{-3t}\sin(3t) + 10.$$

Looking closely at this function, we see the first two terms will decay over time (as a result of the negative exponent in the exponential function). Therefore, the capacitor eventually approaches a steady-state charge of 10 C.

7.21 Find the charge on the capacitor in an *RLC* series circuit where $L = 1/5$ H, $R = 2/5\Omega,$ $C = 1/2$ F, and $E(t) = 50$ V. Assume the initial charge on the capacitor is 0 C and the initial current is 4 A.

7.3 EXERCISES

86. A mass weighing 4 lb stretches a spring 8 in. Find the equation of motion if the spring is released from the equilibrium position with a downward velocity of 12 ft/sec. What is the period and frequency of the motion?

87. A mass weighing 2 lb stretches a spring 2 ft. Find the equation of motion if the spring is released from 2 in. below the equilibrium position with an upward velocity of 8 ft/sec. What is the period and frequency of the motion?

88. A 100-g mass stretches a spring 0.1 m. Find the equation of motion of the mass if it is released from rest from a position 20 cm below the equilibrium position. What is the frequency of this motion?

89. A 400-g mass stretches a spring 5 cm. Find the equation of motion of the mass if it is released from rest from a position 15 cm below the equilibrium position. What is the frequency of this motion?

90. A block has a mass of 9 kg and is attached to a vertical spring with a spring constant of 0.25 N/m. The block is stretched 0.75 m below its equilibrium position and released.

a. Find the position function $x(t)$ of the block.
b. Find the period and frequency of the vibration.
c. Sketch a graph of $x(t)$.
d. At what time does the block first pass through the equilibrium position?

91. A block has a mass of 5 kg and is attached to a vertical spring with a spring constant of 20 N/m. The block is released from the equilibrium position with a downward velocity of 10 m/sec.

a. Find the position function $x(t)$ of the block.
b. Find the period and frequency of the vibration.
c. Sketch a graph of $x(t)$.
d. At what time does the block first pass through the equilibrium position?

92. A 1-kg mass is attached to a vertical spring with a spring constant of 21 N/m. The resistance in the spring-mass system is equal to 10 times the instantaneous velocity of the mass.

a. Find the equation of motion if the mass is released from a position 2 m below its equilibrium position with a downward velocity of 2 m/sec.
b. Graph the solution and determine whether the motion is overdamped, critically damped, or underdamped.

93. An 800-lb weight (25 slugs) is attached to a vertical spring with a spring constant of 226 lb/ft. The system is immersed in a medium that imparts a damping force equal to 10 times the instantaneous velocity of the mass.

a. Find the equation of motion if it is released from a position 20 ft below its equilibrium position with a downward velocity of 41 ft/sec.
b. Graph the solution and determine whether the motion is overdamped, critically damped, or underdamped.

94. A 9-kg mass is attached to a vertical spring with a spring constant of 16 N/m. The system is immersed in a medium that imparts a damping force equal to 24 times the instantaneous velocity of the mass.

a. Find the equation of motion if it is released from its equilibrium position with an upward velocity of 4 m/sec.
b. Graph the solution and determine whether the motion is overdamped, critically damped, or underdamped.

95. A 1-kg mass stretches a spring 6.25 cm. The resistance in the spring-mass system is equal to eight times the instantaneous velocity of the mass.

a. Find the equation of motion if the mass is released from a position 5 m below its equilibrium position with an upward velocity of 10 m/sec.
b. Determine whether the motion is overdamped, critically damped, or underdamped.

96. A 32-lb weight (1 slug) stretches a vertical spring 4 in. The resistance in the spring-mass system is equal to four times the instantaneous velocity of the mass.

a. Find the equation of motion if it is released from its equilibrium position with a downward velocity of 12 ft/sec.
b. Determine whether the motion is overdamped, critically damped, or underdamped.

97. A 64-lb weight is attached to a vertical spring with a spring constant of 4.625 lb/ft. The resistance in the spring-mass system is equal to the instantaneous velocity. The weight is set in motion from a position 1 ft below its equilibrium position with an upward velocity of 2 ft/sec. Is the mass above or below the equation position at the end of π sec? By what distance?

98. A mass that weighs 8 lb stretches a spring 6 inches. The system is acted on by an external force of $8\sin 8t$ lb. If the mass is pulled down 3 inches and then released, determine the position of the mass at any time.

99. A mass that weighs 6 lb stretches a spring 3 in. The system is acted on by an external force of $8\sin(4t)$ lb. If the mass is pulled down 1 inch and then released, determine the position of the mass at any time.

100. Find the charge on the capacitor in an *RLC* series circuit where $L = 40$ H, $R = 30\Omega$, $C = 1/200$ F, and $E(t) = 200$ V. Assume the initial charge on the capacitor is 7 C and the initial current is 0 A.

101. Find the charge on the capacitor in an *RLC* series circuit where $L = 2$ H, $R = 24\Omega$, $C = 0.005$ F, and $E(t) = 12\sin 10t$ V. Assume the initial charge on the capacitor is 0.001 C and the initial current is 0 A.

102. A series circuit consists of a device where $L = 1$ H, $R = 20\Omega$, $C = 0.002$ F, and $E(t) = 12$ V. If the initial charge and current are both zero, find the charge and current at time *t*.

103. A series circuit consists of a device where $L = \frac{1}{2}$ H, $R = 10\Omega$, $C = \frac{1}{50}$ F, and $E(t) = 250$ V. If the initial charge on the capacitor is 0 C and the initial current is 18 A, find the charge and current at time *t*.

7.4 | Series Solutions of Differential Equations

Learning Objectives

7.4.1 Use power series to solve first-order and second-order differential equations.

In **Introduction to Power Series (http://cnx.org/content/m53760/latest/)** , we studied how functions can be represented as power series, $y(x) = \sum_{n=0}^{\infty} a_n x^n$. We also saw that we can find series representations of the derivatives of such functions by differentiating the power series term by term. This gives $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. In some cases, these power series representations can be used to find solutions to differential equations.

Be aware that this subject is given only a very brief treatment in this text. Most introductory differential equations textbooks include an entire chapter on power series solutions. This text has only a single section on the topic, so several important issues are not addressed here, particularly issues related to existence of solutions. The examples and exercises in this section were chosen for which power solutions exist. However, it is not always the case that power solutions exist. Those of you interested in a more rigorous treatment of this topic should consult a differential equations text.

Problem-Solving Strategy: Finding Power Series Solutions to Differential Equations

1. Assume the differential equation has a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
2. Differentiate the power series term by term to get $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.
3. Substitute the power series expressions into the differential equation.
4. Re-index sums as necessary to combine terms and simplify the expression.
5. Equate coefficients of like powers of x to determine values for the coefficients a_n in the power series.
6. Substitute the coefficients back into the power series and write the solution.

Example 7.25

Series Solutions to Differential Equations

Find a power series solution for the following differential equations.

a. $y'' - y = 0$

b. $(x^2 - 1)y'' + 6xy' + 4y = -4$

Solution

a. Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$ (step 1). Then, $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ (step 2). We want to find values for the coefficients a_n such that

$$\begin{aligned} y'' - y &= 0 \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n &= 0 \text{ (step 3).} \end{aligned}$$

We want the indices on our sums to match so that we can express them using a single summation. That is, we want to rewrite the first summation so that it starts with $n = 0$.

To re-index the first term, replace *n* with $n + 2$ inside the sum, and change the lower summation limit to $n = 0$. We get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n\right] x^n &= 0 \text{ (step 4).} \end{aligned}$$

Because power series expansions of functions are unique, this equation can be true only if the coefficients of each power of *x* are zero. So we have

$$(n+2)(n+1) a_{n+2} - a_n = 0 \text{ for } n = 0, 1, 2, \ldots.$$

This recurrence relationship allows us to express each coefficient a_n in terms of the coefficient two terms earlier. This yields one expression for even values of *n* and another expression for odd values of *n*. Looking first at the equations involving even values of *n*, we see that

$$\begin{aligned} a_2 &= \frac{a_0}{2} \\ a_4 &= \frac{a_2}{4 \cdot 3} = \frac{a_0}{4!} \\ a_6 &= \frac{a_4}{6 \cdot 5} = \frac{a_0}{6!} \\ &\vdots. \end{aligned}$$

Thus, in general, when *n* is even, $a_n = \frac{a_0}{n!}$ (step 5).

For the equations involving odd values of *n*, we see that

$$\begin{aligned} a_3 &= \frac{a_1}{3 \cdot 2} = \frac{a_1}{3!} \\ a_5 &= \frac{a_3}{5 \cdot 4} = \frac{a_1}{5!} \\ a_7 &= \frac{a_5}{7 \cdot 6} = \frac{a_1}{7!} \\ &\vdots. \end{aligned}$$

Therefore, in general, when *n* is odd, $a_n = \frac{a_1}{n!}$ (step 5 continued).

Putting this together, we have

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots. \end{aligned}$$

Re-indexing the sums to account for the even and odd values of *n* separately, we obtain

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{1}{(2k)!}x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}x^{2k+1} \text{ (step 6).}$$

Analysis for part a.

As expected for a second-order differential equation, this solution depends on two arbitrary constants. However, note that our differential equation is a constant-coefficient differential equation, yet the power series solution does not appear to have the familiar form (containing exponential functions) that we are used to seeing. Furthermore, since $y(x) = c_1 e^x + c_2 e^{-x}$ is the general solution to this equation, we must be able to write any solution in this form, and it is not clear whether the power series solution we just found can, in fact, be written in that form.

Fortunately, after writing the power series representations of e^x and e^{-x}, and doing some algebra, we find that if we choose

$$c_0 = \frac{(a_0 + a_1)}{2}, \quad c_1 = \frac{(a_0 - a_1)}{2},$$

we then have $a_0 = c_0 + c_1$ and $a_1 = c_0 - c_1$, and

$$\begin{aligned} y(x) &= a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots \\ &= (c_0 + c_1) + (c_0 - c_1)x + \frac{(c_0 + c_1)}{2}x^2 + \frac{(c_0 - c_1)}{3!}x^3 + \frac{(c_0 + c_1)}{4!}x^4 + \frac{(c_0 - c_1)}{5!}x^5 + \cdots \\ &= c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} + c_1 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\ &= c_0 e^x + c_1 e^{-x}. \end{aligned}$$

So we have, in fact, found the same general solution. Note that this choice of c_1 and c_2 is not obvious. This is a case when we know what the answer should be, and have essentially "reverse-engineered" our choice of coefficients.

b. Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$ (step 1). Then, $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ (step 2). We want to find values for the coefficients a_n such that

$$\begin{aligned} (x^2-1)y'' + 6xy' + 4y &= -4 \\ \left(x^2-1\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 6x\sum_{n=1}^{\infty} na_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n &= -4 \\ x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 6x\sum_{n=1}^{\infty} na_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n &= -4. \end{aligned}$$

Taking the external factors inside the summations, we get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 6na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = -4 \text{ (step 3).}$$

Now, in the first summation, we see that when $n = 0$ or $n = 1$, the term evaluates to zero, so we can add these terms back into our sum to get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n = \sum_{n=0}^{\infty} n(n-1)a_n x^n.$$

Similarly, in the third term, we see that when $n = 0$, the expression evaluates to zero, so we can add that term back in as well. We have

$$\sum_{n=1}^{\infty} 6na_n x^n = \sum_{n=0}^{\infty} 6na_n x^n.$$

Then, we need only shift the indices in our second term. We get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Thus, we have

$$\begin{aligned} \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 6na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n &= -4 \text{ (step 4).} \\ \sum_{n=0}^{\infty} \left[n(n-1)a_n - (n+2)(n+1)a_{n+2} + 6na_n + 4a_n\right]x^n &= -4 \\ \sum_{n=0}^{\infty} \left[(n^2-n)a_n + 6na_n + 4a_n - (n+2)(n+1)a_{n+2}\right]x^n &= -4 \\ \sum_{n=0}^{\infty} \left[n^2 a_n + 5na_n + 4a_n - (n+2)(n+1)a_{n+2}\right]x^n &= -4 \\ \sum_{n=0}^{\infty} \left[(n^2+5n+4)a_n - (n+2)(n+1)a_{n+2}\right]x^n &= -4 \\ \sum_{n=0}^{\infty} \left[(n+4)(n+1)a_n - (n+2)(n+1)a_{n+2}\right]x^n &= -4 \end{aligned}$$

Looking at the coefficients of each power of x, we see that the constant term must be equal to -4, and the coefficients of all other powers of x must be zero. Then, looking first at the constant term,

$$\begin{aligned} 4a_0 - 2a_2 &= -4 \\ a_2 &= 2a_0 + 2 \text{ (step 3).} \end{aligned}$$

For $n \geq 1,$ we have

$$\begin{aligned} (n+4)(n+1)a_n - (n+2)(n+1)a_{n+2} &= 0 \\ (n+1)\left[(n+4)a_n - (n+2)a_{n+2}\right] &= 0. \end{aligned}$$

Since $n \geq 1,$ $n + 1 \neq 0,$ we see that

$$(n+4)a_n - (n+2)a_{n+2} = 0$$

and thus

$$a_{n+2} = \frac{n+4}{n+2}a_n.$$

For even values of n, we have

$$\begin{aligned} a_4 &= \frac{6}{4}\left(2a_0 + 2\right) = 3a_0 + 3 \\ a_6 &= \frac{8}{6}\left(3a_0 + 3\right) = 4a_0 + 4 \\ &\vdots. \end{aligned}$$

In general, $a_{2k} = (k+1)\left(a_0 + 1\right)$ (step 5).

For odd values of n, we have

$$\begin{aligned} a_3 &= \frac{5}{3}a_1 \\ a_5 &= \frac{7}{5}a_3 = \frac{7}{3}a_1 \\ a_7 &= \frac{9}{7}a_5 = \frac{9}{3}a_1 = 3a_1 \\ &\vdots. \end{aligned}$$

In general, $a_{2k+1} = \frac{2k+3}{3}a_1$ (step 5 continued).

Putting this together, we have

$$y(x) = \sum_{k=0}^{\infty}(k+1)(a_0+1)x^{2k} + \sum_{k=0}^{\infty}\left(\frac{2k+3}{3}\right)a_1 x^{2k+1} \text{ (step 6).}$$

7.22 Find a power series solution for the following differential equations.

a. $y' + 2xy = 0$

b. $(x+1)y' = 3y$

We close this section with a brief introduction to Bessel functions. Complete treatment of Bessel functions is well beyond the scope of this course, but we get a little taste of the topic here so we can see how series solutions to differential equations are used in real-world applications. The Bessel equation of order n is given by

$$x^2 y'' + xy' + (x^2 - n^2)y = 0.$$

This equation arises in many physical applications, particularly those involving cylindrical coordinates, such as the vibration of a circular drum head and transient heating or cooling of a cylinder. In the next example, we find a power series solution to the Bessel equation of order 0.

Example 7.26

Power Series Solution to the Bessel Equation

Find a power series solution to the Bessel equation of order 0 and graph the solution.

Solution

The Bessel equation of order 0 is given by

$$x^2 y'' + xy' + x^2 y = 0.$$

We assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Substituting this into the differential equation, we get

$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$	Substitution.
$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$	Bring external factors within sums.
$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$	Re-index third sum.
$\sum_{n=2}^{\infty} n(n-1) a_n x^n + a_1 x + \sum_{n=2}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$	Separate $n = 1$ term from second sum.
$a_1 x + \sum_{n=2}^{\infty} [n(n-1) a_n + n a_n + a_{n-2}] x^n = 0$	Collect summation terms.
$a_1 x + \sum_{n=2}^{\infty} [(n^2 - n) a_n + n a_n + a_{n-2}] x^n = 0$	Multiply through in fir t term.
$a_1 x + \sum_{n=2}^{\infty} [n^2 a_n + a_{n-2}] x^n = 0.$	Simplify.

Then, $a_1 = 0,$ and for $n \geq 2,$

$$\begin{aligned} n^2 a_n + a_{n-2} &= 0 \\ a_n &= -\frac{1}{n^2} a_{n-2}. \end{aligned}$$

Because $a_1 = 0,$ all odd terms are zero. Then, for even values of n, we have

$$\begin{aligned} a_2 &= -\frac{1}{2^2}a_0 \\ a_4 &= -\frac{1}{4^2}a_2 = \frac{1}{4^2 \cdot 2^2}a_0. \\ a_6 &= -\frac{1}{6^2}a_4 = -\frac{1}{6^2 \cdot 4^2 \cdot 2^2}a_0 \end{aligned}$$

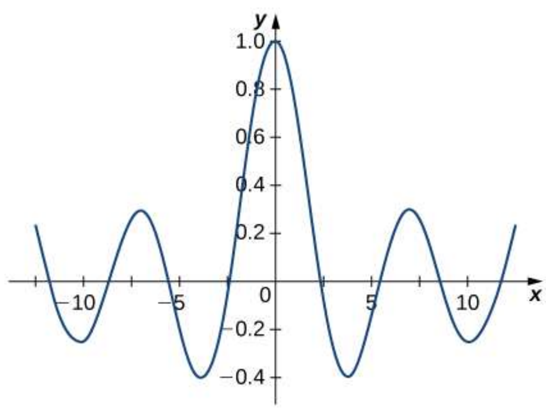
In general,

$$a_{2k} = \frac{(-1)^k}{(2)^{2k}(k!)^2}a_0.$$

Thus, we have

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2)^{2k}(k!)^2}x^{2k}. \tag{7.10}$$

The graph appears below.



7.23 Verify that the expression found in **Example 7.26** is a solution to the Bessel equation of order 0.

7.4 EXERCISES

Find a power series solution for the following differential equations.

104. $y'' + 6y' = 0$

105. $5y'' + y' = 0$

106. $y'' + 25y = 0$

107. $y'' - y = 0$

108. $2y' + y = 0$

109. $y' - 2xy = 0$

110. $(x - 7)y' + 2y = 0$

111. $y'' - xy' - y = 0$

112. $\left(1 + x^2\right)y'' - 4xy' + 6y = 0$

113. $x^2 y'' - xy' - 3y = 0$

114. $y'' - 8y' = 0, \quad y(0) = -2, \quad y'(0) = 10$

115. $y'' - 2xy = 0, \quad y(0) = 1, \quad y'(0) = -3$

116. The differential equation $x^2 y'' + xy' + \left(x^2 - 1\right)y = 0$ is a Bessel equation of order 1. Use a power series of the form $y = \sum_{n=0}^{\infty} a_n x^n$ to find the solution.

CHAPTER 7 REVIEW

KEY TERMS

boundary conditions the conditions that give the state of a system at different times, such as the position of a spring-mass system at two different times

boundary-value problem a differential equation with associated boundary conditions

characteristic equation the equation $a\lambda^2 + b\lambda + c = 0$ for the differential equation $ay'' + by' + cy = 0$

complementary equation for the nonhomogeneous linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x),$$

the associated homogeneous equation, called the *complementary equation*, is

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

homogeneous linear equation a second-order differential equation that can be written in the form $a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x),$ but $r(x) = 0$ for every value of x

linearly dependent a set of functions $f_1(x),\ f_2(x),\ldots,f_n(x)$ for which there are constants $c_1,\ c_2,\ldots c_n,$ not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for all x in the interval of interest

linearly independent a set of functions $f_1(x),\ f_2(x),\ldots,f_n(x)$ for which there are no constants $c_1,\ c_2,\ldots c_n,$ such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for all x in the interval of interest

method of undetermined coefficients a method that involves making a guess about the form of the particular solution, then solving for the coefficients in the guess

method of variation of parameters a method that involves looking for particular solutions in the form $y_p(x) = u(x)y_1(x) + v(x)y_2(x),$ where y_1 and y_2 are linearly independent solutions to the complementary equations, and then solving a system of equations to find $u(x)$ and $v(x)$

nonhomogeneous linear equation a second-order differential equation that can be written in the form $a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x),$ but $r(x) \neq 0$ for some value of x

particular solution a solution $y_p(x)$ of a differential equation that contains no arbitrary constants

***RLC* series circuit** a complete electrical path consisting of a resistor, an inductor, and a capacitor; a second-order, constant-coefficient differential equation can be used to model the charge on the capacitor in an *RLC* series circuit

simple harmonic motion motion described by the equation $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t),$ as exhibited by an undamped spring-mass system in which the mass continues to oscillate indefinitely

steady-state solution a solution to a nonhomogeneous differential equation related to the forcing function; in the long term, the solution approaches the steady-state solution

KEY EQUATIONS

- **Linear second-order differential equation**
  $a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$

- **Second-order equation with constant coefficients**
  $ay'' + by' + cy = 0$

- **Complementary equation**
  $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

- **General solution to a nonhomogeneous linear differential equation**
  $y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- **Equation of simple harmonic motion**
  $x'' + \omega^2 x = 0$
- **Solution for simple harmonic motion**
  $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$
- **Alternative form of solution for SHM**
  $x(t) = A \sin(\omega t + \phi)$
- **Forced harmonic motion**
  $mx'' + bx' + kx = f(t)$
- **Charge in a *RLC* series circuit**
  $L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

KEY CONCEPTS

7.1 Second-Order Linear Equations

- Second-order differential equations can be classified as linear or nonlinear, homogeneous or nonhomogeneous.
- To find a general solution for a homogeneous second-order differential equation, we must find two linearly independent solutions. If $y_1(x)$ and $y_2(x)$ are linearly independent solutions to a second-order, linear, homogeneous differential equation, then the general solution is given by
$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$
- To solve homogeneous second-order differential equations with constant coefficients, find the roots of the characteristic equation. The form of the general solution varies depending on whether the characteristic equation has distinct, real roots; a single, repeated real root; or complex conjugate roots.
- Initial conditions or boundary conditions can then be used to find the specific solution to a differential equation that satisfies those conditions, except when there is no solution or infinitely many solutions.

7.2 Nonhomogeneous Linear Equations

- To solve a nonhomogeneous linear second-order differential equation, first find the general solution to the complementary equation, then find a particular solution to the nonhomogeneous equation.
- Let $y_p(x)$ be any particular solution to the nonhomogeneous linear differential equation
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x),$$
and let $c_1 y_1(x) + c_2 y_2(x)$ denote the general solution to the complementary equation. Then, the general solution to the nonhomogeneous equation is given by
$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$
- When $r(x)$ is a combination of polynomials, exponential functions, sines, and cosines, use the method of undetermined coefficients to find the particular solution. To use this method, assume a solution in the same form as $r(x)$, multiplying by x as necessary until the assumed solution is linearly independent of the general solution to the complementary equation. Then, substitute the assumed solution into the differential equation to find values for the coefficients.
- When $r(x)$ is *not* a combination of polynomials, exponential functions, or sines and cosines, use the method of variation of parameters to find the particular solution. This method involves using Cramer's rule or another suitable

technique to find functions $u'(x)$ and $v'(x)$ satisfying

$$\begin{aligned} u' y_1 + v' y_2 &= 0 \\ u' y_1{}' + v' y_2{}' &= r(x). \end{aligned}$$

Then, $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ is a particular solution to the differential equation.

7.3 Applications

- Second-order constant-coefficient differential equations can be used to model spring-mass systems.
- An examination of the forces on a spring-mass system results in a differential equation of the form

  $$mx'' + bx' + kx = f(t),$$

  where m represents the mass, b is the coefficient of the damping force, k is the spring constant, and $f(t)$ represents any net external forces on the system.
- If $b = 0,$ there is no damping force acting on the system, and simple harmonic motion results. If $b \neq 0,$ the behavior of the system depends on whether $b^2 - 4mk > 0,$ $b^2 - 4mk = 0,$ or $b^2 - 4mk < 0.$
- If $b^2 - 4mk > 0,$ the system is overdamped and does not exhibit oscillatory behavior.
- If $b^2 - 4mk = 0,$ the system is critically damped. It does not exhibit oscillatory behavior, but any slight reduction in the damping would result in oscillatory behavior.
- If $b^2 - 4mk < 0,$ the system is underdamped. It exhibits oscillatory behavior, but the amplitude of the oscillations decreases over time.
- If $f(t) \neq 0,$ the solution to the differential equation is the sum of a transient solution and a steady-state solution. The steady-state solution governs the long-term behavior of the system.
- The charge on the capacitor in an *RLC* series circuit can also be modeled with a second-order constant-coefficient differential equation of the form

  $$L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t),$$

  where *L* is the inductance, *R* is the resistance, *C* is the capacitance, and $E(t)$ is the voltage source.

7.4 Series Solutions of Differential Equations

- Power series representations of functions can sometimes be used to find solutions to differential equations.
- Differentiate the power series term by term and substitute into the differential equation to find relationships between the power series coefficients.

CHAPTER 7 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample.

117. If y and z are both solutions to $y'' + 2y' + y = 0,$ then $y + z$ is also a solution.

118. The following system of algebraic equations has a unique solution:

$$\begin{aligned} 6z_1 + 3z_2 &= 8 \\ 4z_1 + 2z_2 &= 4. \end{aligned}$$

119. $y = e^x \cos(3x) + e^x \sin(2x)$ is a solution to the second-order differential equation $y'' + 2y' + 10 = 0.$

120. To find the particular solution to a second-order differential equation, you need one initial condition.

Classify the differential equation. Determine the order, whether it is linear and, if linear, whether the differential equation is homogeneous or nonhomogeneous. If the equation is second-order homogeneous and linear, find the characteristic equation.

121. $y'' - 2y = 0$

122. $y'' - 3y + 2y = \cos(t)$

123. $\left(\frac{dy}{dt}\right)^2 + yy' = 1$

124. $\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \sin^2(t)y = e^t$

For the following problems, find the general solution.

125. $y'' + 9y = 0$

126. $y'' + 2y' + y = 0$

127. $y'' - 2y' + 10y = 4x$

128. $y'' = \cos(x) + 2y' + y$

129. $y'' + 5y + y = x + e^{2x}$

130. $y'' = 3y' + xe^{-x}$

131. $y'' - x^2 = -3y' - \frac{9}{4}y + 3x$

132. $y'' = 2\cos x + y' - y$

For the following problems, find the solution to the initial-value problem, if possible.

133. $y'' + 4y' + 6y = 0, \quad y(0) = 0, \quad y'(0) = \sqrt{2}$

134. $y'' = 3y - \cos(x), \quad y(0) = \frac{9}{4}, \quad y'(0) = 0$

For the following problems, find the solution to the boundary-value problem.

135. $4y' = -6y + 2y'', \quad y(0) = 0, \quad y(1) = 1$

136. $y'' = 3x - y - y', \quad y(0) = -3, \quad y(1) = 0$

For the following problem, set up and solve the differential equation.

137. The motion of a swinging pendulum for small angles θ can be approximated by $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$, where θ is the angle the pendulum makes with respect to a vertical line, g is the acceleration resulting from gravity, and L is the length of the pendulum. Find the equation describing the angle of the pendulum at time t, assuming an initial displacement of θ_0 and an initial velocity of zero.

The following problems consider the "beats" that occur when the forcing term of a differential equation causes "slow" and "fast" amplitudes. Consider the general differential equation $ay'' + by = \cos(\omega t)$ that governs undamped motion. Assume that $\sqrt{\frac{b}{a}} \neq \omega$.

138. Find the general solution to this equation (*Hint:* call $\omega_0 = \sqrt{b/a}$).

139. Assuming the system starts from rest, show that the particular solution can be written as $y = \frac{2}{a\left(\omega_0{}^2 - \omega^2\right)}\sin\left(\frac{\omega_0 - \omega t}{2}\right)\sin\left(\frac{\omega_0 + \omega t}{2}\right)$.

140. **[T]** Using your solutions derived earlier, plot the solution to the system $2y'' + 9y = \cos(2t)$ over the interval $t = [-50, 50]$. Find, analytically, the period of the fast and slow amplitudes.

For the following problem, set up and solve the differential equations.

141. An opera singer is attempting to shatter a glass by singing a particular note. The vibrations of the glass can be modeled by $y'' + ay = \cos(bt)$, where $y'' + ay = 0$ represents the natural frequency of the glass and the singer is forcing the vibrations at $\cos(bt)$. For what value b would the singer be able to break that glass? (*Note*: in order for the glass to break, the oscillations would need to get higher and higher.)

APPENDIX A | TABLE OF INTEGRALS

Basic Integrals

1. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \; n \neq -1$

2. $\int \frac{du}{u} = \ln|u| + C$

3. $\int e^u \, du = e^u + C$

4. $\int a^u \, du = \frac{a^u}{\ln a} + C$

5. $\int \sin u \, du = -\cos u + C$

6. $\int \cos u \, du = \sin u + C$

7. $\int \sec^2 u \, du = \tan u + C$

8. $\int \csc^2 u \, du = -\cot u + C$

9. $\int \sec u \tan u \, du = \sec u + C$

10. $\int \csc u \cot u \, du = -\csc u + C$

11. $\int \tan u \, du = \ln|\sec u| + C$

12. $\int \cot u \, du = \ln|\sin u| + C$

13. $\int \sec u \, du = \ln|\sec u + \tan u| + C$

14. $\int \csc u \, du = \ln|\csc u - \cot u| + C$

15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$

16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$

17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$

Trigonometric Integrals

18. $\int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

19. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

20. $\int \tan^2 u\,du = \tan u - u + C$

21. $\int \cot^2 u\,du = -\cot u - u + C$

22. $\int \sin^3 u\,du = -\frac{1}{3}\left(2 + \sin^2 u\right)\cos u + C$

23. $\int \cos^3 u\,du = \frac{1}{3}\left(2 + \cos^2 u\right)\sin u + C$

24. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\,du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\,du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

29. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

30. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

31. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

32. $\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

33. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

34. $\int \sin au\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u\sin u\,du = \sin u - u\cos u + C$

38. $\int u\cos u\,du = \cos u + u\sin u + C$

39. $\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

40. $\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

41. $$\begin{aligned}\int \sin^n u\cos^m u\,du &= -\frac{\sin^{n-1} u\cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u\cos^m u\,du \\ &= \frac{\sin^{n+1} u\cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u\cos^{m-2} u\,du\end{aligned}$$

Exponential and Logarithmic Integrals

42. $\int ue^{au}\,du = \frac{1}{a^2}(au - 1)e^{au} + C$

43. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1} e^{au}\,du$

44. $\int e^{au} \sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a \sin bu - b \cos bu) + C$

45. $\int e^{au} \cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a \cos bu + b \sin bu) + C$

46. $\int \ln u\,du = u \ln u - u + C$

47. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u \ln u}\,du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\,du = \cosh u + C$

50. $\int \cosh u\,du = \sinh u + C$

51. $\int \tanh u\,du = \ln \cosh u + C$

52. $\int \coth u\,du = \ln|\sinh u| + C$

53. $\int \operatorname{sech} u\,du = \tan^{-1} |\sinh u| + C$

54. $\int \operatorname{csch} u\,du = \ln\left|\tanh \frac{1}{2}u\right| + C$

55. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

56. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

57. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

58. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \sin^{-1} u\,du = u \sin^{-1} u + \sqrt{1 - u^2} + C$

60. $\int \cos^{-1} u\,du = u \cos^{-1} u - \sqrt{1 - u^2} + C$

61. $\int \tan^{-1} u\,du = u \tan^{-1} u - \frac{1}{2}\ln\left(1 + u^2\right) + C$

62. $\int u \sin^{-1} u\,du = \frac{2u^2 - 1}{4}\sin^{-1} u + \frac{u\sqrt{1 - u^2}}{4} + C$

63. $\int u \cos^{-1} u\, du = \frac{2u^2 - 1}{4}\cos^{-1} u - \frac{u\sqrt{1 - u^2}}{4} + C$

64. $\int u \tan^{-1} u\, du = \frac{u^2 + 1}{2}\tan^{-1} u - \frac{u}{2} + C$

65. $\int u^n \sin^{-1} u\, du = \frac{1}{n + 1}\left[u^{n + 1} \sin^{-1} u - \int \frac{u^{n + 1}\, du}{\sqrt{1 - u^2}}\right], n \neq -1$

66. $\int u^n \cos^{-1} u\, du = \frac{1}{n + 1}\left[u^{n + 1} \cos^{-1} u + \int \frac{u^{n + 1}\, du}{\sqrt{1 - u^2}}\right], n \neq -1$

67. $\int u^n \tan^{-1} u\, du = \frac{1}{n + 1}\left[u^{n + 1} \tan^{-1} u - \int \frac{u^{n + 1}\, du}{1 + u^2}\right], n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2 + u^2}\, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

69. $\int u^2 \sqrt{a^2 + u^2}\, du = \frac{u}{8}\left(a^2 + 2u^2\right)\sqrt{a^2 + u^2} - \frac{a^4}{8}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

70. $\int \frac{\sqrt{a^2 + u^2}}{u}\, du = \sqrt{a^2 + u^2} - a \ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2 + u^2}}{u^2}\, du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2 + u^2}} = \ln\left(u + \sqrt{a^2 + u^2}\right) + C$

73. $\int \frac{u^2\, du}{\sqrt{a^2 + u^2}} = \frac{u}{2}\left(\sqrt{a^2 + u^2}\right) - \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2 + u^2} + a}{u}\right| + C$

75. $\int \frac{du}{u^2 \sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C$

76. $\int \frac{du}{\left(a^2 + u^2\right)^{3/2}} = \frac{u}{a^2 \sqrt{a^2 + u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

78. $\int u^2 \sqrt{u^2 - a^2}\, du = \frac{u}{8}\left(2u^2 - a^2\right)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

79. $\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a\cos^{-1} \frac{a}{|u|} + C$

80. $\int \frac{\sqrt{u^2 - a^2}}{u^2}\, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

81. $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

82. $\int \frac{u^2\,du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

83. $\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

84. $\int \frac{du}{\left(u^2 - a^2\right)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

86. $\int u^2\sqrt{a^2 - u^2}\,du = \frac{u}{8}\left(2u^2 - a^2\right)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

87. $\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

88. $\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

89. $\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{u}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

90. $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

91. $\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

92. $\int \left(a^2 - u^2\right)^{3/2}\,du = -\frac{u}{8}\left(2u^2 - 5a^2\right)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

93. $\int \frac{du}{\left(a^2 - u^2\right)^{3/2}} = -\frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Integrals Involving $2au - u^2$, $a > 0$

94. $\int \sqrt{2au - u^2}\,du = \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

95. $\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}\left(\frac{a - u}{a}\right) + C$

96. $\int u\sqrt{2au - u^2}\,du = \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a - u}{a}\right) + C$

97. $\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$

Integrals Involving $a + bu$, $a \neq 0$

98. $\displaystyle\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(a + bu - a\ln|a+bu|\right) + C$

99. $\displaystyle\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

100. $\displaystyle\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

101. $\displaystyle\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

102. $\displaystyle\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

103. $\displaystyle\int \frac{u\,du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

104. $\displaystyle\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

105. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

106. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

107. $\displaystyle\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}\left(8a^2 + 3b^2u^2 - 4abu\right)\sqrt{a+bu} + C$

108. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$

$\displaystyle\qquad = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$

109. $\displaystyle\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int\frac{du}{u\sqrt{a+bu}}$

110. $\displaystyle\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int\frac{du}{u\sqrt{a+bu}}$

111. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

112. $\displaystyle\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int\frac{u^{n-1}\,du}{\sqrt{a+bu}}$

113. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int\frac{du}{u^{n-1}\sqrt{a+bu}}$

APPENDIX B | TABLE OF DERIVATIVES

General Formulas

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

4. $\frac{d}{dx}(x^n) = nx^{n-1}$, for real numbers n

5. $\frac{d}{dx}(cf(x)) = cf'(x)$

6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$

10. $\frac{d}{dx}(\tan x) = \sec^2 x$

11. $\frac{d}{dx}(\sec x) = \sec x \tan x$

12. $\frac{d}{dx}(\cos x) = -\sin x$

13. $\frac{d}{dx}(\cot x) = -\csc^2 x$

14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$

16. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2}$

17. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2 - 1}}$

18. $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1 - x^2}}$

19. $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1 + x^2}$

20. $\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2 - 1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$

22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

23. $\frac{d}{dx}(b^x) = b^x \ln b$

24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$

26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

28. $\frac{d}{dx}(\cosh x) = \sinh x$

29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2 + 1}}$

32. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1 - x^2}(|x| < 1)$

33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1 - x^2}} \quad (0 < x < 1)$

34. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2 - 1}} \quad (x > 1)$

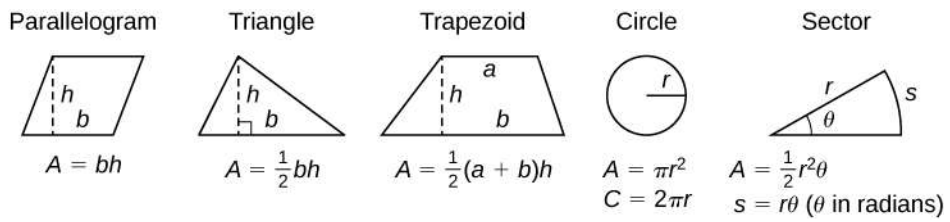
35. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1 - x^2} \quad (|x| > 1)$

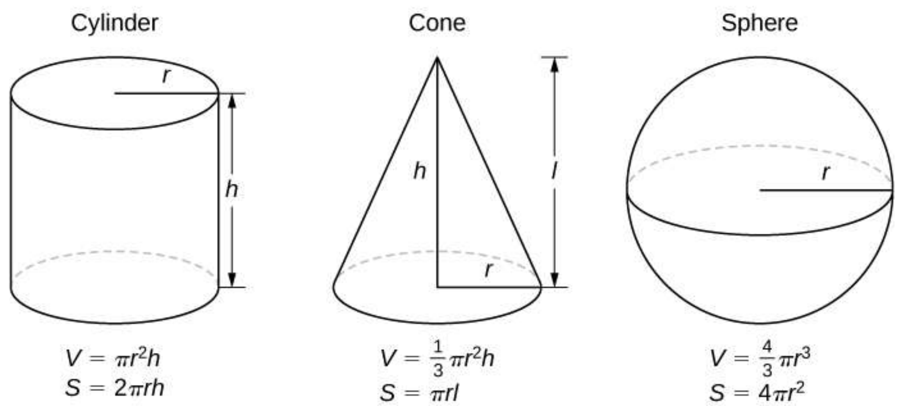
36. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = -\frac{1}{|x|\sqrt{1 + x^2}} (x \neq 0)$

APPENDIX C | REVIEW OF PRE-CALCULUS

Formulas from Geometry

$A = \text{area}, \quad V = \text{Volume}, \text{ and } \; S = \text{lateral surface area}$





Formulas from Algebra

Laws of Exponents

$$x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n} \qquad (x^m)^n = x^{mn}$$

$$x^{-n} = \frac{1}{x^n} \qquad (xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^{1/n} = \sqrt[n]{x} \qquad \sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

Special Factorizations

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)\left(x^2 - xy + y^2\right)$$

$$x^3 - y^3 = (x-y)\left(x^2 + xy + y^2\right)$$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ca}}{2a}$.

