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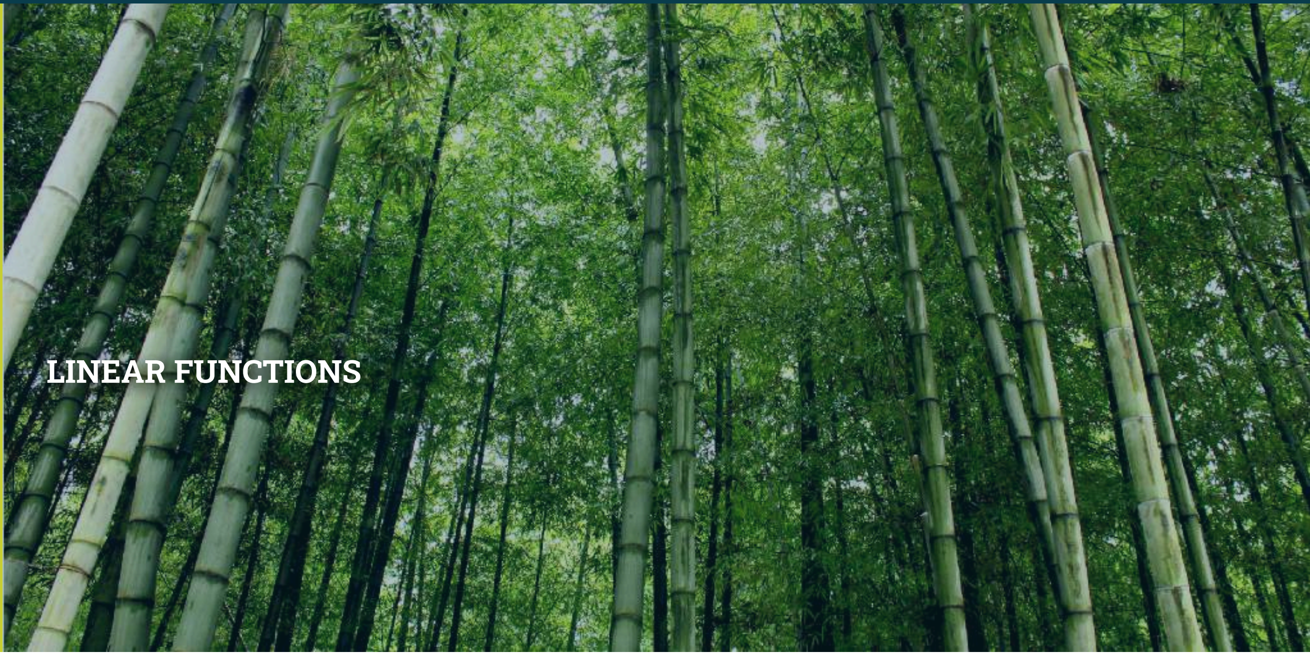

4 LINEAR FUNCTIONS

A bamboo forest in China (credit: "JFXie"/Flickr)

Chapter Outline

4.1 Linear Functions
4.2 Modeling with Linear Functions
4.3 Fitting Linear Models to Data

Introduction to Linear Functions

Imagine placing a plant in the ground one day and finding that it has doubled its height just a few days later. Although it may seem incredible, this can happen with certain types of bamboo species. These members of the grass family are the fastest-growing plants in the world. One species of bamboo has been observed to grow nearly 1.5 inches every hour. [1] In a twenty-four hour period, this bamboo plant grows about 36 inches, or an incredible 3 feet! A constant rate of change, such as the growth cycle of this bamboo plant, is a linear function.

Recall from Functions and Function Notation that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

4.1 Linear Functions

Learning Objectives

In this section, you will:

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Interpret slope as a rate of change.
- Write and interpret an equation for a linear function.
- Graph linear functions.
- Determine whether lines are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

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[1] http://www.guinnessworldrecords.com/records-3000/fastest-growing-plant/

Figure 1 Shanghai MagLev Train (credit: "kanegen"/Flickr)

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train (Figure 1). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes[2].

Suppose a maglev train travels a long distance, and maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train's distance from the station at a given point in time.

Representing Linear Functions

The function describing the train's motion is a linear function, which is defined as a function with a constant rate of change. This is a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train's motion as a function using each method.

Representing a Linear Function in Word Form

Let's begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train's distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the slope-intercept form of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

$$\begin{array}{ll} \text{Equation form} & y = mx + b \\ \text{Function notation} & f(x) = mx + b \end{array}$$

In the example of the train, we might use the notation $D(t)$ where the total distance D is a function of the time t. The rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$$D(t) = 83t + 250$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

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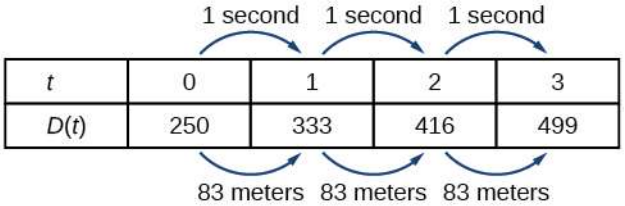
[2] http://www.chinahighlights.com/shanghai/transportation/maglev-train.htm



t	0	1	2	3
$D(t)$	250	333	416	499

Figure 2 Tabular representation of the function D showing selected input and output values

Q&A **Can the input in the previous example be any real number?**

No. The input represents time so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph as represented in Figure 3. Notice the graph is a line**.** When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or slope of the line. The point at which the input value is zero is the vertical intercept, or y-intercept, of the line. We can see from the graph that the y-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

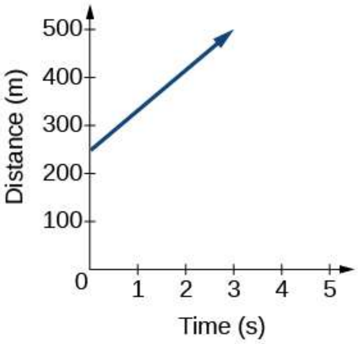


Figure 3 The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1$. Ask yourself what numbers can be input to the function. In other words, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Linear Function

A **linear function** is a function whose graph is a line. Linear functions can be written in the **slope-intercept form** of a line

$$f(x) = mx + b$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or slope of the function. The y-intercept is at $(0, b)$.

EXAMPLE 1

Using a Linear Function to Find the Pressure on a Diver

The pressure, P, in pounds per square inch (PSI) on the diver in Figure 4 depends upon her depth below the water surface, d, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.

Figure 4 (credit: Ilse Reijs and Jan-Noud Hutten)

Solution

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining Whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an increasing function has a positive slope. A line with a positive slope slants upward from left to right as in Figure 5**(a)**. For a decreasing function, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 5**(b)**. If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 5**(c)**.

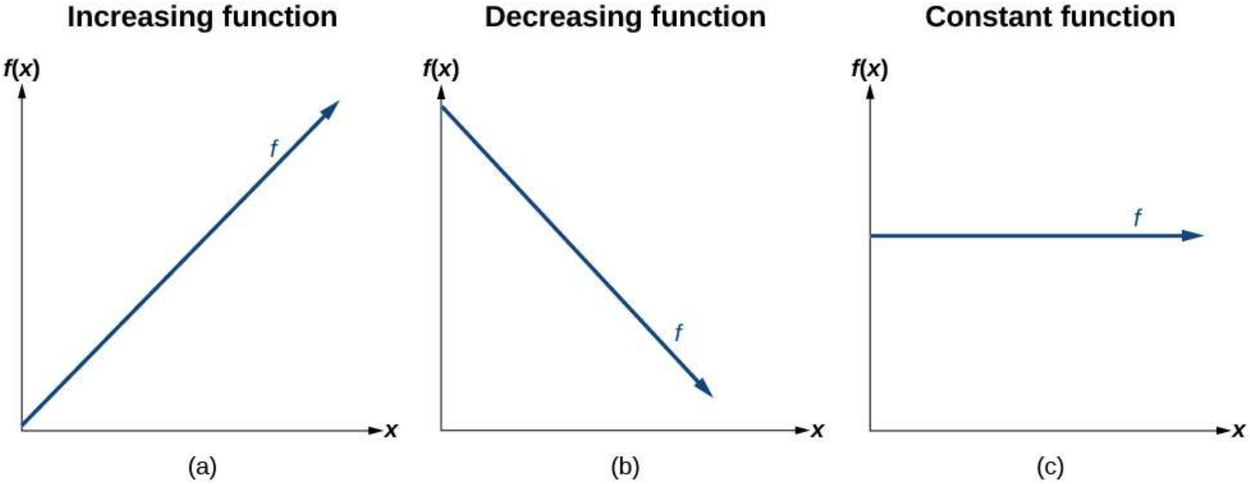


Figure 5

Increasing and Decreasing Functions

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

$f(x) = mx + b$ is an increasing function if $m > 0$.
$f(x) = mx + b$ is a decreasing function if $m < 0$.
$f(x) = mx + b$ is a constant function if $m = 0$.

EXAMPLE 2

Deciding Whether a Function Is Increasing, Decreasing, or Constant

Studies from the early 2010s indicated that teens sent about 60 texts a day, while more recent data indicates much higher messaging rates among all users, particularly considering the various apps with which people can communicate.[3]. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

ⓐ The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
ⓑ A person has a limit of 500 texts per month in their data plan. The input is the number of days, and output is the total number of texts remaining for the month.
ⓒ A person has an unlimited number of texts in their data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

✓ **Solution**

Analyze each function.

ⓐ The function can be represented as $f(x) = 60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
ⓑ The function can be represented as $f(x) = 500 - 60x$ where x is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
ⓒ The cost function can be represented as $f(x) = 50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

Interpreting Slope as a Rate of Change

In the examples we have seen so far, the slope was provided to us. However, we often need to calculate the slope given input and output values. Recall that given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2 —which can be represented by a set of points, (x_1, y_1) and (x_2, y_2) —we can calculate the slope m.

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that in function notation we can obtain two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Figure 6 indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness, or slant. The greater the absolute value of the slope, the steeper the slant is.

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[3] http://www.cbsnews.com/8301-501465_162-57400228-501465/teens-are-sending-60-texts-a-day-study-says/

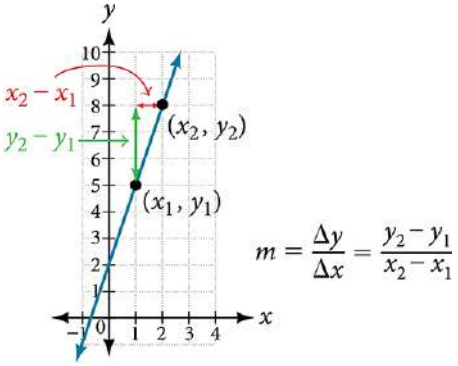


Figure 6 The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the (x_2, y_2) and which is the (x_1, y_1), as long as each calculation is started with the elements from the same coordinate pair.

Q&A **Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?**

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Calculate Slope

The slope, or rate of change, of a function m can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

HOW TO

Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

EXAMPLE 3

Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3, -2)$ and $(8, 1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution

The coordinate pairs are $(3, -2)$ and $(8, 1)$. To find the rate of change, we divide the change in output by the change in input.

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1 - (-2)}{8 - 3} = \frac{3}{5}$$

We could also write the slope as $m = 0.6$. The function is increasing because $m > 0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used. Note that if we had reversed them, we would have obtained the same slope.

$$m = \frac{(-2) - (1)}{3 - 8} = \frac{-3}{-5} = \frac{3}{5}$$

TRY IT #1 If $f(x)$ is a linear function, and $(2, 3)$ and $(0, 4)$ are points on the line, find the slope. Is this function increasing or decreasing?

EXAMPLE 4

Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

Solution

The rate of change relates the change in population to the change in time. The population increased by $27,800 - 23,400 = 4400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4{,}400 \text{ people}}{4 \text{ years}} = 1{,}100 \frac{\text{people}}{\text{year}}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

TRY IT #2 The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

Writing and Interpreting an Equation for a Linear Function

Recall from Equations and Inequalities that we wrote equations in both the slope-intercept form and the point-slope form. Now we can choose which method to use to write equations for linear functions based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in Figure 7.

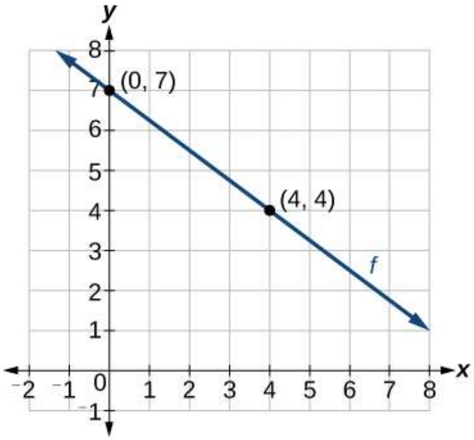


Figure 7

We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let’s choose $(0, 7)$ and $(4, 4)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-7}{4-0} \\ &= -\frac{3}{4} \end{aligned}$$

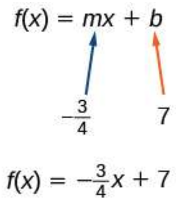
Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -\frac{3}{4}(x - 4) \end{aligned}$$

If we want to rewrite the equation in the slope-intercept form, we would find

$$\begin{aligned} y - 4 &= -\frac{3}{4}(x - 4) \\ y - 4 &= -\frac{3}{4}x + 3 \\ y &= -\frac{3}{4}x + 7 \end{aligned}$$

If we want to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the y-axis when the output value is 7. Therefore, $b = 7$. We now have the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.



So the function is $f(x) = -\frac{3}{4}x + 7$, and the linear equation would be $y = -\frac{3}{4}x + 7$.

HOW TO

Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the y-axis to identify the y-intercept by visual inspection.
4. Substitute the slope and y-intercept into the slope-intercept form of a line equation.

EXAMPLE 5

Writing an Equation for a Linear Function

Write an equation for a linear function given a graph of f shown in Figure 8.

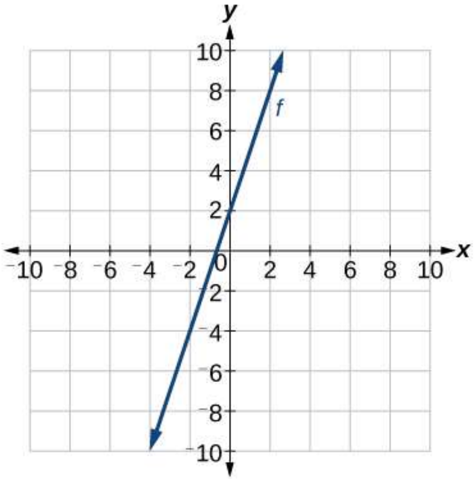


Figure 8

Solution

Identify two points on the line, such as $(0, 2)$ and $(-2, -4)$. Use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4-2}{-2-0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= 3(x - (-2)) \\ y + 4 &= 3(x + 2) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y + 4 &= 3(x + 2) \\ y + 4 &= 3x + 6 \\ y &= 3x + 2 \end{aligned}$$

Analysis

This makes sense because we can see from Figure 9 that the line crosses the *y*-axis at the point $(0,\ \ 2)$, which is the *y*-intercept, so $b = 2$.

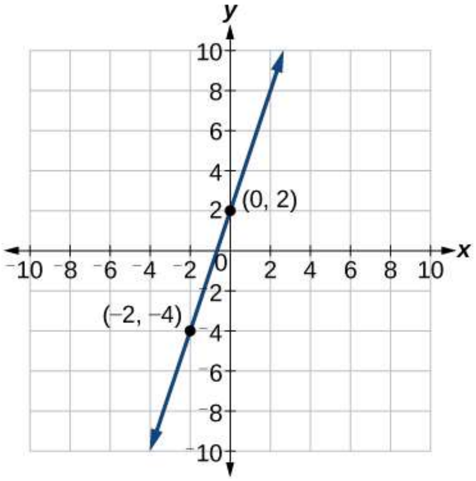


Figure 9

EXAMPLE 6

Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

Solution

The fixed cost is present every month, $1,250. The costs that can vary include the cost to produce each item, which is $37.50. The variable cost, called the marginal cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs, represented by $C(x) = 1250 + 37.5x$.

Analysis

If Ben produces 100 items in a month, his monthly cost is found by substituting 100 for x.

$$\begin{aligned} C(100) &= 1250 + 37.5(100) \\ &= 5000 \end{aligned}$$

So his monthly cost would be $5,000.

EXAMPLE 7

Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3) = -2$, and $f(8) = 1$, find an equation for the function in slope-intercept form.

Solution

We can write the given points using coordinates.

$$\begin{aligned} f(3) &= -2 \rightarrow (3, -2) \\ f(8) &= 1 \rightarrow (8, 1) \end{aligned}$$

We can then use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-(-2)}{8-3} \\ &= \frac{3}{5} \end{aligned}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= \tfrac{3}{5}(x - 3) \end{aligned}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y + 2 &= \tfrac{3}{5}(x - 3) \\ y + 2 &= \tfrac{3}{5}x - \tfrac{9}{5} \\ y &= \tfrac{3}{5}x - \tfrac{19}{5} \end{aligned}$$

TRY IT #3 If $f(x)$ is a linear function, with $f(2) = -11$, and $f(4) = -25$, write an equation for the function in slope-intercept form.

Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

HOW TO

Given a linear function f and the initial value and rate of change, evaluate $f(c)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

EXAMPLE 8

Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his collection as a function of time, t, the number of months. How many songs will he own at the end of one year?

Solution

The initial value for this function is 200 because he currently owns 200 songs, so $N(0) = 200$, which means that $b = 200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m = 15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.

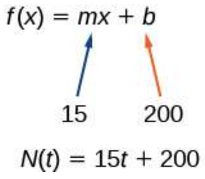


Figure 10

We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have at the end of one year (12 months). In other words, we can evaluate the function at $t = 12$.

$$\begin{aligned} N(12) &= 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

EXAMPLE 9

Using a Linear Function to Calculate Salary Based on Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya’s weekly income I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned $760 for the week. The week before, he sold 5 new policies and earned $920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$\begin{aligned} m &= \frac{920-760}{5-3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by $160 when the number of policies increased by 2, so the rate of change is $80 per policy. Therefore, Ilya earns a commission of $80 for each policy sold during the week.

We can then solve for the initial value.

$$\begin{aligned} I(n) &= 80n + b \\ 760 &= 80(3) + b \qquad \text{when } n = 3, I(3) = 760 \\ 760 &- 80(3) = b \\ 520 &= b \end{aligned}$$

The value of b is the starting value for the function and represents Ilya’s income when $n = 0$, or when no new policies are sold. We can interpret this as Ilya’s base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya’s base salary is $520 per week and he earns an additional $80 commission for each policy sold.

EXAMPLE 10

Using Tabular Form to Write an Equation for a Linear Function

Table 1 relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

number of weeks, *w*	0	2	4	6
number of rats, *P(w)*	1000	1080	1160	1240

Table 1

Solution

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for m, we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w) = 40w + 1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

$$\begin{aligned} m &= \frac{1240-1080}{6-2} \\ &= \frac{160}{4} \\ &= 40 \end{aligned}$$

Q&A **Is the initial value always provided in a table of values like Table 1?**

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for b.

TRY IT #4 A new plant food was introduced to a young tree to test its effect on the height of the tree. Table 2 shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

x	0	2	4	8	12
***H*(*x*)**	12.5	13.5	14.5	16.5	18.5

Table 2

Graphing Linear Functions

Now that we've seen and interpreted graphs of linear functions, let's take a look at how to create the graphs. There are three basic methods of graphing linear functions. The first is by plotting points and then drawing a line through the points. The second is by using the *y*-intercept and slope. And the third method is by using transformations of the identity function $f(x) = x$.

Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x) = 2x$, we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point $(1, 2)$. Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point $(2, 4)$. Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

HOW TO

Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

EXAMPLE 11

Graphing by Plotting Points

Graph $f(x) = -\frac{2}{3}x + 5$ by plotting points.

Solution

Begin by choosing input values. This function includes a fraction with a denominator of 3, so let's choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value, and use the output value to identify coordinate pairs.

$$\begin{array}{ll} x = 0 & f(0) = -\frac{2}{3}(0) + 5 = 5 \Rightarrow (0, 5) \\ x = 3 & f(3) = -\frac{2}{3}(3) + 5 = 3 \Rightarrow (3, 3) \\ x = 6 & f(6) = -\frac{2}{3}(6) + 5 = 1 \Rightarrow (6, 1) \end{array}$$

Plot the coordinate pairs and draw a line through the points. Figure 11 represents the graph of the function $f(x) = -\frac{2}{3}x + 5$.

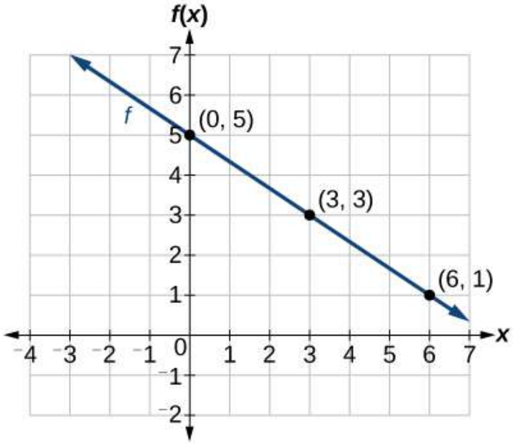


Figure 11 The graph of the linear function $f(x) = -\frac{2}{3}x + 5$.

Analysis

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative, constant rate of change in the equation for the function.

TRY IT #5 Graph $f(x) = -\frac{3}{4}x + 6$ by plotting points.

Graphing a Function Using *y*-intercept and Slope

Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its *y*-intercept, which is the point at which the input value is zero. To find the *y*-intercept, we can set $x = 0$ in the equation.

The other characteristic of the linear function is its slope**.**

Let's consider the following function.

$$f(x) = \frac{1}{2}x + 1$$

The slope is $\frac{1}{2}$. Because the slope is positive, we know the graph will slant upward from left to right. The *y*-intercept is the point on the graph when $x = 0$. The graph crosses the *y*-axis at $(0, 1)$. Now we know the slope and the *y*-intercept. We can begin graphing by plotting the point $(0, 1)$. We know that the slope is the change in the *y*-coordinate over the change in the *x*-coordinate. This is commonly referred to as rise over run, $m = \frac{\text{rise}}{\text{run}}$. From our example, we have $m = \frac{1}{2}$, which means that the rise is 1 and the run is 2. So starting from our *y*-intercept $(0, 1)$, we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in Figure 12.

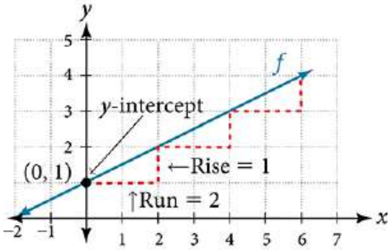


Figure 12

Graphical Interpretation of a Linear Function

In the equation $f(x) = mx + b$

- b is the *y*-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the *y*-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Q&A **Do all linear functions have *y*-intercepts?**

Yes. All linear functions cross the y-axis and therefore have y-intercepts. (Note: *A vertical line is parallel to the y-axis does not have a y-intercept, but it is not a function.*)

HOW TO

Given the equation for a linear function, graph the function using the *y*-intercept and slope.

1. Evaluate the function at an input value of zero to find the *y*-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the *y*-intercept.
4. Use $\frac{\text{rise}}{\text{run}}$ to determine at least two more points on the line.
5. Sketch the line that passes through the points.

EXAMPLE 12

Graphing by Using the *y*-intercept and Slope

Graph $f(x) = -\frac{2}{3}x + 5$ using the *y*-intercept and slope.

Solution

Evaluate the function at $x = 0$ to find the *y*-intercept. The output value when $x = 0$ is 5, so the graph will cross the *y*-axis at $(0, 5)$.

According to the equation for the function, the slope of the line is $-\frac{2}{3}$. This tells us that for each vertical decrease in the "rise" of -2 units, the "run" increases by 3 units in the horizontal direction. We can now graph the function by first plotting the *y*-intercept on the graph in Figure 13. From the initial value $(0, 5)$ we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then drawing a line through the points.

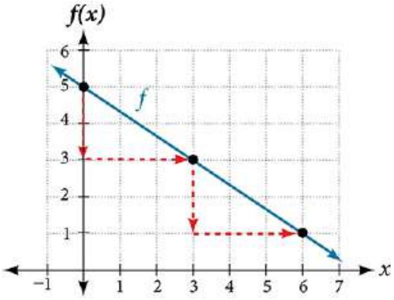


Figure 13 Graph of $f(x) = -2/3x + 5$ and shows how to calculate the rise over run for the slope.

Analysis

The graph slants downward from left to right, which means it has a negative slope as expected.

TRY IT #6 Find a point on the graph we drew in Example 12 that has a negative x-value.

Graphing a Function Using Transformations

Another option for graphing is to use a transformation of the identity function $f(x) = x$. A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

Vertical Stretch or Compression

In the equation $f(x) = mx$, the m is acting as the vertical stretch or compression of the identity function. When m is negative, there is also a vertical reflection of the graph. Notice in Figure 14 that multiplying the equation of $f(x) = x$ by m stretches the graph of f by a factor of m units if $m > 1$ and compresses the graph of f by a factor of m units if $0 < m < 1$. This means the larger the absolute value of m, the steeper the slope.

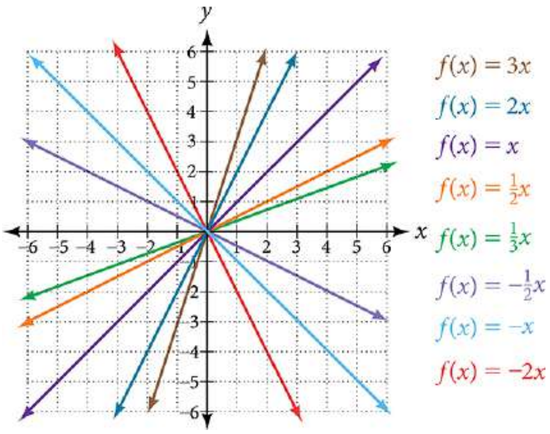


Figure 14 Vertical stretches and compressions and reflections on the function $f(x) = x$

Vertical Shift

In $f(x) = mx + b$, the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Notice in Figure 15 that adding a value of b to the equation of $f(x) = x$ shifts the graph of f a total of b units up if b is positive and $|b|$ units down if b is negative.

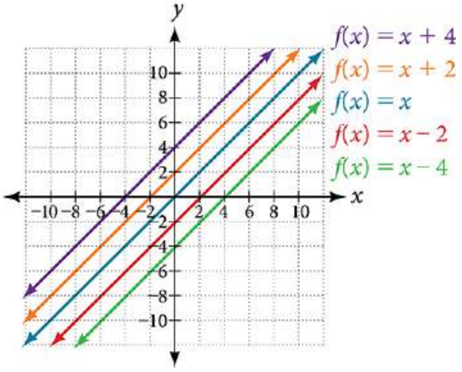


Figure 15 This graph illustrates vertical shifts of the function $f(x) = x$.

Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

HOW TO

Given the equation of a linear function, use transformations to graph the linear function in the form $f(x) = mx + b$.

1. Graph $f(x) = x$.
2. Vertically stretch or compress the graph by a factor m.
3. Shift the graph up or down b units.

EXAMPLE 13

Graphing by Using Transformations

Graph $f(x) = \frac{1}{2}x - 3$ using transformations.

Solution

The equation for the function shows that $m = \frac{1}{2}$ so the identity function is vertically compressed by $\frac{1}{2}$. The equation for the function also shows that $b = -3$ so the identity function is vertically shifted down 3 units. First, graph the identity function, and show the vertical compression as in Figure 16.

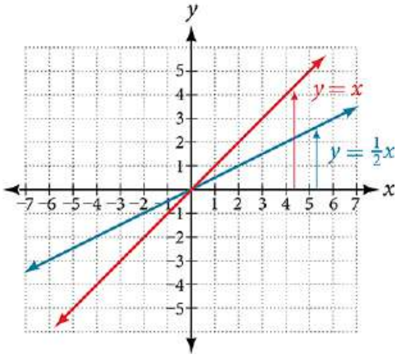


Figure 16 The function, $y = x$, compressed by a factor of $\frac{1}{2}$.

Then show the vertical shift as in Figure 17.

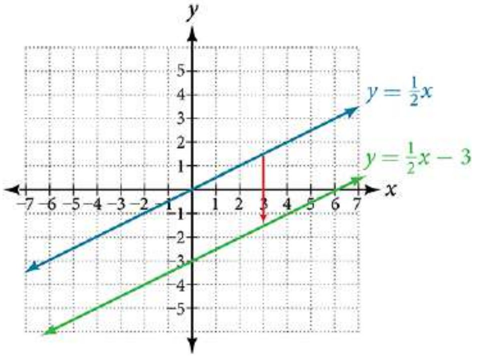


Figure 17 The function $y = \frac{1}{2}x$, shifted down 3 units.

TRY IT #7 Graph $f(x) = 4 + 2x$ using transformations.

Q&A **In Example 15, could we have sketched the graph by reversing the order of the transformations?**

No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following the order: Let the input be 2.

$$\begin{aligned} f(2) &= \tfrac{1}{2}(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Writing the Equation for a Function from the Graph of a Line

Earlier, we wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at Figure 18. We can see right away that the graph crosses the *y*-axis at the point $(0, 4)$ so this is the *y*-intercept.

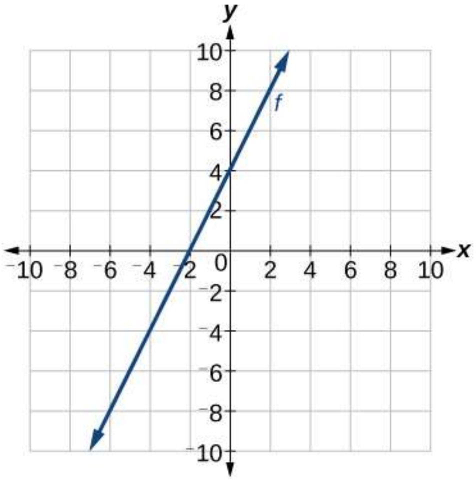


Figure 18

Then we can calculate the slope by finding the rise and run. We can choose any two points, but let’s look at the point $(-2, 0)$. To get from this point to the *y*-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be

$$m = \frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2$$

Substituting the slope and *y*-intercept into the slope-intercept form of a line gives

$$y = 2x + 4$$

HOW TO

Given a graph of linear function, find the equation to describe the function.

1. Identify the *y*-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the *y*-intercept and slope into the slope-intercept form of a line.

EXAMPLE 14

Matching Linear Functions to Their Graphs

Match each equation of the linear functions with one of the lines in Figure 19.

ⓐ $f(x) = 2x + 3$ ⓑ $g(x) = 2x - 3$ ⓒ $h(x) = -2x + 3$ ⓓ $j(x) = \frac{1}{2}x + 3$

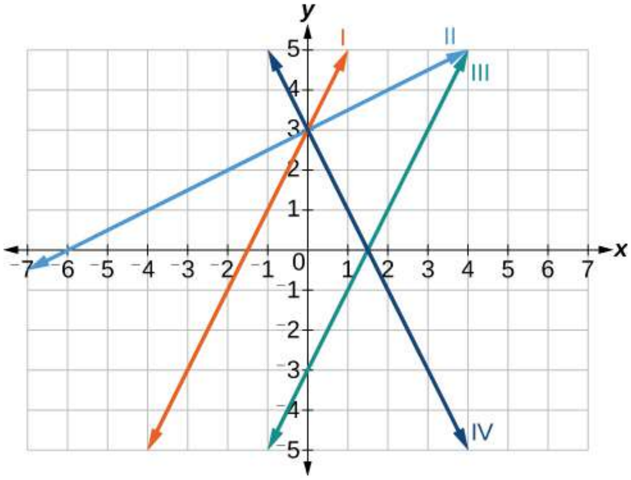


Figure 19

Solution

Analyze the information for each function.

ⓐ This function has a slope of 2 and a *y*-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function g has the same slope, but a different *y*-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through $(0, 3)$ so f must be represented by line I.

ⓑ This function also has a slope of 2, but a *y*-intercept of -3. It must pass through the point $(0, -3)$ and slant upward from left to right. It must be represented by line III.

ⓒ This function has a slope of –2 and a *y*-intercept of 3. This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.

ⓓ This function has a slope of $\frac{1}{2}$ and a *y*-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. Lines I and II pass through $(0, 3)$, but the slope of j is less than the slope of f so the line for j must be flatter. This function is represented by Line II.

Now we can re-label the lines as in Figure 20.

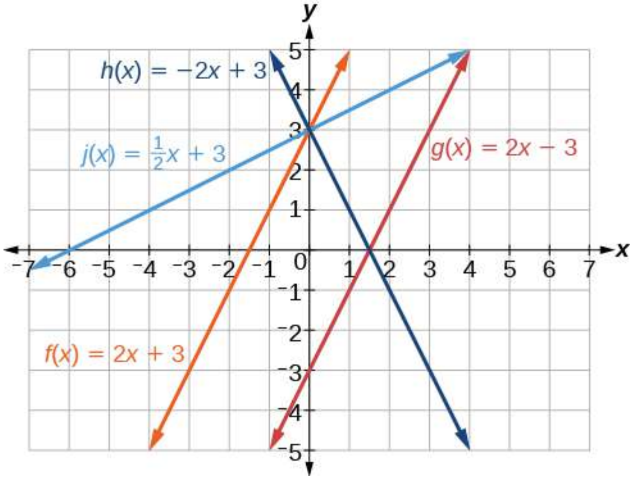


Figure 20

Finding the *x*-intercept of a Line

So far we have been finding the *y*-intercepts of a function: the point at which the graph of the function crosses the *y*-axis. Recall that a function may also have an *x*-intercept, which is the *x*-coordinate of the point where the graph of the function crosses the *x*-axis. In other words, it is the input value when the output value is zero.

To find the *x*-intercept, set a function $f(x)$ equal to zero and solve for the value of x. For example, consider the function shown.

$$f(x) = 3x - 6$$

Set the function equal to 0 and solve for x.

$$\begin{aligned} 0 &= 3x - 6 \\ 6 &= 3x \\ 2 &= x \\ x &= 2 \end{aligned}$$

The graph of the function crosses the *x*-axis at the point $(2, 0)$.

Q&A **Do all linear functions have *x*-intercepts?**

*No. However, linear functions of the form $y = c$, where c is a nonzero real number are the only examples of linear functions with no x-intercept. For example, $y = 5$ is a horizontal line 5 units above the x-axis. This function has no x-intercepts, as shown in **Figure 21**.*

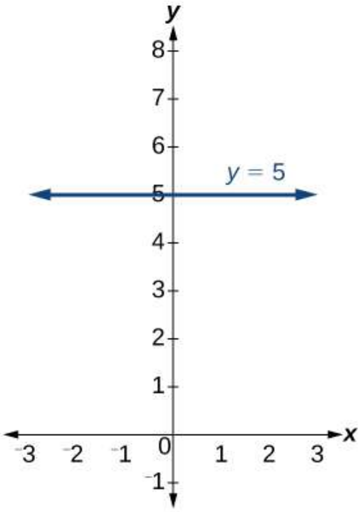


Figure 21

x-intercept

The *x*-intercept of the function is value of x when $f(x) = 0$. It can be solved by the equation $0 = mx + b$.

EXAMPLE 15

Finding an *x*-intercept

Find the *x*-intercept of $f(x) = \frac{1}{2}x - 3$.

Solution

Set the function equal to zero to solve for x.

$$\begin{aligned} 0 &= \tfrac{1}{2}x - 3 \\ 3 &= \tfrac{1}{2}x \\ 6 &= x \\ x &= 6 \end{aligned}$$

The graph crosses the *x*-axis at the point $(6, 0)$.

Analysis

A graph of the function is shown in Figure 22. We can see that the *x*-intercept is $(6, 0)$ as we expected.

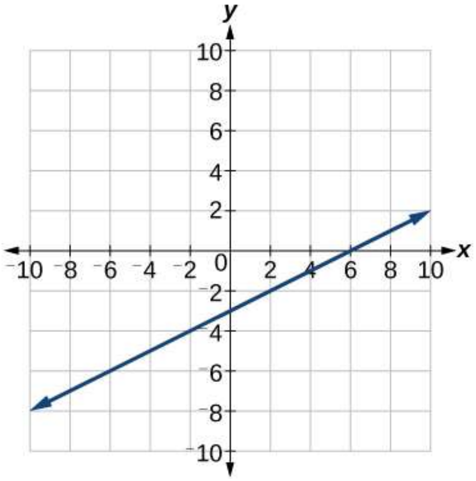
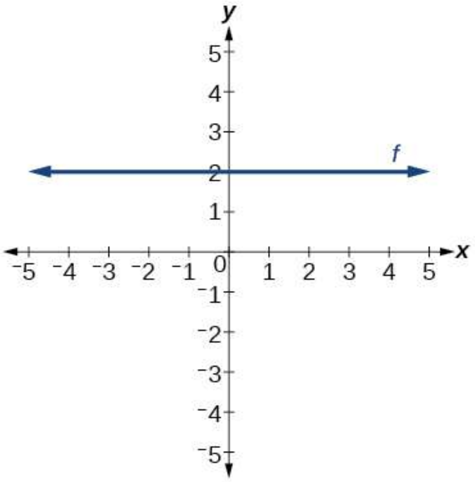


Figure 22

TRY IT #8 Find the *x*-intercept of $f(x) = \frac{1}{4}x - 4$.

Describing Horizontal and Vertical Lines

There are two special cases of lines on a graph—horizontal and vertical lines. A horizontal line indicates a constant output, or *y*-value. In Figure 23, we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m = 0$ in the equation $f(x) = mx + b$, the equation simplifies to $f(x) = b$. In other words, the value of the function is a constant. This graph represents the function $f(x) = 2$.



x	−4	−2	0	2	4
y	2	2	2	2	2

Figure 23 A horizontal line representing the function $f(x) = 2$

A vertical line indicates a constant input, or *x*-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.

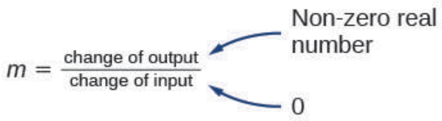
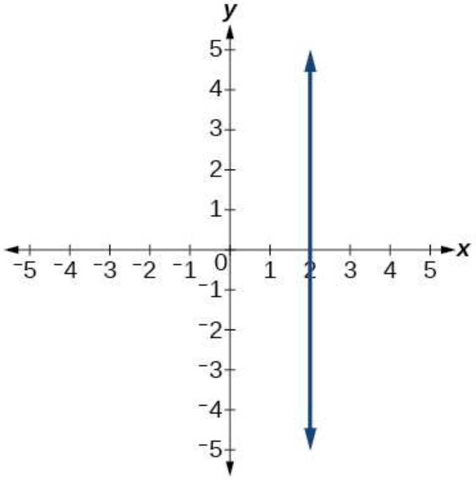


Figure 24 Example of how a line has a vertical slope. 0 in the denominator of the slope.

A vertical line, such as the one in Figure 25, has an *x*-intercept, but no *y*-intercept unless it's the line $x = 0$. This graph represents the line $x = 2$.



x	2	2	2	2	2
y	−4	−2	0	2	4

Figure 25 The vertical line, $x = 2$, which does not represent a function

Horizontal and Vertical Lines

Lines can be horizontal or vertical.

A **horizontal line** is a line defined by an equation in the form $f(x) = b$.

A **vertical line** is a line defined by an equation in the form $x = a$.

EXAMPLE 16

Writing the Equation of a Horizontal Line

Write the equation of the line graphed in Figure 26.

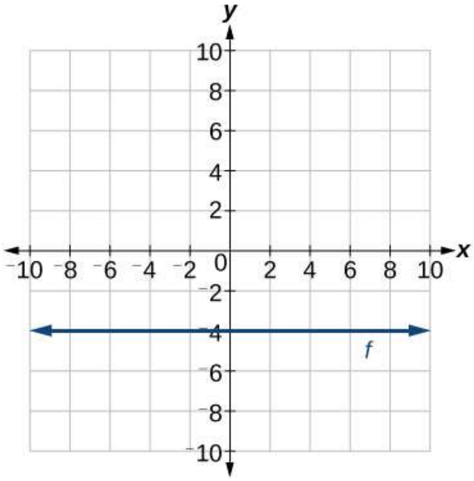


Figure 26

Solution

For any x-value, the y-value is -4, so the equation is $y = -4$.

EXAMPLE 17

Writing the Equation of a Vertical Line

Write the equation of the line graphed in Figure 27.

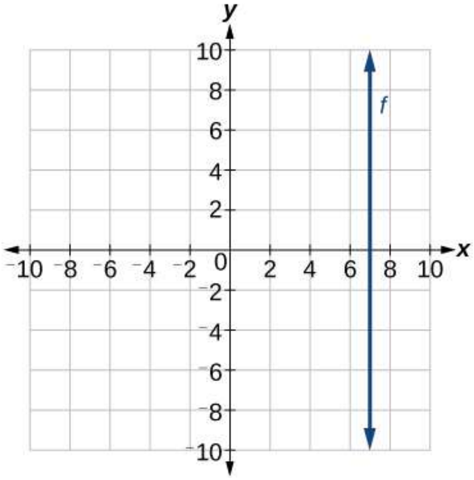


Figure 27

Solution

The constant x-value is 7, so the equation is $x = 7$.

Determining Whether Lines are Parallel or Perpendicular

The two lines in Figure 28 are parallel lines: they will never intersect. They have exactly the same steepness, which means their slopes are identical. The only difference between the two lines is the y-intercept. If we shifted one line vertically toward the other, they would become coincident.

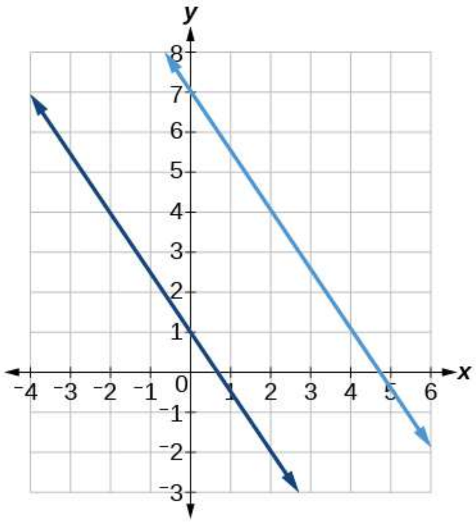


Figure 28 Parallel lines

We can determine from their equations whether two lines are parallel by comparing their slopes. If the slopes are the same and the *y*-intercepts are different, the lines are parallel. If the slopes are different, the lines are not parallel.

$$\left.\begin{array}{l} f(x) = -2x + 6 \\ f(x) = -2x - 4 \end{array}\right\} \text{parallel} \qquad \left.\begin{array}{l} f(x) = 3x + 2 \\ f(x) = 2x + 2 \end{array}\right\} \text{not parallel}$$

Unlike parallel lines, perpendicular lines do intersect. Their intersection forms a right, or 90-degree, angle. The two lines in Figure 29 are perpendicular.

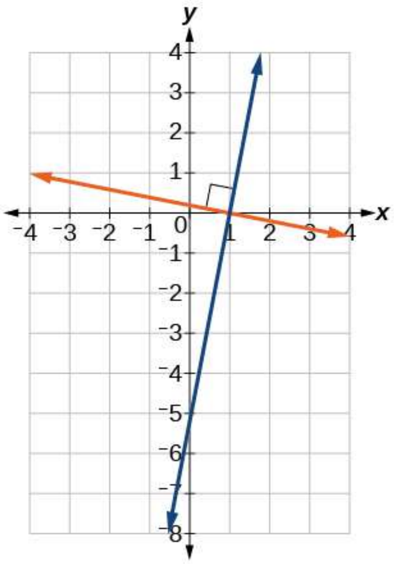


Figure 29 Perpendicular lines

Perpendicular lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1. So, if m_1 and m_2 are negative reciprocals of one another, they can be multiplied together to yield -1.

$$m_1 m_2 = -1$$

To find the reciprocal of a number, divide 1 by the number. So the reciprocal of 8 is $\frac{1}{8}$, and the reciprocal of $\frac{1}{8}$ is 8. To find the negative reciprocal, first find the reciprocal and then change the sign.

As with parallel lines, we can determine whether two lines are perpendicular by comparing their slopes, assuming that the lines are neither horizontal nor vertical. The slope of each line below is the negative reciprocal of the other so the lines are perpendicular.

$$\begin{array}{rcll} f(x) & = & \frac{1}{4}x+2 & \text{negative reciprocal of } \frac{1}{4} \text{ is } -4 \\ f(x) & = & -4x+3 & \text{negative reciprocal of } -4 \text{ is } \frac{1}{4} \end{array}$$

The product of the slopes is –1.

$$-4\left(\frac{1}{4}\right)=-1$$

Parallel and Perpendicular Lines

Two lines are **parallel lines** if they do not intersect. The slopes of the lines are the same.

$$f(x)=m_1x+b_1 \text{ and } g(x)=m_2x+b_2 \text{ are parallel if and only if } m_1=m_2$$

If and only if $b_1=b_2$ and $m_1=m_2$, we say the lines coincide. Coincident lines are the same line.

Two lines are **perpendicular lines** if they intersect to form a right angle.

$$f(x)=m_1x+b_1 \text{ and } g(x)=m_2x+b_2 \text{ are perpendicular if and only if}$$

$$m_1m_2=-1, \text{so } m_2=-\frac{1}{m_1}$$

EXAMPLE 18

Identifying Parallel and Perpendicular Lines

Given the functions below, identify the functions whose graphs are a pair of parallel lines and a pair of perpendicular lines.

$$\begin{array}{rclrcl} f(x) & = & 2x+3 & h(x) & = & -2x+2 \\ g(x) & = & \frac{1}{2}x-4 & j(x) & = & 2x-6 \end{array}$$

Solution

Parallel lines have the same slope. Because the functions $f(x)=2x+3$ and $j(x)=2x-6$ each have a slope of 2, they represent parallel lines. Perpendicular lines have negative reciprocal slopes. Because −2 and $\frac{1}{2}$ are negative reciprocals, the functions $g(x)=\frac{1}{2}x-4$ and $h(x)=-2x+2$ represent perpendicular lines.

Analysis

A graph of the lines is shown in Figure 30.

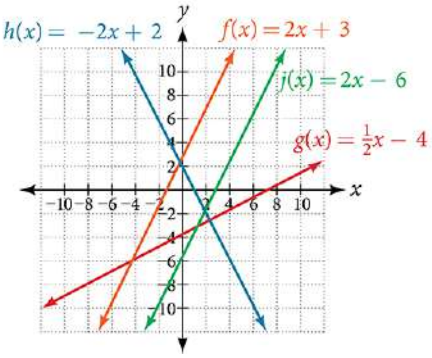


Figure 30

The graph shows that the lines $f(x)=2x+3$ and $j(x)=2x–6$ are parallel, and the lines $g(x)=\frac{1}{2}x–4$ and $h(x)=-2x+2$ are perpendicular.

Writing the Equation of a Line Parallel or Perpendicular to a Given Line

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

Writing Equations of Parallel Lines

Suppose for example, we are given the equation shown.

$$f(x) = 3x + 1$$

We know that the slope of the line formed by the function is 3. We also know that the y-intercept is $(0, 1)$. Any other line with a slope of 3 will be parallel to $f(x)$. So the lines formed by all of the following functions will be parallel to $f(x)$.

$$\begin{array}{rcl} g(x) & = & 3x + 6 \\ h(x) & = & 3x + 1 \\ p(x) & = & 3x + \frac{2}{3} \end{array}$$

Suppose then we want to write the equation of a line that is parallel to f and passes through the point $(1, 7)$. This type of problem is often described as a point-slope problem because we have a point and a slope. In our example, we know that the slope is 3. We need to determine which value of b will give the correct line. We can begin with the point-slope form of an equation for a line, and then rewrite it in the slope-intercept form.

$$\begin{array}{rcl} y - y_1 & = & m(x - x_1) \\ y - 7 & = & 3(x - 1) \\ y - 7 & = & 3x - 3 \\ y & = & 3x + 4 \end{array}$$

So $g(x) = 3x + 4$ is parallel to $f(x) = 3x + 1$ and passes through the point $(1, 7)$.

HOW TO

Given the equation of a function and a point through which its graph passes, write the equation of a line parallel to the given line that passes through the given point.

1. Find the slope of the function.
2. Substitute the given values into either the general point-slope equation or the slope-intercept equation for a line.
3. Simplify.

EXAMPLE 19

Finding a Line Parallel to a Given Line

Find a line parallel to the graph of $f(x) = 3x + 6$ that passes through the point $(3, 0)$.

Solution

The slope of the given line is 3. If we choose the slope-intercept form, we can substitute $m = 3$, $x = 3$, and $f(x) = 0$ into the slope-intercept form to find the y-intercept.

$$\begin{array}{rcl} g(x) & = & 3x + b \\ 0 & = & 3(3) + b \\ b & = & -9 \end{array}$$

The line parallel to $f(x)$ that passes through $(3, 0)$ is $g(x) = 3x - 9$.

Analysis

We can confirm that the two lines are parallel by graphing them. Figure 31 shows that the two lines will never intersect.

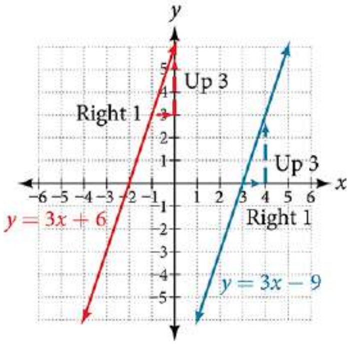


Figure 31

Writing Equations of Perpendicular Lines

We can use a very similar process to write the equation for a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the function shown.

$$f(x) = 2x + 4$$

The slope of the line is 2, and its negative reciprocal is $-\frac{1}{2}$. Any function with a slope of $-\frac{1}{2}$ will be perpendicular to $f(x)$. So the lines formed by all of the following functions will be perpendicular to $f(x)$.

$$\begin{array}{rcl} g(x) & = & -\frac{1}{2}x + 4 \\ h(x) & = & -\frac{1}{2}x + 2 \\ p(x) & = & -\frac{1}{2}x - \frac{1}{2} \end{array}$$

As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose then we want to write the equation of a line that is perpendicular to $f(x)$ and passes through the point $(4, 0)$. We already know that the slope is $-\frac{1}{2}$. Now we can use the point to find the *y*-intercept by substituting the given values into the slope-intercept form of a line and solving for b.

$$\begin{array}{rcl} g(x) & = & mx + b \\ 0 & = & -\frac{1}{2}(4) + b \\ 0 & = & -2 + b \\ 2 & = & b \\ b & = & 2 \end{array}$$

The equation for the function with a slope of $-\frac{1}{2}$ and a *y*-intercept of 2 is

$$g(x) = -\frac{1}{2}x + 2$$

So $g(x) = -\frac{1}{2}x + 2$ is perpendicular to $f(x) = 2x + 4$ and passes through the point $(4, 0)$. Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

Q&A **A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not –1. Doesn't this fact contradict the definition of perpendicular lines?**

No. For two perpendicular linear functions, the product of their slopes is –1. However, a vertical line is not a function so the definition is not contradicted.

HOW TO

Given the equation of a function and a point through which its graph passes, write the equation of a line

perpendicular to the given line.

1. Find the slope of the function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for x and y from the coordinate pair provided into $g(x) = mx + b$.
4. Solve for b.
5. Write the equation of the line.

EXAMPLE 20

Finding the Equation of a Perpendicular Line

Find the equation of a line perpendicular to $f(x) = 3x + 3$ that passes through the point $(3, 0)$.

Solution

The original line has slope $m = 3$, so the slope of the perpendicular line will be its negative reciprocal, or $-\frac{1}{3}$. Using this slope and the given point, we can find the equation of the line.

$$\begin{aligned} g(x) &= -\tfrac{1}{3}x + b \\ 0 &= -\tfrac{1}{3}(3) + b \\ 1 &= b \\ b &= 1 \end{aligned}$$

The line perpendicular to $f(x)$ that passes through $(3, 0)$ is $g(x) = -\frac{1}{3}x + 1$.

Analysis

A graph of the two lines is shown in Figure 32.

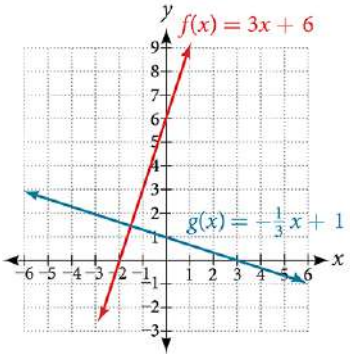


Figure 32

Note that that if we graph perpendicular lines on a graphing calculator using standard zoom, the lines may not appear to be perpendicular. Adjusting the window will make it possible to zoom in further to see the intersection more closely.

TRY IT #9 Given the function $h(x) = 2x - 4$, write an equation for the line passing through $(0, 0)$ that is

ⓐ parallel to $h(x)$ ⓑ perpendicular to $h(x)$

HOW TO

Given two points on a line and a third point, write the equation of the perpendicular line that passes through the point.

1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.

3. Use the slope-intercept form or point-slope form to write the equation by substituting the known values.
4. Simplify.

EXAMPLE 21

Finding the Equation of a Line Perpendicular to a Given Line Passing through a Point

A line passes through the points $(-2, 6)$ and $(4, 5)$. Find the equation of a perpendicular line that passes through the point $(4, 5)$.

Solution

From the two points of the given line, we can calculate the slope of that line.

$$\begin{aligned} m_1 &= \frac{5-6}{4-(-2)} \\ &= \frac{-1}{6} \\ &= -\frac{1}{6} \end{aligned}$$

Find the negative reciprocal of the slope.

$$\begin{aligned} m_2 &= \frac{-1}{-\frac{1}{6}} \\ &= -1\left(-\frac{6}{1}\right) \\ &= 6 \end{aligned}$$

We can then solve for the *y*-intercept of the line passing through the point $(4, 5)$.

$$\begin{aligned} g(x) &= 6x + b \\ 5 &= 6(4) + b \\ 5 &= 24 + b \\ -19 &= b \\ b &= -19 \end{aligned}$$

The equation for the line that is perpendicular to the line passing through the two given points and also passes through point $(4, 5)$ is

$$y = 6x - 19$$

TRY IT #10 A line passes through the points, $(-2, -15)$ and $(2, -3)$. Find the equation of a perpendicular line that passes through the point, $(6, 4)$.

MEDIA

Access this online resource for additional instruction and practice with linear functions.

Linear Functions (http://openstax.org/l/linearfunctions)
Finding Input of Function from the Output and Graph (http://openstax.org/l/findinginput)
Graphing Functions using Tables (http://openstax.org/l/graphwithtable)

4.1 SECTION EXERCISES

Verbal

1. Terry is skiing down a steep hill. Terry's elevation, $E(t)$, in feet after t seconds is given by $E(t) = 3000 - 70t$. Write a complete sentence describing Terry's starting elevation and how it is changing over time.

2. Jessica is walking home from a friend's house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate in miles per hour?

3. A boat is 100 miles away from the marina, sailing directly toward it at 10 miles per hour. Write an equation for the distance of the boat from the marina after t hours.

4. If the graphs of two linear functions are perpendicular, describe the relationship between the slopes and the y-intercepts.

5. If a horizontal line has the equation $f(x) = a$ and a vertical line has the equation $x = a$, what is the point of intersection? Explain why what you found is the point of intersection.

Algebraic

For the following exercises, determine whether the equation of the curve can be written as a linear function.

6. $y = \frac{1}{4}x + 6$

7. $y = 3x - 5$

8. $y = 3x^2 - 2$

9. $3x + 5y = 15$

10. $3x^2 + 5y = 15$

11. $3x + 5y^2 = 15$

12. $-2x^2 + 3y^2 = 6$

13. $-\frac{x-3}{5} = 2y$

For the following exercises, determine whether each function is increasing or decreasing.

14. $f(x) = 4x + 3$

15. $g(x) = 5x + 6$

16. $a(x) = 5 - 2x$

17. $b(x) = 8 - 3x$

18. $h(x) = -2x + 4$

19. $k(x) = -4x + 1$

20. $j(x) = \frac{1}{2}x - 3$

21. $p(x) = \frac{1}{4}x - 5$

22. $n(x) = -\frac{1}{3}x - 2$

23. $m(x) = -\frac{3}{8}x + 3$

For the following exercises, find the slope of the line that passes through the two given points.

24. $(2, 4)$ and $(4, 10)$

25. $(1, 5)$ and $(4, 11)$

26. $(-1, 4)$ and $(5, 2)$

27. $(8, -2)$ and $(4, 6)$

28. $(6, 11)$ and $(-4, 3)$

For the following exercises, given each set of information, find a linear equation satisfying the conditions, if possible.

29. $f(-5) = -4$, and $f(5) = 2$

30. $f(-1) = 4$, and $f(5) = 1$

31. Passes through $(2, 4)$ and $(4, 10)$

32. Passes through $(1, 5)$ and $(4, 11)$

33. Passes through $(-1, 4)$ and $(5, 2)$

34. Passes through $(-2, 8)$ and $(4, 6)$

35. *x* intercept at $(-2, 0)$ and *y* intercept at $(0, -3)$

36. *x* intercept at $(-5, 0)$ and *y* intercept at $(0, 4)$

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither.

37. $\begin{array}{l} 4x - 7y = 10 \\ 7x + 4y = 1 \end{array}$

38. $\begin{array}{l} 3y + x = 12 \\ -y = 8x + 1 \end{array}$

39. $\begin{array}{l} 3y + 4x = 12 \\ -6y = 8x + 1 \end{array}$

40. $\begin{array}{l} 6x - 9y = 10 \\ 3x + 2y = 1 \end{array}$

For the following exercises, find the x- and y-intercepts of each equation.

41. $f(x) = -x + 2$

42. $g(x) = 2x + 4$

43. $h(x) = 3x - 5$

44. $k(x) = -5x + 1$

45. $-2x + 5y = 20$

46. $7x + 2y = 56$

For the following exercises, use the descriptions of each pair of lines given below to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

47. Line 1: Passes through $(0, 6)$ and $(3, -24)$

Line 2: Passes through $(-1, 19)$ and $(8, -71)$

48. Line 1: Passes through $(-8, -55)$ and $(10, 89)$

Line 2: Passes through $(9, -44)$ and $(4, -14)$

49. Line 1: Passes through $(2, 3)$ and $(4, -1)$

Line 2: Passes through $(6, 3)$ and $(8, 5)$

50. Line 1: Passes through $(1, 7)$ and $(5, 5)$

Line 2: Passes through $(-1, -3)$ and $(1, 1)$

51. Line 1: Passes through $(2, 5)$ and $(5, -1)$

Line 2: Passes through $(-3, 7)$ and $(3, -5)$

For the following exercises, write an equation for the line described.

52. Write an equation for a line parallel to $f(x) = -5x - 3$ and passing through the point $(2, -12)$.

53. Write an equation for a line parallel to $g(x) = 3x - 1$ and passing through the point $(4, 9)$.

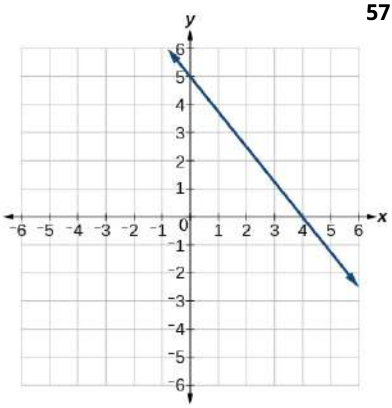
54. Write an equation for a line perpendicular to $h(t) = -2t + 4$ and passing through the point $(-4, -1)$.

55. Write an equation for a line perpendicular to $p(t) = 3t + 4$ and passing through the point $(3, 1)$.

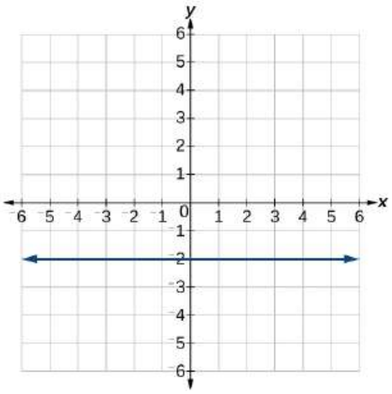
Graphical

For the following exercises, find the slope of the line graphed.

56.

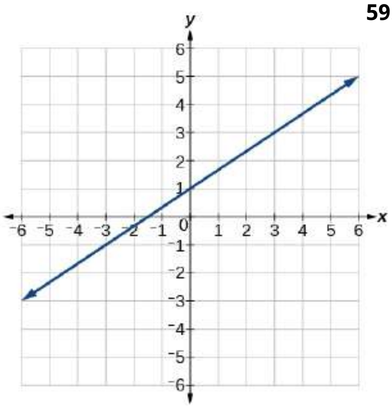


57.



For the following exercises, write an equation for the line graphed.

58.



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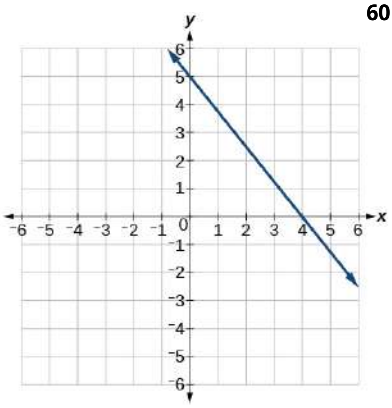
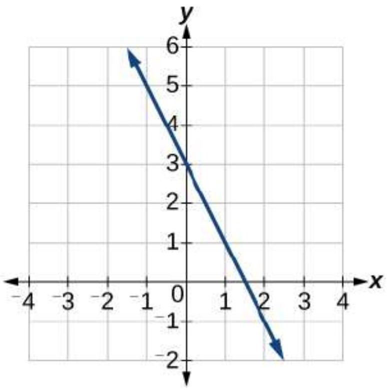


60.



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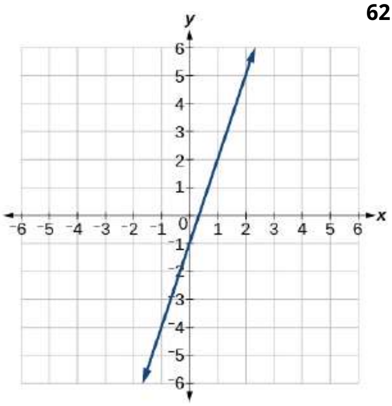
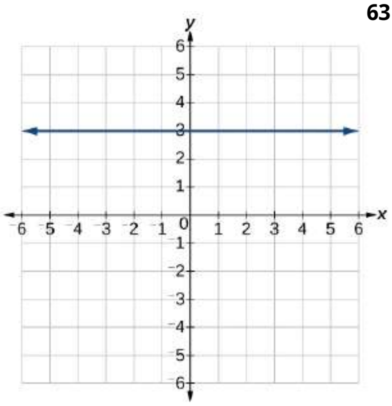
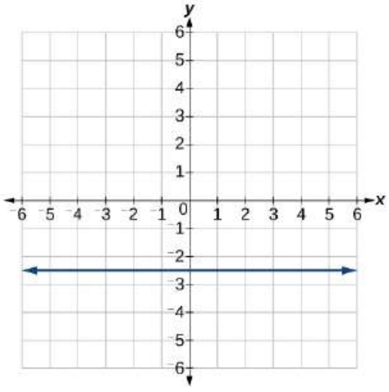


62.



63.



For the following exercises, match the given linear equation with its graph in Figure 33.

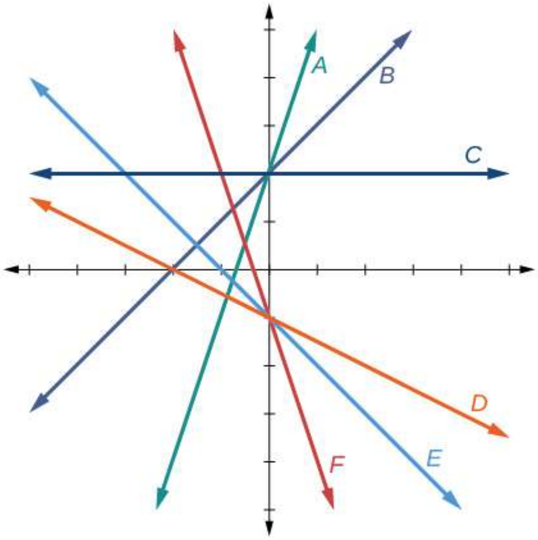


Figure 33

64. $f(x) = -x - 1$

65. $f(x) = -3x - 1$

66. $f(x) = -\frac{1}{2}x - 1$

67. $f(x) = 2$

68. $f(x) = 2 + x$

69. $f(x) = 3x + 2$

For the following exercises, sketch a line with the given features.

70. An x-intercept of $(-4, 0)$ and y-intercept of $(0, -2)$

71. An x-intercept $(-2, 0)$ and y-intercept of $(0, 4)$

72. A y-intercept of $(0, 7)$ and slope $-\frac{3}{2}$

73. A y-intercept of $(0, 3)$ and slope $\frac{2}{5}$

74. Passing through the points $(-6, -2)$ and $(6, -6)$

75. Passing through the points $(-3, -4)$ and $(3, 0)$

For the following exercises, sketch the graph of each equation.

76. $f(x) = -2x - 1$

77. $f(x) = -3x + 2$

78. $f(x) = \frac{1}{3}x + 2$

79. $f(x) = \frac{2}{3}x - 3$

80. $f(t) = 3 + 2t$

81. $p(t) = -2 + 3t$

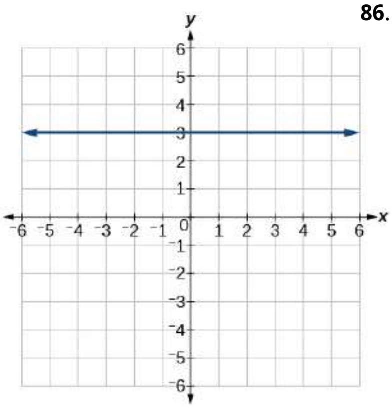
82. $x = 3$

83. $x = -2$

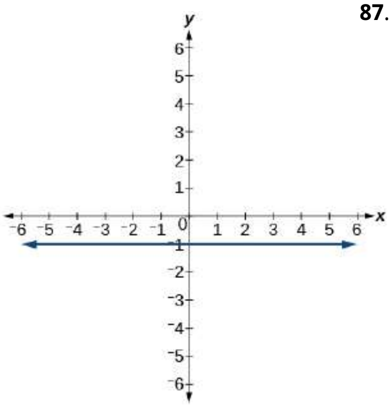
84. $r(x) = 4$

For the following exercises, write the equation of the line shown in the graph.

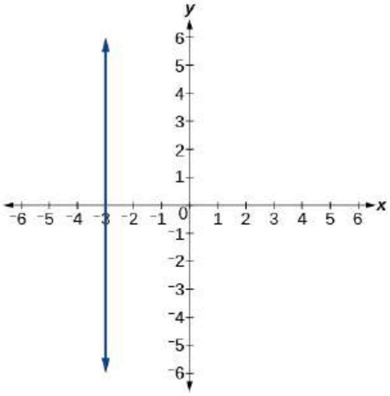
85.



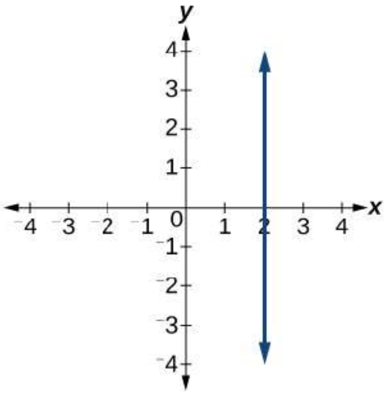
86.



87.



88.



Numeric

For the following exercises, which of the tables could represent a linear function? For each that could be linear, find a linear equation that models the data.

89.

x	0	5	10	15
$g(x)$	5	–10	–25	–40

90.

x	0	5	10	15
$h(x)$	5	30	105	230

91.

x	0	5	10	15
$f(x)$	–5	20	45	70

92.

x	5	10	20	25
$k(x)$	13	28	58	73

93.

x	0	2	4	6
$g(x)$	6	–19	–44	–69

94.

x	2	4	8	10
$h(x)$	13	23	43	53

95.

x	2	4	6	8
$f(x)$	–4	16	36	56

96.

x	0	2	6	8
$k(x)$	6	31	106	231

Technology

For the following exercises, use a calculator or graphing technology to complete the task.

97. If f is a linear function, $f(0.1) = 11.5$, and $f(0.4) = -5.9$, find an equation for the function.

98. Graph the function f on a domain of $[-10, 10] : f(x) = 0.02x - 0.01$. Enter the function in a graphing utility. For the viewing window, set the minimum value of x to be -10 and the maximum value of x to be 10.

99. Graph the function f on a domain of $[-10, 10] : fx) = 2, 500x + 4, 000$

100. Table 3 shows the input, w, and output, k, for a linear function k.

ⓐ Fill in the missing values of the table.
ⓑ Write the linear function k, round to 3 decimal places.

w	–10	5.5	67.5	*b*
k	30	–26	*a*	–44

Table 3

101. Table 4 shows the input, p, and output, q, for a linear function q.

ⓐ Fill in the missing values of the table.
ⓑ Write the linear function k.

p	0.5	0.8	12	*b*
q	400	700	*a*	1,000,000

Table 4

102. Graph the linear function f on a domain of $[-10, 10]$ for the function whose slope is $\frac{1}{8}$ and *y*-intercept is $\frac{31}{16}$. Label the points for the input values of -10 and 10.

103. Graph the linear function f on a domain of $[-0.1, 0.1]$ for the function whose slope is 75 and *y*-intercept is -22.5. Label the points for the input values of -0.1 and 0.1.

104. Graph the linear function f where $f(x) = ax + b$ on the same set of axes on a domain of $[-4, 4]$ for the following values of a and b.

ⓐ $a = 2; b = 3$
ⓑ $a = 2; b = 4$
ⓒ $a = 2; b = -4$
ⓓ $a = 2; b = -5$

Extensions

105. Find the value of x if a linear function goes through the following points and has the following slope: $(x, 2), (-4, 6), \quad m = 3$

106. Find the value of *y* if a linear function goes through the following points and has the following slope: $(10, y), (25, 100), \quad m = -5$

107. Find the equation of the line that passes through the following points:

$(a, \quad b)$ and $(a, \quad b + 1)$

108. Find the equation of the line that passes through the following points:

$(2a, b)$ and $(a, b + 1)$

109. Find the equation of the line that passes through the following points:

$(a, 0)$ and (c, d)

110. Find the equation of the line parallel to the line $g(x) = -0.01x+2.01$ through the point $(1, 2)$.

111. Find the equation of the line perpendicular to the line $g(x) = -0.01x+2.01$ through the point $(1, 2)$.

For the following exercises, use the functions $f(x) = -0.1x+200 \text{ and } g(x) = 20x + 0.1.$

112. Find the point of intersection of the lines f and g.

113. Where is $f(x)$ greater than $g(x)$? Where is $g(x)$ greater than $f(x)$?

Real-World Applications

114. At noon, a barista notices that they have $20 in her tip jar. If they maks an average of $0.50 from each customer, how much will the barista have in the tip jar if they serve n more customers during the shift?

115. A gym membership with two personal training sessions costs $125, while gym membership with five personal training sessions costs $260. What is cost per session?

116. A clothing business finds there is a linear relationship between the number of shirts, n, it can sell and the price, p, it can charge per shirt. In particular, historical data shows that 1,000 shirts can be sold at a price of $\$30$, while 3,000 shirts can be sold at a price of $22. Find a linear equation in the form $p(n) = mn + b$ that gives the price p they can charge for n shirts.

117. A phone company charges for service according to the formula: $C(n) = 24 + 0.1n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

118. A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y, each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationships in the form $y = mn + b$ that gives the yield when n stalks are planted.

119. A city's population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the rate of growth of the population and make a statement about the population rate of change in people per year.

120. A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1,700 people each year. Write an equation, $P(t)$, for the population t years after 2003.

121. Suppose that average annual income (in dollars) for the years 1990 through 1999 is given by the linear function: $I(x) = 1054x + 23{,}286$, where x is the number of years after 1990. Which of the following interprets the slope in the context of the problem?

ⓐ As of 1990, average annual income was \$23,286.
ⓑ In the ten-year period from 1990–1999, average annual income increased by a total of \$1,054.
ⓒ Each year in the decade of the 1990s, average annual income increased by \$1,054.
ⓓ Average annual income rose to a level of \$23,286 by the end of 1999.

122. When temperature is 0 degrees Celsius, the Fahrenheit temperature is 32. When the Celsius temperature is 100, the corresponding Fahrenheit temperature is 212. Express the Fahrenheit temperature as a linear function of C, the Celsius temperature, $F(C)$.

ⓐ Find the rate of change of Fahrenheit temperature for each unit change temperature of Celsius.
ⓑ Find and interpret $F(28)$.
ⓒ Find and interpret $F(-40)$.

4.2 Modeling with Linear Functions

Learning Objectives

In this section, you will:

- Build linear models from verbal descriptions.
- Model a set of data with a linear function.

Figure 1 (credit: EEK Photography/Flickr)

Elan is a college student who plans to spend a summer in Seattle. Elan has saved \$3,500 for their trip and anticipates spending \$400 each week on rent, food, and activities. How can we write a linear model to represent this situation? What would be the *x*-intercept, and what can Elan learn from it? To answer these and related questions, we can create a model using a linear function. Models such as this one can be extremely useful for analyzing relationships and making predictions based on those relationships. In this section, we will explore examples of linear function models.

Building Linear Models from Verbal Descriptions

When building linear models to solve problems involving quantities with a constant rate of change, we typically follow the same problem strategies that we would use for any type of function. Let’s briefly review them:

1. Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2. Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
3. Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
5. When needed, write a formula for the function.
6. Solve or evaluate the function using the formula.
7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8. Clearly convey your result using appropriate units, and answer in full sentences when necessary.

Now let’s take a look at the student in Seattle. In Elan’s situation, there are two changing quantities: time and money. The amount of money they have remaining while on vacation depends on how long they stay. We can use this information to define our variables, including units.

Output: M, money remaining, in dollars
Input: t, time, in weeks

So, the amount of money remaining depends on the number of weeks: $M(t)$.

We can also identify the initial value and the rate of change.

Initial Value: She saved \$3,500, so \$3,500 is the initial value for M.
Rate of Change: She anticipates spending \$400 each week, so $-$ \$400 per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The rate of change is constant, so we can start with the linear model $M(t) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = \underset{\substack{\uparrow \\ -400}}{m}t + \underset{\substack{\uparrow \\ 3500}}{b}$$

$$M(t) = -400t + 3500$$

To find the t-intercept (horizontal axis intercept), we set the output to zero, and solve for the input.

$$\begin{aligned} 0 &= -400t + 3500 \\ t &= \tfrac{3500}{400} \\ &= 8.75 \end{aligned}$$

The t-intercept (horizontal axis intercept) is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Elan will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn’t make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before Elan saved \$3,500, but the scenario discussed poses the question once they saved \$3,500 because this is when the trip and subsequent spending starts. It is also likely that this model is not valid after the t-intercept (horizontal axis intercept), unless Elan uses a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \leq t \leq 8.75$.

In this example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

Using a Given Intercept to Build a Model

Some real-world problems provide the vertical axis intercept, which is the constant or initial value. Once the vertical axis intercept is known, the *t*-intercept (horizontal axis intercept) can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is $1,000. She plans to pay $250 per month until her balance is $0. The *y*-intercept is the initial amount of her debt, or $1,000. The rate of change, or slope, is -$250 per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$\begin{aligned} f(x) &= mx + b \\ &= -250x + 1000 \end{aligned}$$

Now we can set the function equal to 0, and solve for x to find the *x*-intercept.

$$\begin{aligned} 0 &= -250x + 1000 \\ 1000 &= 250x \\ 4 &= x \\ x &= 4 \end{aligned}$$

The *x*-intercept is the number of months it takes her to reach a balance of $0. The *x*-intercept is 4 months, so it will take Hannah four months to pay off her loan.

Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

HOW TO

Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given *x*-value.
6. Use the model to identify an *x*-value that results in a given *y*-value.
7. Answer the question posed.

EXAMPLE 1

Using a Linear Model to Investigate a Town’s Population

A town’s population has been growing linearly. In 2004, the population was 6,200. By 2009, the population had grown to 8,100. Assume this trend continues.

ⓐ Predict the population in 2013. ⓑ Identify the year in which the population will reach 15,000.

Solution

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the *y*-intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004.

$$\text{Input: } t, \text{years since 2004}$$
$$\text{Output: } P(t), \text{the town's population}$$

To predict the population in 2013 ($t = 9$), we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.

$$m = \frac{\text{change in output}}{\text{change in input}}$$

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the *y*-intercept of the function. The year 2009 would correspond to $t = 5$, giving the point $(5, 8100)$.

The two coordinate pairs are $(0, 6200)$ and $(5, 8100)$. Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.

$$\begin{aligned} m &= \tfrac{8100-6200}{5-0} \\ &= \tfrac{1900}{5} \\ &= 380 \text{ people per year} \end{aligned}$$

We already know the *y*-intercept of the line, so we can immediately write the equation:

$$P(t) = 380t + 6200$$

To predict the population in 2013, we evaluate our function at $t = 9$.

$$\begin{aligned} P(9) &= 380(9) + 6{,}200 \\ &= 9{,}620 \end{aligned}$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$\begin{aligned} 15000 &= 380t + 6200 \\ 8800 &= 380t \\ t &\approx 23.158 \end{aligned}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

> **TRY IT** #1 A company sells doughnuts. They incur a fixed cost of $25,000 for rent, insurance, and other expenses. It costs $0.25 to produce each doughnut.

ⓐ Write a linear model to represent the cost C of the company as a function of x, the number of doughnuts produced.

ⓑ Find and interpret the *y*-intercept.

> **TRY IT** #2 A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

ⓐ Predict the population in 2014.

ⓑ Identify the year in which the population will reach 54,000.

Using a Diagram to Build a Model

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometrical shapes or figures are drawn. Distances are often traced out. If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

EXAMPLE 2

Using a Diagram to Model Distance Walked

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

Solution

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question:

"How long will it take them to be 2 miles apart"?

In this problem, our changing quantities are time and position, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define our input and output variables.

$$\begin{array}{l} \text{Input: } t, \text{time in hours.} \\ \text{Output: } A(t), \text{distance in miles, and } E(t), \text{distance in miles} \end{array}$$

Because it is not obvious how to define our output variable, we'll start by drawing a picture such as Figure 2.

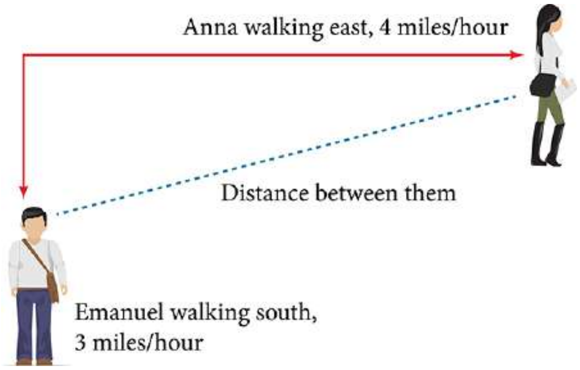


Figure 2

Initial Value: They both start at the same intersection so when $t = 0$, the distance traveled by each person should also be 0. Thus the initial value for each is 0.

Rate of Change: Anna is walking 4 miles per hour and Emanuel is walking 3 miles per hour, which are both rates of change. The slope for A is 4 and the slope for E is 3.

Using those values, we can write formulas for the distance each person has walked.

$$\begin{array}{rcl} A(t) & = & 4t \\ E(t) & = & 3t \end{array}$$

For this problem, the distances from the starting point are important. To notate these, we can define a coordinate system, identifying the "starting point" at the intersection where they both started. Then we can use the variable, A, which we introduced above, to represent Anna's position, and define it to be a measurement from the starting point in the eastward direction. Likewise, can use the variable, E, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system, we specified both the starting point of the measurement and the direction of measure.

We can then define a third variable, D, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the diagram is often helpful, as we can see from Figure 3.

Recall that we need to know how long it takes for D, the distance between them, to equal 2 miles. Notice that for any given input t, the outputs $A(t)$, $E(t)$, and $D(t)$ represent distances.

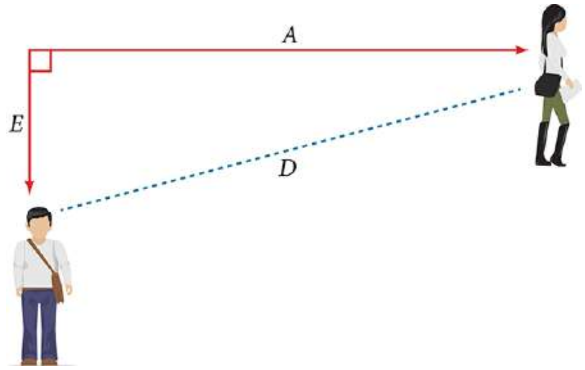


Figure 3

Figure 2 shows us that we can use the Pythagorean Theorem because we have drawn a right angle.

Using the Pythagorean Theorem, we get:

$$\begin{aligned} D(t)^2 &= A(t)^2 + E(t)^2 \\ &= (4t)^2 + (3t)^2 \\ &= 16t^2 + 9t^2 \\ &= 25t^2 \\ D(t) &= \pm\sqrt{25t^2} && \text{Solve for } D(t) \text{ using the square root.} \\ &= \pm 5|t| \end{aligned}$$

In this scenario we are considering only positive values of t, so our distance $D(t)$ will always be positive. We can simplify this answer to $D(t) = 5t$. This means that the distance between Anna and Emanuel is also a linear function. Because D is a linear function, we can now answer the question of when the distance between them will reach 2 miles. We will set the output $D(t) = 2$ and solve for t.

$$\begin{aligned} D(t) &= 2 \\ 5t &= 2 \\ t &= \tfrac{2}{5} = 0.4 \end{aligned}$$

They will fall out of radio contact in 0.4 hour, or 24 minutes.

Q&A **Should I draw diagrams when given information based on a geometric shape?**

Yes. Sketch the figure and label the quantities and unknowns on the sketch.

EXAMPLE 3

Using a Diagram to Model Distance Between Cities

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

Solution

It might help here to draw a picture of the situation. See Figure 4. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts Agritown at coordinates $(30, 10)$, and Eastborough at $(20, 0)$.

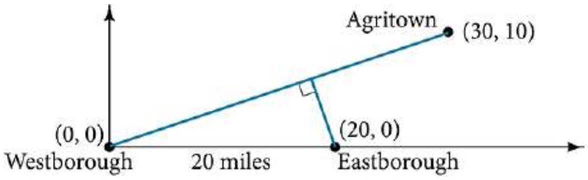


Figure 4

Using this point along with the origin, we can find the slope of the line from Westborough to Agritown.

$$m = \frac{10 - 0}{30 - 0} = \frac{1}{3}$$

Now we can write an equation to describe the road from Westborough to Agritown.

$$W(x) = \frac{1}{3}x$$

From this, we can determine the perpendicular road to Eastborough will have slope $m =-3$. Because the town of Eastborough is at the point (20, 0), we can find the equation.

$$\begin{array}{rcll} E(x) & = & -3x + b & \\ 0 & = & -3(20) + b & \text{Substitute } (20, 0) \text{ into the equation.} \\ b & = & 60 & \\ E(x) & = & -3x + 60 & \end{array}$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$$\begin{array}{rcll} \frac{1}{3}x & = & -3x + 60 & \\ \frac{10}{3}x & = & 60 & \\ 10x & = & 180 & \\ x & = & 18 & \text{Substitute this back into } W(x). \\ y & = & W(18) & \\ & = & \frac{1}{3}(18) & \\ & = & 6 & \end{array}$$

The roads intersect at the point (18, 6). Using the distance formula, we can now find the distance from Westborough to the junction.

$$\begin{array}{rcl} \text{distance} & = & \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ & = & \sqrt{(18 - 0)^2 + (6 - 0)^2} \\ & \approx & 18.974 \text{ miles} \end{array}$$

Analysis

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

TRY IT #3 There is a straight road leading from the town of Timpson to Ashburn 60 miles east and 12 miles north. Partway down the road, it junctions with a second road, perpendicular to the first, leading to the town of Garrison. If the town of Garrison is located 22 miles directly east of the town of Timpson, how far is the road junction from Timpson?

