College Algebra 2e

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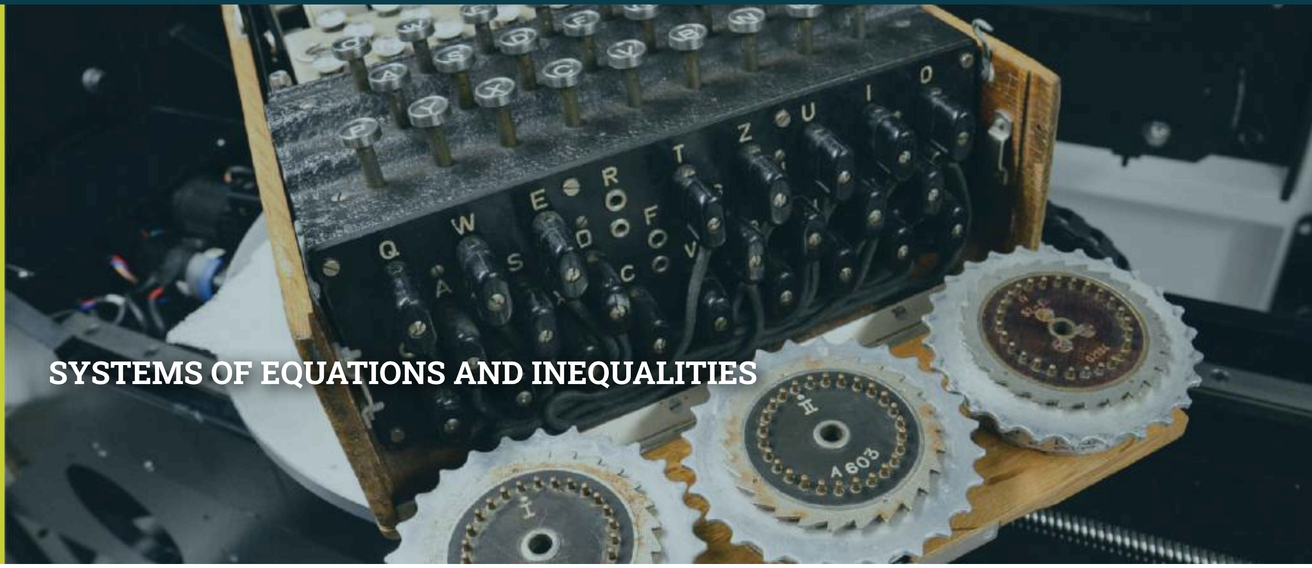


7 SYSTEMS OF EQUATIONS AND INEQUALITIES

Enigma machines like this one were used by government and military officials for enciphering and deciphering top-secret communications during World War II. By varying the combinations of the plugboard and the settings of the rotors, encoders could add complex encryption to their messages. Notice that the three rotors each contain 26 pins, one for each letter of the alphabet; later versions had four and five rotors. (credit: modification of "Enigma Machine" by School of Mathematics, University of Manchester/flickr)

Chapter Outline

Introduction to Systems of Equations and Inequalities

At the start of the Second World War, British military and intelligence officers recognized that defeating Nazi Germany would require the Allies to know what the enemy was planning. This task was complicated by the fact that the German military transmitted all of its communications through a presumably uncrackable code created by a machine called Enigma. The Germans had been encoding their messages with this machine since the early 1930s, and were so confident in its security that they used it for everyday military communications as well as highly important strategic messages. Concerned about the increasing military threat, other European nations began working to decipher the Enigma codes. Poland was the first country to make significant advances when it trained and recruited a new group of codebreakers: math students from Poznań University. With the help of intelligence obtained by French spies, Polish mathematicians, led by Marian Rejewski, were able to decipher initial codes and later to understand the wiring of the machines; eventually they create replicas. However, the German military eventually increased the complexity of the machines by adding additional rotors, requiring a new method of decryption.

The machine attached letters on a keyboard to three, four, or five rotors (depending on the version), each with 26 starting positions that could be set prior to encoding; a decryption code (called a cipher key) essentially conveyed these settings to the message recipient, and allowed people to interpret the message using another Enigma machine. Even with the simpler three-rotor scrambler, there were 17,576 different combinations of starting positions (26 x 26 x 26); plus the machine had numerous other methods of introducing variation. Not long after the war started, the British recruited a team of brilliant codebreakers to crack the Enigma code. The codebreakers, led by Alan Turing, used what they knew about the Enigma machine to build a mechanical computer that could crack the code. And that knowledge of what the Germans were planning proved to be a key part of the ultimate Allied victory of Nazi Germany in 1945.

The Enigma is perhaps the most famous cryptographic device ever known. It stands as an example of the pivotal role cryptography has played in society. Now, technology has moved cryptanalysis to the digital world.

Many ciphers are designed using invertible matrices as the method of message transference, as finding the inverse of a matrix is generally part of the process of decoding. In addition to knowing the matrix and its inverse, the receiver must also know the key that, when used with the matrix inverse, will allow the message to be read.

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions. We will not be breaking any secret codes here, but we will lay the foundation for future courses.

7.1 Systems of Linear Equations: Two Variables

Learning Objectives

In this section, you will:

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

Figure 1 (credit: Thomas Sørenes)

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. A **system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x + y &= 15 \\ 3x - y &= 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair (4, 7) is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$2(4) + (7) = 15 \text{ True}$$
$$3(4) - (7) = 5 \text{ True}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A **consistent system** of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same *y*-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different *y*-intercepts. There are no points common to both lines; hence, there is no solution to the system.

Types of Linear Systems

There are three types of systems of linear equations in two variables, and three types of solutions.

- An **independent system** has exactly one solution pair (x, y) . The point where the two lines intersect is the only solution.
- An **inconsistent system** has no solution. Notice that the two lines are parallel and will never intersect.
- A **dependent system** has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Figure 2 compares graphical representations of each type of system.

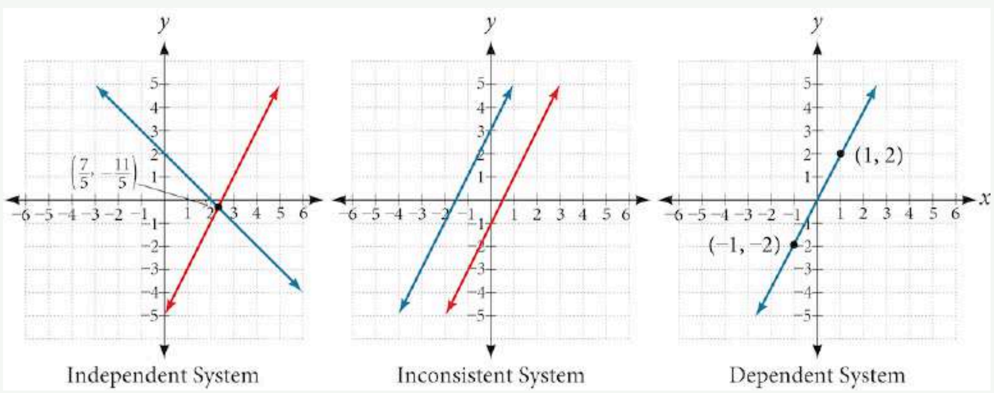


Figure 2

HOW TO

Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution.

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

EXAMPLE 1

Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$\begin{aligned} x + 3y &= 8 \\ 2x - 9 &= y \end{aligned}$$

Solution

Substitute the ordered pair $(5, 1)$ into both equations.

$$\begin{aligned} &(5) + 3(1) = 8 \\ &8 = 8 \qquad \text{True} \\ &2(5) - 9 = (1) \\ &1 = 1 \qquad \text{True} \end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure 3.

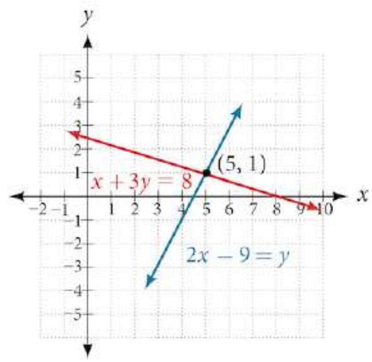


Figure 3

TRY IT #1 Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$\begin{aligned} 5x - 4y &= 20 \\ 2x + 1 &= 3y \end{aligned}$$

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

EXAMPLE 2

Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Solution

Solve the first equation for y.

$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for y.

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in Figure 4.

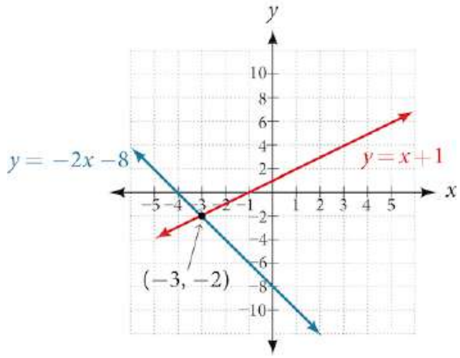


Figure 4

The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3) + (-2) &= -8 \\ -8 &= -8 \qquad \text{True} \\ (-3) - (-2) &= -1 \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

The solution to the system is the ordered pair $(-3,-2)$, so the system is independent.

TRY IT #2 Solve the following system of equations by graphing.

$$\begin{aligned} 2x - 5y &= -25 \\ -4x + 5y &= 35 \end{aligned}$$

Q&A **Can graphing be used if the system is inconsistent or dependent?**

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

HOW TO

Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

EXAMPLE 3

Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{aligned}-x + y &= -5\\ 2x - 5y &= 1\end{aligned}$$

Solution

First, we will solve the first equation for y.

$$\begin{aligned}-x + y &= -5\\ y &= x-5\end{aligned}$$

Now we can substitute the expression $x-5$ for y in the second equation.

$$\begin{aligned}2x - 5y &= 1\\ 2x - 5(x-5) &= 1\\ 2x - 5x + 25 &= 1\\ -3x &= -24\\ x &= 8\end{aligned}$$

Now, we substitute $x = 8$ into the first equation and solve for y.

$$\begin{aligned}-(8) + y &= -5\\ y &= 3\end{aligned}$$

Our solution is $(8, 3)$.

Check the solution by substituting $(8, 3)$ into both equations.

$$\begin{aligned}-x + y &= -5 &&\\ -(8) + (3) &= -5 && \text{True}\\ 2x - 5y &= 1 &&\\ 2(8) - 5(3) &= 1 && \text{True}\end{aligned}$$

TRY IT #3 Solve the following system of equations by substitution.

$$\begin{aligned}x &= y + 3\\ 4 &= 3x-2y\end{aligned}$$

Q&A **Can the substitution method be used to solve any linear system in two variables?**

Yes, but the method works best if one of the equations contains a coefficient of 1 or –1 so that we do not have to deal with fractions.

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the **addition method**. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

HOW TO

Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation

has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.

3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

EXAMPLE 4

Solving a System by the Addition Method

Solve the given system of equations by addition.

$$\begin{aligned} x + 2y &= -1 \\ -x + y &= 3 \end{aligned}$$

Solution

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, –1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{array}{r} x + 2y = -1 \\ \underline{-x + y = 3} \\ 3y = 2 \end{array}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$\begin{aligned} 3y &= 2 \\ y &= \tfrac{2}{3} \end{aligned}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x + y &= 3 \\ -x + \tfrac{2}{3} &= 3 \\ -x &= 3 - \tfrac{2}{3} \\ -x &= \tfrac{7}{3} \\ x &= -\tfrac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} &x + 2y = -1 \\ &\left(-\tfrac{7}{3}\right) + 2\left(\tfrac{2}{3}\right) = \\ &-\tfrac{7}{3} + \tfrac{4}{3} = \\ &-\tfrac{3}{3} = \\ &-1 = -1 \qquad \text{True} \end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 5 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

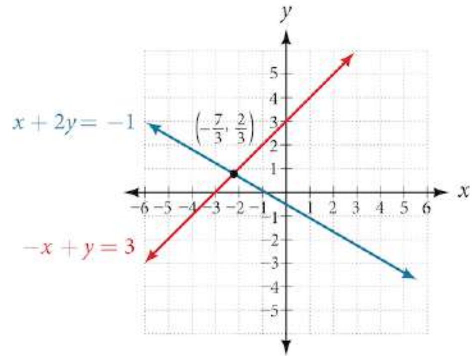


Figure 5

EXAMPLE 5

Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$$3x + 5y = -11$$

$$x - 2y = 11$$

Solution

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3, the x-terms will add to zero.

$$\begin{aligned} x-2y &= 11 & \\ -3(x-2y) &= -3(11) & \text{Multiply both sides by } -3. \\ -3x+6y &= -33 & \text{Use the distributive property.} \end{aligned}$$

Now, let's add them.

$$\begin{array}{r} 3x + 5y = -11 \\ -3x + 6y = -33 \\ \hline 11y = -44 \\ y = -4 \end{array}$$

For the last step, we substitute $y = -4$ into one of the original equations and solve for x.

$$\begin{aligned} 3x + 5y &= -11 \\ 3x + 5(-4) &= -11 \\ 3x - 20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3, -4)$. See Figure 6. Check the solution in the original second equation.

$$\begin{aligned} x - 2y &= 11 & \\ (3) - 2(-4) &= 3 + 8 & \\ 11 &= 11 & \text{True} \end{aligned}$$

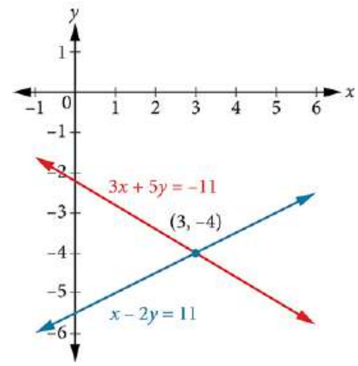


Figure 6

TRY IT #4 Solve the system of equations by addition.

$$\begin{aligned} 2x - 7y &= 2 \\ 3x + y &= -20 \end{aligned}$$

EXAMPLE 6

Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$$\begin{aligned} 2x + 3y &= -16 \\ 5x - 10y &= 30 \end{aligned}$$

Solution

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let’s eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$\begin{aligned} -5(2x + 3y) &= -5(-16) \\ -10x - 15y &= 80 \\ 2(5x - 10y) &= 2(30) \\ 10x - 20y &= 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{aligned} -10x - 15y &= 80 \\ \underline{10x - 20y = 60} \\ -35y &= 140 \\ y &= -4 \end{aligned}$$

Substitute $y = -4$ into the original first equation.

$$\begin{aligned} 2x + 3(-4) &= -16 \\ 2x - 12 &= -16 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

The solution is $(-2, -4)$. Check it in the other equation.

$$\begin{aligned} 5x - 10y &= 30 \\ 5(-2) - 10(-4) &= 30 \\ -10 + 40 &= 30 \\ 30 &= 30 \end{aligned}$$

See Figure 7.

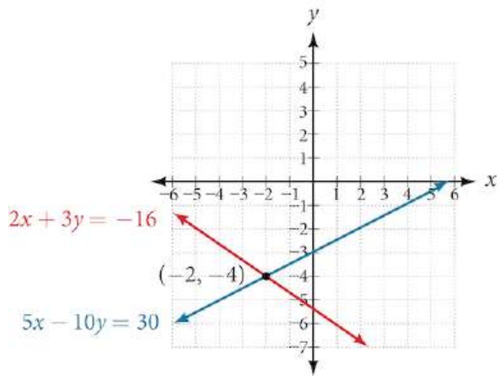


Figure 7

EXAMPLE 7

Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

$$\begin{aligned} \frac{x}{3} + \frac{y}{6} &= 3 \\ \frac{x}{2} - \frac{y}{4} &= 1 \end{aligned}$$

Solution

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$\begin{aligned} 6\left(\frac{x}{3} + \frac{y}{6}\right) &= 6(3) \\ 2x + y &= 18 \\ 4\left(\frac{x}{2} - \frac{y}{4}\right) &= 4(1) \\ 2x - y &= 4 \end{aligned}$$

Now multiply the second equation by -1 so that we can eliminate the x-variable.

$$\begin{aligned} -1(2x - y) &= -1(4) \\ -2x + y &= -4 \end{aligned}$$

Add the two equations to eliminate the x-variable and solve the resulting equation.

$$\begin{array}{l} 2x + y = 18 \\ \underline{-2x + y = -4} \\ 2y = 14 \\ y = 7 \end{array}$$

Substitute $y = 7$ into the first equation.

$$\begin{aligned} 2x + (7) &= 18 \\ 2x &= 11 \\ x &= \frac{11}{2} \\ &= 5.5 \end{aligned}$$

The solution is $\left(\frac{11}{2}, 7\right)$. Check it in the other equation.

$$\begin{aligned} \frac{x}{2} - \frac{y}{4} &= 1 \\ \frac{\frac{11}{2}}{2} - \frac{7}{4} &= 1 \\ \frac{11}{4} - \frac{7}{4} &= 1 \\ \frac{4}{4} &= 1 \end{aligned}$$

TRY IT #5 Solve the system of equations by addition.

$$\begin{aligned} 2x + 3y &= 8 \\ 3x + 5y &= 10 \end{aligned}$$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y -intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

EXAMPLE 8

Solving an Inconsistent System of Equations

Solve the following system of equations.

$$\begin{aligned} x &= 9 - 2y \\ x + 2y &= 13 \end{aligned}$$

Solution

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} x + 2y &= 13 \\ (9 - 2y) + 2y &= 13 \\ 9 + 0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9 - 2y \\ 2y &= -x + 9 \\ y &= -\frac{1}{2}x + \frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x + 2y &= 13 \\ 2y &= -x + 13 \\ y &= -\frac{1}{2}x + \frac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different y-intercepts. Therefore, the lines are parallel and do not intersect.

$$y = -\frac{1}{2}x + \frac{9}{2}$$
$$y = -\frac{1}{2}x + \frac{13}{2}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure 8.

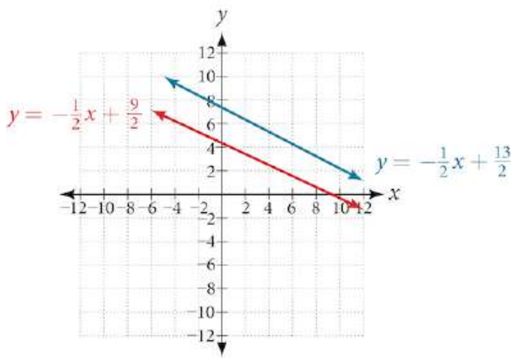


Figure 8

TRY IT #6 Solve the following system of equations in two variables.

$$2y - 2x = 2$$
$$2y - 2x = 6$$

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0 = 0$.

EXAMPLE 9

Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the addition method.

$$x + 3y = 2$$
$$3x + 9y = 6$$

Solution

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x -variable.

$$\begin{aligned} x + 3y &= 2 \\ (-3)(x + 3y) &= (-3)(2) \\ -3x - 9y &= -6 \end{aligned}$$

Now add the equations.

$$\begin{array}{rl} -3x - 9y & = -6 \\ +\ \ 3x + 9y & = 6 \\ \hline 0 & = 0 \end{array}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x + 3y &= 2 \\ 3y &= -x + 2 \\ y &= -\tfrac{1}{3}x + \tfrac{2}{3} \\ 3x + 9y &= 6 \\ 9y &= -3x + 6 \\ y &= -\tfrac{3}{9}x + \tfrac{6}{9} \\ y &= -\tfrac{1}{3}x + \tfrac{2}{3} \end{aligned}$$

See Figure 9. Notice the results are the same. The general solution to the system is $\left(x\ ,-\frac{1}{3}x + \frac{2}{3}\right)$.

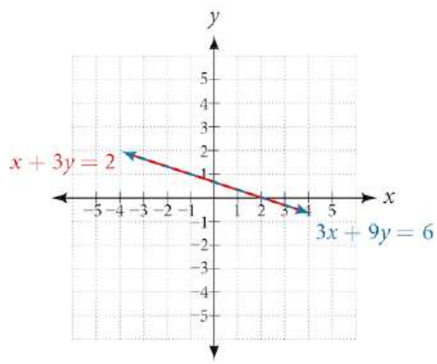


Figure 9

TRY IT #7 Solve the following system of equations in two variables.

$$\begin{aligned} y - 2x &= 5 \\ -3y + 6x &= -15 \end{aligned}$$

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's **revenue function** is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where $x =$ quantity and $p =$ price. The revenue function is shown in orange in Figure 10.

The **cost function** is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 10. The x -axis represents quantity in hundreds of units. The y-axis represents either cost or revenue in hundreds of dollars.