Binomial Theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n,$$

where $\binom{n}{k} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1} = \dfrac{n!}{k!(n-k)!}$

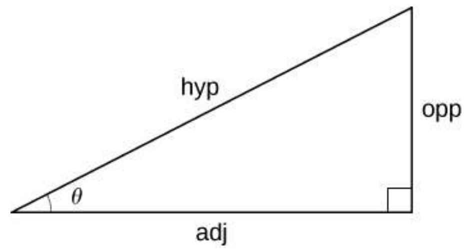
Formulas from Trigonometry

Right-Angle Trigonometry

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$



Trigonometric Functions of Important Angles

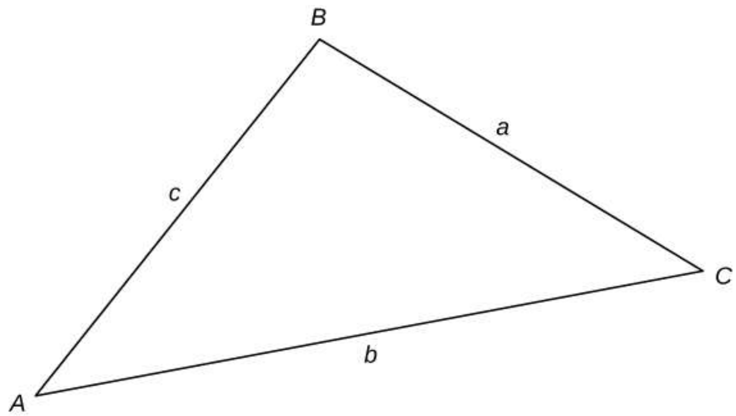
θ	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
90°	$\pi/2$	1	0	—

Fundamental Identities

$$
\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 & \sin(-\theta) &= -\sin\theta \\
1 + \tan^2\theta &= \sec^2\theta & \cos(-\theta) &= \cos\theta \\
1 + \cot^2\theta &= \csc^2\theta & \tan(-\theta) &= -\tan\theta \\
\sin\left(\frac{\pi}{2} - \theta\right) &= \cos\theta & \sin(\theta + 2\pi) &= \sin\theta \\
\cos\left(\frac{\pi}{2} - \theta\right) &= \sin\theta & \cos(\theta + 2\pi) &= \cos\theta \\
\tan\left(\frac{\pi}{2} - \theta\right) &= \cot\theta & \tan(\theta + \pi) &= \tan\theta
\end{aligned}
$$

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Law of Cosines

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \\
b^2 &= a^2 + c^2 - 2ac\cos B \\
c^2 &= a^2 + b^2 - 2ab\cos C
\end{aligned}
$$

Addition and Subtraction Formulas

$$
\begin{aligned}
\sin(x + y) &= \sin x\cos y + \cos x\sin y \\
\sin(x - y) &= \sin x\cos y - \cos x\sin y \\
\cos(x + y) &= \cos x\cos y - \sin x\sin y \\
\cos(x - y) &= \cos x\cos y + \sin x\sin y \\
\tan(x + y) &= \frac{\tan x + \tan y}{1 - \tan x\tan y} \\
\tan(x - y) &= \frac{\tan x - \tan y}{1 + \tan x\tan y}
\end{aligned}
$$

Double-Angle Formulas

$$
\begin{aligned}
\sin 2x &= 2\sin x\cos x \\
\cos 2x &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \\
\tan 2x &= \frac{2\tan x}{1 - \tan^2 x}
\end{aligned}
$$

Half-Angle Formulas

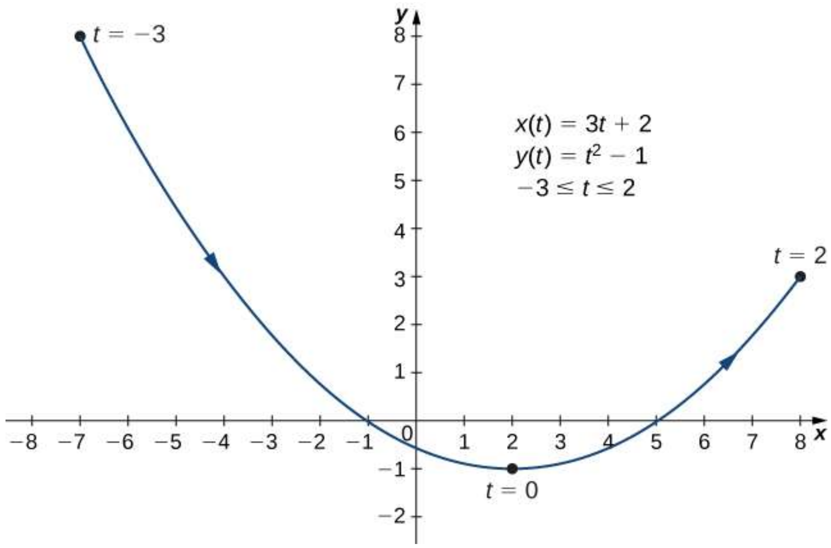
$$\begin{aligned} \sin^2 x &= \frac{1 - \cos 2x}{2} \\ \cos^2 x &= \frac{1 + \cos 2x}{2} \end{aligned}$$

ANSWER KEY

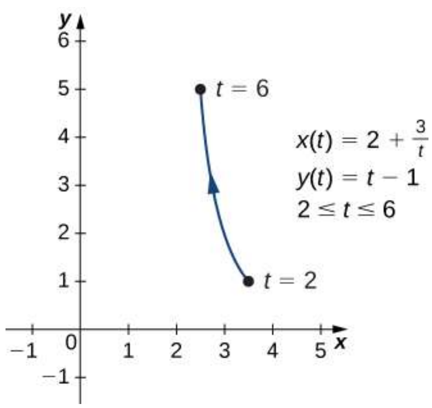
Chapter 1

Checkpoint

1.1.



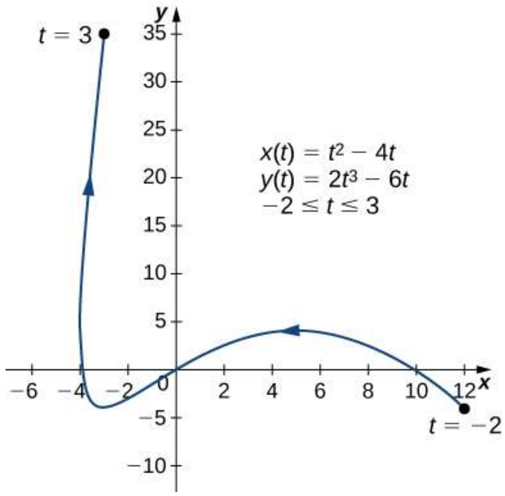
1.2. $x = 2 + \frac{3}{y+1}$, or $y = -1 + \frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $(2, -1)$.



1.3. One possibility is $x(t) = t, \quad y(t) = t^2 + 2t.$ Another possibility is $x(t) = 2t - 3, \quad y(t) = (2t-3)^2 + 2(2t-3) = 4t^2 - 8t + 3.$ There are, in fact, an infinite number of possibilities.

1.4. $x'(t) = 2t - 4$ and $y'(t) = 6t^2 - 6,$ so $\frac{dy}{dx} = \frac{6t^2-6}{2t-4} = \frac{3t^2-3}{t-2}.$

This expression is undefined when $t = 2$ and equal to zero when $t = \pm 1.$



1.5. The equation of the tangent line is $y = 24x + 100.$

1.6. $\frac{d^2 y}{dx^2} = \frac{3t^2 - 12t + 3}{2(t-2)^3}.$ Critical points $(5,\ 4),\ (-3,\ -4),$ and $(-4,\ 6).$

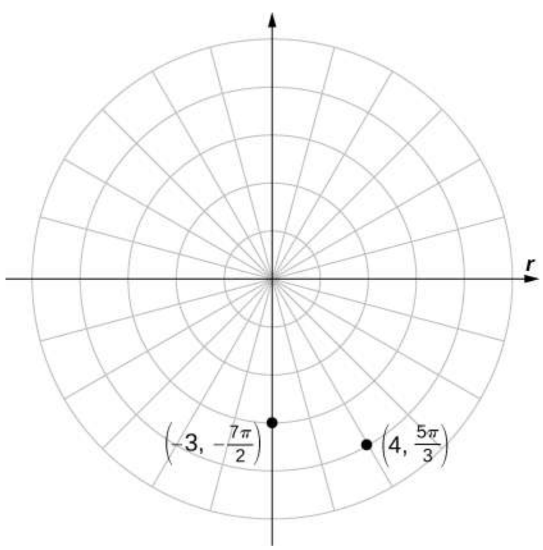
1.7. $A = 3\pi$ (Note that the integral formula actually yields a negative answer. This is due to the fact that $x(t)$ is a decreasing function over the interval $[0,\ 2\pi];$ that is, the curve is traced from right to left.)

1.8. $s = 2\left(10^{3/2} - 2^{3/2}\right) \approx 57.589$

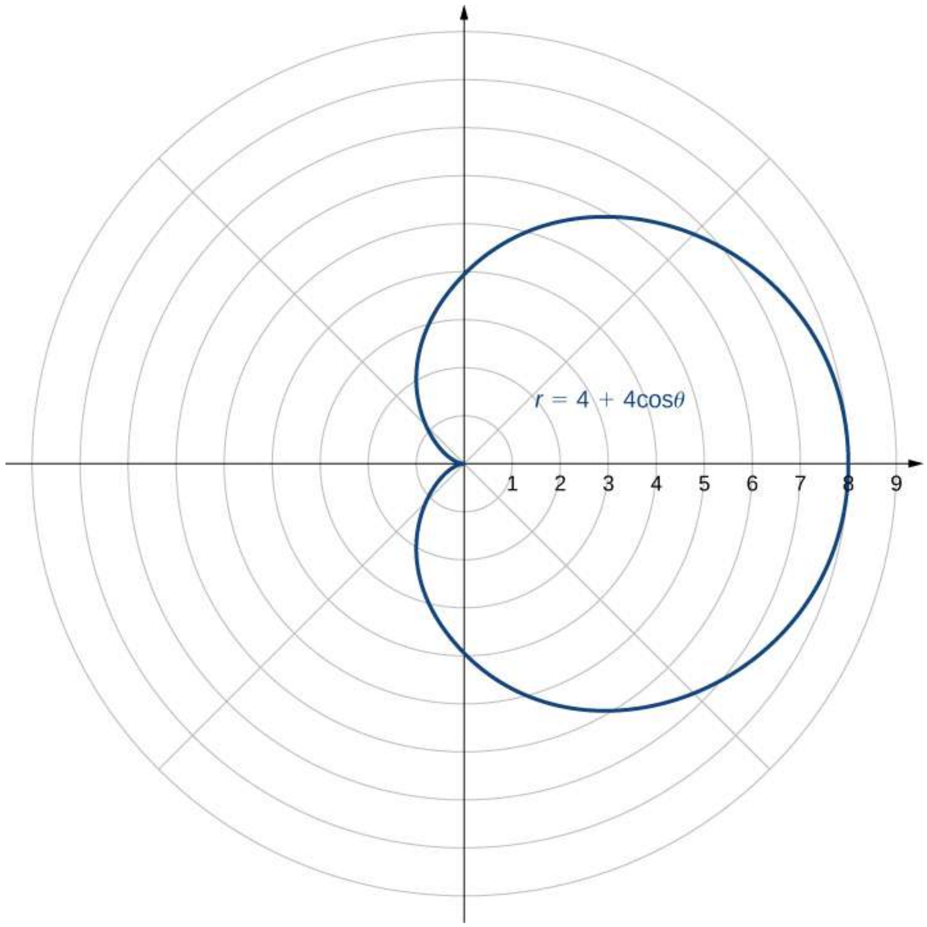
1.9. $A = \frac{\pi\left(494\sqrt{13} + 128\right)}{1215}$

1.10. $\left(8\sqrt{2},\ \frac{5\pi}{4}\right)$ and $\left(-2,\ 2\sqrt{3}\right)$

1.11.



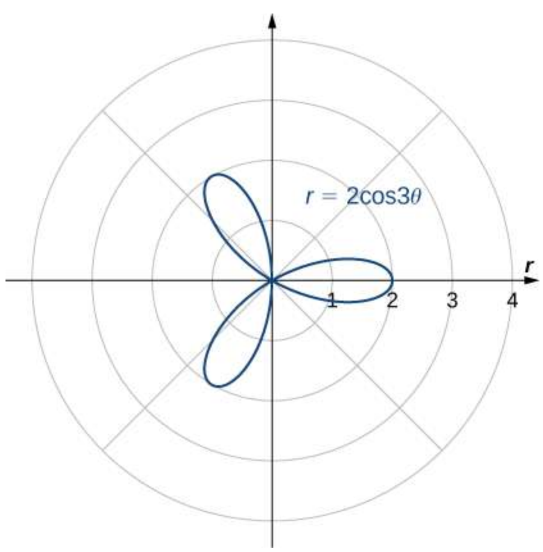
1.12.



The name of this shape is a cardioid, which we will study further later in this section.

1.13. $y = x^2$, which is the equation of a parabola opening upward.

1.14. Symmetric with respect to the polar axis.

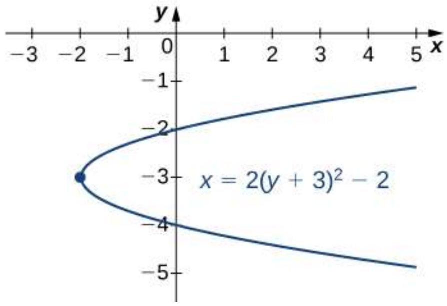


1.15. $A = 3\pi/2$

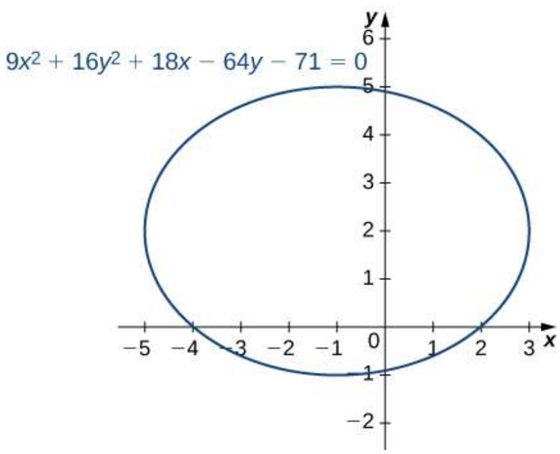
1.16. $A = \frac{4\pi}{3} + 4\sqrt{3}$

1.17. $s = 3\pi$

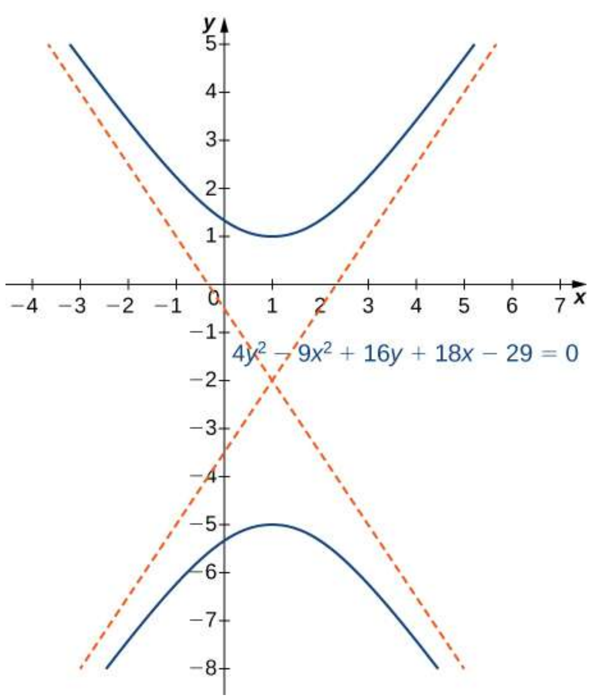
1.18. $x = 2(y+3)^2 - 2$



1.19. $\frac{(x+1)^2}{16} + \frac{(y-2)^2}{9} = 1$

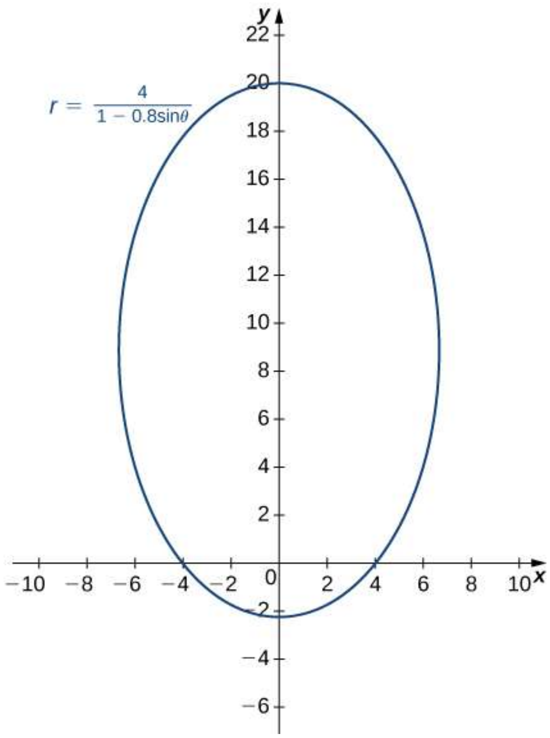


1.20. $\frac{(y+2)^2}{9} - \frac{(x-1)^2}{4} = 1.$ This is a vertical hyperbola. Asymptotes $y = -2 \pm \frac{3}{2}(x-1).$



1.21. $e = \frac{c}{a} = \frac{\sqrt{74}}{7} \approx 1.229$

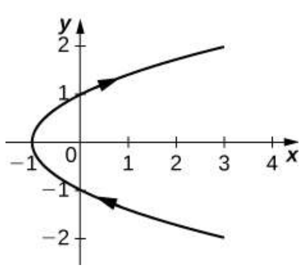
1.22. Here $e = 0.8$ and $p = 5.$ This conic section is an ellipse.



1.23. The conic is a hyperbola and the angle of rotation of the axes is $\theta = 22.5°.$

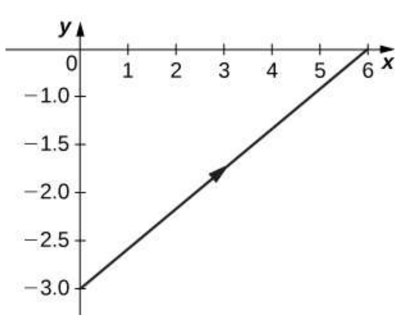
Section Exercises

1.



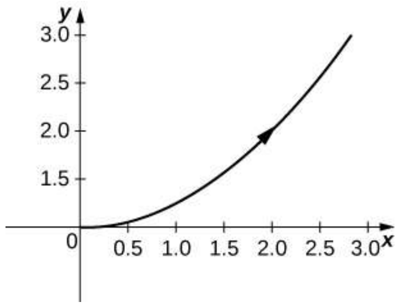
orientation: bottom to top

3.



orientation: left to right

5. $y = \frac{x^2}{4} + 1$



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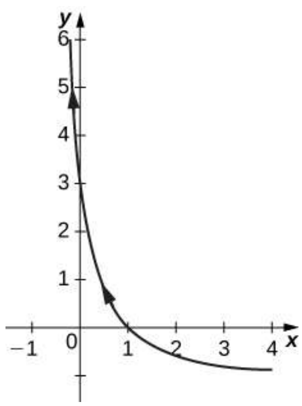
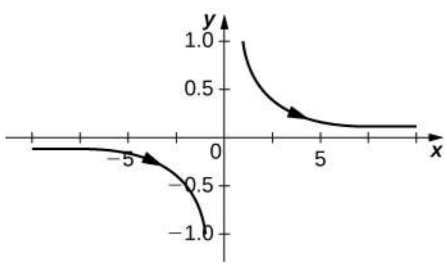
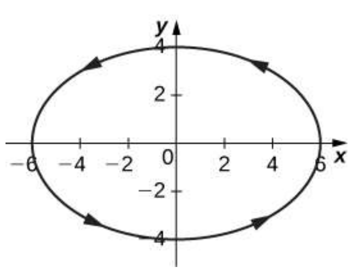


9.



11.



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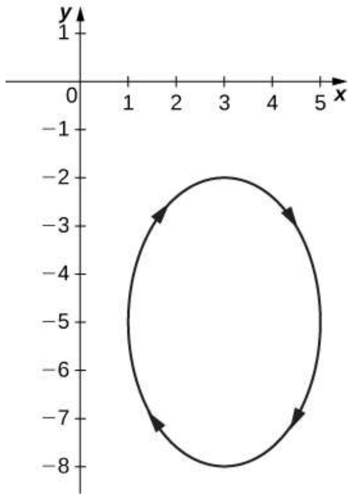

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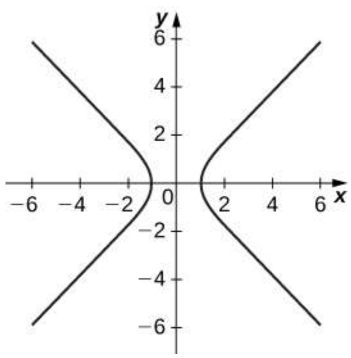
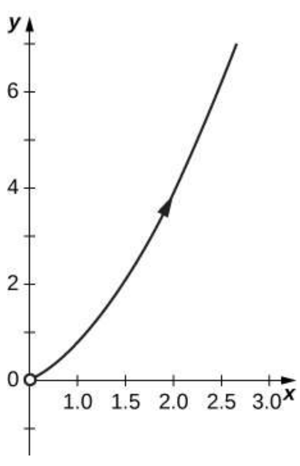


Asymptotes are $y = x$ and $y = -x$

17.



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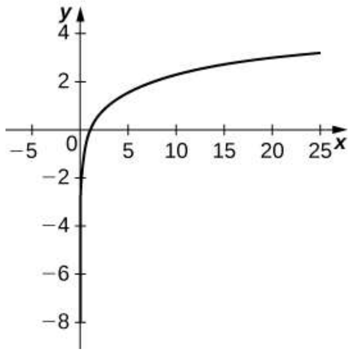


21. $x = 4y^2 - 1;$ domain: $x \in [1, \infty).$

23. $\frac{x^2}{16} + \frac{y^2}{9} = 1;$ domain $x \in [-4, 4].$

25. $y = 3x + 2;$ domain: all real numbers.

27. $(x - 1)^2 + (y - 3)^2 = 1;$ domain: $x \in [0, 2].$

29. $y = \sqrt{x^2 - 1};$ domain: $x \in [-1, 1].$

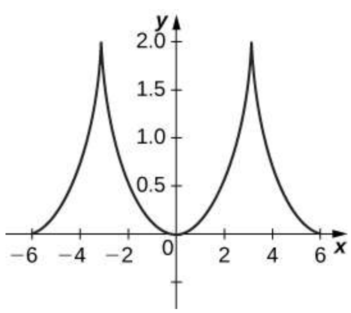
31. $y^2 = \frac{1 - x}{2};$ domain: $x \in [2, \infty) \cup (-\infty, -2].$

33. $y = \ln x;$ domain: $x \in (0, \infty).$

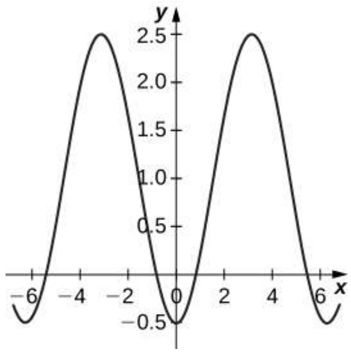
35. $y = \ln x;$ domain: $x \in (0, \infty).$

37. $x^2 + y^2 = 4;$ domain: $x \in [-2, 2].$

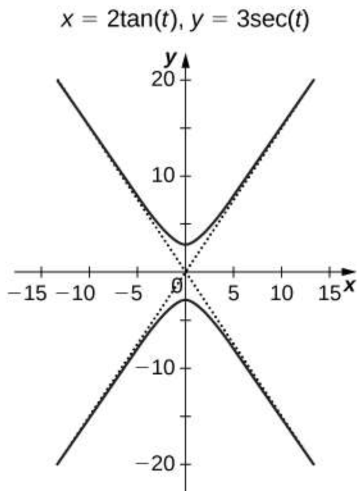
39. line
41. parabola
43. circle
45. ellipse
47. hyperbola
51. The equations represent a cycloid.



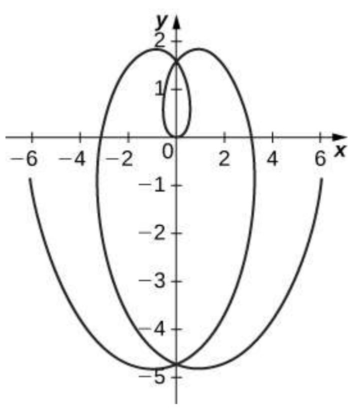
53.



55. 22,092 meters at approximately 51 seconds.
57.



59.



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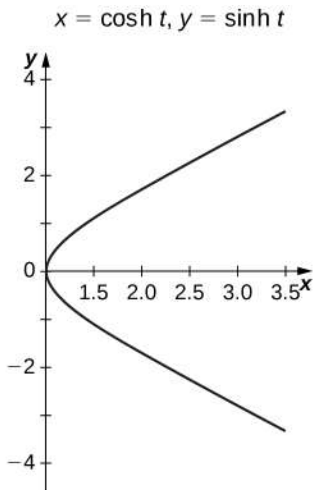


63. 0

65. $\frac{-3}{5}$

67. $\text{Slope} = 0; \; y = 8.$

69. Slope is undefined; $x = 2$.

71. $t = \arctan(-2); \; \left(\frac{4}{\sqrt{5}}, \frac{-8}{\sqrt{5}}\right).$

73. No points possible; undefined expression.

75. $y = -\left(\frac{2}{e}\right)x + 3$

77. $y = 2x - 7$

79. $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

81. $\frac{dy}{dx} = -\tan(t)$

83. $\frac{dy}{dx} = \frac{3}{4}$ and $\frac{d^2 y}{dx^2} = 0,$ so the curve is neither concave up nor concave down at $t = 3$. Therefore the graph is linear and has a constant slope but no concavity.

85. $\frac{dy}{dx} = 4, \; \frac{d^2 y}{dx^2} = -6\sqrt{3};$ the curve is concave down at $\theta = \frac{\pi}{6}$.

87. No horizontal tangents. Vertical tangents at $(1, 0), (-1, 0)$.

89. $-\sec^3(\pi t)$

91. Horizontal $(0, -9);$ vertical $(\pm 2, -6)$.

93. 1

95. 0

97. 4

99. Concave up on $t > 0$.

101. 1

103. $\frac{3\pi}{2}$

105. $6\pi a^2$

107. $2\pi ab$

109. $\frac{1}{3}(2\sqrt{2} - 1)$

111. 7.075

113. $6a$

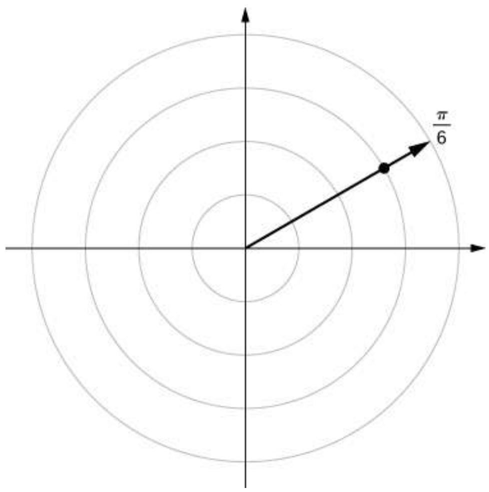
115. $6\sqrt{2}$

119. $\dfrac{2\pi\left(247\sqrt{13}+64\right)}{1215}$

121. 59.101

123. $\dfrac{8\pi}{3}(17\sqrt{17}-1)$

125.



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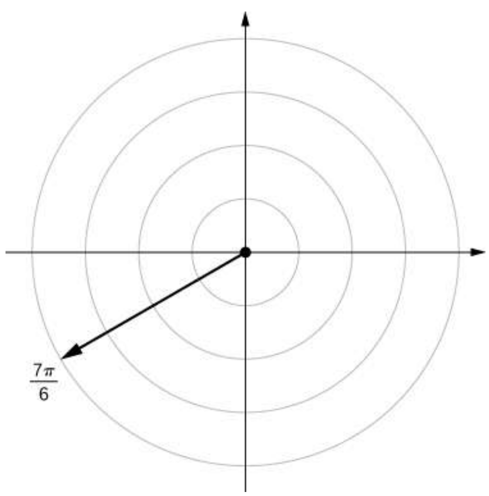

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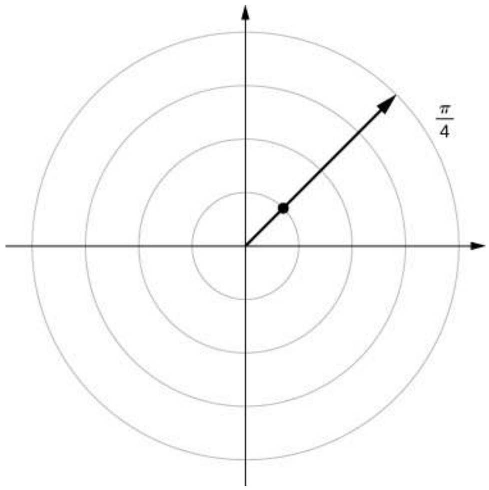
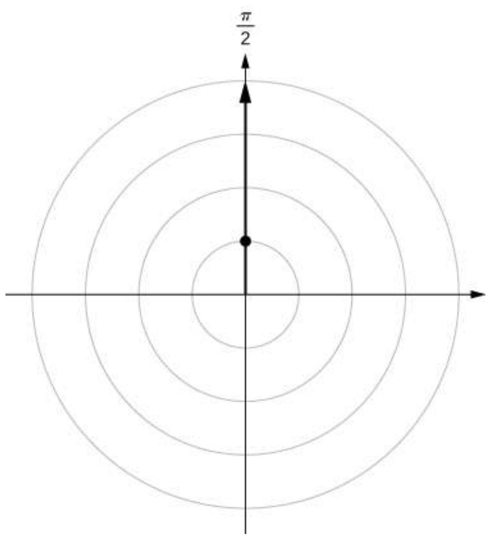


131.



133. $B\left(3, \frac{-\pi}{3}\right)$ $B\left(-3, \frac{2\pi}{3}\right)$

135. $D\left(5, \frac{7\pi}{6}\right)D\left(-5, \frac{\pi}{6}\right)$

137. $(5, -0.927)$ $(-5, -0.927 + \pi)$

139. $(10, -0.927)(-10, -0.927 + \pi)$

141. $\left(2\sqrt{3}, -0.524\right)\left(-2\sqrt{3}, -0.524 + \pi\right)$

143. $\left(-\sqrt{3}, \quad -1\right)$

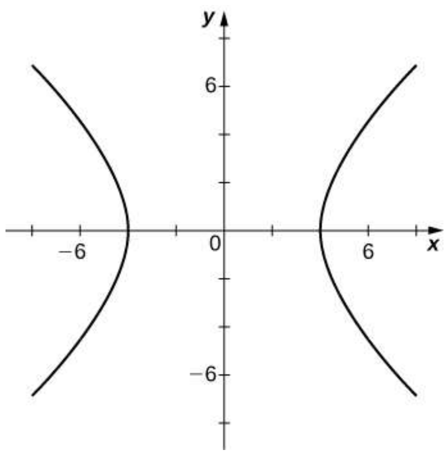
145. $\left(-\frac{\sqrt{3}}{2}, \quad \frac{-1}{2}\right)$

147. $(0, \quad 0)$

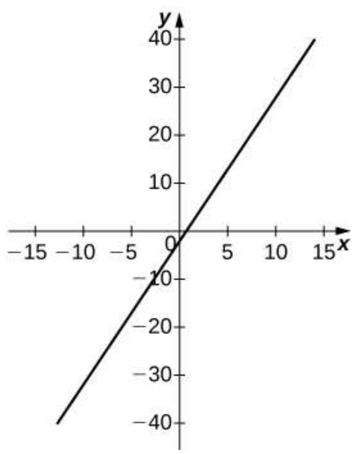
149. Symmetry with respect to the *x*-axis, *y*-axis, and origin.
151. Symmetric with respect to *x*-axis only.
153. Symmetry with respect to *x*-axis only.
155. Line $y = x$

157. $y = 1$

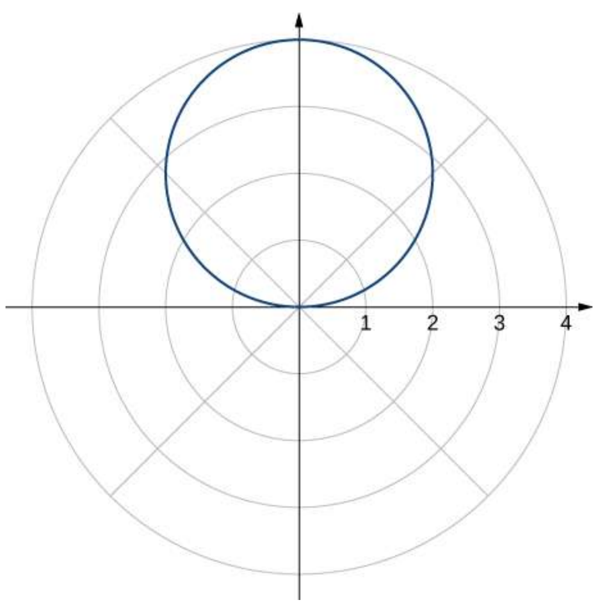
159. Hyperbola; polar form $r^2 \cos(2\theta) = 16$ or $r^2 = 16 \sec \theta$.



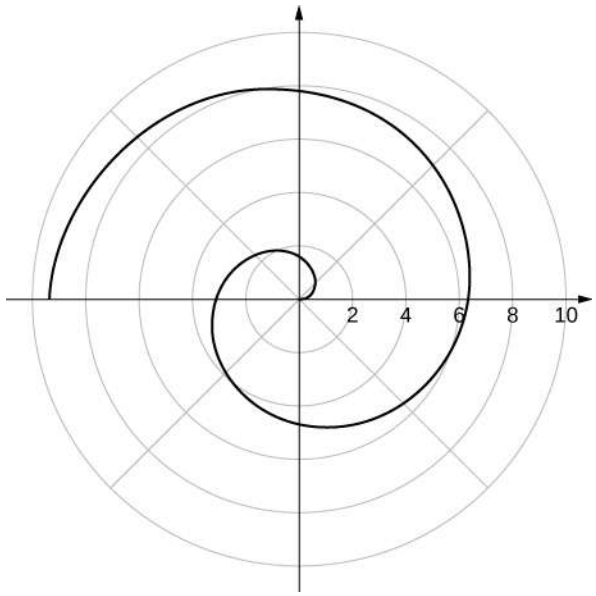
161. $r = \dfrac{2}{3\cos\theta - \sin\theta}$



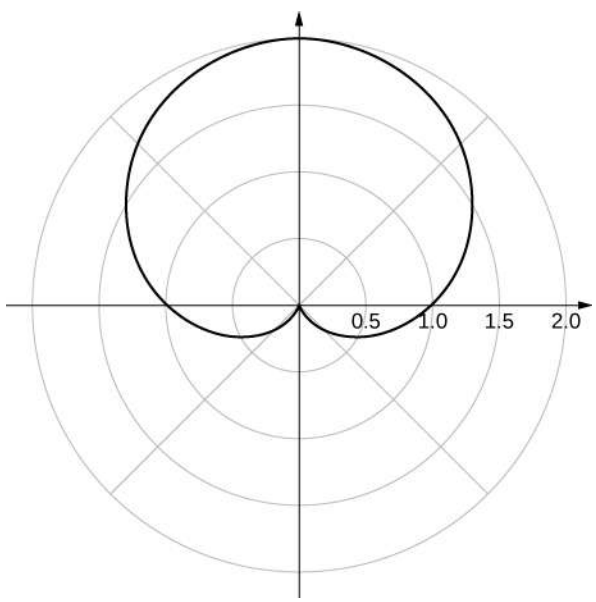
163. $x^2 + y^2 = 4y$



165. $x \tan\sqrt{x^2 + y^2} = y$

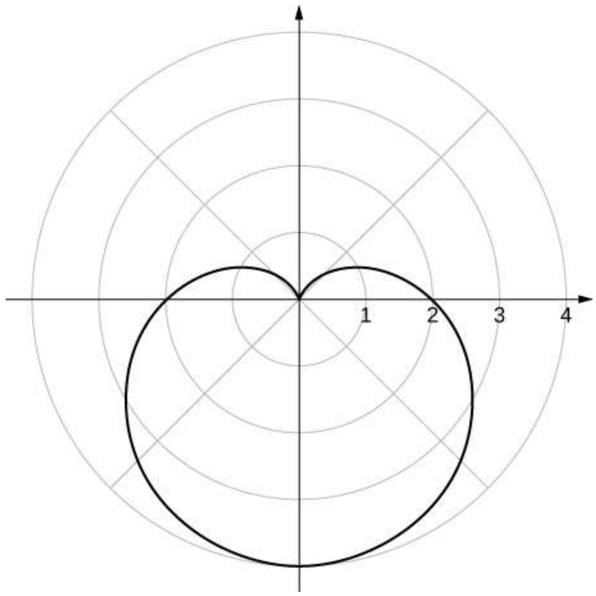


167.



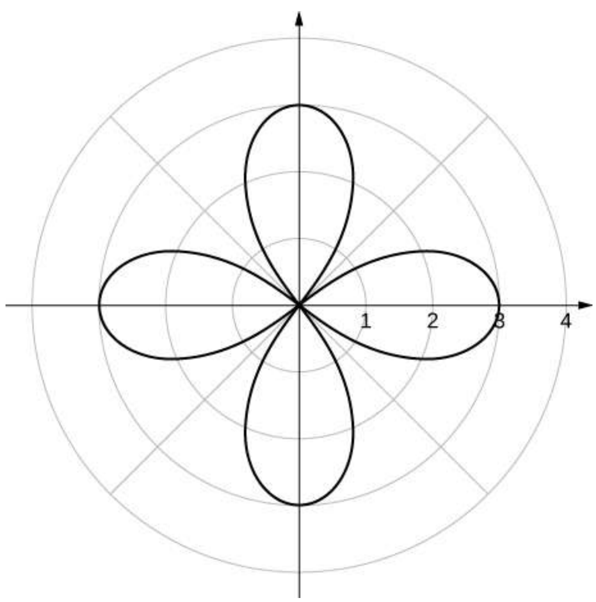
y-axis symmetry

169.



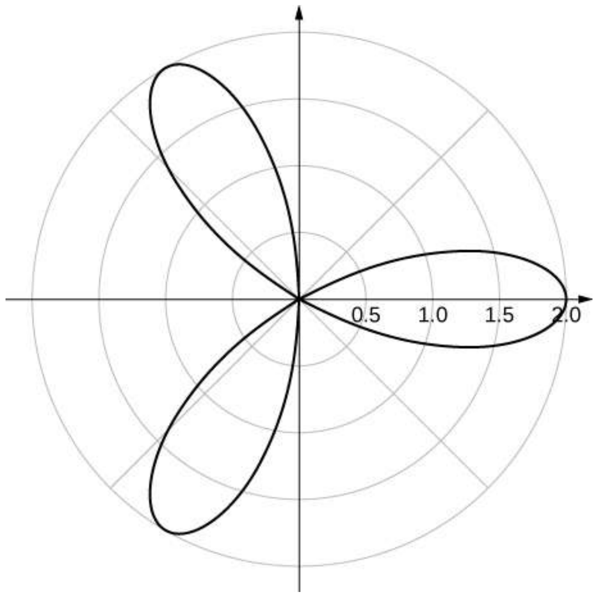
y-axis symmetry

171.

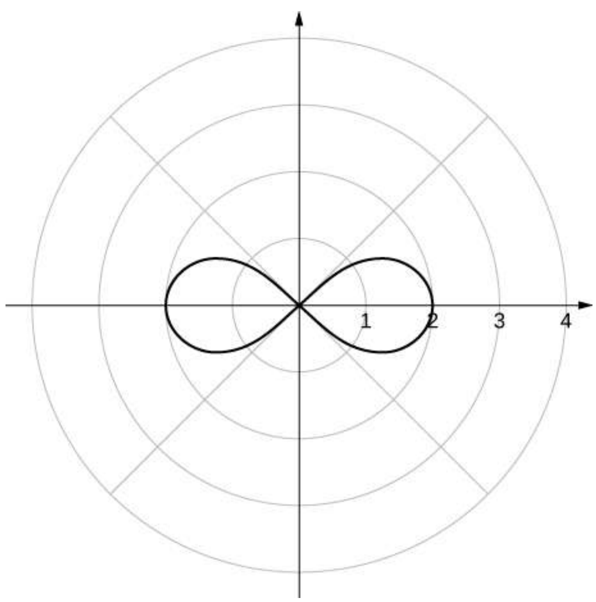


x- and *y*-axis symmetry and symmetry about the pole

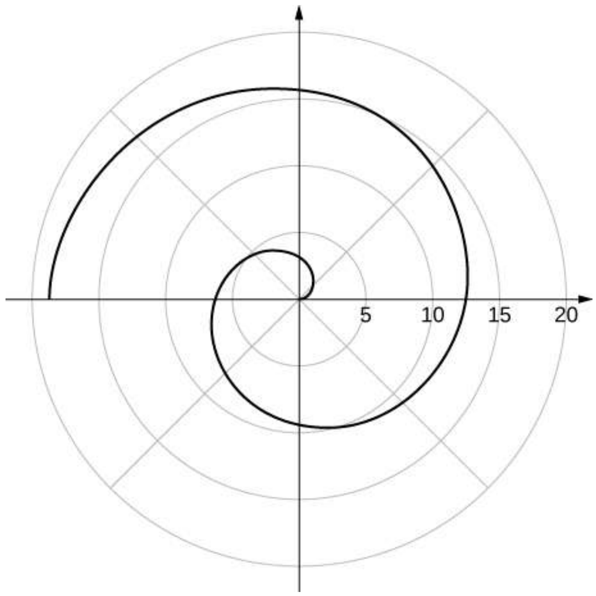
173.



x-axis symmetry
175.