Modeling a Set of Data with Linear Functions

Real-world situations including two or more linear functions may be modeled with a system of linear equations. Remember, when solving a system of linear equations, we are looking for points the two lines have in common. Typically, there are three types of answers possible, as shown in Figure 5.

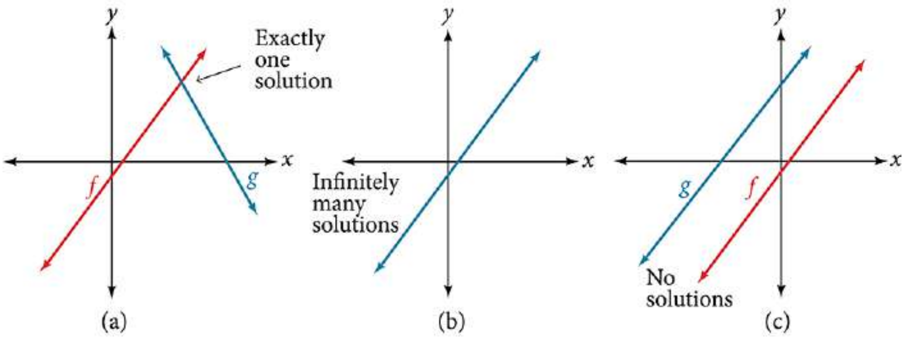


Figure 5

HOW TO

Given a situation that represents a system of linear equations, write the system of equations and identify the solution.

1. Identify the input and output of each linear model.
2. Identify the slope and y-intercept of each linear model.
3. Find the solution by setting the two linear functions equal to another and solving for x, or find the point of intersection on a graph.

EXAMPLE 4

Building a System of Linear Models to Choose a Truck Rental Company

Jamal is choosing between two truck-rental companies. The first, Keep on Trucking, Inc., charges an up-front fee of $20, then 59 cents a mile. The second, Move It Your Way, charges an up-front fee of $16, then 63 cents a mile[4]. When will Keep on Trucking, Inc. be the better choice for Jamal?

Solution

The two important quantities in this problem are the cost and the number of miles driven. Because we have two companies to consider, we will define two functions in Table 1.

Input	d, distance driven in miles
Outputs	$K(d)$: cost, in dollars, for renting from Keep on Trucking $M\ (d)$ cost, in dollars, for renting from Move It Your Way
Initial Value	Up-front fee: $K\ (0) = 20$ and $M\ (0) = 16$
Rate of Change	$K(d) = \$0.59$ /mile and $P(d) = \$0.63$ /mile

Table 1

A linear function is of the form $f(x) = mx + b$. Using the rates of change and initial charges, we can write the equations

$$\begin{aligned} K(d) &= 0.59d + 20 \\ M(d) &= 0.63d + 16 \end{aligned}$$

Using these equations, we can determine when Keep on Trucking, Inc., will be the better choice. Because all we have to make that decision from is the costs, we are looking for when Move It Your Way, will cost less, or when $K(d) < M(d)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, and then see where

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[4] Rates retrieved Aug 2, 2010 from http://www.budgettruck.com and http://www.uhaul.com/

the $K(d)$ function is smaller.

These graphs are sketched in Figure 6, with $K(d)$ in blue.

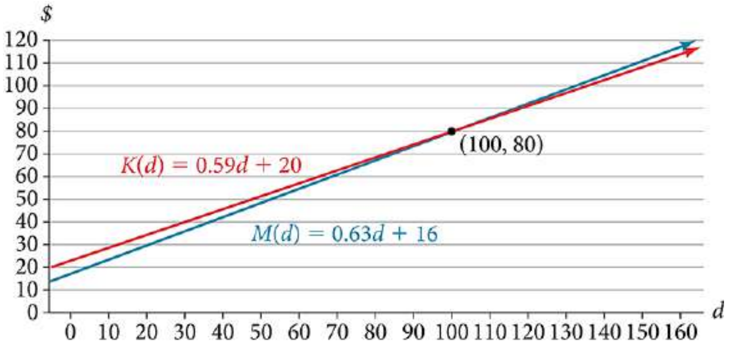

Figure 6

To find the intersection, we set the equations equal and solve:

$$\begin{aligned} K(d) &= M(d) \\ 0.59d + 20 &= 0.63d + 16 \\ 4 &= 0.04d \\ 100 &= d \\ d &= 100 \end{aligned}$$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that $K(d)$ is growing at a slower rate, we can conclude that Keep on Trucking, Inc. will be the cheaper price when more than 100 miles are driven, that is $d > 100$.

MEDIA

Access this online resource for additional instruction and practice with linear function models.

Interpreting a Linear Function (http://openstax.org/l/interpretlinear)

4.2 SECTION EXERCISES

Verbal

1. Explain how to find the input variable in a word problem that uses a linear function.

2. Explain how to find the output variable in a word problem that uses a linear function.

3. Explain how to interpret the initial value in a word problem that uses a linear function.

4. Explain how to determine the slope in a word problem that uses a linear function.

Algebraic

5. Find the area of a parallelogram bounded by the y-axis, the line $x = 3$, the line $f(x) = 1 + 2x$, and the line parallel to $f(x)$ passing through $(2, 7)$.

6. Find the area of a triangle bounded by the x-axis, the line $f(x) = 12 – \frac{1}{3}x$, and the line perpendicular to $f(x)$ that passes through the origin.

7. Find the area of a triangle bounded by the y-axis, the line $f(x) = 9 – \frac{6}{7}x$, and the line perpendicular to $f(x)$ that passes through the origin.

8. Find the area of a parallelogram bounded by the x-axis, the line $g(x) = 2$, the line $f(x) = 3x$, and the line parallel to $f(x)$ passing through $(6, 1)$.

For the following exercises, consider this scenario: A town's population has been decreasing at a constant rate. In 2010 the population was 5,900. By 2012 the population had dropped to 4,700. Assume this trend continues.

9. Predict the population in 2016.

10. Identify the year in which the population will reach 0.

For the following exercises, consider this scenario: A town's population has been increased at a constant rate. In 2010 the population was 46,020. By 2012 the population had increased to 52,070. Assume this trend continues.

11. Predict the population in 2016.

12. Identify the year in which the population will reach 75,000.

For the following exercises, consider this scenario: A town has an initial population of 75,000. It grows at a constant rate of 2,500 per year for 5 years.

13. Find the linear function that models the town's population P as a function of the year, t, where t is the number of years since the model began.

14. Find a reasonable domain and range for the function P.

15. If the function P is graphed, find and interpret the x- and y-intercepts.

16. If the function P is graphed, find and interpret the slope of the function.

17. When will the population reach 100,000?

18. What is the population in the year 12 years from the onset of the model?

For the following exercises, consider this scenario: The weight of a newborn is 7.5 pounds. The baby gained one-half pound a month for its first year.

19. Find the linear function that models the baby's weight W as a function of the age of the baby, in months, t.

20. Find a reasonable domain and range for the function W.

21. If the function W is graphed, find and interpret the x- and y-intercepts.

22. If the function W is graphed, find and interpret the slope of the function.

23. When did the baby weight 10.4 pounds?

24. What is the output when the input is 6.2?

For the following exercises, consider this scenario: The number of people afflicted with the common cold in the winter months steadily decreased by 205 each year from 2005 until 2010. In 2005, 12,025 people were inflicted.

25. Find the linear function that models the number of people inflicted with the common cold C as a function of the year, t.

26. Find a reasonable domain and range for the function C.

27. If the function C is graphed, find and interpret the x- and y-intercepts.

28. If the function C is graphed, find and interpret the slope of the function.

29. When will the output reach 0?

30. In what year will the number of people be 9,700?

Graphical

For the following exercises, use the graph in Figure 7, which shows the profit, y, in thousands of dollars, of a company in a given year, t, where t represents the number of years since 1980.

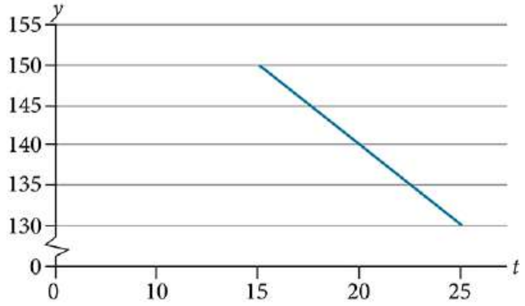

Figure 7

31. Find the linear function y, where y depends on t, the number of years since 1980.

32. Find and interpret the y-intercept.

33. Find and interpret the x-intercept.

34. Find and interpret the slope.

For the following exercises, use the graph in Figure 8, which shows the profit, y, in thousands of dollars, of a company in a given year, t, where t represents the number of years since 1980.

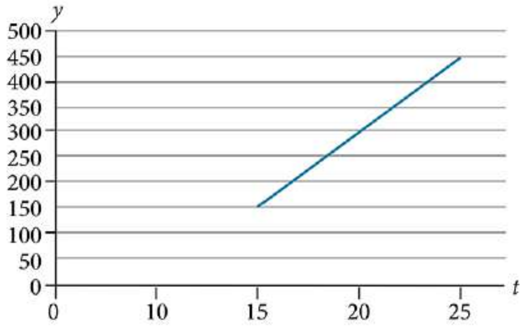

Figure 8

35. Find the linear function y, where y depends on t, the number of years since 1980.

36. Find and interpret the y-intercept.

37. Find and interpret the x-intercept.

38. Find and interpret the slope.

Numeric

For the following exercises, use the median home values in Mississippi and Hawaii (adjusted for inflation) shown in Table 2. Assume that the house values are changing linearly.

Year	Mississippi	Hawaii
1950	$25,200	$74,400
2000	$71,400	$272,700

Table 2

39. In which state have home values increased at a higher rate?

40. If these trends were to continue, what would be the median home value in Mississippi in 2010?

41. If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

For the following exercises, use the median home values in Indiana and Alabama (adjusted for inflation) shown in Table 3. Assume that the house values are changing linearly.

Year	Indiana	Alabama
1950	$37,700	$27,100
2000	$94,300	$85,100

Table 3

42. In which state have home values increased at a higher rate?

43. If these trends were to continue, what would be the median home value in Indiana in 2010?

44. If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

Real-World Applications

45. In 2004, a school population was 1001. By 2008 the population had grown to 1697. Assume the population is changing linearly.

ⓐ How much did the population grow between the year 2004 and 2008?
ⓑ How long did it take the population to grow from 1001 students to 1697 students?
ⓒ What is the average population growth per year?
ⓓ What was the population in the year 2000?
ⓔ Find an equation for the population, P, of the school t years after 2000.
ⓕ Using your equation, predict the population of the school in 2011.

46. In 2003, a town's population was 1431. By 2007 the population had grown to 2134. Assume the population is changing linearly.

ⓐ How much did the population grow between the year 2003 and 2007?
ⓑ How long did it take the population to grow from 1431 people to 2134 people?
ⓒ What is the average population growth per year?
ⓓ What was the population in the year 2000?
ⓔ Find an equation for the population, P, of the town t years after 2000.
ⓕ Using your equation, predict the population of the town in 2014.

47. A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used for voice or video calling. If a customer uses 410 minutes, the monthly cost will be $71.50. If the customer uses 720 minutes, the monthly cost will be $118.

ⓐ Find a linear equation for the monthly cost of the cell plan as a function of x, the number of monthly minutes used.
ⓑ Interpret the slope and y-intercept of the equation.
ⓒ Use your equation to find the total monthly cost if 687 minutes are used.

48. A phone company has a monthly cellular data plan where a customer pays a flat monthly fee of $10 and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be $11.20. If the customer uses 130 MB, the monthly cost will be $17.80.

ⓐ Find a linear equation for the monthly cost of the data plan as a function of x, the number of MB used.
ⓑ Interpret the slope and y-intercept of the equation.
ⓒ Use your equation to find the total monthly cost if 250 MB are used.

49. In 1991, the moose population in a park was measured to be 4,360. By 1999, the population was measured again to be 5,880. Assume the population continues to change linearly.

ⓐ Find a formula for the moose population, P since 1990.
ⓑ What does your model predict the moose population to be in 2003?

50. In 2003, the owl population in a park was measured to be 340. By 2007, the population was measured again to be 285. The population changes linearly. Let the input be years since 2003.

ⓐ Find a formula for the owl population, P. Let the input be years since 2003.
ⓑ What does your model predict the owl population to be in 2012?

51. The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010 and is being depleted by about 2.1 billion cubic feet each year.

ⓐ Give a linear equation for the remaining federal helium reserves, R, in terms of t, the number of years since 2010.
ⓑ In 2015, what will the helium reserves be?
ⓒ If the rate of depletion doesn't change, in what year will the Federal Helium Reserve be depleted?

52. Suppose the world's oil reserves in 2014 are 1,820 billion barrels. If, on average, the total reserves are decreasing by 25 billion barrels of oil each year:

ⓐ Give a linear equation for the remaining oil reserves, R, in terms of t, the number of years since now.
ⓑ Seven years from now, what will the oil reserves be?
ⓒ If the rate at which the reserves are decreasing is constant, when will the world's oil reserves be depleted?

53. You are choosing between two different prepaid cell phone plans. The first plan charges a rate of 26 cents per minute. The second plan charges a monthly fee of $19.95 *plus* 11 cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

54. You are choosing between two different window washing companies. The first charges $5 per window. The second charges a base fee of $40 plus $3 per window. How many windows would you need to have for the second company to be preferable?

55. When hired at a new job selling jewelry, you are given two pay options:

Option A: Base salary of $17,000 a year with a commission of 12% of your sales

Option B: Base salary of $20,000 a year with a commission of 5% of your sales

How much jewelry would you need to sell for option A to produce a larger income?

56. When hired at a new job selling electronics, you are given two pay options:

Option A: Base salary of $14,000 a year with a commission of 10% of your sales

Option B: Base salary of $19,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

57. When hired at a new job selling electronics, you are given two pay options:

Option A: Base salary of $20,000 a year with a commission of 12% of your sales

Option B: Base salary of $26,000 a year with a commission of 3% of your sales

How much electronics would you need to sell for option A to produce a larger income?

58. When hired at a new job selling electronics, you are given two pay options:

Option A: Base salary of $10,000 a year with a commission of 9% of your sales

Option B: Base salary of $20,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

4.3 Fitting Linear Models to Data

Learning Objectives

In this section, you will:

- Draw and interpret scatter diagrams.
- Use a graphing utility to find the line of best fit.
- Distinguish between linear and nonlinear relations.
- Fit a regression line to a set of data and use the linear model to make predictions.

A professor is attempting to identify trends among final exam scores. His class has a mixture of students, so he wonders if there is any relationship between age and final exam scores. One way for him to analyze the scores is by creating a diagram that relates the age of each student to the exam score received. In this section, we will examine one such diagram known as a scatter plot.

Drawing and Interpreting Scatter Plots

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a linear model, or a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions. Figure 1 shows a sample scatter plot.

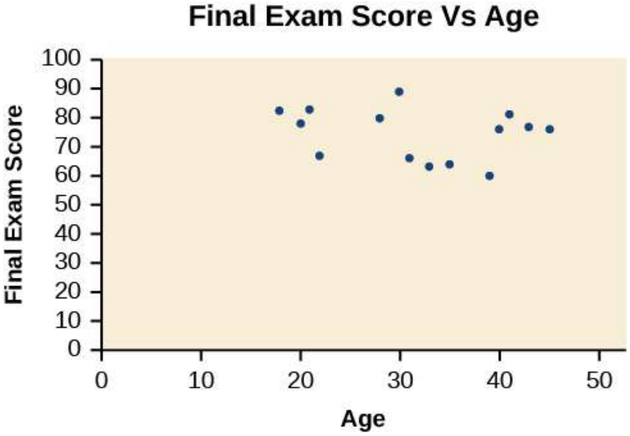

Figure 1 A scatter plot of age and final exam score variables

Notice this scatter plot does *not* indicate a linear relationship. The points do not appear to follow a trend. In other words, there does not appear to be a relationship between the age of the student and the score on the final exam.

EXAMPLE 1

Using a Scatter Plot to Investigate Cricket Chirps

Table 1 shows the number of cricket chirps in 15 seconds, for several different air temperatures, in degrees Fahrenheit[5]. Plot this data, and determine whether the data appears to be linearly related.

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature	80.5	70.5	57	66	68	72	52	73.5	53

Table 1 **Cricket Chirps vs Air Temperature**

Solution

Plotting this data, as depicted in Figure 2 suggests that there may be a trend. We can see from the trend in the data that the number of chirps increases as the temperature increases. The trend appears to be roughly linear, though certainly not perfectly so.

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[5] Selected data from http://classic.globe.gov/fsl/scientistsblog/2007/10/. Retrieved Aug 3, 2010

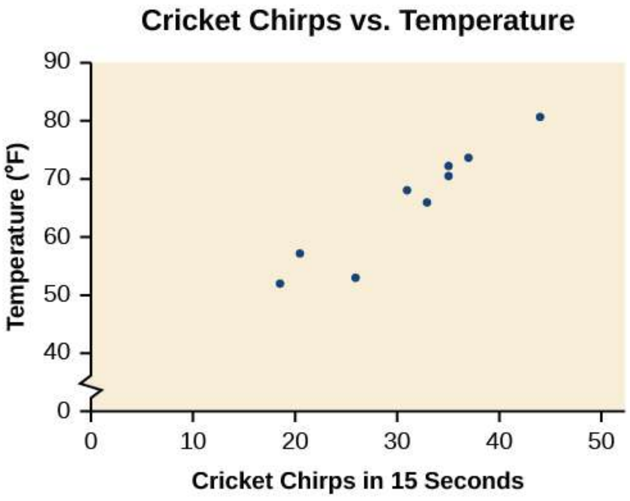


Figure 2

Finding the Line of Best Fit

Once we recognize a need for a linear function to model that data, the natural follow-up question is “what is that linear function?” One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the y-intercept. We can approximate the slope of the line by extending it until we can estimate the $\frac{\text{rise}}{\text{run}}$.

EXAMPLE 2

Finding a Line of Best Fit

Find a linear function that fits the data in Table 1 by “eyeballing” a line that seems to fit.

Solution

On a graph, we could try sketching a line. Using the starting and ending points of our hand drawn line, points (0, 30) and (50, 90), this graph has a slope of

$$\begin{aligned} m &= \frac{60}{50} \\ &= 1.2 \end{aligned}$$

and a y-intercept at 30. This gives an equation of

$$T(c) = 1.2c + 30$$

where c is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit. The resulting equation is represented in Figure 3.

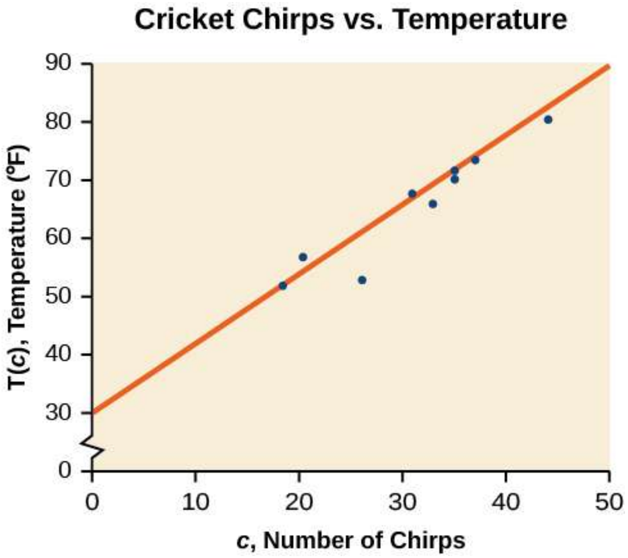


Figure 3

Analysis

This linear equation can then be used to approximate answers to various questions we might ask about the trend.

Recognizing Interpolation or Extrapolation

While the data for most examples does not fall perfectly on the line, the equation is our best guess as to how the relationship will behave outside of the values for which we have data. We use a process known as **interpolation** when we predict a value inside the domain and range of the data. The process of **extrapolation** is used when we predict a value outside the domain and range of the data.

Figure 4 compares the two processes for the cricket-chirp data addressed in Example 2. We can see that interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44. Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

There is a difference between making predictions inside the domain and range of values for which we have data and outside that domain and range. Predicting a value outside of the domain and range has its limitations. When our model no longer applies after a certain point, it is sometimes called **model breakdown**. For example, predicting a cost function for a period of two years may involve examining the data where the input is the time in years and the output is the cost. But if we try to extrapolate a cost when $x = 50$, that is in 50 years, the model would not apply because we could not account for factors fifty years in the future.

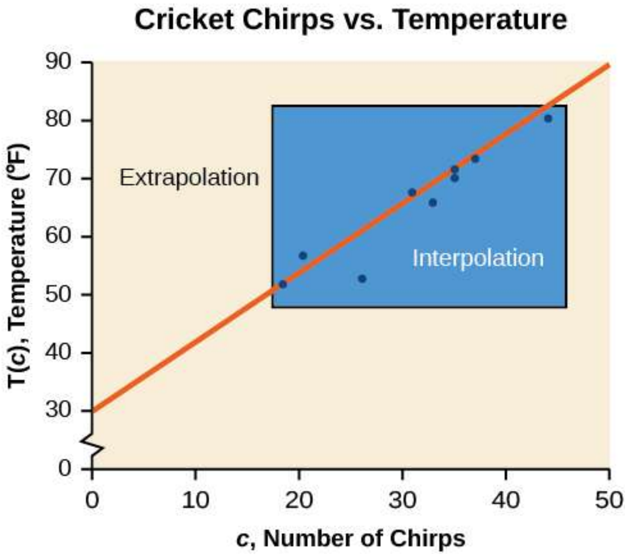


Figure 4 Interpolation occurs within the domain and range of the provided data whereas extrapolation occurs outside.

Interpolation and Extrapolation

Different methods of making predictions are used to analyze data.

The method of **interpolation** involves predicting a value inside the domain and/or range of the data. The method of **extrapolation** involves predicting a value outside the domain and/or range of the data. Model breakdown occurs at the point when the model no longer applies.

EXAMPLE 3

Understanding Interpolation and Extrapolation

Use the cricket data from Table 1 to answer the following questions:

ⓐ Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

ⓑ Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

Solution

ⓐ The number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:

$$\begin{aligned} \mathrm{T}\,(30) &= 30 + 1.2(30) \\ &= 66 \text{ degrees} \end{aligned}$$

Based on the data we have, this value seems reasonable.

ⓑ The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation because 40 is outside the range of our data. Using our model:

$$\begin{aligned} 40 &= 30 + 1.2c \\ 10 &= 1.2c \\ c &\approx 8.33 \end{aligned}$$

We can compare the regions of interpolation and extrapolation using Figure 5.

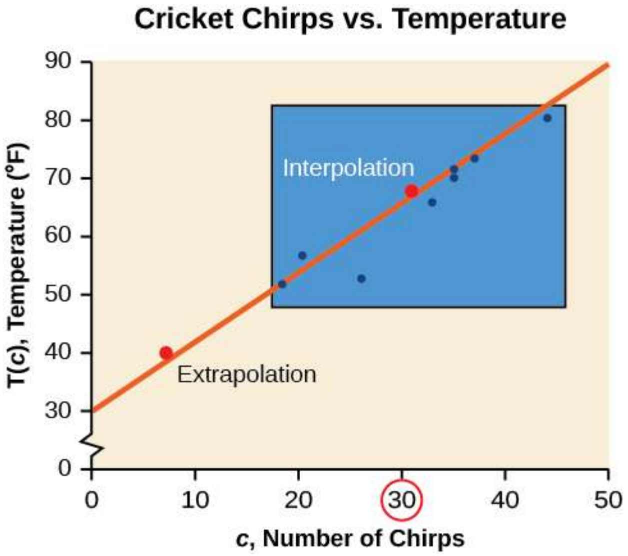


Figure 5

Analysis

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

TRY IT #1 According to the data from Table 1, what temperature can we predict it is if we counted 20 chirps in 15 seconds?

Finding the Line of Best Fit Using a Graphing Utility

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values[6]. One such technique is called **least squares regression** and can be computed by many graphing calculators, spreadsheet software, statistical software, and many web-based calculators[7]. Least squares regression is one means to determine the line that best fits the data, and here we will refer to this method as linear regression.

HOW TO

Given data of input and corresponding outputs from a linear function, find the best fit line using linear regression.

1. Enter the input in List 1 (L1).
2. Enter the output in List 2 (L2).
3. On a graphing utility, select Linear Regression (LinReg).

EXAMPLE 4

Finding a Least Squares Regression Line

Find the least squares regression line using the cricket-chirp data in Table 2.

Solution

1. Enter the input (chirps) in List 1 (L1).

---

[6] Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.

[7] For example, http://www.shodor.org/unchem/math/lls/leastsq.html

2. Enter the output (temperature) in List 2 (L2). See Table 2.

L1	44	35	20.4	33	31	35	18.5	37	26
L2	80.5	70.5	57	66	68	72	52	73.5	53

Table 2

3. On a graphing utility, select Linear Regression (LinReg). Using the cricket chirp data from earlier, with technology we obtain the equation:

$$T(c) = 30.281 + 1.143c$$

Analysis

Notice that this line is quite similar to the equation we “eyeballed” but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

$$\begin{aligned} T(30) &= 30.281 + 1.143(30) \\ &= 64.571 \\ &\approx 64.6 \ \text{ degrees} \end{aligned}$$

The graph of the scatter plot with the least squares regression line is shown in Figure 6.

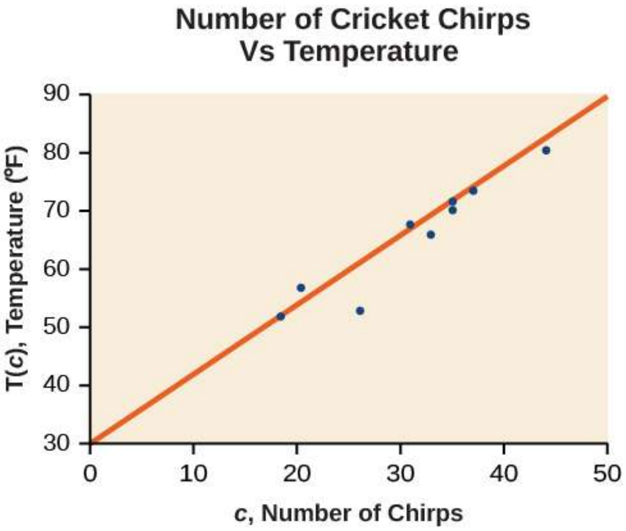


Figure 6

Q&A **Will there ever be a case where two different lines will serve as the best fit for the data?**

No. There is only one best fit line.

Distinguishing Between Linear and Nonlinear Models

As we saw above with the cricket-chirp model, some data exhibit strong linear trends, but other data, like the final exam scores plotted by age, are clearly nonlinear. Most calculators and computer software can also provide us with the correlation coefficient, which is a measure of how closely the line fits the data. Many graphing calculators require the user to turn a "diagnostic on" selection to find the correlation coefficient, which mathematicians label as r The correlation coefficient provides an easy way to get an idea of how close to a line the data falls.

We should compute the correlation coefficient only for data that follows a linear pattern or to determine the degree to which a data set is linear. If the data exhibits a nonlinear pattern, the correlation coefficient for a linear regression is meaningless. To get a sense for the relationship between the value of r and the graph of the data, Figure 7 shows some large data sets with their correlation coefficients. Remember, for all plots, the horizontal axis shows the input and the vertical axis shows the output.

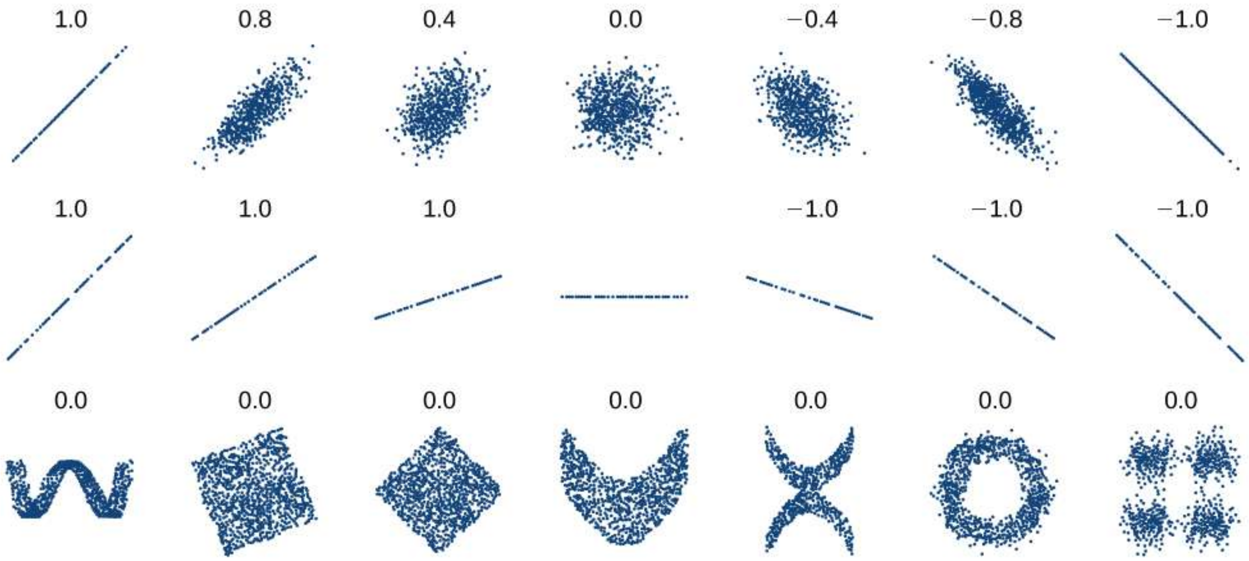


Figure 7 Plotted data and related correlation coefficients. (credit: "DenisBoigelot," Wikimedia Commons)

Correlation Coefficient

The **correlation coefficient** is a value, r, between –1 and 1.

- $r > 0$ suggests a positive (increasing) relationship
- $r < 0$ suggests a negative (decreasing) relationship
- The closer the value is to 0, the more scattered the data.
- The closer the value is to 1 or –1, the less scattered the data is.

EXAMPLE 5

Finding a Correlation Coefficient

Calculate the correlation coefficient for cricket-chirp data in Table 1.

Solution

Because the data appear to follow a linear pattern, we can use technology to calculate r Enter the inputs and corresponding outputs and select the Linear Regression. The calculator will also provide you with the correlation coefficient, $r = 0.9509$. This value is very close to 1, which suggests a strong increasing linear relationship.

Note: For some calculators, the Diagnostics must be turned "on" in order to get the correlation coefficient when linear regression is performed: [2nd]>[0]>[alpha][x–1], then scroll to **DIAGNOSTICSON**.

Fitting a Regression Line to a Set of Data

Once we determine that a set of data is linear using the correlation coefficient, we can use the regression line to make predictions. As we learned above, a regression line is a line that is closest to the data in the scatter plot, which means that only one such line is a best fit for the data.

EXAMPLE 6

Using a Regression Line to Make Predictions

Gasoline consumption in the United States has been steadily increasing. Consumption data from 1994 to 2004 is shown in Table 3.[8] Determine whether the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

---

[8] http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billions of gallons)	113	116	118	119	123	125	126	128	131	133	136

Table 3

The scatter plot of the data, including the least squares regression line, is shown in Figure 8.

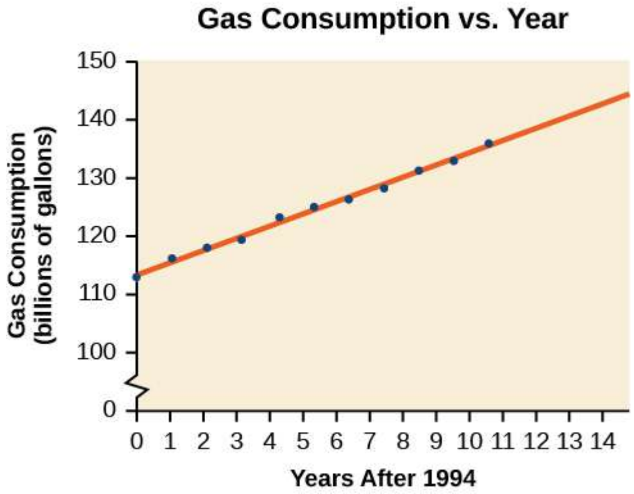


Figure 8

Solution

We can introduce a new input variable, t, representing years since 1994.

The least squares regression equation is:

$$C(t) = 113.318 + 2.209t$$

Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.

Using this to predict consumption in 2008 $(t = 14)$,

$$\begin{aligned} C(14) &= 113.318 + 2.209(14) \\ &= 144.244 \end{aligned}$$

The model predicts 144.244 billion gallons of gasoline consumption in 2008.

TRY IT #2 Use the model we created using technology in Example 6 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

MEDIA

Access these online resources for additional instruction and practice with fitting linear models to data.

Introduction to Regression Analysis (http://openstax.org/l/introregress)
Linear Regression (http://openstax.org/l/linearregress)

4.3 SECTION EXERCISES

Verbal

1. Describe what it means if there is a model breakdown when using a linear model.

2. What is interpolation when using a linear model?

3. What is extrapolation when using a linear model?

4. Explain the difference between a positive and a negative correlation coefficient.

5. Explain how to interpret the absolute value of a correlation coefficient.

Algebraic

6. A regression was run to determine whether there is a relationship between hours of TV watched per day (x) and number of sit-ups a person can do (y). The results of the regression are given below. Use this to predict the number of sit-ups a person who watches 11 hours of TV can do.

$$\begin{aligned} y &= ax + b \\ a &= -1.341 \\ b &= 32.234 \\ r &= -0.896 \end{aligned}$$

7. A regression was run to determine whether there is a relationship between the diameter of a tree (x , in inches) and the tree's age (y , in years). The results of the regression are given below. Use this to predict the age of a tree with diameter 10 inches.

$$\begin{aligned} y &= ax + b \\ a &= 6.301 \\ b &= -1.044 \\ r &= -0.970 \end{aligned}$$

For the following exercises, draw a scatter plot for the data provided. Does the data appear to be linearly related?

8.

0	2	4	6	8	10
–22	–19	–15	–11	–6	–2

9.

1	2	3	4	5	6
46	50	59	75	100	136

10.

100	250	300	450	600	750
12	12.6	13.1	14	14.5	15.2

11.

1	3	5	7	9	11
1	9	28	65	125	216

12. For the following data, draw a scatter plot. If we wanted to know when the population would reach 15,000, would the answer involve interpolation or extrapolation? Eyeball the line, and estimate the answer.

Year	Population
1990	11,500
1995	12,100
2000	12,700
2005	13,000
2010	13,750

13. For the following data, draw a scatter plot. If we wanted to know when the temperature would reach 28°F, would the answer involve interpolation or extrapolation? Eyeball the line and estimate the answer.

Temperature,°F	16	18	20	25	30
Time, seconds	46	50	54	55	62

Graphical

For the following exercises, match each scatterplot with one of the four specified correlations in Figure 9 and Figure 10.

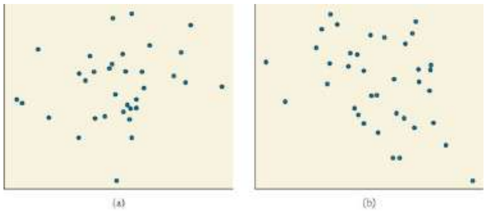


Figure 9

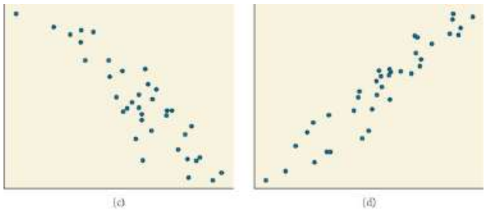


Figure 10

14. $r = 0.95$

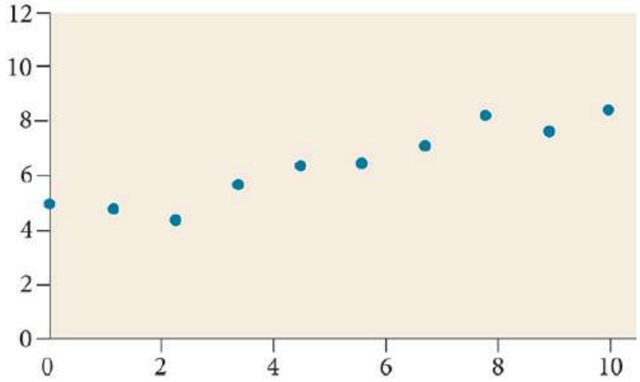
15. $r = -0.89$

16. $r = -0.26$

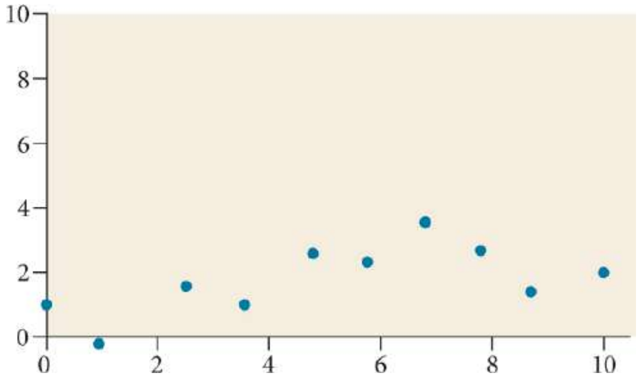
17. $r = -0.39$

For the following exercises, draw a best-fit line for the plotted data.

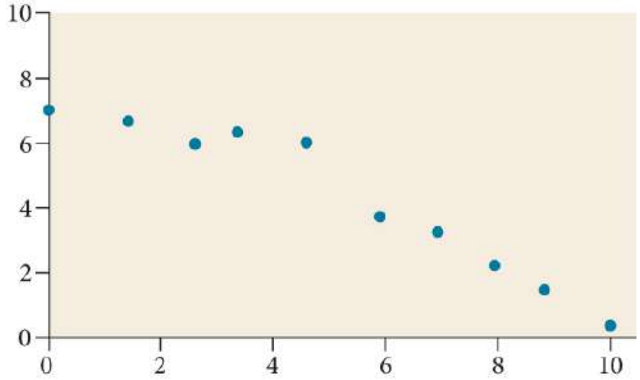
18.



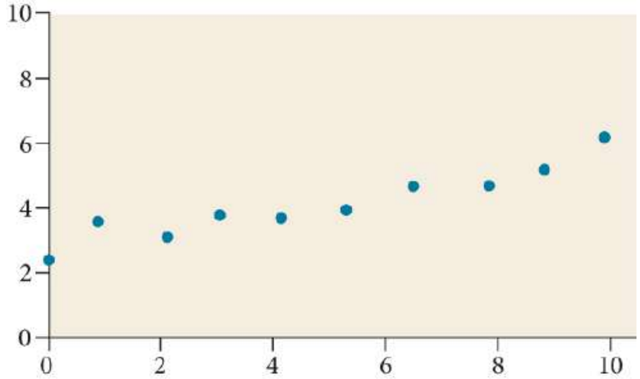
19.



20.



21.



Numeric

22. The U.S. Census tracks the percentage of persons 25 years or older who are college graduates. That data for several years is given in Table 4.[9] Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the percentage exceed 35%?

Year	Percent Graduates
1990	21.3
1992	21.4
1994	22.2
1996	23.6
1998	24.4
2000	25.6
2002	26.7
2004	27.7
2006	28
2008	29.4

Table 4

23. The U.S. import of wine (in hectoliters) for several years is given in Table 5. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will imports exceed 12,000 hectoliters?

Year	Imports
1992	2665
1994	2688
1996	3565
1998	4129
2000	4584
2002	5655
2004	6549
2006	7950
2008	8487
2009	9462

Table 5

24. Table 6 shows the year and the number of people unemployed in a particular city for several years. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the number of unemployed reach 5?

Year	Number Unemployed
1990	750
1992	670
1994	650
1996	605
1998	550
2000	510
2002	460
2004	420
2006	380
2008	320

Table 6

Technology

For the following exercises, use each set of data to calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to 3 decimal places of accuracy.

25.

x	8	15	26	31	56
y	23	41	53	72	103

26.

x	5	7	10	12	15
y	4	12	17	22	24

---

[9] Based on data from http://www.census.gov/hhes/socdemo/education/data/cps/historical/index.html. Accessed 5/1/2014.

27.

x	y	x	y
3	21.9	10	18.54
4	22.22	11	15.76
5	22.74	12	13.68
6	22.26	13	14.1
7	20.78	14	14.02
8	17.6	15	11.94
9	16.52	16	12.76

28.

x	y
4	44.8
5	43.1
6	38.8
7	39
8	38
9	32.7
10	30.1
11	29.3
12	27
13	25.8

29.

x	21	25	30	31	40	50
y	17	11	2	–1	–18	–40

30.

x	y
100	2000
80	1798
60	1589
55	1580
40	1390
20	1202

31.

x	900	988	1000	1010	1200	1205
y	70	80	82	84	105	108

Extensions

32. Graph $f(x) = 0.5x + 10$. Pick a set of five ordered pairs using inputs $x = -2, 1, 5, 6, 9$ and use linear regression to verify that the function is a good fit for the data.

33. Graph $f(x) = -2x - 10$. Pick a set of five ordered pairs using inputs $x = -2, 1, 5, 6, 9$ and use linear regression to verify the function.

For the following exercises, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs shows dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span, (number of units sold, profit) for specific recorded years:

$(46, 1{,}600), (48, 1{,}550), (50, 1{,}505), (52, 1{,}540), (54, 1{,}495).$

34. Use linear regression to determine a function P where the profit in thousands of dollars depends on the number of units sold in hundreds.

35. Find to the nearest tenth and interpret the *x*-intercept.

36. Find to the nearest tenth and interpret the *y*-intercept.

Real-World Applications

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs shows the population and the year over the ten-year span, (population, year) for specific recorded years:

$(2500, 2000), (2650, 2001), (3000, 2003), (3500, 2006), (4200, 2010)$

37. Use linear regression to determine a function y, where the year depends on the population. Round to three decimal places of accuracy.

38. Predict when the population will hit 8,000.

For the following exercises, consider this scenario: The profit of a company increased steadily over a ten-year span. The following ordered pairs show the number of units sold in hundreds and the profit in thousands of over the ten year span, (number of units sold, profit) for specific recorded years:

$(46, 250), (48, 305), (50, 350), (52, 390), (54, 410).$

39. Use linear regression to determine a function *y*, where the profit in thousands of dollars depends on the number of units sold in hundreds.

40. Predict when the profit will exceed one million dollars.

For the following exercises, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs show dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span (number of units sold, profit) for specific recorded years:

$(46, 250), (48, 225), (50, 205), (52, 180), (54, 165).$

41. Use linear regression to determine a function *y*, where the profit in thousands of dollars depends on the number of units sold in hundreds.

42. Predict when the profit will dip below the $25,000 threshold.

Chapter Review

Key Terms

correlation coefficient a value, r, between –1 and 1 that indicates the degree of linear correlation of variables, or how closely a regression line fits a data set.

decreasing linear function a function with a negative slope: If $f(x) = mx + b,\ \text{then } m < 0$

extrapolation predicting a value outside the domain and range of the data

horizontal line a line defined by $f(x) = b$, where b is a real number. The slope of a horizontal line is 0.

increasing linear function a function with a positive slope: If $f(x) = mx + b,\ \text{then } m > 0.$

interpolation predicting a value inside the domain and range of the data

least squares regression a statistical technique for fitting a line to data in a way that minimizes the differences between the line and data values

linear function a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

model breakdown when a model no longer applies after a certain point

parallel lines two or more lines with the same slope

perpendicular lines two lines that intersect at right angles and have slopes that are negative reciprocals of each other

point-slope form the equation for a line that represents a linear function of the form $y - y_1 = m(x - x_1)$

slope the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form the equation for a line that represents a linear function in the form $f(x) = mx + b$

vertical line a line defined by $x = a$, where a is a real number. The slope of a vertical line is undefined.

Key Concepts

4.1 Linear Functions

- Linear functions can be represented in words, function notation, tabular form, and graphical form. See Example 1.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope. A decreasing linear function results in a graph that slants downward from left to right and has a negative slope. A constant linear function results in a graph that is a horizontal line. See Example 2.
- Slope is a rate of change. The slope of a linear function can be calculated by dividing the difference between *y*-values by the difference in corresponding *x*-values of any two points on the line. See Example 3 and Example 4.
- An equation for a linear function can be written from a graph. See Example 5.
- The equation for a linear function can be written if the slope m and initial value b are known. See Example 6 and Example 7.
- A linear function can be used to solve real-world problems given information in different forms. See Example 8, Example 9, and Example 10.
- Linear functions can be graphed by plotting points or by using the *y*-intercept and slope. See Example 11 and Example 12.
- Graphs of linear functions may be transformed by using shifts up, down, left, or right, as well as through stretches, compressions, and reflections. See Example 13.
- The equation for a linear function can be written by interpreting the graph. See Example 14.
- The *x*-intercept is the point at which the graph of a linear function crosses the *x*-axis. See Example 15.
- Horizontal lines are written in the form, $f(x) = b$. See Example 16.
- Vertical lines are written in the form, $x = b$. See Example 17.
- Parallel lines have the same slope. Perpendicular lines have negative reciprocal slopes, assuming neither is vertical. See Example 18.
- A line parallel to another line, passing through a given point, may be found by substituting the slope value of the line and the *x*- and *y*-values of the given point into the equation, $f(x) = mx + b$, and using the b that results. Similarly, the point-slope form of an equation can also be used. See Example 19.
- A line perpendicular to another line, passing through a given point, may be found in the same manner, with the exception of using the negative reciprocal slope. See Example 20 and Example 21.

4.2 Modeling with Linear Functions

- We can use the same problem strategies that we would use for any type of function.
- When modeling and solving a problem, identify the variables and look for key values, including the slope and *y*-intercept. See Example 1.
- Draw a diagram, where appropriate. See Example 2 and Example 3.
- Check for reasonableness of the answer.

- Linear models may be built by identifying or calculating the slope and using the y-intercept.
  - The x-intercept may be found by setting $y = 0$, which is setting the expression $mx + b$ equal to 0.
  - The point of intersection of a system of linear equations is the point where the x- and y-values are the same. See Example 4.
  - A graph of the system may be used to identify the points where one line falls below (or above) the other line.

4.3 Fitting Linear Models to Data

- Scatter plots show the relationship between two sets of data. See Example 1.
- Scatter plots may represent linear or non-linear models.
- The line of best fit may be estimated or calculated, using a calculator or statistical software. See Example 2.
- Interpolation can be used to predict values inside the domain and range of the data, whereas extrapolation can be used to predict values outside the domain and range of the data. See Example 3.
- The correlation coefficient, r, indicates the degree of linear relationship between data. See Example 4.
- A regression line best fits the data. See Example 5.
- The least squares regression line is found by minimizing the squares of the distances of points from a line passing through the data and may be used to make predictions regarding either of the variables. See Example 6.

Exercises

Review Exercises

Linear Functions

1. Determine whether the algebraic equation is linear. $2x + 3y = 7$

2. Determine whether the algebraic equation is linear. $6x^2 - y = 5$

3. Determine whether the function is increasing or decreasing.

$f(x) = 7x - 2$

4. Determine whether the function is increasing or decreasing.

$g(x) = -x + 2$

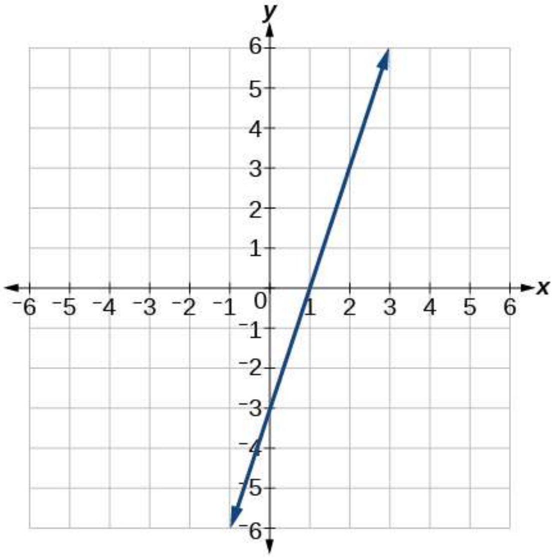
5. Given each set of information, find a linear equation that satisfies the given conditions, if possible.

Passes through $(7, 5)$ and $(3, 17)$

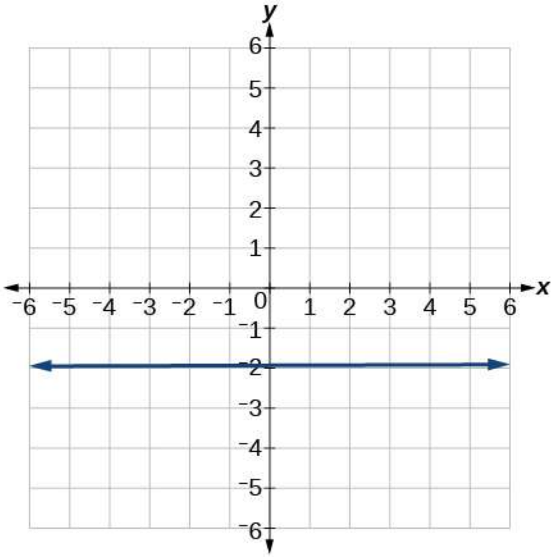
6. Given each set of information, find a linear equation that satisfies the given conditions, if possible.

x-intercept at $(6, 0)$ and y-intercept at $(0, 10)$

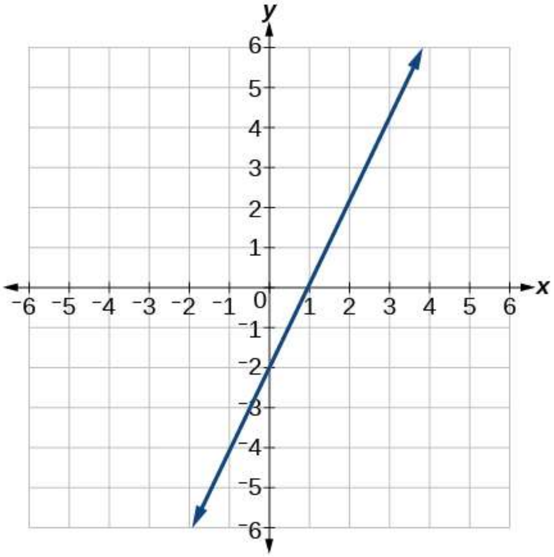
7. Find the slope of the line shown in the graph.



8. Find the slope of the line graphed.



9. Write an equation in slope-intercept form for the line shown.



10. Does the following table represent a linear function? If so, find the linear equation that models the data.

x	–4	0	2	10
g(x)	18	–2	–12	–52

11. Does the following table represent a linear function? If so, find the linear equation that models the data.

x	6	8	12	26
g(x)	–8	–12	–18	–46

12. On June 1st, a company has $4,000,000 profit. If the company then loses 150,000 dollars per day thereafter in the month of June, what is the company's profit *n*th day after June 1st?

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

13. $$\begin{gathered}2x-6y=12\\-x+3y=1\end{gathered}$$

14. $$\begin{gathered}y=\frac{1}{3}x-2\\3x+y=-9\end{gathered}$$

For the following exercises, find the x- and y- intercepts of the given equation

15. $7x+9y=-63$

16. $f(x)=2x-1$

For the following exercises, use the descriptions of the pairs of lines to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

17. Line 1: Passes through $(5,11)$ and $(10,1)$

Line 2: Passes through $(-1,3)$ and $(-5,11)$

18. Line 1: Passes through $(8,-10)$ and $(0,-26)$

Line 2: Passes through $(2,5)$ and $(4,4)$

19. Write an equation for a line perpendicular to $f(x)=5x-1$ and passing through the point (5, 20).

20. Find the equation of a line with a y- intercept of $(0, 2)$ and slope $-\frac{1}{2}$.

21. Sketch a graph of the linear function $f(t) = 2t - 5$.

22. Find the point of intersection for the 2 linear functions: $\begin{array}{c} x = y + 6 \\ 2x - y = 13 \end{array}$.

23. A car rental company offers two plans for renting a car.

Plan A: 25 dollars per day and 10 cents per mile

Plan B: 50 dollars per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

Modeling with Linear Functions

24. Find the area of a triangle bounded by the y axis, the line $f(x) = 10 - 2x$, and the line perpendicular to f that passes through the origin.

25. A town's population increases at a constant rate. In 2010 the population was 55,000. By 2012 the population had increased to 76,000. If this trend continues, predict the population in 2016.

26. The number of people afflicted with the common cold in the winter months dropped steadily by 50 each year since 2004 until 2010. In 2004, 875 people were inflicted.

Find the linear function that models the number of people afflicted with the common cold C as a function of the year, t. When will no one be afflicted?

For the following exercises, use the graph in Figure 1 showing the profit, y, in thousands of dollars, of a company in a given year, x, where x represents years since 1980.

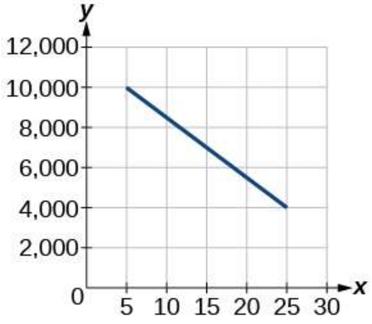


Figure 1

27. Find the linear function y, where y depends on x, the number of years since 1980.

28. Find and interpret the y-intercept.

For the following exercise, consider this scenario: In 2004, a school population was 1,700. By 2012 the population had grown to 2,500.

29. Assume the population is changing linearly.

ⓐ How much did the population grow between the year 2004 and 2012?
ⓑ What is the average population growth per year?
ⓒ Find an equation for the population, P, of the school t years after 2004.

For the following exercises, consider this scenario: In 2000, the moose population in a park was measured to be 6,500. By 2010, the population was measured to be 12,500. Assume the population continues to change linearly.

30. Find a formula for the moose population, P.

31. What does your model predict the moose population to be in 2020?

For the following exercises, consider this scenario: The median home values in subdivisions Pima Central and East Valley (adjusted for inflation) are shown in Table 1. Assume that the house values are changing linearly.

Year	Pima Central	East Valley
1970	32,000	120,250
2010	85,000	150,000

Table 1

32. In which subdivision have home values increased at a higher rate?

33. If these trends were to continue, what would be the median home value in Pima Central in 2015?

Fitting Linear Models to Data

34. Draw a scatter plot for the data in Table 2. Then determine whether the data appears to be linearly related.

0	-105
2	-50
4	1
6	55
8	105
10	160

Table 2

35. Draw a scatter plot for the data in Table 3. If we wanted to know when the population would reach 15,000, would the answer involve interpolation or extrapolation?

Year	Population
1990	5,600
1995	5,950
2000	6,300
2005	6,600
2010	6,900

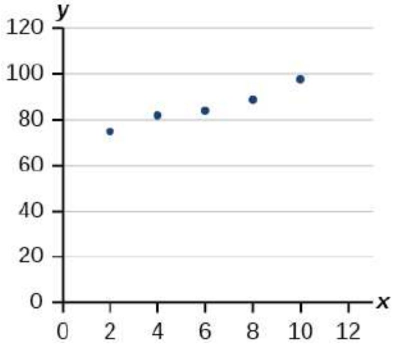
Table 3

36. Eight students were asked to estimate their score on a 10-point quiz. Their estimated and actual scores are given in Table 4. Plot the points, then sketch a line that fits the data.

Predicted	Actual
6	6
7	7
7	8
8	8
7	9
9	10
10	10
10	9

Table 4

37. Draw a best-fit line for the plotted data.



For the following exercises, consider the data in Table 5, which shows the percent of unemployed in a city of people 25 years or older who are college graduates is given below, by year.

Year	2000	2002	2005	2007	2010
Percent Graduates	6.5	7.0	7.4	8.2	9.0

Table 5

38. Determine whether the trend appears to be linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

39. In what year will the percentage exceed 12%?

40. Based on the set of data given in Table 6, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

x	17	20	23	26	29
y	15	25	31	37	40

Table 6

41. Based on the set of data given in Table 7, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

x	10	12	15	18	20
y	36	34	30	28	22

Table 7

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs show the population and the year over the ten-year span (population, year) for specific recorded years:

$(3{,}600, 2000); (4{,}000, 2001); (4{,}700, 2003); (6{,}000, 2006)$

42. Use linear regression to determine a function y, where the year depends on the population, to three decimal places of accuracy.

43. Predict when the population will hit 12,000.

44. What is the correlation coefficient for this model to three decimal places of accuracy?

45. According to the model, what is the population in 2014?

Practice Test

1. Determine whether the following algebraic equation can be written as a linear function. $2x + 3y = 7$

2. Determine whether the following function is increasing or decreasing. $f(x) = -2x + 5$

3. Determine whether the following function is increasing or decreasing. $f(x) = 7x + 9$

4. Find a linear equation that passes through (5, 1) and (3, –9), if possible.

5. Find a linear equation, that has an *x* intercept at (–4, 0) and a *y*-intercept at (0, –6), if possible.

6. Find the slope of the line in Figure 1.

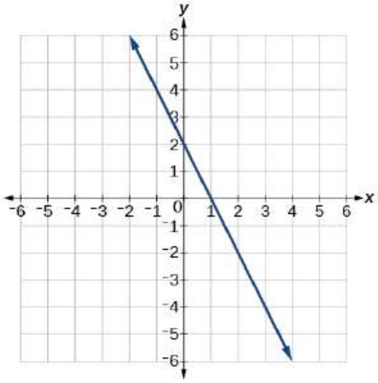


Figure 1

7. Write an equation for line in Figure 2.

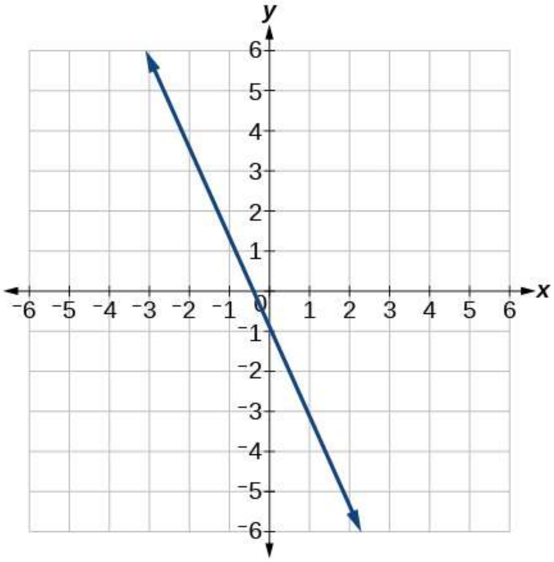


Figure 2

8. Does Table 1 represent a linear function? If so, find a linear equation that models the data.

x	–6	0	2	4
$g(x)$	14	32	38	44

Table 1

9. Does Table 2 represent a linear function? If so, find a linear equation that models the data.

x	1	3	7	11
***g*(*x*)**	4	9	19	12

Table 2

10. At 6 am, an online company has sold 120 items that day. If the company sells an average of 30 items per hour for the remainder of the day, write an expression to represent the number of items that were sold n after 6 am.

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular.

11. $\begin{array}{r} y = \frac{3}{4}x - 9 \\ -4x - 3y = 8 \end{array}$

12. $\begin{array}{r} -2x + y = 3 \\ 3x + \frac{3}{2}y = 5 \end{array}$

13. Find the *x*- and *y*-intercepts of the equation $2x + 7y = -14$.

14. Given below are descriptions of two lines. Find the slopes of Line 1 and Line 2. Is the pair of lines parallel, perpendicular, or neither?

Line 1: Passes through $(-2, -6)$ and $(3, 14)$

Line 2: Passes through $(2, 6)$ and $(4, 14)$

15. Write an equation for a line perpendicular to $f(x) = 4x + 3$ and passing through the point $(8, 10)$.

16. Sketch a line with a *y*-intercept of $(0, 5)$ and slope $-\frac{5}{2}$.

17. Graph of the linear function $f(x) = -x + 6$.

18. For the two linear functions, find the point of intersection: $\begin{array}{r} x = y + 2 \\ 2x - 3y = -1 \end{array}$.

19. A car rental company offers two plans for renting a car.

Plan A: $25 per day and $0.10 per mile

Plan B: $40 per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

20. Find the area of a triangle bounded by the *y* axis, the line $f\,(x) = 12 - 4x$, and the line perpendicular to f that passes through the origin.

21. A town's population increases at a constant rate. In 2010 the population was 65,000. By 2012 the population had increased to 90,000. Assuming this trend continues, predict the population in 2018.

22. The number of people afflicted with the common cold in the winter months dropped steadily by 25 each year since 2002 until 2012. In 2002, 8,040 people were inflicted. Find the linear function that models the number of people afflicted with the common cold C as a function of the year, t. When will less than 6,000 people be afflicted?

For the following exercises, use the graph in Figure 3, showing the profit, y, in thousands of dollars, of a company in a given year, x, where x represents years since 1980.

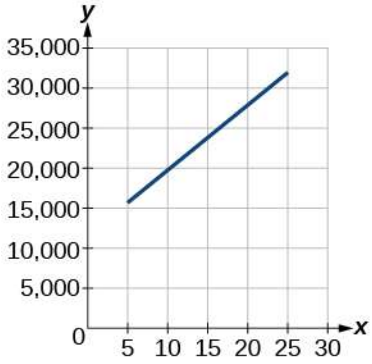


Figure 3

23. Find the linear function y, where y depends on x, the number of years since 1980.

24. Find and interpret the *y*-intercept.

25. In 2004, a school population was 1250. By 2012 the population had dropped to 875. Assume the population is changing linearly.

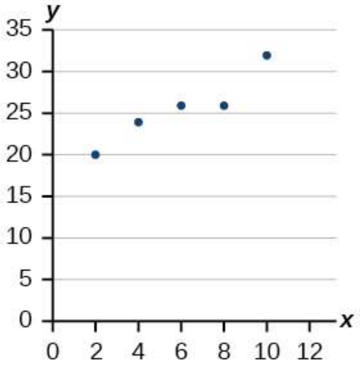
ⓐ How much did the population drop between the year 2004 and 2012?
ⓑ What is the average population decline per year?
ⓒ Find an equation for the population, *P*, of the school *t* years after 2004.

26. Draw a scatter plot for the data provided in Table 3. Then determine whether the data appears to be linearly related.

0	2	4	6	8	10
–450	–200	10	265	500	755

Table 3

27. Draw a best-fit line for the plotted data.



For the following exercises, use Table 4, which shows the percent of unemployed persons 25 years or older who are college graduates in a particular city, by year.

Year	2000	2002	2005	2007	2010
Percent Graduates	8.5	8.0	7.2	6.7	6.4

Table 4

28. Determine whether the trend appears linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

29. In what year will the percentage drop below 4%?

30. Based on the set of data given in Table 5, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient. Round to three decimal places of accuracy.

x	16	18	20	24	26
y	106	110	115	120	125

Table 5

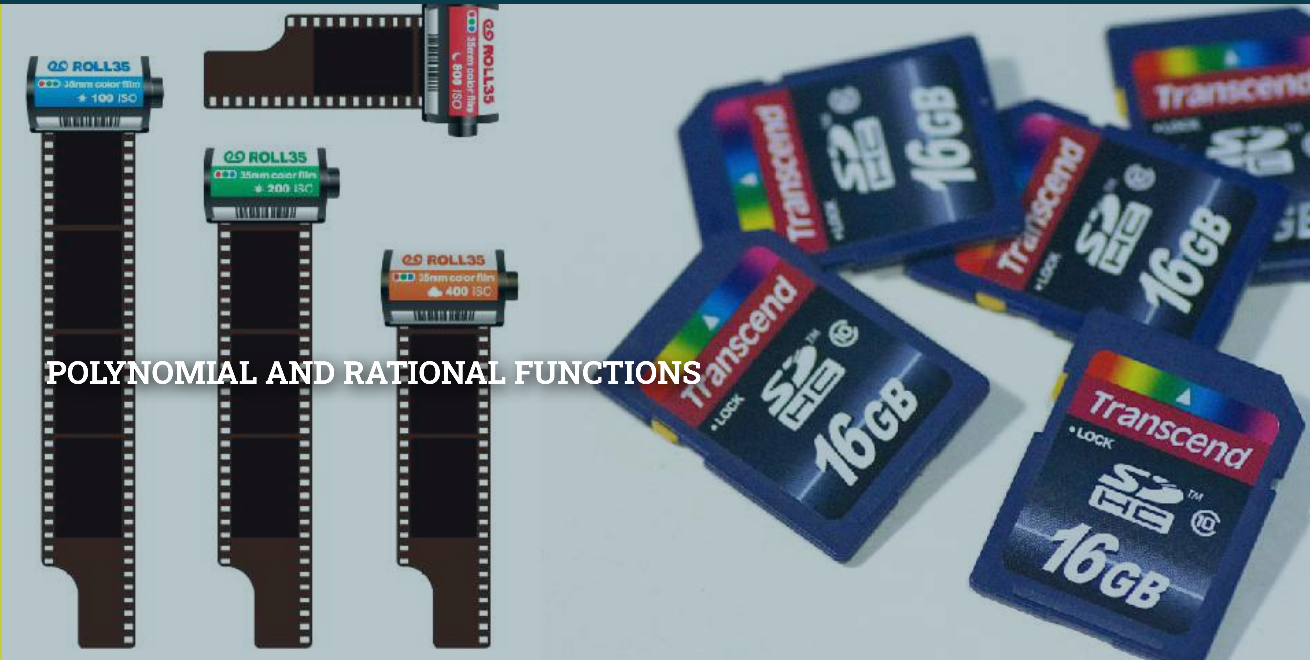
For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs shows the population (in hundreds) and the year over the ten-year span, (population, year) for specific recorded years:

$(4,500, 2000); (4,700, 2001); (5,200, 2003); (5,800, 2006)$

31. Use linear regression to determine a function *y*, where the year depends on the population. Round to three decimal places of accuracy.

32. Predict when the population will hit 20,000.

33. What is the correlation coefficient for this model?



5 POLYNOMIAL AND RATIONAL FUNCTIONS

Whether they think about it in mathematical terms or not, scuba divers must consider the impact of functional relationships in order to remain safe. The gas laws, which are a series of relations and equations that describe the behavior of most gases, play a core role in diving. This diver, near the wreck of a World War II Japanese ocean liner turned troop transport, must remain attentive to gas laws during their dive and as they ascend to the surface. (credit: "Aikoku - Aft Gun": modification of work by montereydiver/flickr)

Chapter Outline

5.1 Quadratic Functions
5.2 Power Functions and Polynomial Functions
5.3 Graphs of Polynomial Functions
5.4 Dividing Polynomials
5.5 Zeros of Polynomial Functions
5.6 Rational Functions
5.7 Inverses and Radical Functions
5.8 Modeling Using Variation

Introduction to Polynomial and Rational Functions

You don't need to dive very deep to feel the effects of pressure. As a person in their neighborhood pool moves eight, ten, twelve feet down, they often feel pain in their ears as a result of water and air pressure differentials. Pressure plays a much greater role at ocean diving depths.

id="scuban">Scuba and free divers are constantly negotiating the effects of pressure in order to experience enjoyable, safe, and productive dives. Gases in a person's respiratory system and diving apparatus interact according to certain physical properties, which upon discovery and evaluation are collectively known as the gas laws. Some are conceptually simple, such as the inverse relationship regarding pressure and volume, and others are more complex. While their formulas seem more straightforward than many you will encounter in this chapter, the gas laws are generally polynomial expressions.

5.1 Quadratic Functions

Learning Objectives

In this section, you will:

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function’s minimum or maximum value.
- Solve problems involving a quadratic function’s minimum or maximum value.

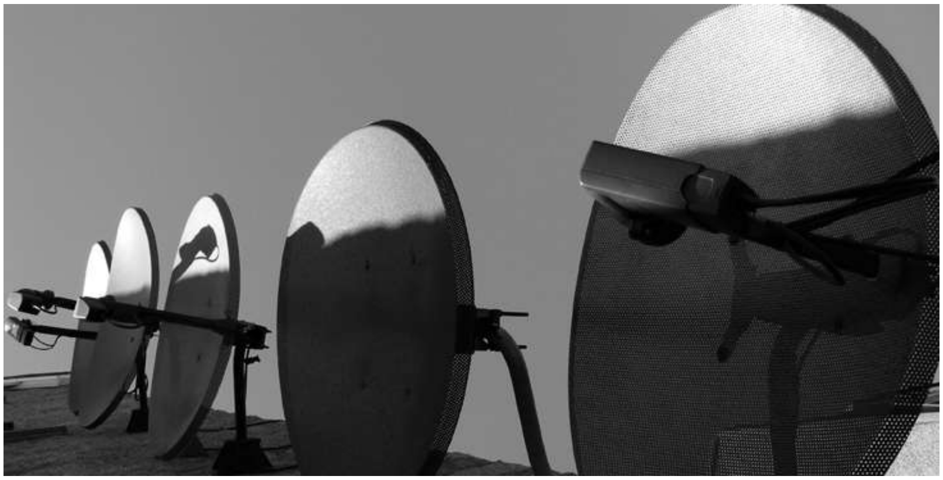


Figure 1 An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)

Curved antennas, such as the ones shown in Figure 1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 2.

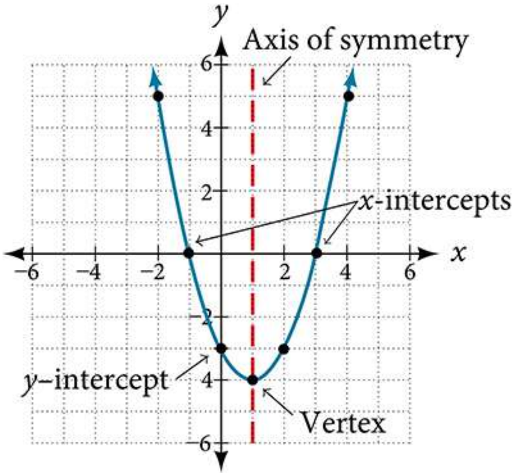


Figure 2

The *y*-intercept is the point at which the parabola crosses the *y*-axis. The *x*-intercepts are the points at which the parabola crosses the *x*-axis. If they exist, the *x*-intercepts represent the **zeros,** or **roots**, of the quadratic function, the values of x at which $y = 0$.

EXAMPLE 1

Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y- intercept of the parabola shown in Figure 3.

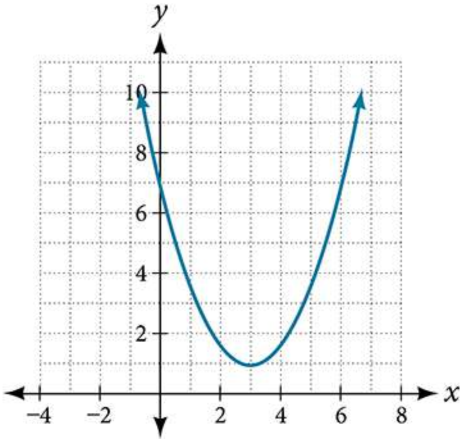


Figure 3

Solution

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x- axis, so it has no zeros. It crosses the y- axis at $(0, 7)$ so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c$$

where $a, b,$ and c are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x- intercepts, or zeros, we find the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 4 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1, b = 4,$ and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x- intercepts, those points where the parabola crosses the x- axis, occur at $(-3, 0)$ and $(-1, 0)$.

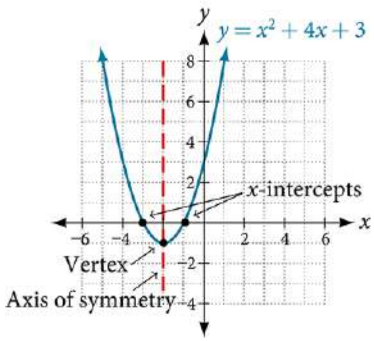


Figure 4

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k$$

where $(h, \quad k)$ is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure 5 represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x– h = x + 2$ in this example, $h = –2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, \quad 4)$.

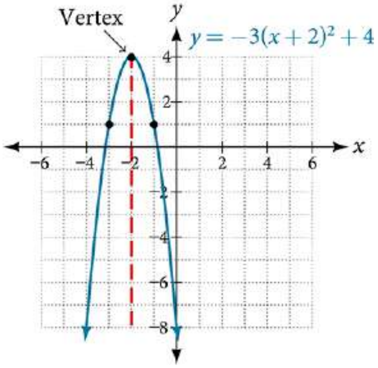


Figure 5

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure 6 is the graph of this basic function.

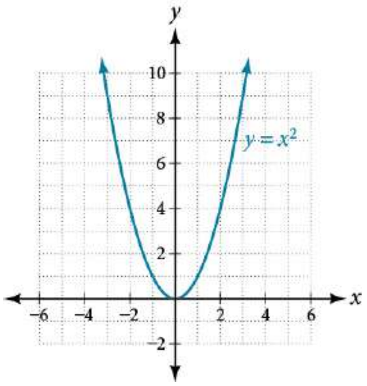


Figure 6

If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 5, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 5, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x- value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x- value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure 5, $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$\begin{aligned} a(x-h)^2 + k &= ax^2 + bx + c \\ ax^2 - 2ahx + (ah^2 + k) &= ax^2 + bx + c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$–2ah = b, \quad \text{so } h = -\frac{b}{2a}$$

This is the axis of symmetry we defined earlier. Setting the constant terms equal:

$$\begin{aligned} ah^2 + k &= c \\ k &= c - ah^2 \\ &= c - a - \left(\tfrac{b}{2a}\right)^2 \\ &= c - \tfrac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h) = k$.

Forms of Quadratic Functions

A quadratic function is a polynomial function of degree two. The graph of a quadratic function is a parabola.

The **general form of a quadratic function** is $f(x) = ax^2 + bx + c$ where a, b, and c are real numbers and $a \neq 0$.

The **standard form of a quadratic function** is $f(x) = a(x-h)^2 + k$ where $a \neq 0$.

The vertex (h, k) is located at

$$h =- \frac{b}{2a}, \quad k = f(h) = f\left(\frac{-b}{2a}\right)$$

HOW TO

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h. Identify the vertical shift of the parabola; this value is k.
2. Substitute the values of the horizontal and vertical shift for h and k. in the function $f(x) = a(x–h)^2 + k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. Expand and simplify to write in general form.

EXAMPLE 2

Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure 7 as a transformation of $f(x) = x^2$, and then expand the formula, and simplify terms to write the equation in general form.

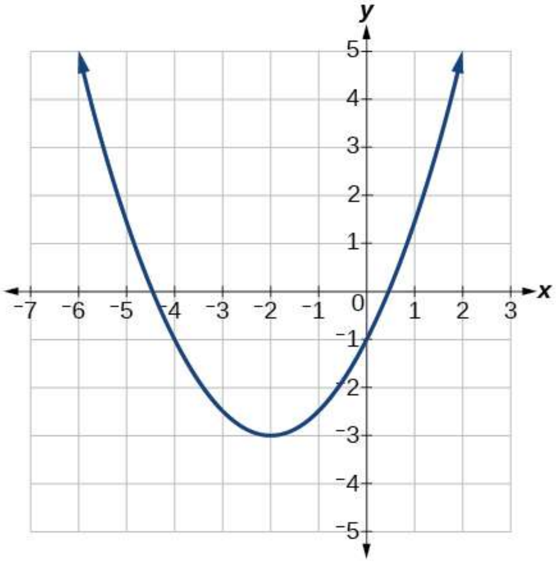


Figure 7

Solution

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x - (-2))^2 - 3 = a(x + 2)^2 - 3.$

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$\begin{array}{rcl} -1 & = & a(0+2)^2 - 3 \\ 2 & = & 4a \\ a & = & \frac{1}{2} \end{array}$$

In standard form, the algebraic model for this graph is $(g)x = \frac{1}{2}(x+2)^2 - 3.$

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{array}{rcl} g(x) & = & \frac{1}{2}(x+2)^2 - 3 \\ & = & \frac{1}{2}(x+2)(x+2) - 3 \\ & = & \frac{1}{2}(x^2 + 4x + 4) - 3 \\ & = & \frac{1}{2}x^2 + 2x + 2 - 3 \\ & = & \frac{1}{2}x^2 + 2x - 1 \end{array}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis

We can check our work using the table feature on a graphing utility. First enter $\mathrm{Y1} = \frac{1}{2}(x+2)^2 - 3$. Next, select TBLSET, then use $\mathrm{TblStart} =\!- 6$ and $\Delta\mathrm{Tbl} = 2$, and select TABLE. See Table 1.

x	−6	−4	−2	0	2
y	5	−1	−3	−1	5

Table 1

The ordered pairs in the table correspond to points on the graph.

TRY IT #1 A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 8. Find an equation for the path of the ball. Does the shooter make the basket?

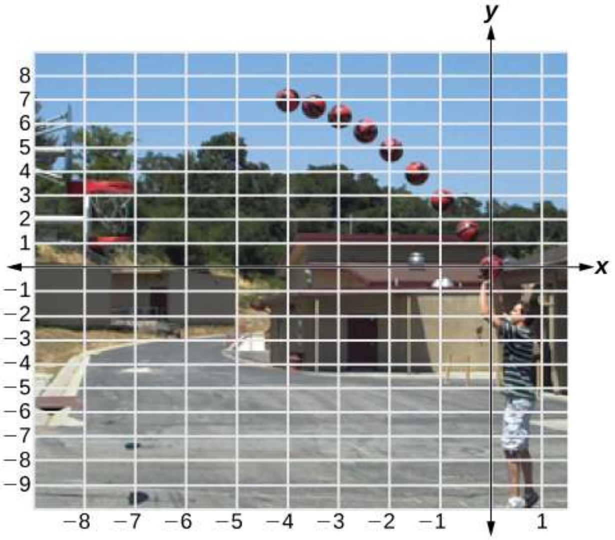


Figure 8 (credit: modification of work by Dan Meyer)

HOW TO

Given a quadratic function in general form, find the vertex of the parabola.

1. Identify $a, b,$ and c.
2. Find h, the x-coordinate of the vertex, by substituting a and b into $h =- \frac{b}{2a}$.
3. Find k, the y-coordinate of the vertex, by evaluating $k = f\left(h\right) = f\left(-\frac{b}{2a}\right)$.

EXAMPLE 3

Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-6}{2(2)} \\ &= \frac{6}{4} \\ &= \frac{3}{2} \end{aligned}$$

The vertical coordinate of the vertex will be at

$$\begin{aligned} k &= f(h) \\ &= f\left(\frac{3}{2}\right) \\ &= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \\ &= \frac{5}{2} \end{aligned}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic. First, find the horizontal coordinate of the vertex. Then find the vertical coordinate of the vertex. Substitute the values into standard form, using the " a " from the general form.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ f(x) &= 2x^2 - 6x + 7 \end{aligned}$$

The standard form of a quadratic function prior to writing the function then becomes the following:

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs, k, and where it occurs, x.

TRY IT #2 Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all *y*-values greater than or equal to the *y*-coordinate at the turning point or less than or equal to the *y*-coordinate at the turning point, depending on whether the parabola opens up or down.

Domain and Range of a Quadratic Function

The domain of any quadratic function is all real numbers unless the context of the function presents some restrictions.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $\left[f\left(-\frac{b}{2a}\right), \infty\right)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f\left(-\frac{b}{2a}\right)$, or $\left(-\infty, f\left(-\frac{b}{2a}\right)\right]$.

The range of a quadratic function written in standard form $f(x) = a(x - h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

HOW TO

Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $\left[k, \infty\right)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $\left(-\infty, k\right]$.

EXAMPLE 4

Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x- value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{9}{2(-5)} \\ &= \frac{9}{10} \end{aligned}$$

The maximum value is given by $f(h)$.

$$\begin{aligned} f\left(\frac{9}{10}\right) &= -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 \\ &= \frac{61}{20} \end{aligned}$$

The range is $f(x) \le \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

TRY IT #3 Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the parabola. We can see the maximum and minimum values in Figure 9.

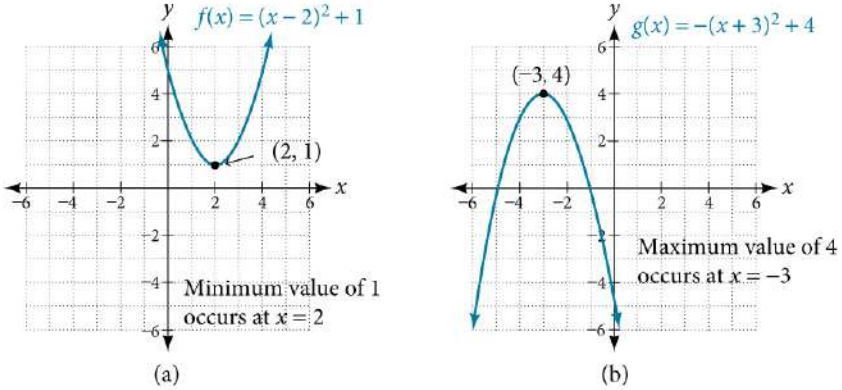


Figure 9

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

EXAMPLE 5

Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

ⓐ Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.

ⓑ What dimensions should she make her garden to maximize the enclosed area?

Solution

Let’s use a diagram such as Figure 10 to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

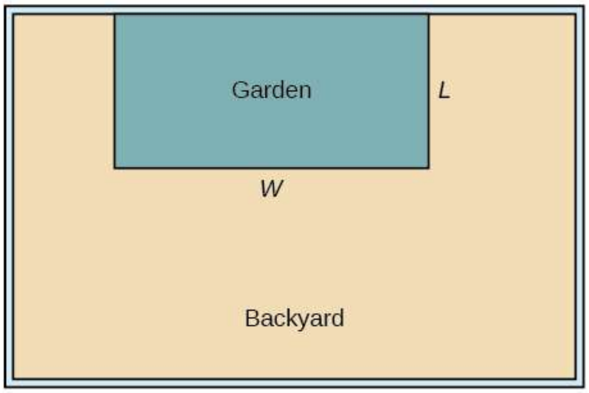


Figure 10

ⓐ We know we have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L.

$$W = 80 - 2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$\begin{aligned} A &= LW = L(80 - 2L) \\ A(L) &= 80L - 2L^2 \end{aligned}$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L) = -2L^2 + 80L.$$

ⓑ The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a = -2$, $b = 80$, and $c = 0$.

To find the vertex:

$$\begin{aligned} h &= -\frac{b}{2a} & & & k &= A(20) \\ &= -\frac{80}{2(-2)} & &\text{and} & &= 80(20) - 2(20)^2 \\ &= 20 & & & &= 800 \end{aligned}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 11.

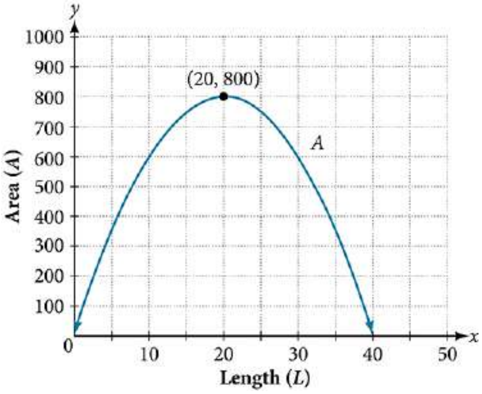


Figure 11

HOW TO

Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for a revenue function.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

EXAMPLE 6

Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

✓ **Solution**

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84{,}000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79{,}000$. From this we can find a linear equation relating the two quantities. The slope will be

$$\begin{aligned} m &= \frac{79{,}000-84{,}000}{32-30} \\ &= \frac{-5{,}000}{2} \\ &= -2{,}500 \end{aligned}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$\begin{aligned} Q &= -2500p + b && \text{Substitute in the point } Q = 84{,}000 \text{ and } p = 30 \\ 84{,}000 &= -2500(30) + b && \text{Solve for } b \\ b &= 159{,}000 \end{aligned}$$

This gives us the linear equation $Q = -2{,}500p + 159{,}000$ relating cost and subscribers. We now return to our revenue equation.

$$\begin{aligned}\text{Revenue} &= pQ \\ \text{Revenue} &= p(-2{,}500p + 159{,}000) \\ \text{Revenue} &= -2{,}500p^2 + 159{,}000p\end{aligned}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$\begin{aligned}h &= -\frac{159{,}000}{2(-2{,}500)} \\ &= 31.8\end{aligned}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\begin{aligned}\text{maximum revenue} &= -2{,}500(31.8)^2 + 159{,}000(31.8) \\ &= 2{,}528{,}100\end{aligned}$$

Analysis

This could also be solved by graphing the quadratic as in Figure 12. We can see the maximum revenue on a graph of the quadratic function.