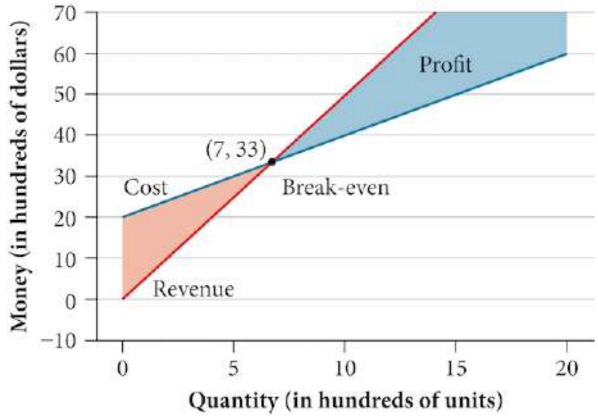


Figure 10

The point at which the two lines intersect is called the **break-even point**. We can see from the graph that if 700 units are produced, the cost is $3,300 and the revenue is also $3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The **profit function** is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

EXAMPLE 10

Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35{,}000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Solution

Write the system of equations using y to replace function notation.

$$\begin{aligned} y &= 0.85x + 35{,}000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35{,}000$ from the first equation into the second equation and solve for x.

$$\begin{aligned} 0.85x + 35{,}000 &= 1.55x \\ 35{,}000 &= 0.7x \\ 50{,}000 &= x \end{aligned}$$

Then, we substitute $x = 50{,}000$ into either the cost function or the revenue function.

$$1.55\,(50{,}000) = 77{,}500$$

The break-even point is $(50{,}000, 77{,}500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$\begin{aligned} P(x) &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

The profit function is $P(x) = 0.7x{-}35{,}000$.

Analysis

The cost to produce 50,000 units is $77,500, and the revenue from the sales of 50,000 units is also $77,500. To make a profit, the business must produce and sell more than 50,000 units. See Figure 11.

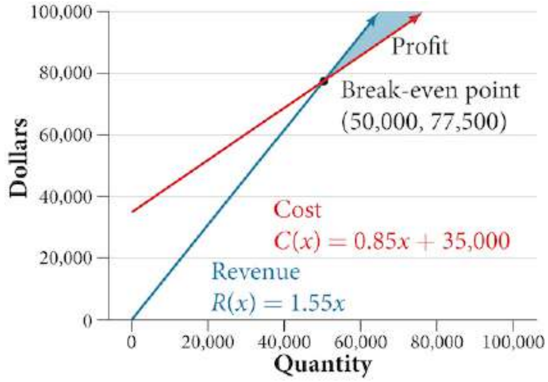


Figure 11

We see from the graph in Figure 12 that the profit function has a negative value until $x = 50{,}000$, when the graph crosses the *x*-axis. Then, the graph emerges into positive *y*-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

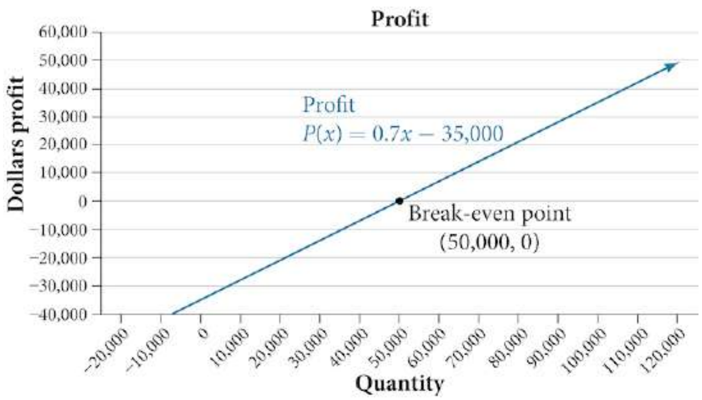


Figure 12

EXAMPLE 11

Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2{,}000$ and the total gate revenue is $\$70{,}000$. How many children and how many adults bought tickets?

Solution

Let *c* = the number of children and *a* = the number of adults in attendance.

The total number of people is $2{,}000$. We can use this to write an equation for the number of people at the circus that day.

$$c + a = 2{,}000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is $\$70{,}000$. We can use this to write an equation for the revenue.

$$25c + 50a = 70{,}000$$

We now have a system of linear equations in two variables.

$$\begin{aligned} c + a &= 2{,}000 \\ 25c + 50a &= 70{,}000 \end{aligned}$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.

$$c + a = 2{,}000$$
$$a = 2{,}000 - c$$

Substitute the expression $2{,}000 - c$ in the second equation for a and solve for c.

$$\begin{aligned} 25c + 50(2{,}000 - c) &= 70{,}000 \\ 25c + 100{,}000 - 50c &= 70{,}000 \\ -25c &= -30{,}000 \\ c &= 1{,}200 \end{aligned}$$

Substitute $c = 1{,}200$ into the first equation to solve for a.

$$\begin{aligned} 1{,}200 + a &= 2{,}000 \\ a &= 800 \end{aligned}$$

We find that $1{,}200$ children and 800 adults bought tickets to the circus that day.

TRY IT #8 Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1{,}650$ meal tickets were bought for a total of $\$14{,}200$, how many children and how many adults bought meal tickets?

MEDIA

Access these online resources for additional instruction and practice with systems of linear equations.

Solving Systems of Equations Using Substitution (http://openstax.org/l/syssubst)
Solving Systems of Equations Using Elimination (http://openstax.org/l/syselim)
Applications of Systems of Equations (http://openstax.org/l/sysapp)

7.1 SECTION EXERCISES

Verbal

1. Can a system of linear equations have exactly two solutions? Explain why or why not.

2. If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company's profit margins.

3. If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?

4. If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?

5. Given a system of equations, explain at least two different methods of solving that system.

Algebraic

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

6. $\begin{aligned} 5x - y &= 4 \\ x + 6y &= 2 \end{aligned}$ and $(4, 0)$

7. $\begin{aligned} -3x - 5y &= 13 \\ -x + 4y &= 10 \end{aligned}$ and $(-6, 1)$

8. $\begin{aligned} 3x + 7y &= 1 \\ 2x + 4y &= 0 \end{aligned}$ and $(2, 3)$

9. $\begin{aligned} -2x + 5y &= 7 \\ 2x + 9y &= 7 \end{aligned}$ and $(-1, 1)$

10. $\begin{aligned} x + 8y &= 43 \\ 3x - 2y &= -1 \end{aligned}$ and $(3, 5)$

For the following exercises, solve each system by substitution.

11. $\begin{aligned} x + 3y &= 5 \\ 2x + 3y &= 4 \end{aligned}$

12. $\begin{aligned} 3x - 2y &= 18 \\ 5x + 10y &= -10 \end{aligned}$

13. $\begin{aligned} 4x + 2y &= -10 \\ 3x + 9y &= 0 \end{aligned}$

14. $\begin{aligned} 2x + 4y &= -3.8 \\ 9x - 5y &= 1.3 \end{aligned}$

15. $\begin{aligned} -2x + 3y &= 1.2 \\ -3x - 6y &= 1.8 \end{aligned}$

16. $\begin{aligned} x - 0.2y &= 1 \\ -10x + 2y &= 5 \end{aligned}$

17. $\begin{aligned} 3x + 5y &= 9 \\ 30x + 50y &= -90 \end{aligned}$

18. $\begin{aligned} -3x + y &= 2 \\ 12x - 4y &= -8 \end{aligned}$

19. $\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{3}y &= 16 \\ \tfrac{1}{6}x + \tfrac{1}{4}y &= 9 \end{aligned}$

20. $\begin{aligned} -\tfrac{1}{4}x + \tfrac{3}{2}y &= 11 \\ -\tfrac{1}{8}x + \tfrac{1}{3}y &= 3 \end{aligned}$

For the following exercises, solve each system by addition.

21. $\begin{aligned} -2x + 5y &= -42 \\ 7x + 2y &= 30 \end{aligned}$

22. $\begin{aligned} 6x - 5y &= -34 \\ 2x + 6y &= 4 \end{aligned}$

23. $\begin{aligned} 5x - y &= -2.6 \\ -4x - 6y &= 1.4 \end{aligned}$

24. $\begin{aligned} 7x - 2y &= 3 \\ 4x + 5y &= 3.25 \end{aligned}$

25. $\begin{aligned} -\mathrm{x} + 2y &= -1 \\ 5x - 10y &= 6 \end{aligned}$

26. $\begin{aligned} 7x + 6y &= 2 \\ -28x - 24y &= -8 \end{aligned}$

27. $\begin{aligned} \tfrac{5}{6}x + \tfrac{1}{4}y &= 0 \\ \tfrac{1}{8}x - \tfrac{1}{2}y &= -\tfrac{43}{120} \end{aligned}$

28. $\begin{aligned} \tfrac{1}{3}x + \tfrac{1}{9}y &= \tfrac{2}{9} \\ -\tfrac{1}{2}x + \tfrac{4}{5}y &= -\tfrac{1}{3} \end{aligned}$

29. $\begin{aligned} -0.2x + 0.4y &= 0.6 \\ x - 2y &= -3 \end{aligned}$

30. $\begin{aligned} -0.1x + 0.2y &= 0.6 \\ 5x - 10y &= 1 \end{aligned}$

For the following exercises, solve each system by any method.

31. $\begin{aligned} 5x + 9y &= 16 \\ x + 2y &= 4 \end{aligned}$

32. $\begin{aligned} 6x - 8y &= -0.6 \\ 3x + 2y &= 0.9 \end{aligned}$

33. $\begin{aligned} 5x - 2y &= 2.25 \\ 7x - 4y &= 3 \end{aligned}$

34. $\begin{aligned} x - \tfrac{5}{12}y &= -\tfrac{55}{12} \\ -6x + \tfrac{5}{2}y &= \tfrac{55}{2} \end{aligned}$

35. $\begin{aligned} 7x - 4y &= \tfrac{7}{6} \\ 2x + 4y &= \tfrac{1}{3} \end{aligned}$

36. $\begin{aligned} 3x + 6y &= 11 \\ 2x + 4y &= 9 \end{aligned}$

37. $$\begin{aligned} \frac{7}{3}x - \frac{1}{6}y &= 2 \\ -\frac{21}{6}x + \frac{3}{12}y &= -3 \end{aligned}$$

38. $$\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &= \frac{1}{3} \\ \frac{3}{2}x + \frac{1}{4}y &= -\frac{1}{8} \end{aligned}$$

39. $$\begin{aligned} 2.2x + 1.3y &= -0.1 \\ 4.2x + 4.2y &= 2.1 \end{aligned}$$

40. $$\begin{aligned} 0.1x + 0.2y &= 2 \\ 0.35x - 0.3y &= 0 \end{aligned}$$

Graphical

For the following exercises, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

41. $$\begin{aligned} 3x - y &= 0.6 \\ x - 2y &= 1.3 \end{aligned}$$

42. $$\begin{aligned} -x + 2y &= 4 \\ 2x - 4y &= 1 \end{aligned}$$

43. $$\begin{aligned} x + 2y &= 7 \\ 2x + 6y &= 12 \end{aligned}$$

44. $$\begin{aligned} 3x - 5y &= 7 \\ x - 2y &= 3 \end{aligned}$$

45. $$\begin{aligned} 3x - 2y &= 5 \\ -9x + 6y &= -15 \end{aligned}$$

Technology

For the following exercises, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

46. $$\begin{aligned} 0.1x + 0.2y &= 0.3 \\ -0.3x + 0.5y &= 1 \end{aligned}$$

47. $$\begin{aligned} -0.01x + 0.12y &= 0.62 \\ 0.15x + 0.20y &= 0.52 \end{aligned}$$

48. $$\begin{aligned} 0.5x + 0.3y &= 4 \\ 0.25x - 0.9y &= 0.46 \end{aligned}$$

49. $$\begin{aligned} 0.15x + 0.27y &= 0.39 \\ -0.34x + 0.56y &= 1.8 \end{aligned}$$

50. $$\begin{aligned} -0.71x + 0.92y &= 0.13 \\ 0.83x + 0.05y &= 2.1 \end{aligned}$$

Extensions

For the following exercises, solve each system in terms of $A, B, C, D, E,$ and F where $A – F$ are nonzero numbers. Note that $A \neq B$ and $AE \neq BD$.

51. $$\begin{aligned} x + y &= A \\ x - y &= B \end{aligned}$$

52. $$\begin{aligned} x + Ay &= 1 \\ x + By &= 1 \end{aligned}$$

53. $$\begin{aligned} Ax + y &= 0 \\ Bx + y &= 1 \end{aligned}$$

54. $$\begin{aligned} Ax + By &= C \\ x + y &= 1 \end{aligned}$$

55. $$\begin{aligned} Ax + By &= C \\ Dx + Ey &= F \end{aligned}$$

Real-World Applications

For the following exercises, solve for the desired quantity.

56. A stuffed animal business has a total cost of production $C = 12x + 30$ and a revenue function $R = 20x$. Find the break-even point.

57. An Ethiopian restaurant has a cost of production $C(x) = 11x + 120$ and a revenue function $R(x) = 5x$. When does the company start to turn a profit?

58. A cell phone factory has a cost of production $C(x) = 150x + 10,000$ and a revenue function $R(x) = 200x$. What is the break-even point?

59. A musician charges $C(x) = 64x + 20{,}000$ where x is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

60. A guitar factory has a cost of production $C(x) = 75x + 50{,}000$. If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the following exercises, use a system of linear equations with two variables and two equations to solve.

61. Find two numbers whose sum is 28 and difference is 13.

62. A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.

63. The startup cost for a restaurant is $120,000, and each meal costs $10 for the restaurant to make. If each meal is then sold for $15, after how many meals does the restaurant break even?

64. A moving company charges a flat rate of $150, and an additional $5 for each box. If a taxi service would charge $20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

65. A total of 1,595 first- and second-year college students gathered at a pep rally. The number of first-years exceeded the number of second-years by 15. How many students from each year group were in attendance?

66. 276 students enrolled in an introductory chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.

67. There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?

68. A jeep and a pickup truck enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the pickup did, and traveled 7 mph slower than the pickup. After 2 hours from the time the pickup entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control and retained the same speed.

69. If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?

70. An investor earned triple the profits of what they earned last year. If they made $500,000.48 total for both years, how much did the investor earn in profits each year?

71. An investor invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?

72. If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and the other that pays $2\frac{7}{8}\%$ interest, and the investor earns $737.50 annual interest, how much was invested in each account?

73. If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?

74. Blu-rays cost $5.96 more than regular DVDs at All Bets Are Off Electronics. How much would 6 Blu-rays and 2 DVDs cost if 5 Blu-rays and 2 DVDs cost $127.73?

75. A store clerk sold 60 pairs of sneakers. The high-tops sold for $98.99 and the low-tops sold for $129.99. If the receipts for the two types of sales totaled $6,404.40, how many of each type of sneaker were sold?

76. A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was $12.50, and the price for an adult ticket was $16.00. The register confirms that $5,075 was taken in. How many student tickets and adult tickets were sold?

77. Admission into an amusement park for 4 children and 2 adults is $116.90. For 6 children and 3 adults, the admission is $175.35. Assuming a different price for children and adults, what is the price of the child’s ticket and the price of the adult ticket?

7.2 Systems of Linear Equations: Three Variables

Learning Objectives

In this section, you will:

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

Figure 1 (credit: “Elembis,” Wikimedia Commons)

Jordi received an inheritance of $12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. Jordi invested $4,000 more in municipal funds than in municipal bonds. He earned $670 in interest the first year. How much did Jordi invest in each type of fund?

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visualization.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z) , which we call an ordered triple. A system in upper triangular form looks like the following:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for z, and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The **solution set** to a three-by-three system is an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

Number of Possible Solutions

Figure 2 and Figure 3 illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a **solution set** consisting of an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

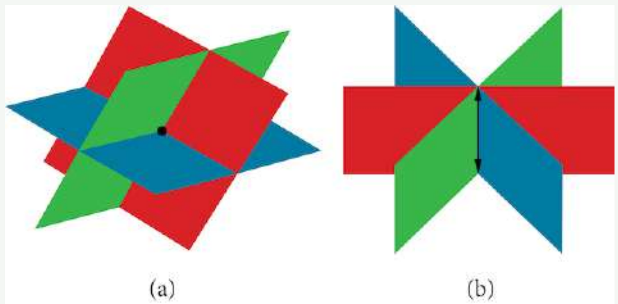


Figure 2 (a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.

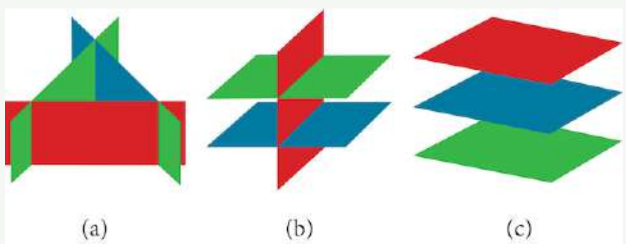


Figure 3 All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

EXAMPLE 1

Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

Solution

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$$\begin{aligned} x + y + z &= 2 \\ (3) + (-2) + (1) &= 2 \\ &\text{True} \end{aligned} \qquad \begin{aligned} 6x - 4y + 5z &= 31 \\ 6(3) - 4(-2) + 5(1) &= 31 \\ 18 + 8 + 5 &= 31 \\ &\text{True} \end{aligned} \qquad \begin{aligned} 5x + 2y + 2z &= 13 \\ 5(3) + 2(-2) + 2(1) &= 13 \\ 15 - 4 + 2 &= 13 \\ &\text{True} \end{aligned}$$

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

HOW TO

Given a linear system of three equations, solve for three unknowns.

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

EXAMPLE 2

Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ -x + 3y - z &= -6 \quad (2) \\ 2x - 5y + 5z &= 17 \quad (3) \end{aligned}$$

Solution

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

$$\begin{array}{rl} x - 2y + 3z = 9 & (1) \\ -x + 3y - z = -6 & (2) \\ \hline y + 2z = 3 & (3) \end{array}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.

$$\begin{array}{rl} -2x + 4y - 6z = -18 & (1) \text{ multiplied by } -2 \\ 2x - 5y + 5z = 17 & (3) \\ \hline -y - z = -1 & (5) \end{array}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.

$$\begin{array}{rl} y + 2z = 3 & (4) \\ -y - z = -1 & (5) \\ \hline z = 2 & (6) \end{array}$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ y + 2z &= 3 \quad (4) \\ z &= 2 \quad (6) \end{aligned}$$

Next, we back-substitute $z = 2$ into equation (4) and solve for y.

$$\begin{aligned} y + 2(2) &= 3 \\ y + 4 &= 3 \\ y &= -1 \end{aligned}$$

Finally, we can back-substitute $z = 2$ and $y = -1$ into equation (1). This will yield the solution for x.

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 \\ x + 2 + 6 &= 9 \\ x &= 1 \end{aligned}$$

The solution is the ordered triple $(1, -1, 2)$. See Figure 4.

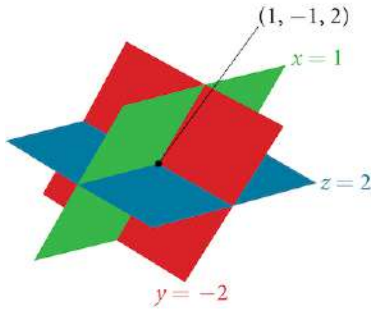


Figure 4

EXAMPLE 3

Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, Jordi invested his inheritance of $12,000 in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. Jordi invested $4,000 more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $670. How much did he invest in each type of fund?

Solution

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$\begin{aligned} x &= \text{amount invested in money-market fund} \\ y &= \text{amount invested in municipal bonds} \\ z &= \text{amount invested in mutual funds} \end{aligned}$$

The first equation indicates that the sum of the three principal amounts is $12,000.

$$x + y + z = 12{,}000$$

We form the second equation according to the information that Jordi invested $4,000 more in mutual funds than he invested in municipal bonds.

$$z = y + 4{,}000$$

The third equation shows that the total amount of interest earned from each fund equals $670.

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.

$$\begin{aligned} x + y + z &= 12{,}000 \\ -y + z &= 4{,}000 \\ 0.03x + 0.04y + 0.07z &= 670 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,

$$\begin{aligned} x + y + z &= 12{,}000 \quad (1) \\ -y + z &= 4{,}000 \quad (2) \\ 3x + 4y + 7z &= 67{,}000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

$$\begin{aligned} x + y + z &= 12{,}000 \\ 3x + 4y + 7z &= 67{,}000 \\ -y + z &= 4{,}000 \end{aligned}$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.

$$\begin{aligned} x + y + z &= 12{,}000 \\ y + 4z &= 31{,}000 \\ -y + z &= 4{,}000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

$$\begin{aligned} x + y + z &= 12{,}000 \\ y + 4z &= 31{,}000 \\ 5z &= 35{,}000 \end{aligned}$$

Step 4. Solve for z in equation (3). Back-substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.

$$\begin{aligned} 5z &= 35{,}000 \\ z &= 7{,}000 \end{aligned}$$

$$\begin{aligned} y + 4(7{,}000) &= 31{,}000 \\ y &= 3{,}000 \end{aligned}$$

$$\begin{aligned} x + 3{,}000 + 7{,}000 &= 12{,}000 \\ x &= 2{,}000 \end{aligned}$$

Jordi invested \$2,000 in a money-market fund, \$3,000 in municipal bonds, and \$7,000 in mutual funds.

TRY IT #1 Solve the system of equations in three variables.

$$\begin{aligned} 2x + y - 2z &= -1 \\ 3x - 3y - z &= 5 \\ x - 2y + 3z &= 6 \end{aligned}$$

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3 = 7$ or some other contradiction.