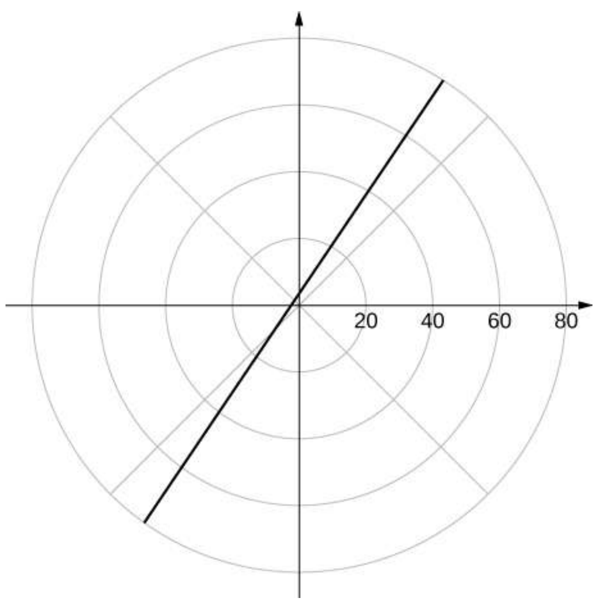


x- and *y*-axis symmetry and symmetry about the pole
177.



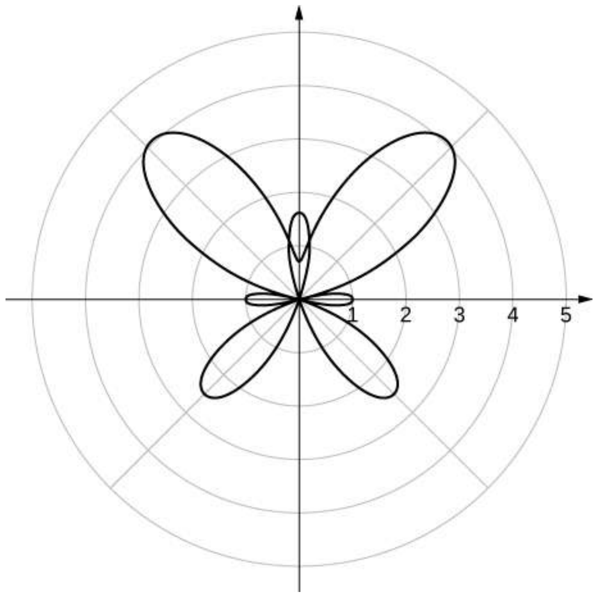
no symmetry

179.



a line

181.



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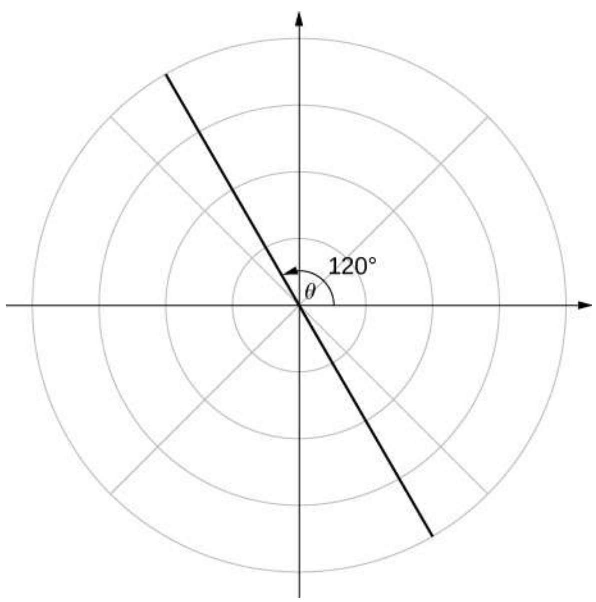

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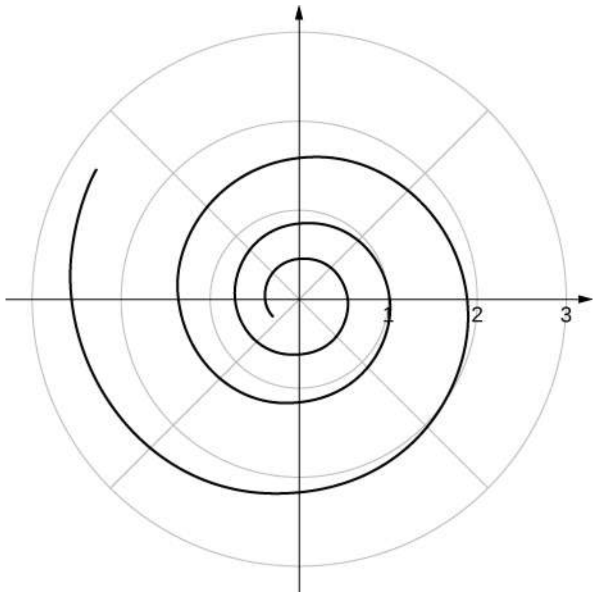


187. Answers vary. One possibility is the spiral lines become closer together and the total number of spirals increases.

189. $\frac{9}{2}\int_0^{\pi} \sin^2\theta\, d\theta$

191. $32\int_0^{\pi/2} \sin^2(2\theta)d\theta$

193. $\frac{1}{2}\int_{\pi}^{2\pi} (1-\sin\theta)^2\, d\theta$

195. $\int_{\sin^{-1}(2/3)}^{\pi/2} (2-3\sin\theta)^2 d\theta$

197. $\int_0^{\pi}(1-2\cos\theta)^2\, d\theta - \int_0^{\pi/3}(1-2\cos\theta)^2 d\theta$

199. $4\int_0^{\pi/3} d\theta + 16\int_{\pi/3}^{\pi/2}\left(\cos^2\theta\right)d\theta$

201. 9π

203. $\frac{9\pi}{4}$

205. $\frac{9\pi}{8}$

207. $\frac{18\pi - 27\sqrt{3}}{2}$

209. $\frac{4}{3}\left(4\pi - 3\sqrt{3}\right)$

211. $\frac{3}{2}\left(4\pi - 3\sqrt{3}\right)$

213. $2\pi - 4$

215. $\int_0^{2\pi}\sqrt{(1+\sin\theta)^2 + \cos^2\theta}d\theta$

217. $\sqrt{2}\int_0^1 e^{\theta}\, d\theta$

219. $\frac{\sqrt{10}}{3}\left(e^6 - 1\right)$

221. 32

223. 6.238
225. 2
227. 4.39

229. $A = \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{2}$ and $\frac{1}{2}\int_0^{\pi}(1 + 2\sin\theta\cos\theta)d\theta = \frac{\pi}{2}$

231. $C = 2\pi\left(\frac{3}{2}\right) = 3\pi$ and $\int_0^{\pi} 3d\theta = 3\pi$

233. $C = 2\pi(5) = 10\pi$ and $\int_0^{\pi} 10\, d\theta = 10\pi$

235. $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$

237. The slope is $\frac{1}{\sqrt{3}}$.

239. The slope is 0.

241. At $(4,\ 0),$ the slope is undefined. At $\left(-4,\ \frac{\pi}{2}\right),$ the slope is 0.

243. The slope is undefined at $\theta = \frac{\pi}{4}$.

245. Slope = −1.

247. Slope is $\frac{-2}{\pi}$.

249. Calculator answer: −0.836.

251. Horizontal tangent at $\left(\pm\sqrt{2},\ \frac{\pi}{6}\right),$ $\left(\pm\sqrt{2},\ -\frac{\pi}{6}\right)$.

253. Horizontal tangents at $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$. Vertical tangents at $\frac{\pi}{6}, \frac{5\pi}{6}$ and also at the pole $(0,\ 0)$.

255. $y^2 = 16x$

257. $x^2 = 2y$

259. $x^2 = -4(y - 3)$

261. $(x + 3)^2 = 8(y - 3)$

263. $\frac{x^2}{16} + \frac{y^2}{12} = 1$

265. $\frac{x^2}{13} + \frac{y^2}{4} = 1$

267. $\frac{(y-1)^2}{16} + \frac{(x+3)^2}{12} = 1$

269. $\frac{x^2}{16} + \frac{y^2}{12} = 1$

271. $\frac{x^2}{25} - \frac{y^2}{11} = 1$

273. $\frac{x^2}{7} - \frac{y^2}{9} = 1$

275. $\frac{(y+2)^2}{4} - \frac{(x+2)^2}{32} = 1$

277. $\frac{x^2}{4} - \frac{y^2}{32} = 1$

279. $e = 1,$ parabola

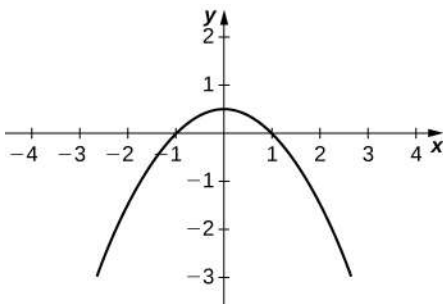
281. $e = \frac{1}{2},$ ellipse

283. $e = 3,$ hyperbola

285. $r = \dfrac{4}{5 + \cos\theta}$

287. $r = \dfrac{4}{1 + 2\sin\theta}$

289.



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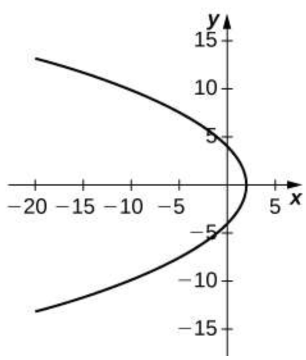
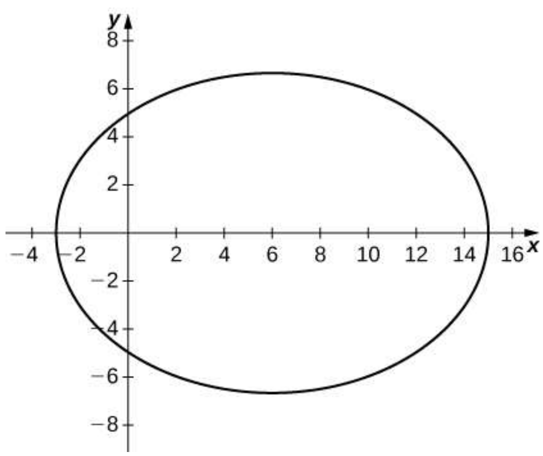
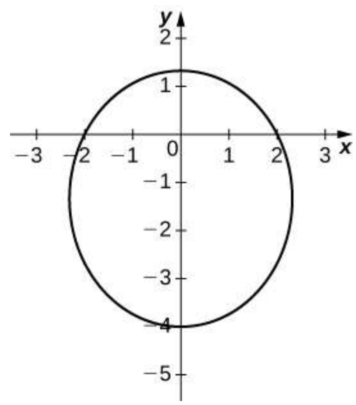


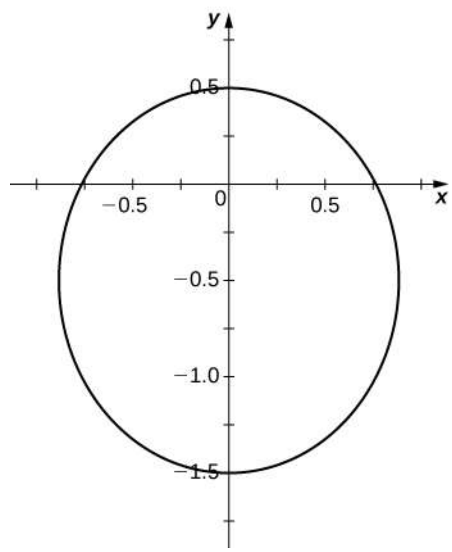
293.



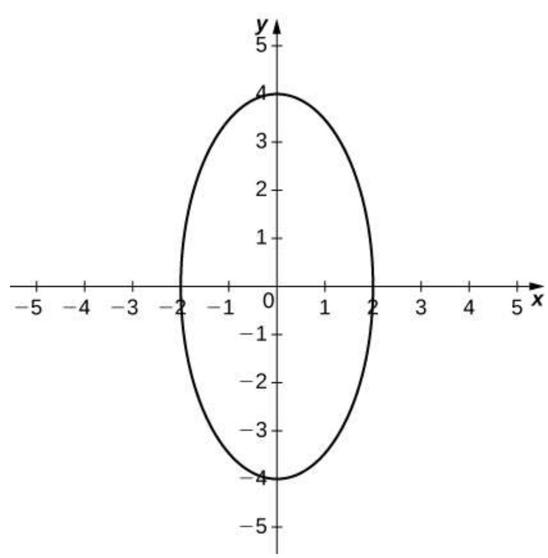
295.


y
2
1
−3
−2
−1
0
1
2
3
x
−1
−2
−3
−4
−5

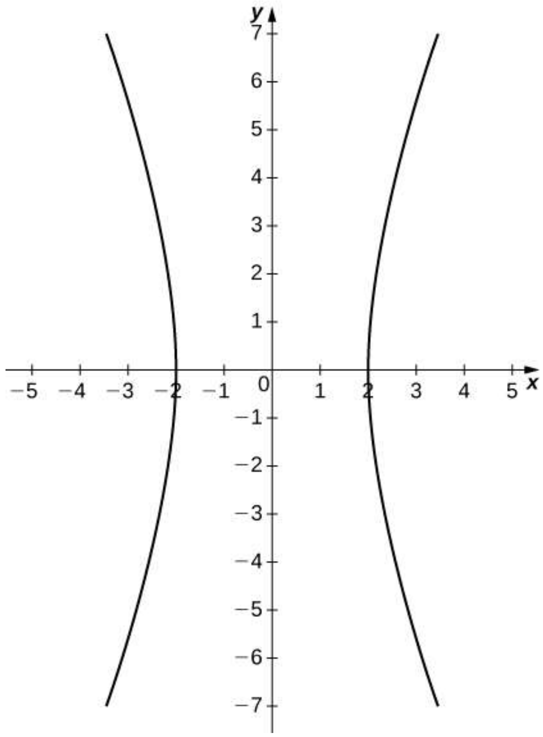
297.


y
0.5
−0.5
0
0.5
x
−0.5
−1.0
−1.5

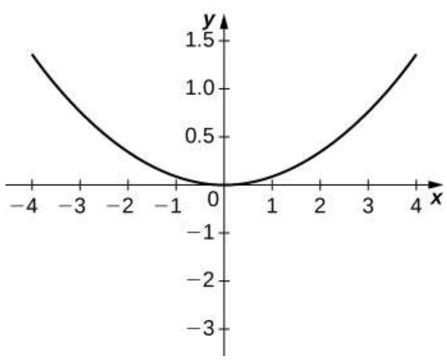
299.


y
5
4
3
2
1
−5
−4
−3
−2
−1
0
1
2
3
4
5
x
−1
−2
−3
−4
−5

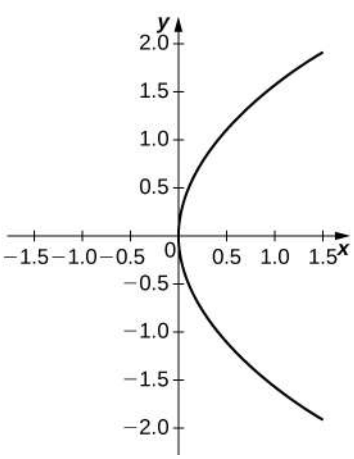
301.



303.



305.

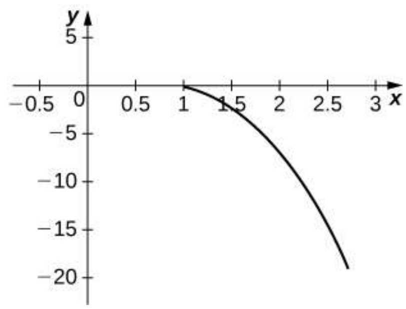


307. Hyperbola
309. Ellipse

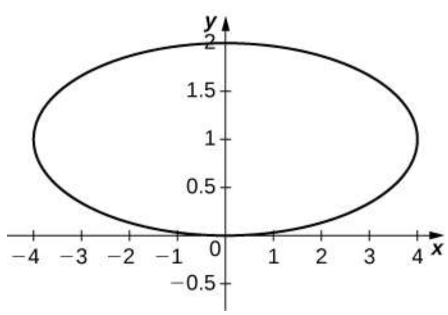
311. Ellipse
313. At the point 2.25 feet above the vertex.
315. 0.5625 feet
317. Length is 96 feet and height is approximately 26.53 feet.
319. $r = \dfrac{2.616}{1 + 0.995\cos\theta}$
321. $r = \dfrac{5.192}{1 + 0.0484\cos\theta}$

Review Exercises

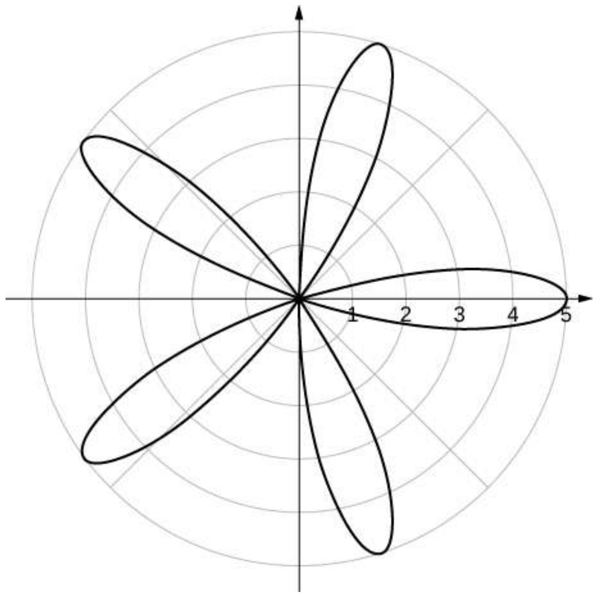
323. True.
325. False. Imagine $y = t + 1, \quad x = -t + 1.$
327.



$y = 1 - x^3$
329.



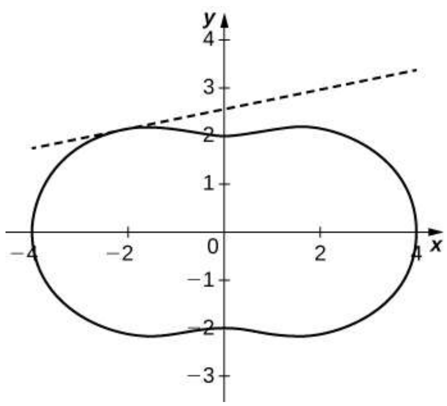
$\dfrac{x^2}{16} + (y - 1)^2 = 1$
331.



Symmetric about polar axis

333. $r^2 = \dfrac{4}{\sin^2\theta - \cos^2\theta}$

335.



$y = \dfrac{3\sqrt{2}}{2} + \dfrac{1}{5}\left(x + \dfrac{3\sqrt{2}}{2}\right)$

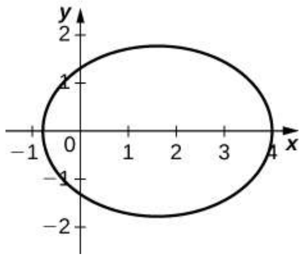
337. $\dfrac{e^2}{2}$

339. $9\sqrt{10}$

341. $(y+5)^2 = -8x + 32$

343. $\dfrac{(y+1)^2}{16} - \dfrac{(x+2)^2}{9} = 1$

345. $e = \dfrac{2}{3},$ ellipse

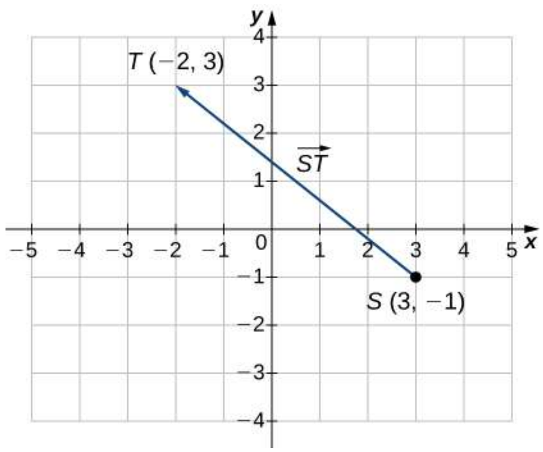


347. $\frac{y^2}{19.03^2} + \frac{x^2}{19.63^2} = 1, \quad e = 0.2447$

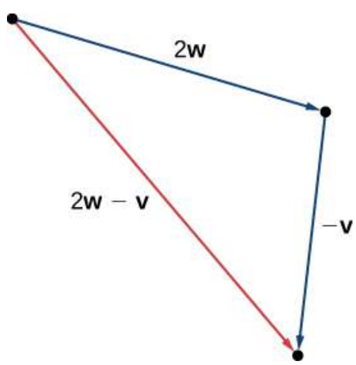
Chapter 2

Checkpoint

2.1.



2.2.



2.3. Vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{e}$ are equivalent.

2.4. $\langle 3, 7 \rangle$

2.5. a. $\| \mathbf{a} \| = 5\sqrt{2}$, b. $\mathbf{b} = \langle -4, -3 \rangle$, c. $3\mathbf{a} - 4\mathbf{b} = \langle 37, 15 \rangle$

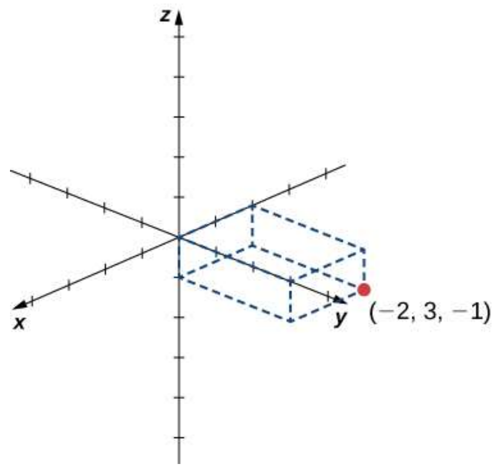
2.7. $\mathbf{v} = \langle -5, 5\sqrt{3} \rangle$

2.8. $\langle -\frac{45}{\sqrt{85}}, -\frac{10}{\sqrt{85}} \rangle$

2.9. $\mathbf{a} = 16\mathbf{i} - 11\mathbf{j}, \quad \mathbf{b} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$

2.10. Approximately 516 mph

2.11.



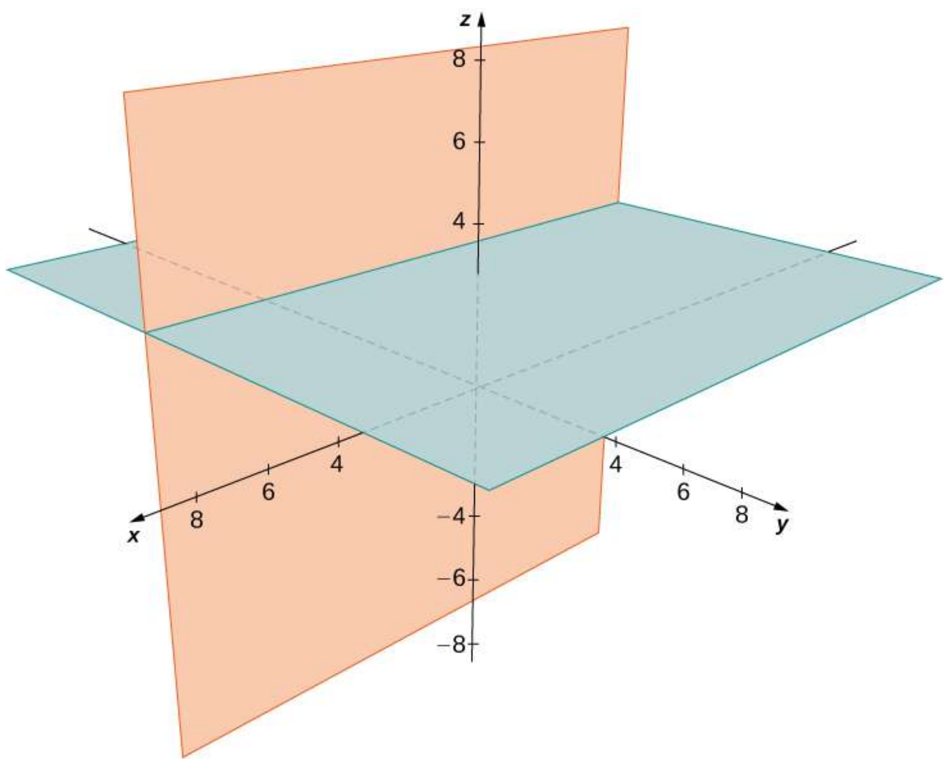
2.12. $5\sqrt{2}$

2.13. $z = -4$

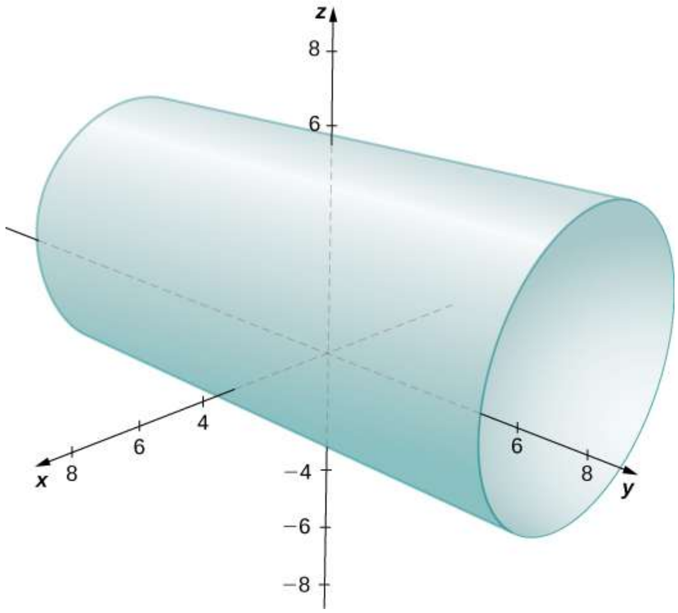
2.14. $(x+2)^2 + (y-4)^2 + (z+5)^2 = 52$

2.15. $x^2 + (y-2)^2 + (z+2)^2 = 14$

2.16. The set of points forms the two planes $y = -2$ and $z = 3$.



2.17. A cylinder of radius 4 centered on the line with $x = 0$ and $z = 2$.



2.18. $\overrightarrow{ST} = \langle -1, -9, 1 \rangle = -\mathbf{i} - 9\mathbf{j} + \mathbf{k}$

2.19. $\langle \frac{1}{3\sqrt{10}}, -\frac{5}{3\sqrt{10}}, \frac{8}{3\sqrt{10}} \rangle$

2.20. $\mathbf{v} = \langle 16\sqrt{2}, 12\sqrt{2}, 20\sqrt{2} \rangle$

2.21. 7

2.22. a. $(\mathbf{r} \cdot \mathbf{p})\mathbf{q} = \langle 12, -12, 12 \rangle$; b. $\| \mathbf{p} \|^2 = 53$

2.23. $\theta \approx 0.22$ rad

2.24. $x = 5$

2.25. a. $\alpha \approx 1.04$ rad; b. $\beta \approx 2.58$ rad; c. $\gamma \approx 1.40$ rad

2.26. Sales = $15,685.50; profit = $14,073.15

2.27. $\mathbf{v} = \mathbf{p} + \mathbf{q},$ where $\mathbf{p} = \frac{18}{5}\mathbf{i} + \frac{9}{5}\mathbf{j}$ and $\mathbf{q} = \frac{7}{5}\mathbf{i} - \frac{14}{5}\mathbf{j}$

2.28. 21 knots

2.29. 150 ft-lb

2.30. $\mathbf{i} - 9\mathbf{j} + 2\mathbf{k}$

2.31. Up (the positive z-direction)

2.32. $-\mathbf{i}$

2.33. $-\mathbf{k}$

2.34. 16

2.35. 40

2.36. $8\mathbf{i} - 35\mathbf{j} + 2\mathbf{k}$

2.37. $\langle \frac{-3}{\sqrt{194}}, \frac{-13}{\sqrt{194}}, \frac{4}{\sqrt{194}} \rangle$

2.38. $6\sqrt{13}$

2.39. 17

2.40. 8 units^3

2.41. No, the triple scalar product is $-4 \neq 0,$ so the three vectors form the adjacent edges of a parallelepiped. They are not coplanar.

2.42. 20 N

2.43. Possible set of parametric equations: $x = 1 + 4t,\ y = -3 + t,\ z = 2 + 6t;$ related set of symmetric equations: $\frac{x-1}{4} = y + 3 = \frac{z-2}{6}$

2.44. $x = -1 - 7t,\ y = 3 - t,\ z = 6 - 2t,\ 0 \le t \le 1$

2.45. $\sqrt{\frac{10}{7}}$

2.46. These lines are skew because their direction vectors are not parallel and there is no point $(x,\ y,\ z)$ that lies on both lines.

2.47. $-2(x - 1) + (y + 1) + 3(z - 1) = 0$ or $-2x + y + 3z = 0$

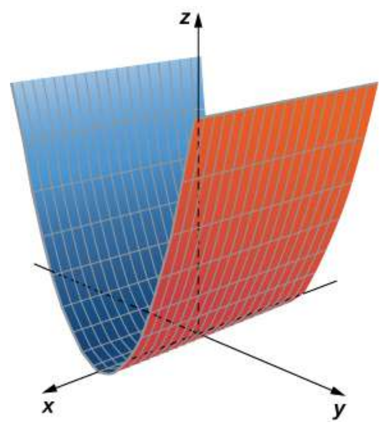
2.48. $\frac{15}{\sqrt{21}}$

2.49. $x = t,\ y = 7 - 3t,\ z = 4 - 2t$

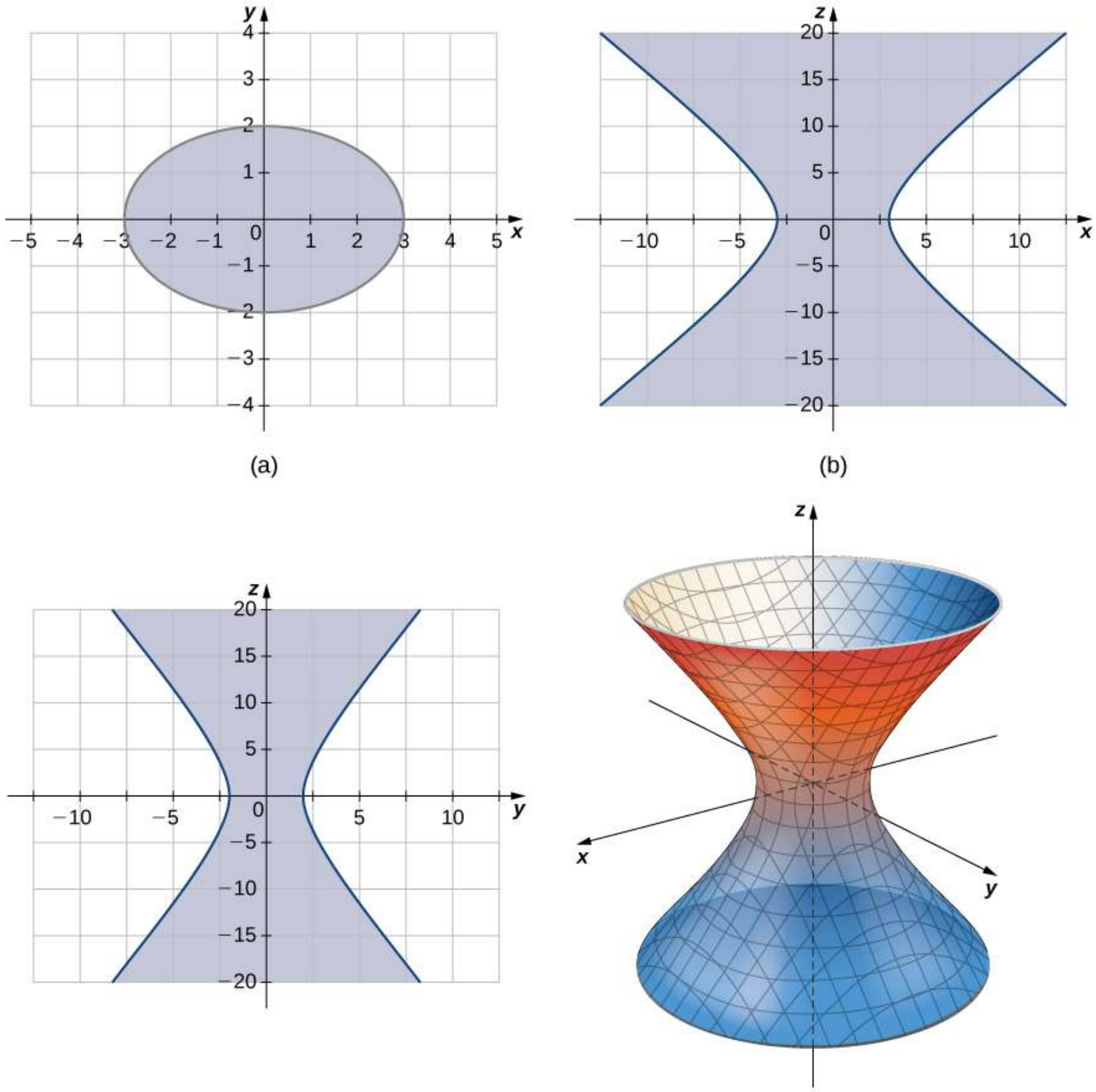
2.50. 1.44 rad

2.51. $\frac{9}{\sqrt{30}}$

2.52.



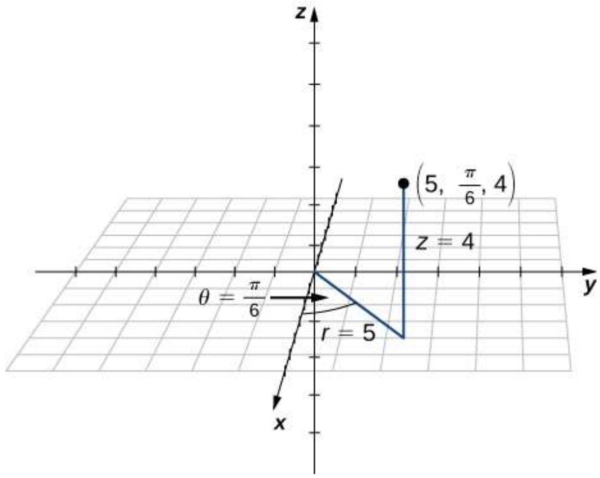
2.53. The traces parallel to the *xy*-plane are ellipses and the traces parallel to the *xz*- and *yz*-planes are hyperbolas. Specifically, the trace in the *xy*-plane is ellipse $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1,$ the trace in the *xz*-plane is hyperbola $\frac{x^2}{3^2} - \frac{z^2}{5^2} = 1,$ and the trace in the *yz*-plane is hyperbola $\frac{y^2}{2^2} - \frac{z^2}{5^2} = 1$ (see the following figure).



(a) (b) (c) (d)

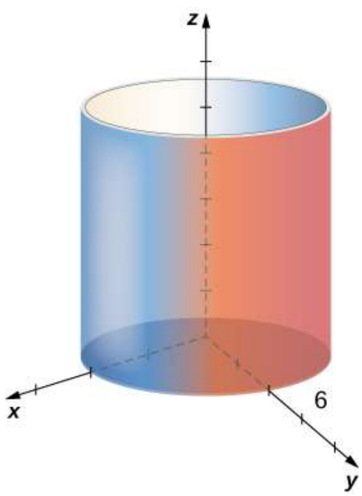
2.54. Hyperboloid of one sheet, centered at $(0,\ 0,\ 1)$

2.55. The rectangular coordinates of the point are $\left(\frac{5\sqrt{3}}{2},\ \frac{5}{2},\ 4\right)$.

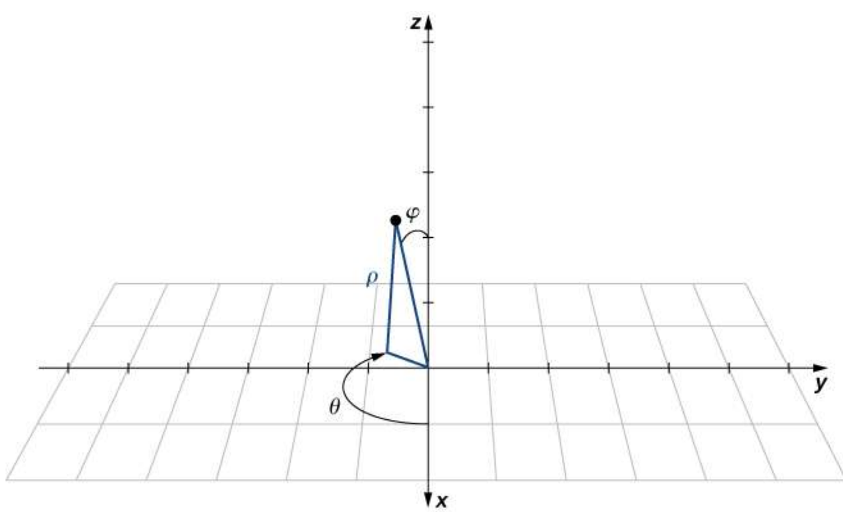


2.56. $\left(8\sqrt{2}, \frac{3\pi}{4}, -7\right)$

2.57. This surface is a cylinder with radius 6.



2.58.



Cartesian: $\left(-\frac{\sqrt{3}}{2},\ -\frac{1}{2}, \sqrt{3}\right)$, cylindrical: $\left(1,\ -\frac{5\pi}{6}, \sqrt{3}\right)$

2.59. a. This is the set of all points 13 units from the origin. This set forms a sphere with radius $13.$ b. This set of points forms a half plane. The angle between the half plane and the positive *x*-axis is $\theta = \frac{2\pi}{3}.$ c. Let P be a point on this surface. The position vector of this point forms an angle of $\varphi = \frac{\pi}{4}$ with the positive *z*-axis, which means that points closer to the origin are closer to the axis. These points form a half-cone.

2.60. $(4000,\ 151°,\ 124°)$

2.61. Spherical coordinates with the origin located at the center of the earth, the *z*-axis aligned with the North Pole, and the *x*-axis aligned with the prime meridian

Section Exercises

1. a. $\overrightarrow{PQ} = \langle 2, 2 \rangle$; b. $\overrightarrow{PQ} = 2\mathbf{i} + 2\mathbf{j}$

3. a. $\overrightarrow{QP} = \langle -2, -2 \rangle$; b. $\overrightarrow{QP} = -2\mathbf{i} - 2\mathbf{j}$

5. a. $\overrightarrow{PQ} + \overrightarrow{PR} = \langle 0, 6 \rangle$; b. $\overrightarrow{PQ} + \overrightarrow{PR} = 6\mathbf{j}$

7. a. $2\overrightarrow{PQ} - 2\overrightarrow{PR} = \langle 8, -4 \rangle$; b. $2\overrightarrow{PQ} - 2\overrightarrow{PR} = 8\mathbf{i} - 4\mathbf{j}$

9. a. $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$; b. $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

11. $\langle \frac{3}{5}, \frac{4}{5} \rangle$

13. $Q(0, 2)$

15. a. $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{a} + \mathbf{b} = \langle 3, 4 \rangle$; b. $\mathbf{a} - \mathbf{b} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{a} - \mathbf{b} = \langle 1, -2 \rangle$; c. Answers will vary; d. $2\mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$, $2\mathbf{a} = \langle 4, 2 \rangle$, $-\mathbf{b} = -\mathbf{i} - 3\mathbf{j}$, $-\mathbf{b} = \langle -1, -3 \rangle$, $2\mathbf{a} - \mathbf{b} = 3\mathbf{i} - \mathbf{j}$, $2\mathbf{a} - \mathbf{b} = \langle 3, -1 \rangle$

17. 15

19. $\lambda = -3$

21. a. $\mathbf{a}(0) = \langle 1, 0 \rangle$, $\mathbf{a}(\pi) = \langle -1, 0 \rangle$; b. Answers may vary; c. Answers may vary

23. Answers may vary

25. $\mathbf{v} = \langle \frac{21}{5}, \frac{28}{5} \rangle$

27. $\mathbf{v} = \langle \frac{21\sqrt{34}}{34},\ -\frac{35\sqrt{34}}{34} \rangle$

29. $\mathbf{u} = \langle \sqrt{3}, 1 \rangle$

31. $\mathbf{u} = \langle 0, 5 \rangle$

33. $\mathbf{u} = \langle -5\sqrt{3}, 5 \rangle$

35. $\theta = \frac{7\pi}{4}$

37. Answers may vary

39. a. $z_0 = f(x_0) + f'(x_0)$; b. $\mathbf{u} = \frac{1}{\sqrt{1 + [f'(x_0)]^2}} \langle 1,\ f'(x_0) \rangle$

43. $D(6, 1)$

45. $\langle 60.62, 35 \rangle$

47. The horizontal and vertical components are 750 ft/sec and 1299.04 ft/sec, respectively.

49. The magnitude of resultant force is 94.71 lb; the direction angle is $13.42°$.

51. The magnitude of the third vector is 60.03 N; the direction angle is $259.38°$.

53. The new ground speed of the airplane is 572.19 mph; the new direction is N41.82E.

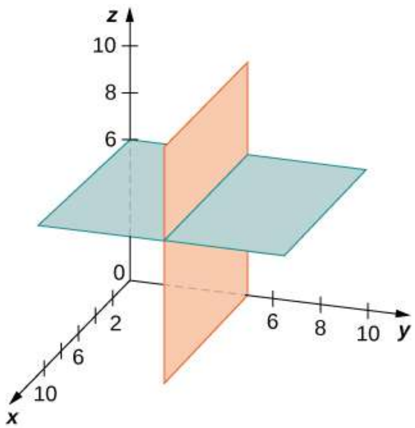
55. $\| \mathbf{T}_1 \| = 30.13$ lb, $\| \mathbf{T}_2 \| = 38.35$ lb

57. $\| \mathbf{v}_1 \| = 750$ lb, $\| \mathbf{v}_2 \| = 1299$ lb

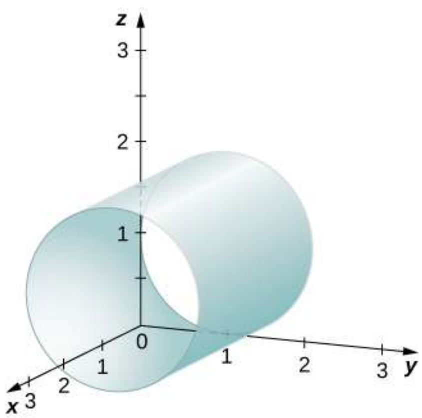
59. The two horizontal and vertical components of the force of tension are 28 lb and 42 lb, respectively.