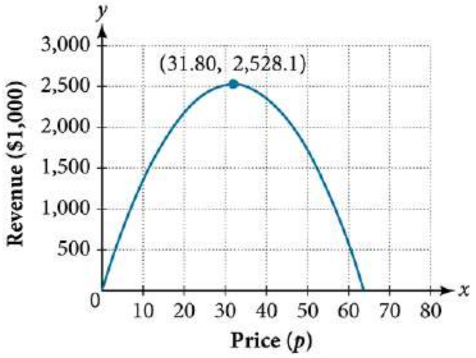


Figure 12

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y- intercept of a quadratic by evaluating the function at an input of zero, and we find the x- intercepts at locations where the output is zero. Notice in Figure 13 that the number of x- intercepts can vary depending upon the location of the graph.

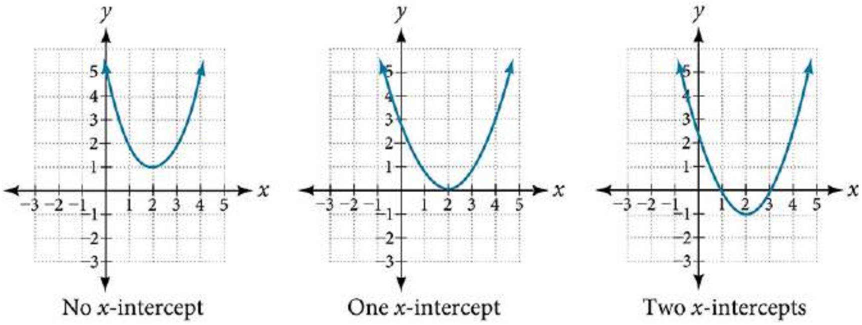


Figure 13 Number of x-intercepts of a parabola

HOW TO

Given a quadratic function $f(x)$, find the y- and x-intercepts.

1. Evaluate $f(0)$ to find the y-intercept.

2. Solve the quadratic equation $f(x) = 0$ to find the *x*-intercepts.

EXAMPLE 7

Finding the *y*- and *x*-Intercepts of a Parabola

Find the *y*- and *x*-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution

We find the *y*-intercept by evaluating $f(0)$.

$$\begin{aligned} f(0) &= 3(0)^2 + 5(0) - 2 \\ &= -2 \end{aligned}$$

So the *y*-intercept is at $(0, -2)$.

For the *x*-intercepts, we find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2)$$

So the *x*-intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.

Analysis

By graphing the function, we can confirm that the graph crosses the *y*-axis at $(0, -2)$. We can also confirm that the graph crosses the *x*-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure 14

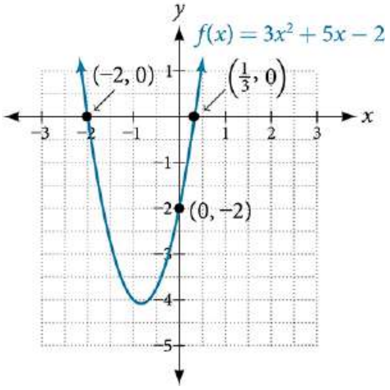


Figure 14

Rewriting Quadratics in Standard Form

In Example 7, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

HOW TO

Given a quadratic function, find the x- intercepts by rewriting in standard form.

1. Substitute a and b into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.
4. Solve for when the output of the function will be zero to find the x- intercepts.

EXAMPLE 8

Finding the *x*-Intercepts of a Parabola

Find the x- intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution

We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x - h)^2 + k$$

We know that $a = 2$. Then we solve for h and k.

$$\begin{aligned} h &= -\tfrac{b}{2a} & \qquad k &= f(-1) \\ &= -\tfrac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x + 1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x + 1)^2 - 6 \\ 6 &= 2(x + 1)^2 \\ 3 &= (x + 1)^2 \\ x + 1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has *x*-intercepts at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$.

We can check our work by graphing the given function on a graphing utility and observing the x- intercepts. See Figure 15.

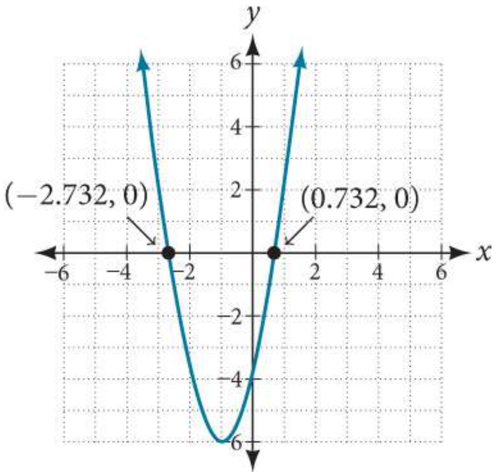


Figure 15

Analysis

We could have achieved the same results using the quadratic formula. Identify $a = 2, b = 4$ and $c = -4$.

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-4\pm\sqrt{4^2-4(2)(-4)}}{2(2)} \\ &= \frac{-4\pm\sqrt{48}}{4} \\ &= \frac{-4\pm\sqrt{3(16)}}{4} \\ &= -1 \pm \sqrt{3} \end{aligned}$$

So the *x*-intercepts occur at $\left(-1 - \sqrt{3}, 0\right)$ and $\left(-1 + \sqrt{3}, 0\right)$.

TRY IT #4 In a Try It, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the *y*- and *x*-intercepts (if any).

EXAMPLE 9

Applying the Vertex and *x*-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

ⓐ When does the ball reach the maximum height? ⓑ What is the maximum height of the ball?

ⓒ When does the ball hit the ground?

Solution

ⓐ The ball reaches the maximum height at the vertex of the parabola.

$$\begin{aligned} h &= -\frac{80}{2(-16)} \\ &= \frac{80}{32} \\ &= \frac{5}{2} \\ &= 2.5 \end{aligned}$$

The ball reaches a maximum height after 2.5 seconds.

ⓑ To find the maximum height, find the y- coordinate of the vertex of the parabola.

$$\begin{aligned} k &= H\left(-\frac{b}{2a}\right) \\ &= H\,(2.5) \\ &= -16(2.5)^2 + 80\,(2.5) + 40 \\ &= 140 \end{aligned}$$

The ball reaches a maximum height of 140 feet.

ⓒ To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$. We use the quadratic formula.

$$\begin{aligned} t &= \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \\ &= \frac{-80 \pm \sqrt{8960}}{-32} \end{aligned}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 16.

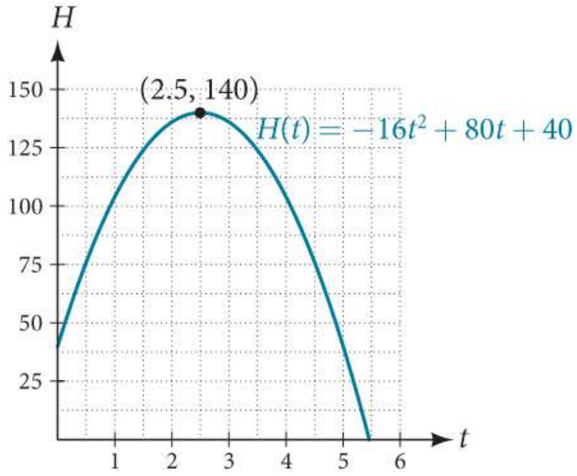


Figure 16

Note that the graph does not represent the physical path of the ball upward and downward. Keep the quantities on each axis in mind while interpreting the graph.

TRY IT #5 A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

ⓐ When does the rock reach the maximum height?

ⓑ What is the maximum height of the rock?

ⓒ When does the rock hit the ocean?

MEDIA

Access these online resources for additional instruction and practice with quadratic equations.

Graphing Quadratic Functions in General Form (http://openstax.org/l/graphquadgen)
Graphing Quadratic Functions in Standard Form (http://openstax.org/l/graphquadstan)
Quadratic Function Review (http://openstax.org/l/quadfuncrev)
Characteristics of a Quadratic Function (http://openstax.org/l/characterquad)

5.1 SECTION EXERCISES

Verbal

1. Explain the advantage of writing a quadratic function in standard form.

2. How can the vertex of a parabola be used in solving real-world problems?

3. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

4. What is another name for the standard form of a quadratic function?

5. What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

Algebraic

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

6. $f(x) = x^2 - 12x + 32$

7. $g(x) = x^2 + 2x - 3$

8. $f(x) = x^2 - x$

9. $f(x) = x^2 + 5x - 2$

10. $h(x) = 2x^2 + 8x - 10$

11. $k(x) = 3x^2 - 6x - 9$

12. $f(x) = 2x^2 - 6x$

13. $f(x) = 3x^2 - 5x - 1$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

14. $y(x) = 2x^2 + 10x + 12$

15. $f(x) = 2x^2 - 10x + 4$

16. $f(x) = -x^2 + 4x + 3$

17. $f(x) = 4x^2 + x - 1$

18. $h(t) = -4t^2 + 6t - 1$

19. $f(x) = \frac{1}{2}x^2 + 3x + 1$

20. $f(x) = -\frac{1}{3}x^2 - 2x + 3$

For the following exercises, determine the domain and range of the quadratic function.

21. $f(x) = (x - 3)^2 + 2$

22. $f(x) = -2(x + 3)^2 - 6$

23. $f(x) = x^2 + 6x + 4$

24. $f(x) = 2x^2 - 4x + 2$

25. $k(x) = 3x^2 - 6x - 9$

For the following exercises, use the vertex (h, k) and a point on the graph (x, y) to find the general form of the equation of the quadratic function.

26. $(h, k) = (2, 0), (x, y) = (4, 4)$

27. $(h, k) = (-2, -1), (x, y) = (-4, 3)$

28. $(h, k) = (0, 1), (x, y) = (2, 5)$

29. $(h, k) = (2, 3), (x, y) = (5, 12)$

30. $(h, k) = (-5, 3), (x, y) = (2, 9)$

31. $(h, k) = (3, 2), (x, y) = (10, 1)$

32. $(h, k) = (0, 1), (x, y) = (1, 0)$

33. $(h, k) = (1, 0), (x, y) = (0, 1)$

Graphical

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

34. $f(x) = x^2 - 2x$

35. $f(x) = x^2 - 6x - 1$

36. $f(x) = x^2 - 5x - 6$

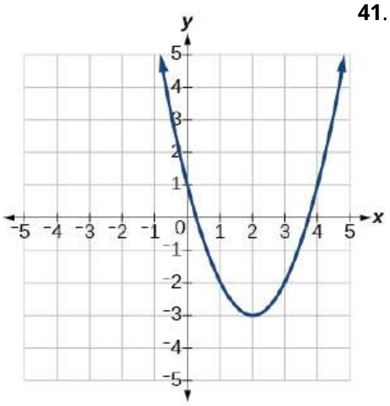
37. $f(x) = x^2 - 7x + 3$

38. $f(x) = -2x^2 + 5x - 8$

39. $f(x) = 4x^2 - 12x - 3$

For the following exercises, write the equation for the graphed quadratic function.

40.



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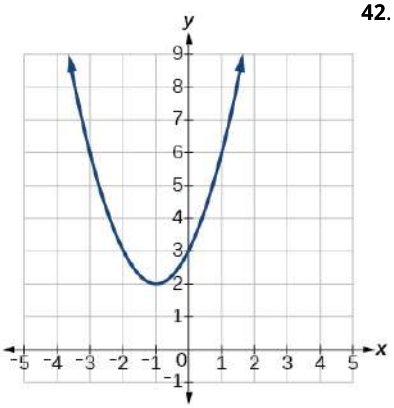

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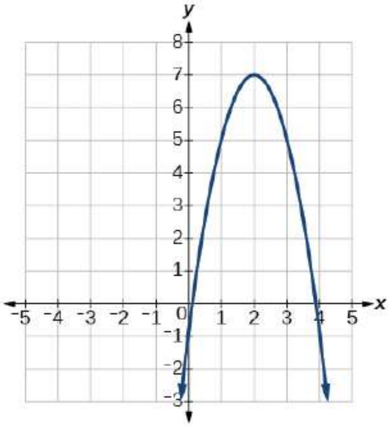

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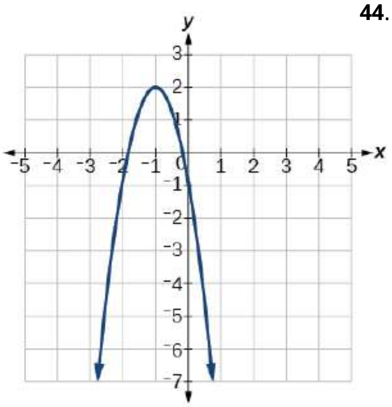

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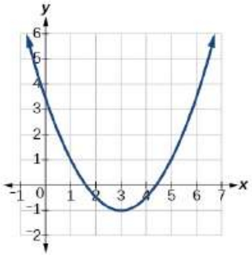

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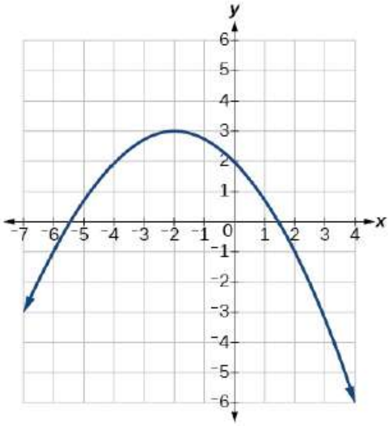


Numeric

For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

46.

x	–2	–1	0	1	2
y	5	2	1	2	5

47.

x	–2	–1	0	1	2
y	1	0	1	4	9

48.

x	–2	–1	0	1	2
y	–2	1	2	1	–2

49.

x	–2	–1	0	1	2
y	–8	–3	0	1	0

50.

x	–2	–1	0	1	2
y	8	2	0	2	8

Technology

For the following exercises, use a calculator to find the answer.

51. Graph on the same set of axes the functions $f(x) = x^2$, $f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$.

What appears to be the effect of changing the coefficient?

52. Graph on the same set of axes $f(x) = x^2$, $f(x) = x^2 + 2$ and $f(x) = x^2$, $f(x) = x^2 + 5$ and $f(x) = x^2 - 3$. What appears to be the effect of adding a constant?

53. Graph on the same set of axes $f(x) = x^2$, $f(x) = (x - 2)^2$, $f(x - 3)^2$, and $f(x) = (x + 4)^2$.

What appears to be the effect of adding or subtracting those numbers?

54. The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x) = \frac{-32}{(80)^2}x^2 + x$ where x is the horizontal distance traveled and $h\,(x)$ is the height in feet. Use the TRACE feature of your calculator to determine the height of the object when it has traveled 100 feet away horizontally.

55. A suspension bridge can be modeled by the quadratic function $h(x) = .0001x^2$ with $-2000 \leq x \leq 2000$ where $|x|$ is the number of feet from the center and $h\,(x)$ is height in feet. Use the TRACE feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

Extensions

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

56. Vertex $(1, -2)$, opens up.

57. Vertex $(-1, 2)$ opens down.

58. Vertex $(-5, 11)$, opens down.

59. Vertex $(-100, 100)$, opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

60. Contains $(1, 1)$ and has shape of $f(x) = 2x^2$. Vertex is on the y- axis.

61. Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the y- axis.

62. Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the y- axis.

63. Contains $(1,-3)$ and has the shape of $f(x)=-x^2$. Vertex is on the y- axis.

64. Contains $(4,3)$ and has the shape of $f(x)=5x^2$. Vertex is on the y- axis.

65. Contains $(1,-6)$ has the shape of $f(x)=3x^2$. Vertex has x-coordinate of -1.

Real-World Applications

66. Find the dimensions of the rectangular dog park producing the greatest enclosed area given 200 feet of fencing.

67. Find the dimensions of the rectangular dog park split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.

68. Find the dimensions of the rectangular dog park producing the greatest enclosed area split into 3 sections of the same size given 500 feet of fencing.

69. Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?

70. Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?

71. Suppose that the price per unit in dollars of a cell phone production is modeled by $p=\$45-0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R=x\cdot p$. Find the production level that will maximize revenue.

72. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t)=-4.9t^2+229t+234$. Find the maximum height the rocket attains.

73. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t)=-4.9t^2+24t+8$. How long does it take to reach maximum height?

74. A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

75. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

5.2 Power Functions and Polynomial Functions

Learning Objectives

In this section, you will:

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.

Figure 1 (credit: Jason Bay, Flickr)

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table 1.

Year	2009	2010	2011	2012	2013
Bird Population	800	897	992	$1,083$	$1,169$

Table 1

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$, where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.

Identifying Power Functions

Before we can understand the bird problem, it will be helpful to understand a different type of function. A **power function** is a function with a single term that is the product of a real number, a **coefficient,** and a variable raised to a fixed real number.

As an example, consider functions for area or volume. The function for the area of a circle with radius r is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Power Function

A **power function** is a function that can be represented in the form

$$f(x) = kx^p$$

where k and p are real numbers, and k is known as the **coefficient**.

Q&A **Is $f(x) = 2^x$ a power function?**

No. A power function contains a variable base raised to a fixed power. This function has a constant base raised to a variable power. This is called an exponential function, not a power function.

EXAMPLE 1

Identifying Power Functions

Which of the following functions are power functions?

$$\begin{array}{rcll} f(x) & = & 1 & \text{Constant function} \\ f(x) & = & x & \text{Identify function} \\ f(x) & = & x^2 & \text{Quadratic function} \\ f(x) & = & x^3 & \text{Cubic function} \\ f(x) & = & \frac{1}{x} & \text{Reciprocal function} \\ f(x) & = & \frac{1}{x^2} & \text{Reciprocal squared function} \\ f(x) & = & \sqrt{x} & \text{Square root function} \\ f(x) & = & \sqrt[3]{x} & \text{Cube root function} \end{array}$$

Solution

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The reciprocal and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and cube root functions are power functions with fractional powers because they can be written as $f(x) = x^{\frac{1}{2}}$ or $f(x) = x^{\frac{1}{3}}$.

TRY IT #1 Which functions are power functions?

$$\begin{array}{rcl} f(x) & = & 2x \cdot 4x^3 \\ g(x) & = & -x^5 + 5x^3 \\ h(x) & = & \frac{2x^5 - 1}{3x^2 + 4} \end{array}$$

Identifying End Behavior of Power Functions

Figure 2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

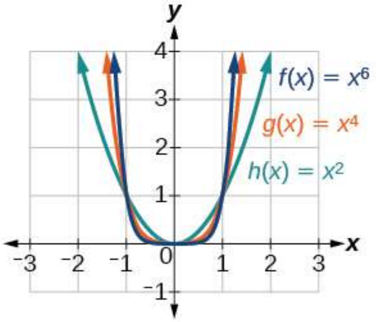


Figure 2 Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that " x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the positive even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty, \ f(x) \to \infty$$

Figure 3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

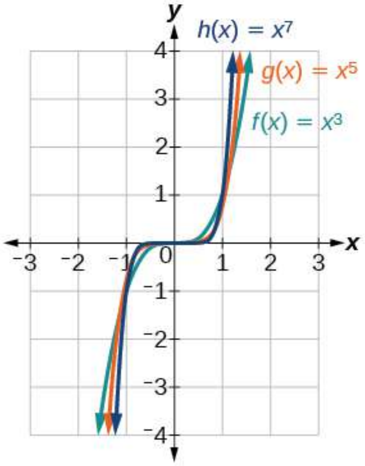


Figure 3 Odd-power functions

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 2 we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y- axis. In Figure 3 we see that odd functions of the form $f(x) = x^n$, n odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty, \ f(x) \to -\infty$$

$$\text{as } x \to \infty, \ f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small ($x \to -\infty$) and get very large ($x \to \infty$) is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.

Figure 4 shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

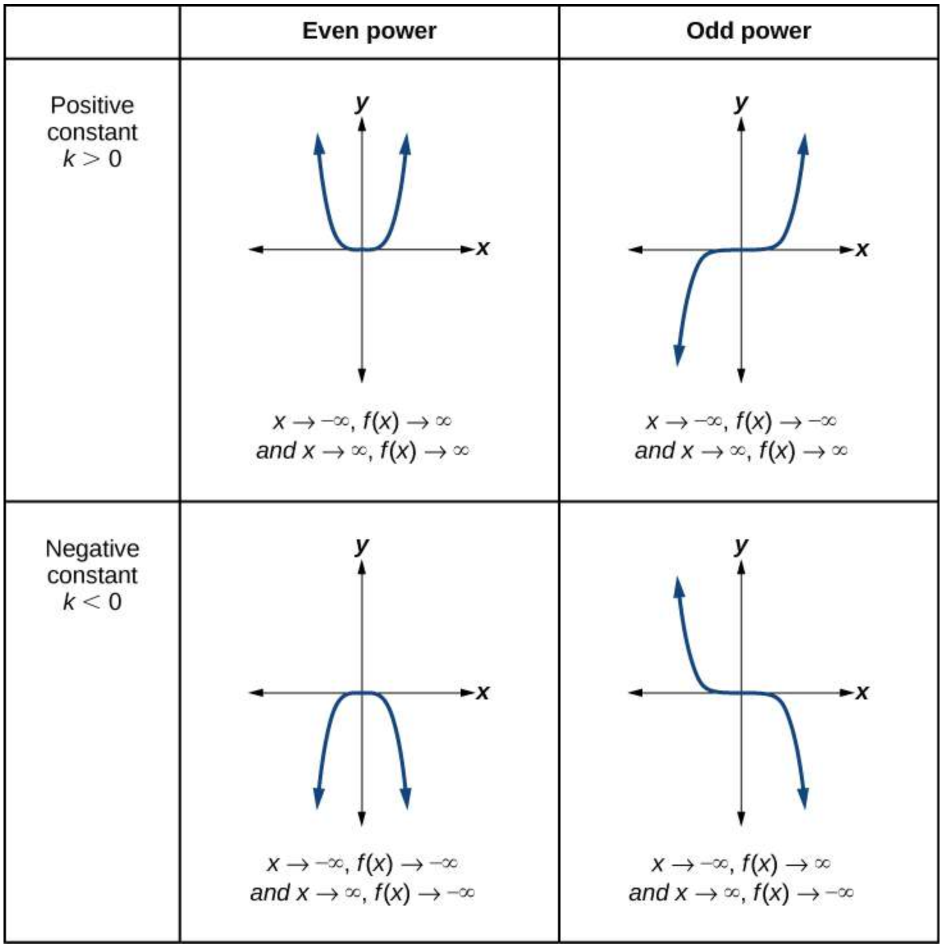


Figure 4

HOW TO

Given a power function $f(x) = kx^n$ where n is a non-negative integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure 4 to identify the end behavior.

EXAMPLE 2

Identifying the End Behavior of a Power Function

Describe the end behavior of the graph of $f(x) = x^8$.

Solution

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x \to \infty$, $f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty,\ f(x) \to \infty$. We can graphically represent the function as shown in Figure 5.

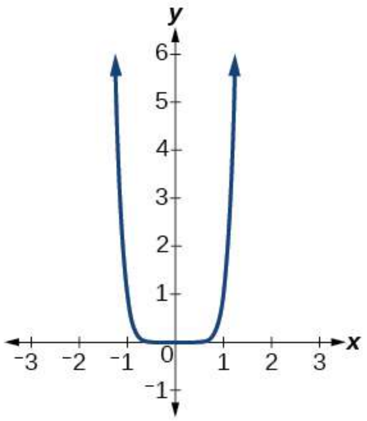


Figure 5

EXAMPLE 3

Identifying the End Behavior of a Power Function.

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution

The exponent of the power function is 9 (an odd number). Because the coefficient is -1 (negative), the graph is the reflection about the x- axis of the graph of $f(x) = x^9$. Figure 6 shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty, \; f(x) \to \infty$$

$$\text{as } x \to \infty, \; f(x) \to -\infty$$

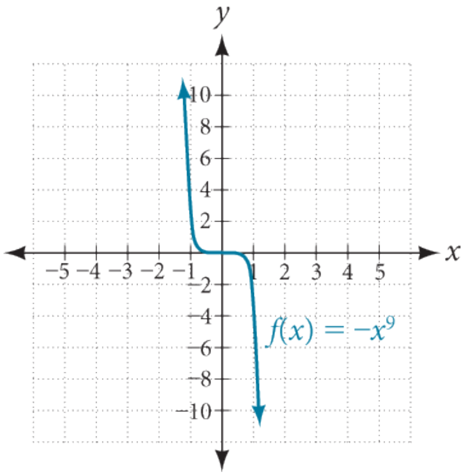


Figure 6

Analysis

We can check our work by using the table feature on a graphing utility.

x	$f(x)$
−10	1,000,000,000

Table 2

x	$f(x)$
–5	1,953,125
0	0
5	−1,953,125
10	−1,000,000,000

Table 2

We can see from Table 2 that, when we substitute very small values for x, the output is very large, and when we substitute very large values for x, the output is very small (meaning that it is a very large negative value).

TRY IT #2 Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius r of the spill depends on the number of weeks w that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a **polynomial function**. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Polynomial Functions

Let n be a non-negative integer. A **polynomial function** is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$$

This is called the general form of a polynomial function. Each a_i is a coefficient and can be any real number, but $a_n \neq$. Each expression $a_i x^i$ is a **term of a polynomial function**.

EXAMPLE 4

Identifying Polynomial Functions

Which of the following are polynomial functions?

$$\begin{aligned} f(x) &= 2x^3 \cdot 3x + 4 \\ g(x) &= -x(x^2 - 4) \\ h(x) &= 5\sqrt{x+2} \end{aligned}$$

Solution

The first two functions are examples of polynomial functions because they can be written in the form $f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0$, where the powers are non-negative integers and the coefficients are real numbers.

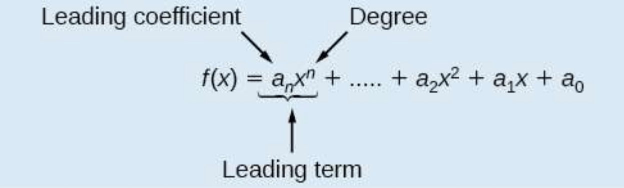
- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The **leading term** is the term containing the highest power of the variable, or the term with the highest degree. The **leading coefficient** is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.



When a polynomial is written in this way, we say that it is in general form.

HOW TO

Given a polynomial function, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree of the function.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

EXAMPLE 5

Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$\begin{aligned} f(x) &= 3 + 2x^2 - 4x^3 \\ g(t) &= 5t^5 - 2t^3 + 7t \\ h(p) &= 6p - p^3 - 2 \end{aligned}$$

Solution

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, -p^3. The leading coefficient is the coefficient of that term, -1.

TRY IT #3 Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the power function consisting of the leading term. See Table 3.

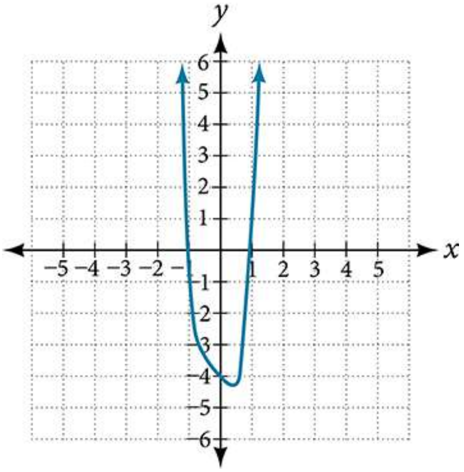
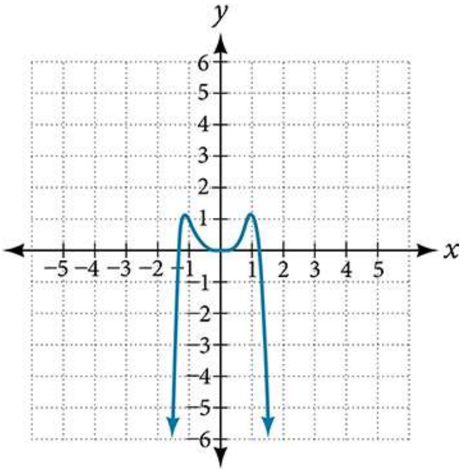
Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	

Table 3

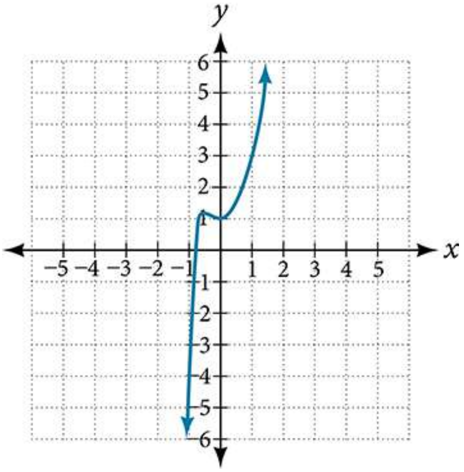
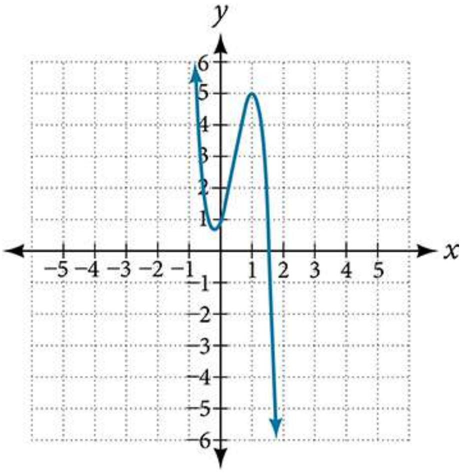
Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Table 3

EXAMPLE 6

Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 7.

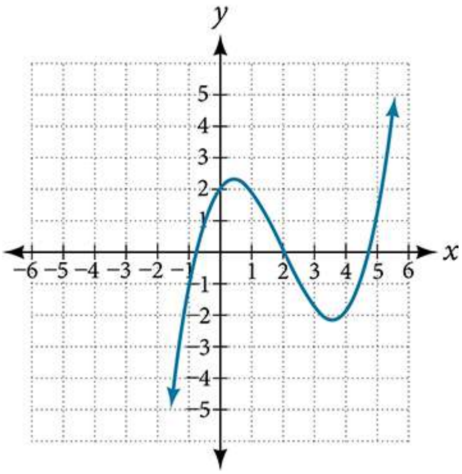


Figure 7

Solution

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

$$\text{as } x \to \infty,\ f(x) \to \infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

TRY IT #4 Describe the end behavior, and determine a possible degree of the polynomial function in Figure 8.

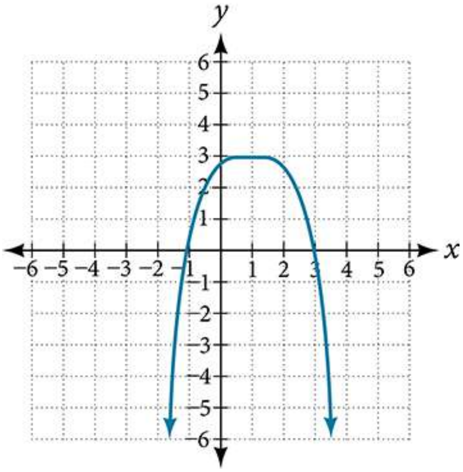


Figure 8

EXAMPLE 7

Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2\left(x^2+3x-4\right) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

$$\text{as } x \to \infty,\ f(x) \to -\infty$$

TRY IT #5 Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the “middle” of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the *y*-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one *y*-intercept $(0, a_0)$. The *x*-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one *x*-intercept. See Figure 9.

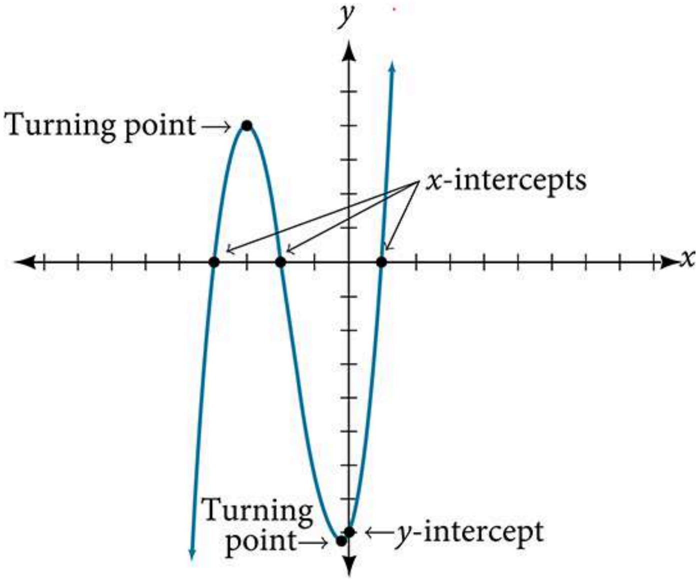


Figure 9

Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The *y*-intercept is the point at which the function has an input value of zero. The *x*-intercepts are the points at which the output value is zero.

HOW TO

Given a polynomial function, determine the intercepts.

1. Determine the *y*-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the *x*-intercepts by solving for the input values that yield an output value of zero.

EXAMPLE 8

Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the *y*- and *x*-intercepts.

Solution

The *y*-intercept occurs when the input is zero so substitute 0 for x.

$$\begin{aligned} f(0) &= f(0) = (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The *y*-intercept is (0, 8).

The *x*-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{array}{rclcrclcrcl} x-2 &=& 0 & \text{or} & x+1 &=& 0 & \text{or} & x-4 &=& 0 \\ x &=& 2 & \text{or} & x &=& -1 & \text{or} & x &=& 4 \end{array}$$

The *x*-intercepts are $(2,0), (-1,0)$, and $(4,0)$.

We can see these intercepts on the graph of the function shown in Figure 10.

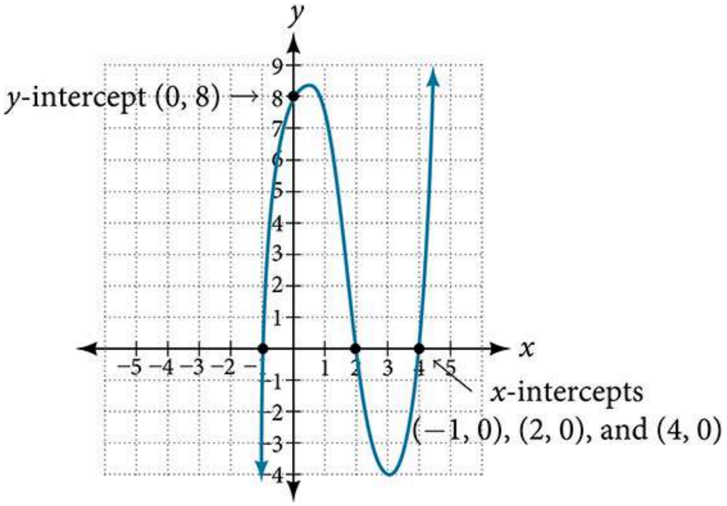


Figure 10

EXAMPLE 9

Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the *y*- and *x*-intercepts.

Solution

The *y*-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4 - 4(0)^2 - 45 \\ &= -45 \end{aligned}$$

The *y*-intercept is $(0, -45)$.

The *x*-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4 - 4x^2 - 45 \\ &= \left(x^2 - 9\right)\left(x^2 + 5\right) \\ &= (x - 3)(x + 3)\left(x^2 + 5\right) \end{aligned}$$

$$0 = (x - 3)(x + 3)\left(x^2 + 5\right)$$

$$\begin{array}{lllll} x - 3 = 0 & \text{or} & x + 3 = 0 & \text{or} & x^2 + 5 = 0 \\ x = 3 & \text{or} & x = -3 & \text{or} & \text{(no real solution)} \end{array}$$

The *x*-intercepts are $(3, 0)$ and $(-3, 0)$.

We can see these intercepts on the graph of the function shown in Figure 11. We can see that the function is even because $f(x) = f(-x)$.

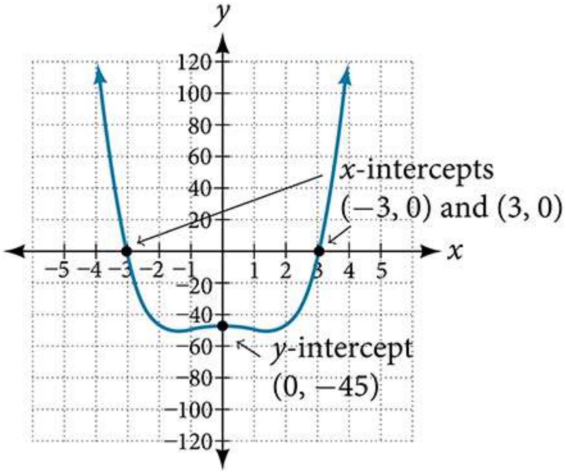


Figure 11

TRY IT #6 Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the *y*- and *x*-intercepts.

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of *x*-intercepts and the number of turning points. A polynomial function of *n*th degree is the product of *n* factors, so it will have at most *n* roots or zeros, or *x*-intercepts. The graph of the polynomial function of degree *n* must have at most $n– 1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A **smooth curve** is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Intercepts and Turning Points of Polynomials

A polynomial of degree n will have, at most, n *x*-intercepts and $n - 1$ turning points.

EXAMPLE 10

Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum number of *x*-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

Solution

The polynomial has a degree of 10, so there are at most 10 *x*-intercepts and at most 9 turning points.

TRY IT #7 Without graphing the function, determine the maximum number of *x*-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$.

EXAMPLE 11

Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 12 based on its intercepts and turning points?

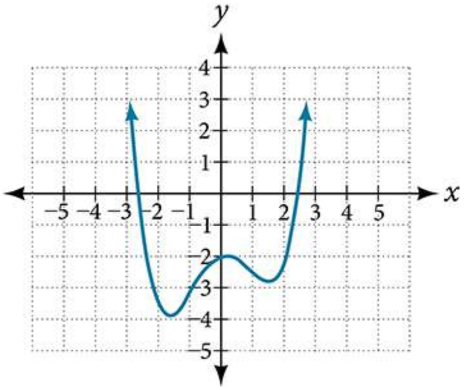


Figure 12

Solution

The end behavior of the graph tells us this is the graph of an even-degree polynomial. See Figure 13.

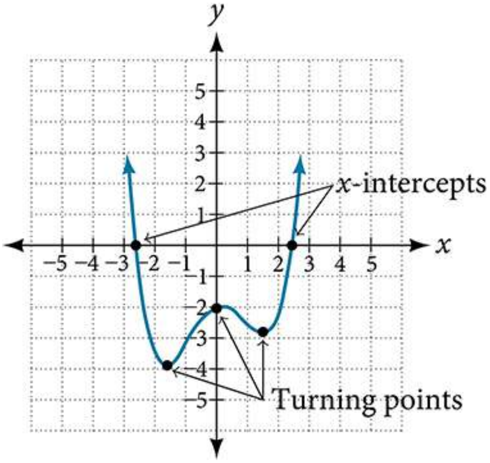


Figure 13

The graph has 2 *x*-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

TRY IT #8 What can we conclude about the polynomial represented by the graph shown in Figure 14 based on its intercepts and turning points?

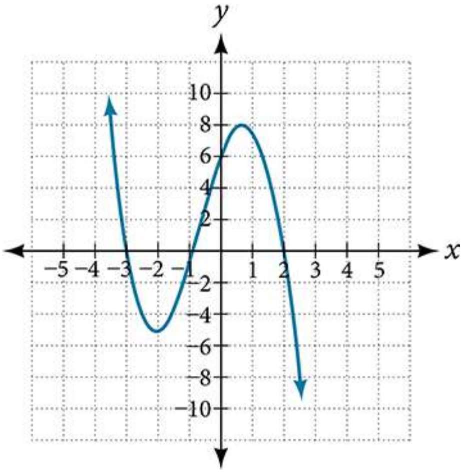


Figure 14

EXAMPLE 12

Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x\,(x+3)\,(x-4)$, determine the local behavior.

Solution

The *y*-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4 \\ &= 0 \end{aligned}$$

The *y*-intercept is $(0, 0)$.

The *x*-intercepts are found by determining the zeros of the function.

$$0 = -4x(x+3)(x-4)$$

$$\begin{array}{rclcrclcrcl} x &=& 0 & \text{or} & x+3 &=& 0 & \text{or} & x-4 &=& 0 \\ x &=& 0 & \text{or} & x &=& -3 & \text{or} & x &=& 4 \end{array}$$

The *x*-intercepts are $(0, 0)$, $(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

TRY IT #9 Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

MEDIA

Access these online resources for additional instruction and practice with power and polynomial functions.

Find Key Information about a Given Polynomial Function (http://openstax.org/l/keyinfopoly)
End Behavior of a Polynomial Function (http://openstax.org/l/endbehavior)
Turning Points and x- intercepts of Polynomial Functions (http://openstax.org/l/turningpoints)
Least Possible Degree of a Polynomial Function (http://openstax.org/l/leastposdegree)

5.2 SECTION EXERCISES

Verbal

1. Explain the difference between the coefficient of a power function and its degree.

2. If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?

3. In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.

4. What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?

5. What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty,\ \ f(x) \to -\infty$ and as $x \to \infty,\ \ f(x) \to -\infty$.

Algebraic

For the following exercises, identify the function as a power function, a polynomial function, or neither.

6. $f(x) = x^5$

7. $f(x) = \left(x^2\right)^3$

8. $f(x) = x - x^4$

9. $f(x) = \frac{x^2}{x^2 - 1}$

10. $f(x) = 2x\,(x+2)\,(x-1)^2$

11. $f(x) = 3^{x+1}$

For the following exercises, find the degree and leading coefficient for the given polynomial.

12. $-3x^4$

13. $7 - 2x^2$

14. $-2x^2 - 3x^5 + x - 6$

15. $x\left(4 - x^2\right)(2x+1)$

16. $x^2(2x-3)^2$

For the following exercises, determine the end behavior of the functions.

17. $f\,(x) = x^4$

18. $f\,(x) = x^3$

19. $f\,(x) = -x^4$

20. $f\,(x) = -x^9$

21. $f(x) = -2x^4 - 3x^2 + x - 1$

22. $f(x) = 3x^2 + x - 2$

23. $f(x) = x^2\left(2x^3 - x + 1\right)$

24. $f(x) = (2-x)^7$

For the following exercises, find the intercepts of the functions.

25. $f\,(t) = 2\,(t-1)\,(t+2)\,(t-3)$

26. $g\,(n) = -2\,(3n-1)\,(2n+1)$

27. $f(x) = x^4 - 16$

28. $f(x) = x^3 + 27$

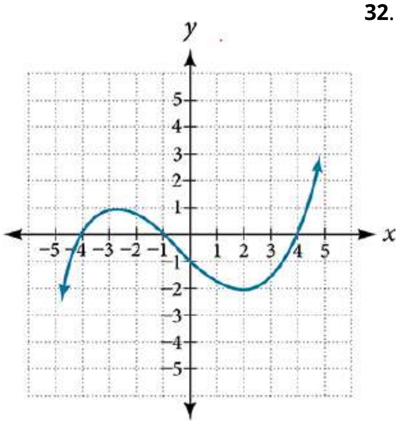
29. $f(x) = x\left(x^2 - 2x - 8\right)$

30. $f(x) = (x+3)\left(4x^2 - 1\right)$

Graphical

For the following exercises, determine the least possible degree of the polynomial function shown.

31.



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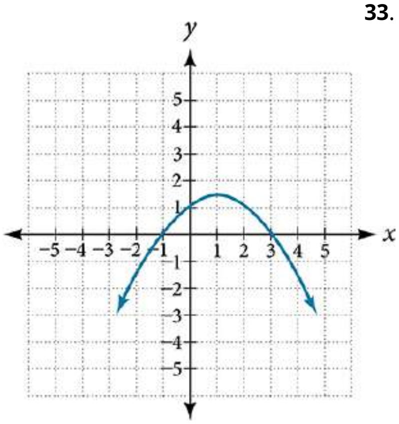

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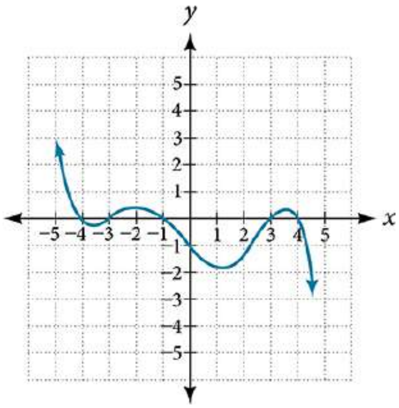

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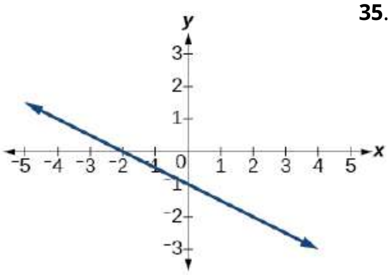

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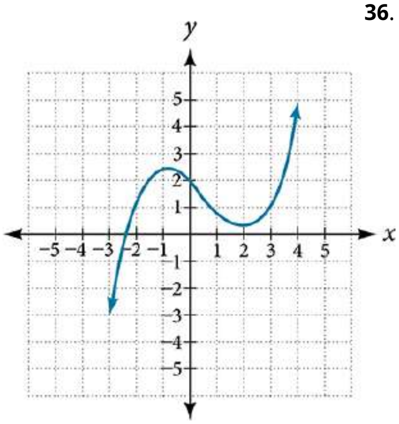

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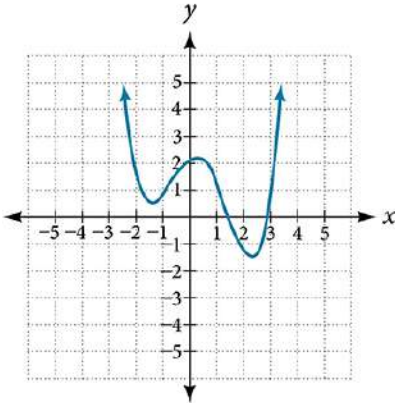

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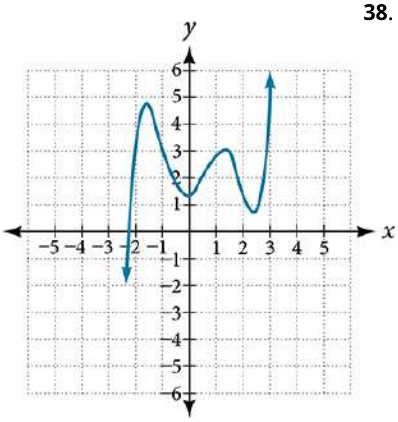

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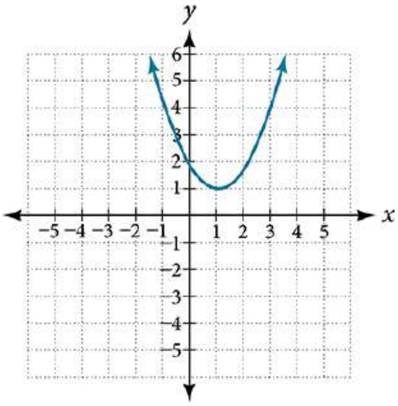
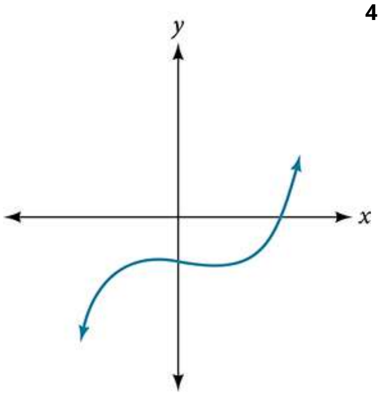


For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

39.



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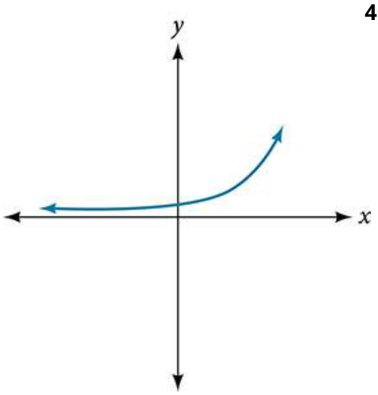

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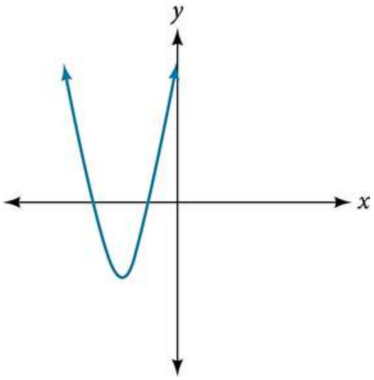

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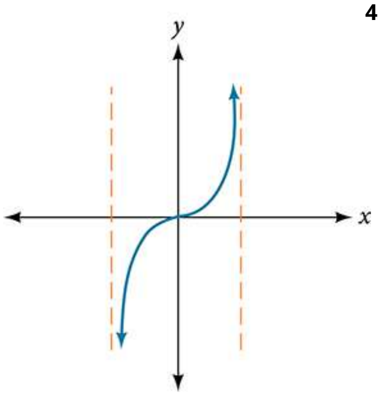

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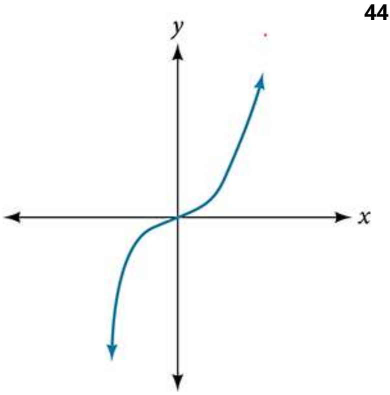

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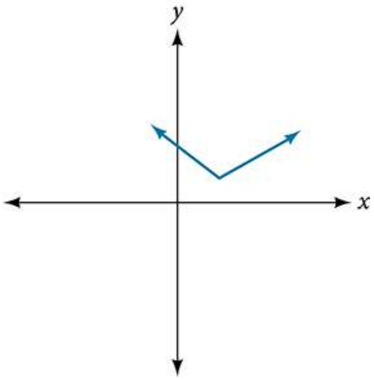

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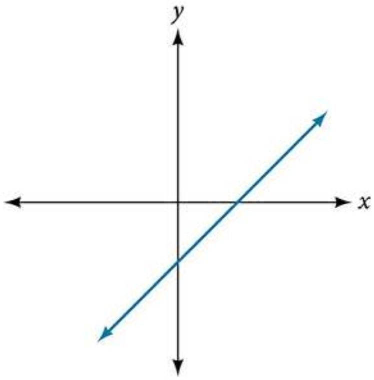


Numeric

For the following exercises, make a table to confirm the end behavior of the function.

46. $f(x) = -x^3$

47. $f(x) = x^4 - 5x^2$

48. $f(x) = x^2(1 - x)^2$

49. $f(x) = (x - 1)(x - 2)(3 - x)$

50. $f(x) = \frac{x^5}{10} - x^4$

Technology

For the following exercises, graph the polynomial functions using a calculator. Based on the graph, determine the intercepts and the end behavior.

51. $f(x) = x^3(x-2)$

52. $f(x) = x(x-3)(x+3)$

53. $f(x) = x(14-2x)(10-2x)$

54. $f(x) = x(14-2x)(10-2x)^2$

55. $f(x) = x^3 - 16x$

56. $f(x) = x^3 - 27$

57. $f(x) = x^4 - 81$

58. $f(x) = -x^3 + x^2 + 2x$

59. $f(x) = x^3 - 2x^2 - 15x$

60. $f(x) = x^3 - 0.01x$

Extensions

For the following exercises, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or –1. There may be more than one correct answer.

61. The y- intercept is $(0,-4)$. The x- intercepts are $(-2,0)$, $(2,0)$. Degree is 2.

End behavior: as $x \to -\infty$, $f(x) \to \infty$; as $x \to \infty$, $f(x) \to \infty$.

62. The y- intercept is $(0,9)$. The x- intercepts are $(-3,0)$, $(3,0)$. Degree is 2.

End behavior: as $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$.

63. The y- intercept is $(0,0)$. The x- intercepts are $(0,0)$, $(2,0)$. Degree is 3.

End behavior: as $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.

64. The y- intercept is $(0,1)$. The x- intercept is $(1,0)$. Degree is 3.

End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$.

65. The y- intercept is $(0,1)$. There is no x- intercept. Degree is 4.

End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$.

Real-World Applications

For the following exercises, use the written statements to construct a polynomial function that represents the required information.

66. An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of 20 meters per day. Express the area of the circle as a function of d, the number of days elapsed.

67. A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of m, the number of minutes elapsed.

68. A rectangle has a length of 10 inches and a width of 6 inches. If the length is increased by x inches and the width increased by twice that amount, express the area of the rectangle as a function of x.

69. An open box is to be constructed by cutting out square corners of x- inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of x.

70. A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width (x).

5.3 Graphs of Polynomial Functions

Learning Objectives

In this section, you will:

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in Table 1.

Year	2006	2007	2008	2009	2010	2011	2012	2013
Revenues	52.4	52.8	51.2	49.5	48.6	48.6	48.7	47.1

Table 1

The revenue can be modeled by the polynomial function

$$R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332$$

where R represents the revenue in millions of dollars and t represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing? These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure 1 shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.

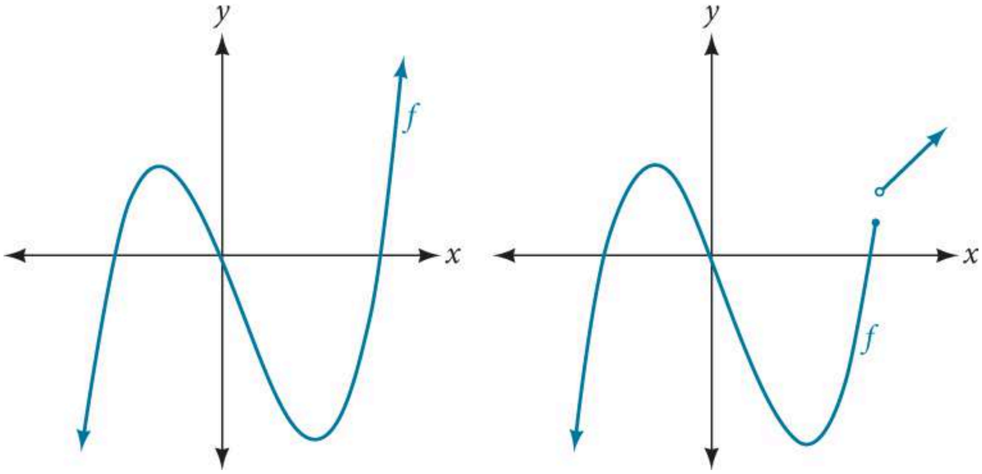


Figure 1

EXAMPLE 1

Recognizing Polynomial Functions

Which of the graphs in Figure 2 represents a polynomial function?

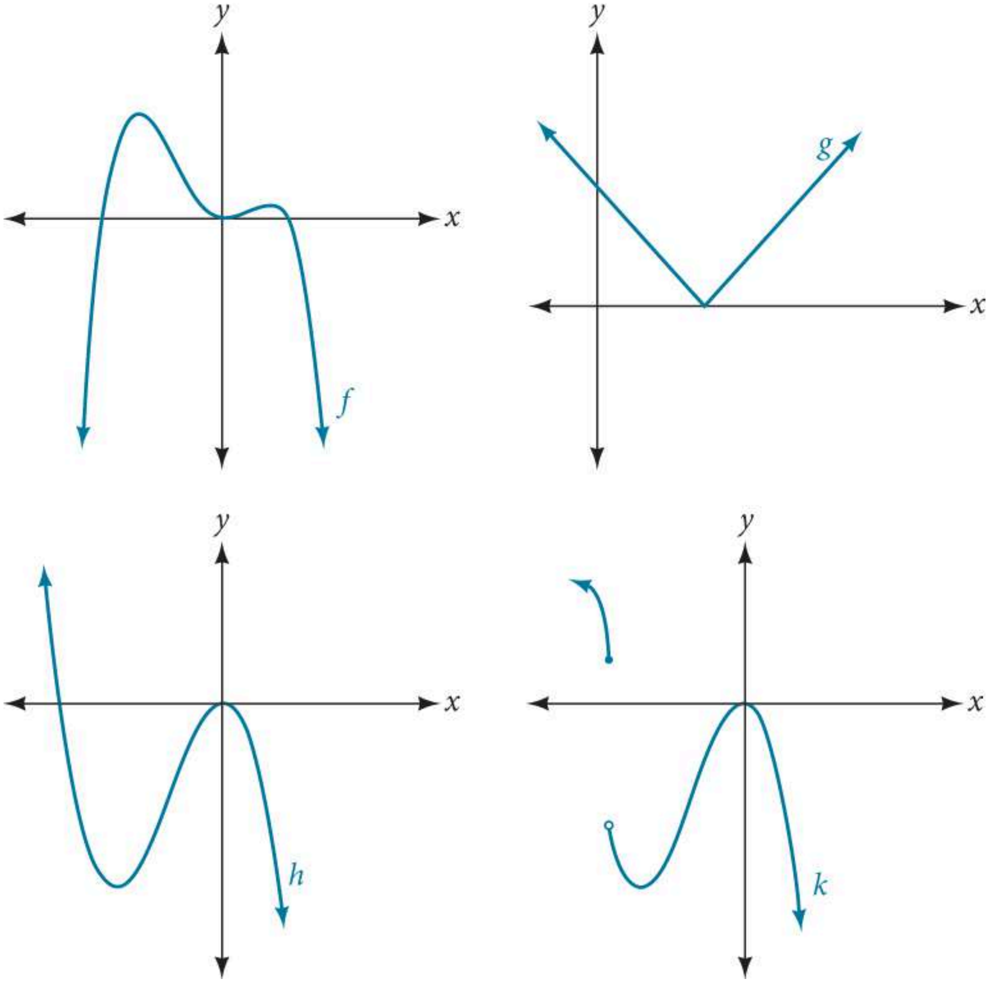


Figure 2

Solution

The graphs of f and h are graphs of polynomial functions. They are smooth and continuous.

The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Q&A **Do all polynomial functions have as their domain all real numbers?**

Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f\,(x) = 0$ are called zeros of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros**.**

We can use this method to find x- intercepts because at the x- intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to determine the intercepts.

HOW TO

Given a polynomial function f, find the *x*-intercepts by factoring.

1. Set $f\,(x) = 0$.
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x- intercepts.

EXAMPLE 2

Finding the *x*-Intercepts of a Polynomial Function by Factoring

Find the *x*-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution

We can attempt to factor this polynomial to find solutions for $f\,(x) = 0$.

$$\begin{aligned} x^6 - 3x^4 + 2x^2 &= 0 && \text{Factor out the greatest common factor.} \\ x^2\left(x^4 - 3x^2 + 2\right) &= 0 && \text{Factor the trinomial.} \\ x^2\left(x^2 - 1\right)\left(x^2 - 2\right) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{array}{rcl c rcl c rcl} & & & & \left(x^2 - 1\right) & = & 0 & & \left(x^2 - 2\right) & = & 0 \\ x^2 & = & 0 & \text{or} & x^2 & = & 1 & \text{or} & x^2 & = & 2 \\ x & = & 0 & & x & = & \pm 1 & & x & = & \pm\sqrt{2} \end{array}$$

This gives us five *x*-intercepts: $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(\sqrt{2}, 0)$, and $(-\sqrt{2}, 0)$. See Figure 3. We can see that this is an even function because it is symmetric about the *y*-axis.

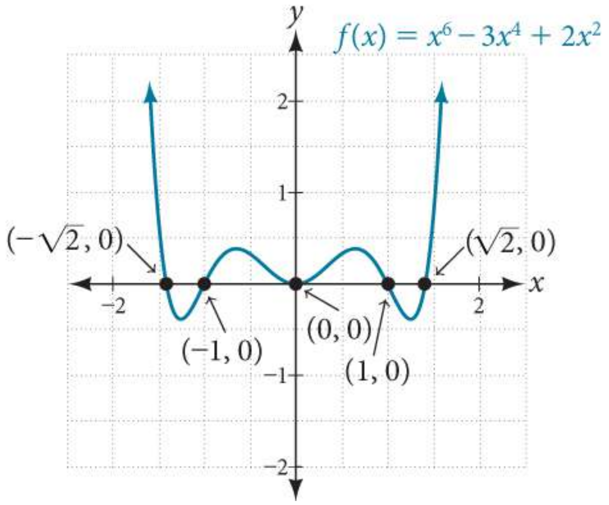


Figure 3

EXAMPLE 3

Finding the *x*-Intercepts of a Polynomial Function by Factoring

Find the *x*-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

Solution

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 && \text{Factor by grouping.} \\ x^2(x-5) - (x-5) &= 0 && \text{Factor out the common factor.} \\ \left(x^2 - 1\right)(x-5) &= 0 && \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x + 1 &= 0 & \text{or} \quad x - 1 &= 0 & \text{or} \quad x - 5 &= 0 \\ x &= -1 & x &= 1 & x &= 5 \end{aligned}$$

There are three *x*-intercepts: $(-1, 0)$, $(1, 0)$, and $(5, 0)$. See Figure 4.

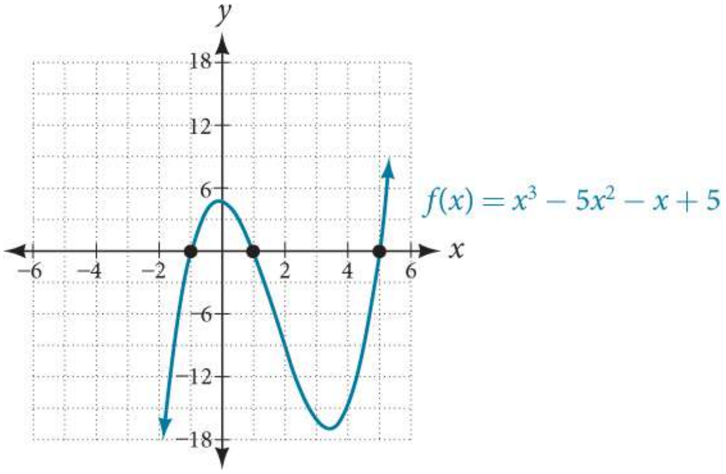


Figure 4

EXAMPLE 4

Finding the *y*- and *x*-Intercepts of a Polynomial in Factored Form

Find the *y*- and *x*-intercepts of $g(x) = (x-2)^2(2x+3)$.

Solution

The *y*-intercept can be found by evaluating $g(0)$.

$$\begin{aligned} g(0) &= (0-2)^2(2(0)+3) \\ &= 12 \end{aligned}$$

So the *y*-intercept is $(0, 12)$.

The *x*-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3) = 0$$

$$\begin{aligned} (x-2)^2 &= 0 & \qquad & & (2x+3) &= 0 \\ x-2 &= 0 & & \text{or} & x &= -\tfrac{3}{2} \\ x &= 2 & & & & \end{aligned}$$

So the *x*-intercepts are $(2, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

Analysis

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in Figure 5.

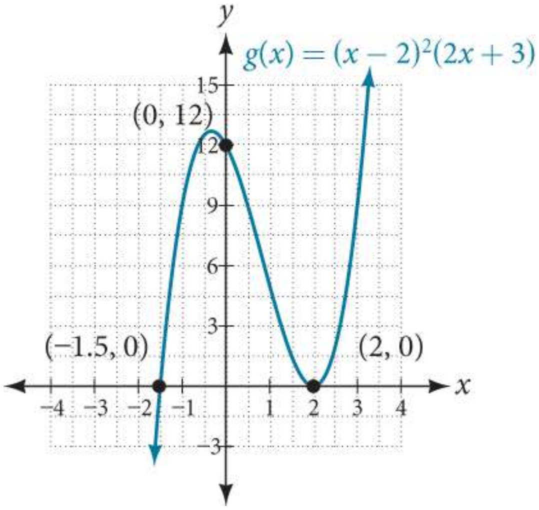


Figure 5

EXAMPLE 5

Finding the *x*-Intercepts of a Polynomial Function Using a Graph

Find the *x*-intercepts of $h(x) = x^3 + 4x^2 + x - 6$.

Solution

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 6, it appears that there are *x*-intercepts at $x = -3, -2,$ and 1.

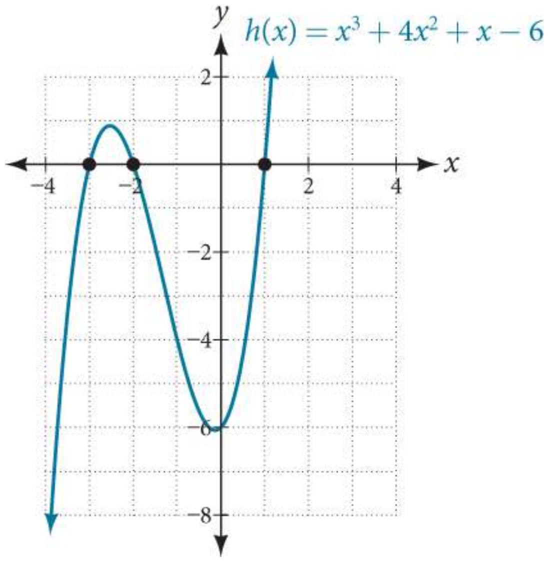


Figure 6

We can check whether these are correct by substituting these values for x and verifying that

$$h(-3) = h(-2) = h(1) = 0$$

Since $h(x) = x^3 + 4x^2 + x - 6$, we have:

$$\begin{aligned} h(-3) &= (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0 \\ h(-2) &= (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0 \\ h(1) &= (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0 \end{aligned}$$

Each *x*-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$\begin{aligned} h(x) &= x^3 + 4x^2 + x - 6 \\ &= (x+3)(x+2)(x-1) \end{aligned}$$

TRY IT #1 Find the *y*- and *x*-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

Identifying Zeros and Their Multiplicities

Graphs behave differently at various *x*-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and "bounce" off.

Suppose, for example, we graph the function shown.

$$f(x) = (x+3)(x-2)^2(x+1)^3$$

Notice in Figure 7 that the behavior of the function at each of the *x*-intercepts is different.

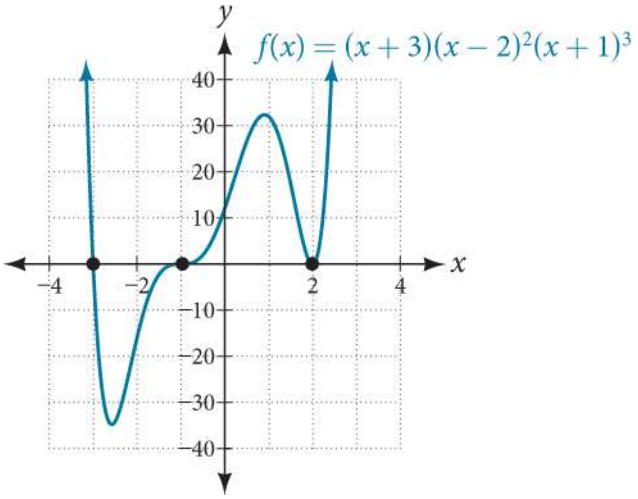


Figure 7 Identifying the behavior of the graph at an *x*-intercept by examining the multiplicity of the zero.

The *x*-intercept $x = -3$ is the solution of equation $(x + 3) = 0$. The graph passes directly through the *x*-intercept at $x = -3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The *x*-intercept $x = 2$ is the repeated solution of equation $(x - 2)^2 = 0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.

$$(x - 2)^2 = (x - 2)(x - 2)$$

The factor is repeated, that is, the factor $(x - 2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x = 2$, has multiplicity 2 because the factor $(x - 2)$ occurs twice.

The *x*-intercept $x = -1$ is the repeated solution of factor $(x + 1)^3 = 0$. The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near the intercept as the toolkit function $f(x) = x^3$. We call this a triple zero, or a zero with multiplicity 3.

For zeros with even multiplicities, the graphs *touch* or are tangent to the *x*-axis. For zeros with odd multiplicities, the graphs *cross* or intersect the *x*-axis. See Figure 8 for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

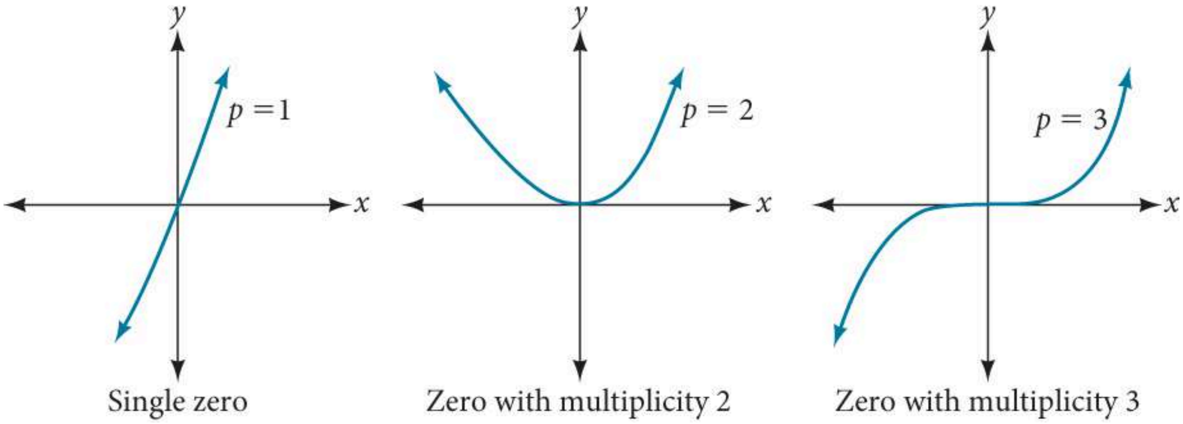


Figure 8

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the *x*-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the *x*-axis.

Graphical Behavior of Polynomials at x-Intercepts

If a polynomial contains a factor of the form $(x - h)^p$, the behavior near the x- intercept h is determined by the power p. We say that $x = h$ is a zero of **multiplicity** p.

The graph of a polynomial function will touch the x-axis at zeros with even multiplicities. The graph will cross the x-axis at zeros with odd multiplicities.

The sum of the multiplicities is the degree of the polynomial function.

HOW TO

Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities.

1. If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the x-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is n.

EXAMPLE 6

Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure 9 to identify the zeros of the function and their possible multiplicities.

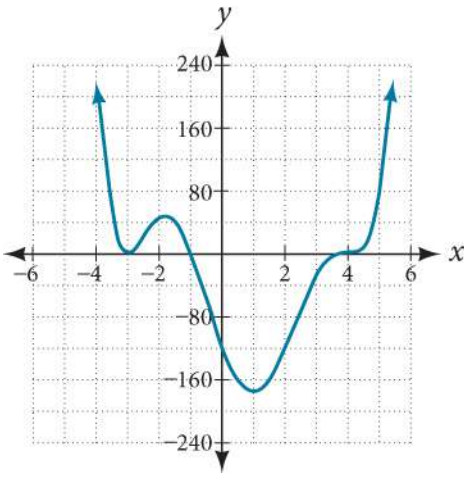


Figure 9

Solution

The polynomial function is of degree 6. The sum of the multiplicities must be 6.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of -3 most likely has multiplicity 2.

The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is a single zero of multiplicity 1.

The last zero occurs at $x = 4$. The graph crosses the x-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is likely 3 and that the sum of the multiplicities is 6.

TRY IT #2 Use the graph of the function of degree 9 in Figure 10 to identify the zeros of the function and their multiplicities.

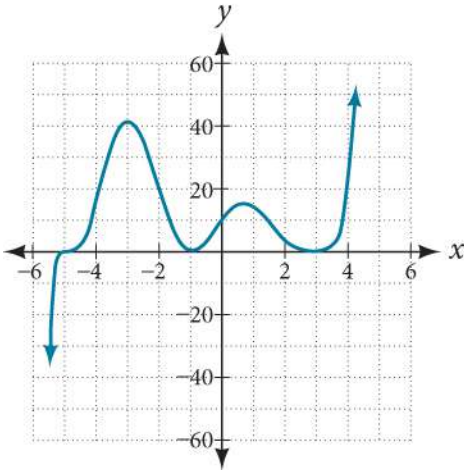


Figure 10

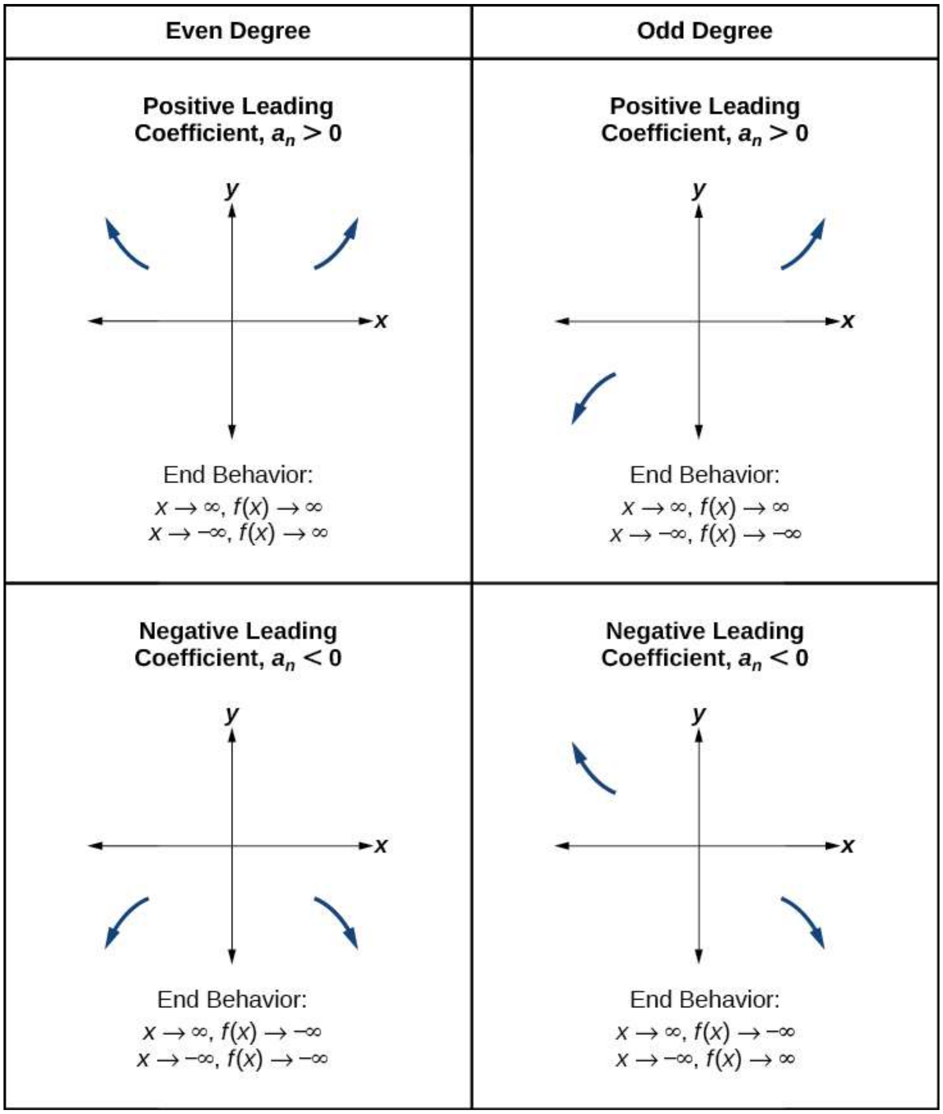
Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

will either ultimately rise or fall as x increases without bound and will either rise or fall as x decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the *end behavior* of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure 11 summarizes all four cases.



Even Degree	Odd Degree
Positive Leading Coefficient, $a_n > 0$ End Behavior: $x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to \infty$	**Positive Leading Coefficient, $a_n > 0$** End Behavior: $x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to -\infty$
Negative Leading Coefficient, $a_n < 0$ End Behavior: $x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to -\infty$	**Negative Leading Coefficient, $a_n < 0$** End Behavior: $x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to \infty$

Figure 11

Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function’s local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure 12. The graph has three turning points.

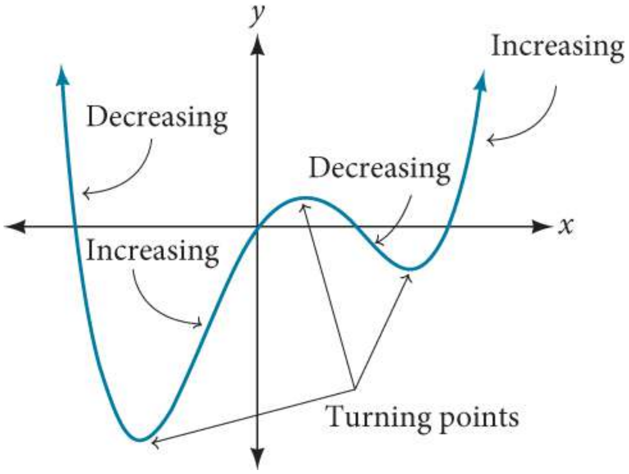


Figure 12

This function f is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

Interpreting Turning Points

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).

A polynomial of degree n will have at most $n - 1$ turning points.

EXAMPLE 7

Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum number of turning points of each polynomial function.

ⓐ $f(x) = -x^3 + 4x^5 - 3x^2 + 1$ ⓑ $f(x) = -(x-1)^2\left(1 + 2x^2\right)$

✓ **Solution**

ⓐ

First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum number of turning points is $5 - 1 = 4$.

ⓑ

First, identify the leading term of the polynomial function if the function were expanded.

$$f(x) = -(x-1)^2\ (1 + 2x^2)$$

$$a_n = -(x^2)\ (2x^2)\ -2x^4$$

Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum number of turning points is $4 - 1 = 3$.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

HOW TO

Given a polynomial function, sketch the graph.

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y- axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x- intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

EXAMPLE 8

Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution

This graph has two *x*-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this *x*-intercept. At $x = 5$, the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The *y*-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The *y*-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure 13.

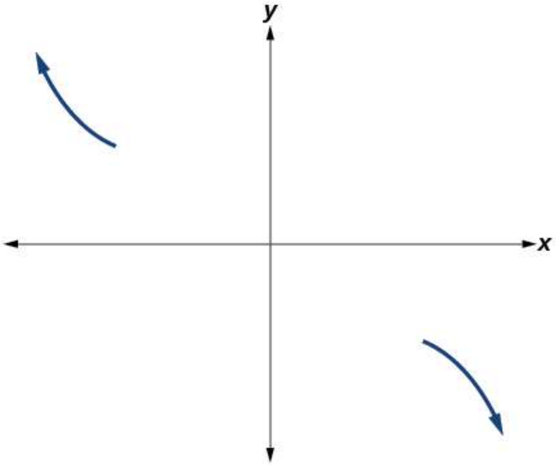


Figure 13

To sketch this, we consider that:

- As $x \to -\infty$ the function $f(x) \to \infty$, so we know the graph starts in the second quadrant and is decreasing toward the x- axis.

- Since $f(-x) = -2(-x+3)^2\,(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of the *x*-axis, so the function must start increasing.
  At $(0, 90)$, the graph crosses the *y*-axis at the *y*-intercept. See Figure 14.

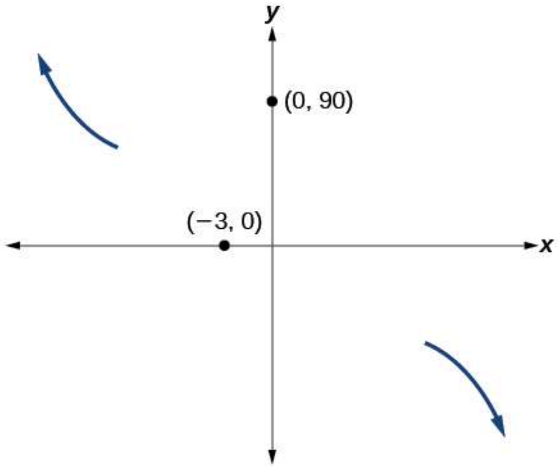


Figure 14

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5, 0)$. See Figure 15.

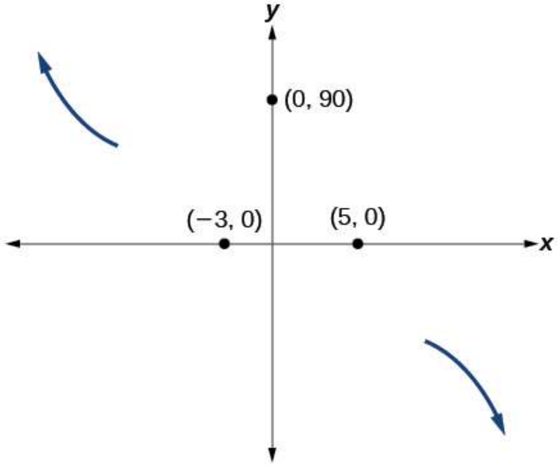


Figure 15

As $x \to \infty$ the function $f(x) \to -\infty$, so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure 16, and verify that the resulting graph looks like our sketch in Figure 15.

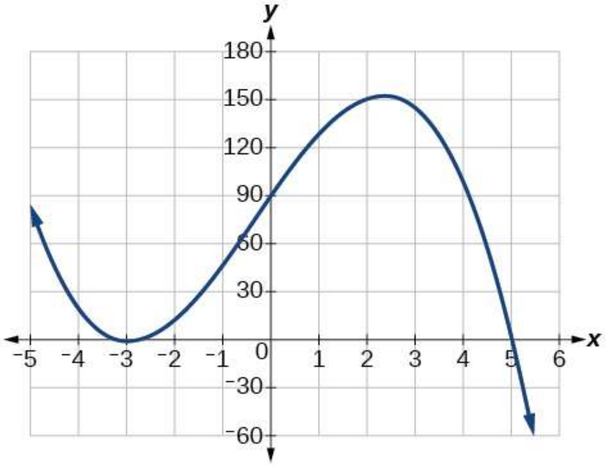


Figure 16 The complete graph of the polynomial function $f(x) = -2(x+3)^2(x-5)$

TRY IT #3 Sketch a graph of $f(x) = \frac{1}{4}x(x-1)^4(x+3)^3$.

Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the *x*-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$. (While the theorem is intuitive, the proof is actually quite complicated and requires higher mathematics.) We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function f at $x = a$ lies above the x- axis and another point at $x = b$ lies below the x- axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x- axis. Call this point $(c, \ f(c))$. This means that we are assured there is a solution c where $f(c) = 0$.

In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x- axis. Figure 17 shows that there is a zero between a and b.

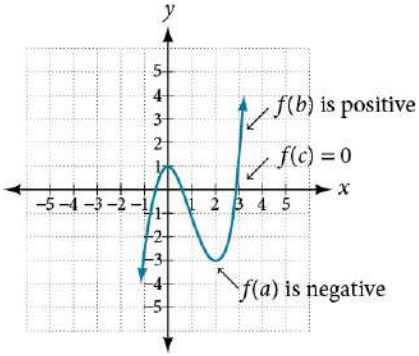


Figure 17 Using the Intermediate Value Theorem to show there exists a zero.

Intermediate Value Theorem

Let f be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

EXAMPLE 9

Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution

As a start, evaluate $f(x)$ at the integer values $x = 1, 2, 3,$ and 4. See Table 2.

x	1	2	3	4
$f(x)$	5	0	–3	2

Table 2

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.

We have shown that there are at least two real zeros between $x = 1$ and $x = 4$.

Analysis

We can also see on the graph of the function in Figure 18 that there are two real zeros between $x = 1$ and $x = 4$.

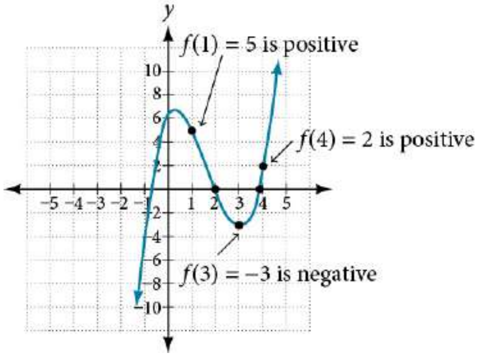


Figure 18

TRY IT #4 Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a polynomial function written in factored form will have an *x*-intercept where each factor is equal to zero, we can form a function that will pass through a set of *x*-intercepts by introducing a corresponding set of factors.