EXAMPLE 4

Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$$\begin{aligned} x - 3y + z &= 4 && (1) \\ -x + 2y - 5z &= 3 && (2) \\ 5x - 13y + 13z &= 8 && (3) \end{aligned}$$

Solution

Looking at the coefficients of x, we can see that we can eliminate x by adding equation (1) to equation (2).

$$\begin{array}{rl} x - 3y + z = 4 & (1) \\ \underline{-x + 2y - 5z = 3} & \underline{(2)} \\ -y - 4z = 7 & (4) \end{array}$$

Next, we multiply equation (1) by -5 and add it to equation (3).

$$\begin{array}{ll} -5x+15y-5z=-20 & (1)\ \text{multiplied by } -5 \\ \underline{5x-13y+13z=8} & \underline{(3)} \\ 2y+8z=-12 & (5) \end{array}$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$\begin{array}{l} -2y-8z=14(4)\ \text{multiplied by } 2 \\ \underline{2y+8z=-12\ (5)} \\ 0=2 \end{array}$$

The final equation $0=2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

TRY IT #2 Solve the system of three equations in three variables.

$$\begin{aligned} x+y+z&=2 \\ y-3z&=1 \\ 2x+y+5z&=0 \end{aligned}$$

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

EXAMPLE 5

Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$\begin{aligned} 2x+y-3z&=0 & (1) \\ 4x+2y-6z&=0 & (2) \\ x-y+z&=0 & (3) \end{aligned}$$

Solution

First, we can multiply equation (1) by -2 and add it to equation (2).

$$\begin{array}{l} -4x-2y+6z=0\ \text{equation}(1)\ \text{multiplied by } -2 \\ \underline{4x+2y-6z=0\quad (2)} \\ 0=0 \end{array}$$

We do not need to proceed any further. The result we get is an identity, $0=0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2, and adding it to equation (1). We then perform the same steps as above and find the same result, $0=0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$$\begin{array}{c} 2x+y-3z=0 \\ \underline{x-y+z=0} \\ 3x-2z=0 \end{array}$$

We then solve the resulting equation for z.

$$\begin{aligned} 3x - 2z &= 0 \\ z &= \frac{3}{2}x \end{aligned}$$

We back-substitute the expression for z into one of the equations and solve for y.

$$\begin{aligned} 2x + y - 3\left(\frac{3}{2}x\right) &= 0 \\ 2x + y - \frac{9}{2}x &= 0 \\ y &= \frac{9}{2}x - 2x \\ y &= \frac{5}{2}x \end{aligned}$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

Analysis

As shown in Figure 5, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

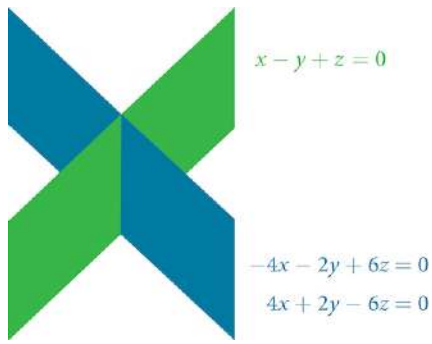


Figure 5

Q&A **Does the generic solution to a dependent system always have to be written in terms of x?**

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

TRY IT #3 Solve the following system.

$$\begin{aligned} x + y + z &= 7 \\ 3x - 2y - z &= 4 \\ x + 6y + 5z &= 24 \end{aligned}$$

MEDIA

Access these online resources for additional instruction and practice with systems of equations in three variables.

Ex 1: System of Three Equations with Three Unknowns Using Elimination (http://openstax.org/l/systhree)
Ex. 2: System of Three Equations with Three Unknowns Using Elimination (http://openstax.org/l/systhelim)

7.2 SECTION EXERCISES

Verbal

1. Can a linear system of three equations have exactly two solutions? Explain why or why not

2. If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.

3. If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.

4. Using the method of addition, is there only one way to solve the system?

5. Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

Algebraic

For the following exercises, determine whether the ordered triple given is the solution to the system of equations.

6. $\begin{array}{r} 2x - 6y + 6z = -12 \\ x + 4y + 5z = -1 \\ -x + 2y + 3z = -1 \end{array}$ and $(0, 1, -1)$

7. $\begin{array}{r} 6x - y + 3z = 6 \\ 3x + 5y + 2z = 0 \\ x + y = 0 \end{array}$ and $(3, -3, -5)$

8. $\begin{array}{r} 6x - 7y + z = 2 \\ -x - y + 3z = 4 \\ 2x + y - z = 1 \end{array}$ and $(4, 2, -6)$

9. $\begin{array}{r} x - y = 0 \\ x - z = 5 \\ x - y + z = -1 \end{array}$ and $(4, 4, -1)$

10. $\begin{array}{r} -x - y + 2z = 3 \\ 5x + 8y - 3z = 4 \\ -x + 3y - 5z = -5 \end{array}$ and $(4, 1, -7)$

For the following exercises, solve each system by elimination.

11. $\begin{array}{r} 3x - 4y + 2z = -15 \\ 2x + 4y + z = 16 \\ 2x + 3y + 5z = 20 \end{array}$

12. $\begin{array}{r} 5x - 2y + 3z = 20 \\ 2x - 4y - 3z = -9 \\ x + 6y - 8z = 21 \end{array}$

13. $\begin{array}{r} 5x + 2y + 4z = 9 \\ -3x + 2y + z = 10 \\ 4x - 3y + 5z = -3 \end{array}$

14. $\begin{array}{r} 4x - 3y + 5z = 31 \\ -x + 2y + 4z = 20 \\ x + 5y - 2z = -29 \end{array}$

15. $\begin{array}{r} 5x - 2y + 3z = 4 \\ -4x + 6y - 7z = -1 \\ 3x + 2y - z = 4 \end{array}$

16. $\begin{array}{r} 4x + 6y + 9z = 0 \\ -5x + 2y - 6z = 3 \\ 7x - 4y + 3z = -3 \end{array}$

For the following exercises, solve each system by Gaussian elimination.

17. $\begin{array}{r} 2x - y + 3z = 17 \\ -5x + 4y - 2z = -46 \\ 2y + 5z = -7 \end{array}$

18. $\begin{array}{r} 5x - 6y + 3z = 50 \\ -x + 4y = 10 \\ 2x - z = 10 \end{array}$

19. $\begin{array}{r} 2x + 3y - 6z = 1 \\ -4x - 6y + 12z = -2 \\ x + 2y + 5z = 10 \end{array}$

20. $\begin{aligned} 4x + 6y - 2z &= 8 \\ 6x + 9y - 3z &= 12 \\ -2x - 3y + z &= -4 \end{aligned}$

21. $\begin{aligned} 2x + 3y - 4z &= 5 \\ -3x + 2y + z &= 11 \\ -x + 5y + 3z &= 4 \end{aligned}$

22. $\begin{aligned} 10x + 2y - 14z &= 8 \\ -x - 2y - 4z &= -1 \\ -12x - 6y + 6z &= -12 \end{aligned}$

23. $\begin{aligned} x + y + z &= 14 \\ 2y + 3z &= -14 \\ -16y - 24z &= -112 \end{aligned}$

24. $\begin{aligned} 5x - 3y + 4z &= -1 \\ -4x + 2y - 3z &= 0 \\ -x + 5y + 7z &= -11 \end{aligned}$

25. $\begin{aligned} x + y + z &= 0 \\ 2x - y + 3z &= 0 \\ x - z &= 0 \end{aligned}$

26. $\begin{aligned} 3x + 2y - 5z &= 6 \\ 5x - 4y + 3z &= -12 \\ 4x + 5y - 2z &= 15 \end{aligned}$

27. $\begin{aligned} x + y + z &= 0 \\ 2x - y + 3z &= 0 \\ x - z &= 1 \end{aligned}$

28. $\begin{aligned} 3x - \tfrac{1}{2}y - z &= -\tfrac{1}{2} \\ 4x + z &= 3 \\ -x + \tfrac{3}{2}y &= \tfrac{5}{2} \end{aligned}$

29. $\begin{aligned} 6x - 5y + 6z &= 38 \\ \tfrac{1}{5}x - \tfrac{1}{2}y + \tfrac{3}{5}z &= 1 \\ -4x - \tfrac{3}{2}y - z &= -74 \end{aligned}$

30. $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{5}y + \tfrac{2}{5}z &= -\tfrac{13}{10} \\ \tfrac{1}{4}x - \tfrac{2}{5}y - \tfrac{1}{5}z &= -\tfrac{7}{20} \\ -\tfrac{1}{2}x - \tfrac{3}{4}y - \tfrac{1}{2}z &= -\tfrac{5}{4} \end{aligned}$

31. $\begin{aligned} -\tfrac{1}{3}x - \tfrac{1}{2}y - \tfrac{1}{4}z &= \tfrac{3}{4} \\ -\tfrac{1}{2}x - \tfrac{1}{4}y - \tfrac{1}{2}z &= 2 \\ -\tfrac{1}{4}x - \tfrac{3}{4}y - \tfrac{1}{2}z &= -\tfrac{1}{2} \end{aligned}$

32. $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{4}y + \tfrac{3}{4}z &= 0 \\ \tfrac{1}{4}x - \tfrac{1}{10}y + \tfrac{2}{5}z &= -2 \\ \tfrac{1}{8}x + \tfrac{1}{5}y - \tfrac{1}{8}z &= 2 \end{aligned}$

33. $\begin{aligned} \tfrac{4}{5}x - \tfrac{7}{8}y + \tfrac{1}{2}z &= 1 \\ -\tfrac{4}{5}x - \tfrac{3}{4}y + \tfrac{1}{3}z &= -8 \\ -\tfrac{2}{5}x - \tfrac{7}{8}y + \tfrac{1}{2}z &= -5 \end{aligned}$

34. $\begin{aligned} -\tfrac{1}{3}x - \tfrac{1}{8}y + \tfrac{1}{6}z &= -\tfrac{4}{3} \\ -\tfrac{2}{3}x - \tfrac{7}{8}y + \tfrac{1}{3}z &= -\tfrac{23}{3} \\ -\tfrac{1}{3}x - \tfrac{5}{8}y + \tfrac{5}{6}z &= 0 \end{aligned}$

35. $\begin{aligned} -\tfrac{1}{4}x - \tfrac{5}{4}y + \tfrac{5}{2}z &= -5 \\ -\tfrac{1}{2}x - \tfrac{5}{3}y + \tfrac{5}{4}z &= \tfrac{55}{12} \\ -\tfrac{1}{3}x - \tfrac{1}{3}y + \tfrac{1}{3}z &= \tfrac{5}{3} \end{aligned}$

36. $\begin{aligned} \tfrac{1}{40}x + \tfrac{1}{60}y + \tfrac{1}{80}z &= \tfrac{1}{100} \\ -\tfrac{1}{2}x - \tfrac{1}{3}y - \tfrac{1}{4}z &= -\tfrac{1}{5} \\ \tfrac{3}{8}x + \tfrac{3}{12}y + \tfrac{3}{16}z &= \tfrac{3}{20} \end{aligned}$

37. $\begin{aligned} 0.1x - 0.2y + 0.3z &= 2 \\ 0.5x - 0.1y + 0.4z &= 8 \\ 0.7x - 0.2y + 0.3z &= 8 \end{aligned}$

38. $\begin{aligned} 0.2x + 0.1y - 0.3z &= 0.2 \\ 0.8x + 0.4y - 1.2z &= 0.1 \\ 1.6x + 0.8y - 2.4z &= 0.2 \end{aligned}$

39. $\begin{aligned} 1.1x + 0.7y - 3.1z &= -1.79 \\ 2.1x + 0.5y - 1.6z &= -0.13 \\ 0.5x + 0.4y - 0.5z &= -0.07 \end{aligned}$

40. $\begin{aligned} 0.5x - 0.5y + 0.5z &= 10 \\ 0.2x - 0.2y + 0.2z &= 4 \\ 0.1x - 0.1y + 0.1z &= 2 \end{aligned}$

41. $\begin{aligned} 0.1x + 0.2y + 0.3z &= 0.37 \\ 0.1x - 0.2y - 0.3z &= -0.27 \\ 0.5x - 0.1y - 0.3z &= -0.03 \end{aligned}$

42. $\begin{aligned} 0.5x - 0.5y - 0.3z &= 0.13 \\ 0.4x - 0.1y - 0.3z &= 0.11 \\ 0.2x - 0.8y - 0.9z &= -0.32 \end{aligned}$

43. $\begin{aligned} 0.5x + 0.2y - 0.3z &= 1 \\ 0.4x - 0.6y + 0.7z &= 0.8 \\ 0.3x - 0.1y - 0.9z &= 0.6 \end{aligned}$

44. $\begin{aligned} 0.3x + 0.3y + 0.5z &= 0.6 \\ 0.4x + 0.4y + 0.4z &= 1.8 \\ 0.4x + 0.2y + 0.1z &= 1.6 \end{aligned}$

45. $\begin{aligned} 0.8x + 0.8y + 0.8z &= 2.4 \\ 0.3x - 0.5y + 0.2z &= 0 \\ 0.1x + 0.2y + 0.3z &= 0.6 \end{aligned}$

Extensions

For the following exercises, solve the system for x, y, and z.

46.
$$\begin{aligned} x+y+z&=3 \\ \frac{x-1}{2}+\frac{y-3}{2}+\frac{z+1}{2}&=0 \\ \frac{x-2}{3}+\frac{y+4}{3}+\frac{z-3}{3}&=\frac{2}{3} \end{aligned}$$

47.
$$\begin{aligned} 5x-3y-\frac{z+1}{2}&=\frac{1}{2} \\ 6x+\frac{y-9}{2}+2z&=-3 \\ \frac{x+8}{2}-4y+z&=4 \end{aligned}$$

48.
$$\begin{aligned} \frac{x+4}{7}-\frac{y-1}{6}+\frac{z+2}{3}&=1 \\ \frac{x-2}{4}+\frac{y+1}{8}-\frac{z+8}{12}&=0 \\ \frac{x+6}{3}-\frac{y+2}{3}+\frac{z+4}{2}&=3 \end{aligned}$$

49.
$$\begin{aligned} \frac{x-3}{6}+\frac{y+2}{2}-\frac{z-3}{3}&=2 \\ \frac{x+2}{4}+\frac{y-5}{2}+\frac{z+4}{2}&=1 \\ \frac{x+6}{2}-\frac{y-3}{2}+z+1&=9 \end{aligned}$$

50.
$$\begin{aligned} \frac{x-1}{3}+\frac{y+3}{4}+\frac{z+2}{6}&=1 \\ 4x+3y-2z&=11 \\ 0.02x+0.015y-0.01z&=0.065 \end{aligned}$$

Real-World Applications

51. Three even numbers sum up to 108. The smaller is half the larger and the middle number is $\frac{3}{4}$ the larger. What are the three numbers?

52. Three numbers sum up to 147. The smallest number is half the middle number, which is half the largest number. What are the three numbers?

53. At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

54. An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?

55. Your roommate, Shani, offered to buy groceries for you and your other roommate. The total bill was \$82. She forgot to save the individual receipts but remembered that your groceries were \$0.05 cheaper than half of her groceries, and that your other roommate's groceries were \$2.10 more than your groceries. How much was each of your share of the groceries?

56. Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was \$100.75. Your supplies were bought with 5% tax, John's with 8% tax, and your third roommate's with 9% sales tax. The total amount of money spent without taxes is \$93.50. If your supplies before tax were \$1 more than half of what your third roommate's supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.

57. Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?

58. At a carnival, $2,914.25 in receipts were taken at the end of the day. The cost of a child's ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult, and senior citizen tickets were sold?

59. A local band sells out for their concert. They sell all 1,175 tickets for a total purse of $28,112.50. The tickets were priced at $20 for student tickets, $22.50 for children, and $29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?

60. In a bag, a child has 325 coins worth $19.50. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?

61. Last year, at Haven's Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of $140,000. This year, due to inflation, the same cars would cost $151,830. The cost of the BMW increased by 8%, the Jeep by 5%, and the Toyota by 12%. If the price of last year's Jeep was $7,000 less than the price of last year's BMW, what was the price of each of the three cars last year?

62. When his youngest child moved out, Deandre sold his home and made three investments using gains from the sale. He invested $80,500 into three accounts, one that paid 4% simple interest, one that paid $3\frac{1}{8}\%$ simple interest, and one that paid $2\frac{1}{2}\%$ simple interest. He earned $2,670 interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

63. You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn \$34,000 in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?

64. An entrepreneur sells a portion of their business for one hundred thousand dollars and invests it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, the entrepreneur earns \$3,650 in interest. If they invested five times the money in the account that pays 4% compared to 3%, how much did they invest in each account?

65. The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?[1]

66. The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world's produced oil. Saudi Arabia and the United States combined for 22.1% of the world's production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?[2]

67. The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percent of the United States oil imports were from these three countries?[3]

68. The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?[4]

1 "Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

2 "Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

3 "Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

4 "USA: The coming global oil crisis," accessed April 6, 2014, http://www.oilcrisis.com/us/.

69. At one time, in the United States, 398 species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised 55% of the endangered species. Birds accounted for 0.7% more than fish, and fish accounted for 1.5% more than mammals. What percent of the endangered species came from mammals, birds, and fish?

70. Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up 4% less than one-quarter of poultry consumption, and red meat consumption is 18.2% higher than poultry consumption, what are the percentages of meat consumption?[5]

7.3 Systems of Nonlinear Equations and Inequalities: Two Variables

Learning Objectives

In this section, you will:

- Solve a system of nonlinear equations using substitution.
- Solve a system of nonlinear equations using elimination.
- Graph a nonlinear inequality.
- Graph a system of nonlinear inequalities.

Halley's Comet (Figure 1) orbits the sun about once every 75 years. Its path can be considered to be a very elongated ellipse. Other comets follow similar paths in space. These orbital paths can be studied using systems of equations. These systems, however, are different from the ones we considered in the previous section because the equations are not linear.

Figure 1 Halley's Comet (credit: "NASA Blueshift"/Flickr)

In this section, we will consider the intersection of a parabola and a line, a circle and a line, and a circle and an ellipse. The methods for solving systems of nonlinear equations are similar to those for linear equations.

Solving a System of Nonlinear Equations Using Substitution

A **system of nonlinear equations** is a system of two or more equations in two or more variables containing at least one equation that is not linear. Recall that a linear equation can take the form $Ax + By + C = 0$. Any equation that cannot be written in this form in nonlinear. The substitution method we used for linear systems is the same method we will use for nonlinear systems. We solve one equation for one variable and then substitute the result into the second equation to solve for another variable, and so on. There is, however, a variation in the possible outcomes.

Intersection of a Parabola and a Line

There are three possible types of solutions for a system of nonlinear equations involving a parabola and a line.

5 "The United States Meat Industry at a Glance," accessed April 6, 2014, http://www.meatami.com/ht/d/sp/i/47465/pid/47465.

Possible Types of Solutions for Points of Intersection of a Parabola and a Line

Figure 2 illustrates possible solution sets for a system of equations involving a parabola and a line.

- No solution. The line will never intersect the parabola.
- One solution. The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions. The line crosses on the inside of the parabola and intersects the parabola at two points.

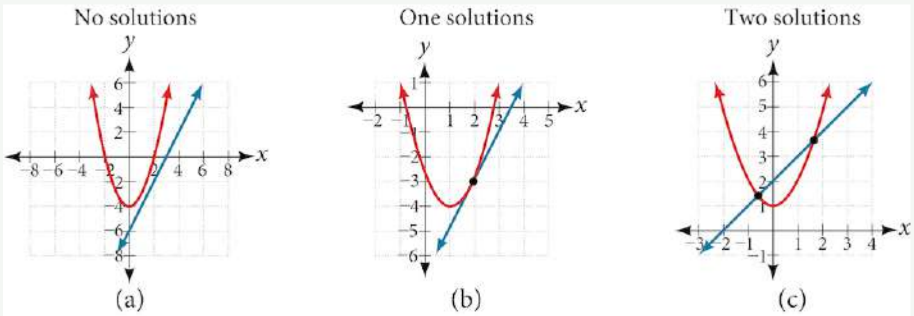


Figure 2

HOW TO

Given a system of equations containing a line and a parabola, find the solution.

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the parabola equation.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

EXAMPLE 1

Solving a System of Nonlinear Equations Representing a Parabola and a Line

Solve the system of equations.

$$\begin{aligned} x - y &= -1 \\ y &= x^2 + 1 \end{aligned}$$

Solution

Solve the first equation for x and then substitute the resulting expression into the second equation.

$$\begin{aligned} x - y &= -1 \\ x &= y - 1 && \text{Solve for } x. \\ \\ y &= x^2 + 1 \\ y &= (y-1)^2 + 1 && \text{Substitute expression for } x. \end{aligned}$$

Expand the equation and set it equal to zero.

$$\begin{aligned} y &= (y-1)^2 + 1 \\ &= (y^2 - 2y + 1) + 1 \\ &= y^2 - 2y + 2 \\ 0 &= y^2 - 3y + 2 \\ &= (y-2)(y-1) \end{aligned}$$

Solving for y gives $y = 2$ and $y = 1$. Next, substitute each value for y into the first equation to solve for x. Always substitute the value into the linear equation to check for extraneous solutions.

$$\begin{aligned} x - y &= -1 \\ x - (2) &= -1 \\ x &= 1 \end{aligned}$$

$$\begin{aligned} x - (1) &= -1 \\ x &= 0 \end{aligned}$$

The solutions are $(1, 2)$ and $(0, 1)$, which can be verified by substituting these (x, y) values into both of the original equations. See Figure 3.