61. a. $(2,\ 0,\ 5),\ (2,\ 0,\ 0),\ (2,\ 3,\ 0),\ (0,\ 3,\ 0),\ (0,\ 3,\ 5),\ (0,\ 0,\ 5);$ b. $\sqrt{38}$

63. A union of two planes: $y = 5$ (a plane parallel to the *xz*-plane) and $z = 6$ (a plane parallel to the *xy*-plane)



65. A cylinder of radius 1 centered on the line $y = 1,\ z = 1$



67. $z = 1$

69. $z = -2$

71. $(x+1)^2 + (y-7)^2 + (z-4)^2 = 16$

73. $(x+3)^2 + (y-3.5)^2 + (z-8)^2 = \frac{29}{4}$

75. Center $C(0,\ 0,\ 2)$ and radius 1

77. a. $\overrightarrow{PQ} = \langle -4,\ -1,\ 2 \rangle$; b. $\overrightarrow{PQ} = -4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

79. a. $\overrightarrow{PQ} = \langle 6,\ -24,\ 24 \rangle$; b. $\overrightarrow{PQ} = 6\mathbf{i} - 24\mathbf{j} + 24\mathbf{k}$

81. $Q(5,\ 2,\ 8)$

83. $\mathbf{a} + \mathbf{b} = \langle -6,\ 4,\ -3 \rangle$, $4\mathbf{a} = \langle -4,\ -8,\ 16 \rangle$, $-5\mathbf{a} + 3\mathbf{b} = \langle -10,\ 28,\ -41 \rangle$

85. $\mathbf{a} + \mathbf{b} = \langle -1,\ 0,\ -1 \rangle$, $4\mathbf{a} = \langle 0,\ 0,\ -4 \rangle$, $-5\mathbf{a} + 3\mathbf{b} = \langle -3,\ 0,\ 5 \rangle$

87. $\|\mathbf{u} - \mathbf{v}\| = \sqrt{38}$, $\|-2\mathbf{u}\| = 2\sqrt{29}$

89. $\|\mathbf{u} - \mathbf{v}\| = 2$, $\|-2\mathbf{u}\| = 2\sqrt{13}$

91. $\mathbf{a} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

93. $\langle \frac{2}{\sqrt{62}},\ -\frac{7}{\sqrt{62}},\ \frac{3}{\sqrt{62}} \rangle$

95. $\langle -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}} \rangle$

97. Equivalent vectors

99. $\mathbf{u} = \langle \frac{70}{\sqrt{59}}, -\frac{10}{\sqrt{59}}, \frac{30}{\sqrt{59}} \rangle$

101. $\mathbf{u} = \langle -\frac{4}{\sqrt{5}}\sin t, -\frac{4}{\sqrt{5}}\cos t, -\frac{2}{\sqrt{5}} \rangle$

103. $\langle \frac{5}{\sqrt{154}}, \frac{15}{\sqrt{154}}, -\frac{60}{\sqrt{154}} \rangle$

105. $\alpha = -\sqrt{7}, \quad \beta = -\sqrt{15}$

111. a. $\mathbf{F} = \langle 30, 40, 0 \rangle$; b. $53°$

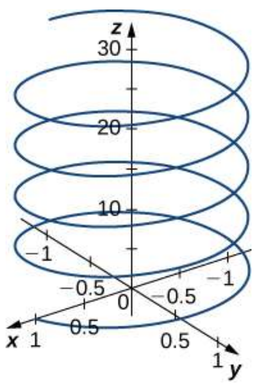
113. $\mathbf{D} = 10\mathbf{k}$

115. $\mathbf{F}_4 = \langle -20, -7, -3 \rangle$

117. a. $\mathbf{F} = -19.6\mathbf{k}, \quad \|\mathbf{F}\| = 19.6$ N; b. $\mathbf{T} = 19.6\mathbf{k}, \quad \|\mathbf{T}\| = 19.6$ N

119. a. $\mathbf{F} = -294\mathbf{k}$ N; b. $\mathbf{F}_1 = \langle -\frac{49\sqrt{3}}{3}, 49, -98 \rangle$, $\mathbf{F}_2 = \langle -\frac{49\sqrt{3}}{3}, -49, -98 \rangle$, and $\mathbf{F}_3 = \langle \frac{98\sqrt{3}}{3}, 0, -98 \rangle$ (each component is expressed in newtons)

121. a. $\mathbf{v}(1) = \langle -0.84, 0.54, 2 \rangle$ (each component is expressed in centimeters per second); $\|\mathbf{v}(1)\| = 2.24$ (expressed in centimeters per second); $\mathbf{a}(1) = \langle -0.54, -0.84, 0 \rangle$ (each component expressed in centimeters per second squared); b.



123. 6

125. 0

127. $(\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \langle -11, -11, 11 \rangle$; $(\mathbf{a}\cdot\mathbf{c})\mathbf{b} = \langle -20, -35, 5 \rangle$

129. $(\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \langle 1, 0, -2 \rangle$; $(\mathbf{a}\cdot\mathbf{c})\mathbf{b} = \langle 1, 0, -1 \rangle$

131. a. $\theta = 2.82$ rad; b. θ is not acute.

133. a. $\theta = \frac{\pi}{4}$ rad; b. θ is acute.

135. $\theta = \frac{\pi}{2}$

137. $\theta = \frac{\pi}{3}$

139. $\theta = 2$ rad

141. Orthogonal

143. Not orthogonal

145. $\mathbf{a} = \langle -\frac{4\alpha}{3}, \alpha \rangle$, where $\alpha \neq 0$ is a real number

147. $\mathbf{u} = -\alpha\mathbf{i} + \alpha\mathbf{j} + \beta\mathbf{k}$, where α and β are real numbers such that $\alpha^2 + \beta^2 \neq 0$

149. $\alpha = -6$

151. a. $\overrightarrow{OP} = 4\mathbf{i} + 5\mathbf{j}, \quad \overrightarrow{OQ} = 5\mathbf{i} - 7\mathbf{j}$; b. $105.8°$

153. $68.33°$

155. **u** and **v** are orthogonal; **v** and **w** are orthogonal.

161. a. $\cos\alpha = \frac{2}{3}, \cos\beta = \frac{2}{3},$ and $\cos\gamma = \frac{1}{3}$; b. $\alpha = 48°, \quad \beta = 48°,$ and $\gamma = 71°$

163. a. $\cos\alpha = -\frac{1}{\sqrt{30}}, \cos\beta = \frac{5}{\sqrt{30}},$ and $\cos\gamma = \frac{2}{\sqrt{30}};$ b. $\alpha = 101°, \quad \beta = 24°,$ and $\gamma = 69°$

167. a. $\mathbf{w} = \langle \frac{80}{29}, \frac{32}{29} \rangle$; b. $\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{16}{\sqrt{29}}$

169. a. $\mathbf{w} = \langle \frac{24}{13}, 0, \frac{16}{13} \rangle$; b. $\text{comp}_{\mathbf{u}}\mathbf{v} = \frac{8}{\sqrt{13}}$

171. a. $\mathbf{w} = \langle \frac{24}{25}, -\frac{18}{25} \rangle$; b. $\mathbf{q} = \langle \frac{51}{25}, \frac{68}{25} \rangle$, $\quad \mathbf{v} = \mathbf{w} + \mathbf{q} = \langle \frac{24}{25}, -\frac{18}{25} \rangle + \langle \frac{51}{25}, \frac{68}{25} \rangle$

173. a. $2\sqrt{2}$; b. $109.47°$

175. $17\text{N} \cdot \text{m}$

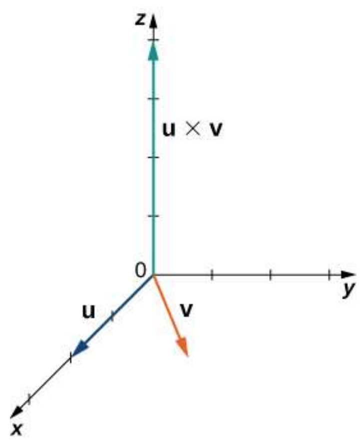
177. $1175 \text{ ft} \cdot \text{lb}$

179. 4330.13 ft-lb

181. a. $\| \mathbf{F}_1 + \mathbf{F}_2 \| = 52.9$ lb; b. The direction angles are $\alpha = 74.5°, \quad \beta = 36.7°,$ and $\gamma = 57.7°$.

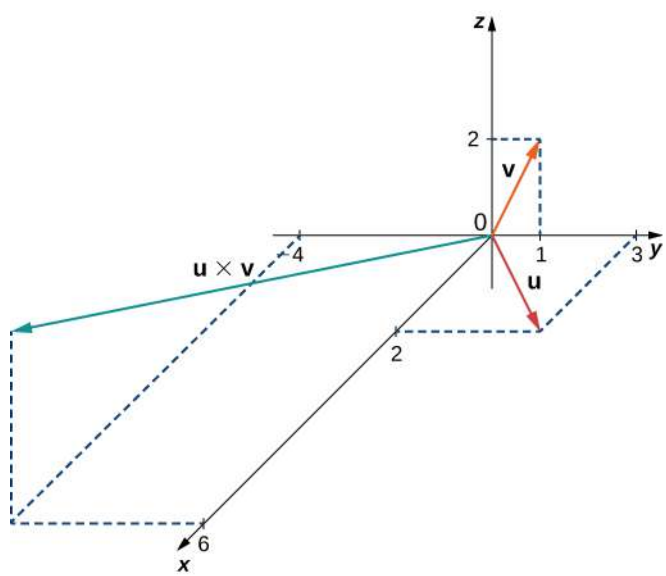
183. a. $\mathbf{u} \times \mathbf{v} = \langle 0, 0, 4 \rangle$;

b.



185. a. $\mathbf{u} \times \mathbf{v} = \langle 6, -4, 2 \rangle$;

b.



187. $-2\mathbf{j} - 4\mathbf{k}$

189. $\mathbf{w} = -\frac{1}{3\sqrt{6}}\mathbf{i} - \frac{7}{3\sqrt{6}}\mathbf{j} - \frac{2}{3\sqrt{6}}\mathbf{k}$

191. $\mathbf{w} = -\frac{4}{\sqrt{21}}\mathbf{i} - \frac{2}{\sqrt{21}}\mathbf{j} - \frac{1}{\sqrt{21}}\mathbf{k}$

193. $\alpha = 10$

197. $-3\mathbf{i} + 11\mathbf{j} + 2\mathbf{k}$

199. $\mathbf{w} = \langle -1, e^t, -e^{-t} \rangle$

201. $-26\mathbf{i} + 17\mathbf{j} + 9\mathbf{k}$

203. $72°$

209. 7

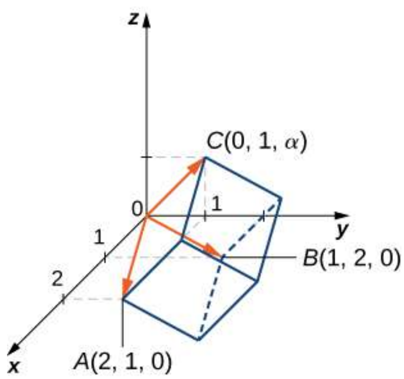
211. a. $5\sqrt{6}$; b. $\frac{5\sqrt{6}}{2}$; c. $\frac{5\sqrt{6}}{\sqrt{59}}$

213. a. 2; b. 2

215. $\mathbf{v} \cdot (\mathbf{u} \times \mathrm{w}) = -1, \quad \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = 1$

217. $\mathbf{a} = \langle 1, 2, 3 \rangle, \quad \mathbf{b} = \langle 0, 2, 5 \rangle, \quad \mathbf{c} = \langle 8, 9, 2 \rangle; \ \mathbf{a} \cdot (\mathrm{b} \times \mathrm{c}) = -9$

219. a. $\alpha = 1$; b. $h = 1$,



225. Yes, $\overrightarrow{AD} = \alpha\overrightarrow{AB} + \beta\overrightarrow{AC}$, where $\alpha = -1$ and $\beta = 1$.

227. $-\mathbf{k}$

229. $\langle 0, \pm 4\sqrt{5}, 2\sqrt{5} \rangle$

233. $\mathbf{w} = \langle w_3 - 1, w_3 + 1, w_3 \rangle$, where w_3 is any real number

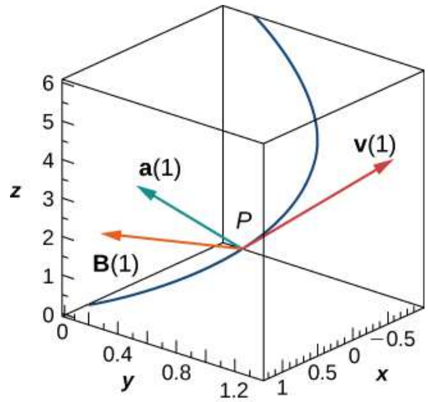
235. 8.66 ft-lb

237. 250 N

239. $\mathbf{F} = 4.8 \times 10^{-15}\,\mathbf{k}\,\mathbf{N}$

241. a. $\mathbf{B}(t) = \langle \frac{2\sin t}{\sqrt{5}}, -\frac{2\cos t}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$;

b.



243. a. $\mathbf{r} = \langle -3, 5, 9 \rangle + t\langle 7, -12, -7 \rangle, \quad t \in \mathbb{R}$; b. $x = -3 + 7t,\ y = 5 - 12t,\ z = 9 - 7t, \quad t \in \mathbb{R}$; c. $\frac{x+3}{7} = \frac{y-5}{-12} = \frac{z-9}{-7}$; d. $x = -3 + 7t,\ y = 5 - 12t,\ z = 9 - 7t, \quad t \in [0, 1]$

245. a. $\mathbf{r} = \langle -1, 0, 5 \rangle + t\langle 5, 0, -2 \rangle, \quad t \in \mathbb{R}$; b. $x = -1 + 5t,\ y = 0,\ z = 5 - 2t, \quad t \in \mathbb{R}$; c. $\frac{x+1}{5} = \frac{z-5}{-2},\ y = 0$; d. $x = -1 + 5t,\ y = 0,\ z = 5 - 2t, \quad t \in [0, 1]$

247. a. $x = 1 + t,\ y = -2 + 2t,\ z = 3 + 3t, \quad t \in \mathbb{R}$; b. $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-3}{3}$; c. $(0, -4, 0)$

249. a. $x = 3 + t,\ y = 1,\ z = 5, \quad t \in \mathbb{R}$; b. $y = 1,\ z = 5$; c. The line does not intersect the *xy*-plane.

251. a. $P(1, 3, 5), \quad v = \langle 1, 1, 4 \rangle$; b. $\sqrt{3}$

253. $\frac{2\sqrt{2}}{\sqrt{3}}$

255. a. Parallel; b. $\frac{\sqrt{2}}{\sqrt{3}}$

259. $(-12, 6, -4)$

261. The lines are skew.

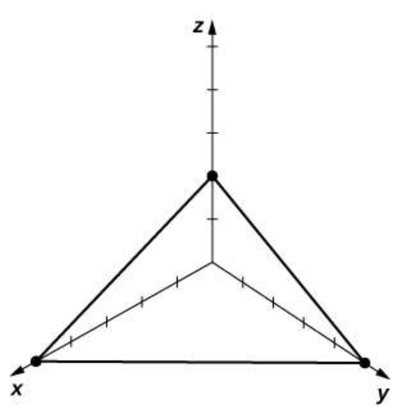
263. The lines are equal.

265. a. $x = 1 + t,\ y = 1 - t,\ z = 1 + 2t, \quad t \in \mathbb{R}$; b. For instance, the line passing through A with direction vector $\mathbf{j} : x = 1,\ z = 1$; c. For instance, the line passing through A and point $(2, 0, 0)$ that belongs to L is a line that intersects; $L : \frac{x-1}{-1} = y - 1 = z - 1$

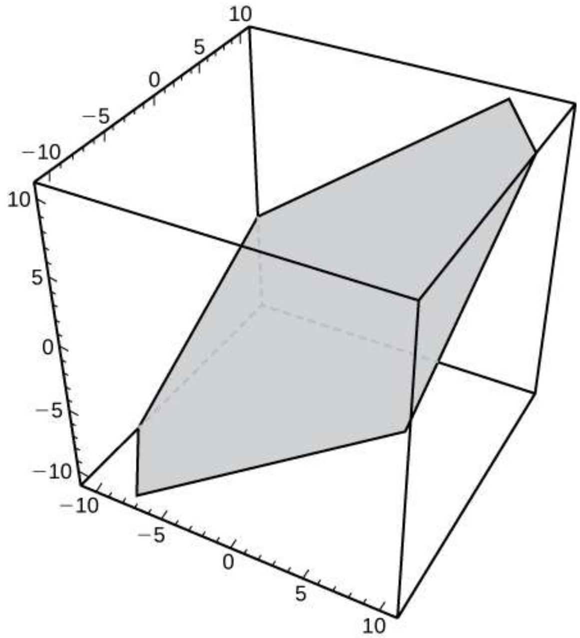
267. a. $3x - 2y + 4z = 0$; b. $3x - 2y + 4z = 0$

269. a. $(x - 1) + 2(y - 2) + 3(z - 3) = 0$; b. $x + 2y + 3z - 14 = 0$

271. a. $\mathbf{n} = 4\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$; b. $(5, 0, 0), \quad (0, 4, 0), \quad$ and $(0, 0, 2)$;
c.



273. a. $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$; b. $(0, 0, 0)$;
c.



275. $(3, 0, 0)$

277. $x = -2 + 2t,\ y = 1 - 3t,\ z = 3 + t, \quad t \in \mathbb{R}$

281. a. $-2y + 3z - 1 = 0$; b. $\langle 0, -2, 3 \rangle \cdot \langle x - 1, y - 1, z - 1 \rangle = 0$; c. $x = 0,\ y = -2t,\ z = 3t, \quad t \in \mathbb{R}$

283. a. Answers may vary; b. $\frac{x-1}{1} = \frac{z-6}{-1},\ y = 4$

285. $2x - 5y - 3z + 15 = 0$

287. The line intersects the plane at point $P(-3, 4, 0)$.

289. $\frac{16}{\sqrt{14}}$

291. a. The planes are neither parallel nor orthogonal; b. $62°$

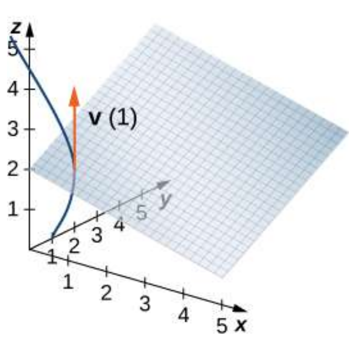
293. a. The planes are parallel.

295. $\frac{1}{\sqrt{6}}$

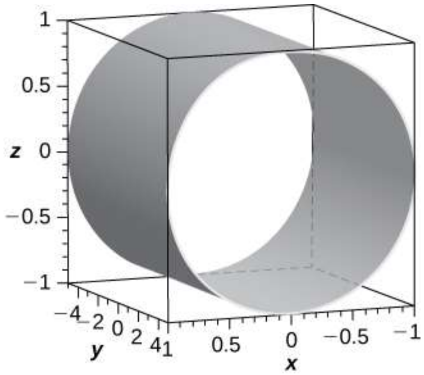
297. a. $\frac{18}{\sqrt{29}}$; b. $P\left(-\frac{51}{29}, \frac{130}{29}, \frac{62}{29}\right)$

299. $4x - 3y = 0$

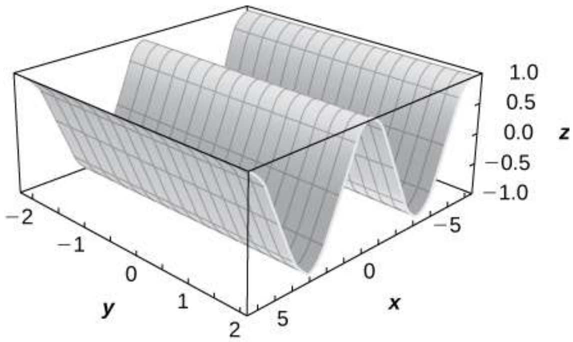
301. a. $\mathbf{v}(1) = \langle \cos 1, -\sin 1, 2 \rangle$; b. $(\cos 1)(x - \sin 1) - (\sin 1)(y - \cos 1) + 2(z - 2) = 0$;
c.



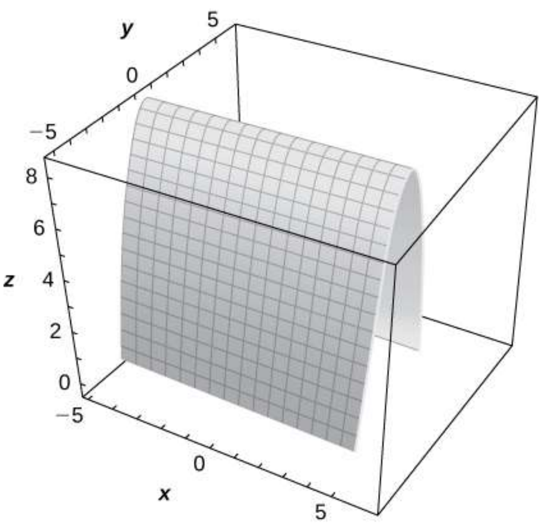
303. The surface is a cylinder with the rulings parallel to the y-axis.



305. The surface is a cylinder with rulings parallel to the *y*-axis.



307. The surface is a cylinder with rulings parallel to the *x*-axis.



309. a. Cylinder; b. The *x*-axis
311. a. Hyperboloid of two sheets; b. The *x*-axis
313. b.
315. d.
317. a.

319. $-\frac{x^2}{9}+\frac{y^2}{\frac{1}{4}}+\frac{z^2}{\frac{1}{4}}=1,$ hyperboloid of one sheet with the *x*-axis as its axis of symmetry

321. $-\frac{x^2}{\frac{10}{3}}+\frac{y^2}{2}-\frac{z^2}{10}=1,$ hyperboloid of two sheets with the *y*-axis as its axis of symmetry

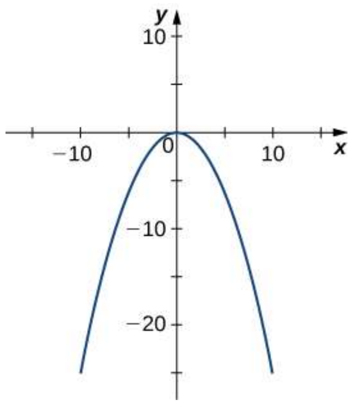
323. $y = -\frac{z^2}{5} + \frac{x^2}{5},$ hyperbolic paraboloid with the *y*-axis as its axis of symmetry

325. $\frac{x^2}{15} + \frac{y^2}{3} + \frac{z^2}{5} = 1,$ ellipsoid

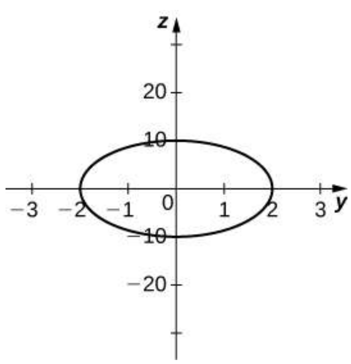
327. $\frac{x^2}{40} + \frac{y^2}{8} - \frac{z^2}{5} = 0,$ elliptic cone with the *z*-axis as its axis of symmetry

329. $x = \frac{y^2}{2} + \frac{z^2}{3},$ elliptic paraboloid with the *x*-axis as its axis of symmetry

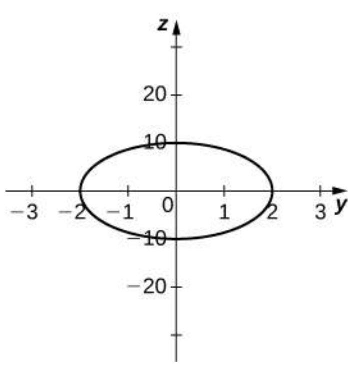
331. Parabola $y = -\frac{x^2}{4},$



333. Ellipse $\frac{y^2}{4} + \frac{z^2}{100} = 1,$



335. Ellipse $\frac{y^2}{4} + \frac{z^2}{100} = 1,$



337. a. Ellipsoid; b. The third equation; c. $\frac{x^2}{100} + \frac{y^2}{400} + \frac{z^2}{225} = 1$

339. a. $\frac{(x+3)^2}{16} + \frac{(z-2)^2}{8} = 1$; b. Cylinder centered at $(-3, 2)$ with rulings parallel to the *y*-axis

341. a. $\frac{(x-3)^2}{4} + (y-2)^2 - (z+2)^2 = 1$; b. Hyperboloid of one sheet centered at $(3, 2, -2)$, with the *z*-axis as its axis of symmetry

343. a. $(x+3)^2 + \frac{y^2}{4} - \frac{z^2}{3} = 0$; b. Elliptic cone centered at $(-3, 0, 0)$, with the *z*-axis as its axis of symmetry

345. $\frac{x^2}{4} + \frac{y^2}{16} + z^2 = 1$

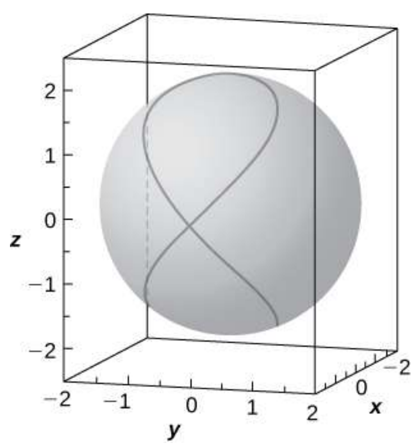
347. $(1, -1, 0)$ and $\left(\frac{13}{3}, 4, \frac{5}{3}\right)$

349. $x^2 + z^2 + 4y = 0$, elliptic paraboloid

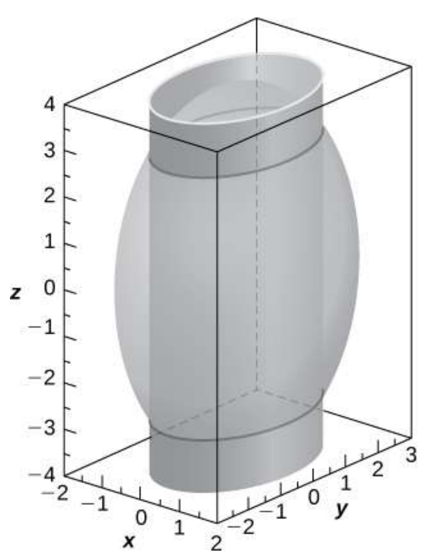
351. $(0, 0, 100)$

355. a. $x = 2 - \frac{z^2}{2}$, $y = \pm \frac{z}{2}\sqrt{4 - z^2}$, where $z \in [-2, 2]$;

b.



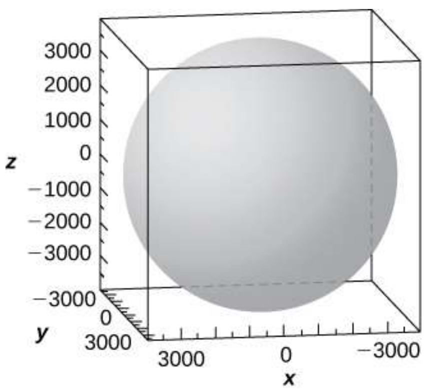
357.



two ellipses of equations $\frac{x^2}{2} + \frac{y^2}{\frac{9}{2}} = 1$ in planes $z = \pm 2\sqrt{2}$

359. a. $\frac{x^2}{3963^2} + \frac{y^2}{3963^2} + \frac{z^2}{3950^2} = 1$;

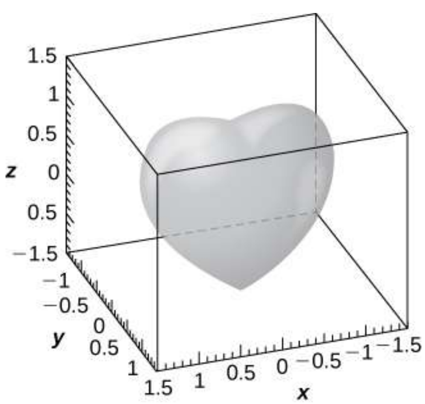
b.



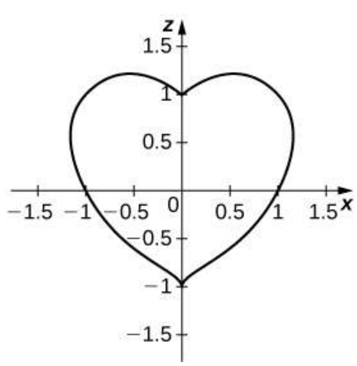
;

c. The intersection curve is the ellipse of equation $\frac{x^2}{3963^2} + \frac{y^2}{3963^2} = \frac{(2950)(4950)}{3950^2}$, and the intersection is an ellipse.; d. The intersection curve is the ellipse of equation $\frac{2y^2}{3963^2} + \frac{z^2}{3950^2} = 1$.

361. a.



b. The intersection curve is $\left(x^2 + z^2 - 1\right)^3 - x^2 z^3 = 0$.



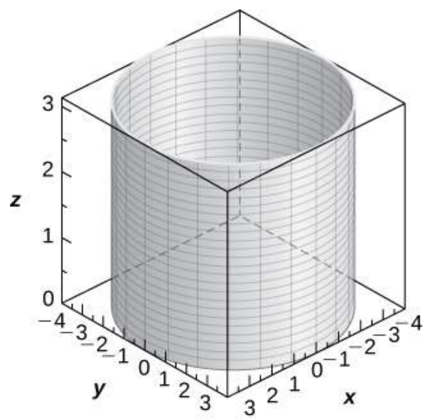
363. $\left(2\sqrt{3},\ 2,\ 3\right)$

365. $\left(-2\sqrt{3},\ -2,\ 3\right)$

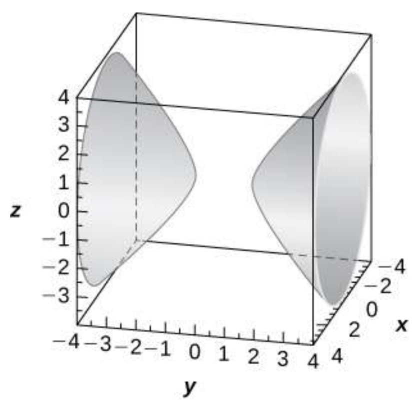
367. $\left(2,\ \frac{\pi}{3},\ 2\right)$

369. $\left(3\sqrt{2},\ -\frac{\pi}{4},\ 7\right)$

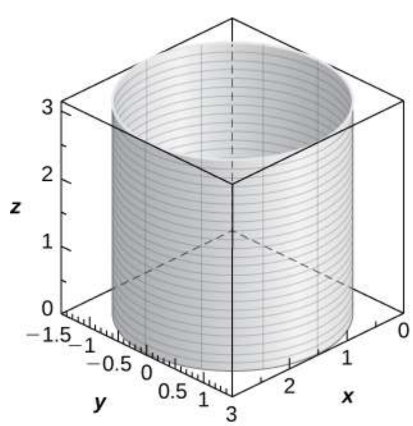
371. A cylinder of equation $x^2 + y^2 = 16,$ with its center at the origin and rulings parallel to the *z*-axis,



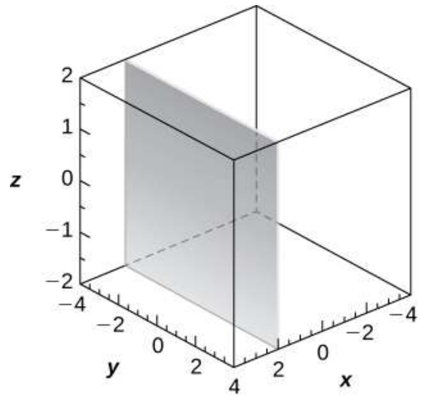
373. Hyperboloid of two sheets of equation $-x^2 + y^2 - z^2 = 1,$ with the *y*-axis as the axis of symmetry,



375. Cylinder of equation $x^2 - 2x + y^2 = 0,$ with a center at $(1,\ 0,\ 0)$ and radius $1,$ with rulings parallel to the *z*-axis,



377. Plane of equation $x = 2,$



379. $z = 3$

381. $r^2 + z^2 = 9$

383. $r = 16\cos\theta,\ r = 0$

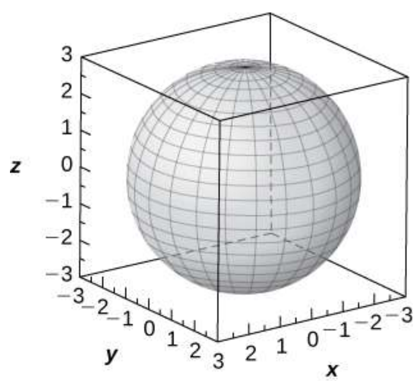
385. $(0,\ 0,\ -3)$

387. $\left(6,\ -6,\ \sqrt{2}\right)$

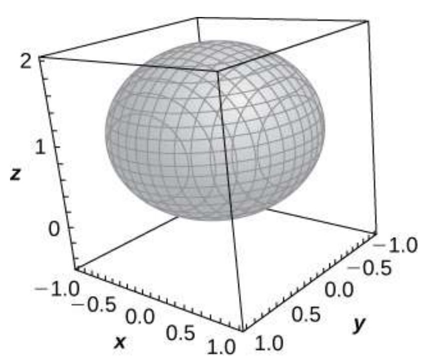
389. $(4,\ 0,\ 90°)$

391. $(3,\ 90°,\ 90°)$

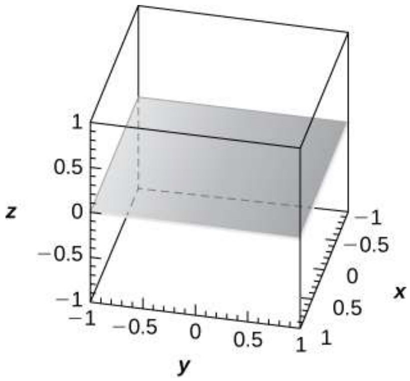
393. Sphere of equation $x^2 + y^2 + z^2 = 9$ centered at the origin with radius 3,



395. Sphere of equation $x^2 + y^2 + (z-1)^2 = 1$ centered at $(0,\ 0,\ 1)$ with radius 1,



397. The *xy*-plane of equation $z = 0$,



399. $\varphi = \frac{\pi}{3}$ or $\varphi = \frac{2\pi}{3}$; Elliptic cone

401. $\rho \cos \varphi = 6$; Plane at $z = 6$

403. $\left(\sqrt{10}, \frac{\pi}{4}, 0.3218\right)$

405. $\left(3\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right)$

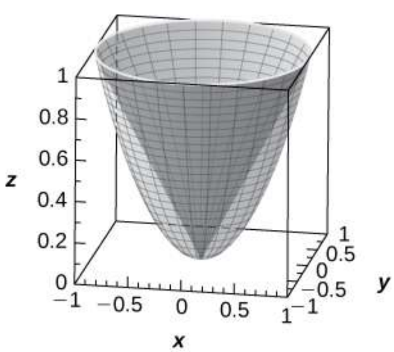
407. $\left(2, -\frac{\pi}{4}, 0\right)$

409. $\left(8, \frac{\pi}{3}, 0\right)$

411. Cartesian system, $\{(x, y, z) | 0 \le x \le a, 0 \le y \le a, 0 \le z \le a\}$

413. Cylindrical system, $\left\{(r, \theta, z) \middle| r^2 + z^2 \le 9, r \ge 3\cos\theta, 0 \le \theta \le 2\pi\right\}$

415. The region is described by the set of points $\left\{(r, \theta, z) \middle| 0 \le r \le 1, 0 \le \theta \le 2\pi, r^2 \le z \le r\right\}$.

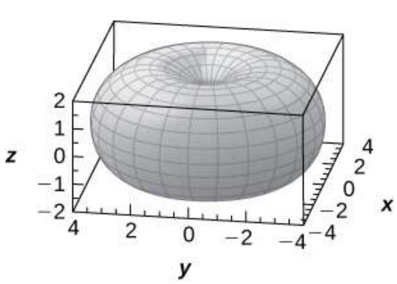


417. $(4000, -77°, 51°)$

419. $43.17°\text{W}, \quad 22.91°\text{S}$

421. a. $\rho = 0, \quad \rho + R^2 - r^2 - 2R\sin\varphi = 0;$

c.



Review Exercises

423. True
425. False
427. a. $\langle 24, -5 \rangle$; b. $\sqrt{85}$; c. Can't dot a vector with a scalar; d. -29

429. $a = \pm 2$

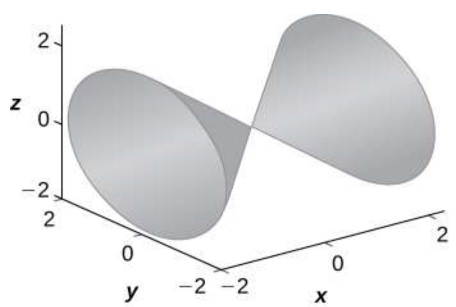
431. $\langle \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \rangle$

433. 27

435. $x = 1 - 3t,\ y = 3 + 3t,\ z = 5 - 8t,\ \mathbf{r}(t) = (1 - 3t)\mathbf{i} + 3(1 + t)\mathbf{j} + (5 - 8t)\mathbf{k}$

437. $-x + 3y + 8z = 43$

439. $x = k$ trace: $k^2 = y^2 + z^2$ is a circle, $y = k$ trace: $x^2 - z^2 = k^2$ is a hyperbola (or a pair of lines if $k = 0$), $z = k$ trace: $x^2 - y^2 = k^2$ is a hyperbola (or a pair of lines if $k = 0$). The surface is a cone.



441. Cylindrical: $z = r^2 - 1$, spherical: $\cos\varphi = \rho\sin^2\varphi - \frac{1}{\rho}$

443. $x^2 - 2x + y^2 + z^2 = 1$, sphere

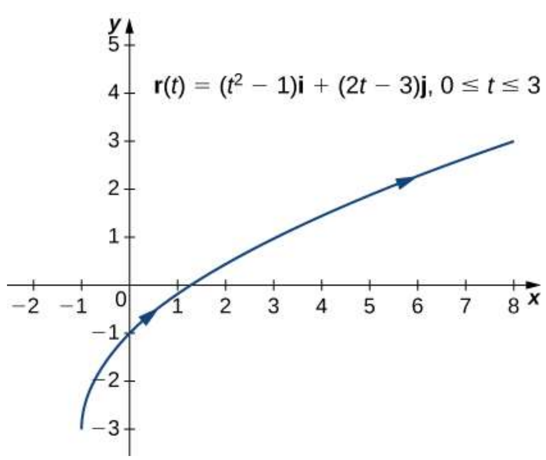
445. 331 N, and 244 N
447. 15 J

449. More, 59.09 J

Chapter 3

Checkpoint

3.1. $\mathbf{r}(0) = \mathbf{j},\ \mathbf{r}(1) = -2\mathbf{i} + 5\mathbf{j},\ \mathbf{r}(-4) = 28\mathbf{i} - 15\mathbf{j}$ The domain of $\mathbf{r}(t) = \left(t^2 - 3t\right)\mathbf{i} + (4t + 1)\mathbf{j}$ is all real numbers.