Factored Form of Polynomials

If a polynomial of lowest degree p has horizontal intercepts at $x = x_1, x_2, \dots, x_n$, then the polynomial can be written in the factored form: $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$ where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the *x*-intercept.

HOW TO

Given a graph of a polynomial function, write a formula for the function.

1. Identify the *x*-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the *x*-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the *y*-intercept may be easiest) to determine the stretch factor.

EXAMPLE 10

Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 19.

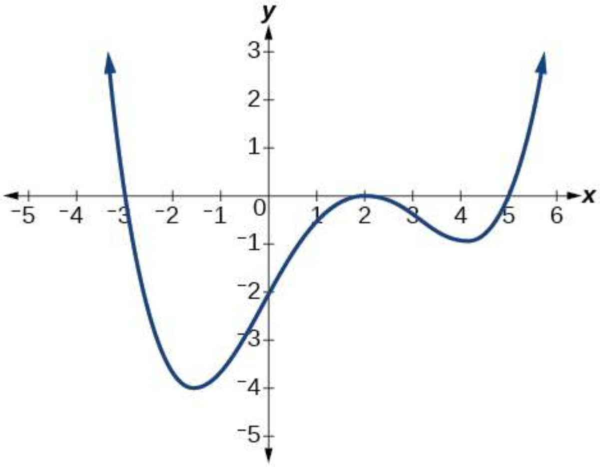


Figure 19

Solution

This graph has three *x*-intercepts: $x = -3, 2,$ and 5. The *y*-intercept is located at $(0, 2)$. At $x = -3$ and $x = 5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us

$$f(x) = a(x+3)(x-2)^2(x-5)$$

To determine the stretch factor, we utilize another point on the graph. We will use the y- intercept $(0, -2)$, to solve for a.

$$\begin{aligned} f(0) &= a(0+3)(0-2)^2(0-5) \\ -2 &= a(0+3)(0-2)^2(0-5) \\ -2 &= -60a \\ a &= \frac{1}{30} \end{aligned}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$.

TRY IT #5 Given the graph shown in Figure 20, write a formula for the function shown.

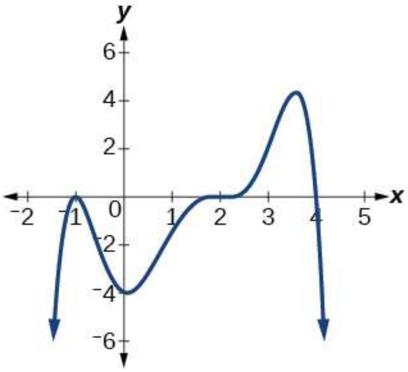


Figure 20

Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

Local and Global Extrema

A local maximum or local minimum at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$. If a function has a local maximum at a, then $f(a) \geq f(x)$ for all x in an open interval around $x = a$. If a function has a local minimum at a, then $f(a) \leq f(x)$ for all x in an open interval around $x = a$.

A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at a, then $f(a) \geq f(x)$ for all x. If a function has a global minimum at a, then $f(a) \leq f(x)$ for all x.

We can see the difference between local and global extrema in Figure 21.

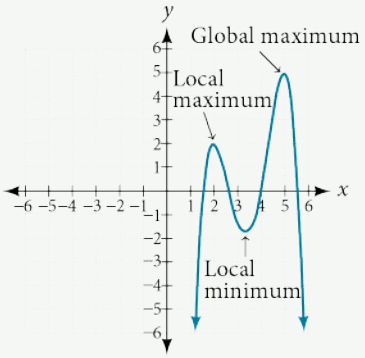


Figure 21

Q&A **Do all polynomial functions have a global minimum or maximum?**

No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ *has neither a global maximum nor a global minimum.*

EXAMPLE 11

Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic and then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

Solution

We will start this problem by drawing a picture like that in Figure 22, labeling the width of the cut-out squares with a variable, w.

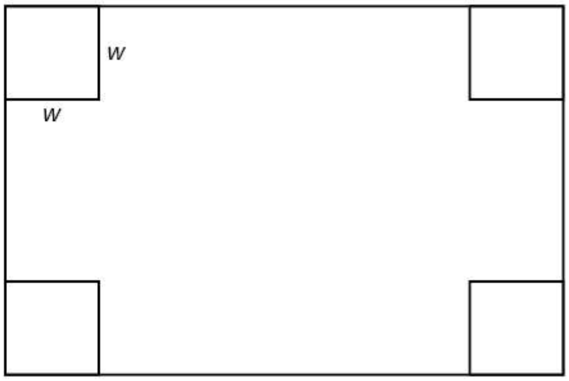


Figure 22

Notice that after a square is cut out from each end, it leaves a $(14 - 2w)$ cm by $(20 - 2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume

$$\begin{array}{rcl} V(w) & = & (20 - 2w)(14 - 2w)w \\ & = & 280w - 68w^2 + 4w^3 \end{array}$$

Notice, since the factors are w, $20–2w$ and $14–2w$, the three zeros are 10, 7, and 0, respectively. Because a height of 0 cm is not reasonable, we consider the only the zeros 10 and 7. The shortest side is 14 and we are cutting off two squares, so values w may take on are greater than zero or less than 7. This means we will restrict the domain of this function to $0 < w < 7$. Using technology to sketch the graph of $V(w)$ on this reasonable domain, we get a graph like that in Figure 23. We can use this graph to estimate the maximum value for the volume, restricted to values for w that are reasonable for this problem—values from 0 to 7.

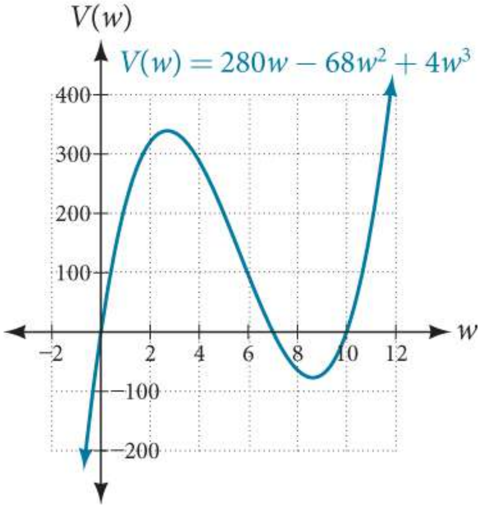


Figure 23

From this graph, we turn our focus to only the portion on the reasonable domain, $[0, \quad 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce Figure 24.

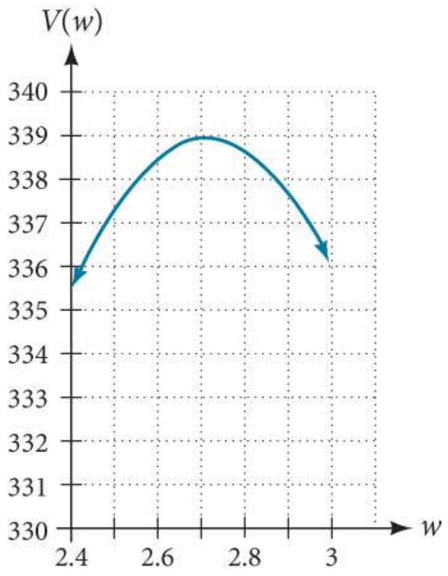


Figure 24

From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

TRY IT #6 Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

MEDIA

Access the following online resource for additional instruction and practice with graphing polynomial functions.

Intermediate Value Theorem (http://openstax.org/l/ivt)

5.3 SECTION EXERCISES

Verbal

1. What is the difference between an x- intercept and a zero of a polynomial function f?

2. If a polynomial function of degree n has n distinct zeros, what do you know about the graph of the function?

3. Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.

4. Explain how the factored form of the polynomial helps us in graphing it.

5. If the graph of a polynomial just touches the x-axis and then changes direction, what can we conclude about the factored form of the polynomial?

Algebraic

For the following exercises, find the x- or t-intercepts of the polynomial functions.

6. $C(t) = 2(t-4)(t+1)(t-6)$

7. $C(t) = 3(t+2)(t-3)(t+5)$

8. $C(t) = 4t(t-2)^2(t+1)$

9. $C(t) = 2t(t-3)(t+1)^2$

10. $C(t) = 2t^4 - 8t^3 + 6t^2$

11. $C(t) = 4t^4 + 12t^3 - 40t^2$

12. $f(x) = x^4 - x^2$

13. $f(x) = x^3 + x^2 - 20x$

14. $f(x) = x^3 + 6x^2 - 7x$

15. $f(x) = x^3 + x^2 - 4x - 4$

16. $f(x) = x^3 + 2x^2 - 9x - 18$

17. $f(x) = 2x^3 - x^2 - 8x + 4$

18. $f(x) = x^6 - 7x^3 - 8$

19. $f(x) = 2x^4 + 6x^2 - 8$

20. $f(x) = x^3 - 3x^2 - x + 3$

21. $f(x) = x^6 - 2x^4 - 3x^2$

22. $f(x) = x^6 - 3x^4 - 4x^2$

23. $f(x) = x^5 - 5x^3 + 4x$

For the following exercises, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

24. $f(x) = x^3 - 9x$, between $x = -4$ and $x = -2$.

25. $f(x) = x^3 - 9x$, between $x = 2$ and $x = 4$.

26. $f(x) = x^5 - 2x$, between $x = 1$ and $x = 2$.

27. $f(x) = -x^4 + 4$, between $x = 1$ and $x = 3$.

28. $f(x) = -2x^3 - x$, between $x = -1$ and $x = 1$.

29. $f(x) = x^3 - 100x + 2$, between $x = 0.01$ and $x = 0.1$

For the following exercises, find the zeros and give the multiplicity of each.

30. $f(x) = (x+2)^3(x-3)^2$

31. $f(x) = x^2(2x+3)^5(x-4)^2$

32. $f(x) = x^3(x-1)^3(x+2)$

33. $f(x) = x^2\left(x^2 + 4x + 4\right)$

34. $f(x) = (2x+1)^3\left(9x^2 - 6x + 1\right)$

35. $f(x) = (3x+2)^5\left(x^2 - 10x + 25\right)$

36. $f(x) = x\left(4x^2 - 12x + 9\right)\left(x^2 + 8x + 16\right)$

37. $f(x) = x^6 - x^5 - 2x^4$

38. $f(x) = 3x^4 + 6x^3 + 3x^2$

39. $f(x) = 4x^5 - 12x^4 + 9x^3$

40. $f(x) = 2x^4\left(x^3 - 4x^2 + 4x\right)$

41. $f(x) = 4x^4\left(9x^4 - 12x^3 + 4x^2\right)$

Graphical

For the following exercises, graph the polynomial functions. Note x- and y- intercepts, multiplicity, and end behavior.

42. $f(x) = (x+3)^2(x-2)$

43. $g(x) = (x+4)(x-1)^2$

44. $h(x) = (x-1)^3(x+3)^2$

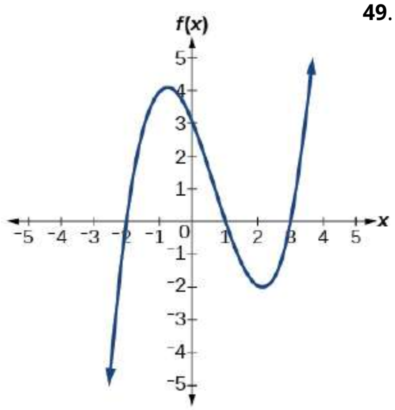
45. $k(x) = (x-3)^3(x-2)^2$

46. $m(x) = -2x(x-1)(x+3)$

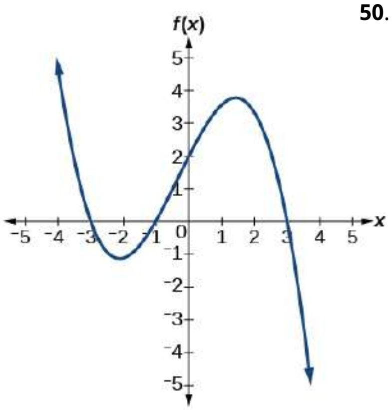
47. $n(x) = -3x(x+2)(x-4)$

For the following exercises, use the graphs to write the formula for a polynomial function of least degree.

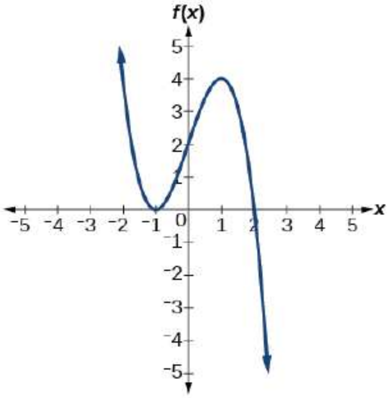
48.



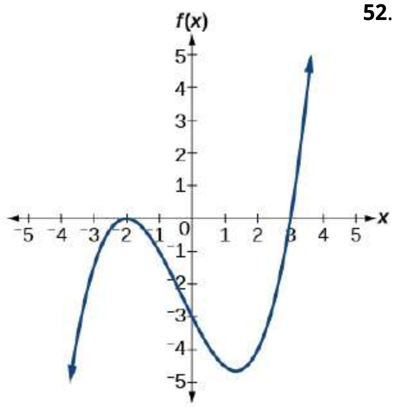
49.



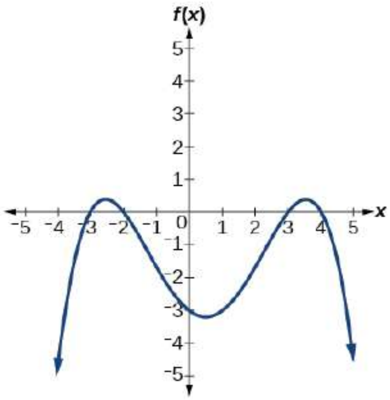
50.



51.

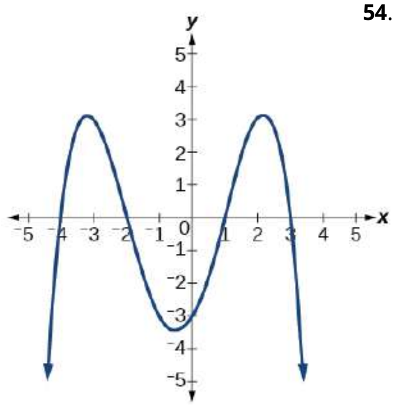


52.

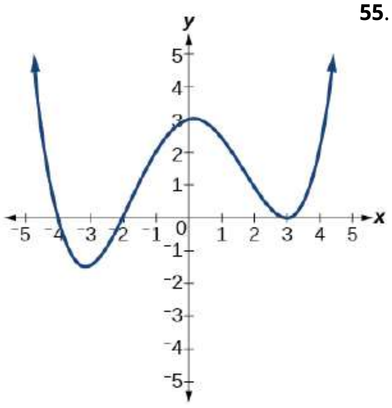


For the following exercises, use the graph to identify zeros and multiplicity.

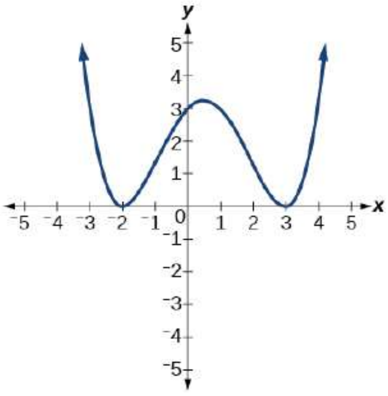
53.



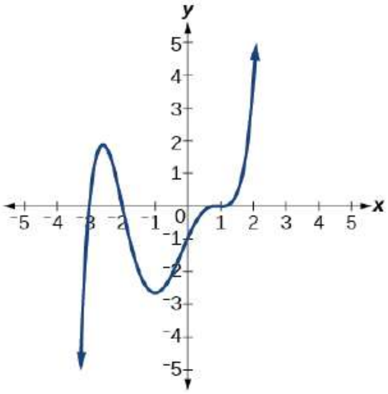
54.



55.



56.



For the following exercises, use the given information about the polynomial graph to write the equation.

57. Degree 3. Zeros at $x = –2$, $x = 1$, and $x = 3$. y-intercept at $(0, –4)$.

58. Degree 3. Zeros at $x = –5$, $x = –2$, and $x = 1$. y-intercept at $(0, 6)$

59. Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = –3$. y-intercept at $(0, 9)$

60. Degree 4. Root of multiplicity 2 at $x = 4$, and a roots of multiplicity 1 at $x = 1$ and $x = –2$. y-intercept at $(0, –3)$.

61. Degree 5. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point $(2, 15)$.

62. Degree 3. Zeros at $x = 4$, $x = 3$, and $x = 2$. y-intercept at $(0, -24)$.

63. Degree 3. Zeros at $x = -3$, $x = -2$ and $x = 1$. y-intercept at $(0, 12)$.

64. Degree 5. Roots of multiplicity 2 at $x = -3$ and $x = 2$ and a root of multiplicity 1 at $x = -2$.

y-intercept at $(0,\ 4)$.

65. Degree 4. Roots of multiplicity 2 at $x = \frac{1}{2}$ and roots of multiplicity 1 at $x = 6$ and $x = -2$.

y-intercept at $(0,18)$.

66. Double zero at $x = -3$ and triple zero at $x = 0$. Passes through the point $(1, 32)$.

Technology

For the following exercises, use a calculator to approximate local minima and maxima or the global minimum and maximum.

67. $f(x) = x^3 - x - 1$

68. $f(x) = 2x^3 - 3x - 1$

69. $f(x) = x^4 + x$

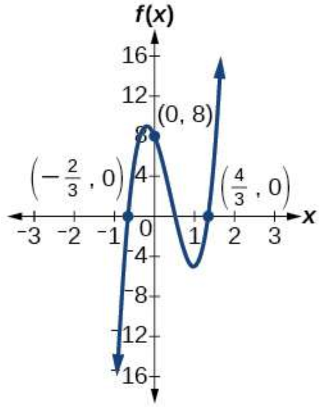
70. $f(x) = -x^4 + 3x - 2$

71. $f(x) = x^4 - x^3 + 1$

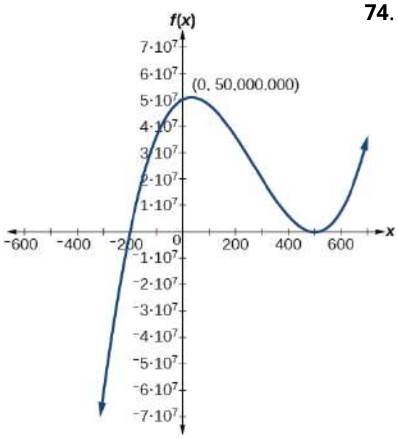
Extensions

For the following exercises, use the graphs to write a polynomial function of least degree.

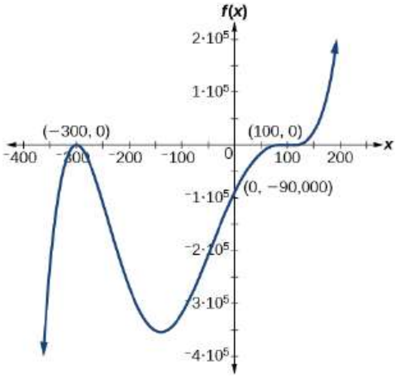
72.



73.



74.



Real-World Applications

For the following exercises, write the polynomial function that models the given situation.

75. A rectangle has a length of 10 units and a width of 8 units. Squares of x by x units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of x.

76. Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of x.

77. A square has sides of 12 units. Squares $x + 1$ by $x + 1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of x.

78. A cylinder has a radius of $x + 2$ units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.

79. A right circular cone has a radius of $3x + 6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ for radius r and height h.

5.4 Dividing Polynomials

Learning Objectives

In this section, you will:

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

Figure 1 Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m.[1] We can easily find the volume using elementary geometry.

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 61.5 \cdot 40 \cdot 30 \\ &= 73{,}800 \end{aligned}$$

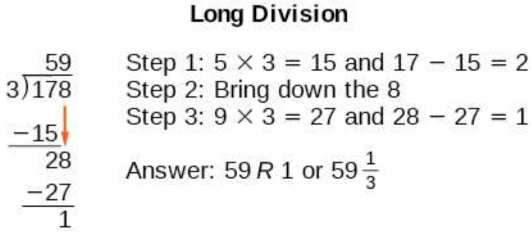
So the volume is 73,800 cubic meters (m^3) . Suppose we knew the volume, length, and width. We could divide to find the height.

$$\begin{aligned} h &= \frac{V}{l \cdot w} \\ &= \frac{73{,}800}{61.5 \cdot 40} \\ &= 30 \end{aligned}$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any, or all, of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$; the width is given by $x - 2$. To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

We are familiar with the long division algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.



Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor} \cdot \text{quotient}) + \text{remainder} \\ 178 &= (3 \cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write

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[1] National Park Service. "Lincoln Memorial Building Statistics." http://www.nps.gov/linc/historyculture/lincoln-memorial-building-statistics.htm. Accessed 4/3/2014

a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$$\begin{array}{rl} x+2\,\overline{)\,2x^3-3x^2+4x+5} & \text{Set up the division problem.} \end{array}$$

$$\begin{array}{rl} 2x^2 & \\ x+2\,\overline{)\,2x^3-3x^2+4x+5} & 2x^3 \text{ divided by } x \text{ is } 2x^2. \end{array}$$

$$\begin{array}{rl} 2x^2 & \\ x+2\,\overline{)\,2x^3-3x^2+4x+5} & \text{Multiply } x+2 \text{ by } 2x^2. \\ \underline{-(2x^3+4x^2)} & \text{Subtract.} \\ -7x^2+4x & \text{Bring down the next term.} \end{array}$$

$$\begin{array}{rl} 2x^2-7x & \\ x+2\,\overline{)\,2x^3-3x^2+4x+5} & \\ \underline{-(2x^3+4x^2)} & \\ -7x^2+4x & -7x^2 \text{ divided by } x \text{ is } -7x. \\ \underline{-(-7x^2-14x)} & \text{Multiply } x+2 \text{ by } -7x. \\ 18x+5 & \text{Subtract. Bring down the next term.} \end{array}$$

$$\begin{array}{rl} 2x^2-7x+18 & \\ x+2\,\overline{)\,2x^3-3x^2+4x+5} & \\ \underline{-(2x^3+4x^2)} & \\ -7x^2+4x & \\ \underline{-(-7x^2-14x)} & \\ 18x+5 & 18x \text{ divided by } x \text{ is } 18. \\ \underline{-(18x+36)} & \text{Multiply } x+2 \text{ by } 18. \\ -31 & \text{Subtract.} \end{array}$$

We have found

$$\frac{2x^3 - 3x^2 + 4x + 5}{x + 2} = 2x^2 - 7x + 18 - \frac{31}{x + 2}$$

or

$$2x^3 - 3x^2 + 4x + 5 = (x + 2)(2x^2 - 7x + 18) - 31$$

We can identify the dividend, the divisor, the quotient, and the remainder.

$$\underset{\substack{\uparrow \\ \text{Dividend}}}{2x^3 - 3x^2 + 4x + 5} = \underset{\substack{\uparrow \\ \text{Divisor}}}{(x + 2)}\ \underset{\substack{\uparrow \\ \text{Quotient}}}{(2x^2 - 7x + 18)} + \underset{\substack{\uparrow \\ \text{Remainder}}}{(-31)}$$

Writing the result in this manner illustrates the Division Algorithm.

The Division Algorithm

The **Division Algorithm** states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

If $r(x) = 0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

HOW TO

Given a polynomial and a binomial, use long division to divide the polynomial by the binomial.

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom binomial from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

EXAMPLE 1

Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2+3x-2$ by $x+1$.

Solution

$$\begin{array}{rl}
x+1\,\overline{)\,5x^2+3x-2} & \text{Set up the division problem.}
\end{array}$$

$$\begin{array}{rl}
5x & \\
x+1\,\overline{)\,5x^2+3x-2} & 5x^2 \text{ divided by } x \text{ is } 5x.
\end{array}$$

$$\begin{array}{rl}
5x & \\
x+1\,\overline{)\,5x^2+3x-2} & \\
\underline{-(5x^2+5x)} & \text{Multiply } x+1 \text{ by } 5x. \\
-2x-2 & \text{Subtract. Bring down the next term.}
\end{array}$$

$$\begin{array}{rl}
5x-2 & \\
x+1\,\overline{)\,5x^2+3x-2} & -2x \text{ divided by } x \text{ is } -2. \\
\underline{-(5x^2+5x)} & \\
-2x-2 & \\
\underline{-(-2x-2)} & \text{Multiply } x+1 \text{ by } -2. \\
0 & \text{Subtract.}
\end{array}$$

The quotient is $5x-2$. The remainder is 0. We write the result as

$$\frac{5x^2+3x-2}{x+1}=5x-2$$

or

$$5x^2+3x-2=(x+1)(5x-2)$$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

EXAMPLE 2

Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3+11x^2-31x+15$ by $3x-2$.

Solution

$$\begin{array}{rl}
2x^2 + 5x - 7 & \\
3x-2\,\overline{)\,6x^3+11x^2-31x+15} & 6x^3 \text{ divided by } 3x \text{ is } 2x^2. \\
\underline{-(6x^3-4x^2)} & \text{Multiply } 3x-2 \text{ by } 2x^2. \\
15x^2-31x & \text{Subtract. Bring down the next term. } 15x^2 \text{ divided by } 3x \text{ is } 5x. \\
\underline{-(15x^2-10x)} & \text{Multiply } 3x-2 \text{ by } 5x. \\
-21x+15 & \text{Subtract. Bring down the next term. } -21x \text{ divided by } 3x \text{ is } -7. \\
\underline{-(-21x+14)} & \text{Multiply } 3x-2 \text{ by } -7. \\
1 & \text{Subtract. The remainder is } 1.
\end{array}$$

There is a remainder of 1. We can express the result as:

$$\frac{6x^3+11x^2-31x+15}{3x-2} = 2x^2+5x-7+\frac{1}{3x-2}$$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x-2)(2x^2+5x-7)+1 = 6x^3+11x^2-31x+15$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1

TRY IT #1 Divide $16x^3-12x^2+20x-3$ by $4x+5$.

Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm.

The final form of the process looked like this:

$$\begin{array}{r}
2x^2-7x+18 \\
x+2\,\overline{)\,2x^3-3x^2+4x+5} \\
\underline{-(2x^3+4x^2)} \\
-7x^2+4x \\
\underline{-(-7x^2-14x)} \\
18x+5 \\
\underline{-(18x+36)} \\
-31
\end{array}$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$\begin{array}{r|rrrr}
2 & 2 & -3 & 4 & 5 \\
 & \underline{-2} & \underline{-4} & & \\
 & & \underline{-7} & 14 & \\
 & & & \underline{18} & -36 \\
 & & & & -31
\end{array}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

We then multiply it by the “divisor” and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2 – 7x + 18$ and the remainder is $–31$. The process will be made more clear in Example 3.

Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x - k$ where k is a real number. In **synthetic division**, only the coefficients are used in the division process.

HOW TO

Given two polynomials, use synthetic division to divide.

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

EXAMPLE 3

Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

Solution

Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & & \\ \hline & & & \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by k.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k. Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x + 12$. The remainder is 0. So $x - 3$ is a factor of the original polynomial.

Analysis

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x - 3)(5x + 12) + 0 = 5x^2 - 3x - 36$$

EXAMPLE 4

Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

Solution

The binomial divisor is $x + 2$ so $k = -2$. Add each column, multiply the result by –2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

The result is $4x^2 + 2x - 10$. The remainder is 0. Thus, $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

Analysis

The graph of the polynomial function $f(x) = 4x^3 + 10x^2 - 6x - 20$ in Figure 2 shows a zero at $x = k = -2$. This confirms that $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

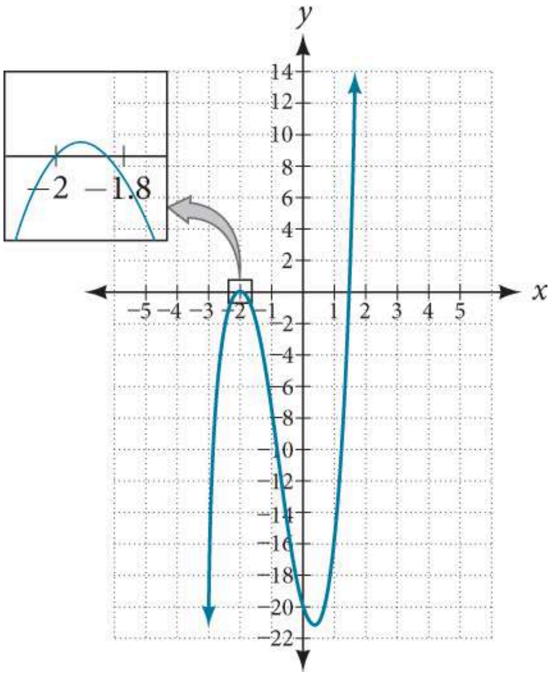


Figure 2

EXAMPLE 5

Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Solution

Notice there is no *x*-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3 + x^2 + 8x + 8 + \frac{2}{x-1}$.

TRY IT #2 Use synthetic division to divide $3x^4 + 18x^3 - 3x + 40$ by $x + 7$.

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

EXAMPLE 6

Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. Find the height, h, of the solid.

Solution

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in Figure 3.

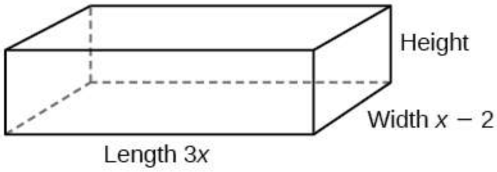


Figure 3

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$\begin{aligned} V &= l \cdot w \cdot h \\ 3x^4 - 3x^3 - 33x^2 + 54x &= 3x \cdot (x - 2) \cdot h \end{aligned}$$

To solve for h, first divide both sides by $3x$.

$$\begin{aligned} \frac{3x \cdot (x-2) \cdot h}{3x} &= \frac{3x^4 - 3x^3 - 33x^2 + 54x}{3x} \\ (x - 2)h &= x^3 - x^2 - 11x + 18 \end{aligned}$$

Now solve for h using synthetic division.

$$h = \frac{x^3 - x^2 - 11x + 18}{x - 2}$$

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The quotient is $x^2 + x - 9$ and the remainder is 0. The height of the solid is $x^2 + x - 9$.

TRY IT #3 The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

MEDIA

Access these online resources for additional instruction and practice with polynomial division.

Dividing a Trinomial by a Binomial Using Long Division (http://openstax.org/l/dividetribild)
Dividing a Polynomial by a Binomial Using Long Division (http://openstax.org/l/dividepolybild)
Ex 2: Dividing a Polynomial by a Binomial Using Synthetic Division (http://openstax.org/l/dividepolybisd2)
Ex 4: Dividing a Polynomial by a Binomial Using Synthetic Division (http://openstax.org/l/dividepolybisd4)

5.4 SECTION EXERCISES

Verbal

1. If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?

2. If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

Algebraic

For the following exercises, use long division to divide. Specify the quotient and the remainder.

3. $\left(x^2+5x-1\right)\div(x-1)$

4. $\left(2x^2-9x-5\right)\div(x-5)$

5. $\left(3x^2+23x+14\right)\div(x+7)$

6. $\left(4x^2-10x+6\right)\div(4x+2)$

7. $\left(6x^2-25x-25\right)\div(6x+5)$

8. $\left(-x^2-1\right)\div(x+1)$

9. $\left(2x^2-3x+2\right)\div(x+2)$

10. $\left(x^3-126\right)\div(x-5)$

11. $\left(3x^2-5x+4\right)\div(3x+1)$

12. $\left(x^3-3x^2+5x-6\right)\div(x-2)$

13. $\left(2x^3+3x^2-4x+15\right)\div(x+3)$

For the following exercises, use synthetic division to find the quotient. Ensure the equation is in the form required by synthetic division. (Hint: divide the dividend and divisor by the coefficient of the linear term in the divisor.)

14. $\left(3x^3-2x^2+x-4\right)\div(x+3)$

15. $\left(2x^3-6x^2-7x+6\right)\div(x-4)$

16. $\left(6x^3-10x^2-7x-15\right)\div(x+1)$

17. $\left(4x^3-12x^2-5x-1\right)\div(2x+1)$

18. $\left(9x^3-9x^2+18x+5\right)\div(3x-1)$

19. $\left(3x^3-2x^2+x-4\right)\div(x+3)$

20. $\left(-6x^3+x^2-4\right)\div(2x-3)$

21. $\left(2x^3+7x^2-13x-3\right)\div(2x-3)$

22. $\left(3x^3-5x^2+2x+3\right)\div(x+2)$

23. $\left(4x^3-5x^2+13\right)\div(x+4)$

24. $\left(x^3-3x+2\right)\div(x+2)$

25. $\left(x^3-21x^2+147x-343\right)\div(x-7)$

26. $\left(x^3-15x^2+75x-125\right)\div(x-5)$

27. $\left(9x^3-x+2\right)\div(3x-1)$

28. $\left(6x^3-x^2+5x+2\right)\div(3x+1)$

29. $\left(x^4+x^3-3x^2-2x+1\right)\div(x+1)$

30. $\left(x^4-3x^2+1\right)\div(x-1)$

31. $\left(x^4+2x^3-3x^2+2x+6\right)\div(x+3)$

32. $\left(x^4-10x^3+37x^2-60x+36\right)\div(x-2)$

33. $\left(x^4-8x^3+24x^2-32x+16\right)\div(x-2)$

34. $\left(x^4+5x^3-3x^2-13x+10\right)\div(x+5)$

35. $\left(x^4-12x^3+54x^2-108x+81\right)\div(x-3)$

36. $\left(4x^4-2x^3-4x+2\right)\div(2x-1)$

37. $\left(4x^4+2x^3-4x^2+2x+2\right)\div(2x+1)$

For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

38. $x - 2, \; 4x^3 - 3x^2 - 8x + 4$

39. $x - 2, \; 3x^4 - 6x^3 - 5x + 10$

40. $x + 3, \; -4x^3 + 5x^2 + 8$

41. $x - 2, \; 4x^4 - 15x^2 - 4$

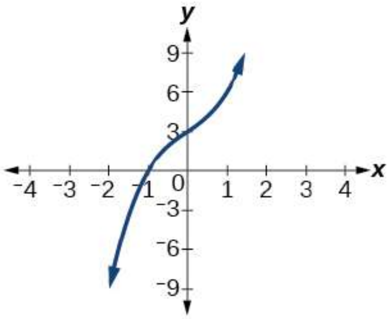
42. $x - \frac{1}{2}, \; 2x^4 - x^3 + 2x - 1$

43. $x + \frac{1}{3}, \; 3x^4 + x^3 - 3x + 1$

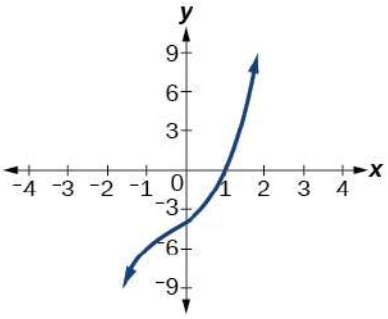
Graphical

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

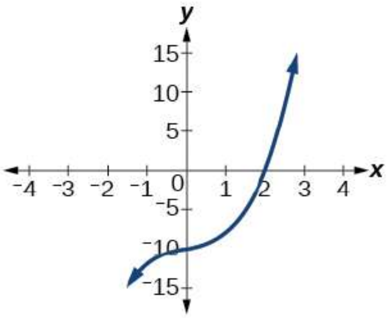
44. Factor is $x^2 - x + 3$



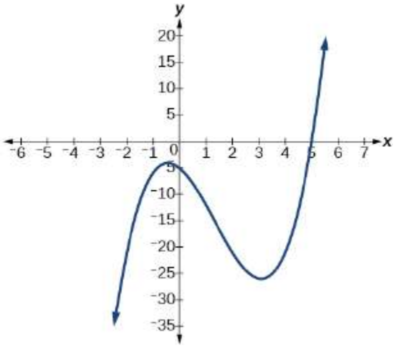
45. Factor is $(x^2 + 2x + 4)$



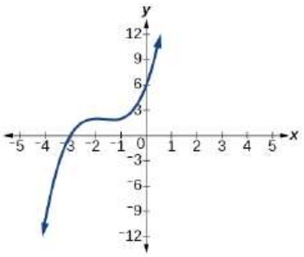
46. Factor is $x^2 + 2x + 5$



47. Factor is $x^2 + x + 1$



48. Factor is $x^2 + 2x + 2$



For the following exercises, use synthetic division to find the quotient and remainder.

49. $\frac{4x^3-33}{x-2}$

50. $\frac{2x^3+25}{x+3}$

51. $\frac{3x^3+2x-5}{x-1}$

52. $\frac{-4x^3-x^2-12}{x+4}$

53. $\frac{x^4-22}{x+2}$

Technology

For the following exercises, use a calculator with CAS to answer the questions.

54. Consider $\frac{x^k-1}{x-1}$ with $k = 1, \; 2, \; 3$. What do you expect the result to be if $k = 4$?

55. Consider $\frac{x^k+1}{x+1}$ for $k = 1, \; 3, \; 5$. What do you expect the result to be if $k = 7$?

56. Consider $\frac{x^4-k^4}{x-k}$ for $k = 1,\ 2,\ 3$. What do you expect the result to be if $k = 4$?

57. Consider $\frac{x^k}{x+1}$ with $k = 1,\ 2,\ 3$. What do you expect the result to be if $k = 4$?

58. Consider $\frac{x^k}{x-1}$ with $k = 1,\ 2,\ 3$. What do you expect the result to be if $k = 4$?

Extensions

For the following exercises, use synthetic division to determine the quotient involving a complex number.

59. $\frac{x+1}{x-i}$

60. $\frac{x^2+1}{x-i}$

61. $\frac{x+1}{x+i}$

62. $\frac{x^2+1}{x+i}$

63. $\frac{x^3+1}{x-i}$

Real-World Applications

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

64. Length is $x + 5$, area is $2x^2 + 9x - 5$.

65. Length is $2x + 5$, area is $4x^3 + 10x^2 + 6x + 15$

66. Length is $3x–4$, area is $6x^4 - 8x^3 + 9x^2 - 9x - 4$

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

67. Volume is $12x^3 + 20x^2 - 21x - 36$, length is $2x + 3$, width is $3x - 4$.

68. Volume is $18x^3 - 21x^2 - 40x + 48$, length is $3x–4$, width is $3x–4$.

69. Volume is $10x^3 + 27x^2 + 2x - 24$, length is $5x–4$, width is $2x + 3$.

70. Volume is $10x^3 + 30x^2 - 8x - 24$, length is 2, width is $x + 3$.

For the following exercises, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

71. Volume is $\pi\left(25x^3 - 65x^2 - 29x - 3\right)$, radius is $5x + 1$.

72. Volume is $\pi\left(4x^3 + 12x^2 - 15x - 50\right)$, radius is $2x + 5$.

73. Volume is $\pi\left(3x^4 + 24x^3 + 46x^2 - 16x - 32\right)$, radius is $x + 4$.

5.5 Zeros of Polynomial Functions

Learning Objectives

In this section, you will:

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zero Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs.
- Solve real-world applications of polynomial equations

A new bakery offers decorated, multi-tiered cakes for display and cutting at Quinceañera and wedding celebrations, as well as sheet cakes for children's birthday parties and other special occasions to serve most of the guests. The bakery wants the volume of a small sheet cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x– k$, the remainder may be found quickly by evaluating the polynomial function at k, that is, $f\,(k)$ Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

and either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. In practice divisors, $d(x)$ will have degrees less than or equal to the degree of $f(x)$. If the divisor, $d(x)$, is $x - k$, this takes the form

$$f(x) = (x - k)q(x) + r$$

Since the divisor $x - k$ is linear, the remainder will be a constant, r. And, if we evaluate this for $x = k$, we have

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= 0 \cdot q(k) + r \\ &= r \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$ by $x - k$.

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is the value $f(k)$.

HOW TO

Given a polynomial function f, evaluate $f\,(x)$ at $x = k$ using the Remainder Theorem.

1. Use synthetic division to divide the polynomial by $x - k$.
2. The remainder is the value $f(k)$.

EXAMPLE 1

Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ at $x = 2$.

Solution

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x - 2$.

$$\begin{array}{r|rrrrr} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array}$$

The remainder is 25. Therefore, $f(2) = 25$.

Analysis

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4 - x^3 - 15x^2 + 2x - 7 \\ f(2) &= 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7 \\ &= 25 \end{aligned}$$

TRY IT #1 Use the Remainder Theorem to evaluate $f(x) = 2x^5 - 3x^4 - 9x^3 + 8x^2 + 2$ at $x = -3$.

Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm.

$$f(x) = (x - k)q(x) + r$$

If k is a zero, then the remainder r is $f(k) = 0$ and $f(x) = (x - k)q(x) + 0$ or $f(x) = (x - k)q(x)$.

Notice, written in this form, $x - k$ is a factor of $f(x)$. We can conclude if k is a zero of $f(x)$, then $x - k$ is a factor of $f(x)$.

Similarly, if $x - k$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x) = (x - k)q(x) + r$ is 0. This tells us that k is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree n in the complex number system will have n zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

The Factor Theorem

According to the **Factor Theorem**, k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$.

HOW TO

Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial.

1. Use synthetic division to divide the polynomial by $(x - k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x - k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

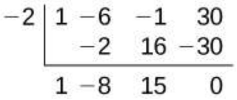
EXAMPLE 2

Using the Factor Theorem to Find the Zeros of a Polynomial Expression

Show that $(x + 2)$ is a factor of $x^3 - 6x^2 - x + 30$. Find the remaining factors. Use the factors to determine the zeros of the polynomial.

Solution

We can use synthetic division to show that $(x + 2)$ is a factor of the polynomial.



$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

The remainder is zero, so $(x + 2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x + 2)\left(x^2 - 8x + 15\right)$$

We can factor the quadratic factor to write the polynomial as

$$(x + 2)(x - 3)(x - 5)$$

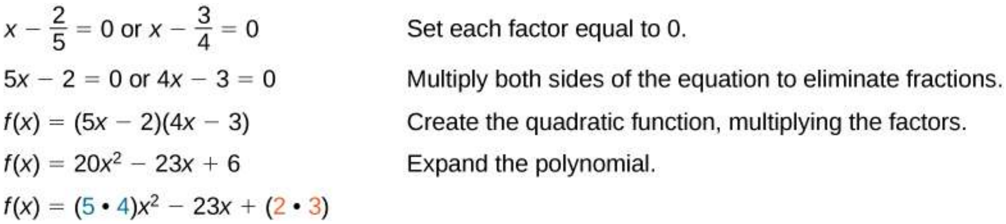
By the Factor Theorem, the zeros of $x^3 - 6x^2 - x + 30$ are –2, 3, and 5.

TRY IT #2 Use the Factor Theorem to find the zeros of $f(x) = x^3 + 4x^2 - 4x - 16$ given that $(x - 2)$ is a factor of the polynomial.

Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading coefficient of the polynomial

Consider a quadratic function with two zeros, $x = \frac{2}{5}$ and $x = \frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.



$x - \frac{2}{5} = 0$ or $x - \frac{3}{4} = 0$	Set each factor equal to 0.
$5x - 2 = 0$ or $4x - 3 = 0$	Multiply both sides of the equation to eliminate fractions.
$f(x) = (5x - 2)(4x - 3)$	Create the quadratic function, multiplying the factors.
$f(x) = 20x^2 - 23x + 6$	Expand the polynomial.
$f(x) = (5 \cdot 4)x^2 - 23x + (2 \cdot 3)$	

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

The Rational Zero Theorem

The **Rational Zero Theorem** states that, if the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

HOW TO

Given a polynomial function $f(x)$, use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f(\frac{p}{q})$.

EXAMPLE 3

Listing All Possible Rational Zeros

List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

Solution

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, –4, and the factors of the leading coefficient, 2.

The constant term is –4; the factors of –4 are $p = \pm1, \pm2, \pm4$.

The leading coefficient is 2; the factors of 2 are $q = \pm1, \pm2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of –4 divided by one of the factors of 2.

$$\frac{p}{q} = \pm\ \frac{1}{1},\ \pm\ \frac{1}{2} \qquad \frac{p}{q} = \pm\ \frac{2}{1}, \pm\ \frac{2}{2} \qquad \frac{p}{q} =\ \pm\ \frac{4}{1}, \pm\ \frac{4}{2}$$

Note that $\frac{2}{2} = 1$ and $\frac{4}{2} = 2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q} = \frac{\text{Factors of the last}}{\text{Factors of the first}} = \pm1, \pm2, \pm4, \pm\frac{1}{2}$$

EXAMPLE 4

Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x) = 2x^3 + x^2 - 4x + 1$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 1 and q is a factor of 2.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of 1}}{\text{factor of 2}} \end{aligned}$$

The factors of 1 are ±1 and the factors of 2 are ±1 and ±2. The possible values for $\frac{p}{q}$ are ±1 and $\pm\frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for x in $f(x)$.

$$\begin{aligned} f(-1) &= 2(-1)^3 + (-1)^2 - 4(-1) + 1 = 4 \\ f(1) &= 2(1)^3 + (1)^2 - 4(1) + 1 = 0 \\ f\left(-\tfrac{1}{2}\right) &= 2\left(-\tfrac{1}{2}\right)^3 + \left(-\tfrac{1}{2}\right)^2 - 4\left(-\tfrac{1}{2}\right) + 1 = 3 \\ f\left(\tfrac{1}{2}\right) &= 2\left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^2 - 4\left(\tfrac{1}{2}\right) + 1 = -\tfrac{1}{2} \end{aligned}$$

Of those, $-1, -\frac{1}{2},$ and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

TRY IT #3 Use the Rational Zero Theorem to find the rational zeros of $f(x) = x^3 - 5x^2 + 2x + 1$.

Finding the Zeros of Polynomial Functions

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use synthetic division repeatedly to determine all of the **zeros** of a polynomial function.

HOW TO

Given a polynomial function f, use synthetic division to find its zeros.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.
4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

EXAMPLE 5

Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x) = 4x^3 - 3x - 1$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –1 and q is a factor of 4.

$$\begin{aligned} \frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of } -1}{\text{factor of } 4} \end{aligned}$$

The factors of -1 are ± 1 and the factors of 4 are $\pm 1, \pm 2,$ and ± 4. The possible values for $\frac{p}{q}$ are $\pm 1,\ \ \pm\frac{1}{2}$, and $\pm\frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let’s begin with 1.

$$\begin{array}{r|rrrr} 1 & 4 & 0 & -3 & -1 \\ & & 4 & 4 & 1 \\ \hline & 4 & 4 & 1 & 0 \end{array}$$

Dividing by $(x - 1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x - 1)(4x^2 + 4x + 1)$$

The quadratic is a perfect square. $f(x)$ can be written as

$$(x - 1)(2x + 1)^2$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$\begin{aligned} 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

Analysis

Look at the graph of the function f in Figure 1. Notice, at $x = -0.5$, the graph bounces off the *x*-axis, indicating the even multiplicity (2,4,6…) for the zero -0.5. At $x = 1$, the graph crosses the *x*-axis, indicating the odd multiplicity (1,3,5…) for the zero $x = 1$.

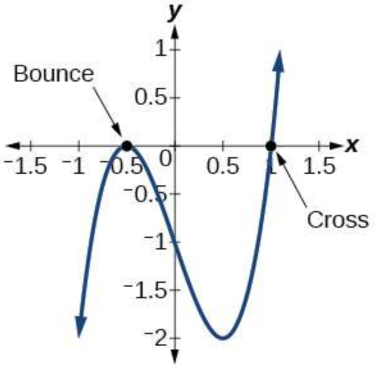


Figure 1

Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose f is a polynomial function of degree four, and $f(x) = 0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it c_1. By the Factor Theorem, we can write $f(x)$ as a product of $x - c_1$ and a polynomial quotient. Since $x - c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it c_2. So we can write the polynomial quotient as a product of $x - c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

The Fundamental Theorem of Algebra

The **Fundamental Theorem of Algebra** states that, if $f(x)$ is a polynomial of degree *n* > *0*, then $f(x)$ has at least one complex zero.

We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n > 0$, and a is a non-zero real number, then $f(x)$ has exactly n linear factors

$$f(x) = a(x - c_1)(x - c_2)...(x - c_n)$$

where $c_1, c_2, ..., c_n$ are complex numbers. Therefore, $f(x)$ has n roots if we allow for multiplicities.

Q&A **Does every polynomial have at least one imaginary zero?**

No. Real numbers are a subset of complex numbers, but not the other way around. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

EXAMPLE 6

Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of $f(x) = 3x^3 + 9x^2 + x + 3$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 3 and q is a factor of 3.

$$\begin{aligned}\frac{p}{q} &= \frac{\text{factor of constant term}}{\text{factor of leading coefficient}} \\ &= \frac{\text{factor of 3}}{\text{factor of 3}}\end{aligned}$$

The factors of 3 are ±1 and ±3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are $\pm3, \pm1, \text{and } \pm\frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let’s begin with –3.

$$\begin{array}{r|rrrr} -3 & 3 & 9 & 1 & 3 \\ & & -9 & 0 & -3 \\ \hline & 3 & 0 & 1 & 0 \end{array}$$

Dividing by $(x+3)$ gives a remainder of 0, so –3 is a zero of the function. The polynomial can be written as

$$(x+3)\left(3x^2+1\right)$$

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$\begin{aligned} 3x^2+1 &= 0 \\ x^2 &= -\tfrac{1}{3} \\ x &= \pm\sqrt{-\tfrac{1}{3}} = \pm\tfrac{i\sqrt{3}}{3} \end{aligned}$$

The zeros of $f(x)$ are –3 and $\pm\frac{i\sqrt{3}}{3}$.

Analysis

Look at the graph of the function f in Figure 2. Notice that, at $x=-3$, the graph crosses the *x*-axis, indicating an odd multiplicity (1) for the zero $x=-3$. Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the *x*-intercepts for the function are shown. So either the multiplicity of $x=-3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x=-3$ is three. Either way, our result is correct.

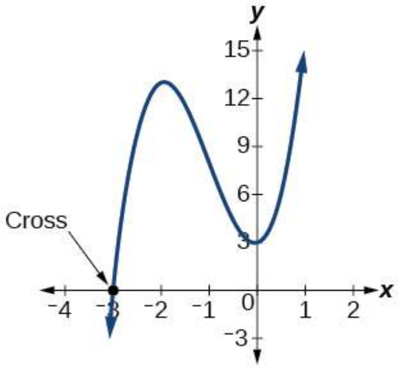


Figure 2

TRY IT #4 Find the zeros of $f(x)=2x^3+5x^2-11x+4$.

Using the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the Fundamental Theorem of Algebra, as we stated above, is that a polynomial function of degree n will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors. The **Linear Factorization Theorem** tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x-c)$, where c is a complex number.

Let f be a polynomial function with real coefficients, and suppose $a+bi$, $b\neq0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x-(a+bi)$ is a factor of $f(x)$. For f to have real coefficients, $x-(a-bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x-(a-bi)$, when multiplied by $x-(a+bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function f with real coefficients has a complex zero $a+bi$, then the complex conjugate $a-bi$ must also be a zero of $f(x)$. This is called the Complex Conjugate Theorem.

Complex Conjugate Theorem

According to the **Linear Factorization Theorem,** a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x - c)$, where c is a complex number.

If the polynomial function f has real coefficients and a complex zero in the form $a + bi$, then the complex conjugate of the zero, $a - bi$, is also a zero.

HOW TO

Given the zeros of a polynomial function f and a point (*c*, *f*(*c*)) on the graph of f, use the Linear Factorization Theorem to find the polynomial function.

1. Use the zeros to construct the linear factors of the polynomial.
2. Multiply the linear factors to expand the polynomial.
3. Substitute $(c, f(c))$ into the function to determine the leading coefficient.
4. Simplify.

EXAMPLE 7

Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of –3, 2, i, such that $f(-2) = 100$.

Solution

Because $x = i$ is a zero, by the Complex Conjugate Theorem $x =- i$ is also a zero. The polynomial must have factors of $(x + 3),\ \ (x - 2),\ \ (x - i)$, and $(x + i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$\begin{array}{rcl} f(x) & = & a(x+3)(x-2)(x-i)(x+i) \\ f(x) & = & a\left(x^2+x-6\right)\left(x^2+1\right) \\ f(x) & = & a\left(x^4+x^3-5x^2+x-6\right) \end{array}$$

We need to find *a* to ensure $f(-2) = 100$. Substitute $x =- 2$ and $f(2) = 100$ into $f(x)$.

$$\begin{array}{rcl} 100 & = & a((-2)^4+(-2)^3-5(-2)^2+(-2)-6) \\ 100 & = & a(-20) \\ -5 & = & a \end{array}$$

So the polynomial function is

$$f(x) = -5\left(x^4+x^3-5x^2+x-6\right)$$

or

$$f(x) = -5x^4-5x^3+25x^2-5x+30$$

Analysis

We found that both i and $-i$ were zeros, but only one of these zeros needed to be given. If i is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of i.

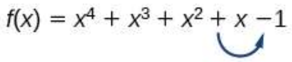
Q&A **If $2 + 3i$ were given as a zero of a polynomial with real coefficients, would $2 - 3i$ also need to be a zero?**

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

TRY IT #5 Find a third degree polynomial with real coefficients that has zeros of 5 and $-2i$ such that $f(1) = 10.$

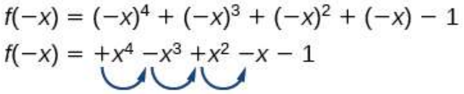
Using Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.



This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.



In this case, $f(-x)$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

Descartes' Rule of Signs

According to **Descartes' Rule of Signs**, if we let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.

EXAMPLE 8

Using Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12.$

Solution

Begin by determining the number of sign changes.

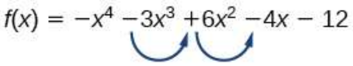


Figure 3

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine $f(-x)$ to determine the number of negative real roots.

$$\begin{aligned} f(-x) &= -(-x)^4 - 3(-x)^3 + 6(-x)^2 - 4(-x) - 12 \\ f(-x) &= -x^4 + 3x^3 + 6x^2 + 4x - 12 \end{aligned}$$

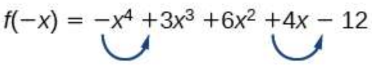


Figure 4

Again, there are two sign changes, so there are either 2 or 0 negative real roots.

There are four possibilities, as we can see in Table 1.

Positive Real Zeros	Negative Real Zeros	Complex Zeros	Total Zeros
2	2	0	4
2	0	2	4
0	2	2	4
0	0	4	4

Table 1

Analysis

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See Figure 5. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

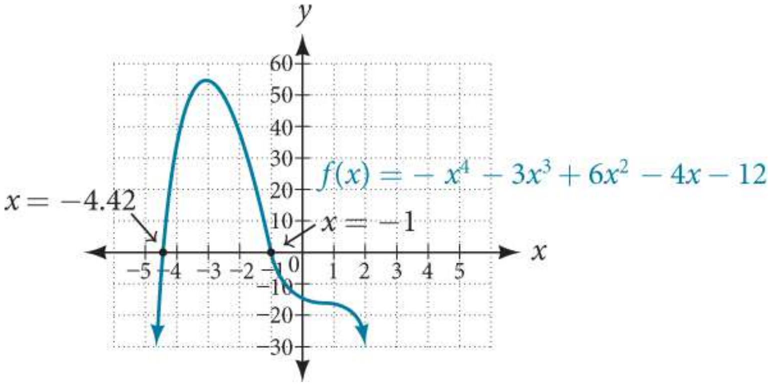


Figure 5

TRY IT #6 Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12$. Use a graph to verify the numbers of positive and negative real zeros for the function.

Solving Real-World Applications

We have now introduced a variety of tools for solving polynomial equations. Let's use these tools to solve the bakery problem from the beginning of the section.

EXAMPLE 9

Solving Polynomial Equations

A new bakery offers decorated, multi-tiered cakes for display and cutting at Quinceañera and wedding celebrations, as well as sheet cakes for children's birthday parties and other special occasions to serve most of the guests. The bakery wants the volume of a small sheet cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

Solution

Begin by writing an equation for the volume of the cake. The volume of a rectangular solid is given by $V = lwh$. We were given that the length must be four inches longer than the width, so we can express the length of the cake as $l = w + 4$. We were given that the height of the cake is one-third of the width, so we can express the height of the cake as $h = \frac{1}{3}w$. Let's write the volume of the cake in terms of width of the cake.

$$\begin{aligned} V &= (w+4)(w)\left(\tfrac{1}{3}w\right) \\ V &= \tfrac{1}{3}w^3 + \tfrac{4}{3}w^2 \end{aligned}$$

Substitute the given volume into this equation.

$$\begin{aligned} 351 &= \tfrac{1}{3}w^3 + \tfrac{4}{3}w^2 && \text{Substitute 351 for } V. \\ 1053 &= w^3 + 4w^2 && \text{Multiply both sides by 3.} \\ 0 &= w^3 + 7w^2 - 1053 && \text{Subtract 1053 from both sides.} \end{aligned}$$

Descartes' rule of signs tells us there is one positive solution. The Rational Zero Theorem tells us that the possible rational zeros are $\pm 1, \ \pm 3, \ \pm 9, \ \pm 13, \ \pm 27, \ \pm 39, \ \pm 81, \ \pm 117, \ \pm 351,$ and ± 1053. We can use synthetic division to test these possible zeros. Only positive numbers make sense as dimensions for a cake, so we need not test any negative values. Let’s begin by testing values that make the most sense as dimensions for a small sheet cake. Use synthetic division to check $x = 1$.

$$\begin{array}{r|rrrr} 1 & 1 & 4 & 0 & -1053 \\ & & 1 & 5 & 5 \\ \hline & 1 & 5 & 5 & -1048 \end{array}$$

Since 1 is not a solution, we will check $x = 3$.

$$\begin{array}{r|rrrr} 3 & 1 & 4 & 0 & -1053 \\ & & 3 & 21 & 63 \\ \hline & 1 & 7 & 21 & -990 \end{array}$$

Since 3 is not a solution either, we will test $x = 9$.

$$\begin{array}{r|rrrr} 9 & 1 & 4 & 0 & -1053 \\ & & 9 & 117 & 1053 \\ \hline & 1 & 13 & 117 & 0 \end{array}$$

Synthetic division gives a remainder of 0, so 9 is a solution to the equation. We can use the relationships between the width and the other dimensions to determine the length and height of the sheet cake pan.

$$l = w + 4 = 9 + 4 = 13 \ \text{ and } h = \frac{1}{3}w = \frac{1}{3}(9) = 3$$

The sheet cake pan should have dimensions 13 inches by 9 inches by 3 inches.

TRY IT #7 A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?

MEDIA

Access these online resources for additional instruction and practice with zeros of polynomial functions.

Real Zeros, Factors, and Graphs of Polynomial Functions (http://openstax.org/l/realzeros)
Complex Factorization Theorem (http://openstax.org/l/factortheorem)
Find the Zeros of a Polynomial Function (http://openstax.org/l/findthezeros)
Find the Zeros of a Polynomial Function 2 (http://openstax.org/l/findthezeros2)
Find the Zeros of a Polynomial Function 3 (http://openstax.org/l/findthezeros3)

5.5 SECTION EXERCISES

Verbal

1. Describe a use for the Remainder Theorem.

2. Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.

3. What is the difference between rational and real zeros?

4. If Descartes' Rule of Signs reveals a no change of signs or one sign of changes, what specific conclusion can be drawn?

5. If synthetic division reveals a zero, why should we try that value again as a possible solution?

Algebraic

For the following exercises, use the Remainder Theorem to find the remainder.

6. $\left(x^4 - 9x^2 + 14\right) \div (x - 2)$

7. $\left(3x^3 - 2x^2 + x - 4\right) \div (x + 3)$

8. $\left(x^4 + 5x^3 - 4x - 17\right) \div (x + 1)$

9. $\left(-3x^2 + 6x + 24\right) \div (x - 4)$

10. $\left(5x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1\right) \div (x + 6)$

11. $\left(x^4 - 1\right) \div (x - 4)$

12. $\left(3x^3 + 4x^2 - 8x + 2\right) \div (x - 3)$

13. $\left(4x^3 + 5x^2 - 2x + 7\right) \div (x + 2)$

For the following exercises, use the Factor Theorem to find all real zeros for the given polynomial function and one factor.

14. $f(x) = 2x^3 - 9x^2 + 13x - 6;\ \ x - 1$

15. $f(x) = 2x^3 + x^2 - 5x + 2;\ \ x + 2$

16. $f(x) = 3x^3 + x^2 - 20x + 12;\ \ x + 3$

17. $f(x) = 2x^3 + 3x^2 + x + 6;\ \ x + 2$

18. $f(x) = -5x^3 + 16x^2 - 9;\ \ x - 3$

19. $x^3 + 3x^2 + 4x + 12;\ \ x + 3$

20. $4x^3 - 7x + 3;\ \ x - 1$

21. $2x^3 + 5x^2 - 12x - 30,\ \ \ 2x + 5$

For the following exercises, use the Rational Zero Theorem to find the real solution(s) to each equation.

22. $x^3 - 3x^2 - 10x + 24 = 0$

23. $2x^3 + 7x^2 - 10x - 24 = 0$

24. $x^3 + 2x^2 - 9x - 18 = 0$

25. $x^3 + 5x^2 - 16x - 80 = 0$

26. $x^3 - 3x^2 - 25x + 75 = 0$

27. $2x^3 - 3x^2 - 32x - 15 = 0$

28. $2x^3 + x^2 - 7x - 6 = 0$

29. $2x^3 - 3x^2 - x + 1 = 0$

30. $3x^3 - x^2 - 11x - 6 = 0$

31. $2x^3 - 5x^2 + 9x - 9 = 0$

32. $2x^3 - 3x^2 + 4x + 3 = 0$

33. $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$

34. $x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$

35. $4x^4 + 4x^3 - 25x^2 - x + 6 = 0$

36. $2x^4 - 3x^3 - 15x^2 + 32x - 12 = 0$

37. $x^4 + 2x^3 - 4x^2 - 10x - 5 = 0$

38. $4x^3 - 3x + 1 = 0$

39. $8x^4 + 26x^3 + 39x^2 + 26x + 6$

For the following exercises, find all complex solutions (real and non-real).

40. $x^3 + x^2 + x + 1 = 0$

41. $x^3 - 8x^2 + 25x - 26 = 0$

42. $x^3 + 13x^2 + 57x + 85 = 0$

43. $3x^3 - 4x^2 + 11x + 10 = 0$

44. $x^4 + 2x^3 + 22x^2 + 50x - 75 = 0$

45. $2x^3 - 3x^2 + 32x + 17 = 0$

Graphical

For the following exercises, use Descartes' Rule to determine the possible number of positive and negative solutions. Confirm with the given graph.

46. $f(x) = x^3 - 1$

47. $f(x) = x^4 - x^2 - 1$

48. $f(x) = x^3 - 2x^2 - 5x + 6$

49. $f(x) = x^3 - 2x^2 + x - 1$

50. $f(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

51. $f(x) = 2x^3 + 37x^2 + 200x + 300$

52. $f(x) = x^3 - 2x^2 - 16x + 32$

53. $f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$

54. $f(x) = 2x^4 - 5x^3 - 14x^2 + 20x + 8$

55. $f(x) = 10x^4 - 21x^2 + 11$

Numeric

For the following exercises, list all possible rational zeros for the functions.

56. $f(x) = x^4 + 3x^3 - 4x + 4$

57. $f(x) = 2x^3 + 3x^2 - 8x + 5$

58. $f(x) = 3x^3 + 5x^2 - 5x + 4$

59. $f(x) = 6x^4 - 10x^2 + 13x + 1$

60. $f(x) = 4x^5 - 10x^4 + 8x^3 + x^2 - 8$

Technology

For the following exercises, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.

61. $f(x) = 6x^3 - 7x^2 + 1$

62. $f(x) = 4x^3 - 4x^2 - 13x - 5$

63. $f(x) = 8x^3 - 6x^2 - 23x + 6$

64. $f(x) = 12x^4 + 55x^3 + 12x^2 - 117x + 54$

65. $f(x) = 16x^4 - 24x^3 + x^2 - 15x + 25$

Extensions

For the following exercises, construct a polynomial function of least degree possible using the given information.

66. Real roots: –1, 1, 3 and $(2, f(2)) = (2, 4)$

67. Real roots: –1, 1 (with multiplicity 2 and 1) and $(2, f(2)) = (2, 4)$

68. Real roots: –2, $\frac{1}{2}$ (with multiplicity 2) and $(-3, f(-3)) = (-3, 5)$

69. Real roots: $-\frac{1}{2}$, 0, $\frac{1}{2}$ and $(-2, f(-2)) = (-2, 6)$

70. Real roots: –4, –1, 1, 4 and $(-2, f(-2)) = (-2, 10)$

Real-World Applications

For the following exercises, find the dimensions of the box described.

71. The length is twice as long as the width. The height is 2 inches greater than the width. The volume is 192 cubic inches.

72. The length, width, and height are consecutive whole numbers. The volume is 120 cubic inches.

73. The length is one inch more than the width, which is one inch more than the height. The volume is 86.625 cubic inches.

74. The length is three times the height and the height is one inch less than the width. The volume is 108 cubic inches.

75. The length is 3 inches more than the width. The width is 2 inches more than the height. The volume is 120 cubic inches.

For the following exercises, find the dimensions of the right circular cylinder described.

76. The radius is 3 inches more than the height. The volume is 16π cubic meters.

77. The height is one less than one half the radius. The volume is 72π cubic meters.

78. The radius and height differ by one meter. The radius is larger and the volume is 48π cubic meters.

79. The radius and height differ by two meters. The height is greater and the volume is 28.125π cubic meters.

80. The radius is $\frac{1}{3}$ meter greater than the height. The volume is $\frac{98}{9}\pi$ cubic meters.

5.6 Rational Functions

Learning Objectives

In this section, you will:

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, x, produced. This is given by the equation $C(x) = 15{,}000x - 0.1x^2 + 1000$. If we want to know the average cost for producing x items, we would divide the cost function by the number of items, x.

The average cost function, which yields the average cost per item for x items produced, is

$$f(x) = \frac{15{,}000x - 0.1x^2 + 1000}{x}$$

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

Using Arrow Notation

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure 1, and notice some of their features.

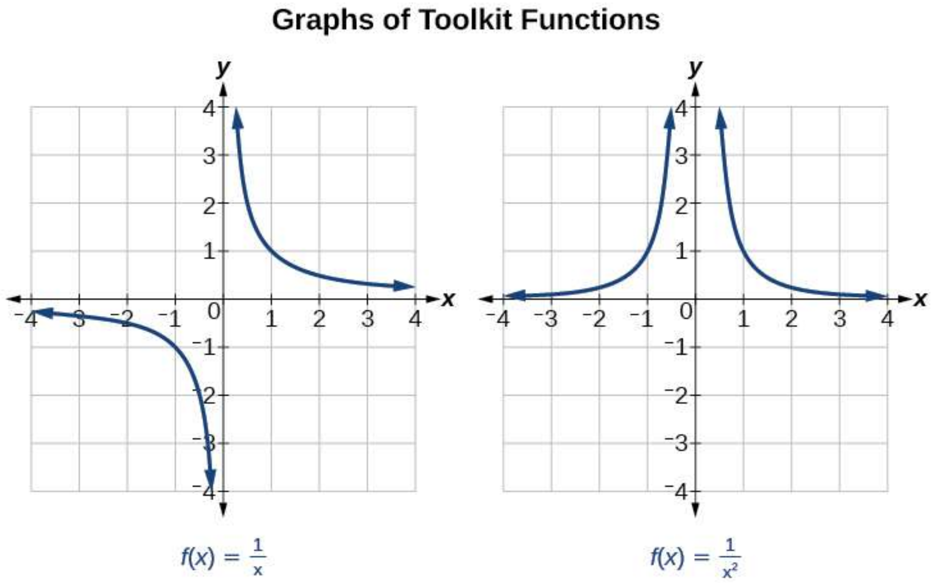


Figure 1

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

1. On the left branch of the graph, the curve approaches the *x*-axis $(y = 0)$ as $x \to -\infty$.
2. As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
3. Finally, on the right branch of the graph, the curves approaches the *x*-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value. See Table 1.

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Table 1

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 2.

x	–0.1	–0.01	–0.001	–0.0001
$f(x) = \frac{1}{x}$	–10	–100	–1000	–10,000

Table 2

We write in arrow notation

$$\text{as } x \to 0^{-}, f(x) \to -\infty$$

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 3.

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	10,000

Table 3

We write in arrow notation

$$\text{As } x \to 0^{+},\ f(x) \to \infty.$$

See Figure 2.

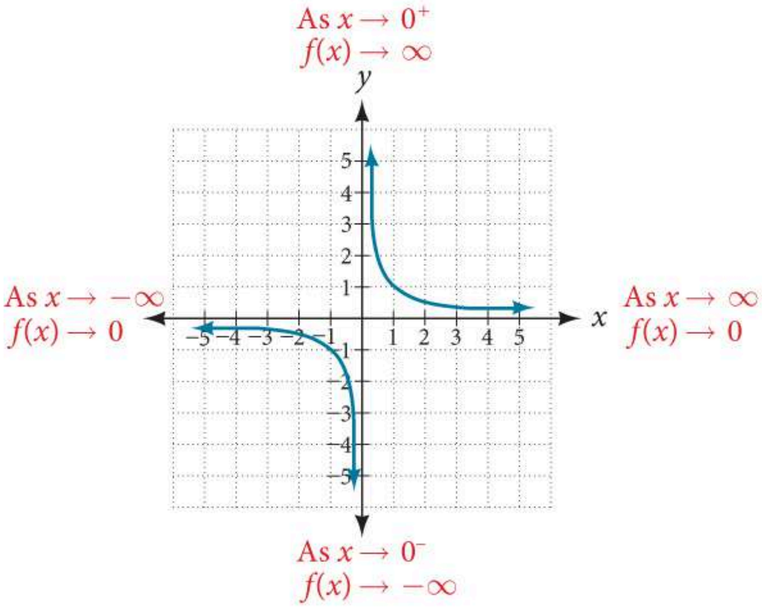


Figure 2

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero. See Figure 3.

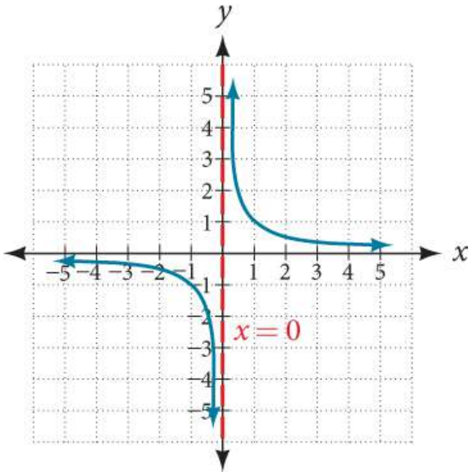


Figure 3

Vertical Asymptote

A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a. We write

$$\text{As } x \to a,\ f(x) \to \pm\infty.$$

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0. See Figure 4. Symbolically, using arrow notation

$$\text{As } x \to \infty,\ f(x) \to 0, \text{and as } x \to -\infty,\ f(x) \to 0.$$

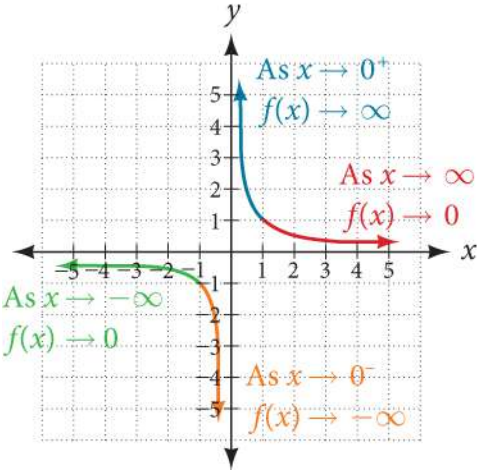


Figure 4

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure 5.

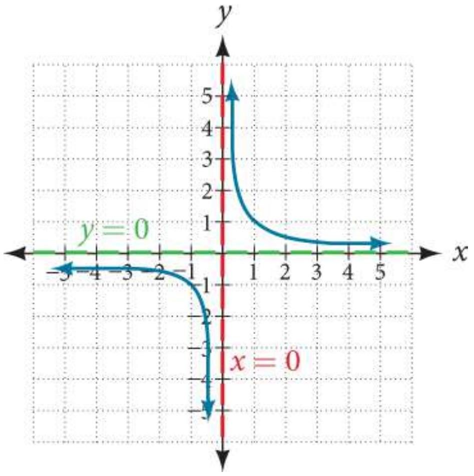


Figure 5

Horizontal Asymptote

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty, \quad f(x) \to b.$$

EXAMPLE 1

Using Arrow Notation

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 6.

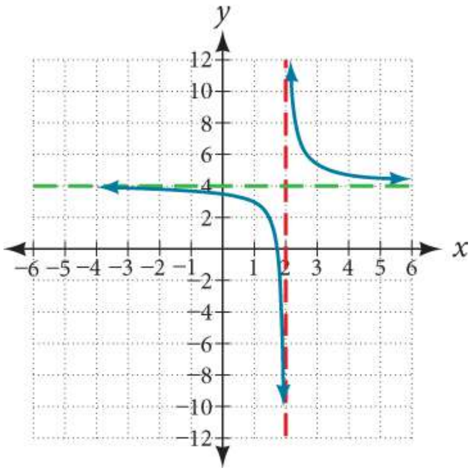


Figure 6

Solution

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

$$\text{As } x \to 2^-, f(x) \to -\infty, \quad \text{and as } x \to 2^+, \quad f(x) \to \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph levels off at 4.

$$\text{As } x \to \infty, \quad f(x) \to 4 \text{ and as } x \to -\infty, \quad f(x) \to 4.$$

TRY IT #1 Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

EXAMPLE 2

Using Transformations to Graph a Rational Function

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2}$$

The graph of the shifted function is displayed in Figure 7.

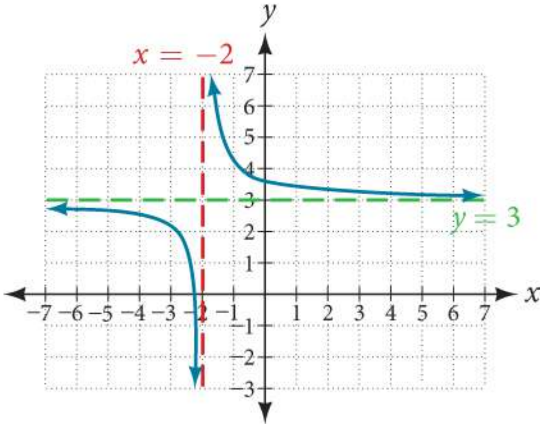


Figure 7

Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \to -2^-,\ f(x) \to -\infty, \text{and as } x \to -2^+,\ f(x) \to \infty.$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \to \pm\infty,\ f(x) \to 3.$$

Analysis

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

TRY IT #2 Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Solving Applied Problems Involving Rational Functions

In Example 2, we shifted a toolkit function in a way that resulted in the function $f(x) = \frac{3x+7}{x+2}$. This is an example of a rational function. A **rational function** is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Rational Function

A **rational function** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1} x^{p-1} + ... + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + ... + b_1 x + b_0}, Q(x) \neq 0$$

EXAMPLE 3

Solving an Applied Problem Involving a Rational Function

After running out of pre-packaged supplies, a nurse in a refugee camp is preparing an intravenous sugar solution for patients in the camp hospital. A large mixing tank currently contains 100 gallons of distilled water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the ratio of sugar to water, in pounds per gallon in the tank after 12 minutes. Is that a greater ratio of sugar to water, in pounds per gallon than at the beginning?

Solution

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\begin{aligned} \text{water: } W(t) &= 100 + 10t \text{ in gallons} \\ \text{sugar: } S(t) &= 5 + 1t \text{ in pounds} \end{aligned}$$

The ratio of sugar to water, in pounds per gallon, C , will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5 + t}{100 + 10t}$$

The ratio of sugar to water, in pounds per gallon after 12 minutes is given by evaluating $C(t)$ at $t = 12$.

$$\begin{aligned} C(12) &= \frac{5+12}{100+10(12)} \\ &= \frac{17}{220} \end{aligned}$$

This means the ratio of sugar to water, in pounds per gallon is 17 pounds of sugar to 220 gallons of water.

At the beginning, the ratio of sugar to water, in pounds per gallon is

$$\begin{aligned} C(0) &= \frac{5+0}{100+10(0)} \\ &= \frac{1}{20} \end{aligned}$$

Since $\frac{17}{220} \approx 0.08 > \frac{1}{20} = 0.05$, the ratio of sugar to water, in pounds per gallon is greater after 12 minutes than at the beginning.

TRY IT #3 There are 1,200 first-year and 1,500 second-year students at a rally at noon. After 12 p.m., 20 first-year students arrive at the rally every five minutes while 15 second-year students leave the rally. Find the ratio of first-year to second-year students at 1 p.m.

Finding the Domains of Rational Functions

A vertical asymptote represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Domain of a Rational Function

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

HOW TO

Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the *x*-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

EXAMPLE 4

Finding the Domain of a Rational Function

Find the domain of $f(x) = \frac{x+3}{x^2-9}$.

Solution

Begin by setting the denominator equal to zero and solving.

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

Analysis

A graph of this function, as shown in Figure 8, confirms that the function is not defined when $x = \pm 3$.

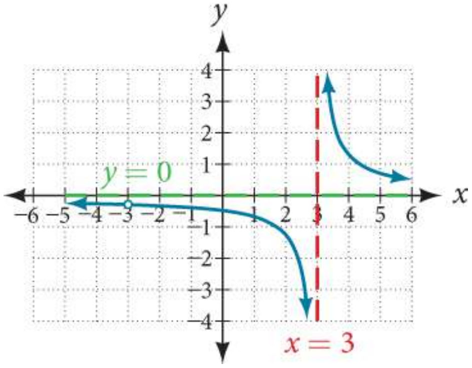


Figure 8

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

TRY IT #4 Find the domain of $f(x) = \frac{4x}{5(x-1)(x-5)}$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

HOW TO

Given a rational function, identify any vertical asymptotes of its graph.

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or "holes."

EXAMPLE 5

Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$\begin{aligned} k(x) &= \frac{5+2x^2}{2-x-x^2} \\ &= \frac{5+2x^2}{(2+x)(1-x)} \end{aligned}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$\begin{aligned} (2+x)(1-x) &= 0 \\ x &= -2, 1 \end{aligned}$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 9 confirms the location of the two vertical asymptotes.

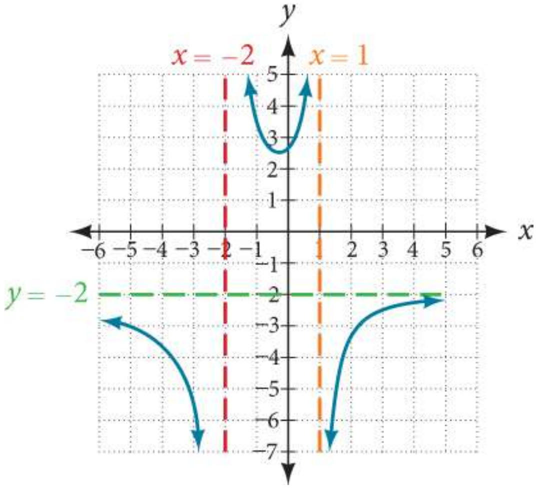


Figure 9

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a **removable discontinuity**.

For example, the function $f(x) = \frac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)\,(x-1)}{(x+1)\,(x-3)}$$

Notice that $x + 1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $x - 3$ is not a factor in both the numerator and denominator. The zero of this factor, $x = 3$, is the vertical asymptote. See Figure 10. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

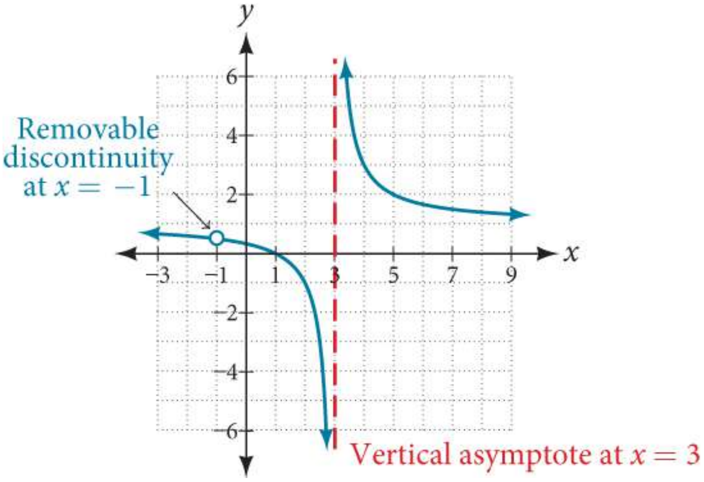


Figure 10

Removable Discontinuities of Rational Functions

A **removable discontinuity** occurs in the graph of a rational function at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

EXAMPLE 6

Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \frac{x-2}{x^2-4}$.

Solution

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x - 2$. The zero for this factor is $x = 2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x + 2$. The zero for this factor is $x = -2$. The vertical asymptote is $x = -2$. See Figure 11.

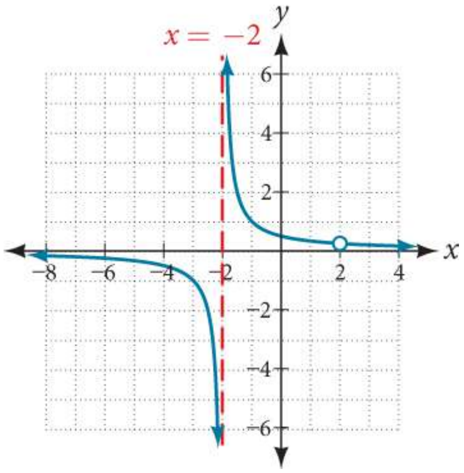


Figure 11

The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole.

TRY IT #5 Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2-25}{x^3-6x^2+5x}$.

Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator > degree of the numerator, there is a horizontal asymptote at $y = 0$.

$$\text{Example: } f(x) = \frac{4x + 2}{x^2 + 4x - 5}$$

In this case, the end behavior is $f(x) \approx \frac{4x}{x^2} = \frac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = \frac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$. See Figure 12. Note that this graph crosses the horizontal asymptote.

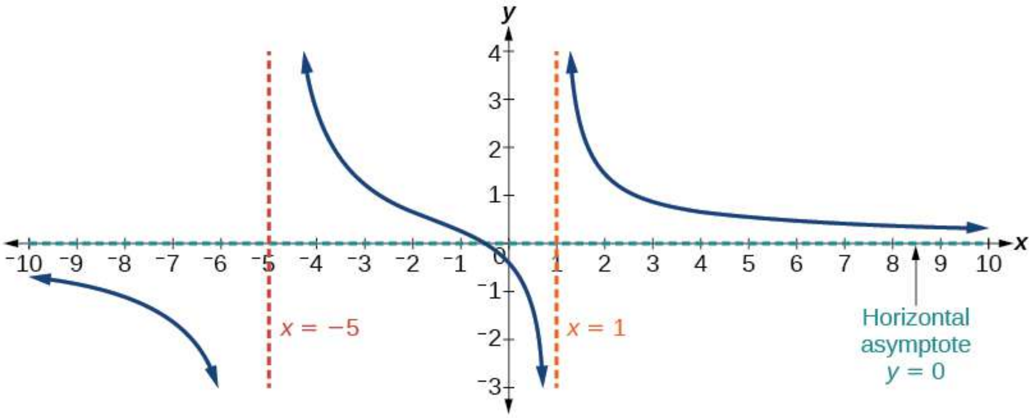


Figure 12 Horizontal asymptote $y = 0$ when $f(x) = \frac{p(x)}{q(x)},\quad q(x) \neq 0\ \text{ where degree of }\ p < \text{degree of } q.$

Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

$$\text{Example: } f(x) = \frac{3x^2 - 2x + 1}{x - 1}$$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x} = 3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ looks like a diagonal line, and since f will behave similarly to g, it will approach a line close to $y = 3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\frac{3x^2-2x+1}{x-1}$. The quotient is $3x + 1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x) = 3x + 1$. See Figure 13.