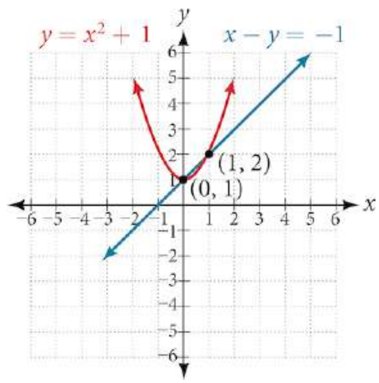


Figure 3

Q&A **Could we have substituted values for y into the second equation to solve for x in Example 1?**

Yes, but because x is squared in the second equation this could give us extraneous solutions for x.

For $y = 1$

$$\begin{aligned} y &= x^2 + 1 \\ 1 &= x^2 + 1 \\ x^2 &= 0 \\ x &= \pm\sqrt{0} = 0 \end{aligned}$$

This gives us the same value as in the solution.

For $y = 2$

$$\begin{aligned} y &= x^2 + 1 \\ 2 &= x^2 + 1 \\ x^2 &= 1 \\ x &= \pm\sqrt{1} = \pm 1 \end{aligned}$$

Notice that -1 is an extraneous solution.

TRY IT #1 Solve the given system of equations by substitution.

$$\begin{aligned} 3x - y &= -2 \\ 2x^2 - y &= 0 \end{aligned}$$

Intersection of a Circle and a Line

Just as with a parabola and a line, there are three possible outcomes when solving a system of equations representing a circle and a line.

Possible Types of Solutions for the Points of Intersection of a Circle and a Line

Figure 4 illustrates possible solution sets for a system of equations involving a circle and a line.

- No solution. The line does not intersect the circle.
- One solution. The line is tangent to the circle and intersects the circle at exactly one point.
- Two solutions. The line crosses the circle and intersects it at two points.

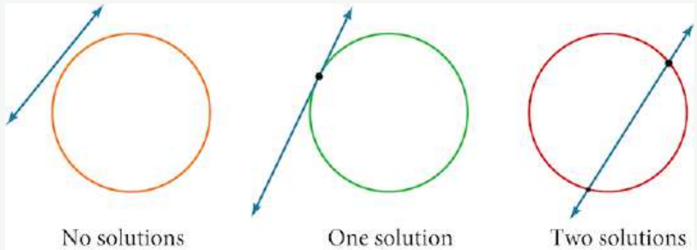


Figure 4

HOW TO

Given a system of equations containing a line and a circle, find the solution.

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the equation for the circle.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

EXAMPLE 2

Finding the Intersection of a Circle and a Line by Substitution

Find the intersection of the given circle and the given line by substitution.

$$\begin{aligned} x^2 + y^2 &= 5 \\ y &= 3x-5 \end{aligned}$$

Solution

One of the equations has already been solved for y. We will substitute $y = 3x-5$ into the equation for the circle.

$$\begin{aligned} x^2 + (3x-5)^2 &= 5 \\ x^2 + 9x^2 - 30x + 25 &= 5 \\ 10x^2 - 30x + 20 &= 0 \end{aligned}$$

Now, we factor and solve for x.

$$\begin{aligned} &10(x^2 - 3x + 2) = 0 \\ &10(x - 2)(x - 1) = 0 \\ &x = 2 \\ &x = 1 \end{aligned}$$

Substitute the two x-values into the original linear equation to solve for y.

$$\begin{aligned} y &= 3(2) - 5 \\ &= 1 \end{aligned}$$

$$\begin{aligned} y &= 3(1) - 5 \\ &= -2 \end{aligned}$$

The line intersects the circle at $(2, 1)$ and $(1, -2)$, which can be verified by substituting these (x, y) values into both of the original equations. See Figure 5.

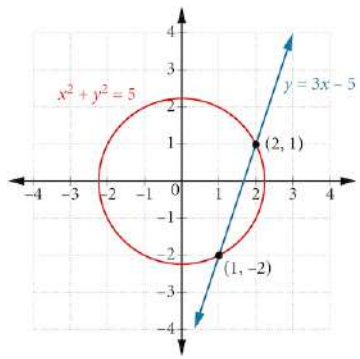


Figure 5

TRY IT #2 Solve the system of nonlinear equations.

$$\begin{aligned} x^2 + y^2 &= 10 \\ x - 3y &= -10 \end{aligned}$$

Solving a System of Nonlinear Equations Using Elimination

We have seen that substitution is often the preferred method when a system of equations includes a linear equation and a nonlinear equation. However, when both equations in the system have like variables of the second degree, solving them using elimination by addition is often easier than substitution. Generally, elimination is a far simpler method when the system involves only two equations in two variables (a two-by-two system), rather than a three-by-three system, as there are fewer steps. As an example, we will investigate the possible types of solutions when solving a system of equations representing a circle and an ellipse.

Possible Types of Solutions for the Points of Intersection of a Circle and an Ellipse

Figure 6 illustrates possible solution sets for a system of equations involving a circle and an ellipse.

- No solution. The circle and ellipse do not intersect. One shape is inside the other or the circle and the ellipse are a distance away from the other.
- One solution. The circle and ellipse are tangent to each other, and intersect at exactly one point.
- Two solutions. The circle and the ellipse intersect at two points.
- Three solutions. The circle and the ellipse intersect at three points.
- Four solutions. The circle and the ellipse intersect at four points.

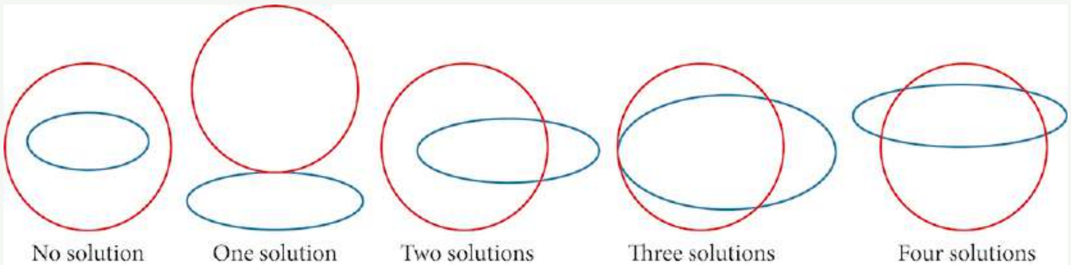


Figure 6

EXAMPLE 3

Solving a System of Nonlinear Equations Representing a Circle and an Ellipse

Solve the system of nonlinear equations.

$$\begin{aligned} x^2 + y^2 &= 26 \quad (1) \\ 3x^2 + 25y^2 &= 100 \quad (2) \end{aligned}$$

Solution

Let's begin by multiplying equation (1) by -3, and adding it to equation (2).

$$\begin{aligned} (-3)(x^2 + y^2) &= (-3)(26) \\ -3x^2 - 3y^2 &= -78 \\ 3x^2 + 25y^2 &= 100 \\ \hline 22y^2 &= 22 \end{aligned}$$

After we add the two equations together, we solve for y.

$$\begin{aligned} y^2 &= 1 \\ y &= \pm\sqrt{1} = \pm 1 \end{aligned}$$

Substitute $y = \pm 1$ into one of the equations and solve for x.

$$\begin{aligned} x^2 + (1)^2 &= 26 \\ x^2 + 1 &= 26 \\ x^2 &= 25 \\ x &= \pm\sqrt{25} = \pm 5 \end{aligned}$$

$$\begin{aligned} x^2 + (-1)^2 &= 26 \\ x^2 + 1 &= 26 \\ x^2 &= 25 = \pm 5 \end{aligned}$$

There are four solutions: $(5, 1)$, $(-5, 1)$, $(5, -1)$, and $(-5, -1)$. See Figure 7.

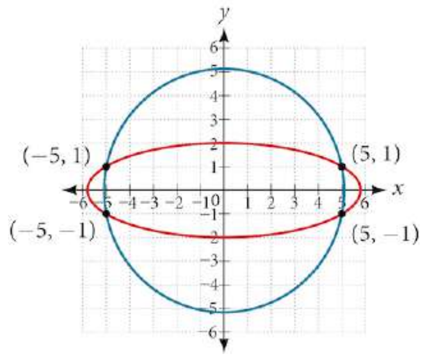


Figure 7

TRY IT #3 Find the solution set for the given system of nonlinear equations.

$$\begin{aligned} 4x^2 + y^2 &= 13 \\ x^2 + y^2 &= 10 \end{aligned}$$

Graphing a Nonlinear Inequality

All of the equations in the systems that we have encountered so far have involved equalities, but we may also encounter systems that involve inequalities. We have already learned to graph linear inequalities by graphing the corresponding equation, and then shading the region represented by the inequality symbol. Now, we will follow similar steps to graph a nonlinear inequality so that we can learn to solve systems of nonlinear inequalities. A **nonlinear inequality** is an inequality containing a nonlinear expression. Graphing a nonlinear inequality is much like graphing a linear inequality.

Recall that when the inequality is greater than, $y > a$, or less than, $y < a$, the graph is drawn with a dashed line. When the inequality is greater than or equal to, $y \geq a$, or less than or equal to, $y \leq a$, the graph is drawn with a solid line. The graphs will create regions in the plane, and we will test each region for a solution. If one point in the region works, the whole region works. That is the region we shade. See Figure 8.

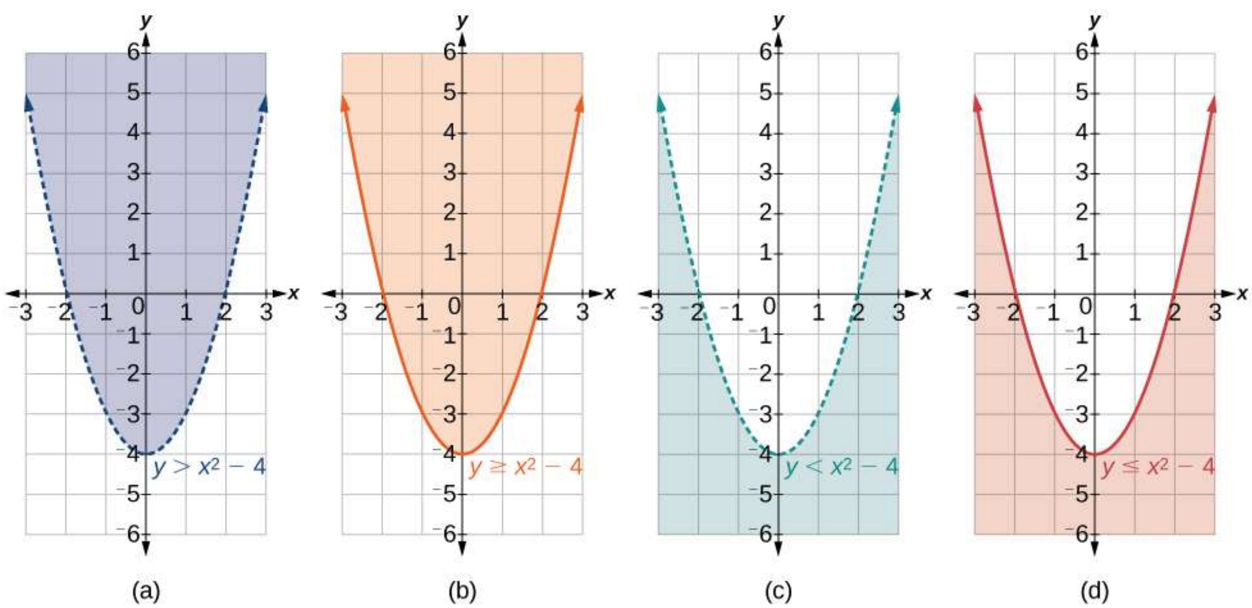


Figure 8 (a) an example of $y > a$; (b) an example of $y \geq a$; (c) an example of $y < a$; (d) an example of $y \leq a$

HOW TO

Given an inequality bounded by a parabola, sketch a graph.

1. Graph the parabola as if it were an equation. This is the boundary for the region that is the solution set.
2. If the boundary is included in the region (the operator is $\leq$ or $\geq$), the parabola is graphed as a solid line.
3. If the boundary is not included in the region (the operator is < or >), the parabola is graphed as a dashed line.
4. Test a point in one of the regions to determine whether it satisfies the inequality statement. If the statement is true, the solution set is the region including the point. If the statement is false, the solution set is the region on the other side of the boundary line.
5. Shade the region representing the solution set.

EXAMPLE 4

Graphing an Inequality for a Parabola

Graph the inequality $y > x^2 + 1$.

Solution

First, graph the corresponding equation $y = x^2 + 1$. Since $y > x^2 + 1$ has a greater than symbol, we draw the graph with a dashed line. Then we choose points to test both inside and outside the parabola. Let's test the points $(0, 2)$ and $(2, 0)$. One point is clearly inside the parabola and the other point is clearly outside.

$$\begin{aligned} y &> x^2 + 1 \\ 2 &> (0)^2 + 1 \\ 2 &> 1 \qquad \text{True} \end{aligned}$$

$$\begin{aligned} 0 &> (2)^2 + 1 \\ 0 &> 5 \qquad \text{False} \end{aligned}$$

The graph is shown in Figure 9. We can see that the solution set consists of all points inside the parabola, but not on the graph itself.

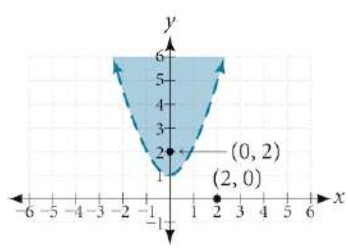


Figure 9

Graphing a System of Nonlinear Inequalities

Now that we have learned to graph nonlinear inequalities, we can learn how to graph systems of nonlinear inequalities. A **system of nonlinear inequalities** is a system of two or more inequalities in two or more variables containing at least one inequality that is not linear. Graphing a system of nonlinear inequalities is similar to graphing a system of linear inequalities. The difference is that our graph may result in more shaded regions that represent a solution than we find in a system of linear inequalities. The solution to a nonlinear system of inequalities is the region of the graph where the shaded regions of the graph of each inequality overlap, or where the regions intersect, called the **feasible region**.

HOW TO

Given a system of nonlinear inequalities, sketch a graph.

1. Find the intersection points by solving the corresponding system of nonlinear equations.
2. Graph the nonlinear equations.
3. Find the shaded regions of each inequality.
4. Identify the feasible region as the intersection of the shaded regions of each inequality or the set of points common to each inequality.

EXAMPLE 5

Graphing a System of Inequalities

Graph the given system of inequalities.

$$\begin{aligned} x^2 - y &\leq 0 \\ 2x^2 + y &\leq 12 \end{aligned}$$

Solution

These two equations are clearly parabolas. We can find the points of intersection by the elimination process: Add both equations and the variable y will be eliminated. Then we solve for x.

$$\begin{aligned} x^2 - y &= 0 \\ 2x^2 + y &= 12 \\ \hline 3x^2 &= 12 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Substitute the x-values into one of the equations and solve for y.

$$\begin{aligned} x^2 - y &= 0 \\ (2)^2 - y &= 0 \\ 4 - y &= 0 \\ y &= 4 \\ \\ (-2)^2 - y &= 0 \\ 4 - y &= 0 \\ y &= 4 \end{aligned}$$

The two points of intersection are $(2, 4)$ and $(-2, 4)$. Notice that the equations can be rewritten as follows.

$$\begin{aligned} &x^2 - y \leq 0 \\ &x^2 \leq y \\ &y \geq x^2 \\ \\ &2x^2 + y \leq 12 \\ &y \leq -2x^2 + 12 \end{aligned}$$

Graph each inequality. See Figure 10. The feasible region is the region between the two equations bounded by $2x^2 + y \leq 12$ on the top and $x^2 - y \leq 0$ on the bottom.

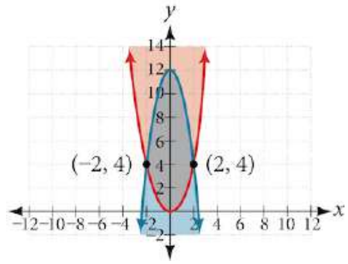


Figure 10

TRY IT #4 Graph the given system of inequalities.

$$\begin{aligned} y &\geq x^2 - 1 \\ x - y &\geq -1 \end{aligned}$$

MEDIA

Access these online resources for additional instruction and practice with nonlinear equations.

[Solve a System of Nonlinear Equations Using Substitution (http://openstax.org/l/nonlinsub)](http://openstax.org/l/nonlinsub)
[Solve a System of Nonlinear Equations Using Elimination (http://openstax.org/l/nonlinelim)](http://openstax.org/l/nonlinelim)

7.3 SECTION EXERCISES

Verbal

1. Explain whether a system of two nonlinear equations can have exactly two solutions. What about exactly three? If not, explain why not. If so, give an example of such a system, in graph form, and explain why your choice gives two or three answers.

2. When graphing an inequality, explain why we only need to test one point to determine whether an entire region is the solution?

3. When you graph a system of inequalities, will there always be a feasible region? If so, explain why. If not, give an example of a graph of inequalities that does not have a feasible region. Why does it not have a feasible region?

4. If you graph a revenue and cost function, explain how to determine in what regions there is profit.

5. If you perform your break-even analysis and there is more than one solution, explain how you would determine which *x*-values are profit and which are not.

Algebraic

For the following exercises, solve the system of nonlinear equations using substitution.

6. $\begin{aligned} x + y &= 4 \\ x^2 + y^2 &= 9 \end{aligned}$

7. $\begin{aligned} y &= x - 3 \\ x^2 + y^2 &= 9 \end{aligned}$

8. $\begin{aligned} y &= x \\ x^2 + y^2 &= 9 \end{aligned}$

9. $\begin{aligned} y &= -x \\ x^2 + y^2 &= 9 \end{aligned}$

10. $\begin{aligned} x &= 2 \\ x^2 - y^2 &= 9 \end{aligned}$

For the following exercises, solve the system of nonlinear equations using elimination.

11. $\begin{aligned} 4x^2 - 9y^2 &= 36 \\ 4x^2 + 9y^2 &= 36 \end{aligned}$

12. $\begin{aligned} x^2 + y^2 &= 25 \\ x^2 - y^2 &= 1 \end{aligned}$

13. $\begin{aligned} 2x^2 + 4y^2 &= 4 \\ 2x^2 - 4y^2 &= 25x - 10 \end{aligned}$

14. $\begin{aligned} y^2 - x^2 &= 9 \\ 3x^2 + 2y^2 &= 8 \end{aligned}$

15. $\begin{aligned} x^2 + y^2 + \tfrac{1}{16} &= 2500 \\ y &= 2x^2 \end{aligned}$

For the following exercises, use any method to solve the system of nonlinear equations.

16. $\begin{aligned} -2x^2 + y &= -5 \\ 6x - y &= 9 \end{aligned}$

17. $\begin{aligned} -x^2 + y &= 2 \\ -x + y &= 2 \end{aligned}$

18. $\begin{aligned} x^2 + y^2 &= 1 \\ y &= 20x^2 - 1 \end{aligned}$

19. $\begin{aligned} x^2 + y^2 &= 1 \\ y &= -x^2 \end{aligned}$

20. $\begin{aligned} 2x^3 - x^2 &= y \\ y &= \tfrac{1}{2} - x \end{aligned}$

21. $\begin{aligned} 9x^2 + 25y^2 &= 225 \\ (x-6)^2 + y^2 &= 1 \end{aligned}$

22. $\begin{aligned} x^4 - x^2 &= y \\ x^2 + y &= 0 \end{aligned}$

23. $\begin{aligned} 2x^3 - x^2 &= y \\ x^2 + y &= 0 \end{aligned}$

For the following exercises, use any method to solve the nonlinear system.

24. $\begin{aligned} x^2 + y^2 &= 9 \\ y &= 3 - x^2 \end{aligned}$

25. $\begin{aligned} x^2 - y^2 &= 9 \\ x &= 3 \end{aligned}$

26. $\begin{aligned} x^2 - y^2 &= 9 \\ y &= 3 \end{aligned}$

27. $\begin{aligned} x^2 - y^2 &= 9 \\ x - y &= 0 \end{aligned}$

28. $\begin{aligned} -x^2 + y &= 2 \\ -4x + y &= -1 \end{aligned}$

29. $\begin{aligned} -x^2 + y &= 2 \\ 2y &= -x \end{aligned}$

30. $\begin{aligned} x^2 + y^2 &= 25 \\ x^2 - y^2 &= 36 \end{aligned}$

31. $\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= x^2 \end{aligned}$

32. $\begin{aligned} 16x^2 - 9y^2 + 144 &= 0 \\ y^2 + x^2 &= 16 \end{aligned}$

33. $\begin{aligned} 3x^2 - y^2 &= 12 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$

34. $\begin{aligned} 3x^2 - y^2 &= 12 \\ (x-1)^2 + y^2 &= 4 \end{aligned}$

35. $\begin{aligned} 3x^2 - y^2 &= 12 \\ x^2 + y^2 &= 16 \end{aligned}$

36. $\begin{aligned} x^2 - y^2 - 6x - 4y - 11 &= 0 \\ -x^2 + y^2 &= 5 \end{aligned}$

37. $\begin{aligned} x^2 + y^2 - 6y &= 7 \\ x^2 + y &= 1 \end{aligned}$

38. $\begin{aligned} x^2 + y^2 &= 6 \\ xy &= 1 \end{aligned}$

Graphical

For the following exercises, graph the inequality.