3.2.

3.3. $\lim\limits_{t \to -2} \mathbf{r}(t) = 3\mathbf{i} - 5\mathbf{j} - \mathbf{k}$

3.4. $\mathbf{r}'(t) = 4t\mathbf{i} + 5\mathbf{j}$

3.5. $\mathbf{r}'(t) = (1 + \ln t)\mathbf{i} + 5e^t\,\mathbf{j} - (\sin t + \cos t)\mathbf{k}$

3.6. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}'(t)] = 8e^{4t}$ $\begin{aligned}&\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{r}(t)]\\ &= -\left(e^{2t}(\cos t + 2\sin t) + \cos 2t\right)\mathbf{i} + \left(e^{2t}(2t+1) - \sin 2t\right)\mathbf{j} + (t\cos t + \sin t - \cos 2t)\mathbf{k}\end{aligned}$

3.7. $\mathbf{T}(t) = \frac{2t}{\sqrt{4t^2+5}}\mathbf{i} + \frac{2}{\sqrt{4t^2+5}}\mathbf{j} + \frac{1}{\sqrt{4t^2+5}}\mathbf{k}$

3.8. $\int_1^3 \left[(2t+4)\mathbf{i} + \left(3t^2 - 4t\right)\mathbf{j}\right]dt = 16\,\mathbf{i} + 10\,\mathbf{j}$

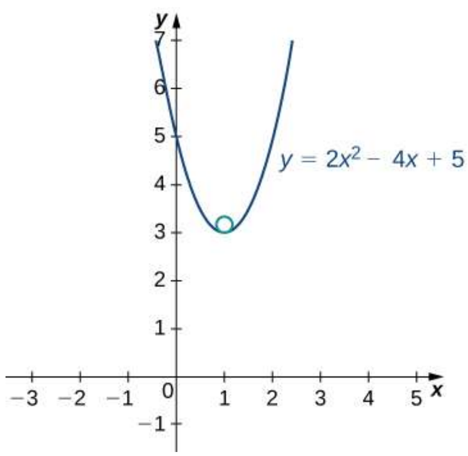
3.9. $\mathbf{r}'(t) = \langle 4t, 4t, 3t^2 \rangle$, so $s = \frac{1}{27}\left(113^{3/2} - 32^{3/2}\right) \approx 37.785$

3.10. $s = 5t$, or $t = s/5$. Substituting this into $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$ gives $\mathbf{r}(s) = \langle 3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5} \rangle$, $s \geq 0$.

3.11. $\kappa = \frac{6}{101^{3/2}} \approx 0.0059$

3.12. $\mathbf{N}(2) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$

3.13. $\kappa = \frac{4}{\left[1 + (4x-4)^2\right]^{3/2}}$ At the point $x = 1$, the curvature is equal to 4. Therefore, the radius of the osculating circle is $\frac{1}{4}$. A graph of this function appears next:



The vertex of this parabola is located at the point $(1, 3)$. Furthermore, the center of the osculating circle is directly above the vertex. Therefore, the coordinates of the center are $\left(1, \frac{13}{4}\right)$. The equation of the osculating circle is $(x-1)^2 + \left(y - \frac{13}{4}\right)^2 = \frac{1}{16}$.

3.14. $\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = (2t-3)\mathbf{i} + 2\mathbf{j} + \mathbf{k}\\ \mathbf{a}(t) &= \mathbf{v}'(t) = 2\mathbf{i}\\ v(t) &= \|\mathbf{r}'(t)\| = \sqrt{(2t-3)^2 + 2^2 + 1^2} = \sqrt{4t^2 - 12t + 14}\end{aligned}$ The units for velocity and speed are feet per second, and the units for acceleration are feet per second squared.

3.15.

a. $$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) = 4\mathbf{i} + 2t\,\mathbf{j}\\ \mathbf{a}(t) &= \mathbf{v}'(t) = 2\mathbf{j}\\ a_{\mathbf{T}} &= \frac{2t}{\sqrt{t^2+4}},\ a_{\mathbf{N}} = \frac{2}{\sqrt{t^2+4}}\end{aligned}$$

b. $a_{\mathbf{T}}(-3) = -\frac{6\sqrt{13}}{13}, a_{\mathbf{N}}(-3) = \frac{2\sqrt{13}}{13}$

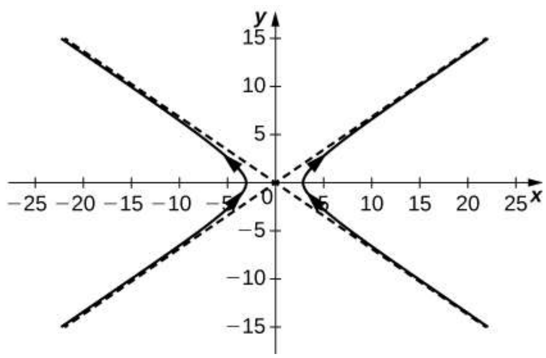
3.16. 967.15 m

3.17. $a = 1.224 \times 10^9 \text{ m} \approx 1{,}224{,}000 \text{ km}$

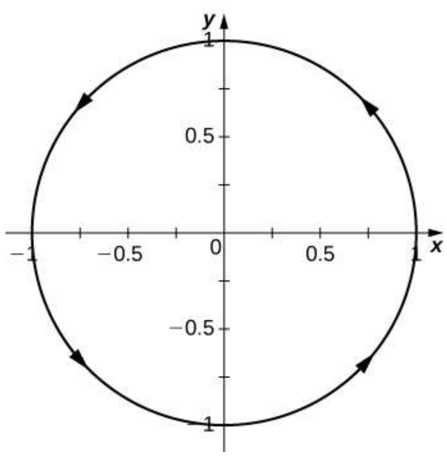
Section Exercises

1. $f(t) = 3\sec t,\ g(t) = 2\tan t$

3.



5. a. $\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$, b. $\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$, c. Yes, the limit as *t* approaches $\pi/3$ is equal to $\mathbf{r}(\pi/3)$, d.



7. a. $\langle e^{\pi/4}, \frac{\sqrt{2}}{2}, \ln\left(\frac{\pi}{4}\right) \rangle$; b. $\langle e^{\pi/4}, \frac{\sqrt{2}}{2}, \ln\left(\frac{\pi}{4}\right) \rangle$; c. Yes

9. $\langle e^{\pi/2}, 1, \ln\left(\frac{\pi}{2}\right) \rangle$

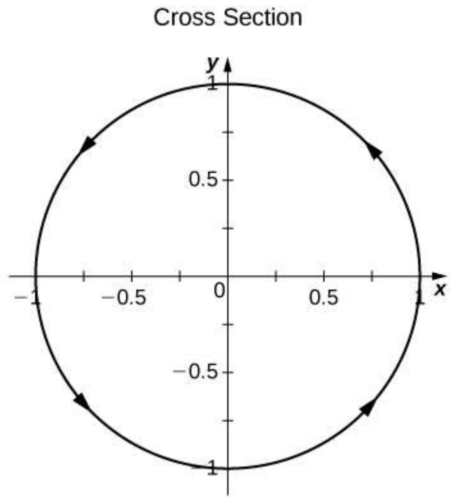
11. $2e^2\,\mathbf{i} + \frac{2}{e^4}\mathbf{j} + 2\mathbf{k}$

13. The limit does not exist because the limit of $\ln(t-1)$ as *t* approaches infinity does not exist.

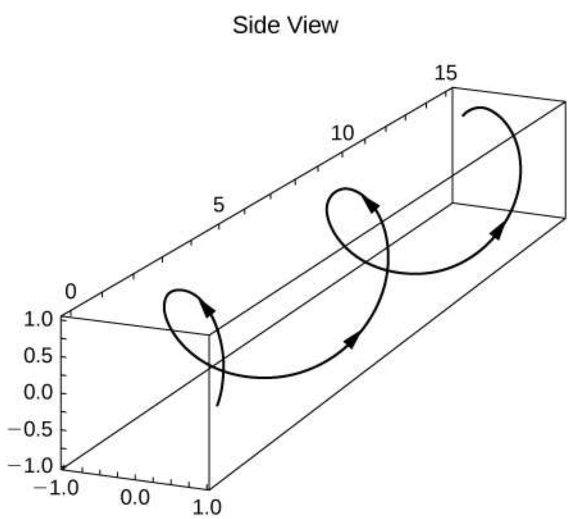
15. $t > 0,\ t \neq (2k+1)\frac{\pi}{2},$ where k is an integer

17. $t > 3,\ t \neq n\pi,$ where *n* is an integer

19.



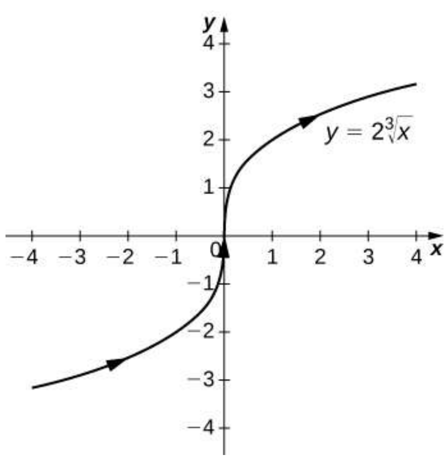
(a)



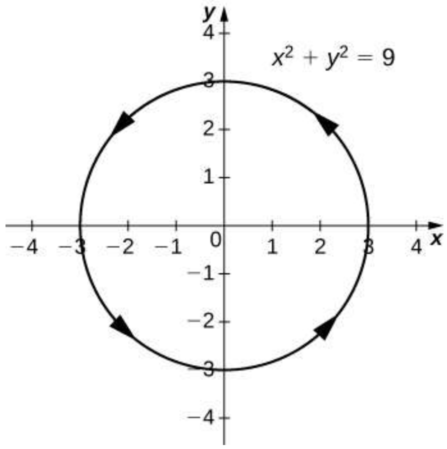
(b)

21. All t such that $t \in (1, \infty)$

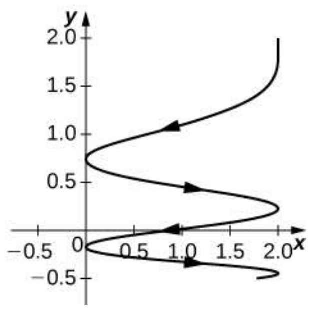
23. $y = 2\sqrt[3]{x},$ a variation of the cube-root function



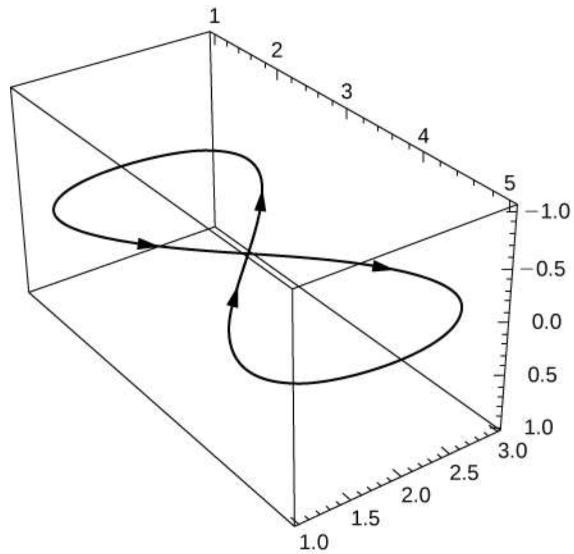
25. $x^2 + y^2 = 9,$ a circle centered at $(0,\ 0)$ with radius 3, and a counterclockwise orientation



27.

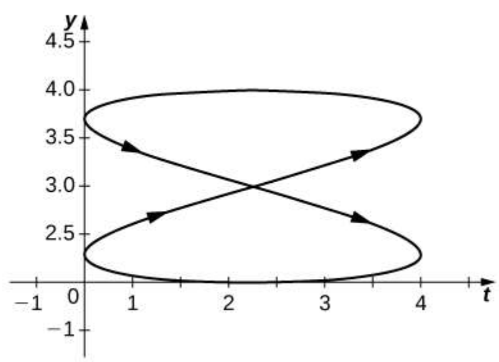


29.



(a)

View in the *yt*-plane



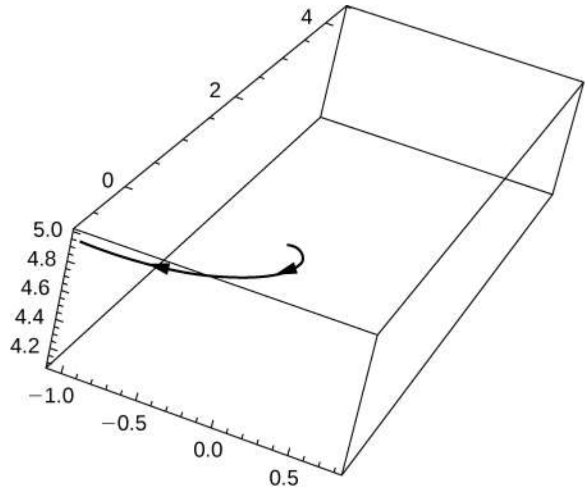
(b)

Find a vector-valued function that traces out the given curve in the indicated direction.

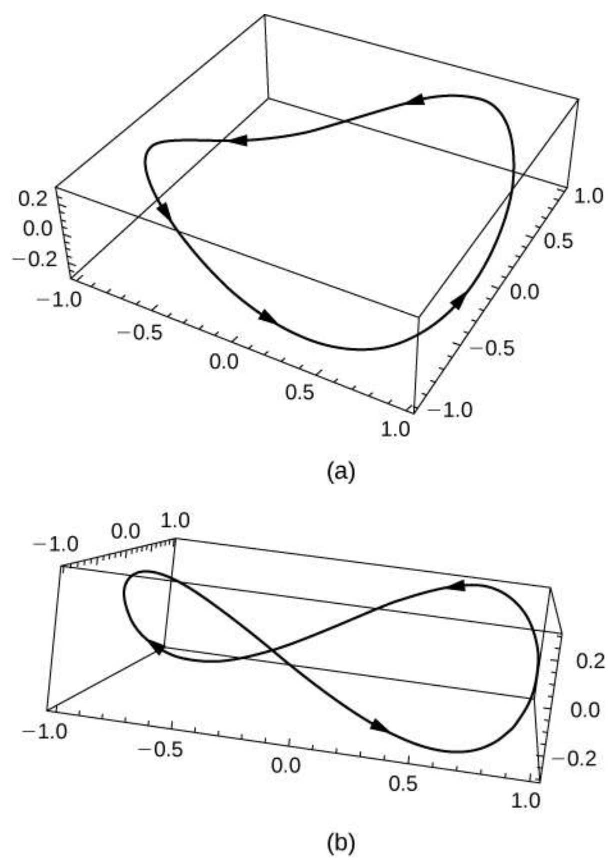
31. For left to right, $y = x^2,$ where t increases

33. $(50, 0, 0)$

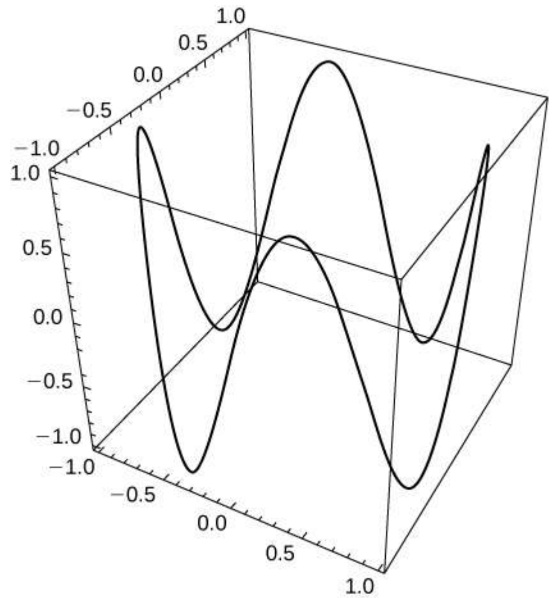
35.



37.



39. One possibility is $r(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin(4t)\,\mathbf{k}$. By increasing the coefficient of t in the third component, the number of turning points will increase.



41. $\langle\, 3t^2,\ 6t,\ \frac{1}{2}t^2 \,\rangle$

43. $\langle\, -e^{-t},\ 3\cos(3t),\ \frac{5}{\sqrt{t}} \,\rangle$

45. $\langle\, 0,\ 0,\ 0 \,\rangle$

47. $\langle\, \frac{-1}{(t+1)^2},\ \frac{1}{1+t^2},\ \frac{3}{t} \,\rangle$

49. $\langle\, 0,\ 12\cos(3t),\ \cos t - t\sin t \,\rangle$

51. $\frac{1}{\sqrt{2}}\langle\, 1,\ -1,\ 0 \,\rangle$

53. $\frac{1}{\sqrt{1060.5625}}\langle\, 6,\ -\frac{3}{4},\ 32 \,\rangle$

55. $\frac{1}{\sqrt{9\sin^2(3t) + 144\cos^2(4t)}}\langle\, 0,\ -3\sin(3t),\ 12\cos(4t) \,\rangle$

57. $\mathbf{T}(t) = \frac{-12}{13}\sin(4t)\,\mathbf{i} + \frac{12}{13}\cos(4t)\,\mathbf{j} + \frac{5}{13}\mathbf{k}$

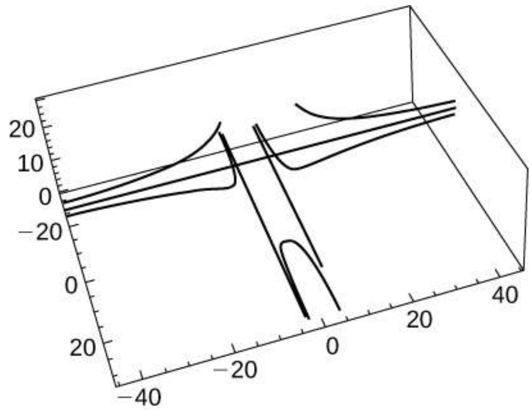
59. $\langle\, 2t,\ 4t^3,\ -8t^7 \,\rangle$

61. $\sin(t) + 2te^t - 4t^3\cos(t) + t\cos(t) + t^2e^t + t^4\sin(t)$

63. $900t^7 + 16t$

65.

a.



b. Undefined or infinite

67. $\mathbf{r}'(t) = -b\omega\sin(\omega t)\,\mathbf{i} + b\omega\cos(\omega t)\,\mathbf{j}$. To show orthogonality, note that $\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$.

69. $0\mathbf{i} + 2\,\mathbf{j} + 4t\,\mathbf{j}$

71. $\frac{1}{3}\left(10^{3/2} - 1\right)$

73.

$$\begin{aligned}
\|\mathbf{v}(t)\| &= k \\
\mathbf{v}(t)\cdot\mathbf{v}(t) &= k \\
\frac{d}{dt}(\mathbf{v}(t)\cdot\mathbf{v}(t)) &= \frac{d}{dt}k = 0 \\
\mathbf{v}(t)\cdot\mathbf{v}'(t) + \mathbf{v}'(t)\cdot\mathbf{v}(t) &= 0 \\
2\,\mathbf{v}(t)\cdot\mathbf{v}'(t) &= 0 \\
\mathbf{v}(t)\cdot\mathbf{v}'(t) &= 0.
\end{aligned}$$

The last statement implies that the velocity and acceleration are perpendicular or orthogonal.

75. $\mathbf{v}(t) = \langle\, 1 - \sin t,\ 1 - \cos t\,\rangle$, speed $= -\ \mathbf{v}(t)\| = \sqrt{4 - 2(\sin t + \cos t)}$

77. $x - 1 = t,\ y - 1 = -t,\ z - 0 = 0$

79. $\mathbf{r}(t) = \langle\, 18,\ 9\,\rangle$ at $t = 3$

81. $\sqrt{593}$

83. $\mathbf{v}(t) = \langle\, -\sin t,\ \cos t,\ 1\,\rangle$

85. $\mathbf{a}(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + 0\,\mathbf{j}$

87. $\mathbf{v}(t) = \langle\, -\sin t,\ 2\cos t,\ 0\,\rangle$

89. $\mathbf{a}(t) = \langle\, -\frac{\sqrt{2}}{2},\ -\sqrt{2},\ 0\,\rangle$

91. $\|\mathbf{v}(t)\| = \sqrt{\sec^4 t + \sec^2 t\tan^2 t} = \sqrt{\sec^2 t(\sec^2 t + \tan^2 t)}$

93. 2

95. $\langle\, 0,\ 2\sin t\left(t - \frac{1}{t}\right) - 2\cos t\left(1 + \frac{1}{t^2}\right),\ 2\sin t\left(1 + \frac{1}{t^2}\right) + 2\cos t\left(t - \frac{2}{t}\right)\,\rangle$

97. $\mathbf{T}(t) = \langle\, \frac{t^2}{\sqrt{t^4 + 1}},\ \frac{-1}{\sqrt{t^4 + 1}}\,\rangle$

99. $\mathbf{T}(t) = \frac{1}{3}\,\langle\, 1,\ 2,\ 2\,\rangle$

101. $\frac{3}{4}\mathbf{i} + \ln(2)\,\mathbf{j} + \left(1 - \frac{1}{e}\right)\mathbf{j}$

103. $8\sqrt{5}$

105. $\frac{1}{54}\left(37^{3/2} - 1\right)$

107. Length $= 2\pi$

109. 6π

111. $e - \frac{1}{e}$

113. $\mathbf{T}(0) = \mathbf{j}, \quad \mathbf{N}(0) = -\mathbf{i}$

115. $\mathbf{T}(t) = \langle 2e^t, e^t\cos t - e^t\sin t, e^t\cos t + e^t\sin t \rangle$

117. $\mathbf{N}(0) = \langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \rangle$

119. $\mathbf{T}(t) = \frac{1}{\sqrt{4t^2+2}} < 1, 2t, 1 >$

121. $\mathbf{T}(t) = \frac{1}{\sqrt{100t^2+13}}(3\mathbf{i} + 10t\mathbf{j} + 2\mathbf{k})$

123. $\mathbf{T}(t) = \frac{1}{\sqrt{9t^4+76t^2+16}}\left(\left[3t^2-4\right]\mathbf{i} + 10t\mathbf{j}\right)$

125. $\mathbf{N}(t) = \langle -\sin t, 0, -\cos t \rangle$

127. Arc-length function: $s(t) = 5t$; r as a parameter of s: $\mathbf{r}(s) = \left(3 - \frac{3s}{5}\right)\mathbf{i} + \frac{4s}{5}\mathbf{j}$

129. $\mathbf{r}(s) = \left(1 + \frac{s}{\sqrt{2}}\right)\sin\left(\ln(1 + \frac{s}{\sqrt{2}})\right)\mathbf{i} + \left(1 + \frac{s}{\sqrt{2}}\right)\cos\left[\ln\left(1 + \frac{s}{\sqrt{2}}\right)\right]\mathbf{j}$

131. The maximum value of the curvature occurs at $x = \sqrt[4]{5}$.

133. $\frac{1}{2}$

135. $\kappa \approx \frac{49.477}{\left(17 + 144t^2\right)^{3/2}}$

137. $\frac{1}{2\sqrt{2}}$

139. The curvature approaches zero.

141. $y = 6x + \pi$ and $x + 6 = 6\pi$

143. $x + 2z = \frac{\pi}{2}$

145. $\frac{a^4b^4}{\left(b^4x^2 + a^4y^2\right)^{3/2}}$

147. $\frac{10\sqrt{10}}{3}$

149. $\frac{38}{3}$

151. The curvature is decreasing over this interval.

153. $\kappa = \frac{6}{x^{2/5}\left(25 + 4x^{6/5}\right)}$

155. $\mathbf{v}(t) = (6t)\mathbf{i} + (2 - \cos(t))\mathbf{j}$

157. $\mathbf{v}(t) = \langle -3\sin t, 3\cos t, 2t \rangle, \quad \mathbf{a}(t) = \langle -3\cos t, -3\sin t, 2 \rangle, \quad \text{speed} = \sqrt{9 + 4t^2}$

159. $\mathbf{v}(t) = -2\sin t\,\mathbf{j} + 3\cos t\,\mathbf{k}, \quad \mathbf{a}(t) = -2\cos t\,\mathbf{j} - 3\sin t\,\mathbf{k}, \quad \text{speed} = \sqrt{4\sin^2(t) + 9^{\cos}(t)}$

161. $\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}, \quad \mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}, \quad \|\mathbf{v}(t)\| \sqrt{e^{2t} + e^{-2t}}$

163. $t = 4$

165. $\mathbf{v}(t) = (\omega - \omega\cos(\omega t))\mathbf{i} + (\omega\sin(\omega t))\mathbf{j}$,

$\mathbf{a}(t) = \left(\omega^2\sin(\omega t)\right)\mathbf{i} + \left(\omega^2\cos(\omega t)\right)\mathbf{j}$,

$\text{speed} = \sqrt{\omega^2 - 2\omega^2\cos(\omega t) + \omega^2\cos^2(\omega t) + \omega^2\sin^2(\omega t)} = \sqrt{2\omega^2(1 - \cos(\omega t))}$

167. $\|\mathbf{v}(t)\| = \sqrt{9 + 4t^2}$

169. $\mathbf{v}(t) = \langle e^{-5t}(\cos t - 5\sin t), -e^{-5t}(\sin t + 5\cos t), -20e^{-5t} \rangle$

171. $\mathbf{a}(t) = \langle e^{-5t}(-\sin t - 5\cos t) - 5e^{-5t}(\cos t - 5\sin t),$
$-e^{-5t}(\cos t - 5\sin t) + 5e^{-5t}(\sin t + 5\cos t), 100e^{-5t} \rangle$

173. 44.185 sec

175. $t = 88.37$ sec

177. 88.37 sec

179. The range is approximately 886.29 m.

181. $\mathbf{v} = 42.16$ m/sec

183. $\mathbf{r}(t) = 0\mathbf{i} + \left(\frac{1}{6}t^3 + 4.5t - \frac{14}{3}\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t + \frac{1}{3}\right)\mathbf{k}$

185. $a_T = 0, \quad a_N = a\omega^2$

187. $a_T = \sqrt{3}e^t, \quad a_N = \sqrt{2}e^t$

189. $a_T = 2t, \quad a_N = 4 + 2t^2$

191. $a_T \frac{6t + 12t^3}{\sqrt{1 + t^4 + t^2}}, \quad a_N = 6\sqrt{\frac{1 + 4t^2 + t^4}{1 + t^2 + t^4}}$

193. $a_T = 0, \quad a_N = 2\sqrt{3}\pi$

195. $\mathbf{r}(t) = \left(\frac{-1}{m}\cos t + c + \frac{1}{m}\right)\mathbf{i} + \left(\frac{-\sin t}{m} + \left(v_0 + \frac{1}{m}\right)t\right)\mathbf{j}$

197. 10.94 km/sec

201. $a_T = 0.43$ m/sec^2,
$a_N = 2.46$ m/sec^2

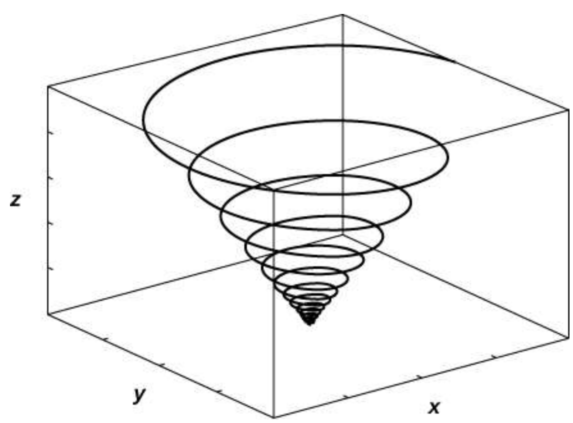
Review Exercises

203. False, $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}(t)] = 0$

205. False, it is $|\mathbf{r}'(t)|$

207. $t < 4, \quad t \neq \frac{n\pi}{2}$

209.



211. $\mathbf{r}(t) = \langle t, 2 - \frac{t^2}{8}, -2 - \frac{t^2}{8} \rangle$

213. $\mathbf{u}'(t) = \langle 2t, 2, 20t^4 \rangle, \quad \mathbf{u}''(t) = \langle 2, 0, 80t^3 \rangle, \quad \frac{d}{dt}[\mathbf{u}'(t) \times \mathbf{u}(t)] = \langle -480t^3 - 160t^4, 24 + 75t^2, 12 + 4t \rangle,$
$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)] = \langle 480t^3 + 160t^4, -24 - 75t^2, -12 - 4t \rangle, \quad \frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{u}'(t)] = 720t^8 - 9600t^3 + 6t^2 + 4,$ unit

tangent vector: $\mathbf{T}(t) = \frac{2t}{\sqrt{400t^8 + 4t^2 + 4}}\mathbf{i} + \frac{2}{\sqrt{400t^8 + 4t^2 + 4}}\mathbf{j} + \frac{20t^4}{\sqrt{400t^8 + 4t^2 + 4}}\mathbf{k}$

215. $\frac{\ln(4)^2}{2}\mathbf{i} + 2\mathbf{j} + \frac{2\left(2 + \sqrt{2}\right)}{\pi}\mathbf{k}$

217. $\frac{\sqrt{37}}{2} + \frac{1}{12}\sinh^{-1}(6)$

219. $\mathbf{r}(t(s)) = \cos\left(\frac{2s}{\sqrt{65}}\right)\mathbf{i} + \frac{8s}{\sqrt{65}}\mathbf{j} - \sin\left(\frac{2s}{\sqrt{65}}\right)\mathbf{k}$

221. $\frac{e^{2t}}{\left(e^{2t} + 1\right)^2}$

223. $a_T = \frac{e^{2t}}{\sqrt{1 + e^{2t}}}, \quad a_N = \frac{\sqrt{2e^{2t} + 4e^{2t}\sin t\cos t + 1}}{\sqrt{1 + e^{2t}}}$

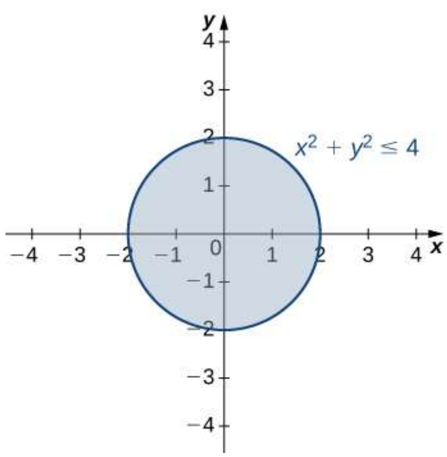
225. $\mathbf{v}(t) = \langle 2t, \frac{1}{t}, \cos(\pi t) \rangle$ m/sec, $\mathbf{a}(t) = \langle 2, -\frac{1}{t^2}, -\sin(\pi t) \rangle$ m/sec^2, speed $= \sqrt{4t^2 + \frac{1}{t^2} + \cos^2(\pi t)}$ m/sec; at $t = 1$, $\mathbf{r}(1) = \langle 1, 0, 0 \rangle$ m, $\mathbf{v}(1) = \langle 2, -1, 1 \rangle$ m/sec, $\mathbf{a}(1) = \langle 2, -1, 0 \rangle$ m/sec^2, and speed $= \sqrt{6}$ m/sec

227. $\mathbf{r}(t) = \mathbf{v}_0 t - \frac{g}{2}t^2\,\mathbf{j}, \quad \mathbf{r}(t) = \langle \mathbf{v}_0(\cos\theta)t, \mathbf{v}_0(\sin\theta)t, -\frac{g}{2}t^2 \rangle$

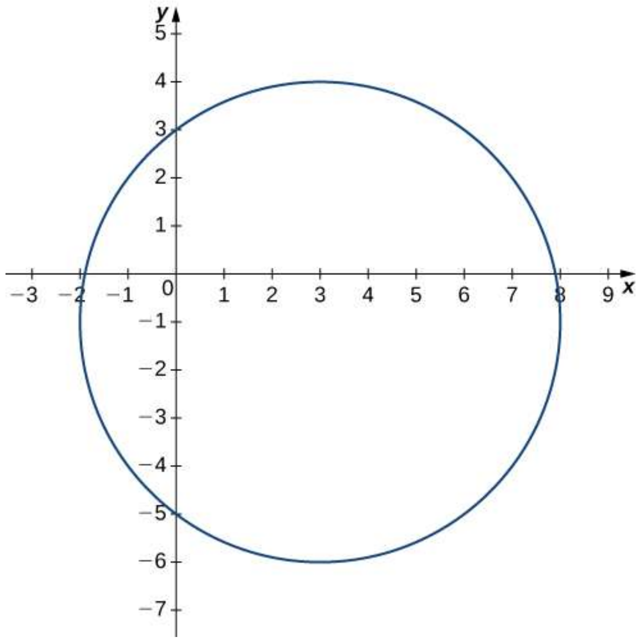
Chapter 4

Checkpoint

4.1. The domain is the shaded circle defined by the inequality $9x^2 + 9y^2 \leq 36$, which has a circle of radius 2 as its boundary. The range is $[0, 6]$.



4.2. The equation of the level curve can be written as $(x - 3)^2 + (y + 1)^2 = 25$, which is a circle with radius 5 centered at $(3, -1)$.



4.3. $z = 3 - (x-1)^2$. This function describes a parabola opening downward in the plane $y = 3$.

4.4. $\text{domain}(h) = \left\{(x, y, t) \in \mathbb{R}^3 \middle| y \geq 4x^2 - 4\right\}$

4.5. $(x-1)^2 + (y+2)^2 + (z-3)^2 = 16$ describes a sphere of radius 4 centered at the point $(1, -2, 3)$.

4.6. $\lim_{(x,y)\to(5,-2)} \sqrt[3]{\dfrac{x^2 - y}{y^2 + x - 1}} = \dfrac{3}{2}$

4.7. If $y = k(x-2) + 1$, then $\lim_{(x,y)\to(2,1)} \dfrac{(x-2)(y-1)}{(x-2)^2 + (y-1)^2} = \dfrac{k}{1+k^2}$. Since the answer depends on k, the limit fails to exist.

4.8. $\lim_{(x,y)\to(5,-2)} \sqrt{29 - x^2 - y^2}$

4.9.

1. The domain of f contains the ordered pair $(2, -3)$ because $f(a, b) = f(2, -3) = \sqrt{16 - 2(2)^2 - (-3)^2} = 3$
2. $\lim_{(x,y)\to(a,b)} f(x, y) = 3$
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) = 3$

4.10. The polynomials $g(x) = 2x^2$ and $h(y) = y^3$ are continuous at every real number; therefore, by the product of continuous functions theorem, $f(x, y) = 2x^2y^3$ is continuous at every point (x, y) in the xy-plane. Furthermore, any constant function is continuous everywhere, so $g(x, y) = 3$ is continuous at every point (x, y) in the xy-plane. Therefore, $f(x, y) = 2x^2y^3 + 3$ is continuous at every point (x, y) in the xy-plane. Last, $h(x) = x^4$ is continuous at every real number x, so by the continuity of composite functions theorem $g(x, y) = \left(2x^2y^3 + 3\right)^4$ is continuous at every point (x, y) in the xy-plane.

4.11. $\lim_{(x,y,z)\to(4,-1,3)} \sqrt{13 - x^2 - 2y^2 + z^2} = 2$

4.12. $\dfrac{\partial f}{\partial x} = 8x + 2y + 3, \quad \dfrac{\partial f}{\partial y} = 2x - 2y - 2$

4.13. $\frac{\partial f}{\partial x} = \left(3x^2 - 6xy^2\right)\sec^2\left(x^3 - 3x^2y^2 + 2y^4\right)$
$\frac{\partial f}{\partial y} = \left(-6x^2y + 8y^3\right)\sec^2\left(x^3 - 3x^2y^2 + 2y^4\right)$

4.14. Using the curves corresponding to $c = -2$ and $c = -3$, we obtain $\left.\frac{\partial f}{\partial y}\right|_{(x,\ y) = (0,\ \sqrt{2})} \approx \frac{f(0,\ \sqrt{3}) - f(0,\ \sqrt{2})}{\sqrt{3} - \sqrt{2}} = \frac{-3 - (-2)}{\sqrt{3} - \sqrt{2}} \cdot \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}} = -\sqrt{3} - \sqrt{2} \approx -3.146.$ The exact answer is $\left.\frac{\partial f}{\partial y}\right|_{(x,\ y) = (0,\ \sqrt{2})} = (-2y|_{(x,\ y) = (0,\ \sqrt{2})} = -2\sqrt{2} \approx -2.828.$

4.15. $\frac{\partial f}{\partial x} = 4x - 8xy + 5z^2 - 6, \quad \frac{\partial f}{\partial y} = -4x^2 + 4y, \quad \frac{\partial f}{\partial z} = 10xz + 3$

4.16. $\frac{\partial f}{\partial x} = 2xy\sec\left(x^2y\right)\tan\left(x^2y\right) - 3x^2yz^2\sec^2\left(x^3yz^2\right)$
$\frac{\partial f}{\partial y} = x^2\sec\left(x^2y\right)\tan\left(x^2y\right) - x^3z^2\sec^2\left(x^3yz^2\right)$
$\frac{\partial f}{\partial z} = -2x^3yz\sec^2\left(x^3yz^2\right)$

4.17. $\frac{\partial^2 f}{\partial x^2} = -9\sin(3x - 2y) - \cos(x + 4y)$
$\frac{\partial^2 f}{\partial x \partial y} = 6\sin(3x - 2y) - 4\cos(x + 4y)$
$\frac{\partial^2 f}{\partial y \partial x} = 6\sin(3x - 2y) - 4\cos(x + 4y)$
$\frac{\partial^2 f}{\partial y^2} = -4\sin(3x - 2y) - 16\cos(x + 4y)$

4.19. $z = 7x + 8y - 3$

4.20. $L(x,\ y) = 6 - 2x + 3y,$ so $L(4.1,\ 0.9) = 6 - 2(4.1) + 3(0.9) = 0.5$ $f(4.1,\ 0.9) = e^{5 - 2(4.1) + 3(0.9)} = e^{-0.5} \approx 0.6065.$

4.21. $f(-1,\ 2) = -19, \quad f_x(-1,\ 2) = 3,\ f_y(-1,\ 2) = -16,\ E(x,\ y) = -4(y - 2)^2.$

$$\begin{aligned}\lim_{(x,\ y) \to (x_0,\ y_0)} \frac{E(x,\ y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} &= \lim_{(x,\ y) \to (-1,\ 2)} \frac{-4(y - 2)^2}{\sqrt{(x + 1)^2 + (y - 2)^2}} \\ &\le \lim_{(x,\ y) \to (-1,\ 2)} \frac{-4\left((x + 1)^2 + (y - 2)^2\right)}{\sqrt{(x + 1)^2 + (y - 2)^2}} \\ &= \lim_{(x,\ y) \to (2,\ -3)} -4\sqrt{(x + 1)^2 + (y - 2)^2} \\ &= 0.\end{aligned}$$

4.22. $$\begin{aligned} dz &= 0.18 \\ \Delta z &= f(1.03,\ -1.02) - f(1,\ -1) = 0.180682\end{aligned}$$

4.23. $$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (2x - 3y)(6\cos 2t) + (-3x + 4y)(-8\sin 2t) \\ &= -92\sin 2t\cos 2t - 72\left(\cos^2 2t - \sin^2 2t\right) \\ &= -46\sin 4t - 72\cos 4t.\end{aligned}$$

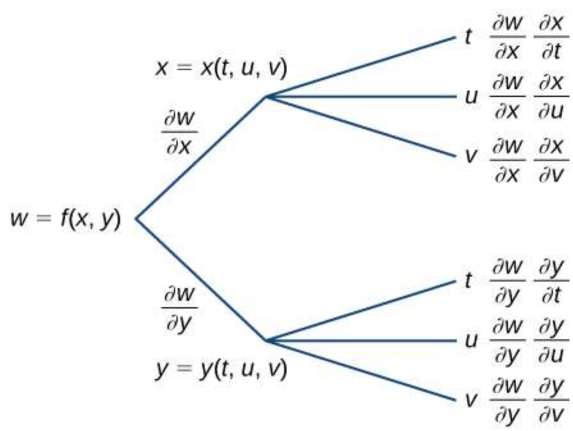
4.24. $\frac{\partial z}{\partial u} = 0, \quad \frac{\partial z}{\partial v} = \frac{-21}{(3\sin 3v + \cos 3v)^2}$

4.25.

$$\frac{\partial w}{\partial u} = 0$$
$$\frac{\partial w}{\partial v} = \frac{15 - 33\sin 3v + 6\cos 3v}{(3 + 2\cos 3v - \sin 3v)^2}$$

4.26.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$



4.27. $\frac{dy}{dx} = \frac{2x + y + 7}{2y - x + 3}\Big|_{(3,\,-2)} = \frac{2(3) + (-2) + 7}{2(-2) - (3) + 3} = -\frac{11}{4}$

Equation of the tangent line: $y = -\frac{11}{4}x + \frac{25}{4}$

4.28.

$$D_{\mathbf{u}} f(x,\ y) = \frac{(6xy - 4y^3 - 4)(1)}{2} + \frac{\left(3x^2 - 12xy^2 + 6y\right)\sqrt{3}}{2}$$
$$D_{\mathbf{u}} f(3,\ 4) = \frac{72 - 256 - 4}{2} + \frac{(27 - 576 + 24)\sqrt{3}}{2} = -94 - \frac{525\sqrt{3}}{2}$$

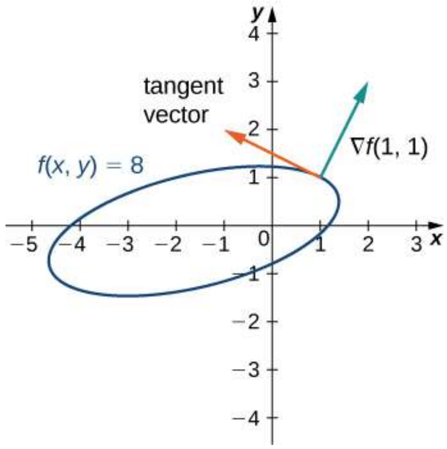
4.29. $\nabla f(x,\ y) = \frac{2x^2 + 2xy + 6y^2}{(2x + y)^2}\mathbf{i} - \frac{x^2 + 12xy + 3y^2}{(2x + y)^2}\mathbf{j}$

4.30. The gradient of g at $(-2,\ 3)$ is $\nabla g(-2,\ 3) = \mathbf{i} + 14\mathbf{j}$. The unit vector that points in the same direction as $\nabla g(-2,\ 3)$ is $\frac{\nabla g(-2,\ 3)}{\|\nabla g(-2,\ 3)\|} = \frac{1}{\sqrt{197}}\mathbf{i} + \frac{14}{\sqrt{197}}\mathbf{j} = \frac{\sqrt{197}}{197}\mathbf{i} + \frac{14\sqrt{197}}{197}\mathbf{j},$ which gives an angle of $\theta = \arcsin\left(\left(14\sqrt{197}\right)/197\right) \approx 1.499$ rad. The maximum value of the directional derivative is $\|\nabla g(-2,\ 3)\| = \sqrt{197}.$

4.31. $\nabla f(x,\ y) = (2x - 2y + 3)\mathbf{i} + (-2x + 10y - 2)\mathbf{j}$

$\nabla f(1,\ 1) = 3\mathbf{i} + 6\mathbf{j}$

Tangent vector: $6\mathbf{i} - 3\mathbf{j}$ or $-6\mathbf{i} + 3\mathbf{j}$



4.32. $$\begin{aligned}\nabla f(x, y, z) &= \frac{2x^2 + 2xy + 6y^2 - 8xz - 2z^2}{(2x + y - 4z)^2}\mathbf{i} - \frac{x^2 + 12xy + 3y^2 - 24yz + z^2}{(2x + y - 4z)^2}\mathbf{j} \\ &+ \frac{4x^2 - 12y^2 - 4z^2 + 4xz + 2yz}{(2x + y - 4z)^2}\mathbf{k}.\end{aligned}$$

4.33. $$\begin{aligned}D_{\mathbf{u}} f(x, y, z) &= -\frac{3}{13}(6x + y + 2z) + \frac{12}{13}(x - 4y + 4z) - \frac{4}{13}(2x + 4y - 2z) \\ D_{\mathbf{u}} f(0, -2, 5) &= \frac{384}{13}\end{aligned}$$

4.34. $(2, -5)$

4.35. $\left(\frac{4}{3}, \frac{1}{3}\right)$ is a saddle point, $\left(-\frac{3}{2}, -\frac{3}{8}\right)$ is a local maximum.