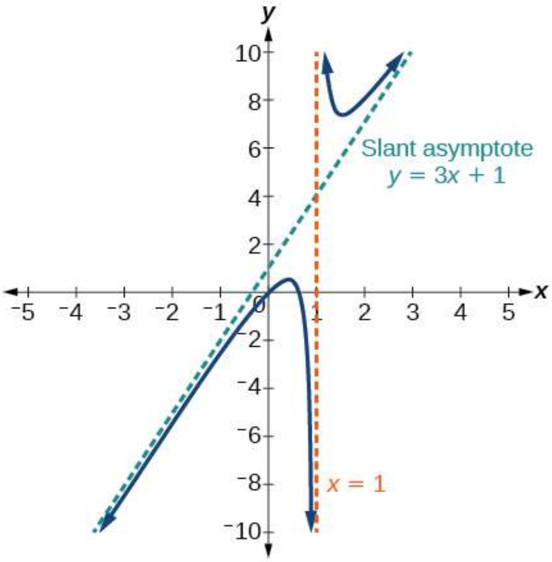


Figure 13 Slant asymptote when $f(x) = \frac{p(x)}{q(x)}, \ q(x) \neq 0$ where degree of $p >$ degree of q by 1.

Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where a_n and b_n are the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \frac{p(x)}{q(x)}, q(x) \neq 0$.

$$\text{Example: } f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See Figure 14. Note that this graph crosses the horizontal asymptote.

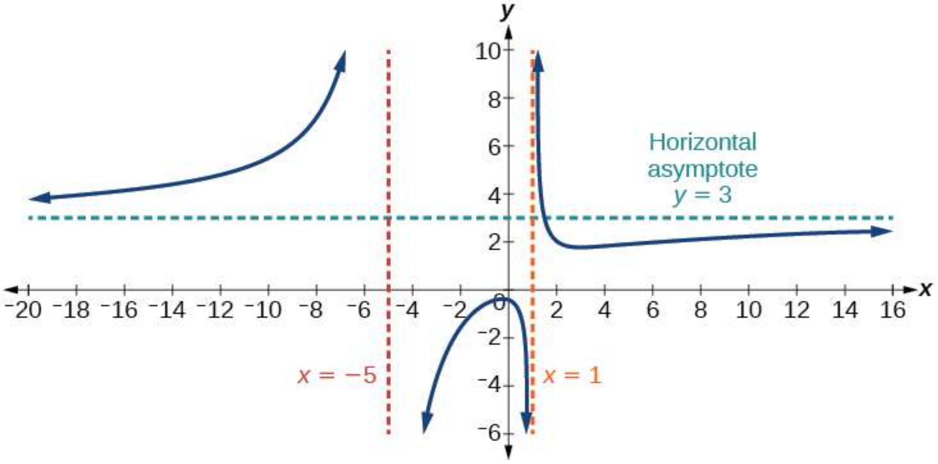


Figure 14 Horizontal asymptote when $f(x) = \frac{p(x)}{q(x)}, \ q(x) \neq 0$ where degree of $p =$ degree of q.

Notice that, while the graph of a rational function will never cross a vertical asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the end behavior of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4,$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, \ f(x) \to \infty$$

Horizontal Asymptotes of Rational Functions

The horizontal asymptote of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator *is less than* degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator *is greater than degree of denominator by one*: no horizontal asymptote; slant asymptote.
- Degree of numerator *is equal to* degree of denominator: horizontal asymptote at ratio of leading coefficients.

EXAMPLE 7

Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

ⓐ $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$ ⓑ $h(x) = \frac{x^2 - 4x + 1}{x + 2}$ ⓒ $k(x) = \frac{x^2 + 4x}{x^3 - 8}$

Solution

For these solutions, we will use $f(x) = \frac{p(x)}{q(x)}, \ q(x) \neq 0.$

ⓐ $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of $p = \text{degree of } q = 3,$ so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3.$

ⓑ $h(x) = \frac{x^2 - 4x + 1}{x + 2}$: The degree of $p = 2$ and degree of $q = 1.$ Since $p > q$ by 1, there is a slant asymptote found at $\frac{x^2 - 4x + 1}{x + 2}.$

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

The quotient is $x - 6$ and the remainder is 13. There is a slant asymptote at $y = x - 6.$

ⓒ $k(x) = \frac{x^2 + 4x}{x^3 - 8}$: The degree of $p = 2 \ < \text{degree of } q = 3,$ so there is a horizontal asymptote $y = 0.$

EXAMPLE 8

Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \frac{5 + t}{100 + 10t}.$

Find the horizontal asymptote and interpret it in context of the problem.

Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty, \; C(t) \to \frac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

EXAMPLE 9

Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2, \text{and } 5$, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty, \; f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 15.

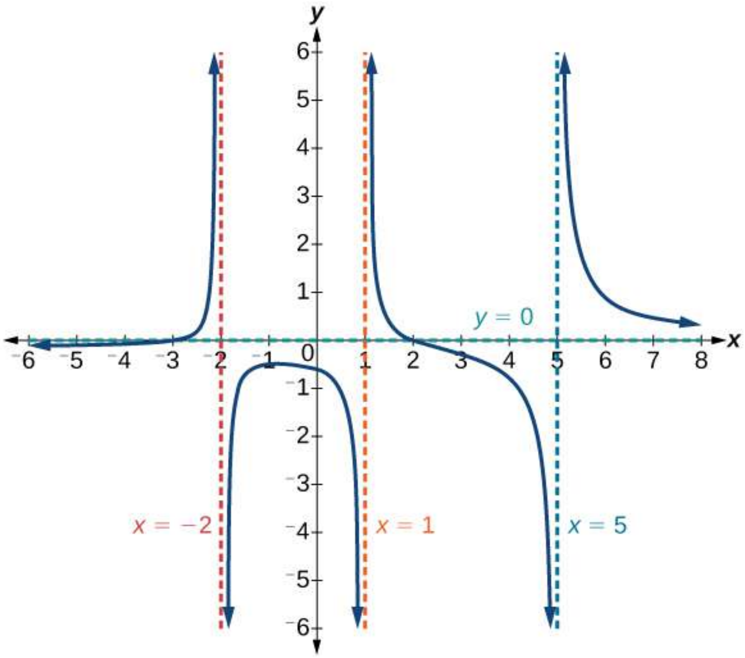


Figure 15

TRY IT #6 Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Intercepts of Rational Functions

A rational function will have a *y*-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a *y*-intercept if the function is not defined at zero.

Likewise, a rational function will have *x*-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, *x*-intercepts can only occur when the numerator of the rational function is equal to zero.

EXAMPLE 10

Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution

We can find the *y*-intercept by evaluating the function at zero

$$\begin{aligned} f(0) &= \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\ &= \frac{-6}{10} \\ &= -\frac{3}{5} \\ &= -0.6 \end{aligned}$$

The *x*-intercepts will occur when the function is equal to zero:

$$\begin{aligned} 0 &= \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} && \text{This is zero when the numerator is zero.} \\ 0 &= (x-2)(x+3) \\ x &= 2, -3 \end{aligned}$$

The *y*-intercept is $(0, -0.6)$, the *x*-intercepts are $(2, 0)$ and $(-3, 0)$. See Figure 16.

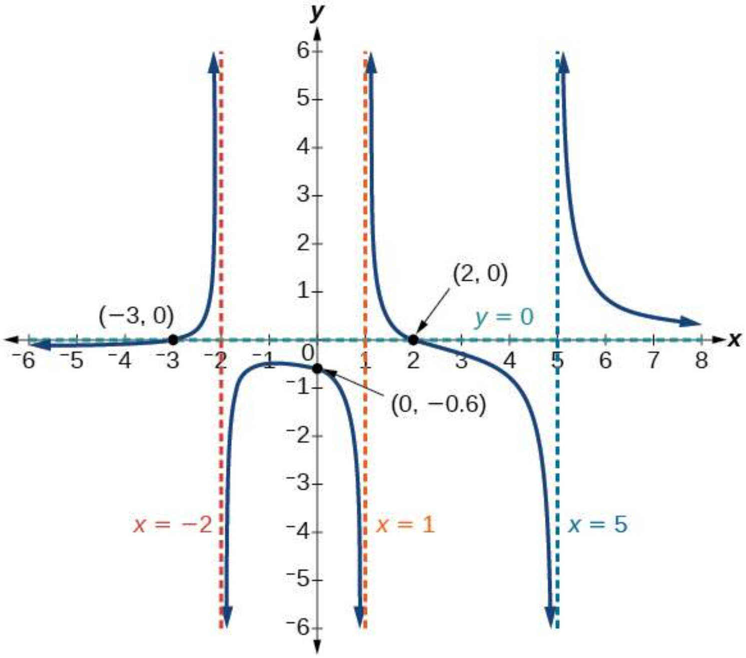


Figure 16

TRY IT #7 Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the *x*- and *y*-intercepts and the horizontal and vertical asymptotes.

Graphing Rational Functions

In Example 9, we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 17.

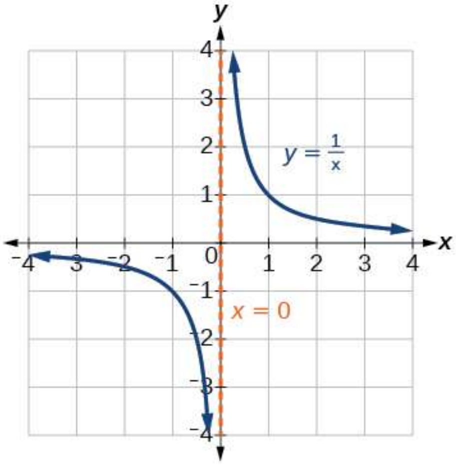


Figure 17

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 18.

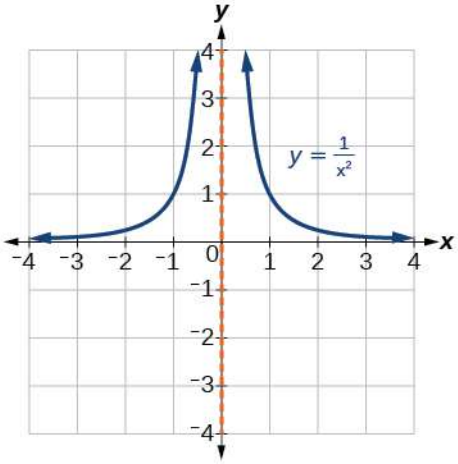


Figure 18

For example, the graph of $f(x) = \frac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 19.

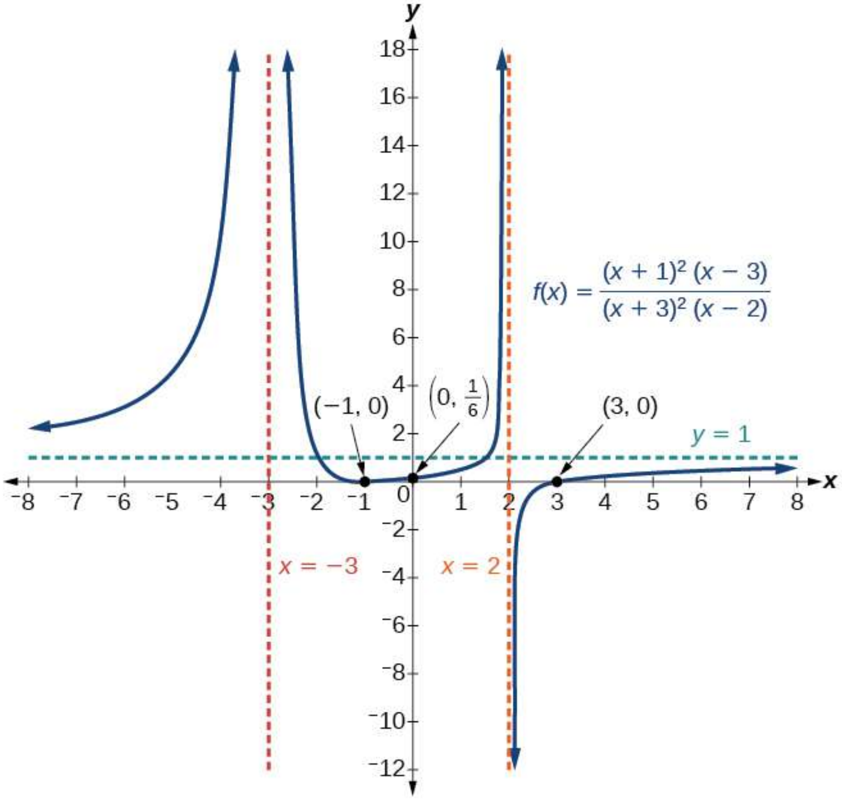


Figure 19

- At the *x*-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the *x*-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side.

HOW TO

Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the *y*-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the *x*-intercepts.
4. Find the multiplicities of the *x*-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

EXAMPLE 11

Graphing a Rational Function

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution

We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the *y*-intercept:

$$\begin{aligned} f(0) &= \frac{(0+2)(0-3)}{(0+1)^2(0-2)} \\ &= 3 \end{aligned}$$

To find the *x*-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find *x*-intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a *y*-intercept at $(0, 3)$ and *x*-intercepts at $(-2, 0)$ and $(3, 0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x + 1 = 0$ and when $x-2 = 0$, giving us vertical asymptotes at $x = -1$ and $x = 2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no *x*-intercepts between the vertical asymptotes, and the *y*-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 20.

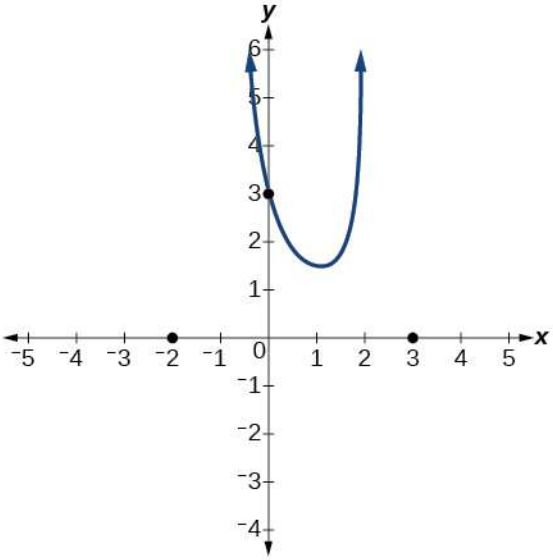


Figure 20

The factor associated with the vertical asymptote at $x = -1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 21. After passing through the *x*-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

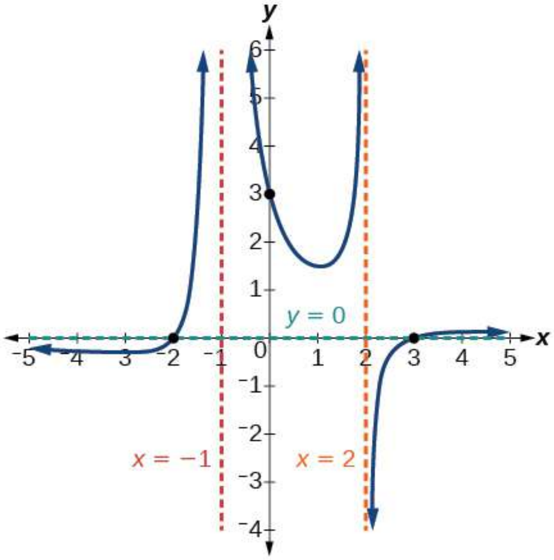


Figure 21

TRY IT #8 Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an *x*-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of *x*-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Writing Rational Functions from Intercepts and Asymptotes

If a rational function has *x*-intercepts at $x = x_1, x_2, ..., x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i = \text{any } v_j$, then the function can be written in the form:

$$f(x) = a\frac{(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}}{(x - v_1)^{q_1}(x - v_2)^{q_2} \cdots (x - v_m)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the *x*-intercept or by the horizontal asymptote if it is nonzero.

HOW TO

Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the *x*-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the “simplest” function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to

determine the factors and their powers.

3. Use any clear point on the graph to find the stretch factor.

EXAMPLE 12

Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 22.

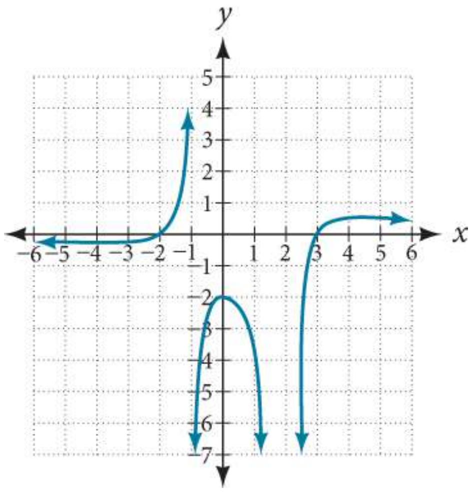


Figure 22

Solution

The graph appears to have *x*-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure 23.

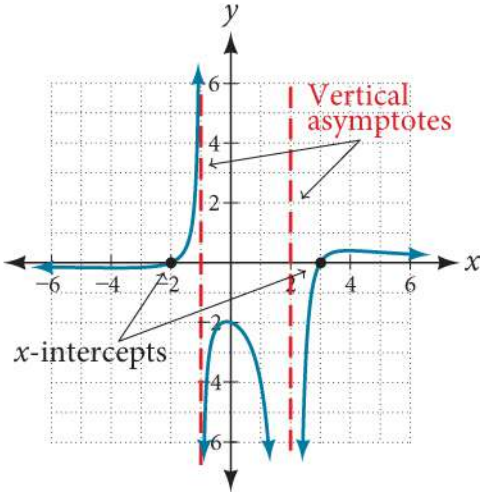


Figure 23

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0, -2)$.

$$\begin{aligned} -2 &= a\frac{(0+2)(0-3)}{(0+1)(0-2)^2} \\ -2 &= a\frac{-6}{4} \\ a &= \frac{-8}{-6} = \frac{4}{3} \end{aligned}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

MEDIA

Access these online resources for additional instruction and practice with rational functions.

Graphing Rational Functions (http://openstax.org/l/graphrational)
Find the Equation of a Rational Function (http://openstax.org/l/equatrational)
Determining Vertical and Horizontal Asymptotes (http://openstax.org/l/asymptote)
Find the Intercepts, Asymptotes, and Hole of a Rational Function (http://openstax.org/l/interasymptote)

5.6 SECTION EXERCISES

Verbal

1. What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?

2. What is the fundamental difference in the graphs of polynomial functions and rational functions?

3. If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?

4. Can a graph of a rational function have no vertical asymptote? If so, how?

5. Can a graph of a rational function have no *x*-intercepts? If so, how?

Algebraic

For the following exercises, find the domain of the rational functions.

6. $f(x) = \frac{x-1}{x+2}$

7. $f(x) = \frac{x+1}{x^2-1}$

8. $f(x) = \frac{x^2+4}{x^2-2x-8}$

9. $f(x) = \frac{x^2+4x-3}{x^4-5x^2+4}$

For the following exercises, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.

10. $f(x) = \frac{4}{x-1}$

11. $f(x) = \frac{2}{5x+2}$

12. $f(x) = \frac{x}{x^2-9}$

13. $f(x) = \frac{x}{x^2+5x-36}$

14. $f(x) = \frac{3+x}{x^3-27}$

15. $f(x) = \frac{3x-4}{x^3-16x}$

16. $f(x) = \frac{x^2-1}{x^3+9x^2+14x}$

17. $f(x) = \frac{x+5}{x^2-25}$

18. $f(x) = \frac{x-4}{x-6}$

19. $f(x) = \frac{4-2x}{3x-1}$

For the following exercises, find the x- and y-intercepts for the functions.

20. $f(x) = \frac{x+5}{x^2+4}$

21. $f(x) = \frac{x}{x^2-x}$

22. $f(x) = \frac{x^2+8x+7}{x^2+11x+30}$

23. $f(x) = \frac{x^2+x+6}{x^2-10x+24}$

24. $f(x) = \frac{94-2x^2}{3x^2-12}$

For the following exercises, describe the local and end behavior of the functions.

25. $f(x) = \frac{x}{2x+1}$

26. $f(x) = \frac{2x}{x-6}$

27. $f(x) = \frac{-2x}{x-6}$

28. $f(x) = \frac{x^2-4x+3}{x^2-4x-5}$

29. $f(x) = \frac{2x^2-32}{6x^2+13x-5}$

For the following exercises, find the slant asymptote of the functions.

30. $f(x) = \frac{24x^2+6x}{2x+1}$

31. $f(x) = \frac{4x^2-10}{2x-4}$

32. $f(x) = \frac{81x^2-18}{3x-2}$

33. $f(x) = \frac{6x^3-5x}{3x^2+4}$

34. $f(x) = \frac{x^2+5x+4}{x-1}$

Graphical

For the following exercises, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

35. The reciprocal function shifted up two units.

36. The reciprocal function shifted down one unit and left three units.

37. The reciprocal squared function shifted to the right 2 units.

38. The reciprocal squared function shifted down 2 units and right 1 unit.

For the following exercises, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.

39. $p(x) = \frac{2x-3}{x+4}$

40. $q(x) = \frac{x-5}{3x-1}$

41. $s(x) = \frac{4}{(x-2)^2}$

42. $r(x) = \frac{5}{(x+1)^2}$

43. $f(x) = \frac{3x^2-14x-5}{3x^2+8x-16}$

44. $g(x) = \frac{2x^2+7x-15}{3x^2-14x+15}$

45. $a(x) = \frac{x^2+2x-3}{x^2-1}$

46. $b(x) = \frac{x^2-x-6}{x^2-4}$

47. $h(x) = \frac{2x^2+x-1}{x-4}$

48. $k(x) = \frac{2x^2-3x-20}{x-5}$

49. $w(x) = \frac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

50. $z(x) = \frac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

For the following exercises, write an equation for a rational function with the given characteristics.

51. Vertical asymptotes at $x = 5$ and $x = -5$, x-intercepts at $(2, 0)$ and $(-1, 0)$, y-intercept at $(0, 4)$

52. Vertical asymptotes at $x = -4$ and $x = -1$, x-intercepts at $(1, 0)$ and $(5, 0)$, y-intercept at $(0, 7)$

53. Vertical asymptotes at $x = -4$ and $x = -5$, x-intercepts at $(4, 0)$ and $(-6, 0)$, Horizontal asymptote at $y = 7$

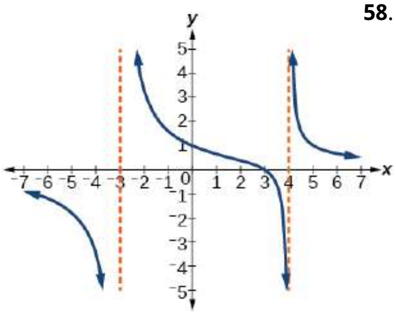
54. Vertical asymptotes at $x = -3$ and $x = 6$, x-intercepts at $(-2, 0)$ and $(1, 0)$, Horizontal asymptote at $y = -2$

55. Vertical asymptote at $x = -1$, Double zero at $x = 2$, y-intercept at $(0, 2)$

56. Vertical asymptote at $x = 3$, Double zero at $x = 1$, y-intercept at $(0, 4)$

For the following exercises, use the graphs to write an equation for the function.

57.



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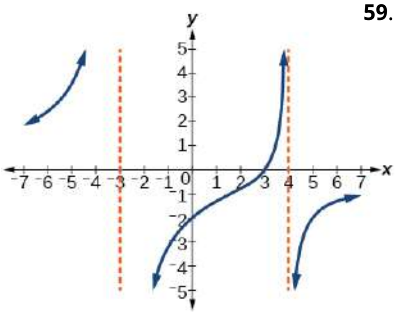

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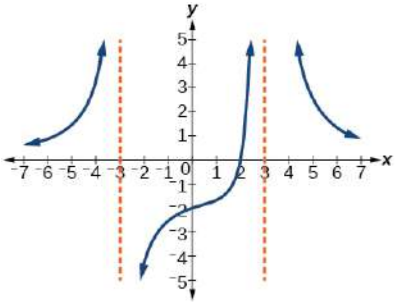

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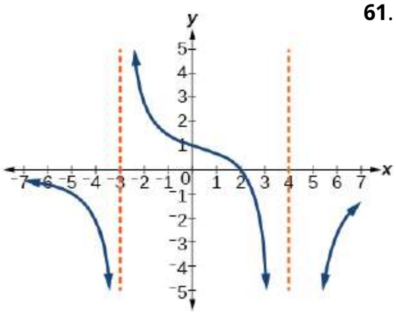

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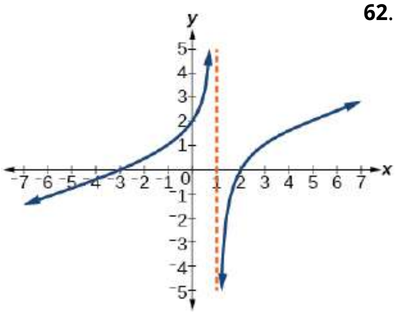

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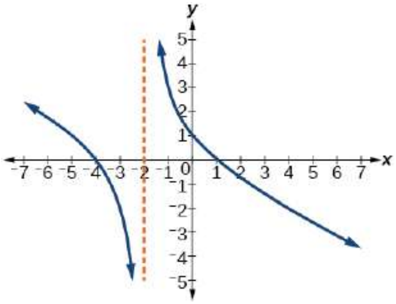

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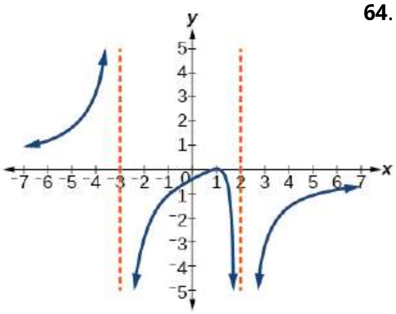

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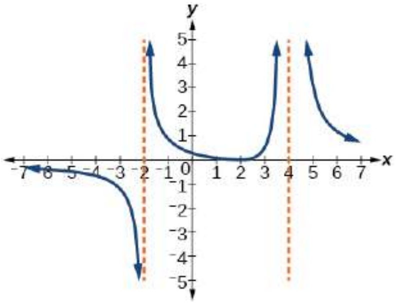


Numeric

For the following exercises, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote

65. $f(x) = \frac{1}{x-2}$

66. $f(x) = \frac{x}{x-3}$

67. $f(x) = \frac{2x}{x+4}$

68. $f(x) = \frac{2x}{(x-3)^2}$

69. $f(x) = \frac{x^2}{x^2+2x+1}$

Technology

For the following exercises, use a calculator to graph $f(x)$. Use the graph to solve $f(x) > 0$.

70. $f(x) = \frac{2}{x+1}$

71. $f(x) = \frac{4}{2x-3}$

72. $f(x) = \frac{2}{(x-1)(x+2)}$

73. $f(x) = \frac{x+2}{(x-1)(x-4)}$

74. $f(x) = \frac{(x+3)^2}{(x-1)^2(x+1)}$

Extensions

For the following exercises, identify the removable discontinuity.

75. $f(x) = \frac{x^2-4}{x-2}$

76. $f(x) = \frac{x^3+1}{x+1}$

77. $f(x) = \frac{x^2+x-6}{x-2}$

78. $f(x) = \frac{2x^2+5x-3}{x+3}$

79. $f(x) = \frac{x^3+x^2}{x+1}$

Real-World Applications

For the following exercises, express a rational function that describes the situation.

80. In the refugee camp hospital, a large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

81. In the refugee camp hospital, a large mixing tank currently contains 300 gallons of water, into which 8 pounds of sugar have been mixed. A tap will open, pouring 20 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

For the following exercises, use the given rational function to answer the question.

82. The concentration C of a drug in a patient's bloodstream t hours after injection is given by $C(t) = \frac{2t}{3+t^2}$. What happens to the concentration of the drug as t increases?

83. The concentration C of a drug in a patient's bloodstream t hours after injection is given by $C(t) = \frac{100t}{2t^2+75}$. Use a calculator to approximate the time when the concentration is highest.

For the following exercises, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.

84. An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let x = length of the side of the base.

85. A rectangular box with a square base is to have a volume of 20 cubic feet. The material for the base costs 30 cents/ square foot. The material for the sides costs 10 cents/square foot. The material for the top costs 20 cents/square foot. Determine the dimensions that will yield minimum cost. Let x = length of the side of the base.

86. A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let x = radius.

87. A right circular cylinder with no top has a volume of 50 cubic meters. Find the radius that will yield minimum surface area. Let x = radius.

88. A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents/square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let x = radius.

5.7 Inverses and Radical Functions

Learning Objectives

In this section, you will:

- Find the inverse of an invertible polynomial function.
- Restrict the domain to find the inverse of a polynomial function.

Park rangers and other trail managers may construct rock piles, stacks, or other arrangements, usually called cairns, to mark trails or other landmarks. (Rangers and environmental scientists discourage hikers from doing the same, in order to avoid confusion and preserve the habitats of plants and animals.) A cairn in the form of a mound of gravel is in the shape of a cone with the height equal to twice the radius.

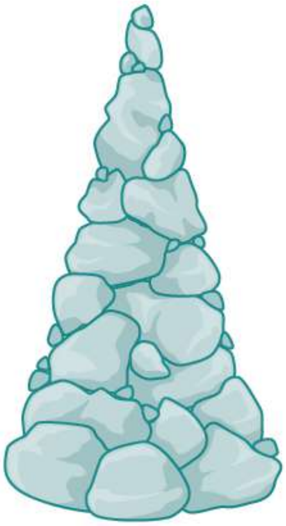


Figure 1

The volume is found using a formula from elementary geometry.

$$\begin{aligned} V &= \tfrac{1}{3}\pi r^2 h \\ &= \tfrac{1}{3}\pi r^2 (2r) \\ &= \tfrac{2}{3}\pi r^3 \end{aligned}$$

We have written the volume V in terms of the radius r. However, in some cases, we may start out with the volume and want to find the radius. For example: A customer purchases 100 cubic feet of gravel to construct a cone shape mound with a height twice the radius. What are the radius and height of the new cone? To answer this question, we use the formula

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This function is the inverse of the formula for V in terms of r.

In this section, we will explore the inverses of polynomial and rational functions and in particular the radical functions we encounter in the process.

Finding the Inverse of a Polynomial Function

Two functions f and g are inverse functions if for every coordinate pair in f, (a, b), there exists a corresponding coordinate pair in the inverse function, g, $(b,\ \ a)$. In other words, the coordinate pairs of the inverse functions have the input and output interchanged. Only one-to-one functions have inverses. Recall that a one-to-one function has a unique output value for each input value and passes the horizontal line test.

For example, suppose the Sustainability Club builds a water runoff collector in the shape of a parabolic trough as shown in Figure 2. We can use the information in the figure to find the surface area of the water in the trough as a function of the depth of the water.

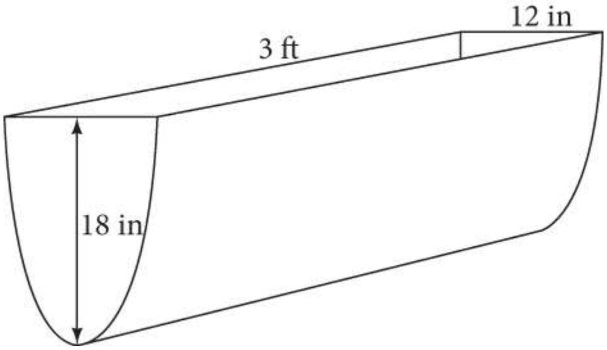


Figure 2

Because it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with x measured horizontally and y measured vertically, with the origin at the vertex of the parabola. See Figure 3.

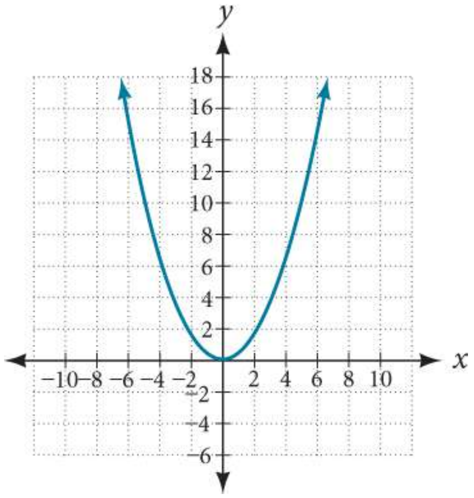


Figure 3

From this we find an equation for the parabolic shape. We placed the origin at the vertex of the parabola, so we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point (6, 18), from which we can solve for the stretch factor a.

$$\begin{aligned} 18 &= a6^2 \\ a &= \tfrac{18}{36} \\ &= \tfrac{1}{2} \end{aligned}$$

Our parabolic cross section has the equation

$$y(x) = \frac{1}{2}x^2$$

We are interested in the surface area of the water, so we must determine the width at the top of the water as a function of the water depth. For any depth y, the width will be given by $2x$, so we need to solve the equation above for x and find the inverse function. However, notice that the original function is not one-to-one, and indeed, given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive x values. On this domain, we can find an inverse by solving for the input variable:

$$\begin{aligned} y &= \tfrac{1}{2}x^2 \\ 2y &= x^2 \\ x &= \pm\sqrt{2y} \end{aligned}$$

This is not a function as written. We are limiting ourselves to positive x values, so we eliminate the negative solution, giving us the inverse function we're looking for.

$$y = \frac{x^2}{2}, \quad x > 0$$

Because x is the distance from the center of the parabola to either side, the entire width of the water at the top will be $2x$. The trough is 3 feet (36 inches) long, so the surface area will then be:

$$\begin{aligned} \text{Area} &= l \cdot w \\ &= 36 \cdot 2x \\ &= 72x \\ &= 72\sqrt{2y} \end{aligned}$$

This example illustrates two important points:

1. When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2. The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power

functions.

Functions involving roots are often called radical functions. While it is not possible to find an inverse of most polynomial functions, some basic polynomials do have inverses. Such functions are called **invertible functions**, and we use the notation $f^{-1}(x)$.

Warning: $f^{-1}(x)$ is not the same as the reciprocal of the function $f(x)$. This use of "–1" is reserved to denote inverse functions. To denote the reciprocal of a function $f(x)$, we would need to write $(f(x))^{-1} = \frac{1}{f(x)}$.

An important relationship between inverse functions is that they "undo" each other. If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. In other words, whatever the function f does to x, f^{-1} undoes it—and vice-versa.

$$f^{-1}(f(x)) = x, \quad \text{for all } x \text{ in the domain of } f$$

and

$$f\left(f^{-1}(x)\right) = x, \quad \text{for all } x \text{ in the domain of } f^{-1}$$

Note that the inverse switches the domain and range of the original function.

Verifying Two Functions Are Inverses of One Another

Two functions, f and g, are inverses of one another if for all x in the domain of f and g.

$$g(f(x)) = f(g(x)) = x$$

HOW TO

Given a polynomial function, find the inverse of the function by restricting the domain in such a way that the new function is one-to-one.

1. Replace $f(x)$ with y.
2. Interchange x and y.
3. Solve for y, and rename the function $f^{-1}(x)$.

EXAMPLE 1

Verifying Inverse Functions

Show that $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses, for $x \neq 0, -1$.

Solution

We must show that $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$.

$$\begin{aligned}
f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{x+1}\right) \\
&= \frac{1}{\frac{1}{x+1}} - 1 \\
&= (x+1) - 1 \\
&= x \\
f(f^{-1}(x)) &= f\left(\frac{1}{x} - 1\right) \\
&= \frac{1}{\left(\frac{1}{x}-1\right)+1} \\
&= \frac{1}{\frac{1}{x}} \\
&= x
\end{aligned}$$

Therefore, $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses.

TRY IT #1 Show that $f(x) = \frac{x+5}{3}$ and $f^{-1}(x) = 3x - 5$ are inverses.

EXAMPLE 2

Finding the Inverse of a Cubic Function

Find the inverse of the function $f(x) = 5x^3 + 1$.

Solution

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x.

$$\begin{aligned} y &= 5x^3 + 1 \\ x &= 5y^3 + 1 \\ x - 1 &= 5y^3 \\ \frac{x-1}{5} &= y^3 \\ f^{-1}(x) &= \sqrt[3]{\frac{x-1}{5}} \end{aligned}$$

Analysis

Look at the graph of f and f^{-1}. Notice that one graph is the reflection of the other about the line $y = x$. This is always the case when graphing a function and its inverse function.

Also, since the method involved interchanging x and y, notice corresponding points. If (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}. Since $(0, 1)$ is on the graph of f, then $(1, 0)$ is on the graph of f^{-1}. Similarly, since $(1, 6)$ is on the graph of f, then $(6, 1)$ is on the graph of f^{-1}. See Figure 4.

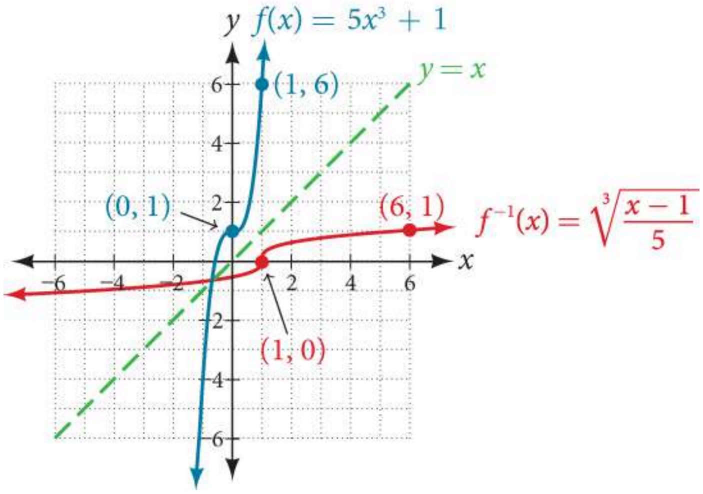


Figure 4

TRY IT #2 Find the inverse function of $f(x) = \sqrt[3]{x+4}$.

Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of cubic functions without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an inverse function. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

Restricting the Domain

If a function is not one-to-one, it cannot have an inverse. If we restrict the domain of the function so that it becomes one-to-one, thus creating a new function, this new function will have an inverse.

HOW TO

Given a polynomial function, restrict the domain of a function that is not one-to-one and then find the inverse.

1. Restrict the domain by determining a domain on which the original function is one-to-one.
2. Replace $f(x)$ with y.
3. Interchange x and y.
4. Solve for y, and rename the function or pair of function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

EXAMPLE 3

Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of f:

ⓐ $f(x) = (x-4)^2,\ x \geq 4$ ⓑ $f(x) = (x-4)^2,\ x \leq 4$

Solution

The original function $f(x) = (x-4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ or $x \leq 4$ on which it is one-to-one. See Figure 5.

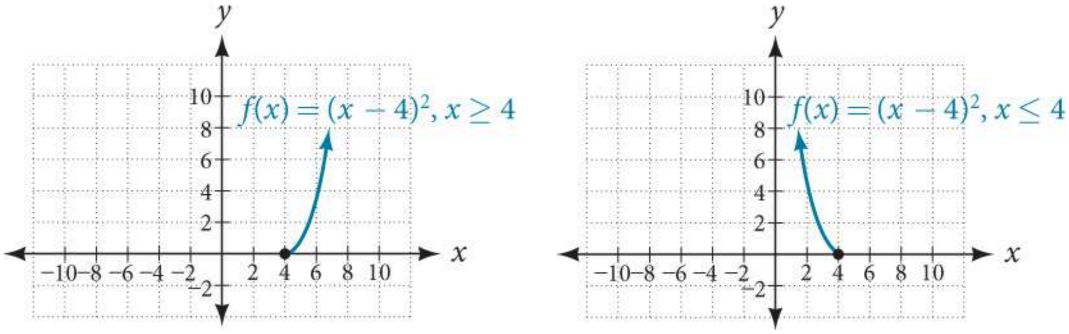


Figure 5

To find the inverse, start by replacing $f(x)$ with the simple variable y.

$$\begin{array}{rcll} y & = & (x-4)^2 & \text{Interchange} x \text{and } y. \\ x & = & (y-4)^2 & \text{Take the square root.} \\ \pm\sqrt{x} & = & y-4 & \text{Add 4 to both sides.} \\ 4 \pm \sqrt{x} & = & y & \end{array}$$

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse. Since we reversed the roles of x and y for the original $f(x)$, we looked at the domain: the values x could assume. When we reversed the roles of x and y, this gave us the values y could assume. For this function, $x \geq 4$, so for the inverse, we should have $y \geq 4$, which is what our inverse function gives.

ⓐ The domain of the original function was restricted to $x \geq 4$, so the outputs of the inverse need to be the same, $f(x) \geq 4$, and we must use the + case:

$$f^{-1}(x) = 4 + \sqrt{x}$$

ⓑ The domain of the original function was restricted to $x \leq 4$, so the outputs of the inverse need to be the same, $f\,(x) \leq 4$, and we must use the – case:

$$f^{-1}(x) = 4 - \sqrt{x}$$

Analysis

On the graphs in Figure 6, we see the original function graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y = x$. The coordinate pair $(4,\ 0)$ is on the graph of f and the coordinate pair $(0,\ 4)$ is on the graph of f^{-1}. For any coordinate pair, if $(a,\ \ b)$ is on the graph of f, then $(b,\ \ a)$ is on the graph of f^{-1}. Finally, observe that the graph of f intersects the graph of f^{-1} on the line $y = x$. Points of intersection for the graphs of f and f^{-1} will always lie on the line $y = x$.

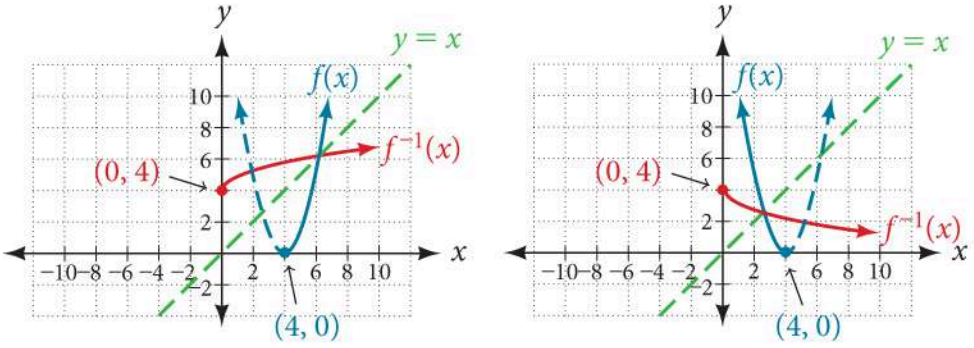


Figure 6

EXAMPLE 4

Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of

$$f(x) = (x - 2)^2 - 3.$$

Solution

We can see this is a parabola with vertex at $(2, -3)$ that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we will use the vertex form of the quadratic. We start by replacing $f(x)$ with a simple variable, y, then solve for x.

$$\begin{array}{rcll}
y & = & (x-2)^2 - 3 & \text{Interchange } x \text{ and } y. \\
x & = & (y-2)^2 - 3 & \text{Add 3 to both sides.} \\
x+3 & = & (y-2)^2 & \text{Take the square root.} \\
\pm\sqrt{x+3} & = & y-2 & \text{Add 2 to both sides.} \\
2 \pm \sqrt{x+3} & = & y & \text{Rename the function.} \\
f^{-1}(x) & = & 2 \pm \sqrt{x+3} &
\end{array}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the + case

$$f^{-1}(x) = 2 + \sqrt{x+3}$$

If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

Analysis

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just have easily opted to restrict the domain on $x \leq 2$, in which case $f^{-1}(x) = 2 - \sqrt{x+3}$. Observe the original function graphed on the same set of axes as its inverse function in Figure 7. Notice that both graphs show symmetry about the line $y = x$. The coordinate pair $(2, \ -3)$ is on the graph of f and the coordinate pair $(-3, \ 2)$ is on the graph of f^{-1}. Observe from the graph of both functions on the same set of axes that

$$\text{domain of } f = \text{range of } f^{-1} = \left[2, \infty\right)$$

and

$$\text{domain of } f^{-1} = \text{range of } f = \left[-3, \infty\right).$$

Finally, observe that the graph of f intersects the graph of f^{-1} along the line $y = x$.

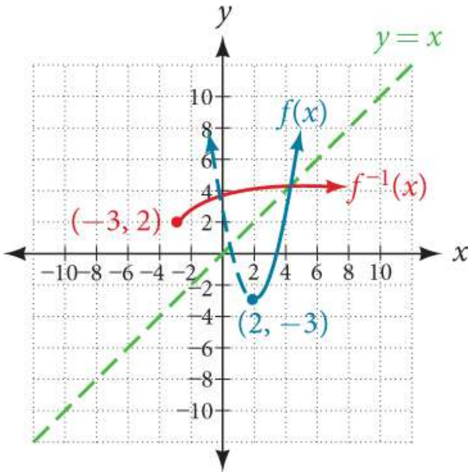


Figure 7

TRY IT #3 Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

Solving Applications of Radical Functions

Notice that the functions from previous examples were all polynomials, and their inverses were radical functions. If we want to find the inverse of a radical function, we will need to restrict the domain of the answer because the range of the original function is limited.

HOW TO

Given a radical function, find the inverse.

1. Determine the range of the original function.
2. Replace $f(x)$ with y, then solve for x.
3. If necessary, restrict the domain of the inverse function to the range of the original function.

EXAMPLE 5

Finding the Inverse of a Radical Function

Restrict the domain of the function $f(x) = \sqrt{x-4}$ and then find the inverse.

Solution

Note that the original function has range $f(x) \geq 0$. Replace $f(x)$ with y, then solve for x.

$$\begin{array}{rcll} y & = & \sqrt{x-4} & \text{Replace } f(x) \text{ with } y. \\ x & = & \sqrt{y-4} & \text{Interchange } x \text{ and } y. \\ x & = & \sqrt{y-4} & \text{Square each side.} \\ x^2 & = & y-4 & \text{Add 4.} \\ x^2+4 & = & y & \text{Rename the function } f^{-1}(x). \\ f^{-1}(x) & = & x^2+4 & \end{array}$$

Recall that the domain of this function must be limited to the range of the original function.

$$f^{-1}(x) = x^2 + 4, x \geq 0$$

Analysis

Notice in Figure 8 that the inverse is a reflection of the original function over the line $y = x$. Because the original function has only positive outputs, the inverse function has only positive inputs.

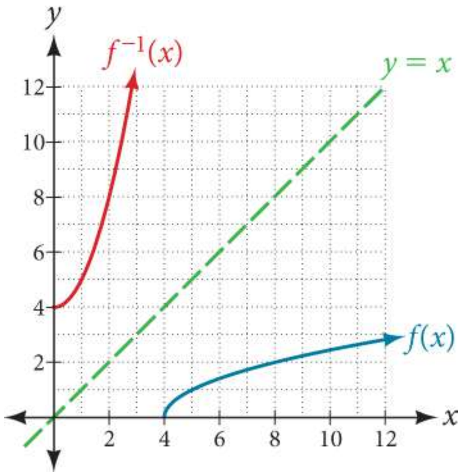


Figure 8

TRY IT #4 Restrict the domain and then find the inverse of the function $f(x) = \sqrt{2x+3}$.

Solving Applications of Radical Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

EXAMPLE 6

Solving an Application with a Cubic Function

Park rangers construct a mound of gravel in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3$$

Find the inverse of the function $V = \frac{2}{3}\pi r^3$ that determines the volume V of a cone and is a function of the radius r. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi = 3.14$.

Solution

Start with the given function for V. Notice that the meaningful domain for the function is $r > 0$ since negative radii would not make sense in this context nor would a radius of 0. Also note the range of the function (hence, the domain of the inverse function) is $V > 0$. Solve for r in terms of V, using the method outlined previously. Note that in real-world applications, we do not swap the variables when finding inverses. Instead, we change which variable is considered to be the independent variable.

$$\begin{aligned} V &= \tfrac{2}{3}\pi r^3 \\ r^3 &= \tfrac{3V}{2\pi} && \text{Solve for } r^3. \\ r &= \sqrt[3]{\tfrac{3V}{2\pi}} && \text{Solve for } r. \end{aligned}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi = 3.14$.

$$\begin{aligned} r &= \sqrt[3]{\tfrac{3V}{2\pi}} \\ &= \sqrt[3]{\tfrac{3\cdot 100}{2\cdot 3.14}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft.

Determining the Domain of a Radical Function Composed with Other Functions

When radical functions are composed with other functions, determining domain can become more complicated.

EXAMPLE 7

Finding the Domain of a Radical Function Composed with a Rational Function

Find the domain of the function $f(x) = \sqrt{\frac{(x+2)(x-3)}{(x-1)}}$.

Solution

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\frac{(x+2)(x-3)}{(x-1)} \geq 0$. The output of a rational function can change signs (change from positive to negative or vice versa) at *x*-intercepts and at vertical asymptotes. For this equation, the graph could change signs at $x = -2,\ 1,\ \text{and } 3$.

To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, for this example we will use a graph as shown in Figure 9.

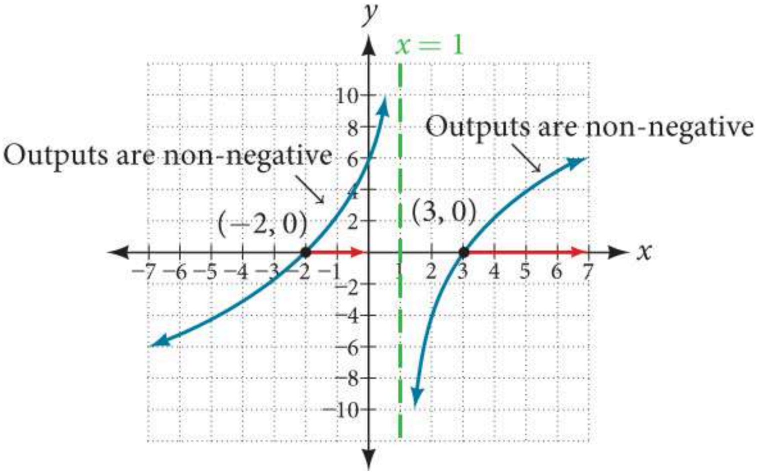


Figure 9

This function has two *x*-intercepts, both of which exhibit linear behavior near the *x*-intercepts. There is one vertical asymptote, corresponding to a linear factor; this behavior is similar to the basic reciprocal toolkit function, and there is no horizontal asymptote because the degree of the numerator is larger than the degree of the denominator. There is a *y*-intercept at $(0, \sqrt{6})$.

From the *y*-intercept and *x*-intercept at $x = -2$, we can sketch the left side of the graph. From the behavior at the asymptote, we can sketch the right side of the graph.

From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f\,(x)$ will be defined. $f\,(x)$ has domain $-2 \leq x < 1 \;\text{ or }\; x \geq 3$, or in interval notation, $[-2, 1) \cup [3, \infty)$.

Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

EXAMPLE 8

Finding the Inverse of a Rational Function

The function $C = \frac{20+0.4n}{100+n}$ represents the concentration C of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for n in terms of C. Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

Solution

We first want the inverse of the function in order to determine how many mL we need for a given concentration. We will solve for n in terms of C.

$$\begin{aligned} C &= \tfrac{20+0.4n}{100+n} \\ C(100+n) &= 20+0.4n \\ 100C+Cn &= 20+0.4n \\ 100C-20 &= 0.4n-Cn \\ 100C-20 &= (0.4n-C)n \\ n &= \tfrac{100C-20}{0.4-C} \end{aligned}$$

Now evaluate this function at 35%, which is $C = 0.35$.

$$\begin{aligned} n &= \tfrac{100(0.35)-20}{0.4-0.35} \\ &= \tfrac{15}{0.05} \\ &= 300 \end{aligned}$$

We can conclude that 300 mL of the 40% solution should be added.

TRY IT #5 Find the inverse of the function $f(x) = \frac{x+3}{x-2}$.

MEDIA

Access these online resources for additional instruction and practice with inverses and radical functions.

Graphing the Basic Square Root Function (http://openstax.org/l/graphsquareroot)
Find the Inverse of a Square Root Function (http://openstax.org/l/inversesquare)
Find the Inverse of a Rational Function (http://openstax.org/l/inverserational)
Find the Inverse of a Rational Function and an Inverse Function Value (http://openstax.org/l/rationalinverse)
Inverse Functions (http://openstax.org/l/inversefunction)

5.7 SECTION EXERCISES

Verbal

1. Explain why we cannot find inverse functions for all polynomial functions.

2. Why must we restrict the domain of a quadratic function when finding its inverse?

3. When finding the inverse of a radical function, what restriction will we need to make?

4. The inverse of a quadratic function will always take what form?

Algebraic

For the following exercises, find the inverse of the function on the given domain.

5. $f(x) = (x-4)^2,\ [4, \infty)$

6. $f(x) = (x+2)^2,\ [-2, \infty)$

7. $f(x) = (x+1)^2 - 3,\ [-1, \infty)$

8. $f(x) = 3x^2 + 5, \quad \left(\infty, 0\right]$

9. $f(x) = 12 - x^2,\ [0, \infty)$

10. $f(x) = 9 - x^2,\ [0, \infty)$

11. $f(x) = 2x^2 + 4,\ [0, \infty)$

For the following exercises, find the inverse of the functions.

12. $f(x) = x^3 + 5$

13. $f(x) = 3x^3 + 1$

14. $f(x) = 4 - x^3$

15. $f(x) = 4 - 2x^3$

For the following exercises, find the inverse of the functions.

16. $f(x) = \sqrt{2x+1}$

17. $f(x) = \sqrt{3-4x}$

18. $f(x) = 9 + \sqrt{4x-4}$

19. $f(x) = \sqrt{6x-8} + 5$

20. $f(x) = 9 + 2\sqrt[3]{x}$

21. $f(x) = 3 - \sqrt[3]{x}$

22. $f(x) = \frac{2}{x+8}$

23. $f(x) = \frac{3}{x-4}$

24. $f(x) = \frac{x+3}{x+7}$

25. $f(x) = \frac{x-2}{x+7}$

26. $f(x) = \frac{3x+4}{5-4x}$

27. $f(x) = \frac{5x+1}{2-5x}$

28. $f(x) = x^2 + 2x,\ [-1, \infty)$

29. $f(x) = x^2 + 4x + 1,\ [-2, \infty)$

30. $f(x) = x^2 - 6x + 3,\ [3, \infty)$

Graphical

For the following exercises, find the inverse of the function and graph both the function and its inverse.

31. $f(x) = x^2 + 2, \quad x \geq 0$

32. $f(x) = 4 - x^2, \quad x \geq 0$

33. $f(x) = (x + 3)^2, \quad x \geq -3$

34. $f(x) = (x - 4)^2, \quad x \geq 4$

35. $f(x) = x^3 + 3$

36. $f(x) = 1 - x^3$

37. $f(x) = x^2 + 4x, \quad x \geq -2$

38. $f(x) = x^2 - 6x + 1, \quad x \geq 3$

39. $f(x) = \frac{2}{x}$

40. $f(x) = \frac{1}{x^2}, \quad x \geq 0$

For the following exercises, use a graph to help determine the domain of the functions.

41. $f(x) = \sqrt{\frac{(x+1)(x-1)}{x}}$

42. $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$

43. $f(x) = \sqrt{\frac{x(x+3)}{x-4}}$

44. $f(x) = \sqrt{\frac{x^2-x-20}{x-2}}$

45. $f(x) = \sqrt{\frac{9-x^2}{x+4}}$

Technology

For the following exercises, use a calculator to graph the function. Then, using the graph, give three points on the graph of the inverse with y-coordinates given.

46. $f(x) = x^3 - x - 2, y = 1, 2, 3$

47. $f(x) = x^3 + x - 2, y = 0, 1, 2$

48. $f(x) = x^3 + 3x - 4, y = 0, 1, 2$

49. $f(x) = x^3 + 8x - 4, y = -1, 0, 1$

50. $f(x) = x^4 + 5x + 1, y = -1, 0, 1$

Extensions

For the following exercises, find the inverse of the functions with a, b, c positive real numbers.

51. $f(x) = ax^3 + b$

52. $f(x) = x^2 + bx$

53. $f(x) = \sqrt{ax^2 - b}$

54. $f(x) = \sqrt[3]{ax + b}$

55. $f(x) = \frac{ax+b}{x+c}$

Real-World Applications

For the following exercises, determine the function described and then use it to answer the question.

56. An object dropped from a height of 200 meters has a height, $h(t)$, in meters after t seconds have lapsed, such that $h(t) = 200 - 4.9t^2$. Express t as a function of height, h, and find the time to reach a height of 50 meters.

57. An object dropped from a height of 600 feet has a height, $h(t)$, in feet after t seconds have elapsed, such that $h(t) = 600 - 16t^2$. Express t as a function of height h, and find the time to reach a height of 400 feet.

58. The volume, V, of a sphere in terms of its radius, r, is given by $V(r) = \frac{4}{3}\pi r^3$. Express r as a function of V, and find the radius of a sphere with volume of 200 cubic feet.

59. The surface area, A, of a sphere in terms of its radius, r, is given by $A(r) = 4\pi r^2$. Express r as a function of A, and find the radius of a sphere with a surface area of 1000 square inches.

60. A container holds 100 mL of a solution that is 25 mL acid. If n mL of a solution that is 60% acid is added, the function $C(n) = \frac{25+.6n}{100+n}$ gives the concentration, C, as a function of the number of mL added, n. Express n as a function of C and determine the number of mL that need to be added to have a solution that is 50% acid.

61. The period T, in seconds, of a simple pendulum as a function of its length l, in feet, is given by $T(l) = 2\pi\sqrt{\frac{l}{32.2}}$. Express l as a function of T and determine the length of a pendulum with period of 2 seconds.

62. The volume of a cylinder , V, in terms of radius, r, and height, h, is given by $V = \pi r^2 h$. If a cylinder has a height of 6 meters, express the radius as a function of V and find the radius of a cylinder with volume of 300 cubic meters.

63. The surface area, A, of a cylinder in terms of its radius, r, and height, h, is given by $A = 2\pi r^2 + 2\pi rh$. If the height of the cylinder is 4 feet, express the radius as a function of A and find the radius if the surface area is 200 square feet.

64. The volume of a right circular cone, V, in terms of its radius, r, and its height, h, is given by $V = \frac{1}{3}\pi r^2 h$. Express r in terms of V if the height of the cone is 12 feet and find the radius of a cone with volume of 50 cubic inches.

65. Consider a cone with height of 30 feet. Express the radius, r, in terms of the volume, V, and find the radius of a cone with volume of 1000 cubic feet.

5.8 Modeling Using Variation

Learning Objectives

In this section, you will:

- Solve direct variation problems.
- Solve inverse variation problems.
- Solve problems involving joint variation.

A pre-owned car dealer has just offered their best candidate, Nicole, a position in sales. The position offers 16% commission on her sales. Her earnings depend on the amount of her sales. For instance, if she sells a vehicle for $4,600, she will earn $736. As she considers the offer, she takes into account the typical price of the dealer's cars, the overall market, and how many she can reasonably expect to sell. In this section, we will look at relationships, such as this one, between earnings, sales, and commission rate.

Solving Direct Variation Problems

In the example above, Nicole’s earnings can be found by multiplying her sales by her commission. The formula $e = 0.16s$ tells us her earnings, e, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See Table 1.

s , sales price	$e = 0.16s$	Interpretation
$4,600	$e = 0.16\,(4{,}600) = 736$	A sale of a $4,600 vehicle results in $736 earnings.
$9,200	$e = 0.16\,(9{,}200) = 1{,}472$	A sale of a $9,200 vehicle results in $1472 earnings.
$18,400	$e = 0.16\,(18{,}400) = 2{,}944$	A sale of a $18,400 vehicle results in $2944 earnings.

Table 1

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from \$4,600 to \$9,200, and we double the earnings from \$736 to \$1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship **varies directly** with the other.

Figure 1 represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the car. The formula $y = kx^n$ is used for direct variation. The value k is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$ and $n = 1$. We saw functions like this one when we discussed power functions.

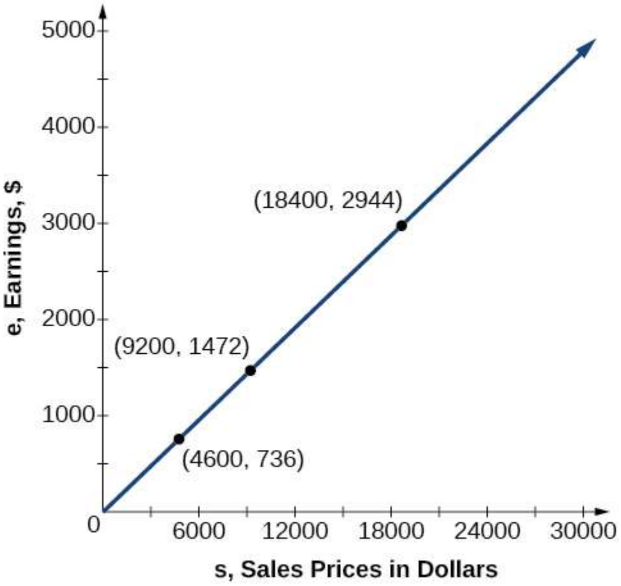


Figure 1

Direct Variation

If x and y are related by an equation of the form

$$y = kx^n$$

then we say that the relationship is **direct variation** and y **varies directly** with, or is proportional to, the nth power of x. In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x^n}$, where k is called the **constant of variation**, which help defines the relationship between the variables.

HOW TO

Given a description of a direct variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to divide y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

EXAMPLE 1

Solving a Direct Variation Problem

The quantity y varies directly with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Solution

The general formula for direct variation with a cube is $y = kx^3$. The constant can be found by dividing y by the cube of x.

$$\begin{aligned} k &= \frac{y}{x^3} \\ &= \frac{25}{2^3} \\ &= \frac{25}{8} \end{aligned}$$

Now use the constant to write an equation that represents this relationship.

$$y = \frac{25}{8}x^3$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \frac{25}{8}(6)^3 \\ &= 675 \end{aligned}$$

Analysis

The graph of this equation is a simple cubic, as shown in Figure 2.

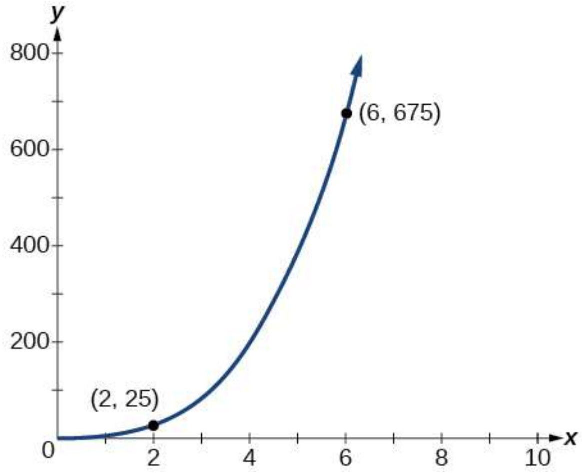


Figure 2

Q&A **Do the graphs of all direct variation equations look like Example 1?**

No. Direct variation equations are power functions—they may be linear, quadratic, cubic, quartic, radical, etc. But all of the graphs pass through $(0,0)$.

TRY IT #1 The quantity y varies directly with the square of x. If $y = 24$ when $x = 3$, find y when x is 4.

Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water's depth. The formula $T = \frac{14,000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers 22% of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F.

If we create Table 2, we observe that, as the depth increases, the water temperature decreases.

d, depth	$T = \frac{14,000}{d}$	Interpretation
500 ft	$\frac{14,000}{500} = 28$	At a depth of 500 ft, the water temperature is 28° F.

Table 2

d, depth	$T = \frac{14{,}000}{d}$	Interpretation
1000 ft	$\frac{14{,}000}{1000} = 14$	At a depth of 1,000 ft, the water temperature is 14° F.
2000 ft	$\frac{14{,}000}{2000} = 7$	At a depth of 2,000 ft, the water temperature is 7° F.

Table 2

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be **inversely proportional** and each term **varies inversely** with the other. Inversely proportional relationships are also called **inverse variations**.

For our example, Figure 3 depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula $y = \frac{k}{x}$ for inverse variation in this case uses $k = 14{,}000$.

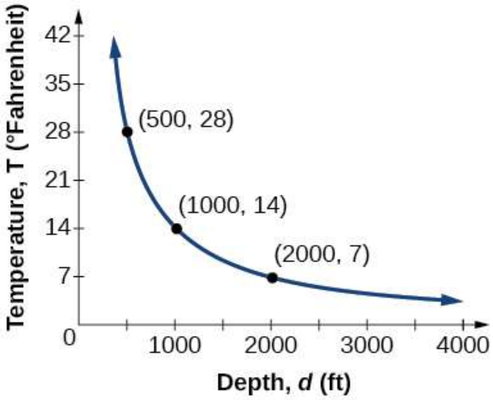


Figure 3

Inverse Variation

If x and y are related by an equation of the form

$$y = \frac{k}{x^n}$$

where k is a nonzero constant, then we say that y **varies inversely** with the nth power of x. In **inversely proportional** relationships, or **inverse variations**, there is a constant multiple $k = x^n y$.

EXAMPLE 2

Writing a Formula for an Inversely Proportional Relationship

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

Solution

Recall that multiplying speed by time gives distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which the tourist drives, then $vt = \text{distance}$. Because the distance is fixed at 100 miles, $vt = 100$ so $t = 100/v$. Because time is a function of velocity, we can write $t(v)$.

$$\begin{aligned} t(v) &= \frac{100}{v} \\ &= 100v^{-1} \end{aligned}$$

We can see that the constant of variation is 100 and, although we can write the relationship using the negative exponent, it is more common to see it written as a fraction. We say that time varies inversely with velocity.

HOW TO

Given a description of an indirect variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to multiply y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

EXAMPLE 3

Solving an Inverse Variation Problem

A quantity y varies inversely with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Solution

The general formula for inverse variation with a cube is $y = \frac{k}{x^3}$. The constant can be found by multiplying y by the cube of x.

$$\begin{aligned} k &= x^3 y \\ &= 2^3 \cdot 25 \\ &= 200 \end{aligned}$$

Now we use the constant to write an equation that represents this relationship.

$$\begin{aligned} y &= \frac{k}{x^3}, \quad k = 200 \\ y &= \frac{200}{x^3} \end{aligned}$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \frac{200}{6^3} \\ &= \frac{25}{27} \end{aligned}$$

Analysis

The graph of this equation is a rational function, as shown in Figure 4.

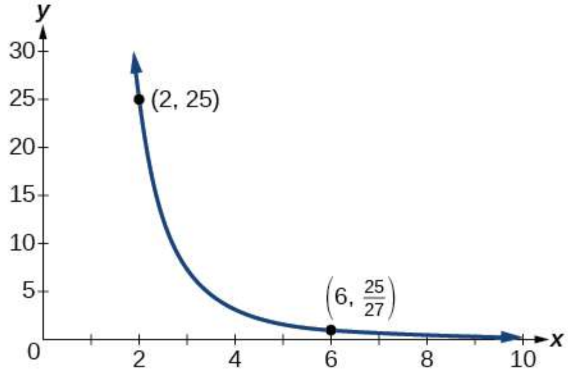


Figure 4

TRY IT #2 A quantity y varies inversely with the square of x. If $y = 8$ when $x = 3$, find y when x is 4.

Solving Problems Involving Joint Variation

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called **joint variation**. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable c, cost, varies jointly with the number of students, n, and the distance, d.

Joint Variation

Joint variation occurs when a variable varies directly or inversely with multiple variables.

For instance, if x varies directly with both y and z, we have $x = kyz$. If x varies directly with y and inversely with z, we have $x = \frac{ky}{z}$. Notice that we only use one constant in a joint variation equation.

EXAMPLE 4

Solving Problems Involving Joint Variation

A quantity x varies directly with the square of y and inversely with the cube root of z. If $x = 6$ when $y = 2$ and $z = 8$, find x when $y = 1$ and $z = 27$.

Solution

Begin by writing an equation to show the relationship between the variables.

$$x = \frac{ky^2}{\sqrt[3]{z}}$$

Substitute $x = 6$, $y = 2$, and $z = 8$ to find the value of the constant k.

$$\begin{aligned} 6 &= \frac{k2^2}{\sqrt[3]{8}} \\ 6 &= \frac{4k}{2} \\ 3 &= k \end{aligned}$$

Now we can substitute the value of the constant into the equation for the relationship.

$$x = \frac{3y^2}{\sqrt[3]{z}}$$

To find x when $y = 1$ and $z = 27$, we will substitute values for y and z into our equation.

$$\begin{aligned} x &= \frac{3(1)^2}{\sqrt[3]{27}} \\ &= 1 \end{aligned}$$

TRY IT #3 A quantity x varies directly with the square of y and inversely with z. If $x = 40$ when $y = 4$ and $z = 2$, find x when $y = 10$ and $z = 25$.

MEDIA

Access these online resources for additional instruction and practice with direct and inverse variation.

Direct Variation (http://openstax.org/l/directvariation)
Inverse Variation (http://openstax.org/l/inversevariatio)
Direct and Inverse Variation (http://openstax.org/l/directinverse)

5.8 SECTION EXERCISES

Verbal

1. What is true of the appearance of graphs that reflect a direct variation between two variables?

2. If two variables vary inversely, what will an equation representing their relationship look like?

3. Is there a limit to the number of variables that can vary jointly? Explain.

Algebraic

For the following exercises, write an equation describing the relationship of the given variables.

4. y varies directly as x and when $x = 6, y = 12.$

5. y varies directly as the square of x and when $x = 4, \quad y = 80.$

6. y varies directly as the square root of x and when $x = 36, \quad y = 24.$

7. y varies directly as the cube of x and when $x = 36, \quad y = 24.$

8. y varies directly as the cube root of x and when $x = 27, \quad y = 15.$

9. y varies directly as the fourth power of x and when $x = 1, \quad y = 6.$

10. y varies inversely as x and when $x = 4, \quad y = 2.$

11. y varies inversely as the square of x and when $x = 3, \quad y = 2.$

12. y varies inversely as the cube of x and when $x = 2, \quad y = 5.$

13. y varies inversely as the fourth power of x and when $x = 3, \quad y = 1.$

14. y varies inversely as the square root of x and when $x = 25, \quad y = 3.$

15. y varies inversely as the cube root of x and when $x = 64, \quad y = 5.$

16. y varies jointly with x and z and when $x = 2$ and $z = 3, \quad y = 36.$

17. y varies jointly as x, z, and w and when $x = 1, \quad z = 2, \quad w = 5,$ then $y = 100.$

18. y varies jointly as the square of x and the square of z and when $x = 3$ and $z = 4$, then $y = 72.$

19. y varies jointly as x and the square root of z and when $x = 2$ and $z = 25$, then $y = 100.$

20. y varies jointly as the square of x the cube of z and the square root of W. When $x = 1, z = 2,$ and $w = 36$, then $y = 48.$

21. y varies jointly as x and z and inversely as w. When $x = 3, \quad z = 5$, and $w = 6$, then $y = 10.$

22. y varies jointly as the square of x and the square root of z and inversely as the cube of w. When $x = 3, z = 4$, and $w = 3$, then $y = 6.$

23. y varies jointly as x and z and inversely as the square root of w and the square of t. When $x = 3, z = 1, w = 25$, and $t = 2$, then $y = 6.$

Numeric

For the following exercises, use the given information to find the unknown value.

24. y varies directly as x. When $x = 3$, then $y = 12$. Find y wneh $x = 20$.

25. y varies directly as the square of x. When $x = 2$, then $y = 16$. Find y when $x = 8$.

26. y varies directly as the cube of x. When $x = 3$, then $y = 5$. Find y when $x = 4$.

27. y varies directly as the square root of x. When $x = 16$, then $y = 4$. Find y when $x = 36$.

28. y varies directly as the cube root of x. When $x = 125$, then $y = 15$. Find y when $x = 1{,}000$.

29. y varies inversely with x. When $x = 3$, then $y = 2$. Find y when $x = 1$.

30. y varies inversely with the square of x. When $x = 4$, then $y = 3$. Find y when $x = 2$.

31. y varies inversely with the cube of x. When $x = 3$, then $y = 1$. Find y when $x = 1$.

32. y varies inversely with the square root of x. When $x = 64$, then $y = 12$. Find y when $x = 36$.

33. y varies inversely with the cube root of x. When $x = 27$, then $y = 5$. Find y when $x = 125$.

34. y varies jointly as $x \text{and} z$. When $x = 4$ and $z = 2$, then $y = 16$. Find y when $x = 3$ and $z = 3$.

35. y varies jointly as $x, z, \text{and} w$. When $x = 2$, $z = 1$, and $w = 12$, then $y = 72$. Find y when $x = 1$, $z = 2$, and $w = 3$.

36. y varies jointly as x and the square of z. When $x = 2$ and $z = 4$, then $y = 144$. Find y when $x = 4$ and $z = 5$.

37. y varies jointly as the square of x and the square root of z. When $x = 2$ and $z = 9$, then $y = 24$. Find y when $x = 3$ and $z = 25$.

38. y varies jointly as x and z and inversely as w. When $x = 5$, $z = 2$, and $w = 20$, then $y = 4$. Find y when $x = 3$ and $z = 8$, and $w = 48$.

39. y varies jointly as the square of x and the cube of z and inversely as the square root of w. When $x = 2$, $z = 2$, and $w = 64$, then $y = 12$. Find y when $x = 1$, $z = 3$, and $w = 4$.

40. y varies jointly as the square of x and of z and inversely as the square root of w and of t. When $x = 2$, $z = 3$, $w = 16$, and $t = 3$, then $y = 1$. Find y when $x = 3$, $z = 2$, $w = 36$, and $t = 5$.

Technology

For the following exercises, use a calculator to graph the equation implied by the given variation.

41. y varies directly with the square of x and when $x = 2, y = 3$.

42. y varies directly as the cube of x and when $x = 2, y = 4$.

43. y varies directly as the square root of x and when $x = 36, y = 2$.

44. y varies inversely with x and when $x = 6, y = 2$.

45. y varies inversely as the square of x and when $x = 1, y = 4$.

Extensions

For the following exercises, use Kepler's Law, which states that the square of the time, T, required for a planet to orbit the Sun varies directly with the cube of the mean distance, a, that the planet is from the Sun.

46. Using Earth's time of 1 year and mean distance of 93 million miles, find the equation relating T and a.

47. Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is 142 million miles.

48. Using Earth's distance of 150 million kilometers, find the equation relating T and a.

49. Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is 108 million kilometers.

50. Using Earth's distance of 1 astronomical unit (A.U.), determine the time for Saturn to orbit the Sun if its mean distance is 9.54 A.U.

Real-World Applications

For the following exercises, use the given information to answer the questions.

51. The distance s that an object falls varies directly with the square of the time, t, of the fall. If an object falls 16 feet in one second, how long for it to fall 144 feet?

52. The velocity v of a falling object varies directly to the time, t, of the fall. If after 2 seconds, the velocity of the object is 64 feet per second, what is the velocity after 5 seconds?

53. The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 24 inches long and vibrates 128 times per second, what is the length of a string that vibrates 64 times per second?

54. The volume of a gas held at constant temperature varies indirectly as the pressure of the gas. If the volume of a gas is 1200 cubic centimeters when the pressure is 200 millimeters of mercury, what is the volume when the pressure is 300 millimeters of mercury?

55. The weight of an object above the surface of Earth varies inversely with the square of the distance from the center of Earth. If a body weighs 50 pounds when it is 3960 miles from Earth's center, what would it weigh it were 3970 miles from Earth's center?

56. The intensity of light measured in foot-candles varies inversely with the square of the distance from the light source. Suppose the intensity of a light bulb is 0.08 foot-candles at a distance of 3 meters. Find the intensity level at 8 meters.

57. The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 amperes, the resistance is 10 ohms. Find the current if the resistance is 12 ohms.

58. The force exerted by the wind on a plane surface varies jointly with the square of the velocity of the wind and with the area of the plane surface. If the area of the surface is 40 square feet surface and the wind velocity is 20 miles per hour, the resulting force is 15 pounds. Find the force on a surface of 65 square feet with a velocity of 30 miles per hour.

59. The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm) and the cube of the diameter. If the shaft of a certain material 3 inches in diameter can transmit 45 hp at 100 rpm, what must the diameter be in order to transmit 60 hp at 150 rpm?

60. The kinetic energy K of a moving object varies jointly with its mass m and the square of its velocity v. If an object weighing 40 kilograms with a velocity of 15 meters per second has a kinetic energy of 1000 joules, find the kinetic energy if the velocity is increased to 20 meters per second.

Chapter Review

Key Terms

arrow notation a way to represent symbolically the local and end behavior of a function by using arrows to indicate that an input or output approaches a value

axis of symmetry a vertical line drawn through the vertex of a parabola, that opens up or down, around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

coefficient a nonzero real number multiplied by a variable raised to an exponent

constant of variation the non-zero value k that helps define the relationship between variables in direct or inverse variation

continuous function a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

degree the highest power of the variable that occurs in a polynomial

Descartes' Rule of Signs a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

direct variation the relationship between two variables that are a constant multiple of each other; as one quantity increases, so does the other

Division Algorithm given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

end behavior the behavior of the graph of a function as the input decreases without bound and increases without bound

Factor Theorem k is a zero of polynomial function $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$

Fundamental Theorem of Algebra a polynomial function with degree greater than 0 has at least one complex zero

general form of a quadratic function the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where $a, b,$ and c are real numbers and $a \neq 0$.

global maximum highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all x.

global minimum lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all x.

horizontal asymptote a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound.

Intermediate Value Theorem for two numbers a and b in the domain of f, if $a < b$ and $f\,(a) \neq f\,(b)$, then the function f takes on every value between $f\,(a)$ and $f\,(b)$; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x- axis

inverse variation the relationship between two variables in which the product of the variables is a constant

inversely proportional a relationship where one quantity is a constant divided by the other quantity; as one quantity increases, the other decreases

invertible function any function that has an inverse function

joint variation a relationship where a variable varies directly or inversely with multiple variables

leading coefficient the coefficient of the leading term

leading term the term containing the highest power of the variable

Linear Factorization Theorem allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x - c)$, where c is a complex number

multiplicity the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x - h)^p$, $x = h$ is a zero of multiplicity p.

polynomial function a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

power function a function that can be represented in the form $f(x) = kx^p$ where k is a constant, the base is a variable, and the exponent, p , is a constant

rational function a function that can be written as the ratio of two polynomials

Rational Zero Theorem the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.

Remainder Theorem if a polynomial $f(x)$ is divided by $x - k$, then the remainder is equal to the value $f(k)$

removable discontinuity a single point at which a function is undefined that, if filled in, would make the function continuous; it appears as a hole on the graph of a function

roots in a given function, the values of x at which $y = 0$, also called zeros

smooth curve a graph with no sharp corners

standard form of a quadratic function the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where $(h, \ k)$ is the vertex

synthetic division a shortcut method that can be used to divide a polynomial by a binomial of the form $x - k$

term of a polynomial function any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0$

turning point the location at which the graph of a function changes direction

varies directly a relationship where one quantity is a constant multiplied by the other quantity

varies inversely a relationship where one quantity is a constant divided by the other quantity

vertex the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertex form of a quadratic function another name for the standard form of a quadratic function

vertical asymptote a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a

zeros in a given function, the values of x at which $y = 0$, also called roots

Key Equations

general form of a quadratic function	$f(x) = ax^2 + bx + c$
standard form of a quadratic function	$f(x) = a(x - h)^2 + k$

general form of a polynomial function	$f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0$

Division Algorithm	$f(x) = d(x)q(x) + r(x)$ where $q(x) \neq 0$

Rational Function	$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1} x^{p-1} + ... + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + ... + b_1 x + b_0}, \ Q(x) \neq 0$

Direct variation	$y = kx^n, \ k$ is a nonzero constant.
Inverse variation	$y = \frac{k}{x^n}, \ k$ is a nonzero constant.

Key Concepts

5.1 Quadratic Functions

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x- intercepts, are the points at which the parabola crosses the x- axis. The y- intercept is the point at which the parabola crosses the y- axis. See Example 1, Example 7, and Example 8.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph. See Example 2.
- The vertex can be found from an equation representing a quadratic function. See Example 3.
- The domain of a quadratic function is all real numbers. The range varies with the function. See Example 4.
- A quadratic function's minimum or maximum value is given by the y- value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue. See Example 5 and Example 6.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems. See Example 9.

5.2 Power Functions and Polynomial Functions

- A power function is a variable base raised to a number power. See Example 1.
- The behavior of a graph as the input decreases beyond bound and increases beyond bound is called the end behavior.
- The end behavior depends on whether the power is even or odd. See Example 2 and Example 3.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power. See Example 4.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient. See Example 5.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function. See Example 6 and Example 7.
- A polynomial of degree n will have at most n x-intercepts and at most $n-1$ turning points. See Example 8, Example 9, Example 10, Example 11, and Example 12.

5.3 Graphs of Polynomial Functions

- Polynomial functions of degree 2 or more are smooth, continuous functions. See Example 1.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero. See Example 2**,** Example 3, and Example 4.
- Another way to find the x- intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x- axis. See Example 5**.**
- The multiplicity of a zero determines how the graph behaves at the x- intercepts. See Example 6**.**
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n-1$ turning points. See Example 7**.**
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points. See Example 8 and Example 10**.**
- Graphing a polynomial function helps to estimate local and global extremas. See Example 11**.**
- The Intermediate Value Theorem tells us that if $f(a) \text{ and } f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c)=0$. See Example 9**.**

5.4 Dividing Polynomials

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree. See Example 1 and Example 2**.**
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x-k$. See Example 3**,** Example 4, and Example 5**.**
- Polynomial division can be used to solve application problems, including area and volume. See Example 6**.**

5.5 Zeros of Polynomial Functions

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x-k$. This is known as the Remainder Theorem. See Example 1.
- According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$. See Example 2.
- According to the Rational Zero Theorem, each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See Example 3 and Example 4.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See Example 5.
- According to the Fundamental Theorem, every polynomial function has at least one complex zero. See Example 6.
- Every polynomial function with degree greater than 0 has at least one complex zero.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x-c)$, where c is a complex number. See Example 7.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less

than the number of sign changes by an even integer. See Example 8.
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division. See Example 9.

5.6 Rational Functions

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See Example 1.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See Example 2.
- Application problems involving rates and concentrations often involve rational functions. See Example 3.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See Example 4.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See Example 5.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See Example 6.
- A rational function’s end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See Example 7, Example 8, Example 9, and Example 10.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See Example 11.
- If a rational function has *x*-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i = \text{any } v_j$, then the function can be written in the form

$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$$

See Example 12.

5.7 Inverses and Radical Functions

- The inverse of a quadratic function is a square root function.
- If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. See Example 1.
- While it is not possible to find an inverse of most polynomial functions, some basic polynomials are invertible. See Example 2.
- To find the inverse of certain functions, we must restrict the function to a domain on which it will be one-to-one. See Example 3 and Example 4.
- When finding the inverse of a radical function, we need a restriction on the domain of the answer. See Example 5 and Example 7.
- Inverse and radical and functions can be used to solve application problems. See Example 6 and Example 8.

5.8 Modeling Using Variation

- A relationship where one quantity is a constant multiplied by another quantity is called direct variation. See Example 1.
- Two variables that are directly proportional to one another will have a constant ratio.
- A relationship where one quantity is a constant divided by another quantity is called inverse variation. See Example 2.
- Two variables that are inversely proportional to one another will have a constant multiple. See Example 3.
- In many problems, a variable varies directly or inversely with multiple variables. We call this type of relationship joint variation. See Example 4.

Exercises

Review Exercises

Quadratic Functions

For the following exercises, write the quadratic function in standard form. Then give the vertex and axes intercepts. Finally, graph the function.

1. $f(x) = x^2 - 4x - 5$

2. $f(x) = -2x^2 - 4x$

For the following exercises, find the equation of the quadratic function using the given information.

3. The vertex is $(-2, 3)$ and a point on the graph is $(3, 6)$.

4. The vertex is $(-3, 6.5)$ and a point on the graph is $(2, 6)$.

For the following exercises, complete the task.

5. A rectangular plot of land is to be enclosed by fencing. One side is along a river and so needs no fence. If the total fencing available is 600 meters, find the dimensions of the plot to have maximum area.

6. An object projected from the ground at a 45 degree angle with initial velocity of 120 feet per second has height, h, in terms of horizontal distance traveled, x, given by $h(x) = \frac{-32}{(120)^2}x^2 + x$. Find the maximum height the object attains.