39. $x^2 + y < 9$

40. $x^2 + y^2 < 4$

For the following exercises, graph the system of inequalities. Label all points of intersection.

41. $x^2 + y < 1$
$y > 2x$

42. $x^2 + y < -5$
$y > 5x + 10$

43. $x^2 + y^2 < 25$
$3x^2 - y^2 > 12$

44. $x^2 - y^2 > -4$
$x^2 + y^2 < 12$

45. $x^2 + 3y^2 > 16$
$3x^2 - y^2 < 1$

Extensions

For the following exercises, graph the inequality.

46. $y \geq e^x$
$y \leq \ln(x) + 5$

47. $y \leq -\log(x)$
$y \leq e^x$

For the following exercises, find the solutions to the nonlinear equations with two variables.

48. $\frac{4}{x^2} + \frac{1}{y^2} = 24$
$\frac{5}{x^2} - \frac{2}{y^2} + 4 = 0$

49. $\frac{6}{x^2} - \frac{1}{y^2} = 8$
$\frac{1}{x^2} - \frac{6}{y^2} = \frac{1}{8}$

50. $x^2 - xy + y^2 - 2 = 0$
$x + 3y = 4$

51. $x^2 - xy - 2y^2 - 6 = 0$
$x^2 + y^2 = 1$

52. $x^2 + 4xy - 2y^2 - 6 = 0$
$x = y + 2$

Technology

For the following exercises, solve the system of inequalities. Use a calculator to graph the system to confirm the answer.

53. $xy < 1$
$y > \sqrt{x}$

54. $x^2 + y < 3$
$y > 2x$

Real-World Applications

For the following exercises, construct a system of nonlinear equations to describe the given behavior, then solve for the requested solutions.

55. Two numbers add up to 300. One number is twice the square of the other number. What are the numbers?

56. The squares of two numbers add to 360. The second number is half the value of the first number squared. What are the numbers?

57. A laptop company has discovered their cost and revenue functions for each day: $C(x) = 3x^2 - 10x + 200$ and $R(x) = -2x^2 + 100x + 50$. If they want to make a profit, what is the range of laptops per day that they should produce? Round to the nearest number which would generate profit.

58. A cell phone company has the following cost and revenue functions: $C(x) = 8x^2 - 600x + 21{,}500$ and $R(x) = -3x^2 + 480x$. What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.

7.4 Partial Fractions

Learning Objectives

In this section, you will:

- Decompose $\frac{P(x)}{Q(x)}$, where $Q(x)$ has only nonrepeated linear factors.
- Decompose $\frac{P(x)}{Q(x)}$, where $Q(x)$ has repeated linear factors.
- Decompose $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a nonrepeated irreducible quadratic factor.
- Decompose $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a repeated irreducible quadratic factor.

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized—the decomposition of rational expressions.

Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

Decomposing $\frac{P(x)}{Q(x)}$ Where *Q(x)* Has Only Nonrepeated Linear Factors

Recall the algebra regarding adding and subtracting rational expressions. These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at partial fraction decomposition, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified rational expression to the original expressions, called the **partial fraction**.

For example, suppose we add the following fractions:

$$\frac{2}{x-3}+\frac{-1}{x+2}$$

We would first need to find a common denominator, $(x+2)(x-3)$.

Next, we would write each expression with this common denominator and find the sum of the terms.

$$\begin{aligned}\frac{2}{x-3}\left(\frac{x+2}{x+2}\right)+\frac{-1}{x+2}\left(\frac{x-3}{x-3}\right) &=\\ \frac{2x+4-x+3}{(x+2)(x-3)} &= \frac{x+7}{x^2-x-6}\end{aligned}$$

Partial fraction decomposition is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.

$$\underset{\text{Simplified sum}}{\frac{x+7}{x^2-x-6}} \qquad = \qquad \underset{\text{Partial fraction decomposition}}{\frac{2}{x-3}+\frac{-1}{x+2}}$$

We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of x^2-x-6 are $(x-3)(x+2)$, the denominators of the decomposed rational expression. So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.

Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has Nonrepeated Linear Factors

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ when $Q(x)$ has nonrepeated linear factors and the degree of $P(x)$ is less than the degree of $Q(x)$ is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1 x + b_1)} + \frac{A_2}{(a_2 x + b_2)} + \frac{A_3}{(a_3 x + b_3)} + \cdots + \frac{A_n}{(a_n x + b_n)}.$$

HOW TO

Given a rational expression with distinct linear factors in the denominator, decompose it.

1. Use a variable for the original numerators, usually $A, B,$ or C, depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use A_n for each numerator
$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1 x + b_1)} + \frac{A_2}{(a_2 x + b_2)} + \cdots + \frac{A_n}{(a_n x + b_n)}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

EXAMPLE 1

Decomposing a Rational Function with Distinct Linear Factors

Decompose the given rational expression with distinct linear factors.

$$\frac{3x}{(x+2)(x-1)}$$

Solution

We will separate the denominator factors and give each numerator a symbolic label, like A, B, or C.

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$$

Multiply both sides of the equation by the common denominator to eliminate the fractions:

$$(x+2)(x-1)\left[\frac{3x}{(x+2)(x-1)}\right] = \cancel{(x+2)}(x-1)\left[\frac{A}{\cancel{(x+2)}}\right] + (x+2)\cancel{(x-1)}\left[\frac{B}{\cancel{(x-1)}}\right]$$

The resulting equation is

$$3x = A(x-1) + B(x+2)$$

Expand the right side of the equation and collect like terms.

$$\begin{aligned} 3x &= Ax - A + Bx + 2B \\ 3x &= (A+B)x - A + 2B \end{aligned}$$

Set up a system of equations associating corresponding coefficients.

$$\begin{aligned} 3 &= \quad A + B \\ 0 &= -A + 2B \end{aligned}$$

Add the two equations and solve for B.

$$\begin{array}{l} 3 = \quad A + B \\ \underline{0 = -A + 2B} \\ 3 \;=\; \quad 0 \quad + 3B \\ 1 = B \end{array}$$

Substitute $B = 1$ into one of the original equations in the system.

$$3 = A + 1$$
$$2 = A$$

Thus, the partial fraction decomposition is

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Another method to use to solve for A or B is by considering the equation that resulted from eliminating the fractions and substituting a value for x that will make either the A- or B-term equal 0. If we let $x = 1$, the A- term becomes 0 and we can simply solve for B.

$$\begin{aligned} 3x &= A(x-1) + B(x+2) \\ 3(1) &= A[(1)-1] + B[(1)+2] \\ 3 &= 0 + 3B \\ 1 &= B \end{aligned}$$

Next, either substitute $B = 1$ into the equation and solve for A, or make the B-term 0 by substituting $x = -2$ into the equation.

$$\begin{aligned} 3x &= A(x-1) + B(x+2) \\ 3(-2) &= A[(-2)-1] + B[(-2)+2] \\ -6 &= -3A + 0 \\ \tfrac{-6}{-3} &= A \\ 2 &= A \end{aligned}$$

We obtain the same values for A and B using either method, so the decompositions are the same using either method.

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Although this method is not seen very often in textbooks, we present it here as an alternative that may make some partial fraction decompositions easier. It is known as the Heaviside method, named after Charles Heaviside, a pioneer in the study of electronics.

TRY IT #1 Find the partial fraction decomposition of the following expression.

$$\frac{x}{(x-3)(x-2)}$$

Decomposing $\frac{P(x)}{Q(x)}$ Where $Q(x)$ Has Repeated Linear Factors

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors. We must remember that we account for repeated factors by writing each factor in increasing powers.

Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has Repeated Linear Factors

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated linear factor occurring n times and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}$$

Write the denominator powers in increasing order.

HOW TO

Given a rational expression with repeated linear factors, decompose it.

1. Use a variable like A, B, or C for the numerators and account for increasing powers of the denominators.
$$\frac{P(x)}{Q(x)} = \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

EXAMPLE 2

Decomposing with Repeated Linear Factors

Decompose the given rational expression with repeated linear factors.

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}$$

Solution

The denominator factors are $x(x-2)^2$. To allow for the repeated factor of $(x-2)$, the decomposition will include three denominators: x, $(x-2)$, and $(x-2)^2$. Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

Next, we multiply both sides by the common denominator.

$$\begin{aligned} x(x-2)^2\left[\frac{-x^2+2x+4}{x(x-2)^2}\right] &= \left[\frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}\right]x(x-2)^2 \\ -x^2+2x+4 &= A(x-2)^2 + Bx(x-2) + Cx \end{aligned}$$

On the right side of the equation, we expand and collect like terms.

$$\begin{aligned} -x^2+2x+4 &= A(x^2-4x+4) + B(x^2-2x) + Cx \\ &= Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx \\ &= (A+B)x^2 + (-4A-2B+C)x + 4A \end{aligned}$$

Next, we compare the coefficients of both sides. This will give the system of equations in three variables:

$$-x^2+2x+4 = (A+B)x^2 + (-4A-2B+C)x + 4A$$

$$\begin{aligned} A+B &= -1 \quad (1) \\ -4A-2B+C &= 2 \quad (2) \\ 4A &= 4 \quad (3) \end{aligned}$$

Solving for A, we have

$$\begin{aligned} 4A &= 4 \\ A &= 1 \end{aligned}$$

Substitute $A = 1$ into equation (1).

$$\begin{aligned} A+B &= -1 \\ (1)+B &= -1 \\ B &= -2 \end{aligned}$$

Then, to solve for C, substitute the values for A and B into equation (2).

$$
\begin{aligned}
-4A-2B+C&=2\\
-4(1)-2(-2)+C&=2\\
-4+4+C&=2\\
C&=2
\end{aligned}
$$

Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}=\frac{1}{x}-\frac{2}{(x-2)}+\frac{2}{(x-2)^2}$$

TRY IT #2 Find the partial fraction decomposition of the expression with repeated linear factors.

$$\frac{6x-11}{(x-1)^2}$$

Decomposing $\frac{P(x)}{Q(x)}$, Where *Q(x)* Has a Nonrepeated Irreducible Quadratic Factor

So far, we have performed partial fraction decomposition with expressions that have had linear factors in the denominator, and we applied numerators $A, B,$ or C representing constants. Now we will look at an example where one of the factors in the denominator is a quadratic expression that does not factor. This is referred to as an irreducible quadratic factor. In cases like this, we use a linear numerator such as $Ax+B, Bx+C,$ etc.

Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has a Nonrepeated Irreducible Quadratic Factor

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ such that $Q(x)$ has a nonrepeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$ is written as

$$\frac{P(x)}{Q(x)}=\frac{A_1x+B_1}{\left(a_1x^2+b_1x+c_1\right)}+\frac{A_2x+B_2}{\left(a_2x^2+b_2x+c_2\right)}+\cdots+\frac{A_nx+B_n}{\left(a_nx^2+b_nx+c_n\right)}$$

The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: $A, B, C,$ and so on.

HOW TO

Given a rational expression where the factors of the denominator are distinct, irreducible quadratic factors, decompose it.

1. Use variables such as $A, B,$ or C for the constant numerators over linear factors, and linear expressions such as $A_1x+B_1, A_2x+B_2,$ etc., for the numerators of each quadratic factor in the denominator.
$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{\left(a_1x^2+b_1x+c_1\right)}+\frac{A_2x+B_2}{\left(a_2x^2+b_2x+c_2\right)}+\cdots+\frac{A_nx+B_n}{\left(a_nx^2+b_nx+c_n\right)}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

EXAMPLE 3

Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Contains a Nonrepeated Irreducible Quadratic Factor

Find a partial fraction decomposition of the given expression.

$$\frac{8x^2 + 12x - 20}{(x+3)\left(x^2 + x + 2\right)}$$

Solution

We have one linear factor and one irreducible quadratic factor in the denominator, so one numerator will be a constant and the other numerator will be a linear expression. Thus,

$$\frac{8x^2 + 12x - 20}{(x+3)\left(x^2 + x + 2\right)} = \frac{A}{(x+3)} + \frac{Bx + C}{\left(x^2 + x + 2\right)}$$

We follow the same steps as in previous problems. First, clear the fractions by multiplying both sides of the equation by the common denominator.

$$\begin{aligned} (x+3)(x^2+x+2)\left[\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}\right] &= \left[\frac{A}{(x+3)} + \frac{Bx+C}{(x^2+x+2)}\right](x+3)(x^2+x+2) \\ 8x^2 + 12x - 20 &= A(x^2+x+2) + (Bx+C)(x+3) \end{aligned}$$

Notice we could easily solve for A by choosing a value for x that will make the $Bx + C$ term equal 0. Let $x = -3$ and substitute it into the equation.

$$\begin{aligned} 8x^2 + 12x - 20 &= A(x^2+x+2) + (Bx+C)(x+3) \\ 8(-3)^2 + 12(-3) - 20 &= A((-3)^2 + (-3) + 2) + (B(-3) + C)((-3) + 3) \\ 16 &= 8A \\ A &= 2 \end{aligned}$$

Now that we know the value of A, substitute it back into the equation. Then expand the right side and collect like terms.

$$\begin{aligned} 8x^2 + 12x - 20 &= 2(x^2+x+2) + (Bx+C)(x+3) \\ 8x^2 + 12x - 20 &= 2x^2 + 2x + 4 + Bx^2 + 3B + Cx + 3C \\ 8x^2 + 12x - 20 &= (2+B)x^2 + (2+3B+C)x + (4+3C) \end{aligned}$$

Setting the coefficients of terms on the right side equal to the coefficients of terms on the left side gives the system of equations.

$$\begin{aligned} 2 + B &= 8 \quad (1) \\ 2 + 3B + C &= 12 \quad (2) \\ 4 + 3C &= -20 \quad (3) \end{aligned}$$

Solve for B using equation (1) and solve for C using equation (3).

$$\begin{aligned} 2 + B &= 8 \quad (1) \\ B &= 6 \end{aligned}$$

$$\begin{aligned} 4 + 3C &= -20 \quad (3) \\ 3C &= -24 \\ C &= -8 \end{aligned}$$

Thus, the partial fraction decomposition of the expression is

$$\frac{8x^2 + 12x - 20}{(x+3)\left(x^2 + x + 2\right)} = \frac{2}{(x+3)} + \frac{6x - 8}{\left(x^2 + x + 2\right)}$$

Q&A ***Could we have just set up a system of equations to solve Example 3?***

Yes, we could have solved it by setting up a system of equations without solving for A first. The expansion on the right would be:

$$8x^2 + 12x - 20 = Ax^2 + Ax + 2A + Bx^2 + 3B + Cx + 3C$$
$$8x^2 + 12x - 20 = (A + B)x^2 + (A + 3B + C)x + (2A + 3C)$$

So the system of equations would be:

$$\begin{aligned} A + B &= 8 \\ A + 3B + C &= 12 \\ 2A + 3C &= -20 \end{aligned}$$

TRY IT #3 Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.

$$\frac{5x^2 - 6x + 7}{(x-1)\left(x^2 + 1\right)}$$

Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Has a Repeated Irreducible Quadratic Factor

Now that we can decompose a simplified rational expression with an irreducible quadratic factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.

Decomposition of $\frac{P(x)}{Q(x)}$ When $Q(x)$ Has a Repeated Irreducible Quadratic Factor

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{\left(ax^2 + bx + c\right)^n} = \frac{A_1x + B_1}{\left(ax^2 + bx + c\right)} + \frac{A_2x + B_2}{\left(ax^2 + bx + c\right)^2} + \frac{A_3x + B_3}{\left(ax^2 + bx + c\right)^3} + \cdots + \frac{A_nx + B_n}{\left(ax^2 + bx + c\right)^n}$$

Write the denominators in increasing powers.

HOW TO

Given a rational expression that has a repeated irreducible factor, decompose it.

1. Use variables like $A, B,$ or C for the constant numerators over linear factors, and linear expressions such as $A_1x + B_1, A_2x + B_2,$ etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as
$$\frac{P(x)}{Q(x)} = \frac{A}{ax + b} + \frac{A_1x + B_1}{(ax^2 + bx + c)} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_n + B_n}{(ax^2 + bx + c)^n}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

EXAMPLE 4

Decomposing a Rational Function with a Repeated Irreducible Quadratic Factor in the Denominator

Decompose the given expression that has a repeated irreducible factor in the denominator.

$$\frac{x^4+x^3+x^2-x+1}{x\left(x^2+1\right)^2}$$

Solution

The factors of the denominator are $x, (x^2+1),$ and $(x^2+1)^2.$ Recall that, when a factor in the denominator is a quadratic that includes at least two terms, the numerator must be of the linear form $Ax+B.$ So, let's begin the decomposition.

$$\frac{x^4+x^3+x^2-x+1}{x\left(x^2+1\right)^2}=\frac{A}{x}+\frac{Bx+C}{\left(x^2+1\right)}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

We eliminate the denominators by multiplying each term by $x\left(x^2+1\right)^2.$ Thus,

$$x^4+x^3+x^2-x+1=A\left(x^2+1\right)^2+(Bx+C)(x)\left(x^2+1\right)+(Dx+E)(x)$$

Expand the right side.

$$\begin{aligned}x^4+x^3+x^2-x+1&=A(x^4+2x^2+1)+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex\\&=Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex\end{aligned}$$

Now we will collect like terms.

$$x^4+x^3+x^2-x+1=(A+B)x^4+(C)x^3+(2A+B+D)x^2+(C+E)x+A$$

Set up the system of equations matching corresponding coefficients on each side of the equal sign.

$$\begin{aligned}A+B&=1\\C&=1\\2A+B+D&=1\\C+E&=-1\\A&=1\end{aligned}$$

We can use substitution from this point. Substitute $A=1$ into the first equation.

$$\begin{aligned}1+B&=1\\B&=0\end{aligned}$$

Substitute $A=1$ and $B=0$ into the third equation.

$$\begin{aligned}2(1)+0+D&=1\\D&=-1\end{aligned}$$

Substitute $C=1$ into the fourth equation.

$$\begin{aligned}1+E&=-1\\E&=-2\end{aligned}$$

Now we have solved for all of the unknowns on the right side of the equal sign. We have $A=1,\ B=0,\ C=1,\ D=-1,$ and $E=-2$. We can write the decomposition as follows:

$$\frac{x^4+x^3+x^2-x+1}{x\left(x^2+1\right)^2}=\frac{1}{x}+\frac{1}{\left(x^2+1\right)}-\frac{x+2}{\left(x^2+1\right)^2}$$

TRY IT #4 Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.

$$\frac{x^3 - 4x^2 + 9x - 5}{\left(x^2 - 2x + 3\right)^2}$$

MEDIA

Access these online resources for additional instruction and practice with partial fractions.

[Partial Fraction Decomposition (http://openstax.org/l/partdecomp)](http://openstax.org/l/partdecomp)
[Partial Fraction Decomposition With Repeated Linear Factors (http://openstax.org/l/partdecomprlf)](http://openstax.org/l/partdecomprlf)
[Partial Fraction Decomposition With Linear and Quadratic Factors (http://openstax.org/l/partdecomlqu)](http://openstax.org/l/partdecomlqu)

7.4 SECTION EXERCISES

Verbal

1. Can any quotient of polynomials be decomposed into at least two partial fractions? If so, explain why, and if not, give an example of such a fraction
2. Can you explain why a partial fraction decomposition is unique? (Hint: Think about it as a system of equations.)
3. Can you explain how to verify a partial fraction decomposition graphically?
4. You are unsure if you correctly decomposed the partial fraction correctly. Explain how you could double-check your answer.
5. Once you have a system of equations generated by the partial fraction decomposition, can you explain another method to solve it? For example if you had $\frac{7x+13}{3x^2+8x+15} = \frac{A}{x+1} + \frac{B}{3x+5}$, we eventually simplify to $7x + 13 = A(3x + 5) + B(x + 1)$. Explain how you could intelligently choose an x -value that will eliminate either A or B and solve for A and B.

Algebraic

For the following exercises, find the decomposition of the partial fraction for the nonrepeating linear factors.

6. $\frac{5x+16}{x^2+10x+24}$
7. $\frac{3x-79}{x^2-5x-24}$
8. $\frac{-x-24}{x^2-2x-24}$
9. $\frac{10x+47}{x^2+7x+10}$
10. $\frac{x}{6x^2+25x+25}$
11. $\frac{32x-11}{20x^2-13x+2}$
12. $\frac{x+1}{x^2+7x+10}$
13. $\frac{5x}{x^2-9}$
14. $\frac{10x}{x^2-25}$
15. $\frac{6x}{x^2-4}$
16. $\frac{2x-3}{x^2-6x+5}$
17. $\frac{4x-1}{x^2-x-6}$
18. $\frac{4x+3}{x^2+8x+15}$
19. $\frac{3x-1}{x^2-5x+6}$

For the following exercises, find the decomposition of the partial fraction for the repeating linear factors.