4.36. The absolute minimum occurs at $(1, 0)$: $f(1, 0) = -1.$ The absolute maximum occurs at $(0, 3)$: $f(0, 3) = 63.$

4.37. f has a maximum value of 976 at the point $(8, 2)$.

4.38. A maximum production level of 13890 occurs with 5625 labor hours and $\$5500$ of total capital input.

4.39. $$\begin{aligned}f\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) &= \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = \sqrt{3} \\ f\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) &= -\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} = -\sqrt{3}.\end{aligned}$$

4.40. $f(2, 1, 2) = 9$ is a minimum.

Section Exercises

1. $17, 72$

3. $20\pi.$ This is the volume when the radius is 2 and the height is $5.$

5. All points in the xy-plane

7. $x < y^2$

9. All real ordered pairs in the xy-plane of the form (a, b)

11. $\{z | 0 \le z \le 4\}$

13. The set $\mathbb{R}$

15. $y^2 - x^2 = 4,$ a hyperbola

17. $4 = x + y,$ a line; $x + y = 0,$ line through the origin

19. $2x - y = 0,$ $2x - y = -2,$ $2x - y = 2;$ three lines

21. $\frac{x}{x + y} = -1,$ $\frac{x}{x + y} = 0,$ $\frac{x}{x + y} = 2$

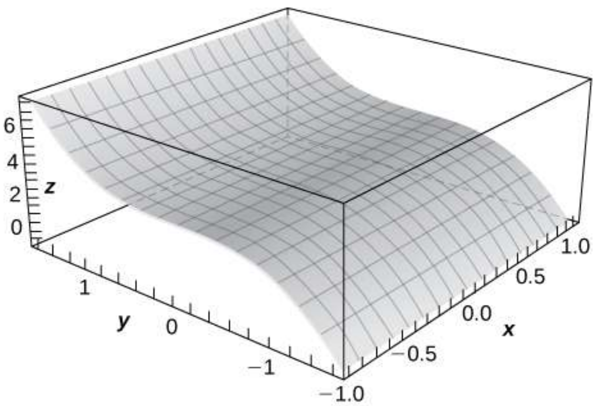
23. $e^{xy} = \frac{1}{2},$ $e^{xy} = 3$

25. $xy - x = -2, \quad xy - x = 0, \; xy - x = 2$

27. $e^{-2}x^2 = y, \quad y = x^2, \; y = e^2 x^2$

29. The level curves are parabolas of the form $y = cx^2 - 2.$

31. $z = 3 + y^3,$ a curve in the zy-plane with rulings parallel to the x-axis

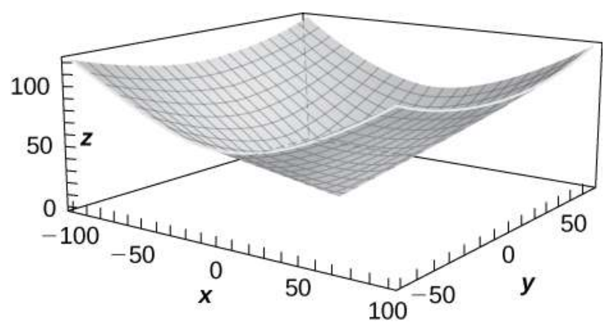


33. $\dfrac{x^2}{25} + \dfrac{y^2}{4} \le 1$

35. $\dfrac{x^2}{9} + \dfrac{y^2}{4} + \dfrac{z^2}{36} < 1$

37. All points in xyz-space

39.



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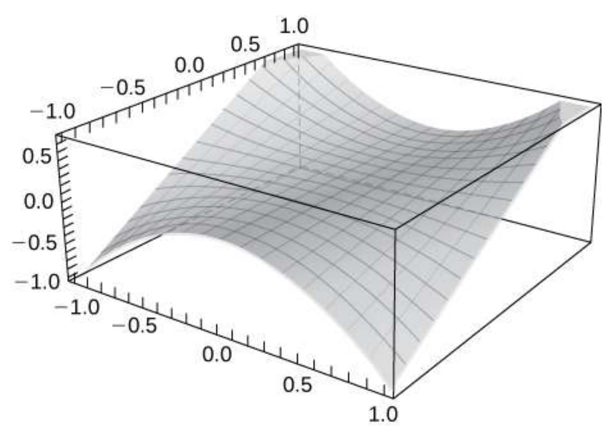

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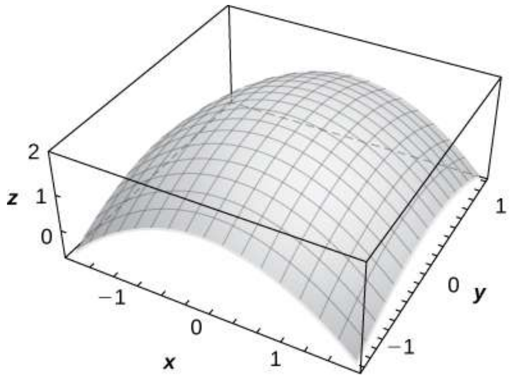

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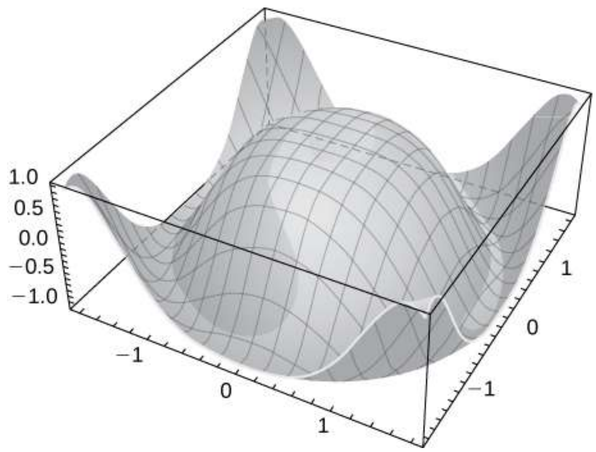
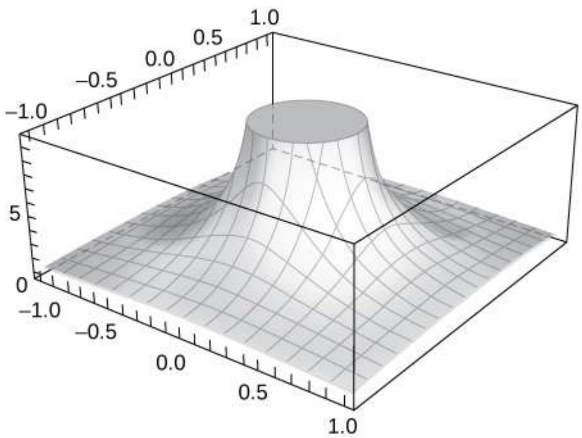
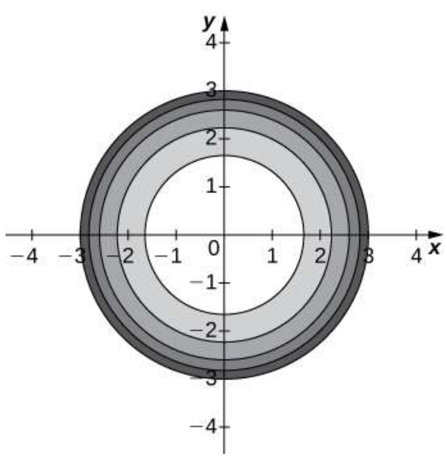


47. The contour lines are circles.
49. $x^2 + y^2 + z^2 = 9,$ a sphere of radius 3
51. $x^2 + y^2 - z^2 = 4,$ a hyperboloid of one sheet
53. $4x^2 + y^2 = 1,$
55. $1 = e^{xy}(x^2 + y^2)$
57. $T(x, y) = \dfrac{k}{x^2 + y^2}$
59. $x^2 + y^2 = \dfrac{k}{40}, \quad x^2 + y^2 = \dfrac{k}{100}.$ The level curves represent circles of radii $\sqrt{10k}/20$ and $\sqrt{k}/10$
61. 2.0
63. $\frac{2}{3}$
65. 1
67. $\frac{1}{2}$
69. $-\frac{1}{2}$
71. e^{-32}
73. 11.0
75. 1.0
77. The limit does not exist because when x and y both approach zero, the function approaches $\ln 0,$ which is undefined (approaches negative infinity).
79. every open disk centered at (x_0, y_0) contains points inside R and outside R

81. 0.0
83. 0.00
85. The limit does not exist.
87. The limit does not exist. The function approaches two different values along different paths.
89. The limit does not exist because the function approaches two different values along the paths.
91. The function f is continuous in the region $y > -x.$
93. The function f is continuous at all points in the xy-plane except at $(0, 0).$
95. The function is continuous at $(0, 0)$ since the limit of the function at $(0, 0)$ is $0,$ the same value of $f(0, 0).$
97. The function is discontinuous at $(0, 0).$ The limit at $(0, 0)$ fails to exist and $g(0, 0)$ does not exist.
99. Since the function $\arctan x$ is continuous over $(-\infty, \infty),$ $g(x, y) = \arctan\left(\frac{xy^2}{x+y}\right)$ is continuous where $z = \frac{xy^2}{x+y}$ is continuous. The inner function z is continuous on all points of the xy-plane except where $y = -x.$ Thus, $g(x, y) = \arctan\left(\frac{xy^2}{x+y}\right)$ is continuous on all points of the coordinate plane *except* at points at which $y = -x.$
101. All points $P(x, y, z)$ in space
103. The graph increases without bound as x and y both approach zero.



105. a.



b. The level curves are circles centered at $(0, 0)$ with radius $9-c.$ c. $x^2+y^2 = 9-c$ d. $z = 3$ e. $\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2+y^2 \le 9\right\}$ f. $\{z | 0 \le z \le 3\}$
107. 1.0
109. $f(g(x, y))$ is continuous at all points (x, y) that are not on the line $2x - 5y = 0.$

111. 2.0

113. $\frac{\partial z}{\partial y} = -3x + 2y$

115. The sign is negative.

117. The partial derivative is zero at the origin.

119. $\frac{\partial z}{\partial y} = -3\sin(3x)\sin(3y)$

121. $\frac{\partial z}{\partial x} = \frac{6x^5}{x^6 + y^4}; \frac{\partial z}{\partial y} = \frac{4y^3}{x^6 + y^4}$

123. $\frac{\partial z}{\partial x} = ye^{xy}; \frac{\partial z}{\partial y} = xe^{xy}$

125. $\frac{\partial z}{\partial x} = 2\sec^2(2x - y), \frac{\partial z}{\partial y} = -\sec^2(2x - y)$

127. $f_x(2, -2) = \frac{1}{4} = f_y(2, -2)$

129. $\frac{\partial z}{\partial x} = -\cos(1)$

131. $f_x = 0, \quad f_y = 0, \quad f_z = 0$

133. a. $V(r, h) = \pi r^2 h$ b. $\frac{\partial V}{\partial r} = 2\pi rh$ c. $\frac{\partial V}{\partial h} = \pi r^2$

135. $f_{xy} = \frac{1}{(x - y)^2}$

137. $\frac{\partial^2 z}{\partial x^2} = 2, \frac{\partial^2 z}{\partial y^2} = 4$

139. $f_{xyy} = f_{yxy} = f_{yyx} = 0$

141.
$$\begin{aligned} \frac{d^2 z}{dx^2} &= -\frac{1}{2}\left(e^y - e^{-y}\right)\sin x \\ \frac{d^2 z}{dy^2} &= \frac{1}{2}\left(e^y - e^{-y}\right)\sin x \\ \frac{d^2 z}{dx^2} + \frac{d^2 z}{dy^2} &= 0 \end{aligned}$$

143. $f_{xyz} = 6y^2 x - 18yz^2$

145. $\left(\frac{1}{4}, \frac{1}{2}\right), (1, 1)$

147. $(0, 0), (0, 2), (\sqrt{3}, -1), \left(-\sqrt{3}, -1\right)$

149. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \sin(y) - e^x \sin y = 0$

151. $c^2 \frac{\partial^2 z}{\partial x^2} = e^{-t}\cos\left(\frac{x}{c}\right)$

153. $\frac{\partial f}{\partial y} = -2x + 7$

155. $\frac{\partial f}{\partial x} = y\cos xy$

159. $\frac{\partial F}{\partial \theta} = 6, \frac{\partial F}{\partial x} = 4 - 3\sqrt{3}$

161. $\frac{\delta f}{\delta x}$ at $(500, 1000) = 172.36$, $\frac{\delta f}{\delta y}$ at $(500, 1000) = 36.93$

163. $\left(\frac{\sqrt{145}}{145}\right)(12\mathbf{i} - \mathbf{k})$

165. Normal vector: $\mathbf{i}+\mathbf{j}$, tangent vector: $\mathbf{i}-\mathbf{j}$

167. Normal vector: $7\mathbf{i}-17\mathbf{j}$, tangent vector: $17\mathbf{i}+7\mathbf{j}$

169. $-1.094\mathbf{i}-0.18238\mathbf{j}$

171. $-36x-6y-z=-39$

173. $z=0$

175. $5x+4y+3z-22=0$

177. $4x-5y+4z=0$

179. $2x+2y-z=0$

181. $-2(x-1)+2(y-2)-(z-1)=0$

183. $x=20t+2,\ y=-4t+1,\ z=-t+18$

185. $x=0,\ y=0,\ z=t$

187. $x-1=2t;\ y-2=-2t;\ z-1=t$

189. The differential of the function $z(x,\ y)=dz=f_x\,dx+f_y\,dy$

191. Using the definition of differentiability, we have $e^{xy}x\approx x+y$.

193. $\Delta z=2x\Delta x+3\Delta y+(\Delta x)^2$. $(\Delta x)^2\to 0$ for small Δx and z satisfies the definition of differentiability.

195. $\Delta z\approx 1.185422$ and $dz\approx 1.108$. They are relatively close.

197. $16\ \text{cm}^3$

199. $\Delta z=$ exact change $=0.6449$, approximate change is $dz=0.65$. The two values are close.

201. 13% or 0.13

203. 0.025

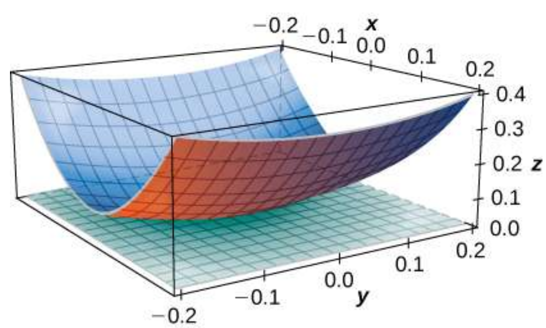
205. 0.3%

207. $2x+\frac{1}{4}y-1$

209. $\frac{1}{2}x+y+\frac{1}{4}\pi-\frac{1}{2}$

211. $\frac{3}{7}x+\frac{2}{7}y+\frac{6}{7}z$

213. $z=0$



215. $\frac{dw}{dt}=y\cos z+x\cos z(2t)-\frac{xy\sin z}{\sqrt{1-t^2}}$

217. $\frac{\partial w}{\partial s}=-30x+4y$, $\frac{\partial w}{\partial t}=10x-16y$

219. $\frac{\partial f}{\partial r}=r\sin(2\theta)$

221. $\frac{df}{dt}=2t+4t^3$

223. $\frac{df}{dt}=-1$

225. $\frac{df}{dt} = 1$

227. $\frac{dw}{dt} = 2e^{2t}$ in both cases

229. $2\sqrt{2}t + \sqrt{2}\pi = \frac{du}{dt}$

231. $\frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy}$

233. $\frac{dy}{dx} = \frac{y - x}{-x + 2y^3}$

235. $\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}$

237. $\frac{dy}{dx} = -\frac{ye^{xy}}{xe^{xy} + e^y(1 + y)}$

239. $\frac{dz}{dt} = 42t^{13}$

241. $\frac{dz}{dt} = -\frac{10}{3}t^{7/3} \times e^{1 - t^{10/3}}$

243. $\frac{\partial z}{\partial u} = \frac{-2 \sin u}{3 \sin v}$ and $\frac{\partial z}{\partial v} = \frac{-2 \cos u \cos v}{3 \sin^2 v}$

245. $\frac{\partial z}{\partial r} = \sqrt{3}e^{\sqrt{3}}, \quad \frac{\partial z}{\partial \theta} = \left(2 - 4\sqrt{3}\right)e^{\sqrt{3}}$

247. $\frac{\partial w}{\partial t} = \cos(xyz) \times yz \times (-3) - \cos(xyz)xze^{1 - t} + \cos(xyz)xy \times 4$

249. $f(tx, ty) = \sqrt{t^2 x^2 + t^2 y^2} = t^1 f(x, y), \quad \frac{\partial f}{\partial y} = x\frac{1}{2}\left(x^2 + y^2\right)^{-1/2} \times 2x + y\frac{1}{2}\left(x^2 + y^2\right)^{-1/2} \times 2y = 1f(x, y)$

251. $\frac{34\pi}{3}$

253. $\frac{dV}{dt} = \frac{1066\pi}{3}\text{cm}^3/\text{min}$

255. $\frac{dA}{dt} = 12 \text{ in.}^2/\text{min}$

257. 2°C/sec

259.
$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial x}{\partial t}\frac{\partial t}{\partial r}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial r}\right) \\ &+ \frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial w}\frac{\partial w}{\partial r} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial r}\right)\end{aligned}$$

261. $-3\sqrt{3}$

263. -1

265. $\frac{2}{\sqrt{6}}$

267. $\sqrt{3}$

269. -1.0

271. $\frac{22}{25}$

273. $\frac{2}{3}$

275. $\frac{-\sqrt{2}(x + y)}{2(x + 2y)^2}$

277. $\frac{e^x\left(y + \sqrt{3}\right)}{2}$

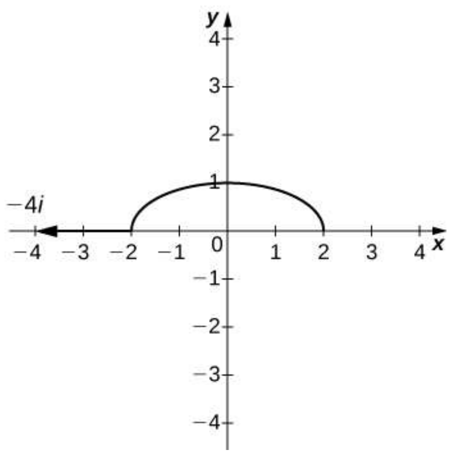
279. $\frac{1+2\sqrt{3}}{2(x+2y)}$

281. $\langle\ 5,\ 4,\ 3\ \rangle$

283. -320

285. $\frac{3}{\sqrt{11}}$

287. $\frac{31}{255}$



289.

291. $\frac{4}{3}\mathbf{i}-3\mathbf{j}$

293. $\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j}+\sqrt{2}\mathbf{k}$

295. $1.6(10^{19})$

297. $\frac{5\sqrt{2}}{99}$

299. $\sqrt{5},\ \langle\ 1,\ 2\ \rangle$

301. $\sqrt{\frac{13}{2}},\ \langle\ -3,\ -2\ \rangle$

303. a. $x+y+z=3,$ b. $x-1=y-1=z-1$

305. a. $x+y-z=1,$ b. $x-1=y=-z$

307. a. $\frac{32}{\sqrt{3}},$ b. $\langle\ 38,\ 6,\ 12\ \rangle$, c. $2\sqrt{406}$

309. $\langle\ u,\ v\ \rangle\ =\ \langle\ \pi\cos(\pi x)\sin(2\pi y),\ 2\pi\sin(\pi x)\cos(2\pi y)\ \rangle$

311. $\left(\frac{2}{3},\ 4\right)$

313. $(0,\ 0)\ \left(\frac{1}{15},\ \frac{1}{15}\right)$

315. Maximum at $(4,\ -1,\ 8)$

317. Relative minimum at $(0,\ 0,\ 1)$

319. The second derivative test fails. Since $x^2y^2>0$ for all *x* and *y* different from zero, and $x^2y^2=0$ when either *x* or *y* equals zero (or both), then the absolute minimum occurs at $(0,\ 0)$.

321. $f\left(-2,\ -\frac{3}{2}\right)=-6$ is a saddle point.

323. $f(0,\ 0)=0;$ $(0,\ 0,\ 0)$ is a saddle point.

325. $f(0,\ 0)=9$ is a local maximum.

327. Relative minimum located at $(2,\ 6)$.

329. $(1,\ -2)$ is a saddle point.

331. $(2,\ 1)$ and $(-2,\ 1)$ are saddle points; $(0,\ 0)$ is a relative minimum.

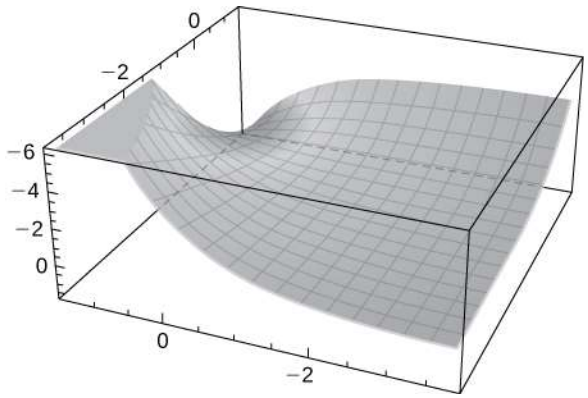
333. $(-1,\ 0)$ is a relative maximum.

335. $(0,\ 0)$ is a saddle point.

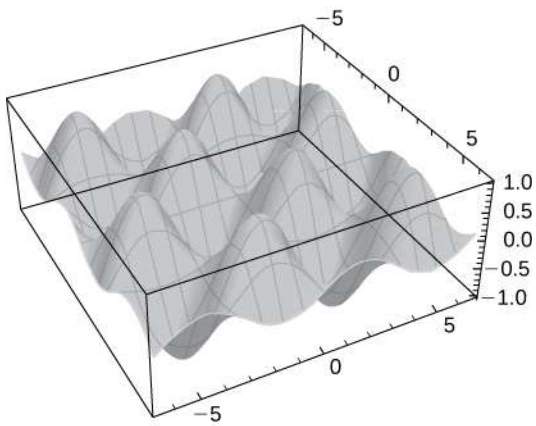
337. The relative maximum is at $(40,\ 40)$.

339. $\left(\frac{1}{4},\ \frac{1}{2}\right)$ is a saddle point and $(1,\ 1)$ is the relative minimum.

341. A saddle point is located at $(0,\ 0)$.



343. There is a saddle point at $(\pi,\ \pi)$, local maxima at $\left(\frac{\pi}{2},\ \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},\ \frac{3\pi}{2}\right)$, and local minima at $\left(\frac{\pi}{2},\ \frac{3\pi}{2}\right)$ and $\left(\frac{3\pi}{2},\ \frac{\pi}{2}\right)$.



345. $(0,\ 1,\ 0)$ is the absolute minimum and $(0,\ -2,\ 9)$ is the absolute maximum.

347. There is an absolute minimum at $(0,\ 1,\ -1)$ and an absolute maximum at $(0,\ -1,\ 1)$.

349. $\left(\sqrt{5},\ 0,\ 0\right), \left(-\sqrt{5},\ 0,\ 0\right)$

351. 18 by 36 by 18 in.

353. $\left(\frac{47}{24},\ \frac{47}{12},\ \frac{235}{24}\right)$

355. $x = 3$ and $y = 6$

357. $V = \frac{64{,}000}{\pi} \approx 20,\ 372\ \text{cm}^3$

359. maximum: $2\frac{\sqrt{3}}{3}$, minimum: $\frac{-2\sqrt{3}}{3}$

361. maximum: $\left(\frac{\sqrt{2}}{2},\ 0,\ \sqrt{2}\right)$, minimum: $\left(\frac{-\sqrt{2}}{2},\ 0,\ -\sqrt{2}\right)$

363. maximum: $\frac{3}{2}$, minimum = 1

365. maxima: $f\left(\frac{3\sqrt{2}}{2}, 2\sqrt{2}\right) = 24, \quad f\left(-\frac{3\sqrt{2}}{2}, -2\sqrt{2}\right) = 24$; minima: $f\left(-\frac{3\sqrt{2}}{2}, 2\sqrt{2}\right) = -24, \quad f\left(\frac{3\sqrt{2}}{2}, -2\sqrt{2}\right) = -24$

367. maximum: $2\sqrt{11}$ at $f\left(\frac{2}{\sqrt{11}}, \frac{6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)$; minimum: $-2\sqrt{11}$ at $f\left(\frac{-2}{\sqrt{11}}, \frac{-6}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$

369. 2.0

371. $19\sqrt{2}$

373. $\left(\frac{1}{\sqrt[3]{2}}, \frac{-1}{\sqrt[3]{2}}\right)$

375. $f(1, 2) = 5$

377. $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

379. minimum: $f(2, 3, 4) = 29$

381. The maximum volume is 4 ft^3. The dimensions are $1 \times 2 \times 2$ ft.

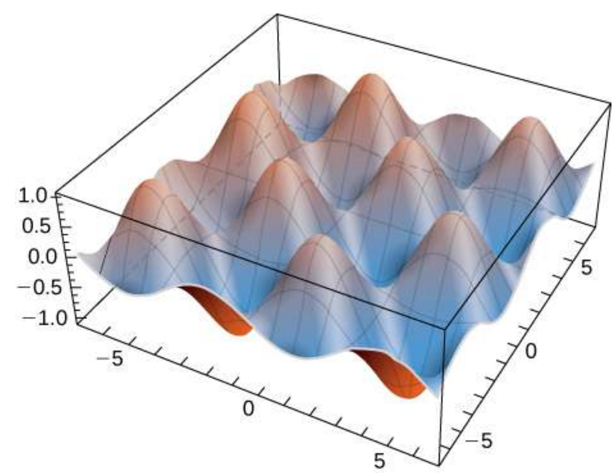
383. $\left(1, \frac{1}{2}, -3\right)$

385. 1.0

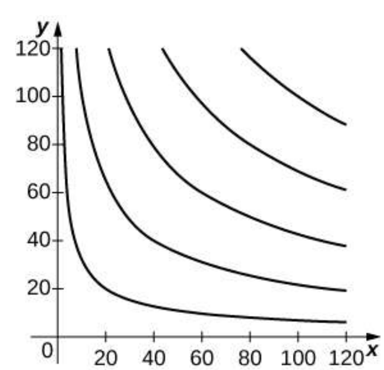
387. $\sqrt{3}$

389. $\left(\frac{2}{5}, \frac{19}{5}\right)$

391. $\frac{1}{2}$



393. Roughly 3365 watches at the critical point $(80, 60)$



Review Exercises

395. True, by Clairaut’s theorem

397. False
399. Answers may vary
401. Does not exist
403. Continuous at all points on the x, y-plane, except where $x^2 + y^2 > 4$.

405. $\frac{\partial u}{\partial x} = 4x^3 - 3y, \quad \frac{\partial u}{\partial y} = -3x, \quad \frac{\partial u}{\partial t} = 2, \quad \frac{\partial u}{\partial t} = 3t^2, \quad \frac{\partial u}{\partial t} = 8x^3 - 6y - 9xt^2$

407. $h_{xx}(x, y, z) = \frac{6xe^{2y}}{z}, \quad h_{xy}(x, y, z) = \frac{6x^2 e^{2y}}{z}, \quad h_{xz}(x, y, z) = -\frac{3x^2 e^{2y}}{z^2}, \quad h_{yx}(x, y, z) = \frac{6x^2 e^{2y}}{z},$ $h_{yy}(x, y, z) = \frac{4x^3 e^{2y}}{z}, \quad h_{yz}(x, y, z) = -\frac{2x^3 e^{2y}}{z^2}, \quad h_{zx}(x, y, z) = -\frac{3x^2 e^{2y}}{z^2}, \quad h_{zy}(x, y, z) = -\frac{2x^3 e^{2y}}{z^2},$ $h_{zz}(x, y, z) = \frac{2x^3 e^{2y}}{z^3}$

409. $z = \frac{1}{9}x - \frac{2}{9}y + \frac{29}{9}$

411. $dz = 4dx - dy, \quad dz(0.1, 0.01) = 0.39, \quad \Delta z = 0.432$

413. $3\sqrt{85}, \ \langle 27, 6 \rangle$

415. $\nabla f(x, y) = -\frac{\sqrt{x} + 2y^2}{2x^2 y}\mathbf{i} + \left(\frac{1}{x} - \frac{1}{\sqrt{x}y^2}\right)\mathbf{j}$

417. maximum: $\frac{16}{3\sqrt{3}}$, minimum: $-\frac{16}{3\sqrt{3}}$

419. 2.3228 cm^3

Chapter 5

Checkpoint

5.1. $V = \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A = 0$

5.2. a. 26 b. Answers may vary.

5.3. $-\frac{1340}{3}$

5.4. $\frac{4 - \ln 5}{\ln 5}$

5.5. $\frac{\pi}{2}$

5.6. Answers to both parts a. and b. may vary.

5.7. Type I and Type II are expressed as $\left\{(x, y) | 0 \le x \le 2, x^2 \le y \le 2x\right\}$ and $\left\{(x, y) | 0 \le y \le 4, \frac{1}{2}y \le x \le \sqrt{y}\right\}$, respectively.

5.8. $\pi/4$

5.9. $\{(x, y) | 0 \le y \le 1, 1 \le x \le e^y\} \cup \{(x, y) | 1 \le y \le e, 1 \le x \le 2\} \cup \left\{(x, y) | e \le y \le e^2, \ln y \le x \le 2\right\}$

5.10. Same as in the example shown.

5.11. $\frac{216}{35}$

5.12. $\frac{e^2}{4} + 10e - \frac{49}{4}$ cubic units

5.13. $\frac{81}{4}$ square units

5.14. $\frac{3}{4}$

5.15. $\frac{\pi}{4}$

5.16. $\frac{55}{72} \approx 0.7638$

5.17. $\frac{14}{3}$

5.18. 8π

5.19. $\pi/8$

5.20. $V = \int_0^{2\pi} \int_0^{2\sqrt{2}} \left(16 - 2r^2\right) r\, dr\, d\theta = 64\pi$ cubic units

5.21. $A = 2 \int_{-\pi/2}^{\pi/6} \int_{1+\sin\theta}^{3-3\sin\theta} r\, dr\, d\theta = 8\pi + 9\sqrt{3}$

5.22. $\frac{\pi}{4}$

5.23. $\iiint_B z \sin x \cos y\, dV = 8$

5.24. $\iiint_E 1 dV = 8\int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} \int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} 1 dz\, dy\, dx = 36\pi.$

5.25. (i) $\int_{z=0}^{z=4} \int_{x=0}^{x=\sqrt{4-z}} \int_{y=x^2}^{y=4-z} f(x, y, z) dy\, dx\, dz,$ (ii) $\int_{y=0}^{y=4} \int_{z=0}^{z=4-y} \int_{x=0}^{x=\sqrt{y}} f(x, y, z) dx\, dz\, dy,$ (iii) $\int_{y=0}^{y=4} \int_{x=0}^{x=\sqrt{y}} \int_{z=0}^{z=4-y} f(x, y, z) dz\, dx\, dy,$ (iv) $\int_{x=0}^{x=2} \int_{y=x^2}^{y=4} \int_{z=0}^{z=4-y} f(x, y, z) dz\, dy\, dx,$ (v) $\int_{x=0}^{x=2} \int_{z=0}^{z=4-x^2} \int_{y=x^2}^{y=4-z} f(x, y, z) dy\, dz\, dx$

5.26. $f_{\text{ave}} = 8$

5.27. 8

5.28. $\iiint_E f(r, \theta, z) r\, dz\, dr\, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=4-r\sin\theta} f(r, \theta, z) r\, dz\, dr\, d\theta.$

5.29. $E = \left\{(r, \theta, z) | 0 \le \theta \le 2\pi,\ 0 \le z \le 1,\ z \le r \le 2 - z^2\right\}$ and $V = \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} \int_{\theta=0}^{\theta=2\pi} r\, d\theta\, dz\, dr.$

5.30. $E_2 = \left\{(r, \theta, z) | 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ r \le z \le \sqrt{4 - r^2}\right\}$ and $V = \int_{r=0}^{r=1} \int_{z=r}^{z=\sqrt{4-r^2}} \int_{\theta=0}^{\theta=2\pi} r\, d\theta\, dz\, dr.$

5.31. $V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\varphi=\pi/3} \int_{\rho=0}^{\rho=2} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

5.32. Rectangular: $\int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\, dy\, dx - \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\, dy\, dx.$

Cylindrical: $\int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r\, dz\, dr\, d\theta.$

Spherical: $\int_{\varphi=\pi/6}^{\varphi=5\pi/6} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\csc\varphi}^{\rho=2} \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi.$

5.33. $\frac{9\pi}{8}$ kg

5.34. $M_x = \frac{81\pi}{64}$ and $M_y = \frac{81\pi}{64}$

5.35. $\bar{x} = \frac{M_y}{m} = \frac{81\pi/64}{9\pi/8} = \frac{9}{8}$ and $\bar{y} = \frac{M_x}{m} = \frac{81\pi/64}{9\pi/8} = \frac{9}{8}$.

5.36. $\bar{x} = \frac{M_y}{m} = \frac{1/20}{1/12} = \frac{3}{5}$ and $\bar{y} = \frac{M_x}{m} = \frac{1/24}{1/12} = \frac{1}{2}$

5.37. $x_c = \frac{M_y}{m} = \frac{1/15}{1/6} = \frac{2}{5}$ and $y_c = \frac{M_x}{m} = \frac{1/12}{1/6} = \frac{1}{2}$

5.38. $I_x = \int_{x=0}^{x=2}\int_{y=0}^{y=x} y^2\sqrt{xy}\,dy\,dx = \frac{64}{35}$ and $I_y = \int_{x=0}^{x=2}\int_{y=0}^{y=x} x^2\sqrt{xy}\,dy\,dx = \frac{64}{35}$. Also, $I_0 = \int_{x=0}^{x=2}\int_{y=0}^{y=x}\left(x^2 + y^2\right)\sqrt{xy}\,dy\,dx = \frac{128}{21}$.

5.39. $R_x = \frac{6\sqrt{35}}{35}$, $R_y = \frac{6\sqrt{15}}{15}$, and $R_0 = \frac{4\sqrt{42}}{7}$.

5.40. $\frac{54}{35} = 1.543$

5.41. $\left(\frac{3}{2}, \frac{9}{8}, \frac{1}{2}\right)$

5.42. The moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $99/35$, $36/7$, and $243/35$, respectively.

5.43. $T^{-1}(x, y) = (u, v)$ where $u = \frac{3x - y}{3}$ and $v = \frac{y}{3}$

5.44. $J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$

5.45. $\int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta$

5.46. $x = \frac{1}{2}(v + u)$ and $y = \frac{1}{2}(v - u)$ and $\int_{-4}^{4}\int_{-2}^{2}\frac{4}{u^2}\left(\frac{1}{2}\right)du\,dv$.

5.47. $\frac{1}{2}(\sin 2 - 2)$

5.48. $\int_0^3\int_0^2\int_1^2\left(\frac{v}{3} + \frac{vw}{3u}\right)du\,dv\,dw = 2 + \ln 8$

Section Exercises

1. 27.
3. 0.
5. 21.3.
7. a. 28 ft^3 b. 1.75 ft.

9. a. 0.112 b. $f_{\text{ave}} \simeq 0.175$; here $f(0.4, 0.2) \simeq 0.1$, $f(0.2, 0.6) \simeq -0.2$, $f(0.8, 0.2) \simeq 0.6$, and $f(0.8, 0.6) \simeq 0.2$.

11. 2π.

13. 40.

15. $\frac{81}{2} + 39\sqrt[3]{2}$.

17. $e - 1$.

19. $15 - \frac{10\sqrt{2}}{9}$.

21. 0.

23. $(e-1)(1+\sin 1-\cos 1)$.

25. $\frac{3}{4}\ln\left(\frac{5}{3}\right)+2\ln^2 2-\ln 2$.

27. $\frac{1}{8}\left[\left(2\sqrt{3}-3\right)\pi+6\ln 2\right]$.

29. $\frac{1}{4}e^4\left(e^4-1\right)$.

31. $4(e-1)(2-\sqrt{e})$.