Power Functions and Polynomial Functions

For the following exercises, determine if the function is a polynomial function and, if so, give the degree and leading coefficient.

7. $f(x) = 4x^5 - 3x^3 + 2x - 1$

8. $f(x) = 5^{x+1} - x^2$

9. $f(x) = x^2\left(3 - 6x + x^2\right)$

For the following exercises, determine end behavior of the polynomial function.

10. $f(x) = 2x^4 + 3x^3 - 5x^2 + 7$

11. $f(x) = 4x^3 - 6x^2 + 2$

12. $f(x) = 2x^2(1 + 3x - x^2)$

Graphs of Polynomial Functions

For the following exercises, find all zeros of the polynomial function, noting multiplicities.

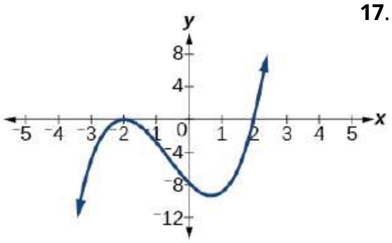
13. $f(x) = (x + 3)^2(2x - 1)(x + 1)^3$

14. $f(x) = x^5 + 4x^4 + 4x^3$

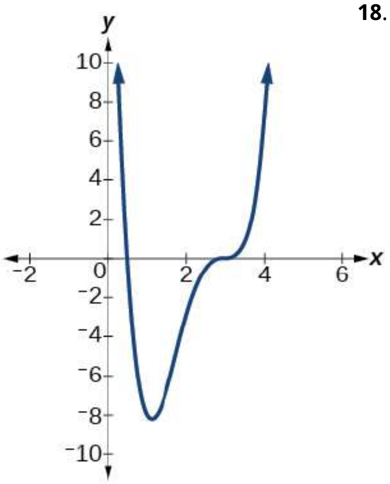
15. $f(x) = x^3 - 4x^2 + x - 4$

For the following exercises, based on the given graph, determine the zeros of the function and note multiplicity.

16.



17.



18. Use the Intermediate Value Theorem to show that at least one zero lies between 2 and 3 for the function $f(x) = x^3 - 5x + 1$

Dividing Polynomials

For the following exercises, use long division to find the quotient and remainder.

19. $\frac{x^3-2x^2+4x+4}{x-2}$

20. $\frac{3x^4-4x^2+4x+8}{x+1}$

For the following exercises, use synthetic division to find the quotient. If the divisor is a factor, then write the factored form.

21. $\frac{x^3-2x^2+5x-1}{x+3}$

22. $\frac{x^3+4x+10}{x-3}$

23. $\frac{2x^3+6x^2-11x-12}{x+4}$

24. $\frac{3x^4+3x^3+2x+2}{x+1}$

Zeros of Polynomial Functions

For the following exercises, use the Rational Zero Theorem to help you solve the polynomial equation.

25. $2x^3 - 3x^2 - 18x - 8 = 0$

26. $3x^3 + 11x^2 + 8x - 4 = 0$

27. $2x^4 - 17x^3 + 46x^2 - 43x + 12 = 0$

28. $4x^4 + 8x^3 + 19x^2 + 32x + 12 = 0$

For the following exercises, use Descartes' Rule of Signs to find the possible number of positive and negative solutions.

29. $x^3 - 3x^2 - 2x + 4 = 0$

30. $2x^4 - x^3 + 4x^2 - 5x + 1 = 0$

Rational Functions

For the following exercises, find the intercepts and the vertical and horizontal asymptotes, and then use them to sketch a graph of the function.

31. $f(x) = \frac{x+2}{x-5}$

32. $f(x) = \frac{x^2+1}{x^2-4}$

33. $f(x) = \frac{3x^2-27}{x^2+x-2}$

34. $f(x) = \frac{x+2}{x^2-9}$

For the following exercises, find the slant asymptote.

35. $f(x) = \frac{x^2-1}{x+2}$

36. $f(x) = \frac{2x^3-x^2+4}{x^2+1}$

Inverses and Radical Functions

For the following exercises, find the inverse of the function with the domain given.

37. $f(x) = (x-2)^2, \quad x \geq 2$

38. $f(x) = (x+4)^2 - 3, \quad x \geq -4$

39. $f(x) = x^2 + 6x - 2, \quad x \geq -3$

40. $f(x) = 2x^3 - 3$

41. $f(x) = \sqrt{4x+5} - 3$

42. $f(x) = \frac{x-3}{2x+1}$

Modeling Using Variation

For the following exercises, find the unknown value.

43. y varies directly as the square of x. If when $x = 3, \quad y = 36$, find y if $x = 4$.

44. y varies inversely as the square root of x If when $x = 25, \quad y = 2$, find y if $x = 4$.

45. y varies jointly as the cube of x and as z. If when $x = 1$ and $z = 2$, $y = 6$, find y if $x = 2$ and $z = 3$.

46. y varies jointly as x and the square of z and inversely as the cube of w. If when $x = 3$, $z = 4$, and $w = 2$, $y = 48$, find y if $x = 4$, $z = 5$, and $w = 3$.

For the following exercises, solve the application problem.

47. The weight of an object above the surface of the earth varies inversely with the distance from the center of the earth. If a person weighs 150 pounds when he is on the surface of the earth (3,960 miles from center), find the weight of the person if he is 20 miles above the surface.

48. The volume V of an ideal gas varies directly with the temperature T and inversely with the pressure P. A cylinder contains oxygen at a temperature of 310 degrees K and a pressure of 18 atmospheres in a volume of 120 liters. Find the pressure if the volume is decreased to 100 liters and the temperature is increased to 320 degrees K.

Practice Test

Give the degree and leading coefficient of the following polynomial function.

1. $f(x) = x^3 \left(3 - 6x - 2x^2\right)$

Determine the end behavior of the polynomial function.

2. $f(x) = 8x^3 - 3x^2 + 2x - 4$

3. $f(x) = -2x^2(4 - 3x - 5x^2)$

Write the quadratic function in standard form. Determine the vertex and axes intercepts and graph the function.

4. $f(x) = x^2 + 2x - 8$

Given information about the graph of a quadratic function, find its equation.

5. Vertex $(2, 0)$ and point on graph $(4, 12)$.

Solve the following application problem.

6. A rectangular field is to be enclosed by fencing. In addition to the enclosing fence, another fence is to divide the field into two parts, running parallel to two sides. If 1,200 feet of fencing is available, find the maximum area that can be enclosed.

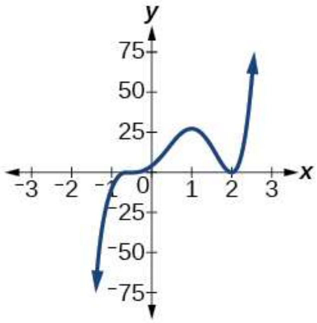
Find all zeros of the following polynomial functions, noting multiplicities.

7. $f(x) = (x - 3)^3(3x - 1)(x - 1)^2$

8. $f(x) = 2x^6 - 12x^5 + 18x^4$

Based on the graph, determine the zeros of the function and multiplicities.

9.



Use long division to find the quotient.

10. $\frac{2x^3+3x-4}{x+2}$

Use synthetic division to find the quotient. If the divisor is a factor, write the factored form.

11. $\frac{x^4+3x^2-4}{x-2}$

12. $\frac{2x^3+5x^2-7x-12}{x+3}$

Use the Rational Zero Theorem to help you find the zeros of the polynomial functions.

13. $f(x) = 2x^3 + 5x^2 - 6x - 9$

14. $f(x) = 4x^4 + 8x^3 + 21x^2 + 17x + 4$

15. $f(x) = 4x^4 + 16x^3 + 13x^2 - 15x - 18$

16. $f(x) = x^5 + 6x^4 + 13x^3 + 14x^2 + 12x + 8$

Given the following information about a polynomial function, find the function.

17. It has a double zero at $x = 3$ and zeros at $x = 1$ and $x = -2$. Its *y*-intercept is $(0, 12)$.

18. It has a zero of multiplicity 3 at $x = \frac{1}{2}$ and another zero at $x = -3$. It contains the point $(1, 8)$.

Use Descartes' Rule of Signs to determine the possible number of positive and negative solutions.

19. $8x^3 - 21x^2 + 6 = 0$

For the following rational functions, find the intercepts and horizontal and vertical asymptotes, and sketch a graph.

20. $f(x) = \frac{x+4}{x^2-2x-3}$

21. $f(x) = \frac{x^2+2x-3}{x^2-4}$

Find the slant asymptote of the rational function.

22. $f(x) = \frac{x^2+3x-3}{x-1}$

Find the inverse of the function.

23. $f(x) = \sqrt{x-2} + 4$

24. $f(x) = 3x^3 - 4$

25. $f(x) = \frac{2x+3}{3x-1}$

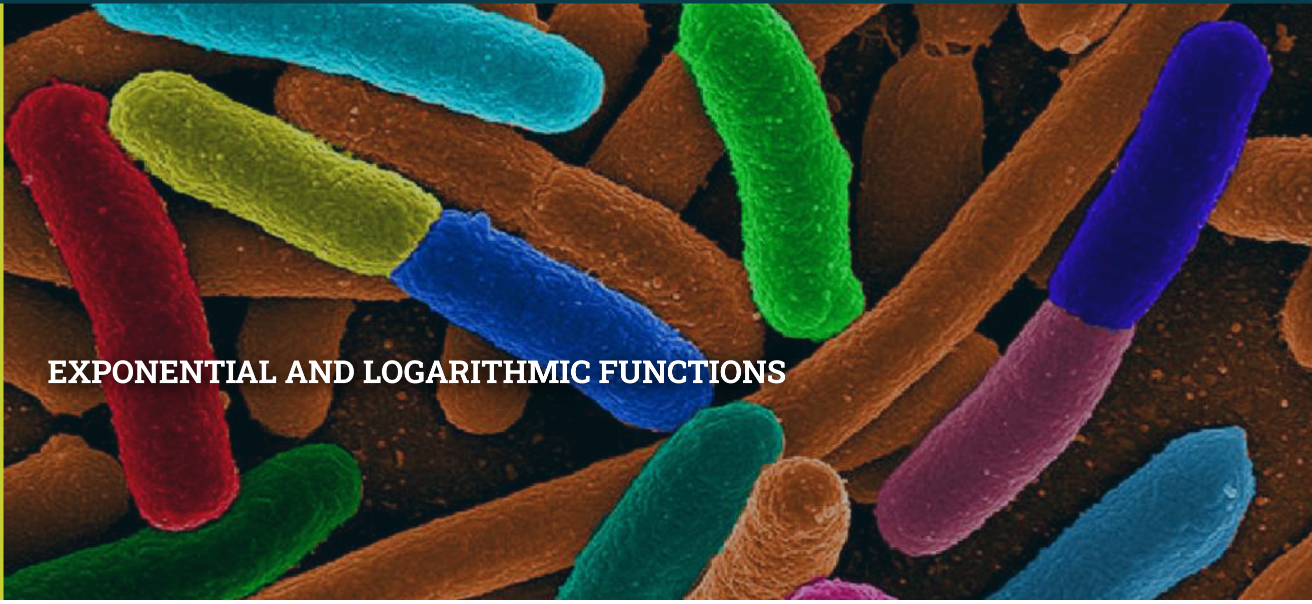
Find the unknown value.

26. y varies inversely as the square of x and when $x = 3$, $y = 2$. Find y if $x = 1$.

27. y varies jointly with x and the cube root of z. If when $x = 2$ and $z = 27$, $y = 12$, find y if $x = 5$ and $z = 8$.

Solve the following application problem.

28. The distance a body falls varies directly as the square of the time it falls. If an object falls 64 feet in 2 seconds, how long will it take to fall 256 feet?



6

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Electron micrograph of *E.Coli* bacteria (credit: “Mattosaurus,” Wikimedia Commons)

Chapter Outline

6.1 Exponential Functions
6.2 Graphs of Exponential Functions
6.3 Logarithmic Functions
6.4 Graphs of Logarithmic Functions
6.5 Logarithmic Properties
6.6 Exponential and Logarithmic Equations
6.7 Exponential and Logarithmic Models
6.8 Fitting Exponential Models to Data

Introduction to Exponential and Logarithmic Functions

Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years.[1]

For simplicity’s sake, suppose we begin with a culture of one bacterial cell that can divide every hour. Table 1 shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

Table 1

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[1] Todar, PhD, Kenneth. Todar's Online Textbook of Bacteriology. http://textbookofbacteriology.net/growth_3.html.

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

6.1 Exponential Functions

Learning Objectives

In this section, you will:

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e .

India is the second most populous country in the world with a population of about 1.39 billion people in 2021. The population is growing at a rate of about 1.2% each year[2] . If this rate continues, the population of India will exceed China’s population by the year 2027. When populations grow rapidly, we often say that the growth is “exponential,” meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.

Identifying Exponential Functions

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by 3 each time the input increases by 1. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2021.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs. See Table 1.

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6

Table 1

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2 http://www.worldometers.info/world-population/. Accessed February 24, 2014.

x	$f(x) = 2^x$	$g(x) = 2x$
4	16	8
5	32	10
6	64	12

Table 1

From Table 1 we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Apparently, the difference between “the same percentage” and “the same amount” is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

The general form of the exponential function is $f(x) = ab^x$, where a is any nonzero number, b is a positive real number not equal to 1.

- If $b > 1$, the function grows at a rate proportional to its size.
- If $0 < b < 1$, the function decays at a rate proportional to its size.

Let’s look at the function $f(x) = 2^x$ from our example. We will create a table (Table 2) to determine the corresponding outputs over an interval in the domain from -3 to 3.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Table 2

Let us examine the graph of f by plotting the ordered pairs we observe on the table in Figure 1, and then make a few observations.

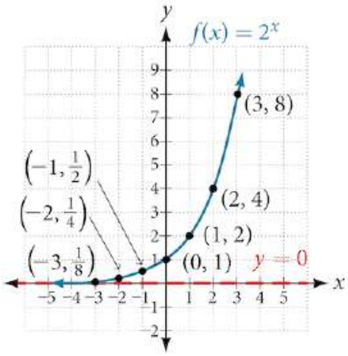

Figure 1

Let’s define the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some its key characteristics.

- the domain is $\left(-\infty, \infty\right)$,
- the range is $\left(0, \infty\right)$,
- as $x \to \infty, f(x) \to \infty$,

- as $x \to -\infty$, $f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the *y*-intercept is 1.

Exponential Function

For any real number x, an exponential function is a function with the form

$$f(x) = ab^x$$

where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The *y*-intercept is $(0, a)$, and the horizontal asymptote is $y = 0$.

EXAMPLE 1

Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base *b* of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

TRY IT #1 Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base b to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{\frac{1}{2}} = \sqrt{-9}$, which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1 :

- Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x.

To evaluate an exponential function with the form $f(x) = b^x$, we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f(x) = 2^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 2^x & \\ f(3) &= 2^3 & \text{Substitute } x = 3. \\ &= 8 & \text{Evaluate the power.} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x) = 30(2)^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 30(2)^x & \\ f(3) &= 30(2)^3 & \text{Substitute } x = 3. \\ &= 30(8) & \text{Simplify the power first.} \\ &= 240 & \text{Multiply.} \end{aligned}$$

Note that if the order of operations were not followed, the result would be incorrect:

$$f(3) = 30(2)^3 \neq 60^3 = 216{,}000$$

EXAMPLE 2

Evaluating Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

Solution

Follow the order of operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} f(x) &= 5(3)^{x+1} & \\ f(2) &= 5(3)^{2+1} & \text{Substitute } x = 2. \\ &= 5(3)^3 & \text{Add the exponents.} \\ &= 5(27) & \text{Simplify the power.} \\ &= 135 & \text{Multiply.} \end{aligned}$$

TRY IT #2 Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term “exponential growth” is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Exponential Growth

A function that models **exponential growth** grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$, an exponential growth function has the form

$$f(x) = ab^x$$

where

- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let’s consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function

$A(x) = 100 + 50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in Table 3.

Year, x	Stores, Company A	Stores, Company B
0	$100 + 50(0) = 100$	$100(1 + 0.5)^0 = 100$
1	$100 + 50(1) = 150$	$100(1 + 0.5)^1 = 150$
2	$100 + 50(2) = 200$	$100(1 + 0.5)^2 = 225$
3	$100 + 50(3) = 250$	$100(1 + 0.5)^3 = 337.5$
x	$A(x) = 100 + 50x$	$B(x) = 100(1 + 0.5)^x$

Table 3

The graphs comparing the number of stores for each company over a five-year period are shown in Figure 2. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

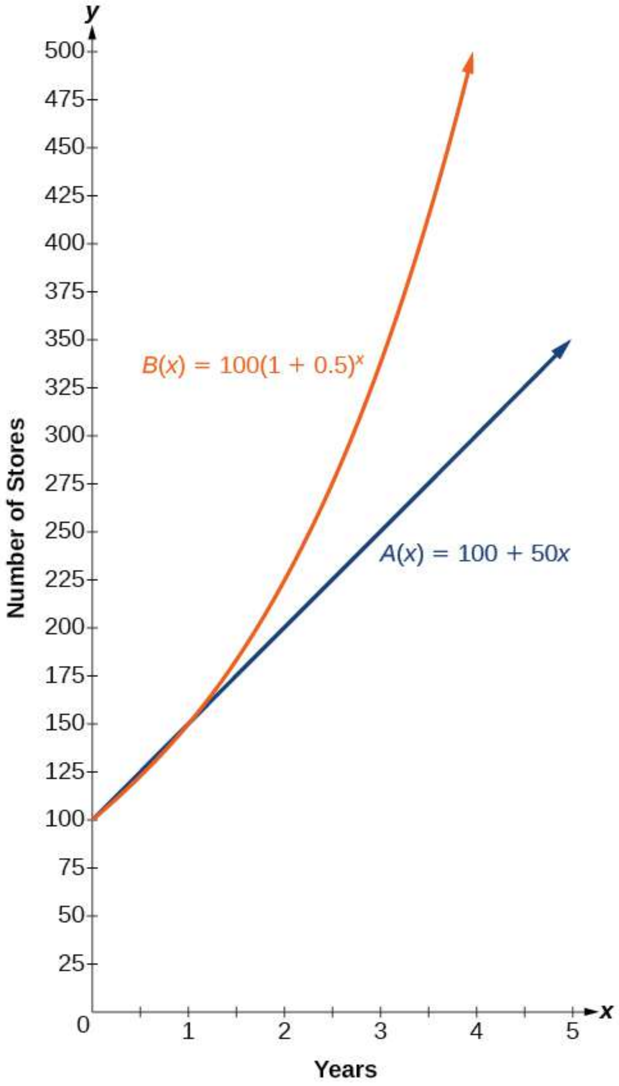

Figure 2 The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is $[0, \infty)$, and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$, where 100 is the initial value, 1.5 is called the *base*, and x is called the *exponent*.

EXAMPLE 3

Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

Solution

To estimate the population in 2031, we evaluate the models for $t = 18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.

TRY IT #3 The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example 3?

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b, and evaluate the function.

HOW TO

Given two data points, write an exponential model.

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

EXAMPLE 4

Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an exponential function $N(t)$ representing the population (N) of deer over time t.

Solution

We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find b :

$$\begin{aligned} N(t) &= 80b^t \\ 180 &= 80b^6 && \text{Substitute using point } (6,\ 180). \\ \frac{9}{4} &= b^6 && \text{Divide and write in lowest terms.} \\ b &= \left(\frac{9}{4}\right)^{\frac{1}{6}} && \text{Isolate } b \text{ using properties of exponents.} \\ b &\approx 1.1447 && \text{Round to 4 decimal places.} \end{aligned}$$

NOTE: *Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.*

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure 3 passes through the initial points given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$, and the range for the function is $[80, \infty)$.

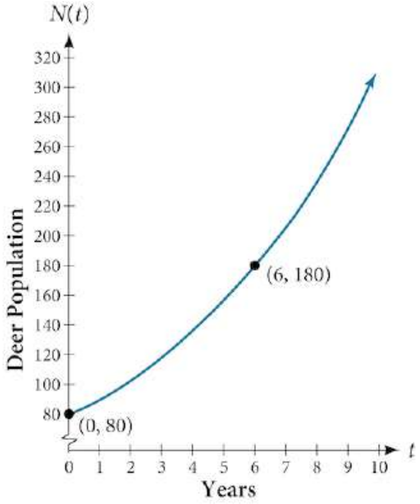

Figure 3 Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, t years after 2006

TRY IT #4 A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

EXAMPLE 5

Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution

Because we don’t have the initial value, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for a in terms of b:

$$6 = ab^{-2}$$

$$\frac{6}{b^{-2}} = a \quad \text{Divide.}$$

$$a = 6b^2 \quad \text{Use properties of exponents to rewrite the denominator.}$$

Substitute a in the second equation, and solve for b :

$$1 = ab^2$$

$$1 = 6b^2b^2 = 6b^4 \quad \text{Substitute } a.$$

$$b = \left(\frac{1}{6}\right)^{\frac{1}{4}} \quad \text{Use properties of exponents to isolate } b.$$

$$b \approx 0.6389 \quad \text{Round 4 decimal places.}$$

Use the value of b in the first equation to solve for the value of a :

$$a = 6b^2 \approx 6(0.6389)^2 \approx 2.4492$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in Figure 4 passes through the initial points given in the problem, $(-2,\ \ 6)$ and $(2,\ \ 1)$. The graph is an example of an exponential decay function.

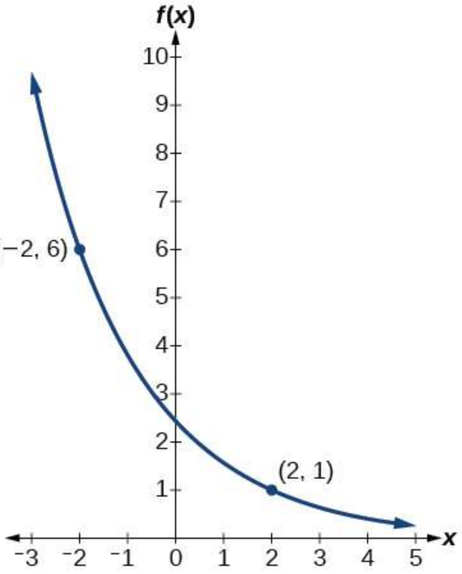

Figure 4 The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.

TRY IT #5 Given the two points $(1, 3)$ and $(2, 4.5)$, find the equation of the exponential function that passes through these two points.

Q&A **Do two points always determine a unique exponential function?**

Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x*, which in many real world cases involves time.*

HOW TO

Given the graph of an exponential function, write its equation.

1. First, identify two points on the graph. Choose the *y*-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the *y*-intercept $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
4. Write the exponential function, $f(x) = a(b)^x$.

EXAMPLE 6

Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in Figure 5.

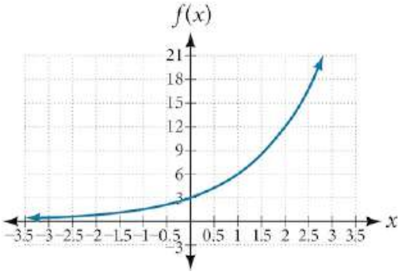

Figure 5

Solution

We can choose the *y*-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$$\begin{array}{rl} y = ab^x & \text{Write the general form of an exponential equation.} \\ y = 3b^x & \text{Substitute the initial value 3 for } a. \\ 12 = 3b^2 & \text{Substitute in 12 for } y \text{ and 2 for } x. \\ 4 = b^2 & \text{Divide by 3.} \\ b = \pm 2 & \text{Take the square root.} \end{array}$$

Because we restrict ourselves to positive values of b, we will use $b = 2$. Substitute a and b into the standard form to yield the equation $f(x) = 3(2)^x$.

TRY IT #6 Find an equation for the exponential function graphed in Figure 6.

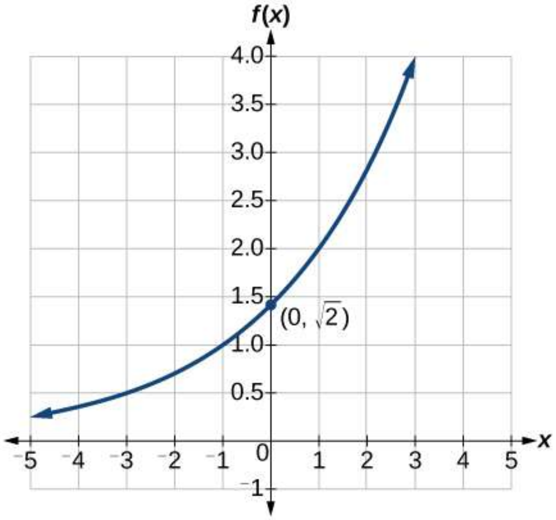

Figure 6

HOW TO

Given two points on the curve of an exponential function, use a graphing calculator to find the equation.

1. Press **[STAT].**
2. Clear any existing entries in columns **L1** or **L2.**
3. In **L1**, enter the *x*-coordinates given.
4. In **L2**, enter the corresponding *y*-coordinates.
5. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential Regression)**, and press **[ENTER].**
6. The screen displays the values of *a* and *b* in the exponential equation $y = a \cdot b^x$.

EXAMPLE 7

Using a Graphing Calculator to Find an Exponential Function

Use a graphing calculator to find the exponential equation that includes the points $(2, 24.8)$ and $(5, 198.4)$.

Solution

Follow the guidelines above. First press **[STAT]**, **[EDIT]**, **[1: Edit…],** and clear the lists **L1** and **L2**. Next, in the **L1** column, enter the *x*-coordinates, 2 and 5. Do the same in the **L2** column for the *y*-coordinates, 24.8 and 198.4.

Now press **[STAT]**, **[CALC]**, **[0: ExpReg]** and press **[ENTER]**. The values $a = 6.2$ and $b = 2$ will be displayed. The exponential equation is $y = 6.2 \cdot 2^x$.

TRY IT #7 Use a graphing calculator to find the exponential equation that includes the points (3, 75.98) and (6, 481.07).

Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, APR r, and number of compounding periods in a year n :

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe Table 4, which shows the result of investing $1,000 at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

Table 4

The Compound Interest Formula

Compound interest can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

EXAMPLE 8

Calculating Compound Interest

If we invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution

Because we are starting with $3,000, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$\begin{aligned} A(t) &= P\left(1 + \tfrac{r}{n}\right)^{nt} && \text{Use the compound interest formula.} \\ A(10) &= 3000\left(1 + \tfrac{0.03}{4}\right)^{4\cdot 10} && \text{Substitute using given values.} \\ &\approx \$4045.05 && \text{Round to two decimal places.} \end{aligned}$$

The account will be worth about $4,045.05 in 10 years.

TRY IT #8 An initial investment of $100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

EXAMPLE 9

Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child’s future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution

The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, P, needed so that the value of the account will be worth $40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for P.

$$\begin{aligned} A(t) &= P\left(1+\frac{r}{n}\right)^{nt} && \text{Use the compound interest formula.} \\ 40{,}000 &= P\left(1+\frac{0.06}{2}\right)^{2(18)} && \text{Substitute using given values } A,\ r,\ n, \text{ and } t. \\ 40{,}000 &= P(1.03)^{36} && \text{Simplify.} \\ \frac{40{,}000}{(1.03)^{36}} &= P && \text{Isolate } P. \\ P &\approx \$13,801 && \text{Divide and round to the nearest dollar.} \end{aligned}$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

TRY IT #9 Refer to Example 9. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

Evaluating Functions with Base *e*

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table 5 shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $1 invested at 100% interest for 1 year, compounded at various frequencies, listed in Table 5.

Frequency	$A(n)=\left(1+\frac{1}{n}\right)^n$	Value
Annually	$\left(1+\frac{1}{1}\right)^1$	$2
Semiannually	$\left(1+\frac{1}{2}\right)^2$	$2.25
Quarterly	$\left(1+\frac{1}{4}\right)^4$	$2.441406
Monthly	$\left(1+\frac{1}{12}\right)^{12}$	$2.613035
Daily	$\left(1+\frac{1}{365}\right)^{365}$	$2.714567
Hourly	$\left(1+\frac{1}{8760}\right)^{8760}$	$2.718127
Once per minute	$\left(1+\frac{1}{525600}\right)^{525600}$	$2.718279
Once per second	$\left(1+\frac{1}{31536000}\right)^{31536000}$	$2.718282

Table 5

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

The Number e

The letter e represents the irrational number

$$\left(1+\frac{1}{n}\right)^n, \text{as } n \text{ increases without bound}$$

The letter e is used as a base for many real-world exponential models. To work with base e, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

EXAMPLE 10

Using a Calculator to Find Powers of e

Calculate $e^{3.14}$. Round to five decimal places.

Solution

On a calculator, press the button labeled $\left[e^x\right]$. The window shows $[e$ ^( ]. Type 3.14 and then close parenthesis, [)]. Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387$. Caution: Many scientific calculators have an “Exp” button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

TRY IT #10 Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, e is used as the base for exponential functions. Exponential models that use e as the base are called *continuous growth or decay models*. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

The Continuous Growth/Decay Formula

For all real numbers t, and all positive numbers a and r, continuous growth or decay is represented by the formula

$$A(t) = ae^{rt}$$

where

- a is the initial value,
- r is the continuous growth rate per unit time,
- and t is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt}$$

where

- P is the principal or the initial invested,
- r is the growth or interest rate per unit time,

- and t is the period or term of the investment.

HOW TO

Given the initial value, rate of growth or decay, and time t, solve a continuous growth or decay function.

1. Use the information in the problem to determine a , the initial value of the function.
2. Use the information in the problem to determine the growth rate r.
   a. If the problem refers to continuous growth, then $r > 0$.
   b. If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time t.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

EXAMPLE 11

Calculating Continuous Growth

A person invested $1,000 in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Solution

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $1,000, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula.} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P,\ r, \text{ and } t. \\ &\approx 1105.17 && \text{Use a calculator to approximate.} \end{aligned}$$

The account is worth $1,105.17 after one year.

TRY IT #11 A person invests $100,000 at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

EXAMPLE 12

Calculating Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

Solution

Since the substance is decaying, the rate, 17.3% , is negative. So, $r = -0.173$. The initial amount of radon-222 was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula.} \\ &= 100e^{-0.173(3)} && \text{Substitute known values for } a,\ r, \text{ and } t. \\ &\approx 59.5115 && \text{Use a calculator to approximate.} \end{aligned}$$

So 59.5115 mg of radon-222 will remain.

TRY IT #12 Using the data in Example 12, how much radon-222 will remain after one year?

MEDIA

Access these online resources for additional instruction and practice with exponential functions.

Exponential Growth Function (http://openstax.org/l/expgrowth)
Compound Interest (http://openstax.org/l/compoundint)

6.1 SECTION EXERCISES

Verbal

1. Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.
2. Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.
3. The Oxford Dictionary defines the word *nominal* as a value that is “stated or expressed but not necessarily corresponding exactly to the real value.”[3] Develop a reasonable argument for why the term *nominal rate* is used to describe the annual percentage rate of an investment account that compounds interest.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

4. The average annual population increase of a pack of wolves is 25.
5. A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.
6. The value of a coin collection has increased by 3.25% annually over the last 20 years.
7. For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.
8. The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$.

For the following exercises, consider this scenario: For each year t, the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)

9. Which forest’s population is growing at a faster rate?
10. Which forest had a greater number of trees initially? By how many?
11. Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?

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[3] Oxford Dictionary. http://oxforddictionaries.com/us/definition/american_english/nomina.

12. Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?

13. Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

14. $y = 300(1-t)^5$

15. $y = 220(1.06)^x$

16. $y = 16.5(1.025)^{\frac{1}{x}}$

17. $y = 11,701(0.97)^t$

For the following exercises, find the formula for an exponential function that passes through the two points given.

18. $(0,6)$ and $(3,750)$

19. $(0,2000)$ and $(2,20)$

20. $\left(-1,\frac{3}{2}\right)$ and $(3,24)$

21. $(-2,6)$ and $(3,1)$

22. $(3,1)$ and $(5,4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

23.

x	1	2	3	4
$f(x)$	70	40	10	-20

24.

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

25.

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

26.

x	1	2	3	4
$f(x)$	10	20	40	80

27.

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

For the following exercises, use the compound interest formula, $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$.

28. After a certain number of years, the value of an investment account is represented by the equation $A = 10,250\left(1+\frac{0.04}{12}\right)^{120}$. What is the value of the account?

29. What was the initial deposit made to the account in the previous exercise?

30. How many years had the account from the previous exercise been accumulating interest?

31. An account is opened with an initial deposit of \$6,500 and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?

32. How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

33. Solve the compound interest formula for the principal, P.

34. Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)

35. How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?

36. Use properties of rational exponents to solve the compound interest formula for the interest rate, r.

37. Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of \$9,000 and was worth \$13,373.53 after 10 years.

38. Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of \$5,500, and was worth \$38,455 after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

39. $y = 3742(e)^{0.75t}$

40. $y = 150(e)^{\frac{3.25}{t}}$

41. $y = 2.25(e)^{-2t}$

42. Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?

43. How much less would the account from Exercise 42 be worth after 30 years if it were compounded monthly instead?

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

44. $f(x) = 2(5)^x$, for $f(-3)$

45. $f(x) = -4^{2x+3}$, for $f(-1)$

46. $f(x) = e^x$, for $f(3)$

47. $f(x) = -2e^{x-1}$, for $f(-1)$

48. $f(x) = 2.7(4)^{-x+1} + 1.5$, for $f(-2)$

49. $f(x) = 1.2e^{2x} - 0.3$, for $f(3)$

50. $f(x) = -\frac{3}{2}(3)^{-x} + \frac{3}{2}$, for $f(2)$

Technology

For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.

51. $(0, 3)$ and $(3, 375)$

52. $(3, 222.62)$ and $(10, 77.456)$

53. $(20, 29.495)$ and $(150, 730.89)$

54. $(5, 2.909)$ and $(13, 0.005)$

55. $(11{,}310.035)$ and $(25{,}356365.2)$

Extensions

56. The *annual percentage yield* (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $\text{APY} = \left(1 + \frac{r}{12}\right)^{12} - 1.$

57. Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.

58. Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that a and b are positive numbers and $b \neq 1.$ Any positive number b can be written as $b = e^n$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number e as a base.

59. In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$, the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f(x) = a(e)^{-nx}$ for some positive number n .

60. The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t) = a(e)^{rt}$, where a is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time t can be calculated with the formula $I(t) = e^{rt} - 1$.

Real-World Applications

61. The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were 23,900 fox counted in the area. What is the fox population predicted to be in the year 2020?

62. A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?

63. In the year 1985, a house was valued at $110,000. By the year 2005, the value had appreciated to $145,000. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

64. A car was valued at $38,000 in the year 2007. By 2013, the value had depreciated to $11,000 If the car’s value continues to drop by the same percentage, what will it be worth by 2017?

65. Jaylen wants to save $54,000 for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?

66. Kyoko has $10,000 that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $15,000 by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (*Hint*: solve the compound interest formula for the interest rate.)

67. Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $13,500. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

68. An investment account with an annual interest rate of 7% was opened with an initial deposit of $4,000 Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

6.2 Graphs of Exponential Functions

Learning Objectives

- Graph exponential functions.
- Graph exponential functions using transformations.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table 1 change as the input increases by 1.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Table 1

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x) = ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

Figure 1 shows the exponential growth function $f(x) = 2^x$.

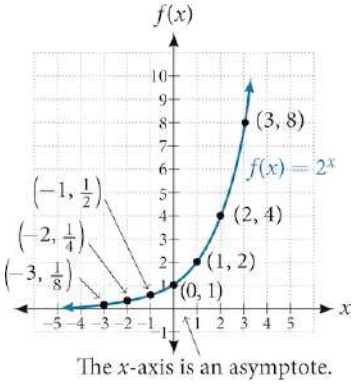


Figure 1 Notice that the graph gets close to the *x*-axis, but never touches it.

The domain of $f(x) = 2^x$ is all real numbers, the range is $\left(0, \infty\right)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of **exponential decay**, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table 2 change as the input increases by 1.

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Table 2

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

Figure 2 shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

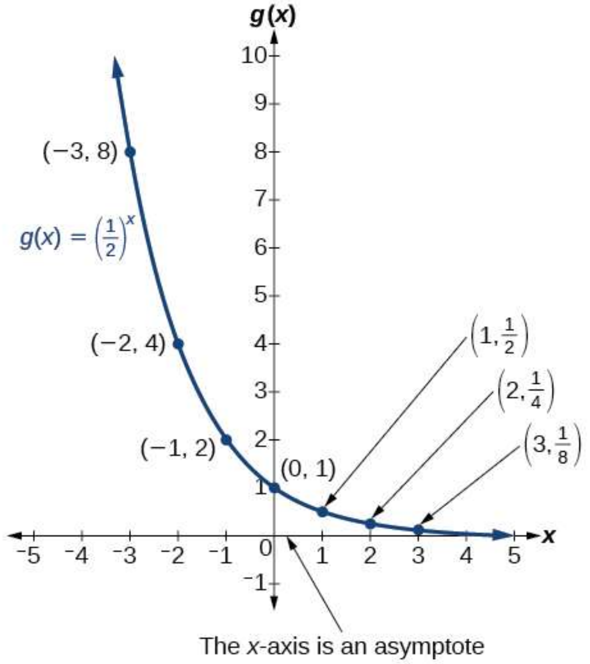


Figure 2

The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $\left(0, \infty\right)$, and the horizontal asymptote is $y = 0$.

Characteristics of the Graph of the Parent Function $f(x) = b^x$

An exponential function with the form $f(x) = b^x$, $b > 0, b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- *x*-intercept: none
- *y*-intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

Figure 3 compares the graphs of exponential growth and decay functions.

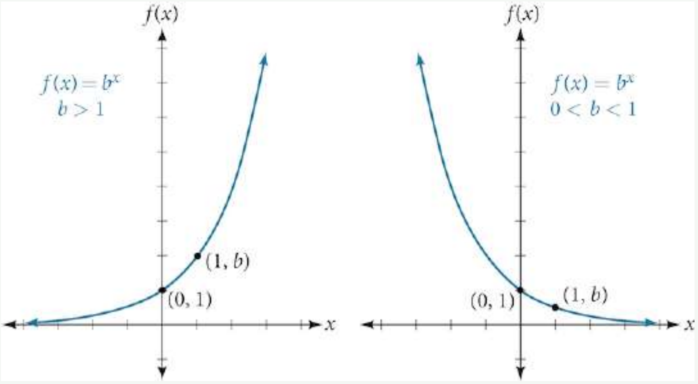


Figure 3

HOW TO

Given an exponential function of the form $f(x) = b^x$, graph the function.

1. Create a table of points.
2. Plot at least 3 point from the table, including the *y*-intercept $(0, 1)$.
3. Draw a smooth curve through the points.
4. State the domain, $\left(-\infty, \infty\right)$, the range, $\left(0, \infty\right)$, and the horizontal asymptote, $y = 0$.

EXAMPLE 1

Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and create a table of points for the graph.

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in Table 3.

x	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	0.0625	0.015625

Table 3

- Plot the *y*-intercept, $(0, 1)$, along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in Figure 4.

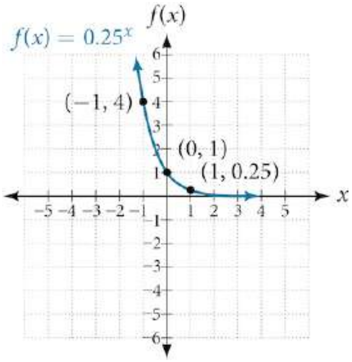


Figure 4

The domain is $\left(-\infty, \infty\right)$; the range is $\left(0, \infty\right)$; the horizontal asymptote is $y = 0$.

TRY IT #1 Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the parent function $f(x) = b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure 5.

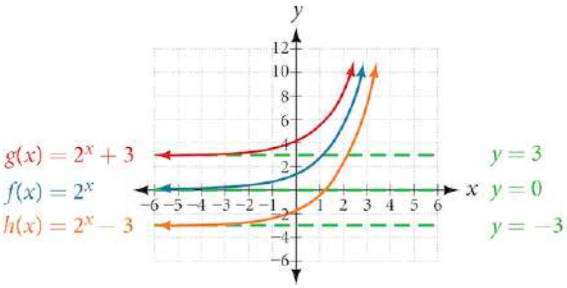


Figure 5

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The y-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The y-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the parent function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure 6.

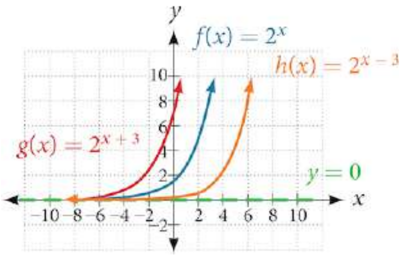


Figure 6

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$, remains unchanged.
- The asymptote, $y = 0$, remains unchanged.
- The y-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$, the y-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)\,2^x$, so the initial value of the function is 8.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$, the y-intercept becomes $\left(0, \frac{1}{8}\right)$. Again, see that $2^{x-3} = \left(\frac{1}{8}\right) 2^x$, so the initial value of the function is $\frac{1}{8}$.

Shifts of the Parent Function $f(x) = b^x$

For any constants c and d, the function $f(x) = b^{x+c} + d$ shifts the parent function $f(x) = b^x$

- vertically d units, in the *same* direction of the sign of d.
- horizontally c units, in the *opposite* direction of the sign of c.
- The y-intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes $\left(d, \infty\right)$.
- The domain, $\left(-\infty, \infty\right)$, remains unchanged.

HOW TO

Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation.

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
3. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
4. State the domain, $\left(-\infty, \infty\right)$, the range, $\left(d, \infty\right)$, and the horizontal asymptote $y = d$.

EXAMPLE 2

Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

Solution

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d =$ -3.

Draw the horizontal asymptote $y = d$, so draw $y = -3$.

Identify the shift as $(-c, d)$, so the shift is $(-1, -3)$.

Shift the graph of $f(x) = b^x$ left 1 units and down 3 units.

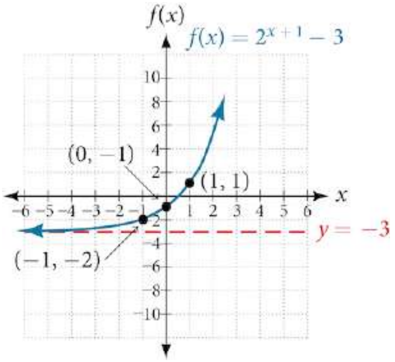


Figure 7

The domain is $\left(-\infty, \infty\right)$; the range is $\left(-3, \infty\right)$; the horizontal asymptote is $y = -3$.

TRY IT #2 Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

HOW TO

Given an equation of the form $f(x) = b^{x+c} + d$ for x, use a graphing calculator to approximate the solution.

- Press **[Y=]**. Enter the given exponential equation in the line headed "**Y_1=**".
- Enter the given value for $f(x)$ in the line headed "**Y_2=**".
- Press **[WINDOW]**. Adjust the y-axis so that it includes the value entered for "**Y_2=**".
- Press **[GRAPH]** to observe the graph of the exponential function along with the line for the specified value of $f(x)$.
- To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of *x* for the indicated value of the function.

EXAMPLE 3

Approximating the Solution of an Exponential Equation

Solve $42 = 1.2(5)^x + 2.8$ graphically. Round to the nearest thousandth.

Solution

Press **[Y=]** and enter $1.2(5)^x + 2.8$ next to **Y_1**=. Then enter 42 next to **Y2=**. For a window, use the values –3 to 3 for x and –5 to 55 for y. Press **[GRAPH]**. The graphs should intersect somewhere near $x = 2$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 2.1661943. (Your answer may be different if you use a different window or use a different value for **Guess?**) To the nearest thousandth, $x \approx 2.166$.

TRY IT #3 Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the parent function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the stretch, using $a = 3$, to get $g(x) = 3(2)^x$ as shown on the left in Figure 8, and the compression, using $a = \frac{1}{3}$, to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure 8.

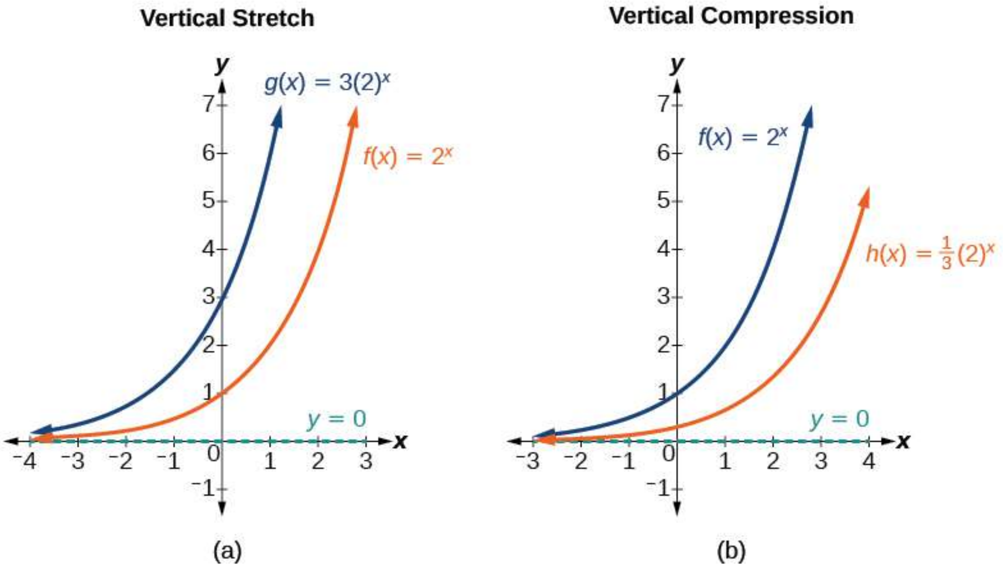


Figure 8 (a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

Stretches and Compressions of the Parent Function $f(x) = b^x$

For any factor $a > 0$, the function $f(x) = a(b)^x$

- is stretched vertically by a factor of a if $|a| > 1$.
- is compressed vertically by a factor of a if $|a| < 1$.
- has a y-intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $\left(0, \infty\right)$, and a domain of $\left(-\infty, \infty\right)$, which are unchanged from the parent function.

EXAMPLE 4

Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the *x*-axis as x increases.
- Since $a = 4$, the graph of $f(x) = \left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Create a table of points as shown in Table 4.

x	-3	-2	-1	0	1	2	3
$f(x) = 4\left(\frac{1}{2}\right)^x$	32	16	8	4	2	1	0.5

Table 4

- Plot the *y*-intercept, $(0, 4)$, along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in Figure 9.

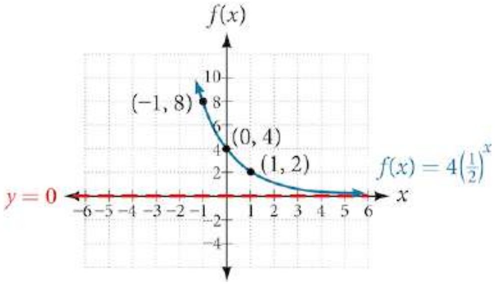


Figure 9

The domain is $\left(-\infty, \infty\right)$; the range is $\left(0, \infty\right)$; the horizontal asymptote is $y = 0$.

TRY IT #4 Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the *x*-axis or the *y*-axis. When we multiply the parent function $f(x) = b^x$ by -1, we get a reflection about the *x*-axis. When we multiply the input by -1, we get a reflection about the *y*-axis. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the *x*-axis, $g(x) = -2^x$, is shown on the left side of Figure 10, and the reflection about the *y*-axis $h(x) = 2^{-x}$, is shown on the right side of Figure 10.

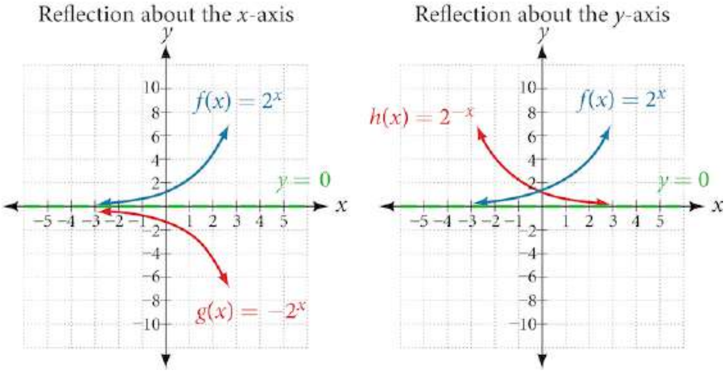


Figure 10 (a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the *y*-axis.

Reflections of the Parent Function $f(x) = b^x$

The function $f(x) = -b^x$

- reflects the parent function $f(x) = b^x$ about the *x*-axis.
- has a *y*-intercept of $(0, -1)$.
- has a range of $\left(-\infty, 0\right)$.
- has a horizontal asymptote at $y = 0$ and domain of $\left(-\infty, \infty\right)$, which are unchanged from the parent function.

The function $f(x) = b^{-x}$

- reflects the parent function $f(x) = b^x$ about the *y*-axis.
- has a *y*-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $\left(0, \infty\right)$, and a domain of $\left(-\infty, \infty\right)$, which are unchanged from the parent function.

EXAMPLE 5

Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = \left(\frac{1}{4}\right)^x$ about the *x*-axis. State its domain, range, and asymptote.

Solution

Since we want to reflect the parent function $f(x) = \left(\frac{1}{4}\right)^x$ about the *x*-axis, we multiply $f(x)$ by -1 to get, $g(x) = -\left(\frac{1}{4}\right)^x$. Next we create a table of points as in Table 5.

x	-3	-2	-1	0	1	2	3
$g(x) = -\left(\frac{1}{4}\right)^x$	-64	-16	-4	-1	-0.25	-0.0625	-0.0156

Table 5

Plot the *y*-intercept, $(0, -1)$, along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

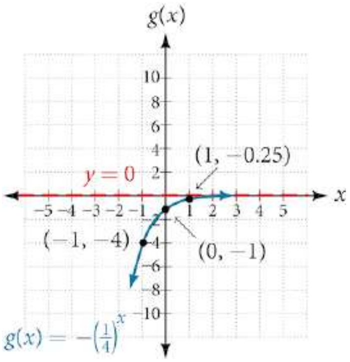


Figure 11

The domain is $\left(-\infty, \infty\right)$; the range is $\left(-\infty, 0\right)$; the horizontal asymptote is $y = 0$.

TRY IT #5 Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the y-axis. State its domain, range, and asymptote.

Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in Table 6 to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $f(x) = b^x$	
Translation	**Form**
Shift • Horizontally c units to the left • Vertically d units up	$f(x) = b^{x+c} + d$
Stretch and Compress • Stretch if $\lvert a\rvert > 1$ • Compression if $0 < \lvert a\rvert < 1$	$f(x) = ab^x$
Reflect about the x-axis	$f(x) = -b^x$
Reflect about the y-axis	$f(x) = b^{-x} = \left(\frac{1}{b}\right)^x$
General equation for all translations	$f(x) = ab^{x+c} + d$

Table 6

Translations of Exponential Functions

A translation of an exponential function has the form

$f(x) = ab^{x+c} + d$

Where the parent function, $y = b^x$, $b > 1$, is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.

- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- shifted vertically d units.
- reflected about the *x*-axis when $a < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

EXAMPLE 6

Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is vertically stretched by a factor of 2 , reflected across the *y*-axis, and then shifted up 4 units.

Solution

We want to find an equation of the general form $f(x) = ab^{x+c} + d$. We use the description provided to find a, b, c, and d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2 , so $a = 2$.
- The function is reflected about the *y*-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$\begin{aligned} f(x) &= ab^{x+c} + d \\ &= 2e^{-x+0} + 4 \\ &= 2e^{-x} + 4 \end{aligned}$$

The domain is $\left(-\infty, \infty\right)$; the range is $\left(4, \infty\right)$; the horizontal asymptote is $y = 4$.

TRY IT #6 Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

- $f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the *x*-axis and then shifted down 2 units.

MEDIA

Access this online resource for additional instruction and practice with graphing exponential functions.

[Graph Exponential Functions (http://openstax.org/l/graphexpfunc)](http://openstax.org/l/graphexpfunc)

6.2 SECTION EXERCISES

Verbal

1. What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?

2. What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

Algebraic

3. The graph of $f(x) = 3^x$ is reflected about the *y*-axis and stretched vertically by a factor of 4. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

4. The graph of $f(x) = \left(\frac{1}{2}\right)^{-x}$ is reflected about the *y*-axis and compressed vertically by a factor of $\frac{1}{5}$. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

5. The graph of $f(x) = 10^x$ is reflected about the *x*-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

6. The graph of $f(x) = (1.68)^x$ is shifted right 3 units, stretched vertically by a factor of 2, reflected about the *x*-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its *y*-intercept (to the nearest thousandth), domain, and range.

7. The graph of $f(x)= -\frac{1}{2}\left(\frac{1}{4}\right)^{x-2} + 4$ is shifted downward 4 units, and then shifted left 2 units, stretched vertically by a factor of 4, and reflected about the *x*-axis. What is the equation of the new function, $g(x)$? State its *y*-intercept, domain, and range.

Graphical

For the following exercises, graph the function and its reflection about the y-axis on the same axes, and give the y-intercept.

8. $f(x) = 3\left(\frac{1}{2}\right)^x$

9. $g(x) = -2(0.25)^x$

10. $h(x) = 6(1.75)^{-x}$

For the following exercises, graph each set of functions on the same axes.

11. $f(x) = 3\left(\frac{1}{4}\right)^x$, $g(x) = 3(2)^x$, and $h(x) = 3(4)^x$

12. $f(x) = \frac{1}{4}(3)^x$, $g(x) = 2(3)^x$, and $h(x) = 4(3)^x$

For the following exercises, match each function with one of the graphs in Figure 12.

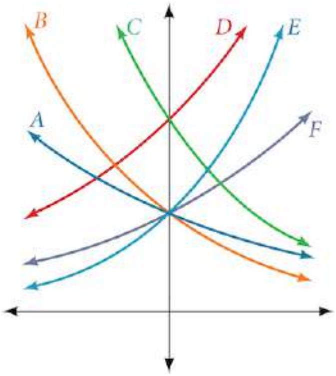


Figure 12

13. $f(x) = 2(0.69)^x$

14. $f(x) = 2(1.28)^x$

15. $f(x) = 2(0.81)^x$

16. $f(x) = 4(1.28)^x$

17. $f(x) = 2(1.59)^x$

18. $f(x) = 4(0.69)^x$

For the following exercises, use the graphs shown in Figure 13. All have the form $f(x) = ab^x$.

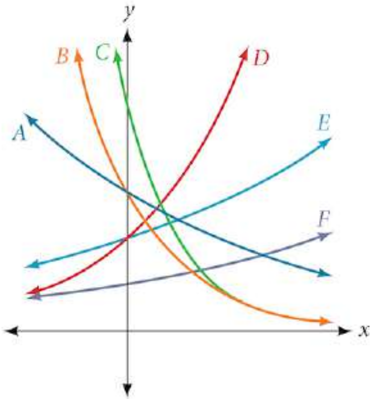


Figure 13

19. Which graph has the largest value for b?

20. Which graph has the smallest value for b?

21. Which graph has the largest value for a?

22. Which graph has the smallest value for a?

For the following exercises, graph the function and its reflection about the x-axis on the same axes.

23. $f(x) = \frac{1}{2}(4)^x$

24. $f(x) = 3(0.75)^x - 1$

25. $f(x) = -4(2)^x + 2$

For the following exercises, graph the transformation of $f(x) = 2^x$. Give the horizontal asymptote, the domain, and the range.

26. $f(x) = 2^{-x}$

27. $h(x) = 2^x + 3$

28. $f(x) = 2^{x-2}$

For the following exercises, describe the end behavior of the graphs of the functions.

29. $f(x) = -5(4)^x - 1$

30. $f(x) = 3\left(\frac{1}{2}\right)^x - 2$

31. $f(x) = 3(4)^{-x} + 2$

For the following exercises, start with the graph of $f(x) = 4^x$. Then write a function that results from the given transformation.

32. Shift $f(x)$ 4 units upward

33. Shift $f(x)$ 3 units downward

34. Shift $f(x)$ 2 units left

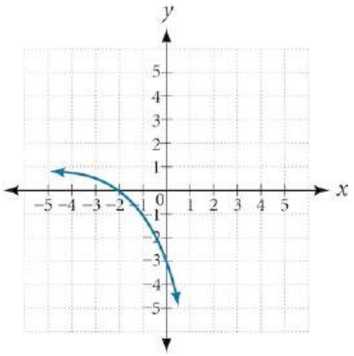
35. Shift $f(x)$ 5 units right

36. Reflect $f(x)$ about the *x*-axis

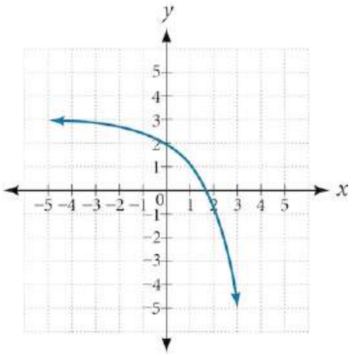
37. Reflect $f(x)$ about the *y*-axis

For the following exercises, each graph is a transformation of $y = 2^x$. Write an equation describing the transformation.

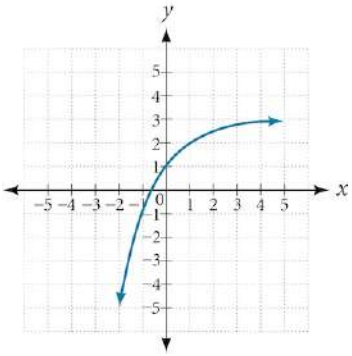
38.



39.

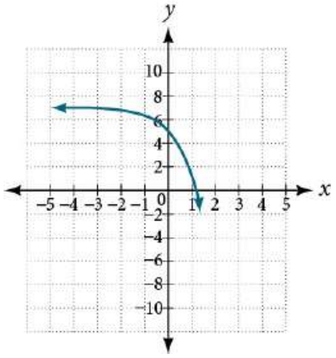


40.

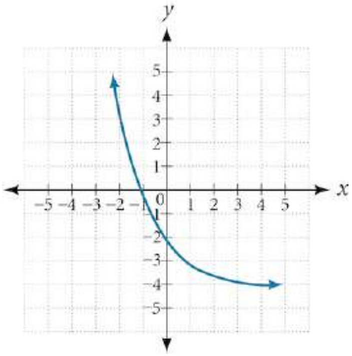


For the following exercises, find an exponential equation for the graph.

41.



42.



Numeric

For the following exercises, evaluate the exponential functions for the indicated value of x.

43. $g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.

44. $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.

45. $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^x + 6$ for $h(-7)$.

Technology

For the following exercises, use a graphing calculator to approximate the solutions of the equation. Round to the nearest thousandth.

46. $-50 = -\left(\frac{1}{2}\right)^{-x}$

47. $116 = \frac{1}{4}\left(\frac{1}{8}\right)^x$

48. $12 = 2(3)^x + 1$

49. $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$

50. $-30 = -4(2)^{x+2} + 2$

Extensions

51. Explore and discuss the graphs of $F(x) = (b)^x$ and $G(x) = \left(\frac{1}{b}\right)^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b}\right)^x$ for any real number $b > 0$.

52. Prove the conjecture made in the previous exercise.

53. Explore and discuss the graphs of $f(x) = 4^x$, $g(x) = 4^{x-2}$, and $h(x) = \left(\frac{1}{16}\right)4^x$. Then make a conjecture about the relationship between the graphs of the functions b^x and $\left(\frac{1}{b^n}\right)b^x$ for any real number *n* and real number $b > 0$.

54. Prove the conjecture made in the previous exercise.

6.3 Logarithmic Functions

Learning Objectives

In this section, you will:

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

Figure 1 Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes[4]. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings,[5] like those shown in Figure 1. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale[6] whereas the Japanese earthquake registered a 9.0.[7]

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4} = 10^4 = 10{,}000$ times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released

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[4] http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/#summary. Accessed 3/4/2013.

[5] http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#summary. Accessed 3/4/2013.

[6] http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/. Accessed 3/4/2013.

[7] http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#details. Accessed 3/4/2013.

from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure 2, to better estimate the solution.

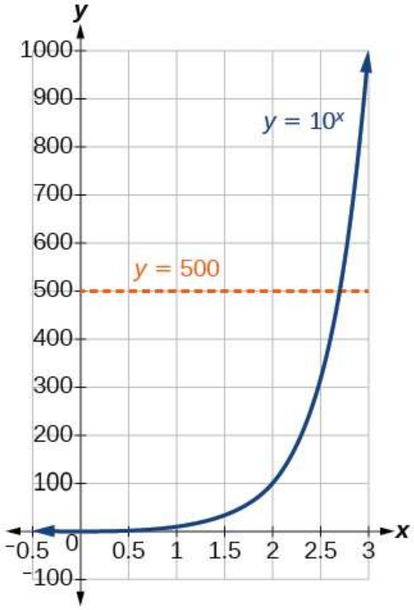


Figure 2

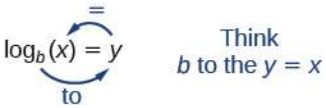
Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure 2 passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, “The logarithm with base b of x is equal to y, ” or, simplified, “log base b of x is y. ” We can also say, “ b raised to the power of y is x, ” because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as “log base 2 of 32 is 5.”

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

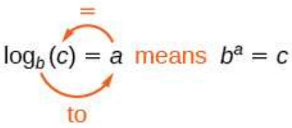
$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:



Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b(x)$ and $y = b^x$ are inverse functions.

Definition of the Logarithmic Function

A **logarithm** base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1,$

$$y = \log_b(x) \quad \text{is equivalent to } b^y = x$$

where,

- we read $\log_b(x)$ as, “the logarithm with base b of x ” or the “log base b of x. "
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

Q&A **Can we take the logarithm of a negative number?**

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

HOW TO

Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form.

1. Examine the equation $y = \log_b\left(x\right)$ and identify $b, y,$ and$x$.
2. Rewrite $\log_b\left(x\right) = y$ as $b^y = x$.

EXAMPLE 1

Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

ⓐ $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$ ⓑ $\log_3(9) = 2$

Solution

First, identify the values of $b, y,$ and$x$. Then, write the equation in the form $b^y = x$.

ⓐ $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$

Here, $b = 6, y = \frac{1}{2},$ and $x = \sqrt{6}$. Therefore, the equation $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$ is equivalent to $6^{\frac{1}{2}} = \sqrt{6}$.

ⓑ $\log_3(9) = 2$

Here, $b = 3, y = 2,$ and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$.

TRY IT #1 Write the following logarithmic equations in exponential form.

ⓐ $\log_{10}(1{,}000{,}000) = 6$ ⓑ $\log_5(25) = 2$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b, exponent x, and output y. Then we write $x = \log_b(y)$.

EXAMPLE 2

Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \frac{1}{10{,}000}$

Solution

First, identify the values of $b, y, \text{and} x$. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$

Here, $b = 2$, $x = 3$, and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$

Here, $b = 5$, $x = 2$, and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \frac{1}{10{,}000}$

Here, $b = 10$, $x = -4$, and $y = \frac{1}{10{,}000}$. Therefore, the equation $10^{-4} = \frac{1}{10{,}000}$ is equivalent to $\log_{10}(\frac{1}{10{,}000}) = -4$.

TRY IT #2 Write the following exponential equations in logarithmic form.

ⓐ $3^2 = 9$ ⓑ $5^3 = 125$ ⓒ $2^{-1} = \frac{1}{2}$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$? " We know $2^2 = 4$ and $3^2 = 9$, so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2$.

HOW TO

Given a logarithm of the form $y = \log_b(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of b : $b^y = x$.
2. Use previous knowledge of powers of b identify y by asking, "To what exponent should b be raised in order to get x? "

EXAMPLE 3

Solving Logarithms Mentally

Solve $y = \log_4 (64)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know

$$4^3 = 64$$

Therefore,

$$\log_4(64) = 3$$

TRY IT #3 Solve $y = \log_{121} (11)$ without using a calculator.

EXAMPLE 4

Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3 \left(\frac{1}{27}\right)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$? "

We know $3^3 = 27$, but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3 \left(\frac{1}{27}\right) = -3$.

TRY IT #4 Evaluate $y = \log_2 \left(\frac{1}{32}\right)$ without using a calculator.

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

Definition of the Common Logarithm

A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \quad \text{is equivalent to } 10^y = x$$

We read $\log(x)$ as, "the logarithm with base 10 of x " or "log base 10 of x. "

The logarithm y is the exponent to which 10 must be raised to get x.

HOW TO

Given a common logarithm of the form $y = \log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of $10 : 10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, “To what exponent must 10 be raised in order to get x? ”

EXAMPLE 5

Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, “To what exponent must 10 be raised in order to get 1000?” We know

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

TRY IT #5 Evaluate $y = \log(1{,}000{,}000)$.

HOW TO

Given a common logarithm with the form $y = \log(x)$, evaluate it using a calculator.

1. Press **[LOG]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

EXAMPLE 6

Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution

- Press **[LOG]**.
- Enter 321, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

$$\begin{array}{ccccc} 100 & < & 321 & < & 1000 \\ 2 & < & 2.5065 & < & 3 \end{array}$$

TRY IT #6 Evaluate $y = \log(123)$ to four decimal places using a calculator.

EXAMPLE 7

Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution

We begin by rewriting the exponential equation in logarithmic form.

$$\begin{aligned} 10^x &= 500 \\ \log(500) &= x \qquad \text{Use the definition of the common log.} \end{aligned}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500, followed by **[ )]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about 2.699.

TRY IT #7 The amount of energy released from one earthquake was $8{,}500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$.

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Definition of the Natural Logarithm

A **natural logarithm** is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \quad \text{is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

HOW TO

Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator.

1. Press **[LN]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

EXAMPLE 8

Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution

- Press **[LN]**.
- Enter 500, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

TRY IT #8 Evaluate $\ln(-500)$.

MEDIA

Access this online resource for additional instruction and practice with logarithms.

Introduction to Logarithms (http://openstax.org/l/intrologarithms)

6.3 SECTION EXERCISES

Verbal

1. What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1$.

2. How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?

3. How can the logarithmic equation $\log_b x = y$ be solved for x using the properties of exponents?

4. Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

5. Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Algebraic

For the following exercises, rewrite each equation in exponential form.

6. $\log_4(q) = m$

7. $\log_a(b) = c$

8. $\log_{16}(y) = x$

9. $\log_x(64) = y$

10. $\log_y(x) = -11$

11. $\log_{15}(a) = b$

12. $\log_y(137) = x$

13. $\log_{13}(142) = a$

14. $\log(v) = t$

15. $\ln(w) = n$

For the following exercises, rewrite each equation in logarithmic form.

16. $4^x = y$

17. $c^d = k$

18. $m^{-7} = n$

19. $19^x = y$

20. $x^{-\frac{10}{13}} = y$

21. $n^4 = 103$

22. $\left(\frac{7}{5}\right)^m = n$

23. $y^x = \frac{39}{100}$

24. $10^a = b$

25. $e^k = h$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

26. $\log_3(x) = 2$

27. $\log_2(x) = -3$

28. $\log_5(x) = 2$

29. $\log_3(x) = 3$

30. $\log_2(x) = 6$

31. $\log_9(x) = \frac{1}{2}$

32. $\log_{18}(x) = 2$

33. $\log_6(x) = -3$

34. $\log(x) = 3$

35. $\ln(x) = 2$

For the following exercises, use the definition of common and natural logarithms to simplify.

36. $\log(100^8)$

37. $10^{\log(32)}$

38. $2\log(.0001)$

39. $e^{\ln(1.06)}$

40. $\ln\left(e^{-5.03}\right)$

41. $e^{\ln(10.125)} + 4$

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

42. $\log_3\left(\frac{1}{27}\right)$

43. $\log_6(\sqrt{6})$

44. $\log_2\left(\frac{1}{8}\right) + 4$

45. $6\log_8(4)$

For the following exercises, evaluate the common logarithmic expression without using a calculator.

46. $\log(10,000)$

47. $\log(0.001)$

48. $\log(1) + 7$

49. $2\log(100^{-3})$

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

50. $\ln(e^{\frac{1}{3}})$

51. $\ln(1)$

52. $\ln(e^{-0.225}) - 3$

53. $25\ln(e^{\frac{2}{5}})$

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

54. $\log(0.04)$

55. $\ln(15)$

56. $\ln\left(\frac{4}{5}\right)$

57. $\log(\sqrt{2})$

58. $\ln(\sqrt{2})$

Extensions

59. Is $x = 0$ in the domain of the function $f(x) = \log(x)$? If so, what is the value of the function when $x = 0$? Verify the result.

60. Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of x? Verify the result.

61. Is there a number x such that $\ln x = 2$? If so, what is that number? Verify the result.

62. Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$? Verify the result.

63. Is the following true: $\frac{\ln\left(e^{1.725}\right)}{\ln(1)} = 1.725$? Verify the result.

Real-World Applications

64. The exposure index EI for a camera is a measurement of the amount of light that hits the image receptor. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the “f-stop” setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?

65. Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?

66. The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula $\log\frac{I_1}{I_2} = M_1 - M_2$ where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0.[8] How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

6.4 Graphs of Logarithmic Functions

Learning Objectives

In this section, you will:

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

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[8] http://earthquake.usgs.gov/earthquakes/world/historical.php. Accessed 3/4/2014.

In Graphs of Exponential Functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure 1 shows this point on the logarithmic graph.

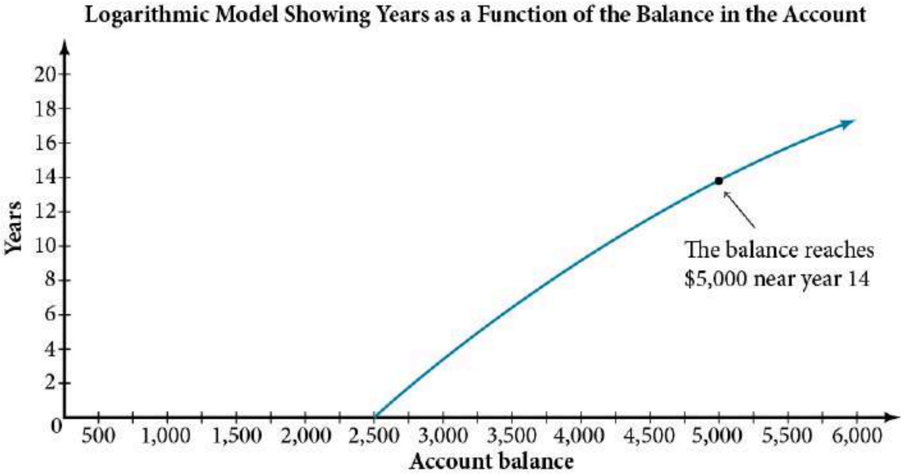


Figure 1

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $\left(-\infty, \infty\right)$.
- The range of y is $\left(0, \infty\right)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $\left(0, \infty\right)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $\left(-\infty, \infty\right)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$, is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{aligned} 2x - 3 &> 0 && \text{Show the argument greater than zero.} \\ 2x &> 3 && \text{Add 3.} \\ x &> 1.5 && \text{Divide by 2.} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $\left(1.5, \infty\right)$.

HOW TO

Given a logarithmic function, identify the domain.

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

EXAMPLE 1

Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$\begin{aligned} x + 3 &> 0 && \text{The input must be positive.} \\ x &> -3 && \text{Subtract 3.} \end{aligned}$$

The domain of $f(x) = \log_2(x + 3)$ is $\left(-3, \infty\right)$.

TRY IT #1 What is the domain of $f(x) = \log_5(x - 2) + 1$?

EXAMPLE 2

Identifying the Domain of a Logarithmic Shift and Reflection

What is the domain of $f(x) = \log(5 - 2x)$?

Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $5–2x > 0$. Solving this inequality,

$$\begin{aligned} 5 - 2x &> 0 && \text{The input must be positive.} \\ -2x &> -5 && \text{Subtract 5.} \\ x &< \tfrac{5}{2} && \text{Divide by } -2 \text{ and switch the inequality.} \end{aligned}$$

The domain of $f(x) = \log(5 - 2x)$ is $\left(-\infty, \frac{5}{2}\right)$.

TRY IT #2 What is the domain of $f(x) = \log(x - 5) + 2$?

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table 1.

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Table 1

Using the inputs and outputs from Table 1, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table 2.

$f(x) = 2^x$	$\left(-3, \frac{1}{8}\right)$	$\left(-2, \frac{1}{4}\right)$	$\left(-1, \frac{1}{2}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$g(x) = \log_2(x)$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$

Table 2

As we'd expect, the *x*- and *y*-coordinates are reversed for the inverse functions. Figure 2 shows the graph of f and g.

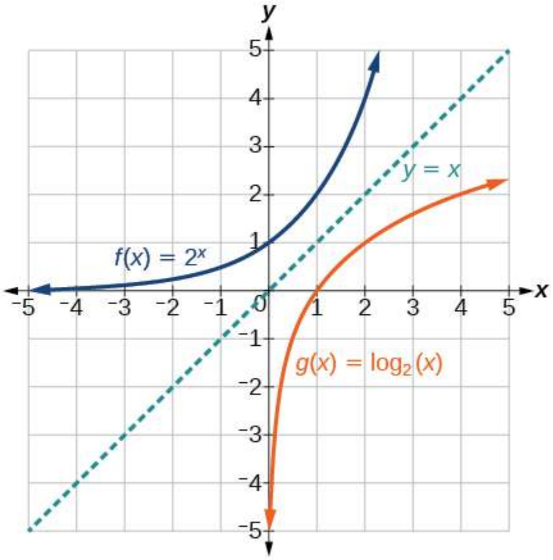


Figure 2 Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a *y*-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an *x*- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $\left(-\infty, \infty\right)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $\left(0, \infty\right)$, is the same as the domain of $g(x) = \log_2(x)$.

Characteristics of the Graph of the Parent Function, $f(x) = \log_b(x)$:

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $\left(-\infty, \infty\right)$
- *x*-intercept: $(1, 0)$ and key point $(b, 1)$
- *y*-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

See Figure 3.

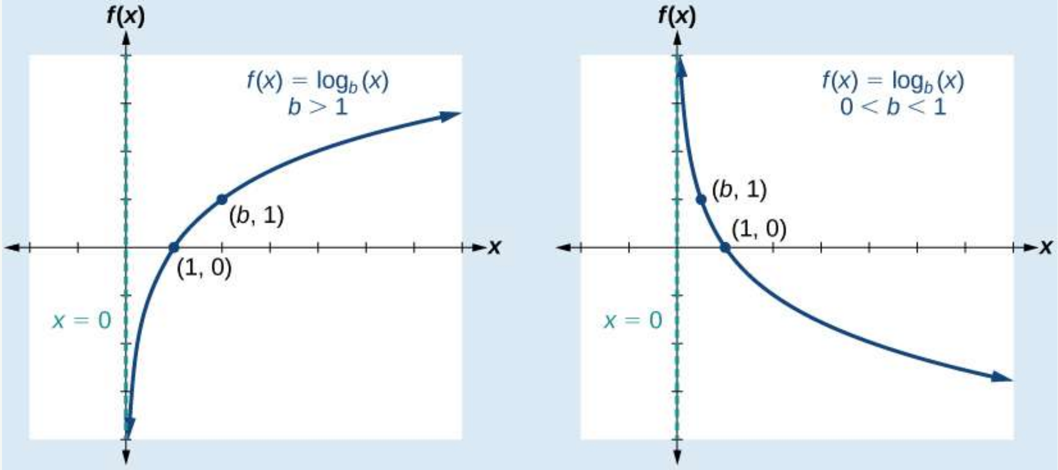


Figure 3

Figure 4 shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (*Note:* recall that the function $\ln(x)$ has base $e \approx 2.718$.)

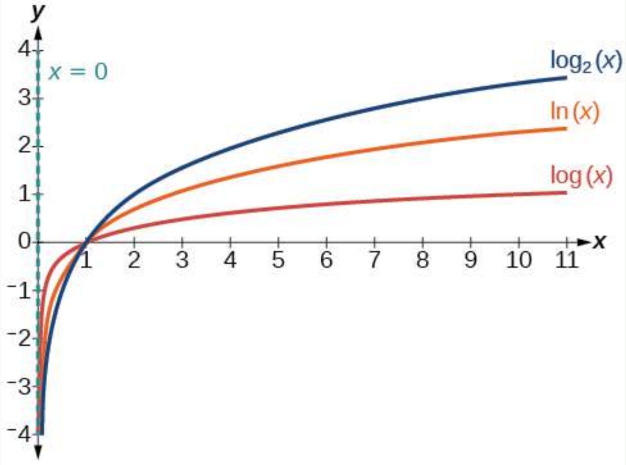


Figure 4 The graphs of three logarithmic functions with different bases, all greater than 1.

HOW TO

Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.

3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $\left(0, \infty\right)$, the range, $\left(-\infty, \infty\right)$, and the vertical asymptote, $x = 0$.

EXAMPLE 3

Graphing a Logarithmic Function with the Form *f*(*x*) = log$_b$(*x*).

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The *x*-intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure 5).

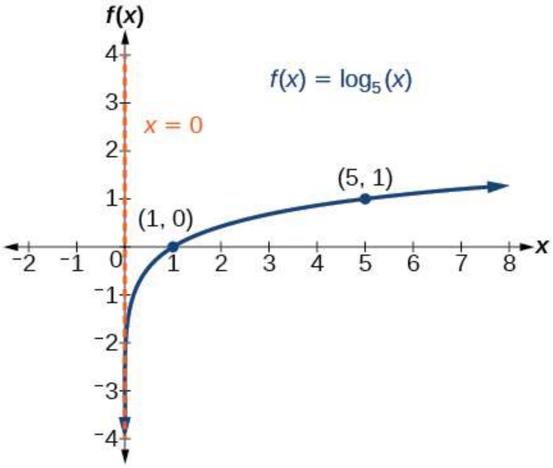


Figure 5

The domain is $\left(0, \infty\right)$, the range is $\left(-\infty, \infty\right)$, and the vertical asymptote is $x = 0$.

TRY IT #3 Graph $f(x) = \log_{\frac{1}{5}}(x)$. State the domain, range, and asymptote.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of *f*(*x*) = log$_b$(*x*)

When a constant c is added to the input of the parent function $f(x) = log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 6.

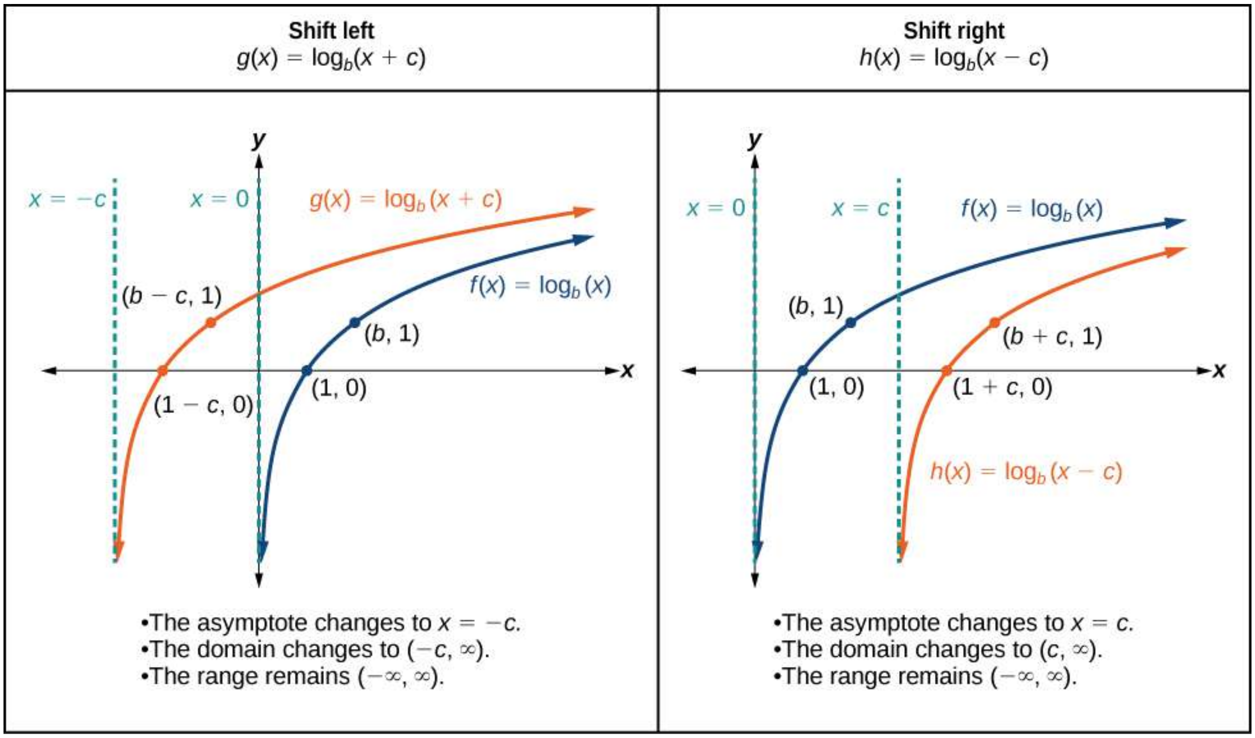


Figure 6

Horizontal Shifts of the Parent Function $f(x) = \log_b(x)$

For any constant c, the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left c units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right c units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $\left(-c, \infty\right)$.
- has range $\left(-\infty, \infty\right)$.

HOW TO

Given a logarithmic function with the form $f(x) = \log_b(x + c)$, graph the translation.

1. Identify the horizontal shift:
   a. If $c > 0$, shift the graph of $f(x) = \log_b(x)$ left c units.
   b. If $c < 0$, shift the graph of $f(x) = \log_b(x)$ right c units.
2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. Label the three points.
5. The Domain is $\left(-c, \infty\right)$, the range is $\left(-\infty, \infty\right)$, and the vertical asymptote is $x = -c$.

EXAMPLE 4

Graphing a Horizontal Shift of the Parent Function $y = \log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x-2)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x-2)$, we notice $x + (-2) = x – 2$.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the parent function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$, $(3, 0)$, and $(5, 1)$.

The domain is $\left(2, \infty\right)$, the range is $\left(-\infty, \infty\right)$, and the vertical asymptote is $x = 2$.