20. $\frac{-5x-19}{(x+4)^2}$
21. $\frac{x}{(x-2)^2}$
22. $\frac{7x+14}{(x+3)^2}$

23. $\frac{-24x-27}{(4x+5)^2}$

24. $\frac{-24x-27}{(6x-7)^2}$

25. $\frac{5-x}{(x-7)^2}$

26. $\frac{5x+14}{2x^2+12x+18}$

27. $\frac{5x^2+20x+8}{2x(x+1)^2}$

28. $\frac{4x^2+55x+25}{5x(3x+5)^2}$

29. $\frac{54x^3+127x^2+80x+16}{2x^2(3x+2)^2}$

30. $\frac{x^3-5x^2+12x+144}{x^2\left(x^2+12x+36\right)}$

For the following exercises, find the decomposition of the partial fraction for the irreducible nonrepeating quadratic factor.

31. $\frac{4x^2+6x+11}{(x+2)\left(x^2+x+3\right)}$

32. $\frac{4x^2+9x+23}{(x-1)\left(x^2+6x+11\right)}$

33. $\frac{-2x^2+10x+4}{(x-1)\left(x^2+3x+8\right)}$

34. $\frac{x^2+3x+1}{(x+1)\left(x^2+5x-2\right)}$

35. $\frac{4x^2+17x-1}{(x+3)\left(x^2+6x+1\right)}$

36. $\frac{4x^2}{(x+5)\left(x^2+7x-5\right)}$

37. $\frac{4x^2+5x+3}{x^3-1}$

38. $\frac{-5x^2+18x-4}{x^3+8}$

39. $\frac{3x^2-7x+33}{x^3+27}$

40. $\frac{x^2+2x+40}{x^3-125}$

41. $\frac{4x^2+4x+12}{8x^3-27}$

42. $\frac{-50x^2+5x-3}{125x^3-1}$

43. $\frac{-2x^3-30x^2+36x+216}{x^4+216x}$

For the following exercises, find the decomposition of the partial fraction for the irreducible repeating quadratic factor.

44. $\frac{3x^3+2x^2+14x+15}{\left(x^2+4\right)^2}$

45. $\frac{x^3+6x^2+5x+9}{\left(x^2+1\right)^2}$

46. $\frac{x^3-x^2+x-1}{\left(x^2-3\right)^2}$

47. $\frac{x^2+5x+5}{(x+2)^2}$

48. $\frac{x^3+2x^2+4x}{\left(x^2+2x+9\right)^2}$

49. $\frac{x^2+25}{\left(x^2+3x+25\right)^2}$

50. $\frac{2x^3+11x^2+7x+70}{\left(2x^2+x+14\right)^2}$

51. $\frac{5x+2}{x\left(x^2+4\right)^2}$

52. $\frac{x^4+x^3+8x^2+6x+36}{x\left(x^2+6\right)^2}$

53. $\frac{2x-9}{\left(x^2-x\right)^2}$

54. $\frac{5x^3-2x+1}{\left(x^2+2x\right)^2}$

Extensions

For the following exercises, find the partial fraction expansion.

55. $\frac{x^2+4}{(x+1)^3}$

56. $\frac{x^3-4x^2+5x+4}{(x-2)^3}$

For the following exercises, perform the operation and then find the partial fraction decomposition.

57. $\frac{7}{x+8}+\frac{5}{x-2}-\frac{x-1}{x^2-6x-16}$

58. $\frac{1}{x-4}-\frac{3}{x+6}-\frac{2x+7}{x^2+2x-24}$

59. $\frac{2x}{x^2-16}-\frac{1-2x}{x^2+6x+8}-\frac{x-5}{x^2-4x}$

7.5 Matrices and Matrix Operations

Learning Objectives

In this section, you will:

- Find the sum and difference of two matrices.
- Find scalar multiples of a matrix.
- Find the product of two matrices.

Figure 1 (credit: "SD Dirk," Flickr)

Two club soccer teams, the Wildcats and the Mud Cats, are hoping to obtain new equipment for an upcoming season. Table 1 shows the needs of both teams.

	Wildcats	Mud Cats
Goals	6	10
Balls	30	24
Jerseys	14	20

Table 1

A goal costs \$300; a ball costs \$10; and a jersey costs \$30. How can we find the total cost for the equipment needed for each team? In this section, we discover a method in which the data in the soccer equipment table can be displayed and used for calculating other information. Then, we will be able to calculate the cost of the equipment.

Finding the Sum and Difference of Two Matrices

To solve a problem like the one described for the soccer teams, we can use a matrix, which is a rectangular array of numbers. A row in a matrix is a set of numbers that are aligned horizontally. A column in a matrix is a set of numbers that are aligned vertically. Each number is an entry, sometimes called an element, of the matrix. Matrices (plural) are enclosed in [ ] or ( ), and are usually named with capital letters. For example, three matrices named A, B, and C are shown below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -5 & 6 \\ 7 & 8 & 2 \end{bmatrix}, C = \begin{bmatrix} -1 & 3 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$$

Describing Matrices

A matrix is often referred to by its size or dimensions: $m \times n$ indicating m rows and n columns. Matrix entries are defined first by row and then by column. For example, to locate the entry in matrix A identified as a_{ij}, we look for the entry in row i, column j. In matrix A, shown below, the entry in row 2, column 3 is a_{23}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A square matrix is a matrix with dimensions $n \times n$, meaning that it has the same number of rows as columns. The 3×3 matrix above is an example of a square matrix.

A row matrix is a matrix consisting of one row with dimensions $1 \times n$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

A column matrix is a matrix consisting of one column with dimensions $m \times 1$.

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

A matrix may be used to represent a system of equations. In these cases, the numbers represent the coefficients of the variables in the system. Matrices often make solving systems of equations easier because they are not encumbered with variables. We will investigate this idea further in the next section, but first we will look at basic matrix operations.

Matrices

A **matrix** is a rectangular array of numbers that is usually named by a capital letter: A, B, C, and so on. Each entry in a matrix is referred to as a_{ij}, such that i represents the row and j represents the column. Matrices are often referred to by their dimensions: $m \times n$ indicating m rows and n columns.

EXAMPLE 1

Finding the Dimensions of the Given Matrix and Locating Entries

Given matrix A :

ⓐ What are the dimensions of matrix A? ⓑ What are the entries at a_{31} and a_{22}?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 7 \\ 3 & 1 & -2 \end{bmatrix}$$

Solution

ⓐ The dimensions are 3×3 because there are three rows and three columns.

ⓑ Entry a_{31} is the number at row 3, column 1, which is 3. The entry a_{22} is the number at row 2, column 2, which is 4. Remember, the row comes first, then the column.

Adding and Subtracting Matrices

We use matrices to list data or to represent systems. Because the entries are numbers, we can perform operations on matrices. We add or subtract matrices by adding or subtracting corresponding entries.

In order to do this, the entries must correspond. Therefore, *addition and subtraction of matrices is only possible when the matrices have the same dimensions*. We can add or subtract a 3×3 matrix and another 3×3 matrix, but we cannot add or subtract a 2×3 matrix and a 3×3 matrix because some entries in one matrix will not have a corresponding entry in the other matrix.

Adding and Subtracting Matrices

Given matrices A and B of like dimensions, addition and subtraction of A and B will produce matrix C or matrix D of the same dimension.

$$A + B = C \text{ such that } a_{ij} + b_{ij} = c_{ij}$$
$$A - B = D \text{ such that } a_{ij} - b_{ij} = d_{ij}$$

Matrix addition is commutative.

$$A + B = B + A$$

It is also associative.

$$(A + B) + C = A + (B + C)$$

EXAMPLE 2

Finding the Sum of Matrices

Find the sum of A and B, given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Solution

Add corresponding entries.

$$\begin{aligned} A + B &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \end{aligned}$$

EXAMPLE 3

Adding Matrix *A* and Matrix *B*

Find the sum of A and B.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix}$$

Solution

Add corresponding entries. Add the entry in row 1, column 1, a_{11}, of matrix A to the entry in row 1, column 1, b_{11}, of B. Continue the pattern until all entries have been added.

$$\begin{aligned} A + B &= \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 4+5 & 1+9 \\ 3+0 & 2+7 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 10 \\ 3 & 9 \end{bmatrix} \end{aligned}$$

EXAMPLE 4

Finding the Difference of Two Matrices

Find the difference of A and B.

$$A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

Solution

We subtract the corresponding entries of each matrix.

$$\begin{aligned} A - B &= \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -2-8 & 3-1 \\ 0-5 & 1-4 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 2 \\ -5 & -3 \end{bmatrix} \end{aligned}$$

EXAMPLE 5

Finding the Sum and Difference of Two 3 x 3 Matrices

Given A and B :

ⓐ Find the sum. ⓑ Find the difference.

$$A = \begin{bmatrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{bmatrix}$$

Solution

ⓐ Add the corresponding entries.

$$\begin{aligned} A + B &= \begin{bmatrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2+6 & -10+10 & -2-2 \\ 14+0 & 12-12 & 10-4 \\ 4-5 & -2+2 & 2-2 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 0 & -4 \\ 14 & 0 & 6 \\ -1 & 0 & 0 \end{bmatrix} \end{aligned}$$

ⓑ Subtract the corresponding entries.

$$\begin{aligned} A - B &= \begin{bmatrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2-6 & -10-10 & -2+2 \\ 14-0 & 12+12 & 10+4 \\ 4+5 & -2-2 & 2+2 \end{bmatrix} \\ &= \begin{bmatrix} -4 & -20 & 0 \\ 14 & 24 & 14 \\ 9 & -4 & 4 \end{bmatrix} \end{aligned}$$

TRY IT #1 Add matrix A and matrix B.

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}$$

Finding Scalar Multiples of a Matrix

Besides adding and subtracting whole matrices, there are many situations in which we need to multiply a matrix by a constant called a scalar. Recall that a scalar is a real number quantity that has magnitude, but not direction. For example, time, temperature, and distance are scalar quantities. The process of scalar multiplication involves multiplying each entry in a matrix by a scalar. A **scalar multiple** is any entry of a matrix that results from scalar multiplication.

Consider a real-world scenario in which a university needs to add to its inventory of computers, computer tables, and chairs in two of the campus labs due to increased enrollment. They estimate that 15% more equipment is needed in both labs. The school's current inventory is displayed in Table 2.

	Lab A	Lab B
Computers	15	27
Computer Tables	16	34
Chairs	16	34

Table 2

Converting the data to a matrix, we have

$$C_{2013} = \begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix}$$

To calculate how much computer equipment will be needed, we multiply all entries in matrix C by 0.15.

$$(0.15)C_{2013} = \begin{bmatrix} (0.15)15 & (0.15)27 \\ (0.15)16 & (0.15)34 \\ (0.15)16 & (0.15)34 \end{bmatrix} = \begin{bmatrix} 2.25 & 4.05 \\ 2.4 & 5.1 \\ 2.4 & 5.1 \end{bmatrix}$$

We must round up to the next integer, so the amount of new equipment needed is

$$\begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix}$$

Adding the two matrices as shown below, we see the new inventory amounts.

$$\begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix}$$

This means

$$C_{2014} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix}$$

Thus, Lab A will have 18 computers, 19 computer tables, and 19 chairs; Lab B will have 32 computers, 40 computer tables, and 40 chairs.

Scalar Multiplication

Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the scalar multiple cA is

$$\begin{aligned} cA &= c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix} \end{aligned}$$

Scalar multiplication is distributive. For the matrices A, B, and C with scalars a and b,

$$a(A + B) = aA + aB$$
$$(a + b)A = aA + bA$$

EXAMPLE 6

Multiplying the Matrix by a Scalar

Multiply matrix A by the scalar 3.

$$A = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

Solution

Multiply each entry in A by the scalar 3.

$$\begin{aligned} 3A &= 3\begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 8 & 3 \cdot 1 \\ 3 \cdot 5 & 3 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 24 & 3 \\ 15 & 12 \end{bmatrix} \end{aligned}$$

TRY IT #2 Given matrix B, find $-2B$ where

$$B = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

EXAMPLE 7

Finding the Sum of Scalar Multiples

Find the sum $3A + 2B$.

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & -4 \end{bmatrix}$$

Solution

First, find $3A$, then $2B$.

$$\begin{aligned} 3A &= \begin{bmatrix} 3 \cdot 1 & 3(-2) & 3 \cdot 0 \\ 3 \cdot 0 & 3(-1) & 3 \cdot 2 \\ 3 \cdot 4 & 3 \cdot 3 & 3(-6) \end{bmatrix} \\ &= \begin{bmatrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 2B &= \begin{bmatrix} 2(-1) & 2\cdot 2 & 2\cdot 1 \\ 2\cdot 0 & 2(-3) & 2\cdot 2 \\ 2\cdot 0 & 2\cdot 1 & 2(-4) \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{bmatrix} \end{aligned}$$

Now, add $3A + 2B$.

$$\begin{aligned} 3A + 2B &= \begin{bmatrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{bmatrix} + \begin{bmatrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{bmatrix} \\ &= \begin{bmatrix} 3-2 & -6+4 & 0+2 \\ 0+0 & -3-6 & 6+4 \\ 12+0 & 9+2 & -18-8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & 2 \\ 0 & -9 & 10 \\ 12 & 11 & -26 \end{bmatrix} \end{aligned}$$

Finding the Product of Two Matrices

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices is only possible when the inner dimensions are the same, meaning that the number of columns of the first matrix is equal to the number of rows of the second matrix. If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product matrix AB is an $m \times n$ matrix. For example, the product AB is possible because the number of columns in A is the same as the number of rows in B. If the inner dimensions do not match, the product is not defined.

$$\begin{array}{ccc} A & \cdot & B \\ 2\times & \underbrace{3 \qquad\qquad 3}_{\text{same}} & \times 3 \end{array}$$

We multiply entries of A with entries of B according to a specific pattern as outlined below. The process of matrix multiplication becomes clearer when working a problem with real numbers.

To obtain the entries in row i of AB, we multiply the entries in row i of A by column j in B and add. For example, given matrices A and B, where the dimensions of A are 2×3 and the dimensions of B are 3×3, the product of AB will be a 2×3 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Multiply and add as follows to obtain the first entry of the product matrix AB.

1. To obtain the entry in row 1, column 1 of AB, multiply the first row in A by the first column in B, and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}\cdot b_{11} + a_{12}\cdot b_{21} + a_{13}\cdot b_{31}$$

2. To obtain the entry in row 1, column 2 of AB, multiply the first row of A by the second column in B, and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11}\cdot b_{12} + a_{12}\cdot b_{22} + a_{13}\cdot b_{32}$$

3. To obtain the entry in row 1, column 3 of AB, multiply the first row of A by the third column in B, and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33}$$

We proceed the same way to obtain the second row of AB. In other words, row 2 of A times column 1 of B; row 2 of A times column 2 of B; row 2 of A times column 3 of B. When complete, the product matrix will be

$$AB = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \end{bmatrix}$$

Properties of Matrix Multiplication

For the matrices A, B, and C the following properties hold.

- Matrix multiplication is associative: $(AB)C = A(BC)$.
- Matrix multiplication is distributive: $$C(A+B) = CA + CB,$$ $$(A+B)C = AC + BC.$$

Note that matrix multiplication is not commutative.

EXAMPLE 8

Multiplying Two Matrices

Multiply matrix A and matrix B.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution

First, we check the dimensions of the matrices. Matrix A has dimensions 2×2 and matrix B has dimensions 2×2. The inner dimensions are the same so we can perform the multiplication. The product will have the dimensions 2×2.

We perform the operations outlined previously.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1(5)+2(7) & 1(6)+2(8) \\ 3(5)+4(7) & 3(6)+4(8) \end{bmatrix} \\ &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \end{aligned}$$

EXAMPLE 9

Multiplying Two Matrices

Given A and B:

ⓐ Find AB. ⓑ Find BA.

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix}$$

Solution

ⓐ As the dimensions of A are 2×3 and the dimensions of B are 3×2, these matrices can be multiplied together because the number of columns in A matches the number of rows in B. The resulting product will be a 2×2 matrix, the number of rows in A by the number of columns in B.

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -1(5)+2(-4)+3(2) & -1(-1)+2(0)+3(3) \\ 4(5)+0(-4)+5(2) & 4(-1)+0(0)+5(3) \end{bmatrix} \\ &= \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \end{aligned}$$

ⓑ The dimensions of B are 3×2 and the dimensions of A are 2×3. The inner dimensions match so the product is defined and will be a 3×3 matrix.

$$\begin{aligned} BA &= \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5(-1)+-1(4) & 5(2)+-1(0) & 5(3)+-1(5) \\ -4(-1)+0(4) & -4(2)+0(0) & -4(3)+0(5) \\ 2(-1)+3(4) & 2(2)+3(0) & 2(3)+3(5) \end{bmatrix} \\ &= \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} \end{aligned}$$

Analysis

Notice that the products AB and BA are not equal.

$$AB = \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \neq \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} = BA$$

This illustrates the fact that matrix multiplication is not commutative.

Q&A **Is it possible for *AB* to be defined but not *BA*?**

Yes, consider a matrix A with dimension 3×4 *and matrix B with dimension* 4×2. *For the product AB the inner dimensions are 4 and the product is defined, but for the product BA the inner dimensions are 2 and 3 so the product is undefined.*

EXAMPLE 10

Using Matrices in Real-World Problems

Let's return to the problem presented at the opening of this section. We have Table 3, representing the equipment needs of two soccer teams.

	Wildcats	Mud Cats
Goals	6	10

Table 3

	Wildcats	Mud Cats
Balls	30	24
Jerseys	14	20

Table 3

We are also given the prices of the equipment, as shown in Table 4.

Goal	$300
Ball	$10
Jersey	$30

Table 4

We will convert the data to matrices. Thus, the equipment need matrix is written as

$$E = \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix}$$

The cost matrix is written as

$$C = \begin{bmatrix} 300 & 10 & 30 \end{bmatrix}$$

We perform matrix multiplication to obtain costs for the equipment.

$$\begin{aligned} CE &= \begin{bmatrix} 300 & 10 & 30 \end{bmatrix} \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix} \\ &= \begin{bmatrix} 300(6) + 10(30) + 30(14) & 300(10) + 10(24) + 30(20) \end{bmatrix} \\ &= \begin{bmatrix} 2{,}520 & 3{,}840 \end{bmatrix} \end{aligned}$$

The total cost for equipment for the Wildcats is $2,520, and the total cost for equipment for the Mud Cats is $3,840.

HOW TO

Given a matrix operation, evaluate using a calculator.

1. Save each matrix as a matrix variable $[A], [B], [C], \ldots$
2. Enter the operation into the calculator, calling up each matrix variable as needed.
3. If the operation is defined, the calculator will present the solution matrix; if the operation is undefined, it will display an error message.

EXAMPLE 11

Using a Calculator to Perform Matrix Operations

Find $AB - C$ given

$$A = \begin{bmatrix} -15 & 25 & 32 \\ 41 & -7 & -28 \\ 10 & 34 & -2 \end{bmatrix}, B = \begin{bmatrix} 45 & 21 & -37 \\ -24 & 52 & 19 \\ 6 & -48 & -31 \end{bmatrix}, \text{and } C = \begin{bmatrix} -100 & -89 & -98 \\ 25 & -56 & 74 \\ -67 & 42 & -75 \end{bmatrix}.$$

Solution

On the matrix page of the calculator, we enter matrix A above as the matrix variable $[A]$, matrix B above as the matrix variable $[B]$, and matrix C above as the matrix variable $[C]$.

On the home screen of the calculator, we type in the problem and call up each matrix variable as needed.

$$[A]\,[B] - [C]$$

The calculator gives us the following matrix.

$$\begin{bmatrix} -983 & -462 & 136 \\ 1,820 & 1,897 & -856 \\ -311 & 2,032 & 413 \end{bmatrix}$$

MEDIA

Access these online resources for additional instruction and practice with matrices and matrix operations.

[Dimensions of a Matrix (http://openstax.org/l/matrixdimen)](http://openstax.org/l/matrixdimen)
[Matrix Addition and Subtraction (http://openstax.org/l/matrixaddsub)](http://openstax.org/l/matrixaddsub)
[Matrix Operations (http://openstax.org/l/matrixoper)](http://openstax.org/l/matrixoper)
[Matrix Multiplication (http://openstax.org/l/matrixmult)](http://openstax.org/l/matrixmult)

7.5 SECTION EXERCISES

Verbal

1. Can we add any two matrices together? If so, explain why; if not, explain why not and give an example of two matrices that cannot be added together.

2. Can we multiply any column matrix by any row matrix? Explain why or why not.

3. Can both the products AB and BA be defined? If so, explain how; if not, explain why.

4. Can any two matrices of the same size be multiplied? If so, explain why, and if not, explain why not and give an example of two matrices of the same size that cannot be multiplied together.

5. Does matrix multiplication commute? That is, does $AB = BA$? If so, prove why it does. If not, explain why it does not.