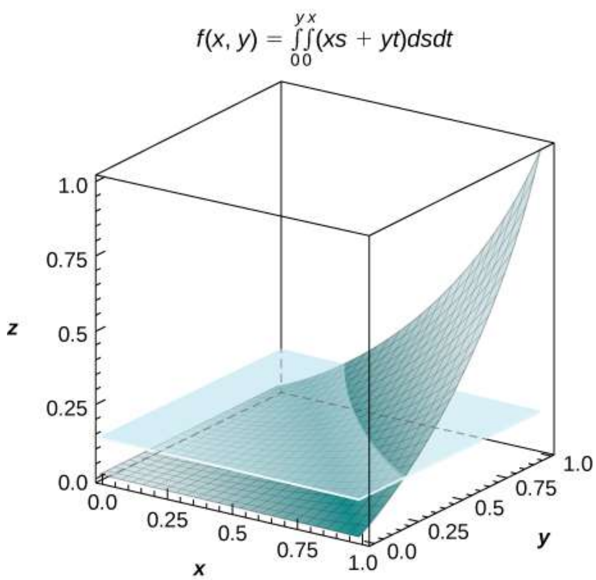
33. $-\frac{\pi}{4}+\ln\left(\frac{5}{4}\right)-\frac{1}{2}\ln 2+\arctan 2$.

35. $\frac{1}{2}$.

37. $\frac{1}{2}(2\cosh 1+\cosh 2-3)$.

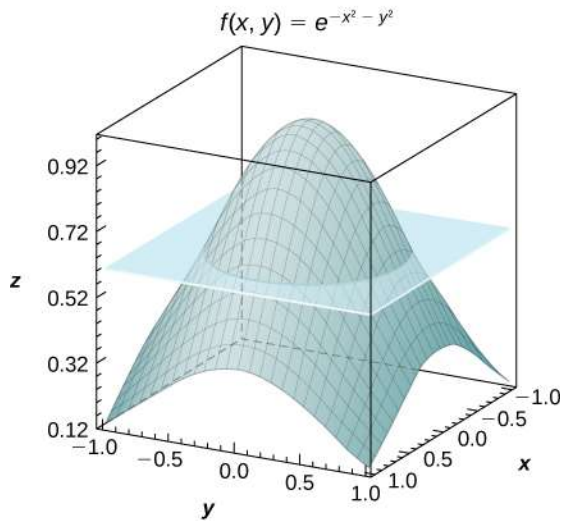
49. a. $f(x, y)=\frac{1}{2}xy\left(x^2+y^2\right)$ b. $V=\int_0^1\int_0^1 f(x, y)dx\,dy=\frac{1}{8}$ c. $f_{\text{ave}}=\frac{1}{8}$;

d.



53. a. For $m=n=2$, $I=4e^{-0.5}\approx 2.43$ b. $f_{\text{ave}}=e^{-0.5}\simeq 0.61$;

c.



55. a. $\frac{2}{n+1} + \frac{1}{4}$ b. $\frac{1}{4}$

59. $56.5°$ F; here $f(x_1^*, y_1^*) = 71,$ $f(x_2^*, y_1^*) = 72,$ $f(x_2^*, y_1^*) = 40,$ $f(x_2^*, y_2^*) = 43,$ where x_i^* and y_j^* are the midpoints of the subintervals of the partitions of $[a, b]$ and $[c, d],$ respectively.

61. $\frac{27}{20}$

63. Type I but not Type II

65. $\frac{\pi}{2}$

67. $\frac{1}{6}(8 + 3\pi)$

69. $\frac{1000}{3}$

71. Type I and Type II

73. The region D is not of Type I: it does not lie between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x).$ The region D is not of Type II: it does not lie between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y).$

75. $\frac{\pi}{2}$

77. 0

79. $\frac{2}{3}$

81. $\frac{41}{20}$

83. -63

85. π

87. a. Answers may vary; b. $\frac{2}{3}$

89. a. Answers may vary; b. $\frac{8}{12}$

91. $\frac{8\pi}{3}$

93. $e - \frac{3}{2}$

95. $\frac{2}{3}$

97. $\int_0^1 \int_{x-1}^{1-x} x\,dy\,dx = \int_{-1}^0 \int_0^{y+1} x\,dx\,dy + \int_0^1 \int_0^{1-y} x\,dxdy = \frac{1}{3}$

99. $\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y\,dx\,dy = \int_1^2 \int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} y\,dy\,dx = 0$

101. $\iint_D \left(x^2 - y^2\right) dA = \int_{-1}^1 \int_{y^4-1}^{1-y^4} \left(x^2 - y^2\right) dx\,dy = \frac{464}{4095}$

103. $\frac{4}{5}$

105. $\frac{5\pi}{32}$

109. 1

111. 2

113. a. $\frac{1}{3}$; b. $\frac{1}{6}$; c. $\frac{1}{6}$

115. a. $\frac{4}{3}$; b. 2π; c. $\frac{6\pi - 4}{3}$

117. 0 and 0.865474; $A(D) = 0.621135$

119. $P[X + Y \le 6] = 1 + \frac{3}{2e^2} - \frac{5}{e^{6/5}} \approx 0.45$; there is a 45% chance that a customer will spend 6 minutes in the drive-thru line.

123. $D = \left\{(r, \theta) | 4 \le r \le 5, \frac{\pi}{2} \le \theta \le \pi\right\}$

125. $D = \left\{(r, \theta) | 0 \le r \le \sqrt{2}, 0 \le \theta \le \pi\right\}$

127. $D = \{(r, \theta) | 0 \le r \le 4 \sin\theta, 0 \le \theta \le \pi\}$

129. $D = \left\{(r, \theta) | 3 \le r \le 5, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\right\}$

131. $D = \left\{(r, \theta) | 3 \le r \le 5, \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}\right\}$

133. $D = \left\{(r, \theta) | 0 \le r \le \tan\theta \sec\theta, 0 \le \theta \le \frac{\pi}{4}\right\}$

135. 0

137. $\frac{63\pi}{16}$

139. $\frac{3367\pi}{18}$

141. $\frac{35\pi^2}{576}$

143. $\frac{7}{576}\pi^2\left(21 - e + e^4\right)$

145. $\frac{5}{4}\ln\left(3 + 2\sqrt{2}\right)$

147. $\frac{1}{6}(2 - \sqrt{2})$

149. $\int_0^{\pi} \int_0^2 r^5\,dr\,d\theta = \frac{32\pi}{3}$

151. $\int_{-\pi/2}^{\pi/2} \int_0^4 r \sin\left(r^2\right) dr\,d\theta = \pi \sin^2 8$

153. $\frac{3\pi}{4}$

155. $\frac{\pi}{2}$

157. $\frac{1}{3}\left(4\pi - 3\sqrt{3}\right)$

159. $\frac{16}{3\pi}$

161. $\frac{\pi}{18}$

163. a. $\frac{2\pi}{3}$; b. $\frac{\pi}{2}$; c. $\frac{\pi}{6}$

165. $\frac{256\pi}{3}\,\text{cm}^3$

167. $\frac{3\pi}{32}$

169. 4π

171. $\frac{\pi}{4}$

173. $\frac{1}{2}\pi e(e-1)$

175. $\sqrt{3} - \frac{\pi}{4}$

177. $\frac{133\pi^3}{864}$

181. 192

183. 0

185. $\int_1^2 \int_2^3 \int_0^1 \left(x^2 + \ln y + z\right) dz\, dx\, dy = \frac{35}{6} + 2\ln 2$

187. $\int_1^3 \int_0^4 \int_{-1}^2 \left(x^2 z + \frac{1}{y}\right) dz\, dx\, dy = 64 + 12\ln 3$

191. $\frac{77}{12}$

193. 2

195. $\frac{439}{120}$

197. 0

199. $-\frac{64}{105}$

201. $\frac{11}{26}$

203. $\frac{113}{450}$

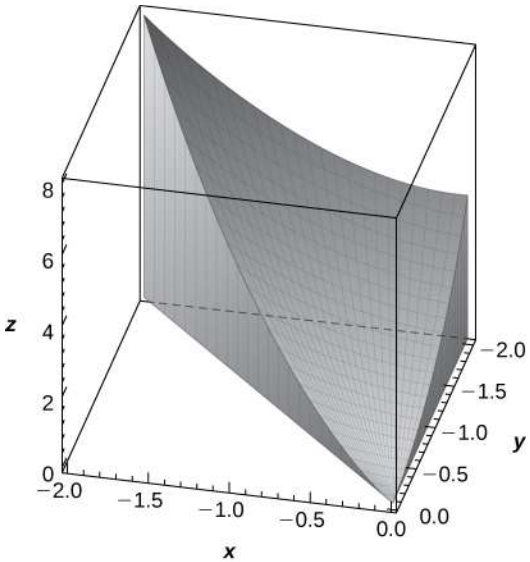
205. $\frac{1}{160}\left(6\sqrt{3} - 41\right)$

207. $\frac{3\pi}{2}$

209. 1250

211. $\int_0^5 \int_{-3}^3 \int_0^{\sqrt{9-y^2}} z\, dz\, dy\, dx = 90$

213. $V = 5.33$



215. $\int_0^1 \int_1^3 \int_2^4 \left(y^2 z^2 + 1\right) dz\, dx\, dy;\quad \int_0^1 \int_1^3 \int_2^4 \left(x^2 y^2 + 1\right) dy\, dz\, dx$

219. $V = \int_{-a}^{a} \int_{-\sqrt{a^2 - z^2}}^{\sqrt{a^2 - z^2}} \int_{\sqrt{x^2 + z^2}}^{a^2} dy\, dx\, dz$

221. $\frac{9}{2}$

223. $\frac{156}{5}$

225. a. Answers may vary; b. $\frac{128}{3}$

227. a. $\int_0^4 \int_0^{\sqrt{r^2 - x^2}} \int_0^{\sqrt{r^2 - x^2 - y^2}} dz\, dy\, dx$; b. $\int_0^2 \int_0^{\sqrt{r^2 - y^2}} \int_0^{\sqrt{r^2 - x^2 - y^2}} dz\, dx\, dy,\quad \int_0^r \int_0^{\sqrt{r^2 - z^2}} \int_0^{\sqrt{r^2 - x^2 - z^2}} dy\, dx\, dz,$ $\int_0^r \int_0^{\sqrt{r^2 - x^2}} \int_0^{\sqrt{r^2 - x^2 - z^2}} dy\, dz\, dx,\quad \int_0^r \int_0^{\sqrt{r^2 - z^2}} \int_0^{\sqrt{r^2 - y^2 - z^2}} dx\, dy\, dz,\quad \int_0^r \int_0^{\sqrt{r^2 - y^2}} \int_0^{\sqrt{r^2 - y^2 - z^2}} dx\, dz\, dy$

229. 3

231. $\frac{250}{3}$

233. $\frac{5}{16} \approx 0.313$

235. $\frac{35}{2}$

241. $\frac{9\pi}{8}$

243. $\frac{1}{8}$

245. $\frac{\pi e^2}{6}$

249. a. $E = \left\{(r, \theta, z) \middle| 0 \le \theta \le \pi,\ 0 \le r \le 4 \sin\theta,\ 0 \le z \le \sqrt{16 - r^2}\right\}$; b. $\int_0^{\pi} \int_0^{4 \sin\theta} \int_0^{\sqrt{16 - r^2}} f(r, \theta, z) r\, dz\, dr\, d\theta$

251. a. $E = \left\{(r, \theta, z) \middle| 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le \sqrt{3}, 9 - r^2 \le z \le 10 - r(\cos\theta + \sin\theta)\right\}$; b. $\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_{9-r^2}^{10-r(\cos\theta+\sin\theta)} f(r, \theta, z) r\, dz\, dr\, d\theta$

253. a. $E = \left\{(r, \theta, z) \middle| 0 \le r \le 3, 0 \le \theta \le \frac{\pi}{2}, 0 \le z \le r\cos\theta + 3\right\}$, $f(r, \theta, z) = \frac{1}{r\cos\theta + 3}$; b. $\int_0^3\int_0^{\pi/2}\int_0^{r\cos\theta+3} \frac{r}{r\cos\theta + 3} dz\, d\theta\, dr = \frac{9\pi}{4}$

255. a. $y = r\cos\theta$, $z = r\sin\theta$, $x = z$, $E = \left\{(r, \theta, z) \middle| 1 \le r \le 3, 0 \le \theta \le 2\pi, 0 \le z \le 1 - r^2\right\}$, $f(r, \theta, z) = z$; b. $\int_1^3\int_0^{2\pi}\int_0^{1-r^2} zr\, dz\, d\theta\, dr = \frac{356\pi}{3}$

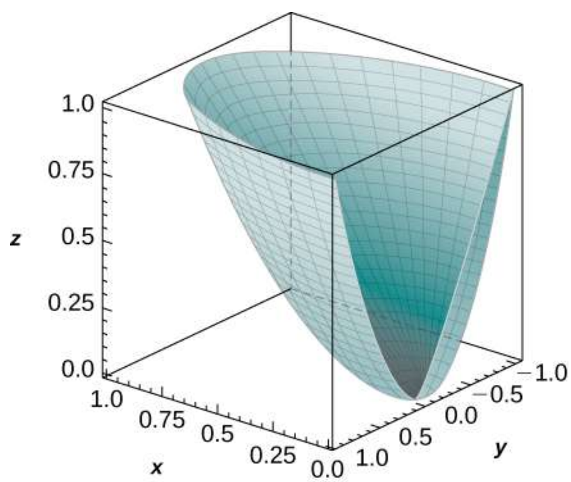
257. π

259. $\frac{\pi}{3}$

261. π

263. $\frac{4\pi}{3}$

265. $V = \frac{\pi}{12} \approx 0.2618$



267. $\int_0^1\int_0^{\pi}\int_{r^2}^{r} zr^2\cos\theta\, dz\, d\theta\, dr$

269. $180\pi\sqrt{10}$

271. $\frac{81\pi(\pi - 2)}{16}$

277. a. $f(\rho, \theta, \varphi) = \rho\sin\varphi(\cos\theta + \sin\theta)$, $E = \left\{(\rho, \theta, \varphi) \middle| 1 \le \rho \le 2, 0 \le \theta \le \pi, 0 \le \varphi \le \frac{\pi}{2}\right\}$; b. $\int_0^{\pi}\int_0^{\pi/2}\int_1^2 \rho^3\cos\varphi\sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{15\pi}{8}$

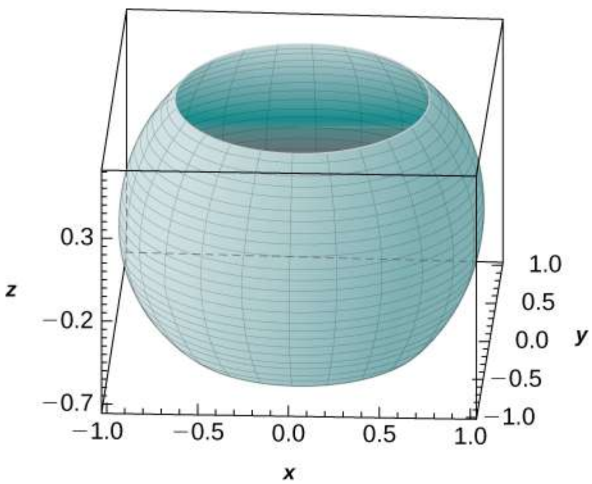
279. a. $f(\rho, \theta, \varphi) = \rho\cos\varphi$; $E = \left\{(\rho, \theta, \varphi) \middle| 0 \le \rho \le 2\cos\varphi, 0 \le \theta \le \frac{\pi}{2}, 0 \le \varphi \le \frac{\pi}{4}\right\}$; b. $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{2\cos\varphi} \rho^3\sin\varphi\cos\varphi\, d\rho\, d\varphi\, d\theta = \frac{7\pi}{24}$

281. $\frac{\pi}{4}$

283. $9\pi(\sqrt{2}-1)$

285. $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^4 \rho^6 \sin\varphi \, d\rho \, d\varphi \, d\theta$

287. $V = \frac{4\pi\sqrt{3}}{3} \approx 7.255$



289. $\frac{343\pi}{32}$

291. $\int_0^{2\pi}\int_2^4 \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta; \quad \int_{\pi/6}^{5\pi/6}\int_0^{2\pi}\int_{2\csc\varphi}^{4} \rho^2 \sin\rho \, d\rho \, d\theta \, d\varphi$

293. $P = \frac{32P_0\pi}{3}$ watts

295. $Q = kr^4\pi\mu C$

297. $\frac{27}{2}$

299. $24\sqrt{2}$

301. 76

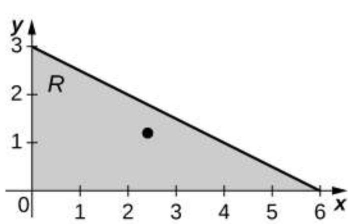
303. 8π

305. $\frac{\pi}{2}$

307. 2

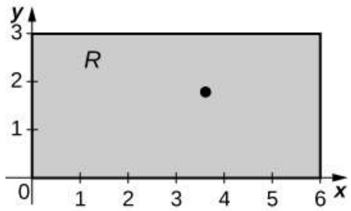
309. a. $M_x = \frac{81}{5}, M_y = \frac{162}{5}$; b. $\bar{x} = \frac{12}{5}, \bar{y} = \frac{6}{5}$;

c.



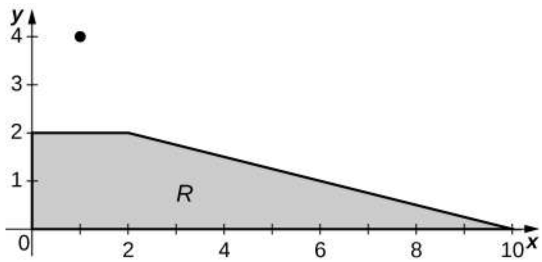
311. a. $M_x = \frac{216\sqrt{2}}{5}, M_y = \frac{432\sqrt{2}}{5}$; b. $\bar{x} = \frac{18}{5}, \bar{y} = \frac{9}{5}$;

c.



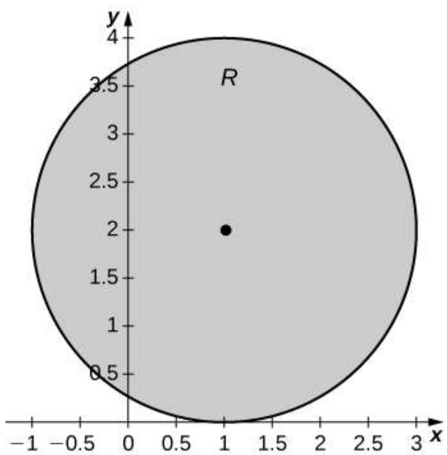
313. a. $M_x = \frac{368}{5}$, $M_y = \frac{1552}{5}$; b. $\overline{x} = \frac{92}{95}$, $\overline{y} = \frac{388}{95}$;

c.



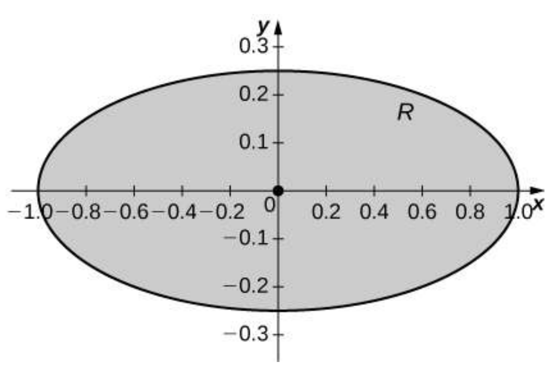
315. a. $M_x = 16\pi$, $M_y = 8\pi$; b. $\overline{x} = 1$, $\overline{y} = 2$;

c.



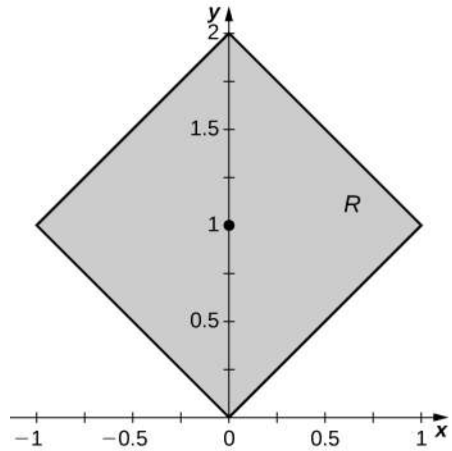
317. a. $M_x = 0$, $M_y = 0$; b. $\overline{x} = 0$, $\overline{y} = 0$;

c.



319. a. $M_x = 2$, $M_y = 0$; b. $\overline{x} = 0$, $\overline{y} = 1$;

c.



321. a. $I_x = \frac{243}{10}$, $I_y = \frac{486}{5}$, and $I_0 = \frac{243}{2}$; b. $R_x = \frac{3\sqrt{5}}{5}$, $R_y = \frac{6\sqrt{5}}{5}$, and $R_0 = 3$

323. a. $I_x = \frac{2592\sqrt{2}}{7}$, $I_y = \frac{648\sqrt{2}}{7}$, and $I_0 = \frac{3240\sqrt{2}}{7}$; b. $R_x = \frac{6\sqrt{21}}{7}$, $R_y = \frac{3\sqrt{21}}{7}$, and $R_0 = \frac{3\sqrt{105}}{7}$

325. a. $I_x = 88$, $I_y = 1560$, and $I_0 = 1648$; b. $R_x = \frac{\sqrt{418}}{19}$, $R_y = \frac{\sqrt{7410}}{19}$, and $R_0 = \frac{2\sqrt{1957}}{19}$

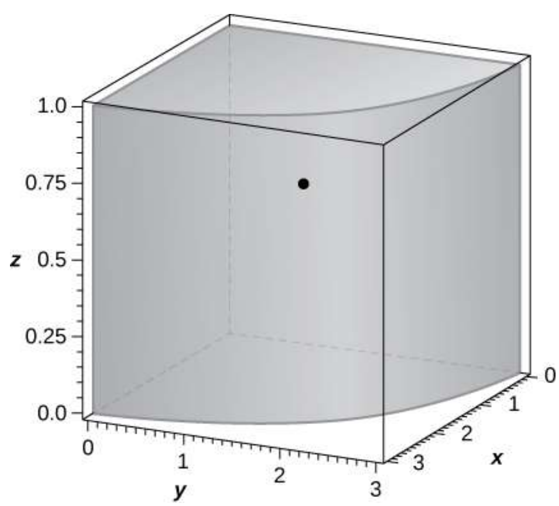
327. a. $I_x = \frac{128\pi}{3}$, $I_y = \frac{56\pi}{3}$, and $I_0 = \frac{184\pi}{3}$; b. $R_x = \frac{4\sqrt{3}}{3}$, $R_y = \frac{\sqrt{21}}{3}$, and $R_0 = \frac{\sqrt{69}}{3}$

329. a. $I_x = \frac{\pi}{32}$, $I_y = \frac{\pi}{8}$, and $I_0 = \frac{5\pi}{32}$; b. $R_x = \frac{1}{4}$, $R_y = \frac{1}{2}$, and $R_0 = \frac{\sqrt{5}}{4}$

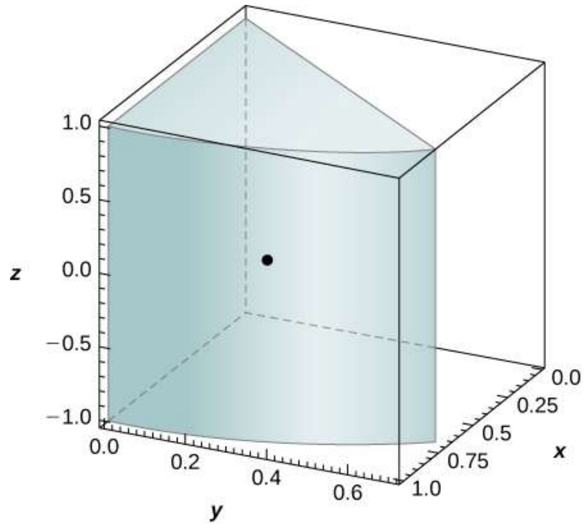
331. a. $I_x = \frac{7}{3}$, $I_y = \frac{1}{3}$, and $I_0 = \frac{8}{3}$; b. $R_x = \frac{\sqrt{42}}{6}$, $R_y = \frac{\sqrt{6}}{6}$, and $R_0 = \frac{2\sqrt{3}}{3}$

333. $m = \frac{1}{3}$

337. a. $m = \frac{9\pi}{4}$; b. $M_{xy} = \frac{3\pi}{2}$, $M_{xz} = \frac{81}{8}$, $M_{yz} = \frac{81}{8}$; c. $\bar{x} = \frac{9}{2\pi}$, $\bar{y} = \frac{9}{2\pi}$, $\bar{z} = \frac{2}{3}$; d. the solid Q and its center of mass are shown in the following figure.



339. a. $\bar{x} = \frac{3\sqrt{2}}{2\pi}$, $\bar{y} = \frac{3(2-\sqrt{2})}{2\pi}$, $\bar{z} = 0$; b. the solid Q and its center of mass are shown in the following figure.



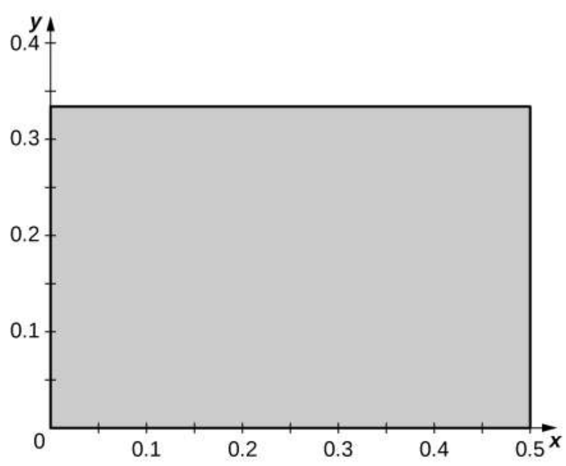
343. $n = -2$

349. a. $\rho(x, y, z) = x^2 + y^2$; b. $\frac{16\pi}{7}$

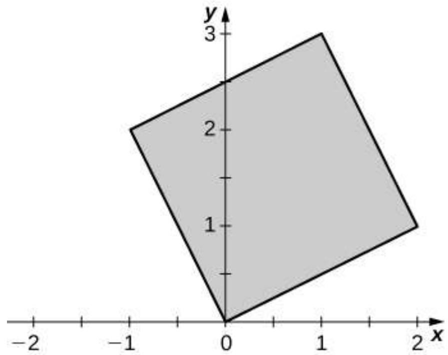
351. $M_{xy} = \pi(f(0) - f(a) + af'(a))$

355. $I_x = I_y = I_z \simeq 0.84$

357. a. $T(u, v) = (g(u, v), h(u, v))$, $x = g(u, v) = \frac{u}{2}$ and $y = h(u, v) = \frac{v}{3}$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = \frac{1}{2}$, $g_v(u, v) = 0$, $h_u(u, v) = 0$ and $h_v(u, v) = \frac{1}{3}$ are continuous on S; b. $T(0, 0) = (0, 0)$, $T(1, 0) = \left(\frac{1}{2}, 0\right)$, $T(0, 1) = \left(0, \frac{1}{3}\right)$, and $T(1, 1) = \left(\frac{1}{2}, \frac{1}{3}\right)$; c. R is the rectangle of vertices $(0, 0)$, $\left(\frac{1}{2}, 0\right)$, $\left(\frac{1}{2}, \frac{1}{3}\right)$, and $\left(0, \frac{1}{3}\right)$ in the xy-plane; the following figure.



359. a. $T(u, v) = (g(u, v), h(u, v))$, $x = g(u, v) = 2u - v$, and $y = h(u, v) = u + 2v$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = 2$, $g_v(u, v) = -1$, $h_u(u, v) = 1$, and $h_v(u, v) = 2$ are continuous on S; b. $T(0, 0) = (0, 0)$, $T(1, 0) = (2, 1)$, $T(0, 1) = (-1, 2)$, and $T(1, 1) = (1, 3)$; c. R is the parallelogram of vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, and $(-1, 2)$ in the xy-plane; see the following figure.



361. a. $T(u, v) = (g(u, v), h(u, v))$, $x = g(u, v) = u^3$, and $y = h(u, v) = v^3$. The functions g and h are continuous and differentiable, and the partial derivatives $g_u(u, v) = 3u^2$, $g_v(u, v) = 0$, $h_u(u, v) = 0$, and $h_v(u, v) = 3v^2$ are continuous on S; b. $T(0, 0) = (0, 0)$, $T(1, 0) = (1, 0)$, $T(0, 1) = (0, 1)$, and $T(1, 1) = (1, 1)$; c. R is the unit square in the xy-plane; see the figure in the answer to the previous exercise.

363. T is not one-to-one: two points of S have the same image. Indeed, $T(-2, 0) = T(2, 0) = (16, 4)$.

365. T is one-to-one: We argue by contradiction. $T(u_1, v_1) = T(u_2, v_2)$ implies $2u_1 - v_1 = 2u_2 - v_2$ and $u_1 = u_2$. Thus, $u_1 = u_2$ and $v_1 = v_2$.

367. T is not one-to-one: $T(1, v, w) = (-1, v, w)$

369. $u = \frac{x - 2y}{3}, v = \frac{x + y}{3}$

371. $u = e^x, v = e^{-x + y}$

373. $u = \frac{x - y + z}{2}, v = \frac{x + y - z}{2}, w = \frac{-x + y + z}{2}$

375. $S = \{(u, v) | u^2 + v^2 \le 1\}$

377. $R = \{(u, v, w) | u^2 - v^2 - w^2 \le 1, w > 0\}$

379. $\frac{3}{2}$

381. -1

383. $2uv$

385. $\frac{v}{u^2}$

387. 2

389. a. $T(u, v) = (2u + v, 3v)$; b. The area of R is

$$A(R) = \int_0^3 \int_{y/3}^{(6 - y)/3} dx\, dy = \int_0^1 \int_0^{1 - u} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dv\, du = \int_0^1 \int_0^{1 - u} 6 dv\, du = 3.$$

391. $-\frac{1}{4}$

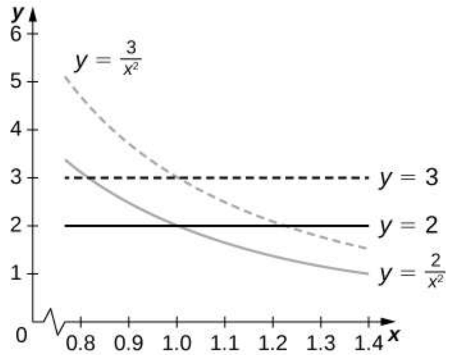
393. $-1 + \cos 2$

395. $\frac{\pi}{15}$

397. $\frac{31}{5}$

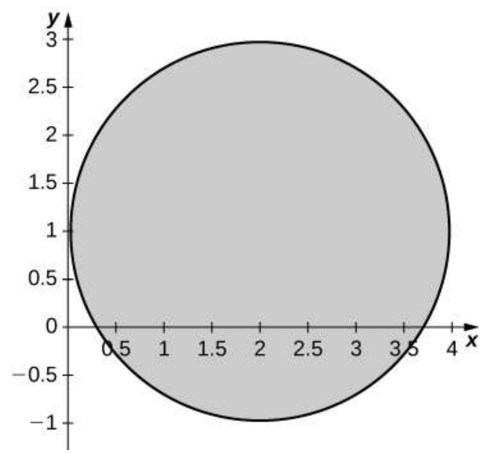
399. $T(r, \theta, z) = (r \cos\theta, r \sin\theta, z)$; $S = [0, 3] \times \left[0, \frac{\pi}{2}\right] \times [0, 1]$ in the $r\theta z$-space

403. The area of R is $10 - 4\sqrt{6}$; the boundary curves of R are graphed in the following figure.



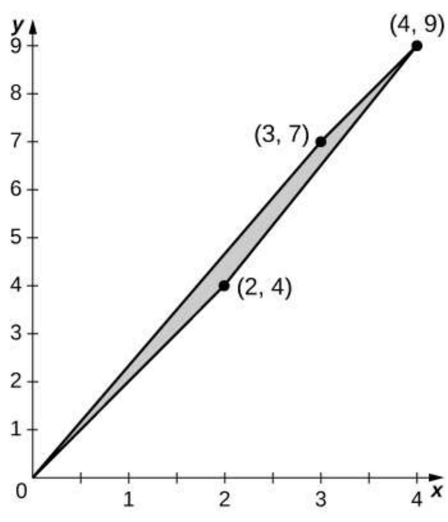
405. 8

409. a. $R = \{(x, y) | y^2 + x^2 - 2y - 4x + 1 \leq 0\}$; b. R is graphed in the following figure;



c. 3.16

411. a. $T_{0, 2} \circ T_{3, 0}(u, v) = (u + 3v, 2u + 7v)$; b. The image S is the quadrilateral of vertices $(0, 0), (3, 7), (2, 4),$ and $(4, 9)$; c. S is graphed in the following figure;



d. $\frac{3}{2}$

413. $\frac{2662}{3\pi} \simeq 282.45 \text{ in}^3$

415. $A(R) \simeq 83{,}999.2$

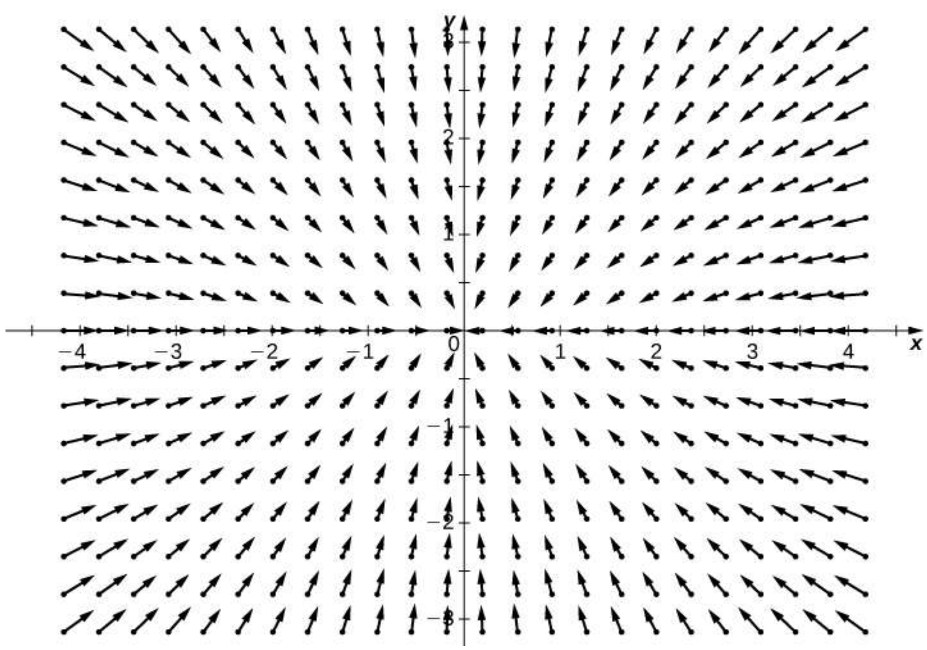
Review Exercises

417. True.
419. False.
421. 0
423. $\frac{1}{4}$
425. 1.475
427. $\frac{52}{3}\pi$
429. $\frac{\pi}{16}$
431. 93.291
433. $\left(\frac{8}{15}, \frac{8}{15}\right)$
435. $\left(0, 0, \frac{8}{5}\right)$
437. $1.452\pi \times 10^{15}$ ft-lb
439. $y = -1.238 \times 10^{-7} x^3 + 0.001196x^2 - 3.666x + 7208;$ average temperature approximately $2800°C$
441. $\frac{\pi}{3}$

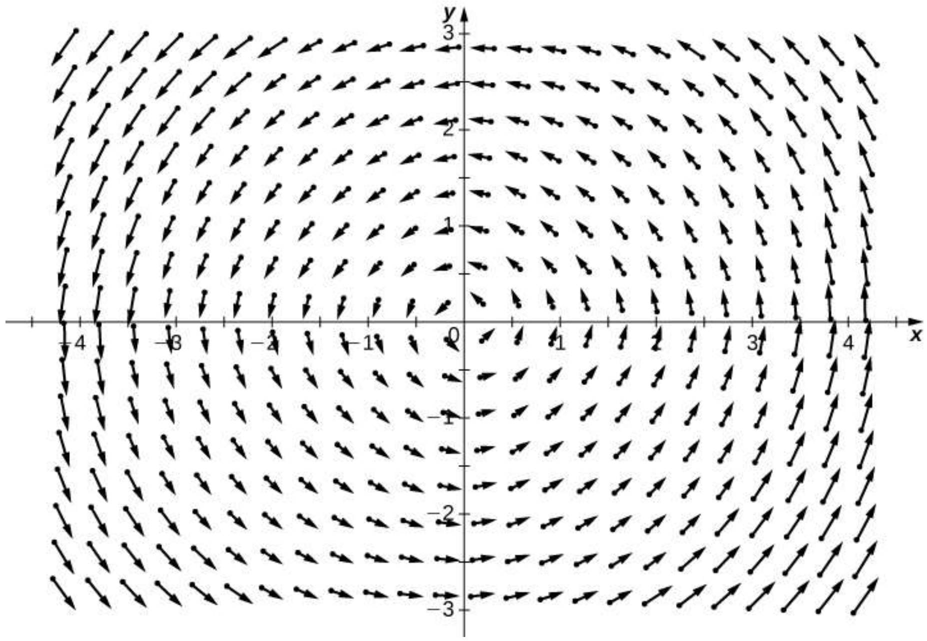
Chapter 6

Checkpoint

6.1. $12\mathbf{i} - \mathbf{j}$
6.2.



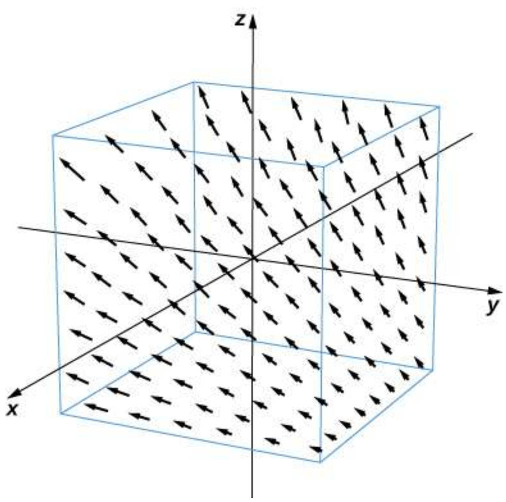
6.3. Rotational



6.4. $\sqrt{65}$ m/sec

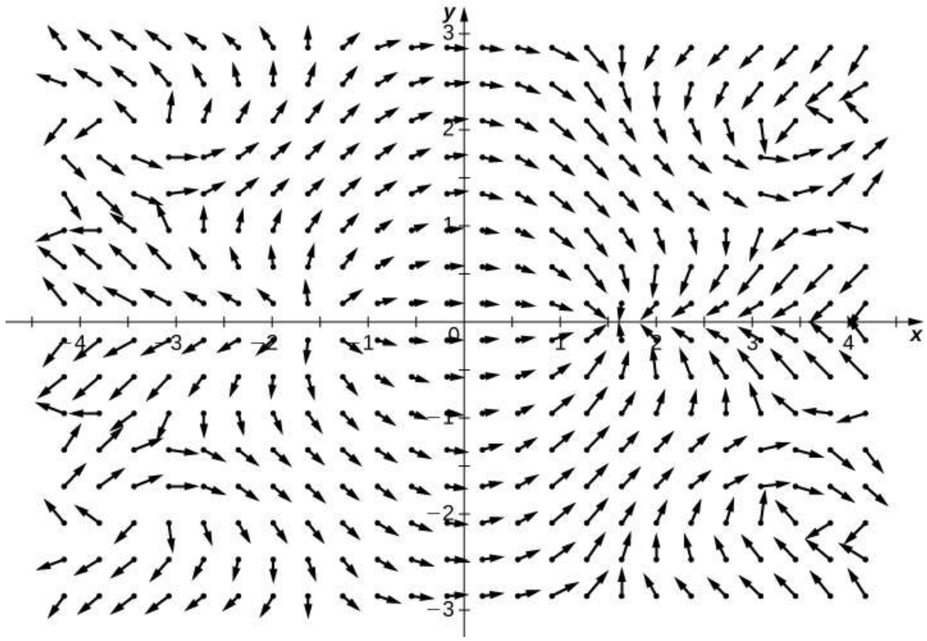
6.5. No.

6.6.



6.7. $1.49063 \times 10^{-18},\ 4.96876 \times 10^{-19},\ 9.93752 \times 10^{-19}\,\mathrm{N}$

6.8.