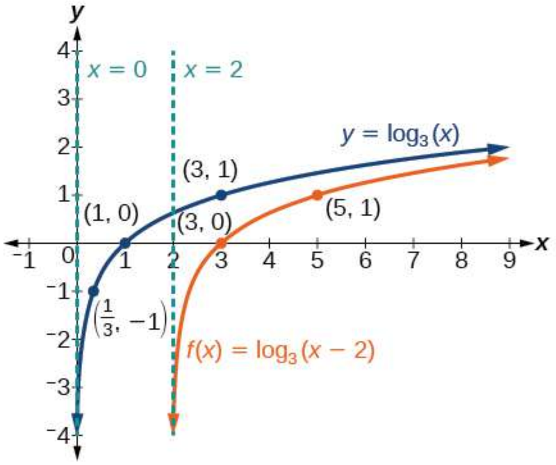
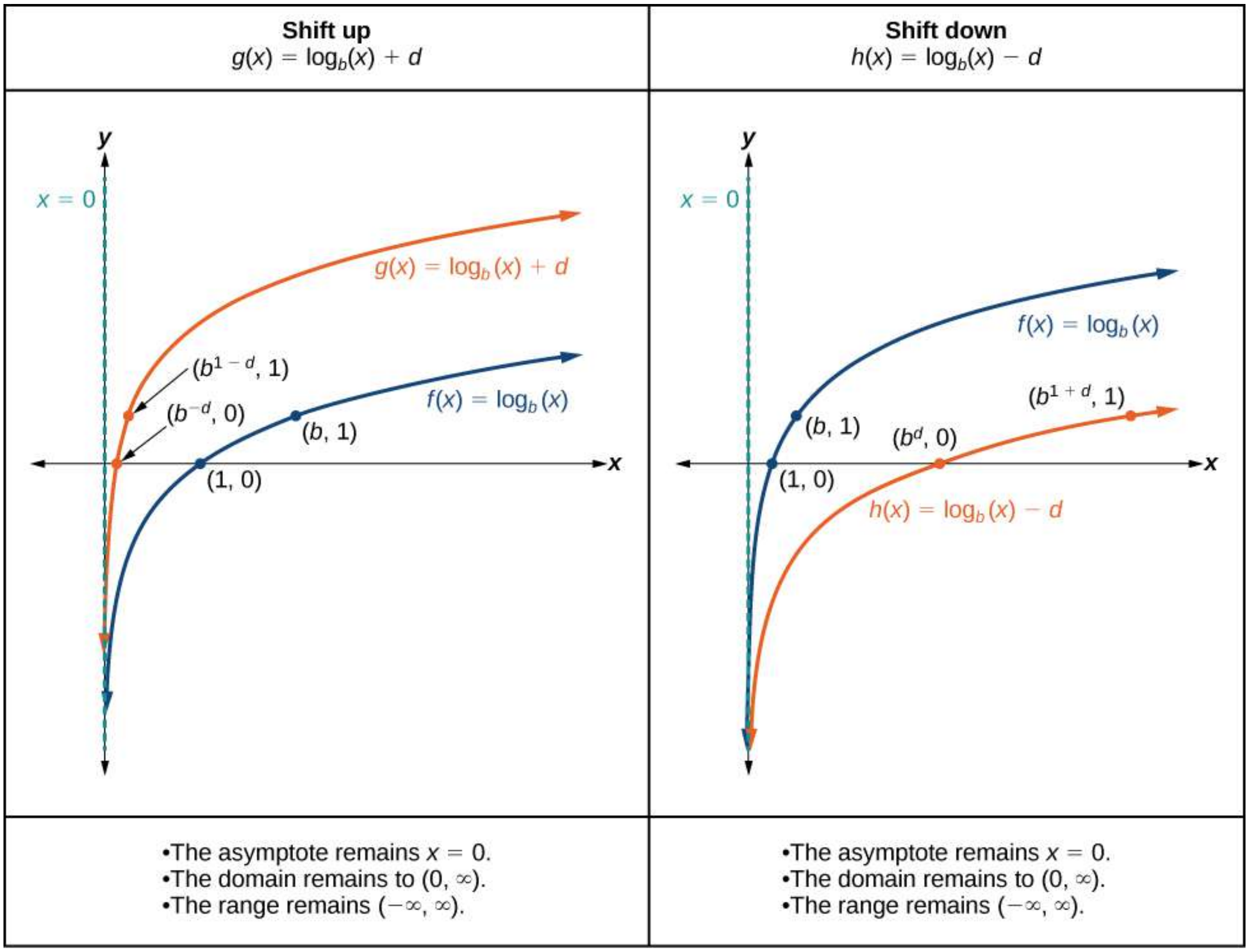


Figure 7

TRY IT #4 Sketch a graph of $f(x) = \log_3(x+4)$ alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Graphing a Vertical Shift of $y = \log_b(x)$

When a constant d is added to the parent function $f(x) = \log_b(x)$, the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$. See Figure 8.



•The asymptote remains $x = 0$.
•The domain remains to $(0, \infty)$.
•The range remains $(-\infty, \infty)$.

•The asymptote remains $x = 0$.
•The domain remains to $(0, \infty)$.
•The range remains $(-\infty, \infty)$.

Figure 8

Vertical Shifts of the Parent Function $y = \log_b(x)$

For any constant d, the function $f(x) = \log_b(x) + d$

- shifts the parent function $y = \log_b(x)$ up d units if $d > 0$.
- shifts the parent function $y = \log_b(x)$ down d units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $\left(0, \infty\right)$.
- has range $\left(-\infty, \infty\right)$.

HOW TO

Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up d units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$ down d units.
2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding d to the y coordinate.

4. Label the three points.
5. The domain is $\left(0,\infty\right)$, the range is $\left(-\infty,\infty\right)$, and the vertical asymptote is $x=0$.

EXAMPLE 5

Graphing a Vertical Shift of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x)=\log_3(x)-2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x)=\log_3(x)-2$, we will notice $d=-2$. Thus $d<0$.

This means we will shift the function $f(x)=\log_3(x)$ down 2 units.

The vertical asymptote is $x=0$.

Consider the three key points from the parent function, $\left(\frac{1}{3},-1\right)$, $(1,0)$, and $(3,1)$.

The new coordinates are found by subtracting 2 from the *y* coordinates.

Label the points $\left(\frac{1}{3},-3\right)$, $(1,-2)$, and $(3,-1)$.

The domain is $\left(0,\infty\right)$, the range is $\left(-\infty,\infty\right)$, and the vertical asymptote is $x=0$.

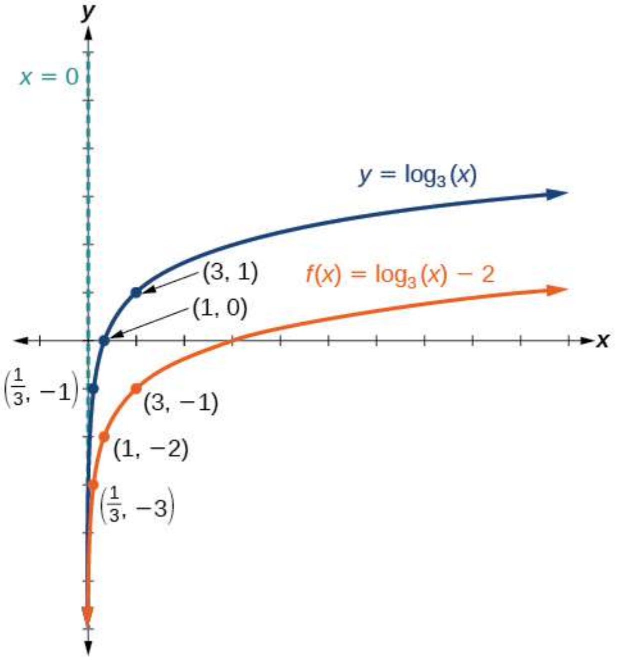


Figure 9

The domain is $\left(0,\infty\right)$, the range is $\left(-\infty,\infty\right)$, and the vertical asymptote is $x=0$.

TRY IT #5 Sketch a graph of $f(x)=\log_2(x)+2$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Graphing Stretches and Compressions of $y = \log_b(x)$

When the parent function $f(x)=\log_b(x)$ is multiplied by a constant $a>0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a>1$ and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$. See Figure 10.

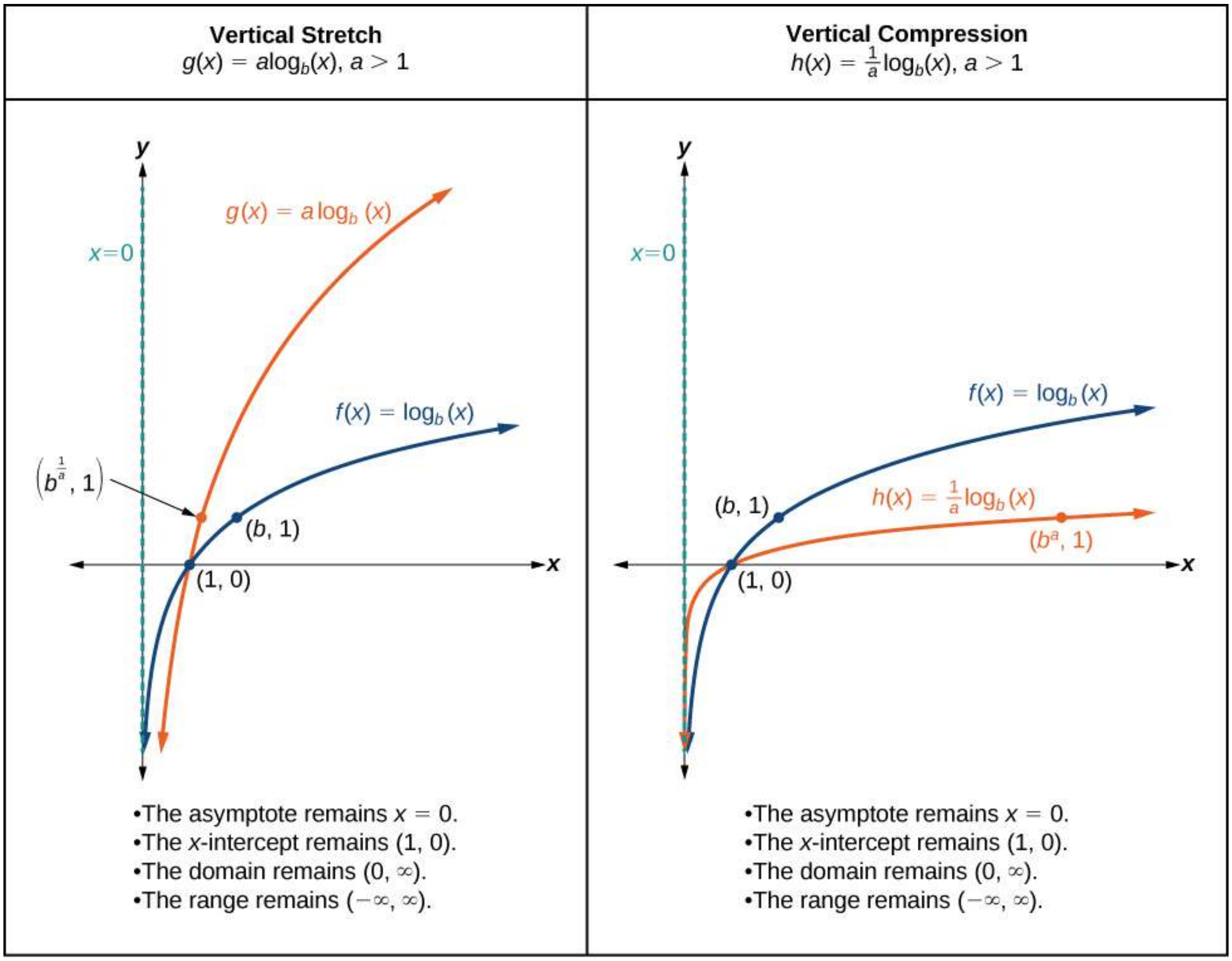


Figure 10

Vertical Stretches and Compressions of the Parent Function $y = \log_b(x)$

For any constant $a > 1$, the function $f(x) = a\log_b(x)$

- stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $a > 1$.
- compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the *x*-intercept $(1, 0)$.
- has domain $\left(0, \infty\right)$.
- has range $\left(-\infty, \infty\right)$.

HOW TO

Given a logarithmic function with the form $f(x) = a\log_b(x)$, $a > 0$, graph the translation.

1. Identify the vertical stretch or compressions:
   - If $|a| > 1$, the graph of $f(x) = \log_b(x)$ is stretched by a factor of a units.
   - If $|a| < 1$, the graph of $f(x) = \log_b(x)$ is compressed by a factor of a units.

2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying the y coordinates by a.
4. Label the three points.
5. The domain is $\left(0, \infty\right)$, the range is $\left(-\infty, \infty\right)$, and the vertical asymptote is $x = 0$.

EXAMPLE 6

Graphing a Stretch or Compression of the Parent Function $y = \log_b(x)$

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = 2\log_4(x)$, we will notice $a = 2$.

This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

The vertical asymptote is $x = 0$.

Consider the three key points from the parent function, $\left(\frac{1}{4}, -1\right)$, $(1, 0)$, and $(4, 1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $\left(\frac{1}{4}, -2\right)$, $(1, 0)$, and $(4, 2)$.

The domain is $\left(0, \infty\right)$, the range is $\left(-\infty, \infty\right)$, and the vertical asymptote is $x = 0$. See Figure 11.

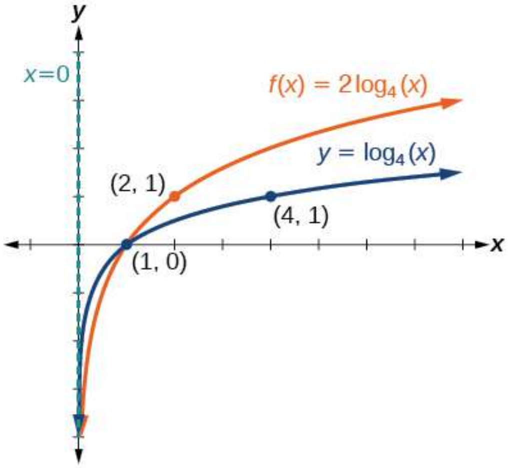


Figure 11

The domain is $\left(0, \infty\right)$, the range is $\left(-\infty, \infty\right)$, and the vertical asymptote is $x = 0$.

TRY IT #6 Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

EXAMPLE 7

Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x + 2)$. State the domain, range, and asymptote.

Solution

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in Figure 12. The vertical asymptote will be shifted to $x = -2$. The x-intercept will be $(-1,0)$. The domain will be $\left(-2, \infty\right)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the x-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

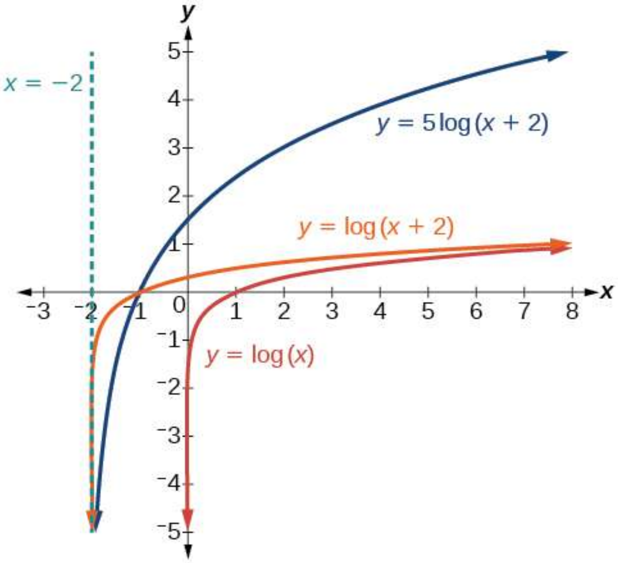


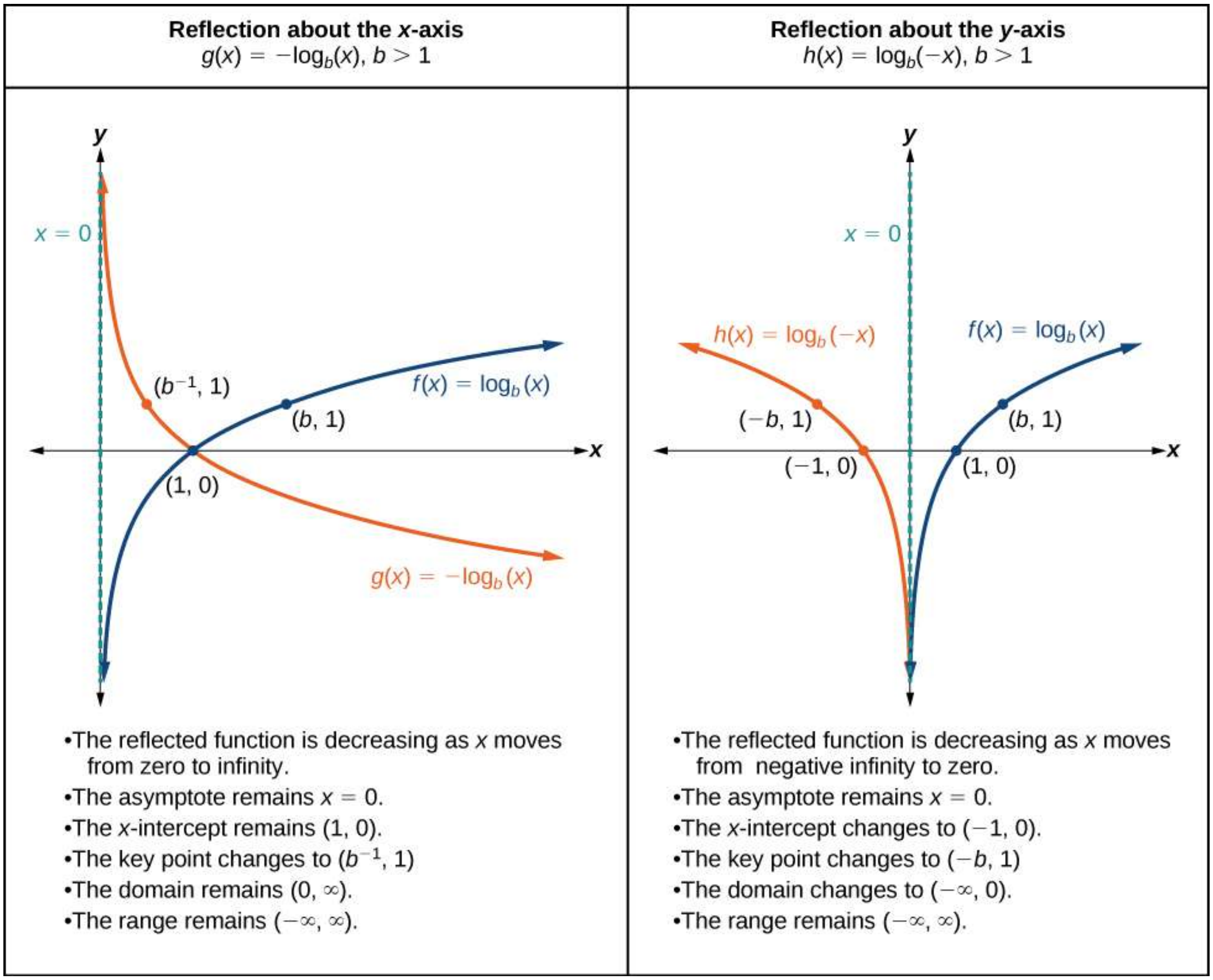
Figure 12

The domain is $\left(-2, \infty\right)$, the range is $\left(-\infty, \infty\right)$, and the vertical asymptote is $x = -2$.

TRY IT #7 Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

Graphing Reflections of $f(x) = \log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by -1, the result is a reflection about the x-axis. When the *input* is multiplied by -1, the result is a reflection about the y-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the x-axis, $g(x) = -\log_b(x)$ and the reflection about the y-axis, $h(x) = \log_b(-x)$.



- The reflected function is decreasing as x moves from zero to infinity.
- The asymptote remains $x = 0$.
- The x-intercept remains $(1, 0)$.
- The key point changes to $(b^{-1}, 1)$
- The domain remains $(0, \infty)$.
- The range remains $(-\infty, \infty)$.

- The reflected function is decreasing as x moves from negative infinity to zero.
- The asymptote remains $x = 0$.
- The x-intercept changes to $(-1, 0)$.
- The key point changes to $(-b, 1)$
- The domain changes to $(-\infty, 0)$.
- The range remains $(-\infty, \infty)$.

Figure 13

Reflections of the Parent Function $y = \log_b(x)$

The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the *x*-axis.
- has domain, $\left(0, \infty\right)$, range, $\left(-\infty, \infty\right)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the *y*-axis.
- has domain $\left(-\infty, 0\right)$.
- has range, $\left(-\infty, \infty\right)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

HOW TO

Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

If $f(x) = -\log_b(x)$	If $f(x) = \log_b(-x)$
1. Draw the vertical asymptote, $x = 0$.	1. Draw the vertical asymptote, $x = 0$.
2. Plot the x-intercept, $(1, 0)$.	2. Plot the x-intercept, $(1, 0)$.
3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the x-axis.	3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the y-axis.
4. Draw a smooth curve through the points.	4. Draw a smooth curve through the points.
5. State the domain, (0, ∞), the range, (−∞, ∞), and the vertical asymptote $x = 0$.	5. State the domain, (−∞, 0) the range, (−∞, ∞) and the vertical asymptote $x = 0$.

Table 3

EXAMPLE 8

Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Before graphing $f(x) = \log(-x)$, identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the parent function is increasing. Since the *input* value is multiplied by -1, f is a reflection of the parent graph about the *y*-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The *x*-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

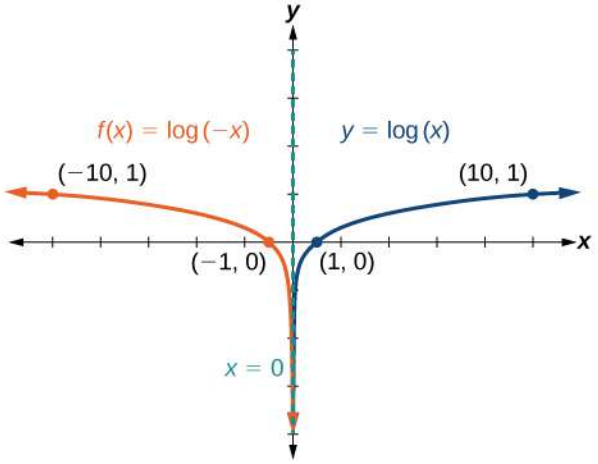


Figure 14

The domain is $\left(-\infty, 0\right)$, the range is $\left(-\infty, \infty\right)$, and the vertical asymptote is $x = 0$.

TRY IT #8 Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

HOW TO

Given a logarithmic equation, use a graphing calculator to approximate solutions.

1. Press **[Y=]**. Enter the given logarithm equation or equations as **Y_1=** and, if needed, **Y_2=**.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of x, for the point(s) of intersection.

EXAMPLE 9

Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x) + 1 = -2\ln(x - 1)$ graphically. Round to the nearest thousandth.

Solution

Press **[Y=]** and enter $4\ln(x) + 1$ next to **Y_1=**. Then enter $-2\ln(x - 1)$ next to **Y_2=**. For a window, use the values 0 to 5 for x and –10 to 10 for y. Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x = 1$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The *x*-coordinate of the point of intersection is displayed as 1.3385297. (Your answer may be different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x \approx 1.339$.

TRY IT #9 Solve $5\log(x + 2) = 4 - \log(x)$ graphically. Round to the nearest thousandth.

Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in Table 4 to arrive at the general equation for translating exponential functions.

Translations of the Parent Function $y = \log_b(x)$

Translation	Form
Shift • Horizontally c units to the left • Vertically d units up	$y = \log_b(x + c) + d$
Stretch and Compress • Stretch if $\lvert a\rvert > 1$ • Compression if $\lvert a\rvert < 1$	$y = a\log_b(x)$
Reflect about the *x*-axis	$y = -\log_b(x)$
Reflect about the *y*-axis	$y = \log_b(-x)$
General equation for all translations	$y = a\log_b(x + c) + d$

Table 4

Translations of Logarithmic Functions

All translations of the parent logarithmic function, $y = \log_b(x)$, have the form

$$f(x) = a\log_b(x + c) + d$$

where the parent function, $y = \log_b(x), b > 1$, is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- reflected about the *x*-axis when $a < 0$.

For $f(x) = \log(-x)$, the graph of the parent function is reflected about the *y*-axis.

EXAMPLE 10

Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x + 4) + 5$?

Solution

The vertical asymptote is at $x = -4$.

Analysis

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x = -4$.

TRY IT #10 What is the vertical asymptote of $f(x) = 3 + \ln(x - 1)$?

EXAMPLE 11

Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in Figure 15.

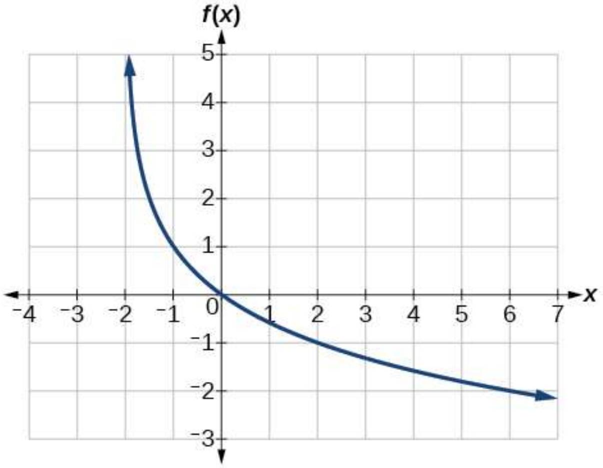


Figure 15

Solution

This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$$f(x) = -a\log(x + 2) + k$$

It appears the graph passes through the points $(–1, 1)$ and $(2, –1)$. Substituting $(–1, 1)$,

$$\begin{aligned} 1 &= -a\log(-1+2)+k && \text{Substitute } (-1,1). \\ 1 &= -a\log(1)+k && \text{Arithmetic.} \\ 1 &= k && \log(1)=0. \end{aligned}$$

Next, substituting in $(2, –1)$,

$$\begin{aligned} -1 &= -a\log(2+2)+1 && \text{Plug in } (2,-1). \\ -2 &= -a\log(4) && \text{Arithmetic.} \\ a &= \frac{2}{\log(4)} && \text{Solve for } a. \end{aligned}$$

This gives us the equation $f(x) =- \frac{2}{\log(4)}\log(x+2)+1$.

Analysis

We can verify this answer by comparing the function values in Table 5 with the points on the graph in Figure 15.

x	−1	0	1	2	3
$f(x)$	1	0	−0.58496	−1	−1.3219
x	4	5	6	7	8
$f(x)$	−1.5850	−1.8074	−2	−2.1699	−2.3219

Table 5

TRY IT #11 Give the equation of the natural logarithm graphed in Figure 16.

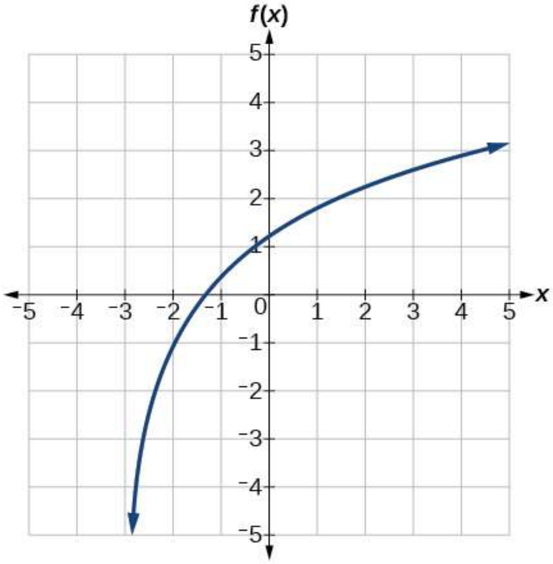


Figure 16

Q&A **Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?**

Yes, if we know the function is a general logarithmic function. For example, look at the graph in Figure 16. The graph approaches $x = -3$ (or thereabouts) more and more closely, so $x = -3$ is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right,

$\{x \mid x > -3\}$. The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is that as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

MEDIA

Access these online resources for additional instruction and practice with graphing logarithms.

Graph an Exponential Function and Logarithmic Function (http://openstax.org/l/graphexplog)
Match Graphs with Exponential and Logarithmic Functions (http://openstax.org/l/matchexplog)
Find the Domain of Logarithmic Functions (http://openstax.org/l/domainlog)

6.4 SECTION EXERCISES

Verbal

1. The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

2. What type(s) of translation(s), if any, affect the range of a logarithmic function?

3. What type(s) of translation(s), if any, affect the domain of a logarithmic function?

4. Consider the general logarithmic function $f(x) = \log_b(x)$. Why can't x be zero?

5. Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

Algebraic

For the following exercises, state the domain and range of the function.

6. $f(x) = \log_3(x + 4)$

7. $h(x) = \ln\left(\frac{1}{2} - x\right)$

8. $g(x) = \log_5(2x + 9) - 2$

9. $h(x) = \ln(4x + 17) - 5$

10. $f(x) = \log_2(12 - 3x) - 3$

For the following exercises, state the domain and the vertical asymptote of the function.

11. $f(x) = \log_b(x - 5)$

12. $g(x) = \ln(3 - x)$

13. $f(x) = \log(3x + 1)$

14. $f(x) = 3\log(-x) + 2$

15. $g(x) = -\ln(3x + 9) - 7$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

16. $f(x) = \ln(2 - x)$

17. $f(x) = \log\left(x - \frac{3}{7}\right)$

18. $h(x) = -\log(3x - 4) + 3$

19. $g(x) = \ln(2x + 6) - 5$

20. $f(x) = \log_3(15 - 5x) + 6$

For the following exercises, state the domain, range, and x- and y-intercepts, if they exist. If they do not exist, write DNE.

21. $h(x) = \log_4(x-1)+1$

22. $f(x) = \log(5x+10)+3$

23. $g(x) = \ln(-x)-2$

24. $f(x) = \log_2(x+2)-5$

25. $h(x) = 3\ln(x)-9$

Graphical

For the following exercises, match each function in Figure 17 with the letter corresponding to its graph.

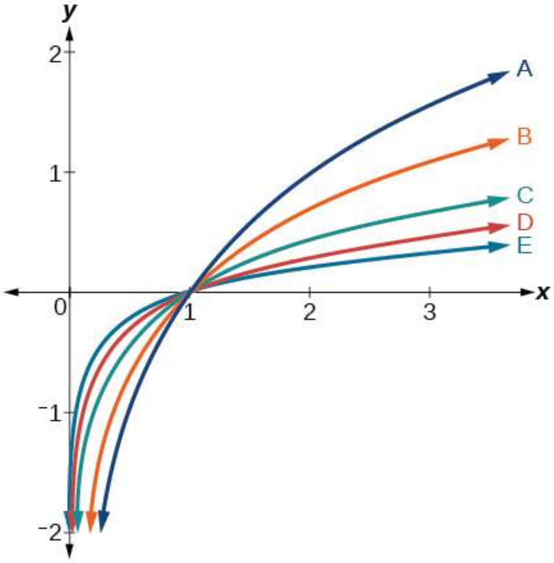


Figure 17

26. $d(x) = \log(x)$

27. $f(x) = \ln(x)$

28. $g(x) = \log_2(x)$

29. $h(x) = \log_5(x)$

30. $j(x) = \log_{25}(x)$

For the following exercises, match each function in Figure 18 with the letter corresponding to its graph.

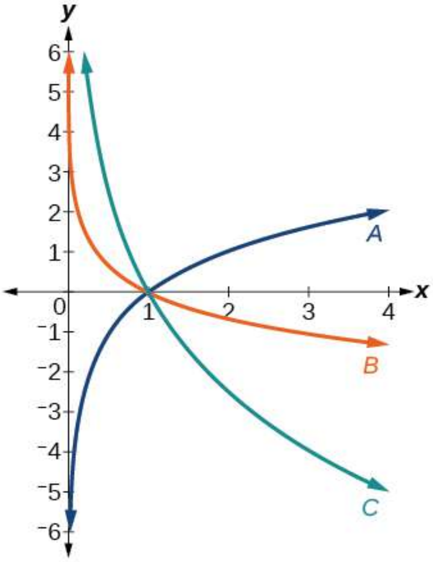


Figure 18

31. $f(x) = \log_{\frac{1}{3}}(x)$

32. $g(x) = \log_2(x)$

33. $h(x) = \log_{\frac{3}{4}}(x)$

For the following exercises, sketch the graphs of each pair of functions on the same axis.

34. $f(x) = \log(x)$ and $g(x) = 10^x$

35. $f(x) = \log(x)$ and $g(x) = \log_{\frac{1}{2}}(x)$

36. $f(x) = \log_4(x)$ and $g(x) = \ln(x)$

37. $f(x) = e^x$ and $g(x) = \ln(x)$

For the following exercises, match each function in Figure 19 with the letter corresponding to its graph.

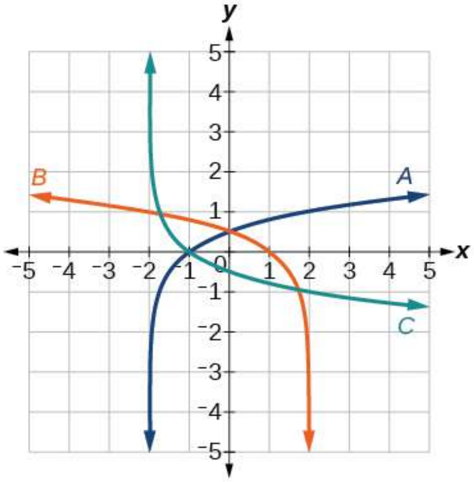


Figure 19

38. $f(x) = \log_4(-x + 2)$

39. $g(x) = -\log_4(x + 2)$

40. $h(x) = \log_4(x + 2)$

For the following exercises, sketch the graph of the indicated function.

41. $f(x) = \log_2(x + 2)$

42. $f(x) = 2\log(x)$

43. $f(x) = \ln(-x)$

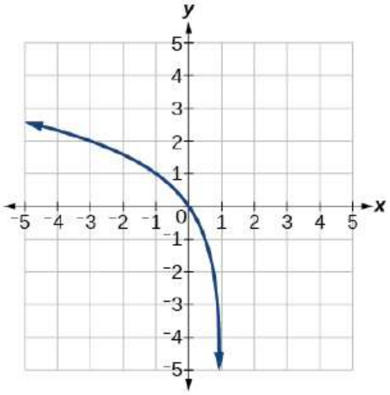
44. $g(x) = \log(4x + 16) + 4$

45. $g(x) = \log(6 - 3x) + 1$

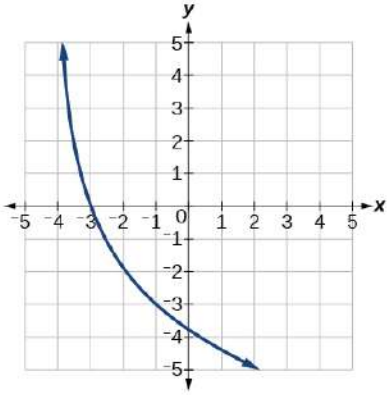
46. $h(x) = -\frac{1}{2}\ln(x + 1) - 3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

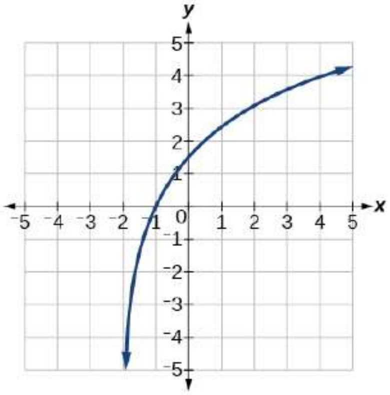
47. Use $y = \log_2(x)$ as the parent function.



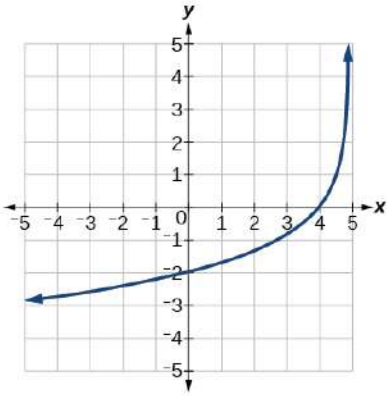
48. Use $f(x) = \log_3(x)$ as the parent function.



49. Use $f(x) = \log_4(x)$ as the parent function.



50. Use $f(x) = \log_5(x)$ as the parent function.



Technology

For the following exercises, use a graphing calculator to find approximate solutions to each equation.

51. $\log(x-1)+2 = \ln(x-1)+2$

52. $\log(2x-3)+2 = -\log(2x-3)+5$

53. $\ln(x-2) = -\ln(x+1)$

54. $2\ln(5x+1) = \frac{1}{2}\ln(-5x)+1$

55. $\frac{1}{3}\log(1-x) = \log(x+1)+\frac{1}{3}$

Extensions

56. Let b be any positive real number such that $b \neq 1$. What must $\log_b 1$ be equal to? Verify the result.

57. Explore and discuss the graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$. Make a conjecture based on the result.

58. Prove the conjecture made in the previous exercise.

59. What is the domain of the function $f(x) = \ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.

60. Use properties of exponents to find the x-intercepts of the function $f(x) = \log\left(x^2 + 4x + 4\right)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.

6.5 Logarithmic Properties

Learning Objectives

In this section, you will:

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

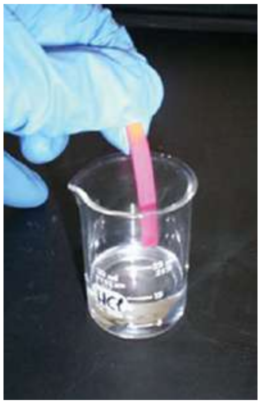


Figure 1 The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan)

In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0 to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be basic. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is basic, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at 25° C)
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or basic, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where H^+ is the concentration of hydrogen ion in the solution

$$\begin{aligned} \text{pH} &= -\log([H^+]) \\ &= \log\left(\frac{1}{[H^+]}\right) \end{aligned}$$

The equivalence of $-\log\left(\left[H^+\right]\right)$ and $\log\left(\frac{1}{[H^+]}\right)$ is one of the logarithm properties we will examine in this section.

Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions “undo” each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0$$
$$\log_b b = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b(b^x) = x$$
$$b^{\log_b x} = x, x > 0$$

For example, to evaluate $\log(100)$, we can rewrite the logarithm as $\log_{10}\left(10^2\right)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}\left(10^2\right) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

$$\log_b M = \log_b N \quad \text{if and only if} \quad M = N$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x+5)$ for x. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for x :

$$\begin{aligned} 3x &= 2x+5 && \text{Set the arguments equal.} \\ x &= 5 && \text{Subtract } 2x. \end{aligned}$$

But what about the equation $\log_3(3x) + \log_3(2x+5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of powers by adding exponents: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for } M \text{ and } N. \\ &= \log_b\left(b^{m+n}\right) && \text{Apply the product rule for exponents.} \\ &= m+n && \text{Apply the inverse property of logs.} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for } m \text{ and } n. \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

The Product Rule for Logarithms

The **product rule for logarithms** can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \quad \text{for } b > 0$$

HOW TO

Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

EXAMPLE 1

Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4))$.

Solution

We begin by factoring the argument completely, expressing 30 as a product of primes.

$$\log_3(30x(3x+4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30x(3x+4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x+4)$$

TRY IT #1 Expand $\log_b(8k)$.

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $\frac{x^a}{x^b} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b\left(\tfrac{M}{N}\right) &= \log_b\left(\tfrac{b^m}{b^n}\right) && \text{Substitute for } M \text{ and } N. \\ &= \log_b\left(b^{m-n}\right) && \text{Apply the quotient rule for exponents.} \\ &= m - n && \text{Apply the inverse property of logs.} \\ &= \log_b(M) - \log_b(N) && \text{Substitute for } m \text{ and } n. \end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned} \log\left(\tfrac{2x^2+6x}{3x+9}\right) &= \log\left(\tfrac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator.} \\ &= \log\left(\tfrac{2x}{3}\right) && \text{Cancel the common factors.} \end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\begin{aligned} \log\left(\tfrac{2x}{3}\right) &= \log(2x) - \log(3) \\ &= \log(2) + \log(x) - \log(3) \end{aligned}$$

The Quotient Rule for Logarithms

The **quotient rule for logarithms** can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

HOW TO

Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

EXAMPLE 2

Using the Quotient Rule for Logarithms

Expand $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

Solution

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2\left(15x(x-1)\right) - \log_2\left((3x+4)(2-x)\right)$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{aligned}\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] - [\log_2(3x+4) + \log_2(2-x)] \\ &= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log_2(2-x)\end{aligned}$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

TRY IT #2 Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned}\log_b\left(x^2\right) &= \log_b(x \cdot x) \\ &= \log_b x + \log_b x \\ &= 2\log_b x\end{aligned}$$

Notice that we used the product rule for logarithms to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \qquad \sqrt{3} = 3^{\frac{1}{2}} \qquad \frac{1}{e} = e^{-1}$$

The Power Rule for Logarithms

The **power rule for logarithms** can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b\left(M^n\right) = n\log_b M$$

HOW TO

Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm.

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

EXAMPLE 3

Expanding a Logarithm with Powers

Expand $\log_2 x^5$.

Solution

The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2\left(x^5\right) = 5\log_2 x$$

TRY IT #3 Expand $\ln x^2$.

EXAMPLE 4

Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3(25)$ using the power rule for logs.

Solution

Expressing the argument as a power, we get $\log_3(25) = \log_3\left(5^2\right)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3\left(5^2\right) = 2\log_3(5)$$

TRY IT #4 Expand $\ln\left(\frac{1}{x^2}\right)$.

EXAMPLE 5

Using the Power Rule in Reverse

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

TRY IT #5 Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called "laws of logs." Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\tfrac{6x}{y}\right) &= \log_b(6x) - \log_b y\\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\tfrac{A}{C}\right) &= \log_b\left(AC^{-1}\right)\\ &= \log_b(A) + \log_b\left(C^{-1}\right)\\ &= \log_b A + (-1)\log_b C\\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

EXAMPLE 6

Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $\ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs.

Solution

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4 y}{7}\right) = \ln\left(x^4 y\right) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln\left(x^4 y\right) - \ln(7) = \ln\left(x^4\right) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln\left(x^4\right) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

TRY IT #6 Expand $\log\left(\frac{x^2 y^3}{z^4}\right)$.

EXAMPLE 7

Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log\left(\sqrt{x}\right)$.

Solution

$$\begin{aligned}\log\left(\sqrt{x}\right) &= \log x^{\left(\frac{1}{2}\right)}\\ &= \tfrac{1}{2}\log x\end{aligned}$$

TRY IT #7 Expand $\ln\left(\sqrt[3]{x^2}\right)$.

Q&A **Can we expand** $\ln\left(x^2+y^2\right)$?

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

EXAMPLE 8

Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Solution

We can expand by applying the Product and Quotient Rules.

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6 64+\log_6 x^3+\log_6(4x+1)-\log_6(2x-1) && \text{Apply the Quotient Rule.}\\ &= \log_6 2^6+\log_6 x^3+\log_6(4x+1)-\log_6(2x-1) && \text{Simplify by writing } 64 \text{ as } 2^6.\\ &= 6\log_6 2+3\log_6 x+\log_6(4x+1)-\log_6(2x-1) && \text{Apply the Power Rule.}\end{aligned}$$

TRY IT #8 Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

HOW TO

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

EXAMPLE 9

Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Solution

Using the product and quotient rules

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$$

This reduces our original expression to

$$\log_3(40) - \log_3(2)$$

Then, using the quotient rule

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

TRY IT #9 Condense $\log 3 - \log 4 + \log 5 - \log 6$.

EXAMPLE 10

Condensing Complex Logarithmic Expressions

Condense $\log_2\left(x^2\right) + \frac{1}{2}\log_2(x-1) - 3\log_2\left((x+3)^2\right)$.

Solution

We apply the power rule first:

$$\log_2\left(x^2\right) + \frac{1}{2}\log_2(x-1) - 3\log_2\left((x+3)^2\right) = \log_2\left(x^2\right) + \log_2\left(\sqrt{x-1}\right) - \log_2\left((x+3)^6\right)$$

Next we apply the product rule to the sum:

$$\log_2\left(x^2\right) + \log_2\left(\sqrt{x-1}\right) - \log_2\left((x+3)^6\right) = \log_2\left(x^2\sqrt{x-1}\right) - \log_2\left((x+3)^6\right)$$

Finally, we apply the quotient rule to the difference:

$$\log_2\left(x^2\sqrt{x-1}\right) - \log_2\left((x+3)^6\right) = \log_2\frac{x^2\sqrt{x-1}}{(x+3)^6}$$

TRY IT #10 Rewrite $\log(5) + 0.5\log(x) - \log(7x-1) + 3\log(x-1)$ as a single logarithm.

EXAMPLE 11

Rewriting as a Single Logarithm

Rewrite $2\log x - 4\log(x+5) + \frac{1}{x}\log(3x+5)$ as a single logarithm.

Solution

We apply the power rule first:

$$2\log x – 4\log(x+5) + \frac{1}{x}\log(3x+5) = \log\left(x^2\right) - \log(x+5)^4 + \log\left((3x+5)^{x^{-1}}\right)$$

Next we rearrange and apply the product rule to the sum:

$$\log\left(x^2\right) - \log(x+5)^4 + \log\left((3x+5)^{x^{-1}}\right)$$

$$= \log\left(x^2\right) + \log\left((3x+5)^{x^{-1}}\right) - \log(x+5)^4$$

$$= \log\left(x^2(3x+5)^{x^{-1}}\right) - \log{(x+5)^4}$$

Finally, we apply the quotient rule to the difference:

$$= \log\left(x^2(3x+5)^{x^{-1}}\right) - \log(x+5)^4 = \log\frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4}$$

TRY IT #11 Condense $4(3\log{(x)} + \log{(x+5)} - \log{(2x+3)})$.

EXAMPLE 12

Applying of the Laws of Logs

Recall that, in chemistry, $\text{pH} = -\log[H^+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$\text{pH} = -\log{(2C)}$$

Using the product rule of logs

$$\text{pH} = -\log{(2C)} = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$, the new pH is

$$\text{pH} = P - \log(2) \approx P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

TRY IT #12 How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the one-to-one property and **power rule for logarithms**.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$, we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$.By exponentiating both sides with base$b$, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{array}{lll} \log_n(b^y) & = \log_n M & \text{Apply the one-to-one property.} \\ y\log_n b & = \log_n M & \text{Apply the power rule for logarithms.} \\ y & = \frac{\log_n M}{\log_n b} & \text{Isolate } y. \\ \log_b M & = \frac{\log_n M}{\log_n b} & \text{Substitute for } y. \end{array}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{array}{lll} \log_5 36 & = \frac{\log(36)}{\log(5)} & \text{Apply the change of base formula using base 10.} \\ & \approx 2.2266 & \text{Use a calculator to evaluate to 4 decimal places.} \end{array}$$

The Change-of-Base Formula

The **change-of-base formula** can be used to evaluate a logarithm with any base.

For any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b}.$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b}$$

and

$$\log_b M = \frac{\log M}{\log b}$$

HOW TO

Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$.

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$, has base e.
2. Rewrite the log as a quotient using the change-of-base formula
   a. The numerator of the quotient will be a logarithm with base n and argument M.
   b. The denominator of the quotient will be a logarithm with base n and argument b.

EXAMPLE 13

Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

Solution

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\begin{aligned} \log_b M &= \frac{\ln M}{\ln b} \\ \log_5 3 &= \frac{\ln 3}{\ln 5} \end{aligned}$$

TRY IT #13 Change $\log_{0.5} 8$ to a quotient of natural logarithms.

Q&A **Can we change common logarithms to natural logarithms?**

Yes. Remember that $\log 9$ *means* $\log_{10} 9$. *So,* $\log 9 = \frac{\ln 9}{\ln 10}$.

EXAMPLE 14

Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

Solution

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$$\begin{array}{ll} \log_2 10 = \frac{\ln 10}{\ln 2} & \text{Apply the change of base formula using base } e. \\ \approx 3.3219 & \text{Use a calculator to evaluate to 4 decimal places.} \end{array}$$

TRY IT #14 Evaluate $\log_5(100)$ using the change-of-base formula.

MEDIA

Access these online resources for additional instruction and practice with laws of logarithms.

The Properties of Logarithms (http://openstax.org/l/proplog)
Expand Logarithmic Expressions (http://openstax.org/l/expandlog)
Evaluate a Natural Logarithmic Expression (http://openstax.org/l/evaluatelog)

6.5 SECTION EXERCISES

Verbal

1. How does the power rule for logarithms help when solving logarithms with the form $\log_b\left(\sqrt[n]{x}\right)$?

2. What does the change-of-base formula do? Why is it useful when using a calculator?

Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

3. $\log_b(7x \cdot 2y)$

4. $\ln(3ab \cdot 5c)$

5. $\log_b\left(\frac{13}{17}\right)$

6. $\log_4\left(\frac{\frac{x}{z}}{w}\right)$

7. $\ln\left(\frac{1}{4^k}\right)$

8. $\log_2(y^x)$

For the following exercises, condense to a single logarithm if possible.

9. $\ln(7) + \ln(x) + \ln(y)$

10. $\log_3(2) + \log_3(a) + \log_3(11) + \log_3(b)$

11. $\log_b(28) - \log_b(7)$

12. $\ln(a) - \ln(d) - \ln(c)$

13. $-\log_b\left(\frac{1}{7}\right)$

14. $\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

15. $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

16. $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$

17. $\log\left(\sqrt{x^3y^{-4}}\right)$

18. $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

19. $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

20. $\log\left(2x^4\right)+\log\left(3x^5\right)$

21. $\ln(6x^9)-\ln(3x^2)$

22. $2\log(x)+3\log(x+1)$

23. $\log(x)-\frac{1}{2}\log(y)+3\log(z)$

24. $4\log_7(c)+\frac{\log_7(a)}{3}+\frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

25. $\log_7(15)$ to base e

26. $\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6)=a$ and $\log_5(11)=b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

27. $\log_{11}(5)$

28. $\log_6(55)$

29. $\log_{11}\left(\frac{6}{11}\right)$

Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

30. $\log_3\left(\frac{1}{9}\right)-3\log_3(3)$

31. $6\log_8(2)+\frac{\log_8(64)}{3\log_8(4)}$

32. $2\log_9(3)-4\log_9(3)+\log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

33. $\log_3(22)$

34. $\log_8(65)$

35. $\log_6(5.38)$

36. $\log_4\left(\frac{15}{2}\right)$

37. $\log_{\frac{1}{2}}(4.7)$

Extensions

38. Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2$. Show the steps for solving.

39. Use the quotient rule for logarithms to find all x values such that $\log_6(x+2)-\log_6(x-3)=1$. Show the steps for solving.

40. Can the power property of logarithms be derived from the power property of exponents using the equation $b^x = m$? If not, explain why. If so, show the derivation.

41. Prove that $\log_b (n) = \frac{1}{\log_n (b)}$ for any positive integers $b > 1$ and $n > 1$.

42. Does $\log_{81} (2401) = \log_3 (7)$? Verify the claim algebraically.

6.6 Exponential and Logarithmic Equations

Learning Objectives

In this section, you will:

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

Figure 1 Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the “rabbit plague.” (credit: Richard Taylor, Flickr)

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers b, S, and T, where $b > 0, \ \ b \neq 1$, $b^S = b^T$ if and only if $S = T$.

In other words, when an exponential equation has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \frac{3^{2x}}{3}$. To solve for x, we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for x:

$3^{4x-7} = \frac{3^{2x}}{3}$

$3^{4x-7} = \frac{3^{2x}}{3^1}$ Rewrite 3 as 3^1.

$3^{4x-7} = 3^{2x-1}$ Use the division property of exponents.

$4x - 7 = 2x - 1$ Apply the one-to-one property of exponents.

$2x = 6$ Subtract $2x$ and add 7 to both sides.

$x = 3$ Divide by 3.

Using the One-to-One Property of Exponential Functions to Solve Exponential Equations

For any algebraic expressions $S$and T, and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T$$

HOW TO

Given an exponential equation with the form $b^S = b^T$, where S and T are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

EXAMPLE 1

Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

Solution

$2^{x-1} = 2^{2x-4}$ The common base is 2.

$x - 1 = 2x - 4$ By the one-to-one property the exponents must be equal.

$x = 3$ Solve for x.

TRY IT #1 Solve $5^{2x} = 5^{3x+2}$.

Rewriting Equations So All Powers Have the Same Base

Sometimes the common base for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x :

$256 = 4^{x-5}$

$2^8 = \left(2^2\right)^{x-5}$ Rewrite each side as a power with base 2.

$2^8 = 2^{2x-10}$ Use the one-to-one property of exponents.

$8 = 2x - 10$ Apply the one-to-one property of exponents.

$18 = 2x$ Add 10 to both sides.

$x = 9$ Divide by 2.

HOW TO

Given an exponential equation with unlike bases, use the one-to-one property to solve it.

1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
3. Use the one-to-one property to set the exponents equal.
4. Solve the resulting equation, $S = T$, for the unknown.

EXAMPLE 2

Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

Solution

$8^{x+2} = 16^{x+1}$	
$\left(2^3\right)^{x+2} = \left(2^4\right)^{x+1}$	Write 8 and 16 as powers of 2.
$2^{3x+6} = 2^{4x+4}$	To take a power of a power, multiply exponents.
$3x + 6 = 4x + 4$	Use the one-to-one property to set the exponents equal.
$x = 2$	Solve for x.

TRY IT #2 Solve $5^{2x} = 25^{3x+2}$.

EXAMPLE 3

Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

Solution

$2^{5x} = 2^{\frac{1}{2}}$	Write the square root of 2 as a power of 2.
$x = \frac{1}{2}$	Use the one-to-one property.
$x = \frac{1}{10}$	Solve for x.

TRY IT #3 Solve $5^x = \sqrt{5}$.

Q&A **Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?**

No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

EXAMPLE 4

Solving an Equation with Positive and Negative Powers

Solve $3^{x+1} = -2$.

Solution

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Analysis

Figure 2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

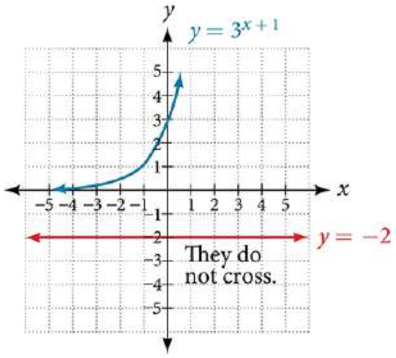


Figure 2

TRY IT #4 Solve $2^x = -100$.

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

HOW TO

Given an exponential equation in which a common base cannot be found, solve for the unknown.

1. Apply the logarithm of both sides of the equation.
   a. If one of the terms in the equation has base 10, use the common logarithm.
   b. If none of the terms in the equation has base 10, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

EXAMPLE 5

Solving an Equation Containing Powers of Different Bases

Solve $5^{x+2} = 4^x$.

Solution

$5^{x+2} = 4^x$	There is no easy way to get the powers to have the same base.
$\ln 5^{x+2} = \ln 4^x$	Take ln of both sides.
$(x+2)\ln 5 = x \ln 4$	Use laws of logs.
$x\ln 5 + 2\ln 5 = x\ln 4$	Use the distributive law.
$x\ln 5 - x\ln 4 = -2\ln 5$	Get terms containing x on one side, terms without x on the other.
$x(\ln 5 - \ln 4) = -2\ln 5$	On the left hand side, factor out an$x$.
$x\ln\left(\frac{5}{4}\right) = \ln\left(\frac{1}{25}\right)$	Use the laws of logs.
$x = \dfrac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$	Divide by the coefficient of x.

TRY IT #5 Solve $2^x = 3^{x+1}$.

Q&A **Is there any way to solve $2^x = 3^x$?**

Yes. The solution is 0.

Equations Containing *e*

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

HOW TO

Given an equation of the form $y = Ae^{kt}$, solve for t.

1. Divide both sides of the equation by A.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by k.

EXAMPLE 6

Solve an Equation of the Form *y* = *Ae*

Solve $100 = 20e^{2t}$.

Solution

$$\begin{aligned}
100 &= 20e^{2t} && \\
5 &= e^{2t} && \text{Divide by the coefficient of the power.} \\
\ln 5 &= 2t && \text{Take ln of both sides. Use the fact that } \ln(x) \text{ and } e^x \text{ are inverse functions.} \\
t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of } t.
\end{aligned}$$

Analysis

Using laws of logs, we can also write this answer in the form $t = \ln\sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

TRY IT #6 Solve $3e^{0.5t} = 11$.

Q&A **Does every equation of the form $y = Ae^{kt}$ have a solution?**

No. There is a solution when $k \neq 0$*, and when* y *and* A *are either both 0 or neither 0, and they have the same sign. An example of an equation with this form that has no solution is* $2 = -3e^t$.

EXAMPLE 7

Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x} + 5 = 12$.

Solution

$4e^{2x} + 5 = 12$	
$4e^{2x} = 7$	Combine like terms.
$e^{2x} = \frac{7}{4}$	Divide by the coefficient of the power.
$2x = \ln\left(\frac{7}{4}\right)$	Take ln of both sides.
$x = \frac{1}{2}\ln\left(\frac{7}{4}\right)$	Solve for x.

TRY IT #7 Solve $3 + e^{2t} = 7e^{2t}$.

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

EXAMPLE 8

Solving Exponential Functions in Quadratic Form

Solve $e^{2x} - e^x = 56$.

Solution

$e^{2x} - e^x = 56$	
$e^{2x} - e^x - 56 = 0$	Get one side of the equation equal to zero.
$(e^x + 7)(e^x - 8) = 0$	Factor by the FOIL method.
$e^x + 7 = 0 \text{ or} e^x - 8 = 0$	If a product is zero, then one factor must be zero.
$e^x = -7 \text{ or } e^x = 8$	Isolate the exponentials.
$e^x = 8$	Reject the equation in which the power equals a negative number.
$x = \ln 8$	Solve the equation in which the power equals a positive number.

Analysis

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^x = -7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

TRY IT #8 Solve $e^{2x} = e^x + 2$.

Q&A **Does every logarithmic equation have a solution?**

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x :

$\log_2(2) + \log_2(3x - 5) = 3$	
$\log_2(2(3x - 5)) = 3$	Apply the product rule of logarithms.
$\log_2(6x - 10) = 3$	Distribute.
$2^3 = 6x - 10$	Apply the definition of a logarithm.
$8 = 6x - 10$	Calculate2^3.
$18 = 6x$	Add 10 to both sides.
$x = 3$	Divide by 6.

Using the Definition of a Logarithm to Solve Logarithmic Equations

For any algebraic expression S and real numbers b and c, where $b > 0,\ \ b \neq 1,$

$$\log_b(S) = c \quad \text{if and only if} \quad b^c = S$$

EXAMPLE 9

Using Algebra to Solve a Logarithmic Equation

Solve $2 \ln x + 3 = 7$.

Solution

$2 \ln x + 3 = 7$	
$2 \ln x = 4$	Subtract 3.
$\ln x = 2$	Divide by 2.
$x = e^2$	Rewrite in exponential form.

TRY IT #9 Solve $6 + \ln x = 10$.

EXAMPLE 10

Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2 \ln(6x) = 7$.

Solution

$2 \ln(6x) = 7$	
$\ln(6x) = \frac{7}{2}$	Divide by 2.
$6x = e^{\left(\frac{7}{2}\right)}$	Use the definition of ln.
$x = \frac{1}{6}e^{\left(\frac{7}{2}\right)}$	Divide by 6.

TRY IT #10 Solve $2 \ln(x + 1) = 10$.

EXAMPLE 11

Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x = 3$.

Solution

$\ln x = 3$	
$x = e^3$	Use the definition of the natural logarithm.

Figure 3 represents the graph of the equation. On the graph, the x-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

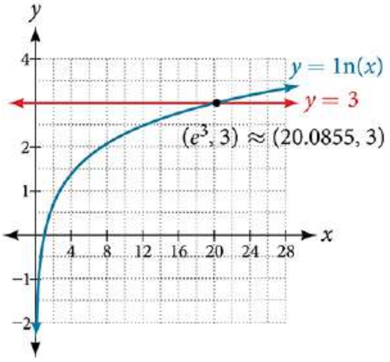


Figure 3 The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately (20.0855, 3).

TRY IT #11 Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for x, and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x :

$$\begin{array}{ll} \log(3x-2) - \log(2) = \log(x+4) & \\ \log\left(\frac{3x-2}{2}\right) = \log(x+4) & \text{Apply the quotient rule of logarithms.} \\ \frac{3x-2}{2} = x + 4 & \text{Apply the one to one property of a logarithm.} \\ 3x - 2 = 2x + 8 & \text{Multiply both sides of the equation by 2.} \\ x = 10 & \text{Subtract } 2x \text{ and add 2.} \end{array}$$

To check the result, substitute $x = 10$ into $\log(3x-2) - \log(2) = \log(x+4)$.

$$\begin{array}{ll} \log(3(10)-2) - \log(2) = \log((10)+4) & \\ \log(28) - \log(2) = \log(14) & \\ \log\left(\frac{28}{2}\right) = \log(14) & \text{The solution checks.} \end{array}$$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

For any algebraic expressions S and T and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \quad \text{if and only if} \quad S = T$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

HOW TO

Given an equation containing logarithms, solve it using the one-to-one property.

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S = \log_b T$.
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S = T$, for the unknown.

EXAMPLE 12

Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x + 3)$.

Solution

$$\begin{array}{ll}
\ln(x^2) = \ln(2x+3) & \\
x^2 = 2x + 3 & \text{Use the one-to-one property of the logarithm.} \\
x^2 - 2x - 3 = 0 & \text{Get zero on one side before factoring.} \\
(x-3)(x+1) = 0 & \text{Factor using FOIL.} \\
x - 3 = 0 \ \text{ or } x + 1 = 0 & \text{If a product is zero, one of the factors must be zero.} \\
x = 3 \ \text{ or } \ x = -1 & \text{Solve for } x.
\end{array}$$

Analysis

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

TRY IT #12 Solve $\ln(x^2) = \ln 1$.

Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its half-life. Table 1 lists the half-life for several of the more common radioactive substances.

Substance	Use	Half-life
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours

Table 1

Substance	Use	Half-life
americium-241	construction	432 years
carbon-14	archeological dating	5,715 years
uranium-235	atomic power	703,800,000 years

Table 1

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$\begin{aligned} A(t) &= A_0 e^{\frac{\ln(0.5)}{T}t} \\ A(t) &= A_0 e^{\ln(0.5)\frac{t}{T}} \\ A(t) &= A_0 (e^{\ln(0.5)})^{\frac{t}{T}} \\ A(t) &= A_0 \left(\tfrac{1}{2}\right)^{\frac{t}{T}} \end{aligned}$$

where

- A_0 is the amount initially present
- T is the half-life of the substance
- t is the time period over which the substance is studied
- $A(t)$ is the amount of the substance present after time t

EXAMPLE 13

Using the Formula for Radioactive Decay to Find the Quantity of a Substance

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

Solution

$$\begin{aligned} y &= 1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} & & \\ 900 &= 1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} & & \text{After 10\% decays, 900 grams are left.} \\ 0.9 &= e^{\frac{\ln(0.5)}{703{,}800{,}000}t} & & \text{Divide by 1000.} \\ \ln(0.9) &= \ln\left(e^{\frac{\ln(0.5)}{703{,}800{,}000}t}\right) & & \text{Take ln of both sides.} \\ \ln(0.9) &= \frac{\ln(0.5)}{703{,}800{,}000}t & & \ln(e^M) = M \\ t &= 703{,}800{,}000 \times \frac{\ln(0.9)}{\ln(0.5)} \text{years} & & \text{Solve for } t. \\ t &\approx 106{,}979{,}777 \text{ years} & & \end{aligned}$$

Analysis

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

TRY IT #13 How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

MEDIA

Access these online resources for additional instruction and practice with exponential and logarithmic equations.

Solving Logarithmic Equations (http://openstax.org/l/solvelogeq)

Solving Exponential Equations with Logarithms (http://openstax.org/l/solveexplog)

6.6 SECTION EXERCISES

Verbal

1. How can an exponential equation be solved?

2. When does an extraneous solution occur? How can an extraneous solution be recognized?

3. When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

Algebraic

For the following exercises, use like bases to solve the exponential equation.

4. $4^{-3v-2} = 4^{-v}$

5. $64 \cdot 4^{3x} = 16$

6. $3^{2x+1} \cdot 3^{x} = 243$

7. $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$

8. $625 \cdot 5^{3x+3} = 125$

9. $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$

10. $\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^{6}$

For the following exercises, use logarithms to solve.

11. $9^{x-10} = 1$

12. $2e^{6x} = 13$

13. $e^{r+10} - 10 = -42$

14. $2 \cdot 10^{9a} = 29$

15. $-8 \cdot 10^{p+7} - 7 = -24$

16. $7e^{3n-5} + 5 = -89$

17. $e^{-3k} + 6 = 44$

18. $-5e^{9x-8} - 8 = -62$

19. $-6e^{9x+8} + 2 = -74$

20. $2^{x+1} = 5^{2x-1}$

21. $e^{2x} - e^{x} - 132 = 0$

22. $7e^{8x+8} - 5 = -95$

23. $10e^{8x+3} + 2 = 8$

24. $4e^{3x+3} - 7 = 53$

25. $8e^{-5x-2} - 4 = -90$

26. $3^{2x+1} = 7^{x-2}$

27. $e^{2x} - e^{x} - 6 = 0$

28. $3e^{3-3x} + 6 = -31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

29. $\log\left(\frac{1}{100}\right) = -2$

30. $\log_{324}(18) = \frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

31. $5\log_{7} n = 10$

32. $-8\log_{9} x = 16$

33. $4 + \log_{2}(9k) = 2$

34. $2\log(8n + 4) + 6 = 10$

35. $10 - 4\ln(9 - 8x) = 6$

For the following exercises, use the one-to-one property of logarithms to solve.

36. $\ln(10-3x)=\ln(-4x)$

37. $\log_{13}(5n-2)=\log_{13}(8-5n)$

38. $\log(x+3)-\log(x)=\log(74)$

39. $\ln(-3x)=\ln\left(x^2-6x\right)$

40. $\log_4(6-m)=\log_4 3m$

41. $\ln(x-2)-\ln(x)=\ln(54)$

42. $\log_9\left(2n^2-14n\right)=\log_9\left(-45+n^2\right)$

43. $\ln\left(x^2-10\right)+\ln(9)=\ln(10)$

For the following exercises, solve each equation for x.

44. $\log(x+12)=\log(x)+\log(12)$

45. $\ln(x)+\ln(x-3)=\ln(7x)$

46. $\log_2(7x+6)=3$

47. $\ln(7)+\ln\left(2-4x^2\right)=\ln(14)$

48. $\log_8(x+6)-\log_8(x)=\log_8(58)$

49. $\ln(3)-\ln(3-3x)=\ln(4)$

50. $\log_3(3x)-\log_3(6)=\log_3(77)$

Graphical

For the following exercises, solve the equation for x, if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.

51. $\log_9(x)-5=-4$

52. $\log_3(x)+3=2$

53. $\ln(3x)=2$

54. $\ln(x-5)=1$

55. $\log(4)+\log(-5x)=2$

56. $-7+\log_3(4-x)=-6$

57. $\ln(4x-10)-6=-5$

58. $\log(4-2x)=\log(-4x)$

59. $\log_{11}\left(-2x^2-7x\right)=\log_{11}(x-2)$

60. $\ln(2x+9)=\ln(-5x)$

61. $\log_9(3-x)=\log_9(4x-8)$

62. $\log\left(x^2+13\right)=\log(7x+3)$

63. $\frac{3}{\log_2(10)}-\log(x-9)=\log(44)$

64. $\ln(x)-\ln(x+3)=\ln(6)$

For the following exercises, solve for the indicated value, and graph the situation showing the solution point.

65. An account with an initial deposit of $\$6,500$ earns 7.25% annual interest, compounded continuously. How much will the account be worth after 20 years?

66. The formula for measuring sound intensity in decibels D is defined by the equation $D=10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0=10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8.3\cdot 10^2$ watts per square meter?

67. The population of a small town is modeled by the equation $P=1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach $20{,}000$?

Technology

For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to 3 decimal places.

68. $1000(1.03)^t = 5000$ using the common log.

69. $e^{5x} = 17$ using the natural log

70. $3(1.04)^{3t} = 8$ using the common log

71. $3^{4x-5} = 38$ using the common log

72. $50e^{-0.12t} = 10$ using the natural log

For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.

73. $7e^{3x-5} + 7.9 = 47$

74. $\ln(3) + \ln(4.4x + 6.8) = 2$

75. $\log(-0.7x - 9) = 1 + 5\log(5)$

76. Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch? (*Hint*: there are 5280 feet in a mile)

77. The magnitude *M* of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules and $E_0 = 10^{4.4}$ is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

Extensions

78. Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b x} = x$.

79. Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

80. Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.

81. Newton's Law of Cooling states that the temperature T of an object at any time t can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$, where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and k is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

6.7 Exponential and Logarithmic Models

Learning Objectives

In this section, you will:

- Model exponential growth and decay.
- Use Newton’s Law of Cooling.
- Use logistic-growth models.
- Choose an appropriate model for data.
- Express an exponential model in base e .

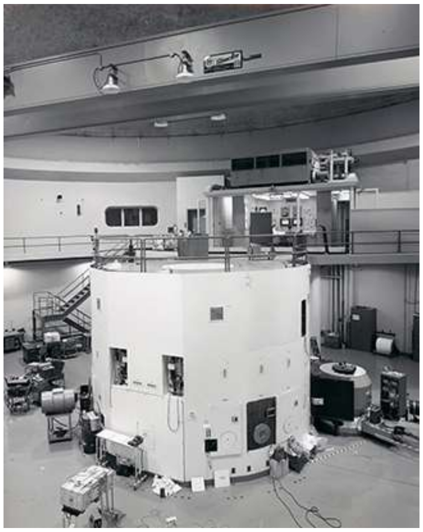


Figure 1 A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus (credit: Georgia Tech Research Institute)

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton’s Law of Cooling.

Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function:

$$y = A_0 e^{kt}$$

where A_0 is equal to the value at time zero, e is Euler’s constant, and k is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double. Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth based on a doubling time. In some applications, however, as we will see when we discuss the logistic equation, the logistic model sometimes fits the data better than the exponential model.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the **exponential decay** model. Again, we have the form $y = A_0 e^{kt}$ where A_0 is the starting value, and e is Euler’s constant. Now k is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating **half-life**, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in Figure 2 and Figure 3. It is important to remember that, although parts of each of the two graphs seem to lie on the *x*-axis, they are really a tiny distance above the *x*-axis.

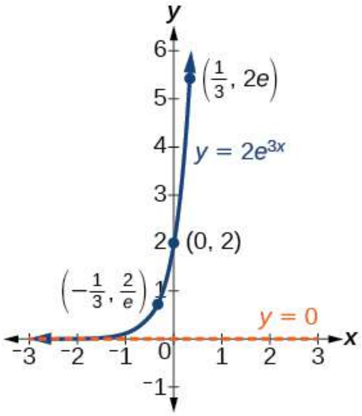


Figure 2 A graph showing exponential growth. The equation is $y = 2e^{3x}$.

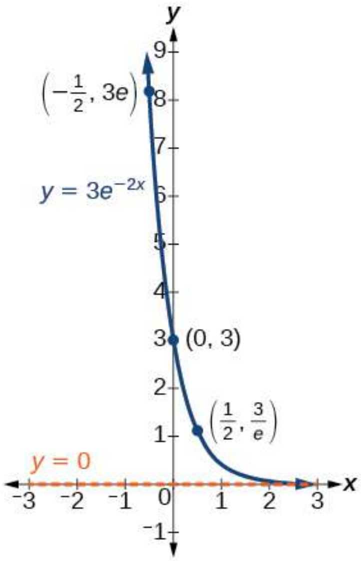


Figure 3 A graph showing exponential decay. The equation is $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude. The **order of magnitude** is the power of ten, when the number is expressed in scientific notation, with one digit to the left of the decimal. For example, the distance to the nearest star, Proxima Centauri, measured in kilometers, is 40,113,497,200,000 kilometers. Expressed in scientific notation, this is $4.01134972 \times 10^{13}$. So, we could describe this number as having order of magnitude 10^{13}.

Characteristics of the Exponential Function, $y = A_0e^{kt}$

An exponential function with the form $y = A_0e^{kt}$ has the following characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x intercept: none
- y-intercept: $(0, A_0)$
- increasing if $k > 0$ (see Figure 4)
- decreasing if $k < 0$ (see Figure 4)

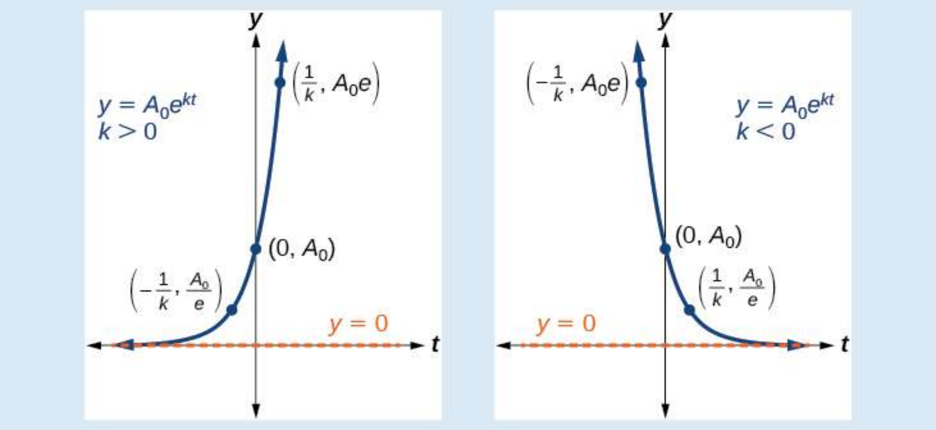


Figure 4 An exponential function models exponential growth when $k > 0$ and exponential decay when $k < 0$.

EXAMPLE 1

Graphing Exponential Growth

A population of bacteria doubles every hour. If the culture started with 10 bacteria, graph the population as a function of time.

Solution

When an amount grows at a fixed percent per unit time, the growth is exponential. To find A_0 we use the fact that A_0 is the amount at time zero, so $A_0 = 10$. To find k, use the fact that after one hour $(t = 1)$ the population doubles from 10 to 20. The formula is derived as follows

$$\begin{aligned} 20 &= 10e^{k\cdot 1} & \\ 2 &= e^k & \text{Divide by 10} \\ \ln 2 &= k & \text{Take the natural logarithm} \end{aligned}$$

so $k = \ln(2)$. Thus the equation we want to graph is $y = 10e^{(\ln 2)t} = 10\left(e^{\ln 2}\right)^t = 10 \cdot 2^t$. The graph is shown in Figure 5.

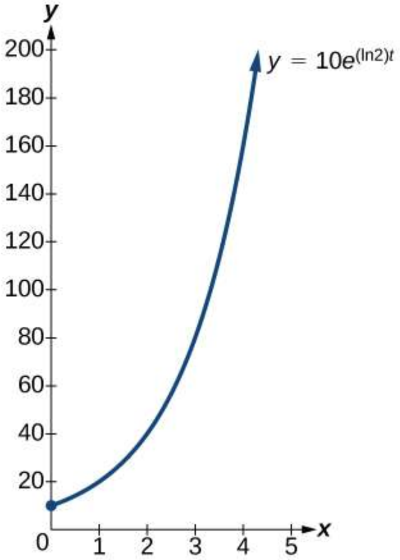


Figure 5 The graph of $y = 10e^{(\ln 2)t}$

Analysis

The population of bacteria after ten hours is 10,240. We could describe this amount is being of the order of magnitude 10^4. The population of bacteria after twenty hours is 10,485,760 which is of the order of magnitude 10^7, so we could say that the population has increased by three orders of magnitude in ten hours.

Half-Life

We now turn to **exponential decay**. One of the common terms associated with exponential decay, as stated above, is **half-life**, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay.

To find the half-life of a function describing exponential decay, solve the following equation:

$$\frac{1}{2}A_0 = A_o e^{kt}$$

We find that the half-life depends only on the constant k and not on the starting quantity A_0.

The formula is derived as follows

$$\begin{aligned}
\tfrac{1}{2}A_0 &= A_o e^{kt} & & \\
\tfrac{1}{2} &= e^{kt} & & \text{Divide by} A_0. \\
\ln\left(\tfrac{1}{2}\right) &= kt & & \text{Take the natural log.} \\
-\ln(2) &= kt & & \text{Apply laws of logarithms.} \\
-\frac{\ln(2)}{k} &= t & & \text{Divide by} k.
\end{aligned}$$

Since t, the time, is positive, k must, as expected, be negative. This gives us the half-life formula

$$t = -\frac{\ln(2)}{k}$$

HOW TO

Given the half-life, find the decay rate.

1. Write $A = A_o e^{kt}$.
2. Replace A by $\frac{1}{2}A_0$ and replace t by the given half-life.
3. Solve to find k. Express k as an exact value (do not round).

Note: *It is also possible to find the decay rate using* $k = -\frac{\ln(2)}{t}$.

EXAMPLE 2

Finding the Function that Describes Radioactive Decay

The half-life of carbon-14 is 5,730 years. Express the amount of carbon-14 remaining as a function of time, t.

Solution

This formula is derived as follows.

$$\begin{aligned}
A &= A_0 e^{kt} & & \text{The continuous growth formula.} \\
0.5A_0 &= A_0 e^{k \cdot 5730} & & \text{Substitute the half-life for} t \text{ and} 0.5A_0 \text{ for} f(t). \\
0.5 &= e^{5730k} & & \text{Divide by} A_0. \\
\ln(0.5) &= 5730k & & \text{Take the natural log of both sides.} \\
k &= \frac{\ln(0.5)}{5730} & & \text{Divide by the coefficient of} k. \\
A &= A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} & & \text{Substitute for} r \text{ in the continuous growth formula.}
\end{aligned}$$

The function that describes this continuous decay is $f(t) = A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$. We observe that the coefficient of t,

$\frac{\ln(0.5)}{5730} \approx -1.2097 \times 10^{-4}$ is negative, as expected in the case of exponential decay.

TRY IT #1 The half-life of plutonium-244 is 80,000,000 years. Find a function that gives the amount of plutonium-244 remaining as a function of time, measured in years.

Radiocarbon Dating

The formula for radioactive decay is important in radiocarbon dating, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about 1% error for plants or animals that died within the last 60,000 years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of 5,730 years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of 12 and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last 60,000 years, using tree rings and other organic samples of known dates—although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is 5,730 years, the formula for the amount of carbon-14 remaining after t years is

$$A \approx A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t}$$

where

- A is the amount of carbon-14 remaining
- A_0 is the amount of carbon-14 when the plant or animal began decaying.

This formula is derived as follows:

$$\begin{array}{ll}
A = A_0 e^{kt} & \text{The continuous growth formula.} \\
0.5A_0 = A_0 e^{k \cdot 5730} & \text{Substitute the half-life for } t \text{ and } 0.5A_0 \text{ for } f(t). \\
0.5 = e^{5730k} & \text{Divide by } A_0. \\
\ln(0.5) = 5730k & \text{Take the natural log of both sides.} \\
k = \frac{\ln(0.5)}{5730} & \text{Divide by the coefficient of } k. \\
A = A_0 e^{\left(\frac{\ln(0.5)}{5730}\right)t} & \text{Substitute for } k \text{ in the continuous growth formula.}
\end{array}$$

To find the age of an object, we solve this equation for t :

$$t = \frac{\ln\left(\frac{A}{A_0}\right)}{-0.000121}$$

Out of necessity, we neglect here the many details that a scientist takes into consideration when doing carbon-14 dating, and we only look at the basic formula. The ratio of carbon-14 to carbon-12 in the atmosphere is approximately 0.0000000001%. Let r be the ratio of carbon-14 to carbon-12 in the organic artifact or fossil to be dated, determined by a method called liquid scintillation. From the equation $A \approx A_0 e^{-0.000121t}$ we know the ratio of the percentage of carbon-14 in the object we are dating to the initial amount of carbon-14 in the object when it was formed is $r = \frac{A}{A_0} \approx e^{-0.000121t}$. We solve this equation for t, to get

$$t = \frac{\ln(r)}{-0.000121}$$

HOW TO

Given the percentage of carbon-14 in an object, determine its age.

1. Express the given percentage of carbon-14 as an equivalent decimal, k.
2. Substitute for k in the equation $t = \frac{\ln(r)}{-0.000121}$ and solve for the age, t.

EXAMPLE 3

Finding the Age of a Bone

A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

Solution

We substitute $20\% = 0.20$ for r in the equation and solve for t :

$$\begin{aligned} t &= \frac{\ln(r)}{-0.000121} && \text{Use the general form of the equation.} \\ &= \frac{\ln(0.20)}{-0.000121} && \text{Substitute for } r. \\ &\approx 13301 && \text{Round to the nearest year.} \end{aligned}$$

The bone fragment is about 13,301 years old.

Analysis

The instruments that measure the percentage of carbon-14 are extremely sensitive and, as we mention above, a scientist will need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about 1%, so this age should be given as $13{,}301 \text{ years} \pm 1\%$ or $13{,}301 \text{ years} \pm 133$ years.

TRY IT #2 Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Calculating Doubling Time

For decaying quantities, we determined how long it took for half of a substance to decay. For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the **doubling time**.

Given the basic exponential growth equation $A = A_0 e^{kt}$, doubling time can be found by solving for when the original quantity has doubled, that is, by solving $2A_0 = A_0 e^{kt}$.

The formula is derived as follows:

$$\begin{aligned} 2A_0 &= A_0 e^{kt} && \\ 2 &= e^{kt} && \text{Divide by} A_0. \\ \ln 2 &= kt && \text{Take the natural logarithm.} \\ t &= \frac{\ln 2}{k} && \text{Divide by the coefficient of} t. \end{aligned}$$

Thus the doubling time is

$$t = \frac{\ln 2}{k}$$

EXAMPLE 4

Finding a Function That Describes Exponential Growth

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

Solution

The formula is derived as follows:

$$\begin{aligned} t &= \frac{\ln 2}{k} && \text{The doubling time formula.} \\ 2 &= \frac{\ln 2}{k} && \text{Use a doubling time of two years.} \\ k &= \frac{\ln 2}{2} && \text{Multiply by } k \text{ and divide by 2.} \\ A &= A_0 e^{\frac{\ln 2}{2} t} && \text{Substitute } k \text{ into the continuous growth formula.} \end{aligned}$$

The function is $A_0 e^{\frac{\ln 2}{2} t}$.

TRY IT #3 Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.

Using Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a vertical shift of the generic **exponential decay** function. This translation leads to **Newton's Law of Cooling**, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature

$$T(t) = ae^{kt} + T_s$$

This formula is derived as follows:

$$\begin{aligned} T(t) &= Ab^{ct} + T_s && \\ T(t) &= Ae^{\ln(b^{ct})} + T_s && \text{Laws of logarithms.} \\ T(t) &= Ae^{ct \ln b} + T_s && \text{Laws of logarithms.} \\ T(t) &= Ae^{kt} + T_s && \text{Rename the constant } c \ \ln\ b, \ \ \text{calling it } k. \end{aligned}$$

Newton's Law of Cooling

The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$$T(t) = Ae^{kt} + T_s$$

where

- t is time
- A is the difference between the initial temperature of the object and the surroundings
- k is a constant, the continuous rate of cooling of the object

HOW TO

Given a set of conditions, apply Newton's Law of Cooling.

1. Set T_s equal to the y-coordinate of the horizontal asymptote (usually the ambient temperature).
2. Substitute the given values into the continuous growth formula $T(t) = Ae^{kt} + T_s$ to find the parameters A and k.
3. Substitute in the desired time to find the temperature or the desired temperature to find the time.

EXAMPLE 5

Using Newton's Law of Cooling

A cheesecake is taken out of the oven with an ideal internal temperature of $165°\mathrm{F}$, and is placed into a $35°\mathrm{F}$ refrigerator. After 10 minutes, the cheesecake has cooled to $150°\mathrm{F}$. If we must wait until the cheesecake has cooled to $70°\mathrm{F}$ before we eat it, how long will we have to wait?

Solution

Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35, following the equation

$$T(t) = Ae^{kt} + 35$$

We know the initial temperature was 165, so $T(0) = 165$.

$$\begin{array}{rll} 165 &= Ae^{k0} + 35 & \text{Substitute } (0, 165). \\ A &= 130 & \text{Solve for } A. \end{array}$$

We were given another data point, $T(10) = 150$, which we can use to solve for k.

$$\begin{array}{ll} 150 = 130e^{k10} + 35 & \text{Substitute (10, 150).} \\ 115 = 130e^{k10} & \text{Subtract 35.} \\ \frac{115}{130} = e^{10k} & \text{Divide by 130.} \\ \ln\left(\frac{115}{130}\right) = 10k & \text{Take the natural log of both sides.} \\ k = \frac{\ln\left(\frac{115}{130}\right)}{10} \approx -0.0123 & \text{Divide by the coefficient of} k. \end{array}$$

This gives us the equation for the cooling of the cheesecake: $T(t) = 130e^{-0.0123t} + 35$.

Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$\begin{array}{ll} 70 = 130e^{-0.0123t} + 35 & \text{Substitute in 70 for} T(t). \\ 35 = 130e^{-0.0123t} & \text{Subtract 35.} \\ \frac{35}{130} = e^{-0.0123t} & \text{Divide by 130.} \\ \ln(\frac{35}{130}) = -0.0123t & \text{Take the natural log of both sides} \\ t = \frac{\ln(\frac{35}{130})}{-0.0123} \approx 106.68 & \text{Divide by the coefficient of} t. \end{array}$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to $70°\mathrm{F}$.

TRY IT #4 A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the exponential growth model is still useful over a short term, before approaching the limiting value.

The **logistic growth model** is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, called the **carrying capacity**. For constants a, b, and c, the logistic growth of a population over time t is represented by the model

$$f(t) = \frac{c}{1 + ae^{-bt}}$$

The graph in Figure 6 shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.