Algebraic

For the following exercises, use the matrices below and perform the matrix addition or subtraction. Indicate if the operation is undefined.

$$A=\begin{bmatrix}1&3\\0&7\end{bmatrix}, B=\begin{bmatrix}2&14\\22&6\end{bmatrix}, C=\begin{bmatrix}1&5\\8&92\\12&6\end{bmatrix}, D=\begin{bmatrix}10&14\\7&2\\5&61\end{bmatrix}, E=\begin{bmatrix}6&12\\14&5\end{bmatrix}, F=\begin{bmatrix}0&9\\78&17\\15&4\end{bmatrix}$$

6. $A+B$

7. $C+D$

8. $A+C$

9. $B-E$

10. $C+F$

11. $D-B$

For the following exercises, use the matrices below to perform scalar multiplication.

$$A=\begin{bmatrix}4&6\\13&12\end{bmatrix}, B=\begin{bmatrix}3&9\\21&12\\0&64\end{bmatrix}, C=\begin{bmatrix}16&3&7&18\\90&5&3&29\end{bmatrix}, D=\begin{bmatrix}18&12&13\\8&14&6\\7&4&21\end{bmatrix}$$

12. $5A$

13. $3B$

14. $-2B$

15. $-4C$

16. $\frac{1}{2}C$

17. $100D$

For the following exercises, use the matrices below to perform matrix multiplication.

$$A=\begin{bmatrix}-1&5\\3&2\end{bmatrix}, B=\begin{bmatrix}3&6&4\\-8&0&12\end{bmatrix}, C=\begin{bmatrix}4&10\\-2&6\\5&9\end{bmatrix}, D=\begin{bmatrix}2&-3&12\\9&3&1\\0&8&-10\end{bmatrix}$$

18. AB

19. BC

20. CA

21. BD

22. DC

23. CB

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed.

$$A=\begin{bmatrix}2&-5\\6&7\end{bmatrix}, B=\begin{bmatrix}-9&6\\-4&2\end{bmatrix}, C=\begin{bmatrix}0&9\\7&1\end{bmatrix}, D=\begin{bmatrix}-8&7&-5\\4&3&2\\0&9&2\end{bmatrix}, E=\begin{bmatrix}4&5&3\\7&-6&-5\\1&0&9\end{bmatrix}$$

24. $A+B-C$

25. $4A+5D$

26. $2C+B$

27. $3D+4E$

28. $C-0.5D$

29. $100D-10E$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2 = A \cdot A$)

$$A=\begin{bmatrix}-10&20\\5&25\end{bmatrix}, B=\begin{bmatrix}40&10\\-20&30\end{bmatrix}, C=\begin{bmatrix}-1&0\\0&-1\\1&0\end{bmatrix}$$

30. AB

31. BA

32. CA

33. BC

34. A^2

35. B^2

36. C^2

37. $B^2 A^2$

38. $A^2 B^2$

39. $(AB)^2$

40. $(BA)^2$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2 = A \cdot A$)

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 1 & -5 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 0.1 \\ 1 & 0.2 \\ -0.5 & 0.3 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & -1 \\ -6 & 7 & 5 \\ 4 & 2 & 1 \end{bmatrix}$$

41. AB

42. BA

43. BD

44. DC

45. D^2

46. A^2

47. D^3

48. $(AB)C$

49. $A(BC)$

Technology

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. Use a calculator to verify your solution.

$$A = \begin{bmatrix} -2 & 0 & 9 \\ 1 & 8 & -3 \\ 0.5 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 3 & 0 \\ -4 & 1 & 6 \\ 8 & 7 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

50. AB

51. BA

52. CA

53. BC

54. ABC

Extensions

For the following exercises, use the matrix below to perform the indicated operation on the given matrix.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

55. B^2

56. B^3

57. B^4

58. B^5

59. Using the above questions, find a formula for B^n. Test the formula for B^{201} and B^{202}, using a calculator.

7.6 Solving Systems with Gaussian Elimination

Learning Objectives

In this section, you will:

- Write the augmented matrix of a system of equations.
- Write the system of equations from an augmented matrix.
- Perform row operations on a matrix.
- Solve a system of linear equations using matrices.

Figure 1 German mathematician Carl Friedrich Gauss (1777–1855).

Carl Friedrich Gauss lived during the late 18th century and early 19th century, but he is still considered one of the most prolific mathematicians in history. His contributions to the science of mathematics and physics span fields such as algebra, number theory, analysis, differential geometry, astronomy, and optics, among others. His discoveries regarding matrix theory changed the way mathematicians have worked for the last two centuries.

We first encountered Gaussian elimination in Systems of Linear Equations: Two Variables. In this section, we will revisit this technique for solving systems, this time using matrices.

Writing the Augmented Matrix of a System of Equations

A matrix can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an **augmented matrix**.

For example, consider the following 2×2 system of equations.

$$\begin{aligned} 3x + 4y &= 7 \\ 4x - 2y &= 5 \end{aligned}$$

We can write this system as an augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 4 & 7 \\ 4 & -2 & 5 \end{array}\right]$$

We can also write a matrix containing just the coefficients. This is called the **coefficient matrix**.

$$\begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}$$

A three-by-three system of equations such as

$$\begin{aligned} 3x - y - z &= 0 \\ x + y &= 5 \\ 2x - 3z &= 2 \end{aligned}$$

has a coefficient matrix

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

and is represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2 \end{array}\right]$$

Notice that the matrix is written so that the variables line up in their own columns: x-terms go in the first column, y-terms in the second column, and z-terms in the third column. It is very important that each equation is written in standard form $ax + by + cz = d$ so that the variables line up. When there is a missing variable term in an equation, the coefficient is 0.

HOW TO

Given a system of equations, write an augmented matrix.

1. Write the coefficients of the x-terms as the numbers down the first column.
2. Write the coefficients of the y-terms as the numbers down the second column.
3. If there are z-terms, write the coefficients as the numbers down the third column.
4. Draw a vertical line and write the constants to the right of the line.

EXAMPLE 1

Writing the Augmented Matrix for a System of Equations

Write the augmented matrix for the given system of equations.

$$\begin{aligned} x + 2y - z &= 3 \\ 2x - y + 2z &= 6 \\ x - 3y + 3z &= 4 \end{aligned}$$

Solution

The augmented matrix displays the coefficients of the variables, and an additional column for the constants.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 6 \\ 1 & -3 & 3 & 4 \end{array}\right]$$

TRY IT #1 Write the augmented matrix of the given system of equations.

$$\begin{aligned} 4x - 3y &= 11 \\ 3x + 2y &= 4 \end{aligned}$$

Writing a System of Equations from an Augmented Matrix

We can use augmented matrices to help us solve systems of equations because they simplify operations when the systems are not encumbered by the variables. However, it is important to understand how to move back and forth between formats in order to make finding solutions smoother and more intuitive. Here, we will use the information in an augmented matrix to write the system of equations in standard form.

EXAMPLE 2

Writing a System of Equations from an Augmented Matrix Form

Find the system of equations from the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{array}\right]$$

Solution

When the columns represent the variables x, y, and z,

$$\left[\begin{array}{ccc|c} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{array}\right] \rightarrow \begin{array}{r} x - 3y - 5z = -2 \\ 2x - 5y - 4z = 5 \\ -3x + 5y + 4z = 6 \end{array}$$

TRY IT #2 Write the system of equations from the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & -1 & 3 & 1 \\ 0 & 1 & 1 & -9 \end{array}\right]$$

Performing Row Operations on a Matrix

Now that we can write systems of equations in augmented matrix form, we will examine the various **row operations** that can be performed on a matrix, such as addition, multiplication by a constant, and interchanging rows.

Performing row operations on a matrix is the method we use for solving a system of equations. In order to solve the system of equations, we want to convert the matrix to **row-echelon form**, in which there are ones down the **main diagonal** from the upper left corner to the lower right corner, and zeros in every position below the main diagonal as shown.

$$\text{Row-echelon form}$$
$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$$

We use row operations corresponding to equation operations to obtain a new matrix that is **row-equivalent** in a simpler form. Here are the guidelines to obtaining row-echelon form.

1. In any nonzero row, the first nonzero number is a 1. It is called a *leading* 1.
2. Any all-zero rows are placed at the bottom on the matrix.
3. Any leading 1 is below and to the right of a previous leading 1.
4. Any column containing a leading 1 has zeros in all other positions in the column.

To solve a system of equations we can perform the following row operations to convert the coefficient matrix to row-echelon form and do back-substitution to find the solution.

1. Interchange rows. (Notation: $R_i \quad \leftrightarrow \quad R_j$)
2. Multiply a row by a constant. (Notation: cR_i)
3. Add the product of a row multiplied by a constant to another row. (Notation: $R_i + cR_j$)

Each of the row operations corresponds to the operations we have already learned to solve systems of equations in three variables. With these operations, there are some key moves that will quickly achieve the goal of writing a matrix in row-echelon form. To obtain a matrix in row-echelon form for finding solutions, we use Gaussian elimination, a method that uses row operations to obtain a 1 as the first entry so that row 1 can be used to convert the remaining rows.

Gaussian Elimination

The **Gaussian elimination** method refers to a strategy used to obtain the row-echelon form of a matrix. The goal is to write matrix A with the number 1 as the entry down the main diagonal and have all zeros below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{After Gaussian elimination}} A = \begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

The first step of the Gaussian strategy includes obtaining a 1 as the first entry, so that row 1 may be used to alter the rows below.

HOW TO

Given an augmented matrix, perform row operations to achieve row-echelon form.

1. The first equation should have a leading coefficient of 1. Interchange rows or multiply by a constant, if necessary.
2. Use row operations to obtain zeros down the first column below the first entry of 1.
3. Use row operations to obtain a 1 in row 2, column 2.
4. Use row operations to obtain zeros down column 2, below the entry of 1.
5. Use row operations to obtain a 1 in row 3, column 3.
6. Continue this process for all rows until there is a 1 in every entry down the main diagonal and there are only zeros below.
7. If any rows contain all zeros, place them at the bottom.

EXAMPLE 3

Solving a 2×2 System by Gaussian Elimination

Solve the given system by Gaussian elimination.

$$\begin{aligned} 2x + 3y &= 6 \\ x - y &= \tfrac{1}{2} \end{aligned}$$

Solution

First, we write this as an augmented matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array}\right]$$

We want a 1 in row 1, column 1. This can be accomplished by interchanging row 1 and row 2.

$$R_1 \leftrightarrow R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{array}\right]$$

We now have a 1 as the first entry in row 1, column 1. Now let's obtain a 0 in row 2, column 1. This can be accomplished by multiplying row 1 by -2, and then adding the result to row 2.

$$-2R_1 + R_2 = R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{array}\right]$$

We only have one more step, to multiply row 2 by $\frac{1}{5}$.

$$\frac{1}{5}R_2 = R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 \end{array}\right]$$

Use back-substitution. The second row of the matrix represents $y = 1$. Back-substitute $y = 1$ into the first equation.

$$\begin{aligned} x - (1) &= \tfrac{1}{2} \\ x &= \tfrac{3}{2} \end{aligned}$$

The solution is the point $\left(\frac{3}{2}, 1\right)$.

TRY IT #3 Solve the given system by Gaussian elimination.

$$\begin{aligned} 4x + 3y &= 11 \\ x - 3y &= -1 \end{aligned}$$

EXAMPLE 4

Using Gaussian Elimination to Solve a System of Equations

Use Gaussian elimination to solve the given 2×2 system of equations.

$$\begin{aligned} 2x + y &= 1 \\ 4x + 2y &= 6 \end{aligned}$$

Solution

Write the system as an augmented matrix.

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Obtain a 1 in row 1, column 1. This can be accomplished by multiplying the first row by $\frac{1}{2}$.

$$\frac{1}{2}R_1 = R_1 \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 4 & 2 & 6 \end{array}\right]$$

Next, we want a 0 in row 2, column 1. Multiply row 1 by -4 and add row 1 to row 2.

$$-4R_1 + R_2 = R_2 \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 4 \end{array}\right]$$

The second row represents the equation $0 = 4$. Therefore, the system is inconsistent and has no solution.

EXAMPLE 5

Solving a Dependent System

Solve the system of equations.

$$\begin{aligned} 3x + 4y &= 12 \\ 6x + 8y &= 24 \end{aligned}$$

Solution

Perform row operations on the augmented matrix to try and achieve row-echelon form.

$$A = \left[\begin{array}{cc|c} 3 & 4 & 12 \\ 6 & 8 & 24 \end{array}\right]$$

$$-\tfrac{1}{2}R_2 + R_1 = R_1 \rightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 6 & 8 & 24 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \rightarrow \left[\begin{array}{cc|c} 6 & 8 & 24 \\ 0 & 0 & 0 \end{array}\right]$$

The matrix ends up with all zeros in the last row: $0y = 0$. Thus, there are an infinite number of solutions and the system is classified as dependent. To find the generic solution, return to one of the original equations and solve for y.

$$\begin{aligned} 3x + 4y &= 12 \\ 4y &= 12 - 3x \\ y &= 3 - \tfrac{3}{4}x \end{aligned}$$

So the solution to this system is $\left(x, 3 - \frac{3}{4}x\right)$.

EXAMPLE 6

Performing Row Operations on a 3×3 Augmented Matrix to Obtain Row-Echelon Form

Perform row operations on the given matrix to obtain row-echelon form.

$$\left[\begin{array}{rrr|r} 1 & -3 & 4 & 3 \\ 2 & -5 & 6 & 6 \\ -3 & 3 & 4 & 6 \end{array}\right]$$

Solution

The first row already has a 1 in row 1, column 1. The next step is to multiply row 1 by −2 and add it to row 2. Then replace row 2 with the result.

$$-2R_1 + R_2 = R_2 \rightarrow \left[\begin{array}{rrr|r} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ -3 & 3 & 4 & 6 \end{array}\right]$$

Next, obtain a zero in row 3, column 1.

$$3R_1 + R_3 = R_3 \rightarrow \left[\begin{array}{rrr|r} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -6 & 16 & 15 \end{array}\right]$$

Next, obtain a zero in row 3, column 2.

$$6R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{rrr|r} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 15 \end{array}\right]$$

The last step is to obtain a 1 in row 3, column 3.

$$\frac{1}{4}R_3 = R_3 \rightarrow \left[\begin{array}{rrr|r} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & \frac{15}{4} \end{array}\right]$$

TRY IT #4 Write the system of equations in row-echelon form.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Solving a System of Linear Equations Using Matrices

We have seen how to write a system of equations with an augmented matrix, and then how to use row operations and back-substitution to obtain row-echelon form. Now, we will take row-echelon form a step farther to solve a 3 by 3 system of linear equations. The general idea is to eliminate all but one variable using row operations and then back-substitute to solve for the other variables.

EXAMPLE 7

Solving a System of Linear Equations Using Matrices

Solve the system of linear equations using matrices.

$$\begin{aligned} x-y+z&=8\\ 2x+3y-z&=-2\\ 3x-2y-9z&=9 \end{aligned}$$

Solution

First, we write the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8\\ 2 & 3 & -1 & -2\\ 3 & -2 & -9 & 9 \end{array}\right]$$

Next, we perform row operations to obtain row-echelon form.

$$-2R_1+R_2=R_2 \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8\\ 0 & 5 & -3 & -18\\ 3 & -2 & -9 & 9 \end{array}\right] \qquad -3R_1+R_3=R_3 \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8\\ 0 & 5 & -3 & -18\\ 0 & 1 & -12 & -15 \end{array}\right]$$

The easiest way to obtain a 1 in row 2 of column 1 is to interchange R_2 and R_3.

$$\text{Interchange } R_2 \text{ and } R_3 \to \left[\begin{array}{rrrr} 1 & -1 & 1 & 8\\ 0 & 1 & -12 & -15\\ 0 & 5 & -3 & -18 \end{array}\right]$$

Then

$$-5R_2+R_3=R_3 \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8\\ 0 & 1 & -12 & -15\\ 0 & 0 & 57 & 57 \end{array}\right] \qquad -\tfrac{1}{57}R_3=R_3 \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8\\ 0 & 1 & -12 & -15\\ 0 & 0 & 1 & 1 \end{array}\right]$$

The last matrix represents the equivalent system.

$$\begin{aligned} x-y+z&=8\\ y-12z&=-15\\ z&=1 \end{aligned}$$

Using back-substitution, we obtain the solution as $(4,-3,1)$.

EXAMPLE 8

Solving a Dependent System of Linear Equations Using Matrices

Solve the following system of linear equations using matrices.

$$\begin{aligned} -x-2y+z&=-1\\ 2x+3y&=2\\ y-2z&=0 \end{aligned}$$

Solution

Write the augmented matrix.

$$\left[\begin{array}{rrr|r} -1 & -2 & 1 & -1\\ 2 & 3 & 0 & 2\\ 0 & 1 & -2 & 0 \end{array}\right]$$

First, multiply row 1 by -1 to get a 1 in row 1, column 1. Then, perform row operations to obtain row-echelon form.

$$-R_1 \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 2 & 3 & 0 & 2 \end{array}\right]$$

$$-2R_1 + R_3 = R_3 \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$R_2 + R_3 = R_3 \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix represents the following system.

$$\begin{aligned} x + 2y - z &= 1 \\ y - 2z &= 0 \\ 0 &= 0 \end{aligned}$$

We see by the identity $0 = 0$ that this is a dependent system with an infinite number of solutions. We then find the generic solution. By solving the second equation for y and substituting it into the first equation we can solve for z in terms of x.

$$\begin{aligned} x + 2y - z &= 1 \\ y &= 2z \end{aligned}$$

$$\begin{aligned} x + 2(2z) - z &= 1 \\ x + 3z &= 1 \\ z &= \tfrac{1-x}{3} \end{aligned}$$

Now we substitute the expression for z into the second equation to solve for y in terms of x.

$$\begin{aligned} y - 2z &= 0 \\ z &= \tfrac{1-x}{3} \end{aligned}$$

$$\begin{aligned} y - 2\left(\tfrac{1-x}{3}\right) &= 0 \\ y &= \tfrac{2-2x}{3} \end{aligned}$$

The generic solution is $\left(x, \frac{2-2x}{3}, \frac{1-x}{3}\right)$.

TRY IT #5 Solve the system using matrices.

$$\begin{aligned} x + 4y - z &= 4 \\ 2x + 5y + 8z &= 15 \\ x + 3y - 3z &= 1 \end{aligned}$$

Q&A **Can any system of linear equations be solved by Gaussian elimination?**

Yes, a system of linear equations of any size can be solved by Gaussian elimination.

HOW TO

Given a system of equations, solve with matrices using a calculator.

1. Save the augmented matrix as a matrix variable $[A], \ [B], \ [C], \ \ldots.$
2. Use the **ref(** function in the calculator, calling up each matrix variable as needed.

EXAMPLE 9

Solving Systems of Equations with Matrices Using a Calculator

Solve the system of equations.

$$\begin{aligned} 5x + 3y + 9z &= -1 \\ -2x + 3y - z &= -2 \\ -x - 4y + 5z &= 1 \end{aligned}$$

Solution

Write the augmented matrix for the system of equations.

$$\left[\begin{array}{rrr|r} 5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & -1 \end{array}\right]$$

On the matrix page of the calculator, enter the augmented matrix above as the matrix variable $[A]$.

$$[A] = \begin{bmatrix} 5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & 1 \end{bmatrix}$$

Use the **ref(** function in the calculator, calling up the matrix variable $[A]$.

$$\text{ref}([A])$$

Evaluate.

$$\begin{bmatrix} 1 & \frac{3}{5} & \frac{9}{5} & \frac{1}{5} \\ 0 & 1 & \frac{13}{21} & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{24}{187} \end{bmatrix} \rightarrow \begin{aligned} x + \tfrac{3}{5}y + \tfrac{9}{5}z &= -\tfrac{1}{5} \\ y + \tfrac{13}{21}z &= -\tfrac{4}{7} \\ z &= -\tfrac{24}{187} \end{aligned}$$

Using back-substitution, the solution is $\left(\frac{61}{187}, -\frac{92}{187}, -\frac{24}{187}\right)$.

EXAMPLE 10

Applying 2 × 2 Matrices to Finance

Carolyn invests a total of \$12,000 in two municipal bonds, one paying 10.5% interest and the other paying 12% interest. The annual interest earned on the two investments last year was \$1,335. How much was invested at each rate?

Solution

We have a system of two equations in two variables. Let $x =$ the amount invested at 10.5% interest, and $y =$ the amount invested at 12% interest.

$$\begin{aligned} x + y &= 12{,}000 \\ 0.105x + 0.12y &= 1{,}335 \end{aligned}$$

As a matrix, we have

$$\left[\begin{array}{cc|c} 1 & 1 & 12{,}000 \\ 0.105 & 0.12 & 1{,}335 \end{array}\right]$$

Multiply row 1 by -0.105 and add the result to row 2.

$$\left[\begin{array}{cc|c} 1 & 1 & 12{,}000 \\ 0 & 0.015 & 75 \end{array}\right]$$

Then,

$$\begin{aligned} 0.015y &= 75 \\ y &= 5{,}000 \end{aligned}$$

So $12{,}000 - 5{,}000 = 7{,}000$.

Thus, \$5,000 was invested at 12% interest and \$7,000 at 10.5% interest.

EXAMPLE 11

Applying 3 × 3 Matrices to Finance

Ava invests a total of \$10,000 in three accounts, one paying 5% interest, another paying 8% interest, and the third paying 9% interest. The annual interest earned on the three investments last year was \$770. The amount invested at 9% was twice the amount invested at 5%. How much was invested at each rate?