6.9. No
6.10. $\nabla f = \mathbf{v}$
6.11. $P_y = x \neq Q_x = -2xy$
6.12. No
6.13. $\sqrt{2}$
6.14. $\frac{1}{3} + \frac{\sqrt{2}}{6} + \frac{3\pi}{4}$
6.15. Both line integrals equal $-\frac{1000\sqrt{30}}{3}$.
6.16. $4\sqrt{17}$
6.17. $\int_C \mathbf{F} \cdot \mathbf{T} ds$
6.18. -26
6.19. 0
6.20. $18\sqrt{2}\pi^2$ kg
6.21. 3/2
6.22. 2π
6.23. 0
6.24. Yes
6.25. The region in the figure is connected. The region in the figure is not simply connected.
6.26. 2
6.27. If C_1 and C_2 represent the two curves, then $\int_{C_1} \mathbf{F} \bullet d\mathbf{r} \neq \int_{C_2} \mathbf{F} \bullet d\mathbf{r}$.
6.28. $f(x, y) = e^x y^3 + xy$
6.29. $f(x, y, z) = 4x^3 + \sin y \cos z + z$
6.30. $f(x, y, z) = \frac{G}{\sqrt{x^2 + y^2 + z^2}}$
6.31. It is conservative.
6.32. -10π
6.33. Negative
6.34. $\frac{45}{2}$

6.35. $\frac{4}{3}$

6.36. $\frac{3\pi}{2}$

6.37. $g(x, y) = -x \cos y$

6.38. No

6.39. 105π

6.40. $y - z^2$

6.41. Yes

6.42. All points on line $y = 1$.

6.43. $-\mathbf{i}$

6.44. $\text{curl } \mathbf{v} = \mathbf{0}$

6.45. No

6.46. Yes

6.47. Cylinder $x^2 + y^2 = 4$

6.48. Cone $x^2 + y^2 = z^2$

6.49. $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 0 < u < \infty, 0 \le v < \frac{\pi}{2}$

6.50. Yes

6.51. ≈ 43.02

6.52. With the standard parameterization of a cylinder, **Equation 6.201** shows that the surface area is $2\pi rh$.

6.53. $2\pi\left(\sqrt{2} + \sinh^{-1}(1)\right)$

6.54. 24

6.55. 0

6.56. $38.401\pi \approx 120.640$

6.57. $\mathbf{N}(x, y) = \left\langle \frac{-y}{\sqrt{1 + x^2 + y^2}}, \frac{-x}{\sqrt{1 + x^2 + y^2}}, \frac{1}{\sqrt{1 + x^2 + y^2}} \right\rangle$

6.58. 0

6.59. 400 kg/sec/m

6.60. $-\frac{440\pi}{3}$

6.61. Both integrals give $-\frac{136}{45}$.

6.62. $-\pi$

6.63. $\frac{3}{2}$

6.64. $\text{curl } \mathbf{E} = \langle x, y, -2z \rangle$

6.65. Both integrals equal 6π.

6.66. 30

6.67. $9 \ln(16)$

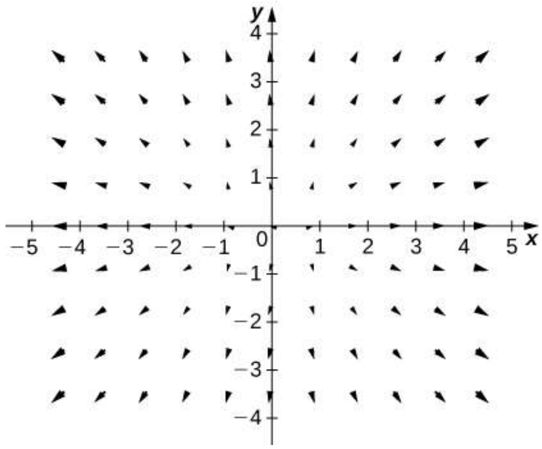
6.68. $\approx 6.777 \times 10^9$

Section Exercises

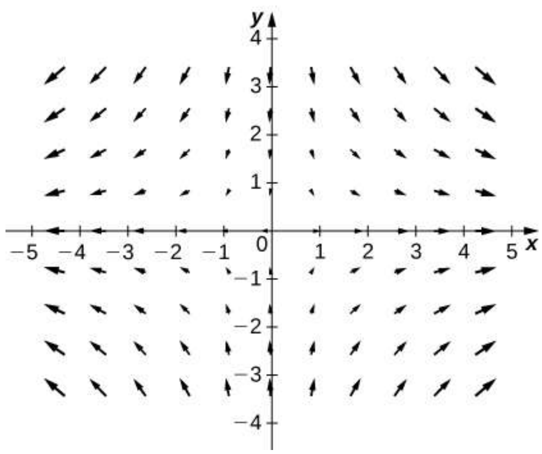
1. Vectors

3. False

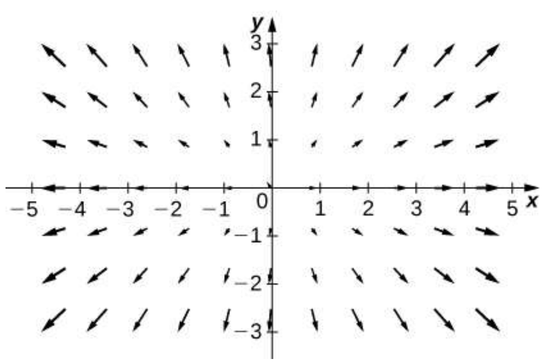
5.


y
x
−5 −4 −3 −2 −1 0 1 2 3 4 5
4 3 2 1 −1 −2 −3 −4

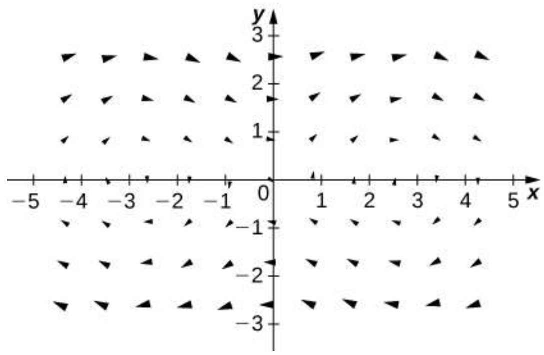
7.


y
x
−5 −4 −3 −2 −1 0 1 2 3 4 5
4 3 2 1 −1 −2 −3 −4

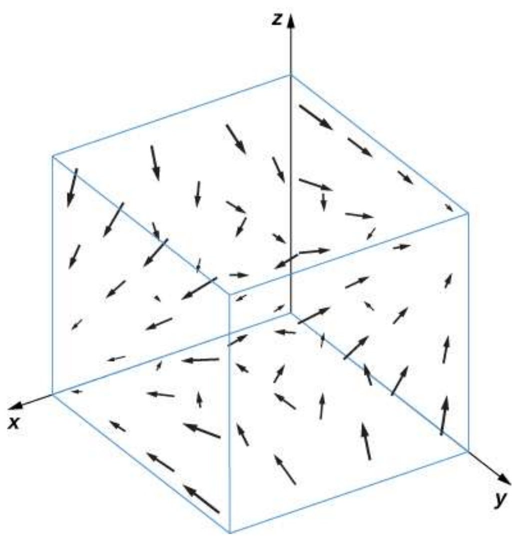
9.


y
x
−5 −4 −3 −2 −1 0 1 2 3 4 5
3 2 1 −1 −2 −3

11.



13.



15. $\mathbf{F}(x,\ y) = \sin(y)\mathbf{i} + (x\cos y - \sin y)\mathbf{j}$

17. $\mathbf{F}(x,\ y,\ z) = (2xy + y)\mathbf{i} + (x^2 + x + 2yz)\mathbf{j} + y^2\mathbf{k}$

19. $\mathbf{F}(x,\ y) = \left(\dfrac{2x}{1 + x^2 + 2y^2}\right)\mathbf{i} + \left(\dfrac{4y}{1 + x^2 + 2y^2}\right)\mathbf{j}$

21. $\mathbf{F}(x,\ y) = \dfrac{(1 - x)\mathbf{i} - y\mathbf{j}}{\sqrt{(1 - x)^2 + y^2}}$

23. $\mathbf{F}(x,\ y) = \dfrac{(y\mathbf{i} - x\mathbf{j})}{\sqrt{x^2 + y^2}}$

25. $\mathbf{F}(x,\ y) = y\mathbf{i} - x\mathbf{j}$

27. $\mathbf{F}(x,\ y) = \dfrac{-10}{\left(x^2 + y^2\right)^{3/2}}(x\mathbf{i} + y\mathbf{j})$

29. $E = \dfrac{c}{|r|^2}r = \dfrac{c}{|r|}\dfrac{r}{|r|}$

31. $\mathbf{c}'(t) = \left(\cos t,\ -\sin t,\ e^{-t}\right) = \mathbf{F}(\mathbf{c}(t))$

33. $\mathbf{H}$

35. d. $-\mathbf{F} + \mathbf{G}$

37. a. $\mathbf{F} + \mathbf{G}$

39. True

41. False

43. False

45. $\int_C (x-y)ds = 10$

47. $\int_C xy^4\,ds = \frac{8192}{5}$

49. $W = 8$

51. $W = \frac{3\pi}{4}$

53. $W = \pi$

55. $\int_C \mathbf{F} \cdot d\mathbf{r} = 4$

57. $\int_C yzdx + xzdy + xydz = -1$

59. $\int_C \left(y^2\right)dx + (x)dy = \frac{245}{6}$

61. $\int_C xydx + ydy = \frac{190}{3}$

63. $\int_C \frac{y}{2x^2 - y^2} ds = \sqrt{2}\ln 5$

65. $W = -66$

67. $W = -10\pi^2$

69. $W = 2$

71. a. $W = 11;$ b. $W = 11;$ c. Yes

73. $W = 2\pi$

75. $\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{25\sqrt{5}+1}{120}$

77. $\int_C y^2\,dx + \left(xy - x^2\right)dy = 6.15$

79. $\int_\gamma xe^y\,ds \approx 7.157$

81. $\int_\gamma \left(y^2 - xy\right)dx \approx -1.379$

83. $\int_C \mathbf{F} \cdot d\mathbf{r} \approx -1.133$

85. $\int_C \mathbf{F} \cdot d\mathbf{r} \approx 22.857$

87. $\text{flu} \;= -\frac{1}{3}$

89. $\text{flu} \;= -20$

91. $\text{flu} \;= 0$

93. $m = 4\pi\rho\sqrt{5}$

95. $W = 0$

97. $W = \frac{k}{2}$

99. True

101. True

103. $\int_C \mathbf{F} \cdot d\mathbf{r} = 24$

105. $\int_C \mathbf{F} \cdot d\mathbf{r} = e - \frac{3\pi}{2}$

107. Not conservative

109. Conservative, $f(x,\ y) = 3x^2 + 5xy + 2y^2$

111. Conservative, $f(x,\ y) = ye^x + x\sin(y)$

113. $\oint_C (2ydx + 2xdy) = 32$

115. $\mathbf{F}(x,\ y) = (10x + 3y)\mathbf{i} + (3x + 10y)\mathbf{j}$

117. $\mathbf{F}$ is not conservative.

119. $\mathbf{F}$ is conservative and a potential function is $f(x,\ y, z) = xye^z$.

121. $\mathbf{F}$ is conservative and a potential function is $f(x,\ y,\ z) = z$.

123. $\mathbf{F}$ is conservative and a potential function is $f(x,\ y, z) = x^2y + y^2z$.

125. $\mathbf{F}$ is conservative and a potential function is $f(x,\ y) = e^{x^2y}$

127. $\int_C \mathbf{F} \cdot dr = e^2 + 1$

129. $\int_C \mathbf{F} \cdot dr = 41$

131. $\oint_{C_1} \mathbf{G} \cdot d\mathbf{r} = -8\pi$

133. $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 7$

135. $\int_C \mathbf{F} \cdot d\mathbf{r} = 150$

137. $\int_C \mathbf{F} \cdot d\mathbf{r} = -1$

139. 4×10^{31} erg

141. $\int_C \mathbf{F} \cdot d\mathbf{s} = 0.4687$

143. circulation $= \pi a^2$ and flu $= 0$

147. $\int_C 2xydx + (x + y)dy = \frac{32}{3}$

149. $\int_C \sin x \cos ydx + (xy + \cos x \sin y)dy = \frac{1}{12}$

151. $\oint_C (-ydx + xdy) = \pi$

153. $\int_C xe^{-2x}dx + \left(x^4 + 2x^2y^2\right)dy = 0$

155. $\oint_C y^3dx - x^3dy = -24\pi$

157. $\oint_C -x^2ydx + xy^2dy = 8\pi$

159. $\oint_C \left(x^2 + y^2\right)dx + 2xydy = 0$

161. $A = 19\pi$

163. $A = \frac{3}{8\pi}$

165. $\int_{C+} \left(y^2 + x^3\right)dx + x^4dy = 0$

167. $A = \frac{9\pi}{8}$

169. $A = \frac{8\sqrt{3}}{5}$

171. $\int_C \left(x^2y - 2xy + y^2\right)ds = 3$

173. $\int_C \frac{xdx + ydy}{x^2 + y^2} = 2\pi$

175. $W = \frac{225}{2}$

177. $W = 12\pi$

179. $W = 2\pi$

181. $\oint_C y^2\, dx + x^2\, dy = \frac{1}{3}$

183. $\int_C \sqrt{1 + x^3} dx + 2xydy = 3$

185. $\int_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4 + 1}\right) dy = 36\pi$

187. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2$

189. $\oint_C (y + x)dx + (x + \sin y)dy = 0$

191. $\oint_C xydx + x^3\, y^3\, dy = \frac{22}{21}$

193. $\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{15\pi}{4}$

195. $\int_C \sin(x + y)dx + \cos(x + y)dy = 4$

197. $\int_C \mathbf{F} \cdot d\mathbf{r} = \pi$

199. $\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = 4$

201. $\oint_C \mathbf{F} \cdot \mathbf{n} ds = 32\pi$

203. $\int_C \left[-y^3 + \sin(xy) + xy\cos(xy)\right] dx + \left[x^3 + x^2 \cos(xy)\right] dy = 4.7124$

205. $\oint_C \left(y + e^{\sqrt{x}}\right) dx + \left(2x + \cos\left(y^2\right)\right) dy = \frac{1}{3}$

207. False

209. True

211. True

213. $\text{curl } \mathbf{F} = \mathbf{i} + x^2\, \mathbf{j} + y^2\, \mathbf{k}$

215. $\text{curl } \mathbf{F} = \left(xz^2 - xy^2\right)\mathbf{i} + \left(x^2\, y - yz^2\right)\mathbf{j} + \left(y^2\, z - x^2\, z\right)\mathbf{k}$

217. $\text{curl } \mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

219. $\text{curl } \mathbf{F} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$

221. $\text{curl } \mathbf{F} = 0$

223. $\text{div } \mathbf{F} = 3yz^2 + 2y \sin z + 2xe^{2z}$

225. $\text{div } \mathbf{F} = 2(x + y + z)$

227. $\text{div } \mathbf{F} = \frac{1}{\sqrt{x^2 + y^2}}$

229. $\text{div } \mathbf{F} = a + b$

231. $\text{div } \mathbf{F} = x + y + z$

233. Harmonic

235. $\text{div } (\mathbf{F} \times \mathbf{G}) = 2z + 3x$

237. $\text{div } \mathbf{F} = 2r^2$

239. $\text{curl } \mathbf{r} = 0$

241. $\text{curl}\,\dfrac{\mathbf{r}}{r^3} = 0$

243. $\text{curl}\,\mathbf{F} = \dfrac{2x}{x^2 + y^2}\mathbf{k}$

245. $\text{div}\,\mathbf{F} = 0$

247. $\text{div}\,\mathbf{F} = 2 - 2e^{-6}$

249. $\text{div}\,\mathbf{F} = 0$

251. $\text{curl}\,\mathbf{F} = \mathbf{j} - 3\mathbf{k}$

253. $\text{curl}\,\mathbf{F} = 2\mathbf{j} - \mathbf{k}$

255. $a = 3$

257. **F** is conservative.

259. $\text{div}\,\mathbf{F} = \cosh x + \sinh y - xy$

261. $(bz - cy)\mathbf{i}(cx - az)\mathbf{j} + (ay - bx)\mathbf{k}$

263. $\text{curl}\,\mathbf{F} = 2\omega$

265. $\mathbf{F} \times \mathbf{G}$ does not have zero divergence.

267. $\nabla \cdot \mathbf{F} = -200k\left[1 + 2\left(x^2 + y^2 + z^2\right)\right]e^{-x^2 + y^2 + z^2}$

269. True

271. True

273. $\mathbf{r}(u, v) = \langle u, v, 2 - 3u + 2v \rangle$ for $-\infty \le u < \infty$ and $-\infty \le v < \infty$.

275. $\mathbf{r}(u, v) = \langle u, v, \frac{1}{3}(16 - 2u + 4v) \rangle$ for $|u| < \infty$ and $|v| < \infty$.

277. $\mathbf{r}(u, v) = \langle 3\cos u, 3\sin u, v \rangle$ for $0 \le u \le \frac{\pi}{2}, 0 \le v \le 3$

279. $A = 87.9646$

281. $\iint_S z\,dS = 8\pi$

283. $\iint_S \left(x^2 + y^2\right)z\,dS = 16\pi$

285. $\iint_S \mathbf{F} \cdot \mathbf{N}\,dS = \frac{4\pi}{3}$

287. $m \approx 13.0639$

289. $m \approx 228.5313$

291. $\iint_S g\,dS = 3\sqrt{4}$

293. $\iint_S \left(x^2 + y - z\right)d\mathbf{S} \approx 0.9617$

295. $\iint_S \left(x^2 + y^2\right)d\mathbf{S} = \frac{4\pi}{3}$

297. $\iint_S x^2 z\,d\mathrm{S} = \frac{1023\sqrt{2\pi}}{5}$

299. $\iint_S (z + y)\,d\mathrm{S} \approx 10.1$

301. $m = \pi a^3$

303. $\iint_S \mathbf{F} \cdot \mathbf{N}\,dS = \frac{13}{24}$

305. $\iint_S \mathbf{F} \cdot \mathbf{N}\,dS = \frac{3}{4}$

307. $\displaystyle\int_0^8 \int_0^6 \left(4 - 3y + \frac{1}{16}y^2 + z\right)\left(\frac{1}{4}\sqrt{17}\right)dz\,dy$

309. $\displaystyle\int_0^2 \int_0^6 \left[x^2 - 2(8 - 4x) + z\right]\sqrt{17}\,dz\,dx$

311. $\iint_S \left(x^2 z + y^2 z\right) dS = \frac{\pi a^5}{2}$

313. $\iint_S x^2 yz dS = 171\sqrt{14}$

315. $\iint_S yz dS = \frac{\sqrt{2}\pi}{4}$

317. $\iint_S (x\mathbf{i} + y\mathbf{j}) \cdot d\mathbf{S} = 16\pi$

319. $m = \frac{\pi a^7}{192}$

321. $F \approx 4.57$ lb.

323. $8\pi a$

325. The net flux is zero.

327. $\iint_S (\text{curl } \mathbf{F} \cdot \mathbf{N}) dS = \pi a^2$

329. $\iint_S (\text{curl } \mathbf{F} \cdot \mathbf{N}) dS = 18\pi$

331. $\iint_S (\text{curl } \mathbf{F} \cdot \mathbf{N}) dS = -8\pi$

333. $\iint_S (\text{curl } \mathbf{F} \cdot \mathbf{N}) dS = 0$

335. $\int_C \mathbf{F} \cdot d\mathbf{S} = 0$

337. $\int_C \mathbf{F} \cdot d\mathbf{S} = -9.4248$

339. $\iint\limits_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0$

341. $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 2.6667$

343. $\iint_S (\text{curl } \mathbf{F} \cdot \mathbf{N}) dS = -\frac{1}{6}$

345. $\int\limits_C \left(\frac{1}{2} y^2 dx + z dy + x dz\right) = -\frac{\pi}{4}$

347. $\iint_S (\text{curl } \mathbf{F} \cdot \mathbf{N}) dS = -3\pi$

349. $\int_C (c\mathbf{k} \times \mathbf{R}) \cdot d\mathbf{S} = 2\pi c$

351. $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0$

353. $\oint \mathbf{F} \cdot d\mathbf{S} = -4$

355. $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0$

357. $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = -36\pi$

359. $\iint_S \text{curl } \mathbf{F} \cdot \mathbf{N} = 0$

361. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$

363. $\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = 84.8230$

365. $A = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 0$

367. $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 2\pi$

369. $C = \pi(\cos \varphi - \sin \varphi)$

371. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 48\pi$

373. $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} = 0$

375. 0

377. $\int_S \mathbf{F} \cdot \mathbf{n} ds = 75.3982$

379. $\int_S \mathbf{F} \cdot \mathbf{n} ds = 127.2345$

381. $\int_S \mathbf{F} \cdot \mathbf{n} ds = 37.6991$

383. $\int_S \mathbf{F} \cdot \mathbf{n} ds = \frac{9\pi a^4}{2}$

385. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{3}$

387. $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$

389. $\iint_S \mathbf{F} \cdot d\mathbf{S} = 241.2743$

391. $\iint_D \mathbf{F} \cdot d\mathbf{S} = -\pi$

393. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{2\pi}{3}$

395. $16\sqrt{6}\pi$

397. $-\frac{128}{3}\pi$

399. -703.7168

401. 20

403. $\iint_S \mathbf{F} \cdot d\mathbf{S} = 8$

405. $\iint_S \mathbf{F} \cdot \mathbf{N} dS = \frac{1}{8}$

407. $\iint_S \| \mathbf{R} \| \mathbf{R} \cdot n\, ds = 4\pi a^4$

409. $\iiint_R z^2\, dV = \frac{4\pi}{15}$

411. $\iint_S \mathbf{F} \cdot d\mathbf{S} = 6.5759$

413. $\iint_S \mathbf{F} \cdot d\mathbf{S} = 21$

415. $\iint_S \mathbf{F} \cdot d\mathbf{S} = 72$

417. $\iint_S \mathbf{F} \cdot d\mathbf{S} = -33.5103$

419. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \pi a^4 b^2$

421. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{5}{2}\pi$

423. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{21\pi}{2}$

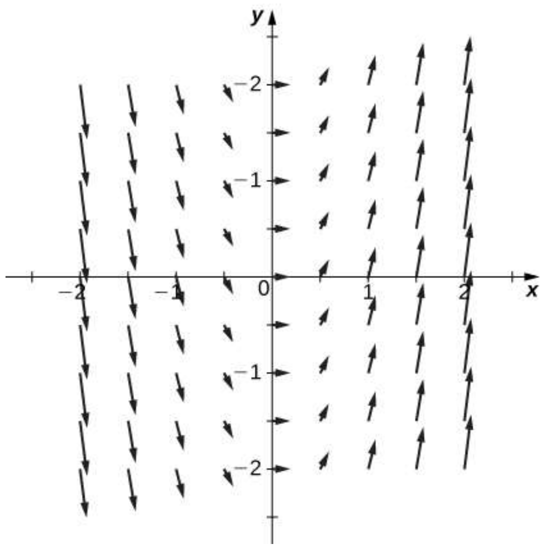
425. $-\left(1 - e^{-1}\right)$

Review Exercises

427. False

429. False

431.



433. Conservative, $f(x, y) = xy - 2e^y$

435. Conservative, $f(x, y, z) = x^2 y + y^2 z + z^2 x$

437. $-\frac{16}{3}$

439. $\frac{32\sqrt{2}}{9}\left(3\sqrt{3} - 1\right)$

441. Divergence: $e^x + xe^{xy} + xye^{xyz}$, curl: $xze^{xyz}\mathbf{i} - yze^{xyz}\mathbf{j} + ye^{xy}\mathbf{k}$

443. -2π

445. $-\pi$

447. $31\pi/2$

449. $\sqrt{2}(2 + \pi)$

451. $2\pi/3$

Chapter 7

Checkpoint

7.1.

a. Nonlinear
b. Linear, nonhomogeneous

7.4. Linearly independent

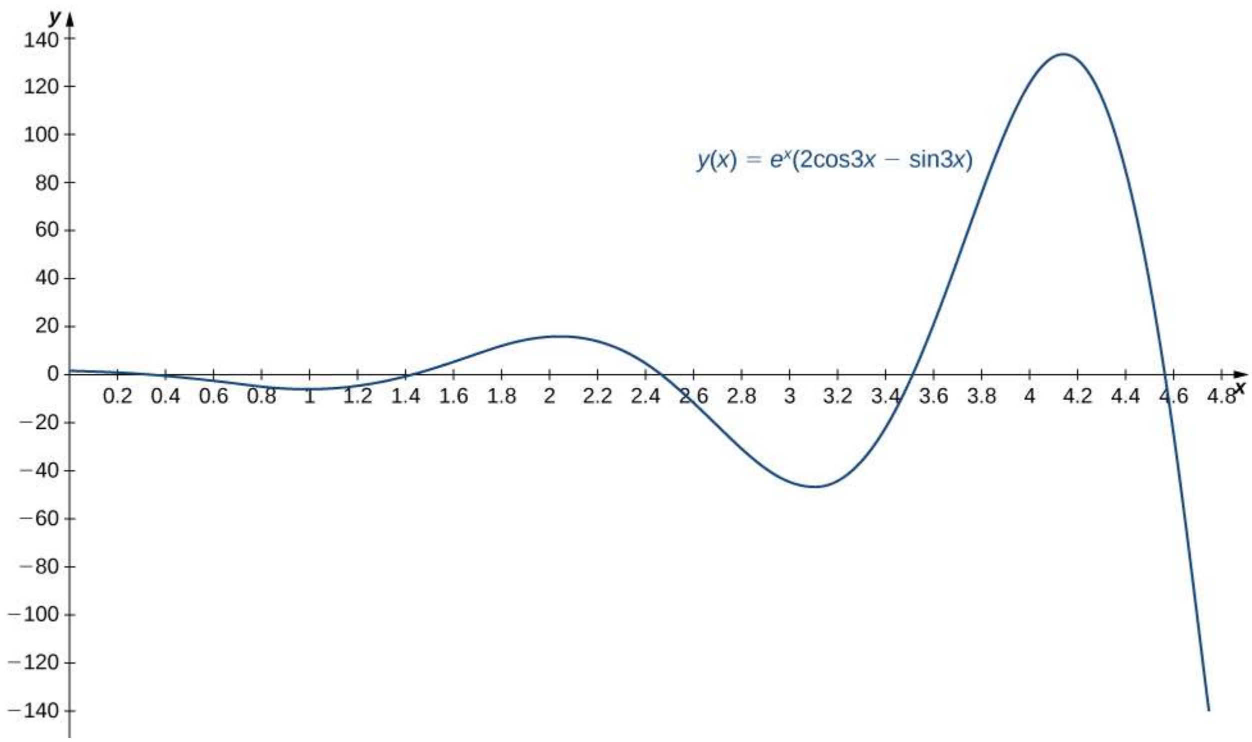
7.5. $y(x) = c_1 e^{3x} + c_2 xe^{3x}$

7.6.

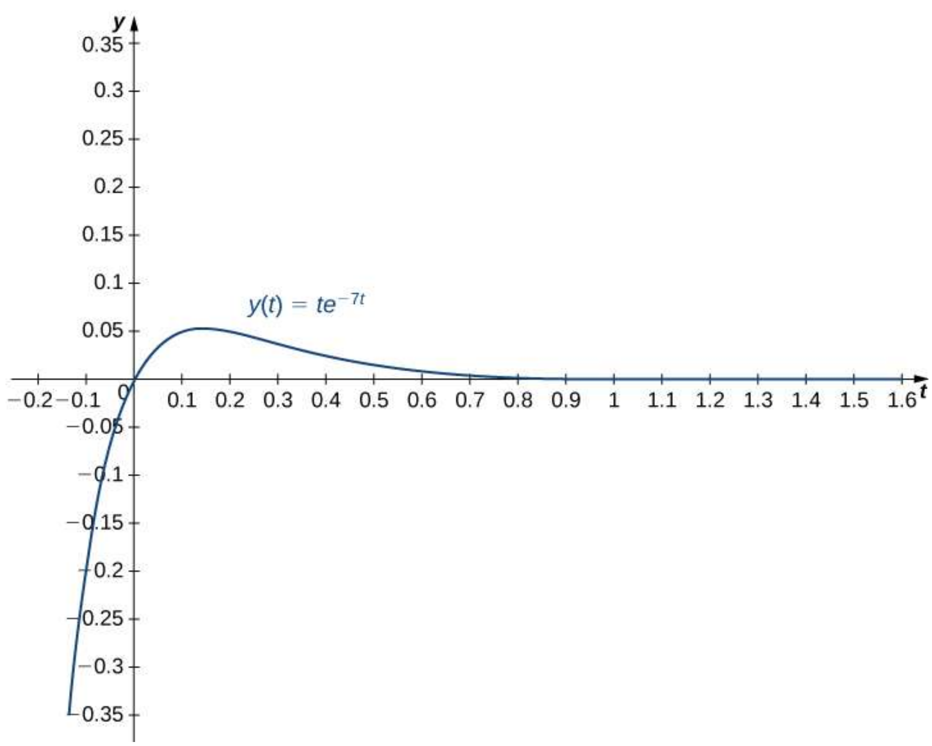
a. $y(x) = e^x\left(c_1 \cos 3x + c_2 \sin 3x\right)$
b. $y(x) = c_1 e^{-7x} + c_2 xe^{-7x}$

7.7. $y(x) = -e^{-2x} + e^{5x}$

7.8. $y(x) = e^x(2\cos 3x - \sin 3x)$



7.9. $y(t) = te^{-7t}$



At time $t = 0.3,$ $y(0.3) = 0.3e^{(-7*0.3)} = 0.3e^{-2.1} \approx 0.0367.$ The mass is 0.0367 ft below equilibrium. At time $t = 0.1,$ $y'(0.1) = 0.3e^{-0.7} \approx 0.1490.$ The mass is moving downward at a speed of 0.1490 ft/sec.

7.10. $y(x) = c_1 e^{-x} + c_2 e^{4x} - 2$

7.11. $y(t) = c_1 e^{2t} + c_2 te^{2t} + \sin t + \cos t$

7.12.

a. $y(x) = c_1 e^{4x} + c_2 e^x - xe^x$

b. $y(t) = c_1 e^{-3t} + c_2 e^{2t} - 5\cos 2t + \sin 2t$

7.13. $z_1 = \dfrac{3x+3}{11x^2}, \quad z_2 = \dfrac{2x+2}{11x}$

7.14.

a. $y(x) = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$

b. $x(t) = c_1 e^t + c_2 te^t + te^t \ln|t|$

7.15. $x(t) = 0.1\cos(14t)$ (in meters); frequency is $\dfrac{14}{2\pi}$ Hz.

7.16. $x(t) = \sqrt{17}\sin(4t + 0.245), \quad \text{frequency} = \dfrac{4}{2\pi} \approx 0.637, \quad A = \sqrt{17}$

7.17. $x(t) = 0.6e^{-2t} - 0.2e^{-6t}$

7.18. $x(t) = \dfrac{1}{2}e^{-8t} + 4te^{-8t}$

7.19. $x(t) = -0.24e^{-2t}\cos(4t) - 0.12e^{-2t}\sin(4t)$

7.20. $x(t) = -\dfrac{1}{2}\cos(4t) + \dfrac{9}{4}\sin(4t) + \dfrac{1}{2}e^{-2t}\cos(4t) - 2e^{-2t}\sin(4t)$

Transient solution: $\dfrac{1}{2}e^{-2t}\cos(4t) - 2e^{-2t}\sin(4t)$

Steady-state solution: $-\dfrac{1}{2}\cos(4t) + \dfrac{9}{4}\sin(4t)$

7.21. $q(t) = -25e^{-t}\cos(3t) - 7e^{-t}\sin(3t) + 25$

7.22.

a. $y(x) = a_0 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} = a_0 e^{-x^2}$

b. $y(x) = a_0(x+1)^3$

Section Exercises

1. linear, homogenous

3. nonlinear

5. linear, homogeneous

11. $y = c_1 e^{5x} + c_2 e^{-2x}$

13. $y = c_1 e^{-2x} + c_2 xe^{-2x}$

15. $y = c_1 e^{5x/2} + c_2 e^{-x}$

17. $y = e^{-x/2}\left(c_1 \cos \dfrac{\sqrt{3}x}{2} + c_2 \sin \dfrac{\sqrt{3}x}{2}\right)$

19. $y = c_1 e^{-11x} + c_2 e^{11x}$

21. $y = c_1 \cos 9x + c_2 \sin 9x$

23. $y = c_1 + c_2 x$

25. $y = c_1 e^{\left((1+\sqrt{22})/3\right)x} + c_2 e^{\left((1-\sqrt{22})/3\right)x}$

27. $y = c_1 e^{-x/6} + c_2 xe^{-x/6}$

29. $y = c_1 + c_2 e^{9x}$

31. $y = -2e^{-2x} + 2e^{-3x}$

33. $y = 3\cos(2x) + 5\sin(2x)$

35. $y = -e^{6x} + 2e^{-5x}$

37. $y = 2e^{-x/5} + \frac{7}{5}xe^{-x/5}$

39. $y = \left(\frac{2}{e^6 - e^{-7}}\right)e^{6x} - \left(\frac{2}{e^6 - e^{-7}}\right)e^{-7x}$

41. No solutions exist.

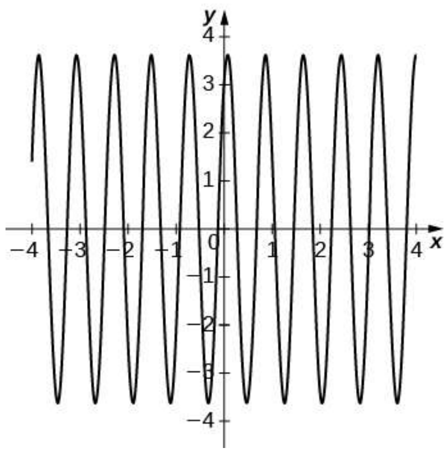
43. $y = 2e^{2x} - \frac{2e^2 + 1}{e^2}xe^{2x}$

45. $y = 4\cos 3x + c_2 \sin 3x,$ infinite y many solutions

47. $5y'' + 19y' - 4y = 0$

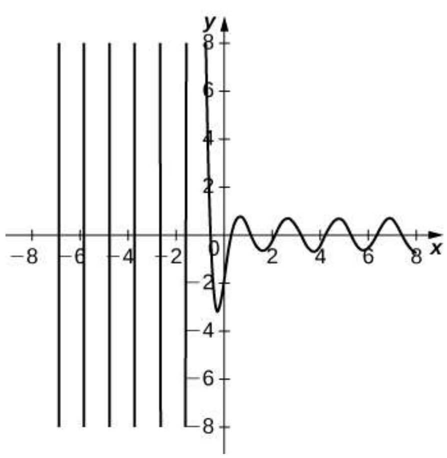
49. a. $y = 3\cos(8x) + 2\sin(8x)$

b.



51. a. $y = e^{(-5/2)x}\left[-2\cos\left(\frac{\sqrt{35}}{2}x\right) + \frac{4\sqrt{35}}{35}\sin\left(\frac{\sqrt{35}}{2}x\right)\right]$

b.



55. $y = c_1 e^{-4x/3} + c_2 e^x - 2$

57. $y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{20}e^{-2x}$

59. $y = c_1 e^{2x} + c_2 xe^{2x} + 2x^2 + 5x$

61. $y = c_1 e^{-x} + c_2 xe^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$

63. $y = c_1 \cos x + c_2 \sin x - \frac{1}{3}x\cos 2x - \frac{5}{9}\sin 2x$

65. $y = c_1 e^{-5x} + c_2 xe^{-5x} + \frac{1}{6}x^3 e^{-5x} + \frac{4}{25}$

67. a. $y_p(x) = Ax^2 + Bx + C$

b. $y_p(x) = -\frac{1}{3}x^2 + \frac{4}{3}x - \frac{35}{9}$

69. a. $y_p(x) = \left(Ax^2 + Bx + C\right)e^{-x}$

b. $y_p(x) = \left(\frac{1}{4}x^2 - \frac{5}{8}x - \frac{33}{32}\right)e^{-x}$

71. a. $y_p(x) = \left(Ax^2 + Bx + C\right)e^x \cos x \ + \left(Dx^2 + Ex + F\right)e^x \sin x$

b. $y_p(x) = \left(-\frac{1}{10}x^2 - \frac{11}{25}x - \frac{27}{250}\right)e^x \cos x \ + \left(-\frac{3}{10}x^2 + \frac{2}{25}x + \frac{39}{250}\right)e^x \sin x$

73. $y = c_1 + c_2 e^{-2x} + \frac{1}{15}e^{3x}$

75. $y = c_1 e^{2x} + c_2 e^{-4x} + xe^{2x}$

77. $y = c_1 e^{3x} + c_2 e^{-3x} - \frac{8x}{9}$

79. $y = c_1 \cos 2x + c_2 \sin 2x - \frac{3}{2}x\cos 2x + \frac{3}{4}\sin 2x \ln(\sin 2x)$

81. $y = -\frac{347}{343} + \frac{4}{343}e^{7x} + \frac{2}{7}x^2 e^{7x} - \frac{4}{49}xe^{7x}$

83. $y = -\frac{57}{25} + \frac{3}{25}e^{5x} + \frac{1}{5}xe^{5x} + \frac{4}{25}e^{-5x}$

85. $y_p = \frac{1}{2} + \frac{10}{3}x^2 \ln x$

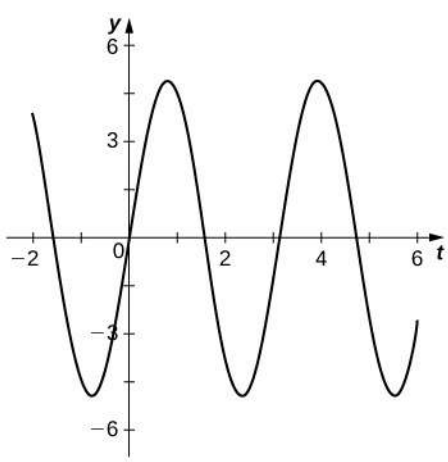
87. $x'' + 16x = 0, \quad x(t) = \frac{1}{6}\cos(4t) - 2\sin(4t),$ period $= \frac{\pi}{2}$ sec, frequency $= \frac{2}{\pi}$ Hz

89. $x'' + 196x = 0, \quad x(t) = 0.15\cos(14t),$ period $= \frac{\pi}{7}$ sec, frequency $= \frac{7}{\pi}$ Hz

91. a. $x(t) = 5\sin(2t)$

b. period $= \pi$ sec, frequency $= \frac{1}{\pi}$ Hz

c.



d. $t = \frac{\pi}{2}$ sec

93. a. $x(t) = e^{-t/5}(20\cos(3t) + 15\sin(3t))$

b. underdamped

95. a. $x(t) = 5e^{-4t} + 10te^{-4t}$

b. critically damped

97. $x(\pi) = \frac{7e^{-\pi/4}}{6}$ ft below

99. $x(t) = \frac{32}{9}\sin(4t) + \cos\left(\sqrt{128}t\right) - \frac{16}{9\sqrt{2}}\sin\left(\sqrt{128}t\right)$

101. $q(t) = e^{-6t}(0.051\cos(8t) + 0.03825\sin(8t)) - \frac{1}{20}\cos(10t)$

103. $q(t) = e^{-10t}(-32t - 5) + 5, \quad I(t) = 2e^{-10t}(160t + 9)$

105. $y = c_0 + 5c_1 \sum_{n=1}^{\infty} \frac{(-x/5)^n}{n!} = c_0 + 5c_1 e^{-x/5}$

107. $y = c_0 \sum_{n=0}^{\infty} \frac{(x)^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{(x)^{2n+1}}{(2n+1)!}$

109. $y = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = c_0 e^{x^2}$

111. $y = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}$

113. $y = c_1 x^3 + \frac{c_2}{x}$

115. $y = 1 - 3x + \frac{2x^3}{3!} - \frac{12x^4}{4!} + \frac{16x^6}{6!} - \frac{120x^7}{7!} + \cdots$

Review Exercises

117. True
119. False
121. second order, linear, homogeneous, $\lambda^2 - 2 = 0$
123. first order, nonlinear, nonhomogeneous
125. $y = c_1 \sin(3x) + c_2 \cos(3x)$

127. $y = c_1 e^x \sin(3x) + c_2 e^x \cos(3x) + \frac{2}{5}x + \frac{2}{25}$

129. $y = c_1 e^{-x} + c_2 e^{-4x} + \frac{x}{4} + \frac{e^{2x}}{18} - \frac{5}{16}$

131. $y = c_1 e^{(-3/2)x} + c_2 xe^{(-3/2)x} + \frac{4}{9}x^2 + \frac{4}{27}x - \frac{16}{27}$

133. $y = e^{-2x}\sin\left(\sqrt{2}x\right)$

135. $y = \frac{e^{1-x}}{e^4 - 1}\left(e^{4x} - 1\right)$

137. $\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right)$

141. $b = \sqrt{a}$

INDEX

Symbols