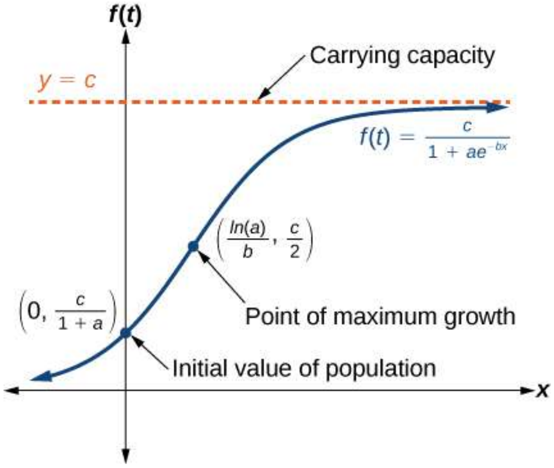


Figure 6

Logistic Growth

The logistic growth model is

$$f(t) = \frac{c}{1 + ae^{-bt}}$$

where

- $\frac{c}{1+a}$ is the initial value
- c is the *carrying capacity*, or *limiting value*
- b is a constant determined by the rate of growth.

EXAMPLE 6

Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of 1,000 people who has the flu. So, in that community, at most 1,000 people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

Solution

We substitute the given data into the logistic growth model

$$f(t) = \frac{c}{1 + ae^{-bt}}$$

Because at most 1,000 people, the entire population of the community, can get the flu, we know the limiting value is $c = 1000$. To find a, we use the formula that the number of cases at time $t = 0$ is $\frac{c}{1+a} = 1$, from which it follows that $a = 999$. This model predicts that, after ten days, the number of people who have had the flu is $f(t) = \frac{1000}{1+999e^{-0.6030x}} \approx 293.8$. Because the actual number must be a whole number (a person has either had the flu or not) we round to 294. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000$.

Analysis

Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.

The graph in Figure 7 gives a good picture of how this model fits the data.

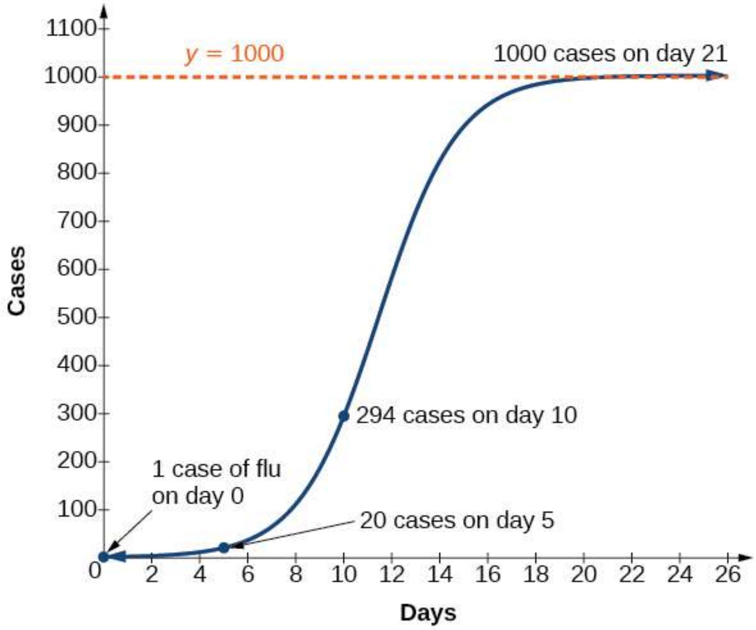


Figure 7 The graph of $f(t) = \frac{1000}{1+999e^{-0.6030x}}$

TRY IT #5 Using the model in Example 6, estimate the number of cases of flu on day 15.

Choosing an Appropriate Model for Data

Now that we have discussed various mathematical models, we need to learn how to choose the appropriate model for the raw data we have. Many factors influence the choice of a mathematical model, among which are experience, scientific laws, and patterns in the data itself. Not all data can be described by elementary functions. Sometimes, a function is chosen that approximates the data over a given interval. For instance, suppose data were gathered on the number of homes bought in the United States from the years 1960 to 2013. After plotting these data in a scatter plot, we notice that the shape of the data from the years 2000 to 2013 follow a logarithmic curve. We could restrict the interval from 2000 to 2010, apply regression analysis using a logarithmic model, and use it to predict the number of home buyers for the year 2015.

Three kinds of functions that are often useful in mathematical models are linear functions, exponential functions, and logarithmic functions. If the data lies on a straight line, or seems to lie approximately along a straight line, a linear model may be best. If the data is non-linear, we often consider an exponential or logarithmic model, though other models, such as quadratic models, may also be considered.

In choosing between an exponential model and a logarithmic model, we look at the way the data curves. This is called the concavity. If we draw a line between two data points, and all (or most) of the data between those two points lies above that line, we say the curve is concave down. We can think of it as a bowl that bends downward and therefore cannot hold water. If all (or most) of the data between those two points lies below the line, we say the curve is concave up. In this case, we can think of a bowl that bends upward and can therefore hold water. An exponential curve, whether rising or falling, whether representing growth or decay, is always concave up away from its horizontal asymptote. A logarithmic curve is always concave away from its vertical asymptote. In the case of positive data, which is the most common case, an exponential curve is always concave up, and a logarithmic curve always concave down.

A logistic curve changes concavity. It starts out concave up and then changes to concave down beyond a certain point, called a point of inflection.

After using the graph to help us choose a type of function to use as a model, we substitute points, and solve to find the parameters. We reduce round-off error by choosing points as far apart as possible.

EXAMPLE 7

Choosing a Mathematical Model

Does a linear, exponential, logarithmic, or logistic model best fit the values listed in Table 1? Find the model, and use a graph to check your choice.

x	1	2	3	4	5	6	7	8	9
y	0	1.386	2.197	2.773	3.219	3.584	3.892	4.159	4.394

Table 1

Solution

First, plot the data on a graph as in Figure 8. For the purpose of graphing, round the data to two decimal places.

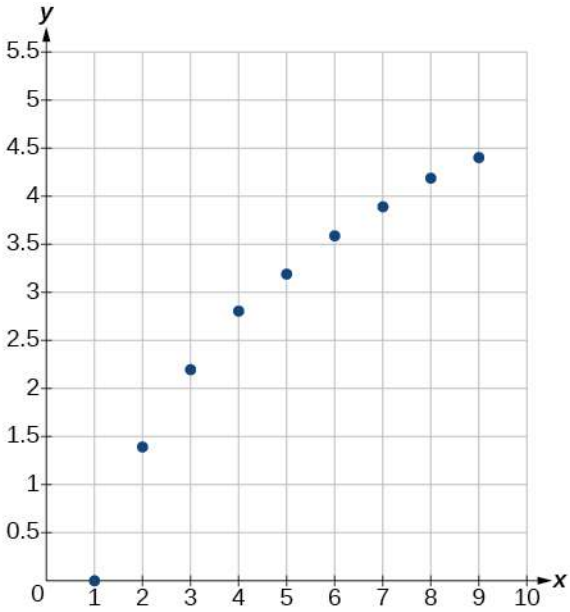


Figure 8

Clearly, the points do not lie on a straight line, so we reject a linear model. If we draw a line between any two of the points, most or all of the points between those two points lie above the line, so the graph is concave down, suggesting a logarithmic model. We can try $y = a\ln(bx)$. Plugging in the first point, $(1{,}0)$, gives $0 = a\ln b$. We reject the case that $a = 0$ (if it were, all outputs would be 0), so we know $\ln(b) = 0$. Thus $b = 1$ and $y = a\ln(\mathrm{x})$. Next we can use the point $(9{,}4.394)$ to solve for a :

$$\begin{aligned} y &= a\ln(x) \\ 4.394 &= a\ln(9) \\ a &= \frac{4.394}{\ln(9)} \end{aligned}$$

Because $a = \frac{4.394}{\ln(9)} \approx 2$, an appropriate model for the data is $y = 2\ln(x)$.

To check the accuracy of the model, we graph the function together with the given points as in Figure 9.

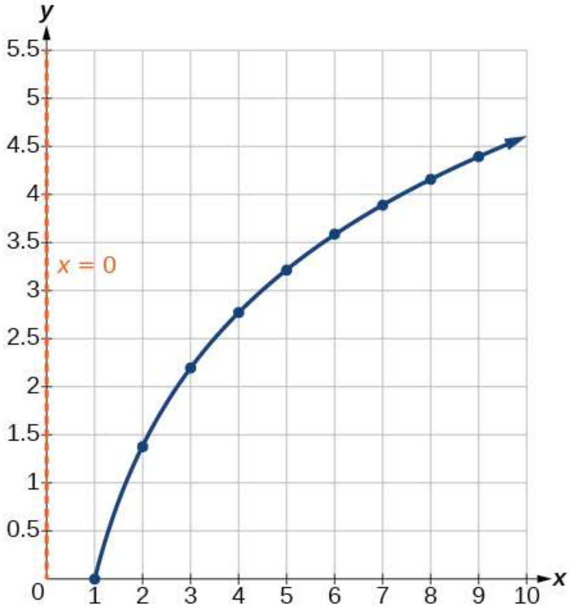


Figure 9 The graph of $y = 2\ln x$.

We can conclude that the model is a good fit to the data.

Compare Figure 9 to the graph of $y = \ln\left(x^2\right)$ shown in Figure 10.

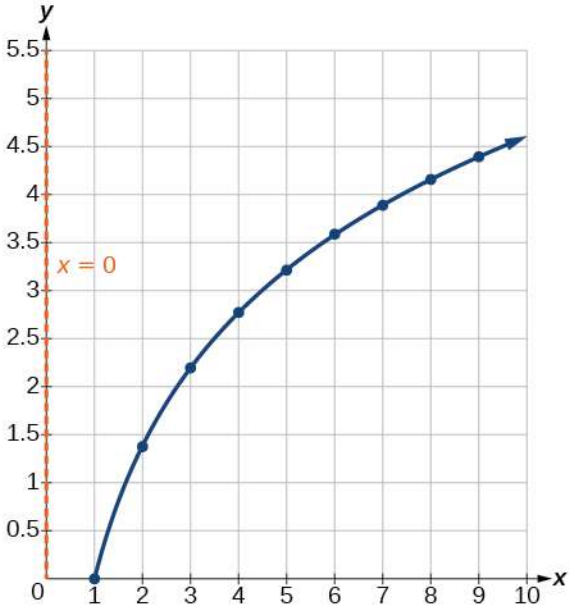


Figure 10 The graph of $y = \ln\left(x^2\right)$

The graphs appear to be identical when $x > 0$. A quick check confirms this conclusion: $y = \ln\left(x^2\right) = 2\ln(x)$ for $x > 0$.

However, if $x < 0$, the graph of $y = \ln\left(x^2\right)$ includes a "extra" branch, as shown in Figure 11. This occurs because, while $y = 2\ln(x)$ cannot have negative values in the domain (as such values would force the argument to be negative), the function $y = \ln\left(x^2\right)$ can have negative domain values.

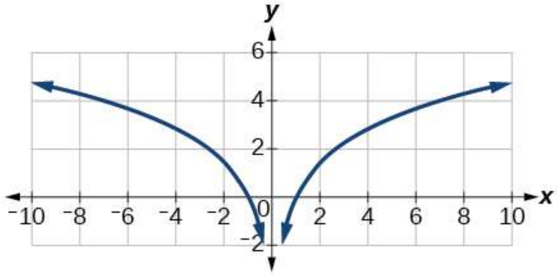


Figure 11

TRY IT #6 Does a linear, exponential, or logarithmic model best fit the data in Table 2? Find the model.

x	1	2	3	4	5	6	7	8	9
y	3.297	5.437	8.963	14.778	24.365	40.172	66.231	109.196	180.034

Table 2

Expressing an Exponential Model in Base e

While powers and logarithms of any base can be used in modeling, the two most common bases are 10 and e. In science and mathematics, the base e is often preferred. We can use laws of exponents and laws of logarithms to change any base to base e.

HOW TO

Given a model with the form $y = ab^x$, change it to the form $y = A_0 e^{kx}$.

1. Rewrite $y = ab^x$ as $y = ae^{\ln(b^x)}$.
2. Use the power rule of logarithms to rewrite y as $y = ae^{x\ln(b)} = ae^{\ln(b)x}$.
3. Note that $a = A_0$ and $k = \ln(b)$ in the equation $y = A_0 e^{kx}$.

EXAMPLE 8

Changing to base *e*

Change the function $y = 2.5(3.1)^x$ so that this same function is written in the form $y = A_0 e^{kx}$.

Solution

The formula is derived as follows

$$\begin{aligned} y &= 2.5(3.1)^x \\ &= 2.5e^{\ln(3.1^x)} && \text{Insert exponential and its inverse.} \\ &= 2.5e^{x\ln 3.1} && \text{Laws of logs.} \\ &= 2.5e^{(\ln 3.1)x} && \text{Commutative law of multiplication} \end{aligned}$$

TRY IT #7 Change the function $y = 3(0.5)^x$ to one having e as the base.

MEDIA

Access these online resources for additional instruction and practice with exponential and logarithmic models.

Logarithm Application – pH (http://openstax.org/l/logph)
Exponential Model – Age Using Half-Life (http://openstax.org/l/expmodelhalf)
Newton’s Law of Cooling (http://openstax.org/l/newtoncooling)
Exponential Growth Given Doubling Time (http://openstax.org/l/expgrowthdbl)
Exponential Growth – Find Initial Amount Given Doubling Time (http://openstax.org/l/initialdouble)

6.7 SECTION EXERCISES

Verbal

1. With what kind of exponential model would *half-life* be associated? What role does half-life play in these models?

2. What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.

3. With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?

4. Define Newton’s Law of Cooling. Then name at least three real-world situations where Newton’s Law of Cooling would be applied.

5. What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

Numeric

6. The temperature of an object in degrees Fahrenheit after *t* minutes is represented by the equation $T(t) = 68e^{-0.0174t} + 72$. To the nearest degree, what is the temperature of the object after one and a half hours?

For the following exercises, use the logistic growth model $f(x) = \frac{150}{1+8e^{-2x}}$.

7. Find and interpret $f(0)$. Round to the nearest tenth.

8. Find and interpret $f(4)$. Round to the nearest tenth.

9. Find the carrying capacity.

10. Graph the model.

11. Determine whether the data from the table could best be represented as a function that is linear, exponential, or logarithmic. Then write a formula for a model that represents the data.

x	$f(x)$
–2	0.694
–1	0.833
0	1
1	1.2
2	1.44
3	1.728
4	2.074
5	2.488

12. Rewrite $f(x) = 1.68(0.65)^x$ as an exponential equation with base e to five decimal places.

Technology

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table could represent a function that is linear, exponential, or logarithmic.

13.

x	$f(x)$
1	2
2	4.079
3	5.296
4	6.159
5	6.828
6	7.375
7	7.838
8	8.238
9	8.592
10	8.908

14.

x	$f(x)$
1	2.4
2	2.88
3	3.456
4	4.147
5	4.977
6	5.972
7	7.166
8	8.6
9	10.32
10	12.383

15.

x	$f(x)$
4	9.429
5	9.972
6	10.415
7	10.79
8	11.115
9	11.401
10	11.657
11	11.889
12	12.101
13	12.295

16.

x	$f(x)$
1.25	5.75
2.25	8.75
3.56	12.68
4.2	14.6
5.65	18.95
6.75	22.25
7.25	23.75
8.6	27.8
9.25	29.75
10.5	33.5

For the following exercises, use a graphing calculator and this scenario: the population of a fish farm in t years is modeled by the equation $P(t) = \frac{1000}{1+9e^{-0.6t}}$.

17. Graph the function.

18. What is the initial population of fish?

19. To the nearest tenth, what is the doubling time for the fish population?

20. To the nearest whole number, what will the fish population be after 2 years?

21. To the nearest tenth, how long will it take for the population to reach 900?

22. What is the carrying capacity for the fish population? Justify your answer using the graph of P.

Extensions

23. A substance has a half-life of 2.045 minutes. If the initial amount of the substance was 132.8 grams, how many half-lives will have passed before the substance decays to 8.3 grams? What is the total time of decay?

24. The formula for an increasing population is given by $P(t) = P_0 e^{rt}$ where P_0 is the initial population and $r > 0$. Derive a general formula for the time t it takes for the population to increase by a factor of *M*.

25. Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. Show each step for solving this equation algebraically for the seismic moment S.

26. What is the *y*-intercept of the logistic growth model $y = \frac{c}{1+ae^{-rx}}$? Show the steps for calculation. What does this point tell us about the population?

27. Prove that $b^x = e^{x\ln(b)}$ for positive $b \neq 1$.

Real-World Applications

For the following exercises, use this scenario: A doctor prescribes 125 milligrams of a therapeutic drug that decays by about 30% each hour.

28. To the nearest hour, what is the half-life of the drug?

29. Write an exponential model representing the amount of the drug remaining in the patient's system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 3 hours. Round to the nearest milligram.

30. Using the model found in the previous exercise, find $f(10)$ and interpret the result. Round to the nearest hundredth.

For the following exercises, use this scenario: A tumor is injected with 0.5 grams of Iodine-125, which has a decay rate of 1.15% per day.

31. To the nearest day, how long will it take for half of the Iodine-125 to decay?

32. Write an exponential model representing the amount of Iodine-125 remaining in the tumor after t days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after 60 days. Round to the nearest tenth of a gram.

33. A scientist begins with 250 grams of a radioactive substance. After 250 minutes, the sample has decayed to 32 grams. Rounding to five decimal places, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?

34. The half-life of Radium-226 is 1590 years. What is the annual decay rate? Express the decimal result to four decimal places and the percentage to two decimal places.

35. The half-life of Erbium-165 is 10.4 hours. What is the hourly decay rate? Express the decimal result to four decimal places and the percentage to two decimal places.

36. A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is 5730 years.)

37. A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was 1350 bacteria. Rounding to five decimal places, write an exponential equation representing this situation. To the nearest whole number, what is the population size after 3 hours?

For the following exercises, use this scenario: A biologist recorded a count of 360 bacteria present in a culture after 5 minutes and 1000 bacteria present after 20 minutes.

38. To the nearest whole number, what was the initial population in the culture?

39. Rounding to six decimal places, write an exponential equation representing this situation. To the nearest minute, how long did it take the population to double?

For the following exercises, use this scenario: A pot of warm soup with an internal temperature of $100°$ *Fahrenheit was taken off the stove to cool in a* $69°$ F *room. After fifteen minutes, the internal temperature of the soup was* $95°$ F.

40. Use Newton's Law of Cooling to write a formula that models this situation.

41. To the nearest minute, how long will it take the soup to cool to $80°$ F?

42. To the nearest degree, what will the temperature be after 2 and a half hours?

For the following exercises, use this scenario: A turkey is taken out of the oven with an internal temperature of $165°\text{F}$ *and is allowed to cool in a* $75°\text{F}$ *room. After half an hour, the internal temperature of the turkey is* $145°\text{F}$.

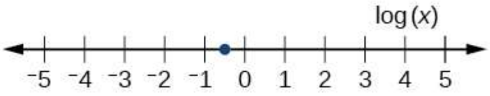
43. Write a formula that models this situation.

44. To the nearest degree, what will the temperature be after 50 minutes?

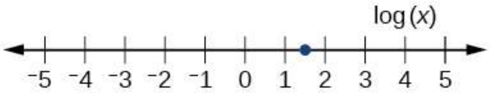
45. To the nearest minute, how long will it take the turkey to cool to $110°$ F?

For the following exercises, find the value of the number shown on each logarithmic scale. Round all answers to the nearest thousandth.

46.



47.



48. Plot each set of approximate values of intensity of sounds on a logarithmic scale: Whisper: $10^{-10}\ \frac{W}{m^2}$, Vacuum: $10^{-4}\ \frac{W}{m^2}$, Jet: $10^{2}\ \frac{W}{m^2}$

49. Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$. One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake. Round to the nearest hundredth.

For the following exercises, use this scenario: The equation $N(t) = \frac{500}{1+49e^{-0.7t}}$ *models the number of people in a town who have heard a rumor after t days.*

50. How many people started the rumor?

51. To the nearest whole number, how many people will have heard the rumor after 3 days?

52. As t increases without bound, what value does $N(t)$ approach? Interpret your answer.

For the following exercise, choose the correct answer choice.

53. A doctor injects a patient with 13 milligrams of radioactive dye that decays exponentially. After 12 minutes, there are 4.75 milligrams of dye remaining in the patient's system. Which is an appropriate model for this situation?

ⓐ $f(t) = 13(0.0805)^t$ ⓑ $f(t) = 13e^{0.9195t}$

ⓒ $f(t) = 13e^{(-0.0839t)}$

ⓓ $f(t) = \frac{4.75}{1+13e^{-0.83925t}}$

6.8 Fitting Exponential Models to Data

Learning Objectives

In this section, you will:

- Build an exponential model from data.
- Build a logarithmic model from data.
- Build a logistic model from data.

In previous sections of this chapter, we were either given a function explicitly to graph or evaluate, or we were given a set of points that were guaranteed to lie on the curve. Then we used algebra to find the equation that fit the points exactly. In this section, we use a modeling technique called *regression analysis* to find a curve that models data collected from real-world observations. With regression analysis, we don't expect all the points to lie perfectly on the curve. The idea is to find a model that best fits the data. Then we use the model to make predictions about future events.

Do not be confused by the word *model*. In mathematics, we often use the terms *function*, *equation*, and *model* interchangeably, even though they each have their own formal definition. The term *model* is typically used to indicate that the equation or function approximates a real-world situation.

We will concentrate on three types of regression models in this section: exponential, logarithmic, and logistic. Having already worked with each of these functions gives us an advantage. Knowing their formal definitions, the behavior of their graphs, and some of their real-world applications gives us the opportunity to deepen our understanding. As each regression model is presented, key features and definitions of its associated function are included for review. Take a moment to rethink each of these functions, reflect on the work we've done so far, and then explore the ways regression is used to model real-world phenomena.

Building an Exponential Model from Data

As we've learned, there are a multitude of situations that can be modeled by exponential functions, such as investment growth, radioactive decay, atmospheric pressure changes, and temperatures of a cooling object. What do these phenomena have in common? For one thing, all the models either increase or decrease as time moves forward. But that's not the whole story. It's the *way* data increase or decrease that helps us determine whether it is best modeled by an exponential equation. Knowing the behavior of exponential functions in general allows us to recognize when to use exponential regression, so let's review exponential growth and decay.

Recall that exponential functions have the form $y = ab^x$ or $y = A_0 e^{kx}$. When performing regression analysis, we use the form most commonly used on graphing utilities, $y = ab^x$. Take a moment to reflect on the characteristics we've already learned about the exponential function $y = ab^x$ (assume $a > 0$) :

- b must be greater than zero and not equal to one.
- The initial value of the model is $y = a$.
    - If $b > 1$, the function models exponential growth. As x increases, the outputs of the model increase slowly at first, but then increase more and more rapidly, without bound.
    - If $0 < b < 1$, the function models exponential decay. As x increases, the outputs for the model decrease rapidly at first and then level off to become asymptotic to the *x*-axis. In other words, the outputs never become equal to or less than zero.

As part of the results, your calculator will display a number known as the *correlation coefficient*, labeled by the variable r, or r^2. (You may have to change the calculator's settings for these to be shown.) The values are an indication of the "goodness of fit" of the regression equation to the data. We more commonly use the value of r^2 instead of r, but the closer either value is to 1, the better the regression equation approximates the data.

Exponential Regression

Exponential regression is used to model situations in which growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero. We use the command "**ExpReg**" on a graphing utility to fit an exponential function to a set of data points. This returns an equation of the form, $y = ab^x$

Note that:

- b must be non-negative.
- when $b > 1$, we have an exponential growth model.

- when $0 < b < 1$, we have an exponential decay model.

HOW TO

Given a set of data, perform exponential regression using a graphing utility.

1. Use the **STAT** then **EDIT** menu to enter given data.
    a. Clear any existing data from the lists.
    b. List the input values in the L1 column.
    c. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the **STATPLOT** feature.
    a. Use **ZOOM** [**9**] to adjust axes to fit the data.
    b. Verify the data follow an exponential pattern.
3. Find the equation that models the data.
    a. Select "**ExpReg**" from the **STAT** then **CALC** menu.
    b. Use the values returned for *a* and *b* to record the model, $y = ab^x$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

EXAMPLE 1

Using Exponential Regression to Fit a Model to Data

In 2007, a university study was published investigating the crash risk of alcohol impaired driving. Data from 2,871 crashes were used to measure the association of a person's blood alcohol level (BAC) with the risk of being in an accident. Table 1 shows results from the study [9]. The *relative risk* is a measure of how many times more likely a person is to crash. So, for example, a person with a BAC of 0.09 is 3.54 times as likely to crash as a person who has not been drinking alcohol.

BAC	0	0.01	0.03	0.05	0.07	0.09
Relative Risk of Crashing	1	1.03	1.06	1.38	2.09	3.54
BAC	0.11	0.13	0.15	0.17	0.19	0.21
Relative Risk of Crashing	6.41	12.6	22.1	39.05	65.32	99.78

Table 1

a. Let x represent the BAC level, and let y represent the corresponding relative risk. Use exponential regression to fit a model to these data.
b. After 6 drinks, a person weighing 160 pounds will have a BAC of about 0.16. How many times more likely is a person with this weight to crash if they drive after having a 6-pack of beer? Round to the nearest hundredth.

Solution

a. Using the **STAT** then **EDIT** menu on a graphing utility, list the **BAC** values in L1 and the relative risk values in L2. Then use the **STATPLOT** feature to verify that the scatterplot follows the exponential pattern shown in Figure 1:

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[9] Source: *Indiana University Center for Studies of Law in Action, 2007*

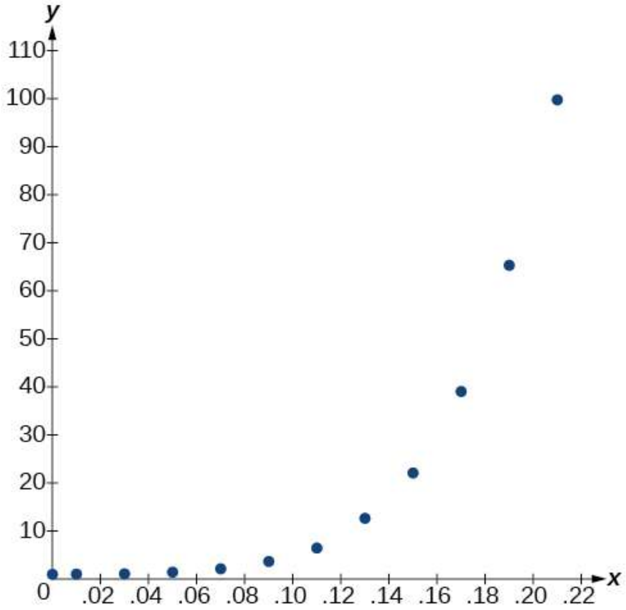

Figure 1

Use the “**ExpReg**” command from the **STAT** then **CALC** menu to obtain the exponential model,

$$y = 0.58304829(2.20720213\text{E}10)^x$$

Converting from scientific notation, we have:

$$y = 0.58304829(22{,}072{,}021{,}300)^x$$

Notice that $r^2 \approx 0.97$ which indicates the model is a good fit to the data. To see this, graph the model in the same window as the scatterplot to verify it is a good fit as shown in Figure 2:

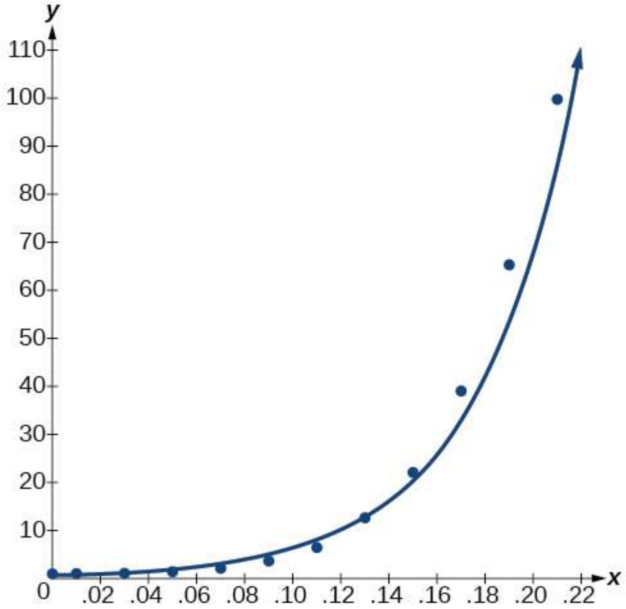

Figure 2

b. Use the model to estimate the risk associated with a BAC of 0.16. Substitute 0.16 for x in the model and solve for y.

$$\begin{array}{lll} y & = 0.58304829(22{,}072{,}021{,}300)^x & \text{Use the regression model found in part (a).} \\ & = 0.58304829(22{,}072{,}021{,}300)^{0.16} & \text{Substitute } 0.16 \text{ for } x. \\ & \approx 26.35 & \text{Round to the nearest hundredth.} \end{array}$$

If a 160-pound person drives after having 6 drinks, they are about 26.35 times more likely to crash than if driving while sober.

TRY IT #1 Table 2 shows a recent graduate's credit card balance each month after graduation.

Month	1	2	3	4	5	6	7	8
Debt ($)	620.00	761.88	899.80	1039.93	1270.63	1589.04	1851.31	2154.92

Table 2

ⓐ Use exponential regression to fit a model to these data.

ⓑ If spending continues at this rate, what will the graduate's credit card debt be one year after graduating?

Q&A **Is it reasonable to assume that an exponential regression model will represent a situation indefinitely?**

No. Remember that models are formed by real-world data gathered for regression. It is usually reasonable to make estimates within the interval of original observation (interpolation). However, when a model is used to make predictions, it is important to use reasoning skills to determine whether the model makes sense for inputs far beyond the original observation interval (extrapolation).

Building a Logarithmic Model from Data

Just as with exponential functions, there are many real-world applications for logarithmic functions: intensity of sound, pH levels of solutions, yields of chemical reactions, production of goods, and growth of infants. As with exponential models, data modeled by logarithmic functions are either always increasing or always decreasing as time moves forward. Again, it is the *way* they increase or decrease that helps us determine whether a logarithmic model is best.

Recall that logarithmic functions increase or decrease rapidly at first, but then steadily slow as time moves on. By reflecting on the characteristics we've already learned about this function, we can better analyze real world situations that reflect this type of growth or decay. When performing logarithmic regression analysis, we use the form of the logarithmic function most commonly used on graphing utilities, $y = a + b\ln(x)$. For this function

- All input values, x, must be greater than zero.
- The point $(1, a)$ is on the graph of the model.
- If $b > 0$, the model is increasing. Growth increases rapidly at first and then steadily slows over time.
- If $b < 0$, the model is decreasing. Decay occurs rapidly at first and then steadily slows over time.

Logarithmic Regression

Logarithmic regression is used to model situations where growth or decay accelerates rapidly at first and then slows over time. We use the command “LnReg” on a graphing utility to fit a logarithmic function to a set of data points. This returns an equation of the form,

$$y = a + b\ln(x)$$

Note that

- all input values, x, must be non-negative.
- when $b > 0$, the model is increasing.
- when $b < 0$, the model is decreasing.

HOW TO

Given a set of data, perform logarithmic regression using a graphing utility.

1. Use the **STAT** then **EDIT** menu to enter given data.
    a. Clear any existing data from the lists.
    b. List the input values in the L1 column.
    c. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the **STATPLOT** feature.
    a. Use **ZOOM** [**9**] to adjust axes to fit the data.
    b. Verify the data follow a logarithmic pattern.
3. Find the equation that models the data.
    a. Select "**LnReg**" from the **STAT** then **CALC** menu.
    b. Use the values returned for *a* and *b* to record the model, $y = a + b\ln(x)$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

EXAMPLE 2

Using Logarithmic Regression to Fit a Model to Data

Due to advances in medicine and higher standards of living, life expectancy has been increasing in most developed countries since the beginning of the 20th century.

Table 3 shows the average life expectancies, in years, of Americans from 1900–2010[10] .

Year	1900	1910	1920	1930	1940	1950
Life Expectancy(Years)	47.3	50.0	54.1	59.7	62.9	68.2
Year	1960	1970	1980	1990	2000	2010
Life Expectancy(Years)	69.7	70.8	73.7	75.4	76.8	78.7

Table 3

ⓐ Let x represent time in decades starting with $x = 1$ for the year 1900, $x = 2$ for the year 1910, and so on. Let y represent the corresponding life expectancy. Use logarithmic regression to fit a model to these data.
ⓑ Use the model to predict the average American life expectancy for the year 2030.

---

[10] Source: *Center for Disease Control and Prevention, 2013*

Solution

ⓐ Using the **STAT** then **EDIT** menu on a graphing utility, list the years using values 1–12 in L1 and the corresponding life expectancy in L2. Then use the **STATPLOT** feature to verify that the scatterplot follows a logarithmic pattern as shown in Figure 3:

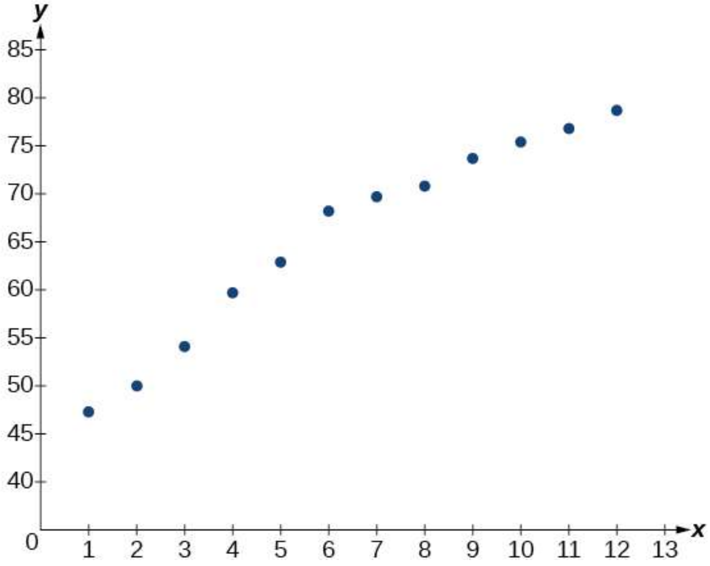

Figure 3

Use the "**LnReg**" command from the **STAT** then **CALC** menu to obtain the logarithmic model,

$$y = 42.52722583 + 13.85752327\ln(x)$$

Next, graph the model in the same window as the scatterplot to verify it is a good fit as shown in Figure 4:

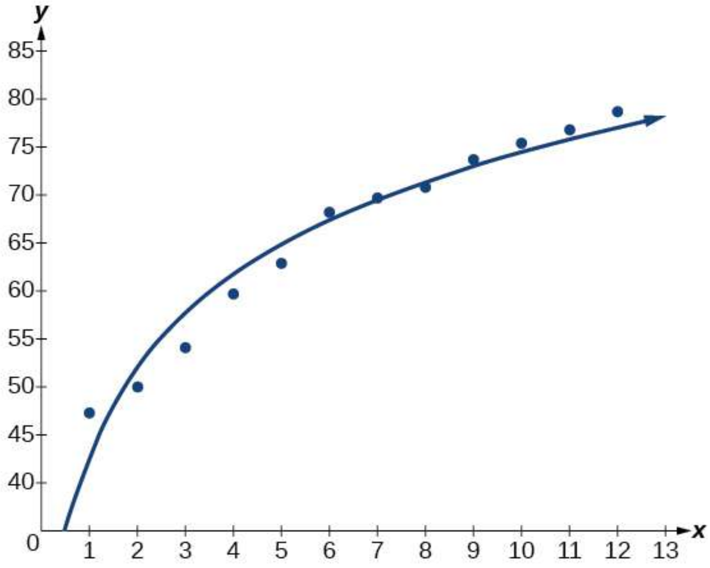

Figure 4

ⓑ To predict the life expectancy of an American in the year 2030, substitute $x = 14$ for the in the model and solve for y :

$$\begin{array}{lll} y & = 42.52722583 + 13.85752327\ln(x) & \text{Use the regression model found in part (a).} \\ & = 42.52722583 + 13.85752327\ln(14) & \text{Substitute 14 for } x. \\ & \approx 79.1 & \text{Round to the nearest tenth.} \end{array}$$

If life expectancy continues to increase at this pace, the average life expectancy of an American will be 79.1 by the year 2030.

TRY IT #2 Sales of a video game released in the year 2000 took off at first, but then steadily slowed as time moved on. Table 4 shows the number of games sold, in thousands, from the years 2000–2010.

Year	2000	2001	2002	2003	2004	2005
Number Sold (thousands)	142	149	154	155	159	161
Year	2006	2007	2008	2009	2010	-
Number Sold (thousands)	163	164	164	166	167	-

Table 4

ⓐ Let x represent time in years starting with $x = 1$ for the year 2000. Let y represent the number of games sold in thousands. Use logarithmic regression to fit a model to these data.

ⓑ If games continue to sell at this rate, how many games will sell in 2015? Round to the nearest thousand.

Building a Logistic Model from Data

Like exponential and logarithmic growth, logistic growth increases over time. One of the most notable differences with logistic growth models is that, at a certain point, growth steadily slows and the function approaches an upper bound, or *limiting value*. Because of this, logistic regression is best for modeling phenomena where there are limits in expansion, such as availability of living space or nutrients.

It is worth pointing out that logistic functions actually model resource-limited exponential growth. There are many examples of this type of growth in real-world situations, including population growth and spread of disease, rumors, and even stains in fabric. When performing logistic regression analysis, we use the form most commonly used on graphing utilities:

$$y = \frac{c}{1 + ae^{-bx}}$$

Recall that:

- $\frac{c}{1+a}$ is the initial value of the model.
- when $b > 0$, the model increases rapidly at first until it reaches its point of maximum growth rate, $\left(\frac{\ln(a)}{b}, \frac{c}{2}\right)$. At that point, growth steadily slows and the function becomes asymptotic to the upper bound $y = c$.
- c is the limiting value, sometimes called the *carrying capacity*, of the model.

Logistic Regression

Logistic regression is used to model situations where growth accelerates rapidly at first and then steadily slows to an upper limit. We use the command “Logistic” on a graphing utility to fit a logistic function to a set of data points. This returns an equation of the form

$$y = \frac{c}{1 + ae^{-bx}}$$

Note that

- The initial value of the model is $\frac{c}{1+a}$.
- Output values for the model grow closer and closer to $y = c$ as time increases.

HOW TO

Given a set of data, perform logistic regression using a graphing utility.

1. Use the **STAT** then **EDIT** menu to enter given data.
    a. Clear any existing data from the lists.
    b. List the input values in the L1 column.
    c. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the **STATPLOT** feature.
    a. Use **ZOOM** [**9**] to adjust axes to fit the data.
    b. Verify the data follow a logistic pattern.
3. Find the equation that models the data.
    a. Select "**Logistic**" from the **STAT** then **CALC** menu.
    b. Use the values returned for a, b, and c to record the model, $y = \frac{c}{1+ae^{-bx}}$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

EXAMPLE 3

Using Logistic Regression to Fit a Model to Data

Mobile telephone service has increased rapidly in America since the mid 1990s. Today, almost all residents have cellular service. Table 5 shows the percentage of Americans with cellular service between the years 1995 and 2012 [11].

Year	Americans with Cellular Service (%)	Year	Americans with Cellular Service (%)
1995	12.69	2004	62.852
1996	16.35	2005	68.63
1997	20.29	2006	76.64
1998	25.08	2007	82.47
1999	30.81	2008	85.68
2000	38.75	2009	89.14
2001	45.00	2010	91.86
2002	49.16	2011	95.28
2003	55.15	2012	98.17

Table 5

ⓐ Let x represent time in years starting with $x = 0$ for the year 1995. Let y represent the corresponding percentage of residents with cellular service. Use logistic regression to fit a model to these data.

ⓑ Use the model to calculate the percentage of Americans with cell service in the year 2013. Round to the nearest tenth of a percent.

ⓒ Discuss the value returned for the upper limit, c. What does this tell you about the model? What would the limiting

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[11] Source: *The World Bank, 2013*

value be if the model were exact?

Solution

ⓐ Using the **STAT** then **EDIT** menu on a graphing utility, list the years using values 0–15 in L1 and the corresponding percentage in L2. Then use the **STATPLOT** feature to verify that the scatterplot follows a logistic pattern as shown in Figure 5:

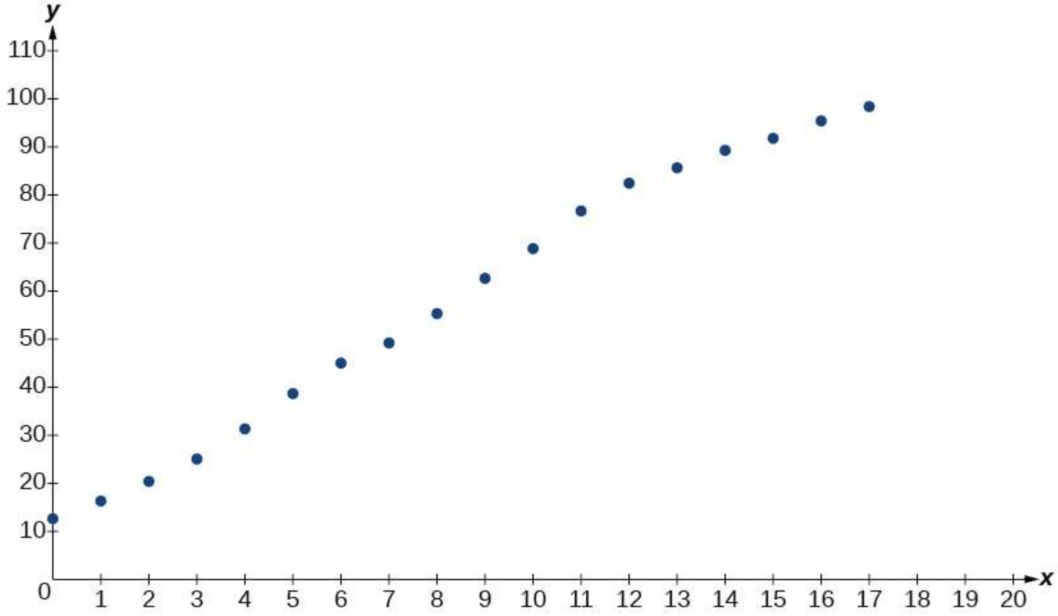


Figure 5

Use the "**Logistic**" command from the **STAT** then **CALC** menu to obtain the logistic model,

$$y = \frac{105.7379526}{1 + 6.88328979e^{-0.2595440013x}}$$

Next, graph the model in the same window as shown in Figure 6 the scatterplot to verify it is a good fit:

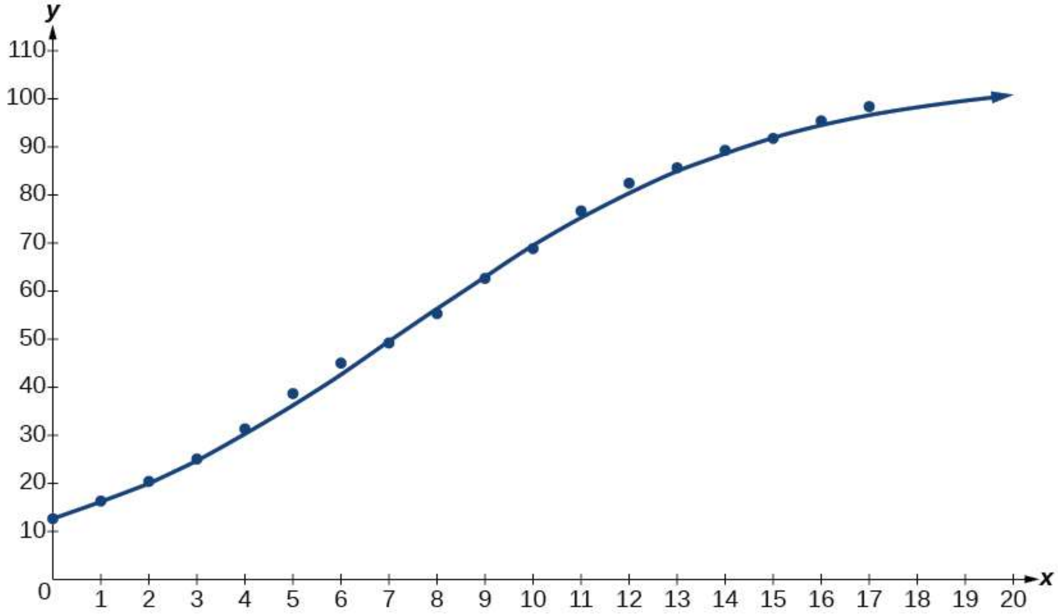


Figure 6

ⓑ

To approximate the percentage of Americans with cellular service in the year 2013, substitute $x = 18$ for the in the model and solve for y :

$$\begin{aligned} y &= \frac{105.7379526}{1+6.88328979e^{-0.2595440013x}} && \text{Use the regression model found in part (a).} \\ &= \frac{105.7379526}{1+6.88328979e^{-0.2595440013(18)}} && \text{Substitute 18 for } x. \\ &\approx 99.3 && \text{Round to the nearest tenth} \end{aligned}$$

According to the model, about 99.3% of Americans had cellular service in 2013.

ⓒ

The model gives a limiting value of about 105. This means that the maximum possible percentage of Americans with cellular service would be 105%, which is impossible. (How could over 100% of a population have cellular service?) If the model were exact, the limiting value would be $c = 100$ and the model’s outputs would get very close to, but never actually reach 100%. After all, there will always be someone out there without cellular service!

TRY IT #3 Table 6 shows the population, in thousands, of harbor seals in the Wadden Sea over the years 1997 to 2012.

Year	Seal Population (Thousands)	Year	Seal Population (Thousands)
1997	3.493	2005	19.590
1998	5.282	2006	21.955
1999	6.357	2007	22.862
2000	9.201	2008	23.869
2001	11.224	2009	24.243
2002	12.964	2010	24.344
2003	16.226	2011	24.919
2004	18.137	2012	25.108

Table 6

ⓐ Let x represent time in years starting with $x = 0$ for the year 1997. Let y represent the number of seals in thousands. Use logistic regression to fit a model to these data.
ⓑ Use the model to predict the seal population for the year 2020.
ⓒ To the nearest whole number, what is the limiting value of this model?

MEDIA

Access this online resource for additional instruction and practice with exponential function models.

Exponential Regression on a Calculator (https://openstax.org/l/pregresscalc)

6.8 SECTION EXERCISES

Verbal

1. What situations are best modeled by a logistic equation? Give an example, and state a case for why the example is a good fit.

2. What is a carrying capacity? What kind of model has a carrying capacity built into its formula? Why does this make sense?

3. What is regression analysis? Describe the process of performing regression analysis on a graphing utility.

4. What might a scatterplot of data points look like if it were best described by a logarithmic model?

5. What does the y-intercept on the graph of a logistic equation correspond to for a population modeled by that equation?

Graphical

For the following exercises, match the given function of best fit with the appropriate scatterplot in Figure 7 through Figure 11. Answer using the letter beneath the matching graph.

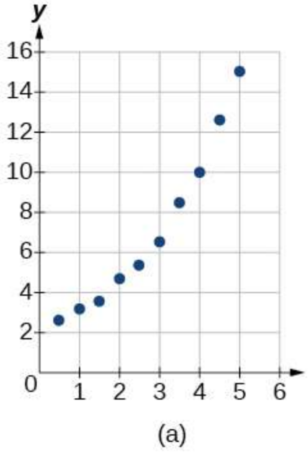


Figure 7

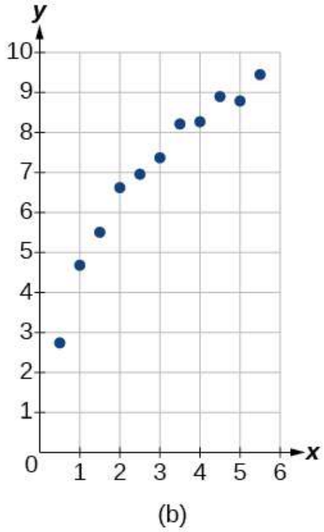


Figure 8

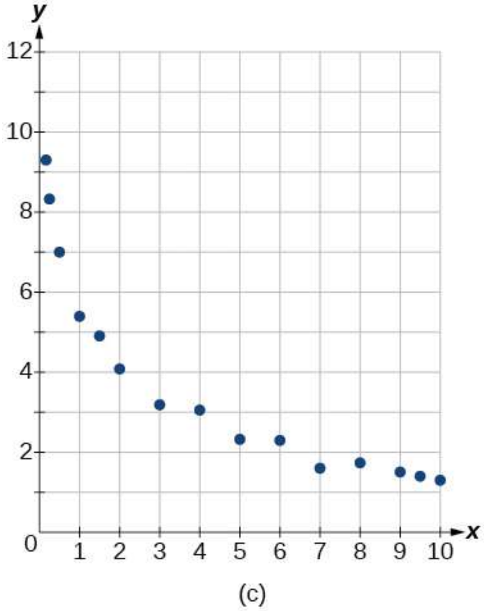


Figure 9

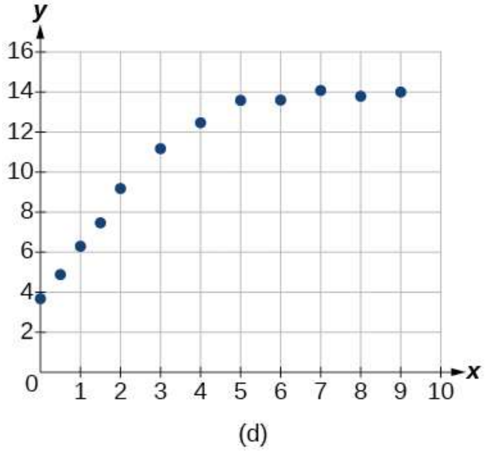


Figure 10

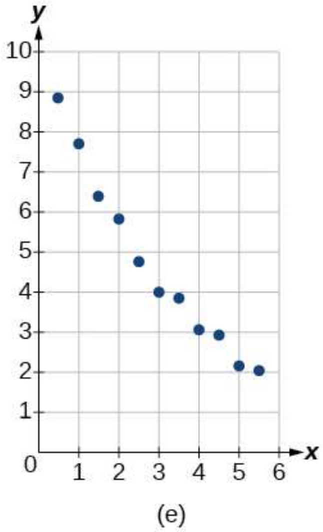


Figure 11

6. $y = 10.209e^{-0.294x}$

7. $y = 5.598 - 1.912\ln(x)$

8. $y = 2.104(1.479)^x$

9. $y = 4.607 + 2.733\ln(x)$

10. $y = \frac{14.005}{1+2.79e^{-0.812x}}$

Numeric

11. To the nearest whole number, what is the initial value of a population modeled by the logistic equation $P(t) = \frac{175}{1+6.995e^{-0.68t}}$? What is the carrying capacity?

12. Rewrite the exponential model $A(t) = 1550(1.085)^x$ as an equivalent model with base e. Express the exponent to four significant digits.

13. A logarithmic model is given by the equation $h(p) = 67.682 - 5.792\ln(p)$. To the nearest hundredth, for what value of p does $h(p) = 62$?

14. A logistic model is given by the equation $P(t) = \frac{90}{1+5e^{-0.42t}}$. To the nearest hundredth, for what value of t does $P(t) = 45$?

15. What is the y-intercept on the graph of the logistic model given in the previous exercise?

Technology

For the following exercises, use this scenario: The population P of a koi pond over x months is modeled by the function $P(x) = \frac{68}{1+16e^{-0.28x}}$.

16. Graph the population model to show the population over a span of 3 years.

17. What was the initial population of koi?

18. How many koi will the pond have after one and a half years?

19. How many months will it take before there are 20 koi in the pond?

20. Use the intersect feature to approximate the number of months it will take before the population of the pond reaches half its carrying capacity.

For the following exercises, use this scenario: The population P of an endangered species habitat for wolves is modeled by the function $P(x) = \frac{558}{1+54.8e^{-0.462x}}$, where x is given in years.

21. Graph the population model to show the population over a span of 10 years.

22. What was the initial population of wolves transported to the habitat?

23. How many wolves will the habitat have after 3 years?

24. How many years will it take before there are 100 wolves in the habitat?

25. Use the intersect feature to approximate the number of years it will take before the population of the habitat reaches half its carrying capacity.

For the following exercises, refer to Table 7.

x	1	2	3	4	5	6
f(x)	1125	1495	2310	3294	4650	6361

Table 7

26. Use a graphing calculator to create a scatter diagram of the data.

27. Use the regression feature to find an exponential function that best fits the data in the table.

28. Write the exponential function as an exponential equation with base e.

29. Graph the exponential equation on the scatter diagram.

30. Use the intersect feature to find the value of x for which $f(x) = 4000$.

For the following exercises, refer to Table 8.

x	1	2	3	4	5	6
f(x)	555	383	307	210	158	122

Table 8

31. Use a graphing calculator to create a scatter diagram of the data.

32. Use the regression feature to find an exponential function that best fits the data in the table.

33. Write the exponential function as an exponential equation with base e.

34. Graph the exponential equation on the scatter diagram.

35. Use the intersect feature to find the value of x for which $f(x) = 250$.

For the following exercises, refer to Table 9.

x	1	2	3	4	5	6
f(x)	5.1	6.3	7.3	7.7	8.1	8.6

Table 9

36. Use a graphing calculator to create a scatter diagram of the data.

37. Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

38. Use the logarithmic function to find the value of the function when $x = 10.$

39. Graph the logarithmic equation on the scatter diagram.

40. Use the intersect feature to find the value of x for which $f(x) = 7.$

For the following exercises, refer to Table 10.

x	1	2	3	4	5	6	7	8
f(x)	7.5	6	5.2	4.3	3.9	3.4	3.1	2.9

Table 10

41. Use a graphing calculator to create a scatter diagram of the data.

42. Use the **LOG**arithm option of the **REG**ression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

43. Use the logarithmic function to find the value of the function when $x = 10.$

44. Graph the logarithmic equation on the scatter diagram.

45. Use the intersect feature to find the value of x for which $f(x) = 8.$

For the following exercises, refer to Table 11.

x	1	2	3	4	5	6	7	8	9	10
f(x)	8.7	12.3	15.4	18.5	20.7	22.5	23.3	24	24.6	24.8

Table 11

46. Use a graphing calculator to create a scatter diagram of the data.

47. Use the LOGISTIC regression option to find a logistic growth model of the form $y = \frac{c}{1+ae^{-bx}}$ that best fits the data in the table.

48. Graph the logistic equation on the scatter diagram.

49. To the nearest whole number, what is the predicted carrying capacity of the model?

50. Use the intersect feature to find the value of x for which the model reaches half its carrying capacity.

For the following exercises, refer to Table 12.

x	0	2	4	5	7	8	10	11	15	17
$f(x)$	12	28.6	52.8	70.3	99.9	112.5	125.8	127.9	135.1	135.9

Table 12

51. Use a graphing calculator to create a scatter diagram of the data.

52. Use the LOGISTIC regression option to find a logistic growth model of the form $y = \frac{c}{1+ae^{-bx}}$ that best fits the data in the table.

53. Graph the logistic equation on the scatter diagram.

54. To the nearest whole number, what is the predicted carrying capacity of the model?

55. Use the intersect feature to find the value of x for which the model reaches half its carrying capacity.

Extensions

56. Recall that the general form of a logistic equation for a population is given by $P(t) = \frac{c}{1+ae^{-bt}}$, such that the initial population at time $t = 0$ is $P(0) = P_0$. Show algebraically that $\frac{c-P(t)}{P(t)} = \frac{c-P_0}{P_0}e^{-bt}$.

57. Use a graphing utility to find an exponential regression formula $f(x)$ and a logarithmic regression formula $g(x)$ for the points $(1.5, 1.5)$ and $(8.5,\ \ 8.5)$. Round all numbers to 6 decimal places. Graph the points and both formulas along with the line $y = x$ on the same axis. Make a conjecture about the relationship of the regression formulas.

58. Verify the conjecture made in the previous exercise. Round all numbers to six decimal places when necessary.

59. Find the inverse function $f^{-1}(x)$ for the logistic function $f(x) = \frac{c}{1+ae^{-bx}}$. Show all steps.

60. Use the result from the previous exercise to graph the logistic model $P(t) = \frac{20}{1+4e^{-0.5t}}$ along with its inverse on the same axis. What are the intercepts and asymptotes of each function?

Chapter Review

Key Terms

annual percentage rate (APR) the yearly interest rate earned by an investment account, also called *nominal rate*

carrying capacity in a logistic model, the limiting value of the output

change-of-base formula a formula for converting a logarithm with any base to a quotient of logarithms with any other base.

common logarithm the exponent to which 10 must be raised to get x; $\log_{10}(x)$ is written simply as $\log(x)$.

compound interest interest earned on the total balance, not just the principal

doubling time the time it takes for a quantity to double

exponential growth a model that grows by a rate proportional to the amount present

extraneous solution a solution introduced while solving an equation that does not satisfy the conditions of the original equation

half-life the length of time it takes for a substance to exponentially decay to half of its original quantity

logarithm the exponent to which b must be raised to get x; written $y = \log_b(x)$

logistic growth model a function of the form $f(x) = \frac{c}{1+ae^{-bx}}$ where $\frac{c}{1+a}$ is the initial value, c is the carrying capacity, or limiting value, and b is a constant determined by the rate of growth

natural logarithm the exponent to which the number e must be raised to get x; $\log_e(x)$ is written as $\ln(x)$.

Newton's Law of Cooling the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature

nominal rate the yearly interest rate earned by an investment account, also called *annual percentage rate*

order of magnitude the power of ten, when a number is expressed in scientific notation, with one non-zero digit to the left of the decimal

power rule for logarithms a rule of logarithms that states that the log of a power is equal to the product of the exponent and the log of its base

product rule for logarithms a rule of logarithms that states that the log of a product is equal to a sum of logarithms

quotient rule for logarithms a rule of logarithms that states that the log of a quotient is equal to a difference of logarithms

Key Equations

definition of the exponential function	$f(x) = b^x, \text{ where } b > 0, b \neq 1$
definition of exponential growth	$f(x) = ab^x, \text{ where } a > 0, b > 0, b \neq 1$
compound interest formula	$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}, \text{ where}$ $A(t)$ is the account value at time t t is the number of years P is the initial investment, often called the principal r is the annual percentage rate (APR), or nominal rate n is the number of compounding periods in one year
continuous growth formula	$A(t) = ae^{rt}, \text{ where}$ t is the number of unit time periods of growth a is the starting amount (in the continuous compounding formula a is replaced with P, the principal) e is the mathematical constant, $e \approx 2.718282$

General Form for the Translation of the Parent Function $f(x) = b^x$	$f(x) = ab^{x+c} + d$

Definition of the logarithmic function	For $x > 0, b > 0, b \neq 1$, $y = \log_b(x)$ if and only if $b^y = x$.
Definition of the common logarithm	For $x > 0$, $y = \log(x)$ if and only if $10^y = x$.
Definition of the natural logarithm	For $x > 0$, $y = \ln(x)$ if and only if $e^y = x$.

General Form for the Translation of the Parent Logarithmic Function $f(x) = \log_b(x)$	$f(x) = a\log_b(x + c) + d$

The Product Rule for Logarithms	$\log_b(MN) = \log_b(M) + \log_b(N)$	
The Quotient Rule for Logarithms	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$	
The Power Rule for Logarithms	$\log_b(M^n) = n\log_b M$	
The Change-of-Base Formula	$\log_b M = \frac{\log_n M}{\log_n b}$	$n > 0, n \neq 1, b \neq 1$

One-to-one property for exponential functions	For any algebraic expressions S and T and any positive real number b, where $b^S = b^T$ if and only if $S = T$.
Definition of a logarithm	For any algebraic expression S and positive real numbers b and c, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$.
One-to-one property for logarithmic functions	For any algebraic expressions S and T and any positive real number b, where $b \neq 1$, $\log_b S = \log_b T$ if and only if $S = T$.

Half-life formula	If $A = A_0 e^{kt}$, $k < 0$, the half-life is $t = -\frac{\ln(2)}{k}$.
Carbon-14 dating	$t = \frac{\ln\left(\frac{A}{A_0}\right)}{-0.000121}$. A_0 is the amount of carbon-14 when the plant or animal died A is the amount of carbon-14 remaining today t is the age of the fossil in years
Doubling time formula	If $A = A_0 e^{kt}$, $k > 0$, the doubling time is $t = \frac{\ln 2}{k}$
Newton’s Law of Cooling	$T(t) = Ae^{kt} + T_s$, where T_s is the ambient temperature, $A = T(0) - T_s$, and k is the continuous rate of cooling.

Key Concepts

6.1 Exponential Functions

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent. See Example 1.

- A function is evaluated by solving at a specific value. See Example 2 and Example 3.
- An exponential model can be found when the growth rate and initial value are known. See Example 4.
- An exponential model can be found when the two data points from the model are known. See Example 5.
- An exponential model can be found using two data points from the graph of the model. See Example 6.
- An exponential model can be found using two data points from the graph and a calculator. See Example 7.
- The value of an account at any time t can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known. See Example 8.
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known. See Example 9.
- The number e is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e \approx 2.718282$.
- Scientific and graphing calculators have the key $\left[e^x\right]$ or $\left[\exp(x)\right]$ for calculating powers of e. See Example 10**.**
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known. See Example 11 and Example 12.

6.2 Graphs of Exponential Functions

- The graph of the function $f(x) = b^x$ has a y-intercept at $(0,\ 1)$, domain $\left(-\infty,\ \infty\right)$, range $\left(0,\ \infty\right)$, and horizontal asymptote $y = 0$. See Example 1.
- If $b > 1$, the function is increasing. The left tail of the graph will approach the asymptote $y = 0$, and the right tail will increase without bound.
- If $0 < b < 1$, the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- The equation $f(x) = b^x + d$ represents a vertical shift of the parent function $f(x) = b^x$.
- The equation $f(x) = b^{x+c}$ represents a horizontal shift of the parent function $f(x) = b^x$. See Example 2.
- Approximate solutions of the equation $f(x) = b^{x+c} + d$ can be found using a graphing calculator. See Example 3.
- The equation $f(x) = ab^x$, where $a > 0$, represents a vertical stretch if $|a| > 1$ or compression if $0 < |a| < 1$ of the parent function $f(x) = b^x$. See Example 4.
- When the parent function $f(x) = b^x$ is multiplied by -1, the result, $f(x) = -b^x$, is a reflection about the x-axis. When the input is multiplied by -1, the result, $f(x) = b^{-x}$, is a reflection about the y-axis. See Example 5.
- All translations of the exponential function can be summarized by the general equation $f(x) = ab^{x+c} + d$. See Table 3.
- Using the general equation $f(x) = ab^{x+c} + d$, we can write the equation of a function given its description. See Example 6.

6.3 Logarithmic Functions

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See Example 1.
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See Example 2.
- Logarithmic functions with base b can be evaluated mentally using previous knowledge of powers of b. See Example 3 and Example 4.
- Common logarithms can be evaluated mentally using previous knowledge of powers of 10. See Example 5**.**
- When common logarithms cannot be evaluated mentally, a calculator can be used. See Example 6**.**
- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator. See Example 7**.**
- Natural logarithms can be evaluated using a calculator Example 8**.**

6.4 Graphs of Logarithmic Functions

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for x. See Example 1 and Example 2
- The graph of the parent function $f(x) = \log_b(x)$ has an x-intercept at $(1, 0)$, domain $\left(0, \infty\right)$, range $\left(-\infty, \infty\right)$, vertical asymptote $x = 0$, and
    - if $b > 1$, the function is increasing.
    - if $0 < b < 1$, the function is decreasing.

See Example 3.

- The equation $f(x) = \log_b(x + c)$ shifts the parent function $y = \log_b(x)$ horizontally
  - left c units if $c > 0.$
  - right c units if $c < 0.$

  See Example 4.
- The equation $f(x) = \log_b(x) + d$ shifts the parent function $y = \log_b(x)$ vertically
  - up d units if $d > 0.$
  - down d units if $d < 0.$

  See Example 5.
- For any constant $a > 0$, the equation $f(x) = a\log_b(x)$
  - stretches the parent function $y = \log_b(x)$ vertically by a factor of a if $|a| > 1.$
  - compresses the parent function $y = \log_b(x)$ vertically by a factor of a if $|a| < 1.$

  See Example 6 and Example 7.
- When the parent function $y = \log_b(x)$ is multiplied by -1, the result is a reflection about the *x*-axis. When the input is multiplied by -1, the result is a reflection about the *y*-axis.
  - The equation $f(x) = -\log_b(x)$ represents a reflection of the parent function about the *x*-axis.
  - The equation $f(x) = \log_b(-x)$ represents a reflection of the parent function about the *y*-axis.

  See Example 8.
  - A graphing calculator may be used to approximate solutions to some logarithmic equations See Example 9.
- All translations of the logarithmic function can be summarized by the general equation $f(x) = a\log_b(x + c) + d$. See Table 4.
- Given an equation with the general form $f(x) = a\log_b(x + c) + d$, we can identify the vertical asymptote $x = -c$ for the transformation. See Example 10.
- Using the general equation $f(x) = a\log_b(x + c) + d$, we can write the equation of a logarithmic function given its graph. See Example 11.

6.5 Logarithmic Properties

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See Example 1.
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See Example 2.
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See Example 3, Example 4, and Example 5.
- We can use the product rule, the quotient rule, and the power rule together to combine or expand a logarithm with a complex input. See Example 6, Example 7, and Example 8.
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See Example 9, Example 10, Example 11, and Example 12.
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See Example 13.
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 and e as the quotient of natural or common logs. That way a calculator can be used to evaluate. See Example 14.

6.6 Exponential and Logarithmic Equations

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown. See Example 1.
- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See Example 2, Example 3, and Example 4.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See Example 5.
- We can solve exponential equations with base e, by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See Example 6 and Example 7.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any

extraneous solutions. See Example 8.

- When given an equation of the form $\log_b(S) = c$, where S is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See Example 9 and Example 10.
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the *x*-value of the intersecting point. See Example 11.
- When given an equation of the form $\log_b S = \log_b T$, where S and T are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See Example 12.
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See Example 13.

6.7 Exponential and Logarithmic Models

- The basic exponential function is $f(x) = ab^x$. If $b > 1$, we have exponential growth; if $0 < b < 1$, we have exponential decay.
- We can also write this formula in terms of continuous growth as $A = A_0 e^{kx}$, where A_0 is the starting value. If A_0 is positive, then we have exponential growth when $k > 0$ and exponential decay when $k < 0$. See Example 1.
- In general, we solve problems involving exponential growth or decay in two steps. First, we set up a model and use the model to find the parameters. Then we use the formula with these parameters to predict growth and decay. See Example 2.
- We can find the age, t, of an organic artifact by measuring the amount, k, of carbon-14 remaining in the artifact and using the formula $t = \frac{\ln(k)}{-0.000121}$ to solve for t. See Example 3.
- Given a substance’s doubling time or half-time, we can find a function that represents its exponential growth or decay. See Example 4.
- We can use Newton’s Law of Cooling to find how long it will take for a cooling object to reach a desired temperature, or to find what temperature an object will be after a given time. See Example 5.
- We can use logistic growth functions to model real-world situations where the rate of growth changes over time, such as population growth, spread of disease, and spread of rumors. See Example 6.
- We can use real-world data gathered over time to observe trends. Knowledge of linear, exponential, logarithmic, and logistic graphs help us to develop models that best fit our data. See Example 7.
- Any exponential function with the form $y = ab^x$ can be rewritten as an equivalent exponential function with the form $y = A_0 e^{kx}$ where $k = \ln b$. See Example 8.

6.8 Fitting Exponential Models to Data

- Exponential regression is used to model situations where growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero.
- We use the command “ExpReg” on a graphing utility to fit function of the form $y = ab^x$ to a set of data points. See Example 1.
- Logarithmic regression is used to model situations where growth or decay accelerates rapidly at first and then slows over time.
- We use the command “LnReg” on a graphing utility to fit a function of the form $y = a + b\ln(x)$ to a set of data points. See Example 2.
- Logistic regression is used to model situations where growth accelerates rapidly at first and then steadily slows as the function approaches an upper limit.
- We use the command “Logistic” on a graphing utility to fit a function of the form $y = \frac{c}{1+ae^{-bx}}$ to a set of data points. See Example 3.

Exercises

Review Exercises

Exponential Functions

1. Determine whether the function $y = 156(0.825)^t$ represents exponential growth, exponential decay, or neither. Explain

2. The population of a herd of deer is represented by the function $A(t) = 205(1.13)^t$, where t is given in years. To the nearest whole number, what will the herd population be after 6 years?

3. Find an exponential equation that passes through the points $(2, 2.25)$ and $(5, 60.75)$.

4. Determine whether Table 1 could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

x	1	2	3	4
f(x)	3	0.9	0.27	0.081

Table 1

5. A retirement account is opened with an initial deposit of $8,500 and earns 8.12% interest compounded monthly. What will the account be worth in 20 years?

6. Hsu-Mei wants to save $5,000 for a down payment on a car. To the nearest dollar, how much will she need to invest in an account now with 7.5% APR, compounded daily, in order to reach her goal in 3 years?

7. Does the equation $y = 2.294e^{-0.654t}$ represent continuous growth, continuous decay, or neither? Explain.

8. Suppose an investment account is opened with an initial deposit of $\$10{,}500$ earning 6.25% interest, compounded continuously. How much will the account be worth after 25 years?

Graphs of Exponential Functions

9. Graph the function $f(x) = 3.5(2)^x$. State the domain and range and give the *y*-intercept.

10. Graph the function $f(x) = 4\left(\frac{1}{8}\right)^x$ and its reflection about the *y*-axis on the same axes, and give the *y*-intercept.

11. The graph of $f(x) = 6.5^x$ is reflected about the y-axis and stretched vertically by a factor of 7. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

12. The graph below shows transformations of the graph of $f(x) = 2^x$. What is the equation for the transformation?

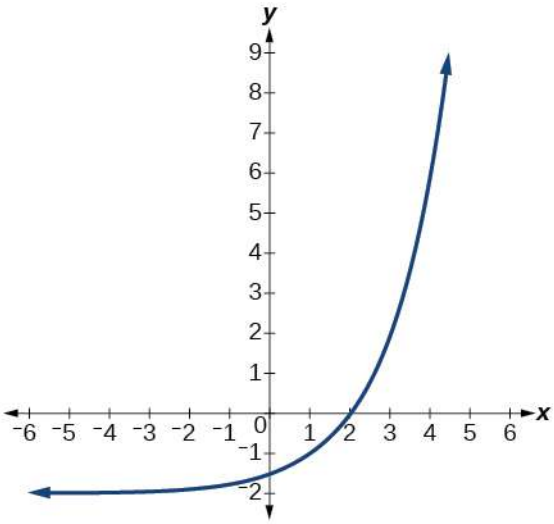


Figure 1

Logarithmic Functions

13. Rewrite $\log_{17}(4913) = x$ as an equivalent exponential equation.

14. Rewrite $\ln(s) = t$ as an equivalent exponential equation.

15. Rewrite $a^{-\frac{2}{5}} = b$ as an equivalent logarithmic equation.

16. Rewrite $e^{-3.5} = h$ as an equivalent logarithmic equation.

17. Solve for x if $\log_{64}(x) = \frac{1}{3}$ by converting the logarithmic equation $\log_{64}(x) = \frac{1}{3}$ to exponential form.

18. Evaluate $\log_5\left(\frac{1}{125}\right)$ without using a calculator.

19. Evaluate $\log(0.000001)$ without using a calculator.

20. Evaluate $\log(4.005)$ using a calculator. Round to the nearest thousandth.

21. Evaluate $\ln\left(e^{-0.8648}\right)$ without using a calculator.

22. Evaluate $\ln\left(\sqrt[3]{18}\right)$ using a calculator. Round to the nearest thousandth.

Graphs of Logarithmic Functions

23. Graph the function $g(x) = \log(7x + 21) - 4$.

24. Graph the function $h(x) = 2\ln(9 - 3x) + 1$.

25. State the domain, vertical asymptote, and end behavior of the function $g(x) = \ln(4x + 20) - 17$.

Logarithmic Properties

26. Rewrite $\ln(7r \cdot 11st)$ in expanded form.

27. Rewrite $\log_8(x) + \log_8(5) + \log_8(y) + \log_8(13)$ in compact form.

28. Rewrite $\log_m \left(\frac{67}{83}\right)$ in expanded form.

29. Rewrite $\ln(z) - \ln(x) - \ln(y)$ in compact form.

30. Rewrite $\ln\left(\frac{1}{x^5}\right)$ as a product.

31. Rewrite $-\log_y\left(\frac{1}{12}\right)$ as a single logarithm.

32. Use properties of logarithms to expand $\log\left(\frac{r^2 s^{11}}{t^{14}}\right)$.

33. Use properties of logarithms to expand $\ln\left(2b\sqrt{\frac{b+1}{b-1}}\right)$.

34. Condense the expression $5\ln(b) + \ln(c) + \frac{\ln(4-a)}{2}$ to a single logarithm.

35. Condense the expression $3\log_7 v + 6\log_7 w - \frac{\log_7 u}{3}$ to a single logarithm.

36. Rewrite $\log_3(12.75)$ to base e.

37. Rewrite $5^{12x-17} = 125$ as a logarithm. Then apply the change of base formula to solve for x using the common log. Round to the nearest thousandth.

Exponential and Logarithmic Equations

38. Solve $216^{3x} \cdot 216^{x} = 36^{3x+2}$ by rewriting each side with a common base.

39. Solve $\frac{125}{\left(\frac{1}{625}\right)^{-x-3}} = 5^3$ by rewriting each side with a common base.

40. Use logarithms to find the exact solution for $7 \cdot 17^{-9x} - 7 = 49$. If there is no solution, write *no solution*.

41. Use logarithms to find the exact solution for $3e^{6n-2} + 1 = -60$. If there is no solution, write *no solution*.

42. Find the exact solution for $5e^{3x} - 4 = 6$. If there is no solution, write *no solution*.

43. Find the exact solution for $2e^{5x-2} - 9 = -56$. If there is no solution, write *no solution*.

44. Find the exact solution for $5^{2x-3} = 7^{x+1}$. If there is no solution, write *no solution*.

45. Find the exact solution for $e^{2x} - e^{x} - 110 = 0$. If there is no solution, write *no solution*.

46. Use the definition of a logarithm to solve. $-5\log_7(10n) = 5$.

47. Use the definition of a logarithm to find the exact solution for $9 + 6\ln(a+3) = 33$.

48. Use the one-to-one property of logarithms to find an exact solution for $\log_8(7) + \log_8(-4x) = \log_8(5)$. If there is no solution, write *no solution*.

49. Use the one-to-one property of logarithms to find an exact solution for $\ln(5) + \ln\left(5x^2 - 5\right) = \ln(56)$. If there is no solution, write *no solution*.

50. The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a large orchestra with a sound intensity of $6.3 \cdot 10^{-3}$ watts per square meter?

51. The population of a city is modeled by the equation $P(t) = 256,114e^{0.25t}$ where t is measured in years. If the city continues to grow at this rate, how many years will it take for the population to reach one million?

52. Find the inverse function f^{-1} for the exponential function $f(x) = 2 \cdot e^{x+1} - 5.$

53. Find the inverse function f^{-1} for the logarithmic function $f(x) = 0.25 \cdot \log_2 \left(x^3 + 1\right).$

Exponential and Logarithmic Models

For the following exercises, use this scenario: A doctor prescribes 300 *milligrams of a therapeutic drug that decays by about* 17% *each hour.*

54. To the nearest minute, what is the half-life of the drug?

55. Write an exponential model representing the amount of the drug remaining in the patient's system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 24 hours. Round to the nearest hundredth of a gram.

For the following exercises, use this scenario: A soup with an internal temperature of $350°$ *Fahrenheit was taken off the stove to cool in a* $71°\text{F}$ *room. After fifteen minutes, the internal temperature of the soup was* $175°\text{F}.$

56. Use Newton's Law of Cooling to write a formula that models this situation.

57. How many minutes will it take the soup to cool to $85°\text{F}?$

For the following exercises, use this scenario: The equation $N(t) = \frac{1200}{1+199e^{-0.625t}}$ *models the number of people in a school who have heard a rumor after* t *days.*

58. How many people started the rumor?

59. To the nearest tenth, how many days will it be before the rumor spreads to half the carrying capacity?

60. What is the carrying capacity?

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table would likely represent a function that is linear, exponential, or logarithmic.

61.

x	*f(x)*
1	3.05
2	4.42
3	6.4
4	9.28
5	13.46
6	19.52
7	28.3
8	41.04
9	59.5
10	86.28

62.

x	*f(x)*
0.5	18.05
1	17
3	15.33
5	14.55
7	14.04
10	13.5
12	13.22
13	13.1
15	12.88
17	12.69
20	12.45

63. Find a formula for an exponential equation that goes through the points $(-2, 100)$ and $(0, 4)$. Then express the formula as an equivalent equation with base *e*.

Fitting Exponential Models to Data

64. What is the carrying capacity for a population modeled by the logistic equation $P(t) = \frac{250{,}000}{1+499e^{-0.45t}}$? What is the initial population for the model?

65. The population of a culture of bacteria is modeled by the logistic equation $P(t) = \frac{14{,}250}{1+29e^{-0.62t}}$, where t is in days. To the nearest tenth, how many days will it take the culture to reach 75% of its carrying capacity?

For the following exercises, use a graphing utility to create a scatter diagram of the data given in the table. Observe the shape of the scatter diagram to determine whether the data is best described by an exponential, logarithmic, or logistic model. Then use the appropriate regression feature to find an equation that models the data. When necessary, round values to five decimal places.

66.

x	*f(x)*
1	409.4
2	260.7
3	170.4
4	110.6
5	74
6	44.7
7	32.4
8	19.5
9	12.7
10	8.1

67.

x	*f(x)*
0.15	36.21
0.25	28.88
0.5	24.39
0.75	18.28
1	16.5
1.5	12.99
2	9.91
2.25	8.57
2.75	7.23
3	5.99
3.5	4.81

68.

x	*f(x)*
0	9
2	22.6
4	44.2
5	62.1
7	96.9
8	113.4
10	133.4
11	137.6
15	148.4
17	149.3

Practice Test

1. The population of a pod of bottlenose dolphins is modeled by the function $A(t) = 8(1.17)^t$, where t is given in years. To the nearest whole number, what will the pod population be after 3 years?

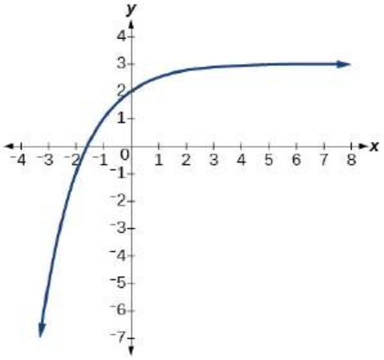
2. Find an exponential equation that passes through the points $(0, 4)$ and $(2, 9)$.

3. Drew wants to save $2,500 to go to the next World Cup. To the nearest dollar, how much will he need to invest in an account now with 6.25% APR, compounding daily, in order to reach his goal in 4 years?

4. An investment account was opened with an initial deposit of \$9,600 and earns 7.4% interest, compounded continuously. How much will the account be worth after 15 years?

5. Graph the function $f(x) = 5(0.5)^{-x}$ and its reflection across the *y*-axis on the same axes, and give the *y*-intercept.

6. The graph shows transformations of the graph of $f(x) = \left(\frac{1}{2}\right)^x$. What is the equation for the transformation?



7. Rewrite $\log_{8.5}(614.125) = a$ as an equivalent exponential equation.

8. Rewrite $e^{\frac{1}{2}} = m$ as an equivalent logarithmic equation.

9. Solve for x by converting the logarithmic equation $log_{\frac{1}{7}}(x) = 2$ to exponential form.

10. Evaluate $\log(10{,}000{,}000)$ without using a calculator.

11. Evaluate $\ln(0.716)$ using a calculator. Round to the nearest thousandth.

12. Graph the function $g(x) = \log(12 - 6x) + 3$.

13. State the domain, vertical asymptote, and end behavior of the function $f(x) = \log_5(39 - 13x) + 7$.

14. Rewrite $\log(17a \cdot 2b)$ as a sum.

15. Rewrite $\log_t(96) - \log_t(8)$ in compact form.

16. Rewrite $\log_8\left(a^{\frac{1}{b}}\right)$ as a product.

17. Use properties of logarithm to expand $\ln\left(y^3 z^2 \cdot \sqrt[3]{x-4}\right)$.

18. Condense the expression $4\ln(c) + \ln(d) + \frac{\ln(a)}{3} + \frac{\ln(b+3)}{3}$ to a single logarithm.

19. Rewrite $16^{3x-5} = 1000$ as a logarithm. Then apply the change of base formula to solve for x using the natural log. Round to the nearest thousandth.

20. Solve $\left(\frac{1}{81}\right)^x \cdot \frac{1}{243} = \left(\frac{1}{9}\right)^{-3x-1}$ by rewriting each side with a common base.

21. Use logarithms to find the exact solution for $-9e^{10a-8} - 5 = -41$. If there is no solution, write *no solution*.

22. Find the exact solution for $10e^{4x+2} + 5 = 56$. If there is no solution, write *no solution*.

23. Find the exact solution for $-5e^{-4x-1} - 4 = 64$. If there is no solution, write *no solution*.

24. Find the exact solution for $2^{x-3} = 6^{2x-1}$. If there is no solution, write *no solution*.

25. Find the exact solution for $e^{2x} - e^{x} - 72 = 0$. If there is no solution, write *no solution*.

26. Use the definition of a logarithm to find the exact solution for $4\log(2n) - 7 = -11$

27. Use the one-to-one property of logarithms to find an exact solution for $\log\left(4x^2 - 10\right) + \log(3) = \log(51)$ If there is no solution, write *no solution*.

28. The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a rock concert with a sound intensity of $4.7 \cdot 10^{-1}$ watts per square meter?

29. A radiation safety officer is working with 112 grams of a radioactive substance. After 17 days, the sample has decayed to 80 grams. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest day, what is the half-life of this substance?

30. Write the formula found in the previous exercise as an equivalent equation with base e. Express the exponent to five significant digits.

31. A bottle of soda with a temperature of $71°$ Fahrenheit was taken off a shelf and placed in a refrigerator with an internal temperature of $35°\text{ F}$. After ten minutes, the internal temperature of the soda was $63°\text{ F}$. Use Newton's Law of Cooling to write a formula that models this situation. To the nearest degree, what will the temperature of the soda be after one hour?

32. The population of a wildlife habitat is modeled by the equation $P(t) = \frac{360}{1+6.2e^{-0.35t}}$, where t is given in years. How many animals were originally transported to the habitat? How many years will it take before the habitat reaches half its capacity?

33. Enter the data from Table 1 into a graphing calculator and graph the resulting scatter plot. Determine whether the data from the table would likely represent a function that is linear, exponential, or logarithmic.

x	***f(x)***
1	3
2	8.55
3	11.79
4	14.09
5	15.88
6	17.33
7	18.57
8	19.64
9	20.58
10	21.42

Table 1

34. The population of a lake of fish is modeled by the logistic equation $P(t) = \frac{16{,}120}{1+25e^{-0.75t}}$, where t is time in years. To the nearest hundredth, how many years will it take the lake to reach 80% of its carrying capacity?

For the following exercises, use a graphing utility to create a scatter diagram of the data given in the table. Observe the shape of the scatter diagram to determine whether the data is best described by an exponential, logarithmic, or logistic model. Then use the appropriate regression feature to find an equation that models the data. When necessary, round values to five decimal places.

35.

x	$f(x)$
1	20
2	21.6
3	29.2
4	36.4
5	46.6
6	55.7
7	72.6
8	87.1
9	107.2
10	138.1

36.

x	$f(x)$
3	13.98
4	17.84
5	20.01
6	22.7
7	24.1
8	26.15
9	27.37
10	28.38
11	29.97
12	31.07
13	31.43

37.

x	$f(x)$
0	2.2
0.5	2.9
1	3.9
1.5	4.8
2	6.4
3	9.3
4	12.3
5	15
6	16.2
7	17.3
8	17.9