Solution

We have a system of three equations in three variables. Let x be the amount invested at 5% interest, let y be the amount invested at 8% interest, and let z be the amount invested at 9% interest. Thus,

$$\begin{aligned} x + y + z &= 10{,}000 \\ 0.05x + 0.08y + 0.09z &= 770 \\ 2x - z &= 0 \end{aligned}$$

As a matrix, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10{,}000 \\ 0.05 & 0.08 & 0.09 & 770 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

Now, we perform Gaussian elimination to achieve row-echelon form.

$$-0.05R_1 + R_2 = R_2 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10{,}000 \\ 0 & 0.03 & 0.04 & 270 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

$$-2R_1 + R_3 = R_3 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10{,}000 \\ 0 & 0.03 & 0.04 & 270 \\ 0 & -2 & -3 & -20{,}000 \end{array}\right]$$

$$\frac{1}{0.03}R_2 = R_2 \to \left[\begin{array}{ccc|c} 0 & 1 & 1 & 10{,}000 \\ 0 & 1 & \frac{4}{3} & 9{,}000 \\ 0 & -2 & -3 & -20{,}000 \end{array}\right]$$

$$2R_2 + R_3 = R_3 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10{,}000 \\ 0 & 1 & \frac{4}{3} & 9{,}000 \\ 0 & 0 & -\frac{1}{3} & -2{,}000 \end{array}\right]$$

The third row tells us $-\frac{1}{3}z = -2{,}000$; thus $z = 6{,}000$.

The second row tells us $y + \frac{4}{3}z = 9{,}000$. Substituting $z = 6{,}000$, we get

$$\begin{aligned} y + \tfrac{4}{3}(6{,}000) &= 9{,}000 \\ y + 8{,}000 &= 9{,}000 \\ y &= 1{,}000 \end{aligned}$$

The first row tells us $x + y + z = 10,000$. Substituting $y = 1,000$ and $z = 6,000$, we get

$$\begin{aligned} x + 1,000 + 6,000 &= 10{,}000 \\ x &= 3{,}000 \end{aligned}$$

The answer is $3,000 invested at 5% interest, $1,000 invested at 8%, and $6,000 invested at 9% interest.

TRY IT #6 A small shoe company took out a loan of $1,500,000 to expand their inventory. Part of the money was borrowed at 7%, part was borrowed at 8%, and part was borrowed at 10%. The amount borrowed at 10% was four times the amount borrowed at 7%, and the annual interest on all three loans was $130,500. Use matrices to find the amount borrowed at each rate.

MEDIA

Access these online resources for additional instruction and practice with solving systems of linear equations using Gaussian elimination.

[Solve a System of Two Equations Using an Augmented Matrix (http://openstax.org/l/system2augmat)](http://openstax.org/l/system2augmat)
[Solve a System of Three Equations Using an Augmented Matrix (http://openstax.org/l/system3augmat)](http://openstax.org/l/system3augmat)
[Augmented Matrices on the Calculator (http://openstax.org/l/augmatcalc)](http://openstax.org/l/augmatcalc)

7.6 SECTION EXERCISES

Verbal

1. Can any system of linear equations be written as an augmented matrix? Explain why or why not. Explain how to write that augmented matrix.

2. Can any matrix be written as a system of linear equations? Explain why or why not. Explain how to write that system of equations.

3. Is there only one correct method of using row operations on a matrix? Try to explain two different row operations possible to solve the augmented matrix $\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 1 & -2 & 6 \end{array}\right]$.

4. Can a matrix whose entry is 0 on the diagonal be solved? Explain why or why not. What would you do to remedy the situation?

5. Can a matrix that has 0 entries for an entire row have one solution? Explain why or why not.

Algebraic

For the following exercises, write the augmented matrix for the linear system.

6. $\begin{aligned} 8x - 37y &= 8 \\ 2x + 12y &= 3 \end{aligned}$

7. $\begin{aligned} 16y &= 4 \\ 9x - y &= 2 \end{aligned}$

8. $\begin{aligned} 3x + 2y + 10z &= 3 \\ -6x + 2y + 5z &= 13 \\ 4x + z &= 18 \end{aligned}$

9. $\begin{aligned} x + 5y + 8z &= 19 \\ 12x + 3y &= 4 \\ 3x + 4y + 9z &= -7 \end{aligned}$

10. $\begin{aligned} 6x + 12y + 16z &= 4 \\ 19x - 5y + 3z &= -9 \\ x + 2y &= -8 \end{aligned}$

For the following exercises, write the linear system from the augmented matrix.

11. $\left[\begin{array}{cc|c} -2 & 5 & 5 \\ 6 & -18 & 26 \end{array}\right]$

12. $\left[\begin{array}{cc|c} 3 & 4 & 10 \\ 10 & 17 & 439 \end{array}\right]$

13. $\left[\begin{array}{ccc|c} 3 & 2 & 0 & 3 \\ -1 & -9 & 4 & -1 \\ 8 & 5 & 7 & 8 \end{array}\right]$

14. $\left[\begin{array}{ccc|c} 8 & 29 & 1 & 43 \\ -1 & 7 & 5 & 38 \\ 0 & 0 & 3 & 10 \end{array}\right]$

15. $\left[\begin{array}{ccc|c} 4 & 5 & -2 & 12 \\ 0 & 1 & 58 & 2 \\ 8 & 7 & -3 & -5 \end{array}\right]$

For the following exercises, solve the system by Gaussian elimination.

16. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right]$

17. $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 0 & 2 \end{array}\right]$

18. $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$

19. $\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 4 & -5 & 6 \end{array}\right]$

20. $\left[\begin{array}{cc|c} -2 & 0 & 1 \\ 0 & 2 & -1 \end{array}\right]$

21. $\begin{aligned} 2x - 3y &= -9 \\ 5x + 4y &= 58 \end{aligned}$

22. $\begin{aligned} 6x + 2y &= -4 \\ 3x + 4y &= -17 \end{aligned}$

23. $\begin{aligned} 2x + 3y &= 12 \\ 4x + y &= 14 \end{aligned}$

24. $\begin{aligned} -4x - 3y &= -2 \\ 3x - 5y &= -13 \end{aligned}$

25. $\begin{aligned} -5x + 8y &= 3 \\ 10x + 6y &= 5 \end{aligned}$

26. $\begin{aligned} 3x + 4y &= 12 \\ -6x - 8y &= -24 \end{aligned}$

27. $\begin{aligned} -60x + 45y &= 12 \\ 20x - 15y &= -4 \end{aligned}$

28. $\begin{aligned} 11x + 10y &= 43 \\ 15x + 20y &= 65 \end{aligned}$

29. $\begin{aligned} 2x - y &= 2 \\ 3x + 2y &= 17 \end{aligned}$

30. $\begin{aligned} -1.06x - 2.25y &= 5.51 \\ -5.03x - 1.08y &= 5.40 \end{aligned}$

31. $\begin{aligned} \tfrac{3}{4}x - \tfrac{3}{5}y &= 4 \\ \tfrac{1}{4}x + \tfrac{2}{3}y &= 1 \end{aligned}$

32. $\begin{aligned} \tfrac{1}{4}x - \tfrac{2}{3}y &= -1 \\ \tfrac{1}{2}x + \tfrac{1}{3}y &= 3 \end{aligned}$

33. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 31 \\ 0 & 1 & 1 & 45 \\ 0 & 0 & 1 & 87 \end{array}\right]$

34. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 50 \\ 1 & 1 & 0 & 20 \\ 0 & 1 & 1 & -90 \end{array}\right]$

35. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{array}\right]$

36. $\left[\begin{array}{ccc|c} -0.1 & 0.3 & -0.1 & 0.2 \\ -0.4 & 0.2 & 0.1 & 0.8 \\ 0.6 & 0.1 & 0.7 & -0.8 \end{array}\right]$

37. $\begin{aligned} -2x + 3y - 2z &= 3 \\ 4x + 2y - z &= 9 \\ 4x - 8y + 2z &= -6 \end{aligned}$

38. $\begin{aligned} x + y - 4z &= -4 \\ 5x - 3y - 2z &= 0 \\ 2x + 6y + 7z &= 30 \end{aligned}$

39. $\begin{aligned} 2x + 3y + 2z &= 1 \\ -4x - 6y - 4z &= -2 \\ 10x + 15y + 10z &= 5 \end{aligned}$

40. $\begin{aligned} x + 2y - z &= 1 \\ -x - 2y + 2z &= -2 \\ 3x + 6y - 3z &= 5 \end{aligned}$

41. $\begin{aligned} x + 2y - z &= 1 \\ -x - 2y + 2z &= -2 \\ 3x + 6y - 3z &= 3 \end{aligned}$

42. $\begin{aligned} x + y &= 2 \\ x + z &= 1 \\ -y - z &= -3 \end{aligned}$

43. $$\begin{aligned} x+y+z&=100 \\ x+2z&=125 \\ -y+2z&=25 \end{aligned}$$

44. $$\begin{aligned} \tfrac{1}{4}x-\tfrac{2}{3}z&=-\tfrac{1}{2} \\ \tfrac{1}{5}x+\tfrac{1}{3}y&=\tfrac{4}{7} \\ \tfrac{1}{5}y-\tfrac{1}{3}z&=\tfrac{2}{9} \end{aligned}$$

45. $$\begin{aligned} -\tfrac{1}{2}x+\tfrac{1}{2}y+\tfrac{1}{7}z&=-\tfrac{53}{14} \\ \tfrac{1}{2}x-\tfrac{1}{2}y+\tfrac{1}{4}z&=3 \\ \tfrac{1}{4}x+\tfrac{1}{5}y+\tfrac{1}{3}z&=\tfrac{23}{15} \end{aligned}$$

46. $$\begin{aligned} -\tfrac{1}{2}x-\tfrac{1}{3}y+\tfrac{1}{4}z&=-\tfrac{29}{6} \\ \tfrac{1}{5}x+\tfrac{1}{6}y-\tfrac{1}{7}z&=\tfrac{431}{210} \\ -\tfrac{1}{8}x+\tfrac{1}{9}y+\tfrac{1}{10}z&=-\tfrac{49}{45} \end{aligned}$$

Extensions

For the following exercises, use Gaussian elimination to solve the system.

47. $$\begin{aligned} \tfrac{x-1}{7}+\tfrac{y-2}{8}+\tfrac{z-3}{4}&=0 \\ x+y+z&=6 \\ \tfrac{x+2}{3}+2y+\tfrac{z-3}{3}&=5 \end{aligned}$$

48. $$\begin{aligned} \tfrac{x-1}{4}-\tfrac{y+1}{4}+3z&=-1 \\ \tfrac{x+5}{2}+\tfrac{y+7}{4}-z&=4 \\ x+y-\tfrac{z-2}{2}&=1 \end{aligned}$$

49. $$\begin{aligned} \tfrac{x-3}{4}-\tfrac{y-1}{3}+2z&=-1 \\ \tfrac{x+5}{2}+\tfrac{y+5}{2}+\tfrac{z+5}{2}&=8 \\ x+y+z&=1 \end{aligned}$$

50. $$\begin{aligned} \tfrac{x-3}{10}+\tfrac{y+3}{2}-2z&=3 \\ \tfrac{x+5}{4}-\tfrac{y-1}{8}+z&=\tfrac{3}{2} \\ \tfrac{x-1}{4}+\tfrac{y+4}{2}+3z&=\tfrac{3}{2} \end{aligned}$$

51. $$\begin{aligned} \tfrac{x-3}{4}-\tfrac{y-1}{3}+2z&=-1 \\ \tfrac{x+5}{2}+\tfrac{y+5}{2}+\tfrac{z+5}{2}&=7 \\ x+y+z&=1 \end{aligned}$$

Real-World Applications

For the following exercises, set up the augmented matrix that describes the situation, and solve for the desired solution.

52. Every day, Angeni's cupcake store sells 5,000 cupcakes in chocolate and vanilla flavors. If the chocolate flavor is 3 times as popular as the vanilla flavor, how many of each cupcake does the store sell per day?

53. At Bakari's competing cupcake store, \$4,520 worth of cupcakes are sold daily. The chocolate cupcakes cost \$2.25 and the red velvet cupcakes cost \$1.75. If the total number of cupcakes sold per day is 2,200, how many of each flavor are sold each day?

54. You invested \$10,000 into two accounts: one that has simple 3% interest, the other with 2.5% interest. If your total interest payment after one year was \$283.50, how much was in each account after the year passed?

55. You invested $2,300 into account 1, and $2,700 into account 2. If the total amount of interest after one year is $254, and account 2 has 1.5 times the interest rate of account 1, what are the interest rates? Assume simple interest rates.

56. Bikes’R’Us manufactures bikes, which sell for $250. It costs the manufacturer $180 per bike, plus a startup fee of $3,500. After how many bikes sold will the manufacturer break even?

57. A major appliance store has agreed to order vacuums from a startup founded by college engineering students. The store would be able to purchase the vacuums for $86 each, with a delivery fee of $9,200, regardless of how many vacuums are sold. If the store needs to start seeing a profit after 230 units are sold, how much should they charge for the vacuums?

58. The three most popular ice cream flavors are chocolate, strawberry, and vanilla, comprising 83% of the flavors sold at an ice cream shop. If vanilla sells 1% more than twice strawberry, and chocolate sells 11% more than vanilla, how much of the total ice cream consumption are the vanilla, chocolate, and strawberry flavors?

59. At an ice cream shop, three flavors are increasing in demand. Last year, banana, pumpkin, and rocky road ice cream made up 12% of total ice cream sales. This year, the same three ice creams made up 16.9% of ice cream sales. The rocky road sales doubled, the banana sales increased by 50%, and the pumpkin sales increased by 20%. If the rocky road ice cream had one less percent of sales than the banana ice cream, find out the percentage of ice cream sales each individual ice cream made last year.

60. A bag of mixed nuts contains cashews, pistachios, and almonds. There are 1,000 total nuts in the bag, and there are 100 less almonds than pistachios. The cashews weigh 3 g, pistachios weigh 4 g, and almonds weigh 5 g. If the bag weighs 3.7 kg, find out how many of each type of nut is in the bag.

61. A bag of mixed nuts contains cashews, pistachios, and almonds. Originally there were 900 nuts in the bag. 30% of the almonds, 20% of the cashews, and 10% of the pistachios were eaten, and now there are 770 nuts left in the bag. Originally, there were 100 more cashews than almonds. Figure out how many of each type of nut was in the bag to begin with.

7.7 Solving Systems with Inverses

Learning Objectives

In this section, you will:

- Find the inverse of a matrix.
- Solve a system of linear equations using an inverse matrix.

Soriya plans to invest \$10,500 into two different bonds to spread out her risk. The first bond has an annual return of 10%, and the second bond has an annual return of 6%. In order to receive an 8.5% return from the two bonds, how much should Soriya invest in each bond? What is the best method to solve this problem?

There are several ways we can solve this problem. As we have seen in previous sections, systems of equations and matrices are useful in solving real-world problems involving finance. After studying this section, we will have the tools to solve the bond problem using the inverse of a matrix.

Finding the Inverse of a Matrix

We know that the multiplicative inverse of a real number a is a^{-1}, and $aa^{-1} = a^{-1}a = \left(\frac{1}{a}\right)a = 1$. For example, $2^{-1} = \frac{1}{2}$ and $\left(\frac{1}{2}\right)2 = 1$. The multiplicative inverse of a matrix is similar in concept, except that the product of matrix A and its inverse A^{-1} equals the identity matrix. The identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by I_n where n represents the dimension of the matrix. Observe the following equations.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The identity matrix acts as a 1 in matrix algebra. For example, $AI = IA = A$.

A matrix that has a multiplicative inverse has the properties

$$AA^{-1} = I$$
$$A^{-1}A = I$$

A matrix that has a multiplicative inverse is called an invertible matrix. Only a square matrix may have a multiplicative inverse, as the reversibility, $AA^{-1} = A^{-1}A = I$, is a requirement. Not all square matrices have an inverse, but if A is invertible, then A^{-1} is unique. We will look at two methods for finding the inverse of a 2×2 matrix and a third method that can be used on both 2×2 and 3×3 matrices.

The Identity Matrix and Multiplicative Inverse

The **identity matrix**, I_n, is a square matrix containing ones down the main diagonal and zeros everywhere else.

$$\underset{2\times 2}{I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad \underset{3\times 3}{I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

If A is an $n \times n$ matrix and B is an $n \times n$ matrix such that $AB = BA = I_n$, then $B = A^{-1}$, the **multiplicative inverse of a matrix** A.

EXAMPLE 1

Showing That the Identity Matrix Acts as a 1

Given matrix *A*, show that $AI = IA = A$.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Solution

Use matrix multiplication to show that the product of A and the identity is equal to the product of the identity and A.

$$AI = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot 1 \\ -2 \cdot 1 + 5 \cdot 0 & -2 \cdot 0 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 0 \cdot (-2) & 1 \cdot 4 + 0 \cdot 5 \\ 0 \cdot 3 + 1 \cdot (-2) & 0 \cdot 4 + 1 \cdot 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

HOW TO

Given two matrices, show that one is the multiplicative inverse of the other.

1. Given matrix A of order $n \times n$ and matrix B of order $n \times n$ multiply AB.
2. If $AB = I$, then find the product BA. If $BA = I$, then $B = A^{-1}$ and $A = B^{-1}$.

EXAMPLE 2

Showing That Matrix *A* Is the Multiplicative Inverse of Matrix *B*

Show that the given matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix}, B = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix}$$

Solution

Multiply AB and BA. If both products equal the identity, then the two matrices are inverses of each other.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \cdot \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-9) + 5(2) & 1(-5) + 5(1) \\ -2(-9) - 9(2) & -2(-5) - 9(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ BA &= \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \\ &= \begin{bmatrix} -9(1) - 5(-2) & -9(5) - 5(-9) \\ 2(1) + 1(-2) & 2(5) + 1(-9) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

A and B are inverses of each other.

TRY IT #1 Show that the following two matrices are inverses of each other.

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}, B = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

Finding the Multiplicative Inverse Using Matrix Multiplication

We can now determine whether two matrices are inverses, but how would we find the inverse of a given matrix? Since we know that the product of a matrix and its inverse is the identity matrix, we can find the inverse of a matrix by setting up an equation using matrix multiplication.

EXAMPLE 3

Finding the Multiplicative Inverse Using Matrix Multiplication

Use matrix multiplication to find the inverse of the given matrix.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Solution

For this method, we multiply A by a matrix containing unknown constants and set it equal to the identity.

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find the product of the two matrices on the left side of the equal sign.

$$\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a{-}2c & 1b{-}2d \\ 2a{-}3c & 2b{-}3d \end{bmatrix}$$

Next, set up a system of equations with the entry in row 1, column 1 of the new matrix equal to the first entry of the identity, 1. Set the entry in row 2, column 1 of the new matrix equal to the corresponding entry of the identity, which is 0.

$$\begin{aligned} 1a{-}2c &= 1 \quad R_1 \\ 2a{-}3c &= 0 \quad R_2 \end{aligned}$$

Using row operations, multiply and add as follows: $(-2)R_1 + R_2 \rightarrow R_2$. Add the equations, and solve for c.

$$\begin{aligned} 1a - 2c &= 1 \\ 0 + 1c &= -2 \\ c &= -2 \end{aligned}$$

Back-substitute to solve for a.

$$\begin{aligned} a{-}2(-2) &= 1 \\ a + 4 &= 1 \\ a &= -3 \end{aligned}$$

Write another system of equations setting the entry in row 1, column 2 of the new matrix equal to the corresponding entry of the identity, 0. Set the entry in row 2, column 2 equal to the corresponding entry of the identity.

$$\begin{aligned} 1b{-}2d &= 0 \quad R_1 \\ 2b{-}3d &= 1 \quad R_2 \end{aligned}$$

Using row operations, multiply and add as follows: $(-2)\ R_1 + R_2 = R_2$. Add the two equations and solve for d.

$$1b{-}2d = 0$$
$$\frac{0{+}1d{=}1}{d{=}1}$$

Once more, back-substitute and solve for b.

$$\begin{aligned} b{-}2(1) &= 0 \\ b{-}2 &= 0 \\ b &= 2 \end{aligned}$$

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

Finding the Multiplicative Inverse by Augmenting with the Identity

Another way to find the multiplicative inverse is by augmenting with the identity. When matrix A is transformed into I, the augmented matrix I transforms into A^{-1}.

For example, given

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

augment A with the identity

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right]$$

Perform row operations with the goal of turning A into the identity.

1. Switch row 1 and row 2.
$$\left[\begin{array}{cc|cc} 5 & 3 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right]$$
2. Multiply row 2 by -2 and add to row 1.
$$\left[\begin{array}{cc|cc} 1 & 1 & -2 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right]$$
3. Multiply row 1 by -2 and add to row 2.
$$\left[\begin{array}{cc|cc} 1 & 1 & -2 & 1 \\ 0 & -1 & 5 & -2 \end{array}\right]$$
4. Add row 2 to row 1.
$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & -1 & 5 & -2 \end{array}\right]$$
5. Multiply row 2 by -1.
$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

The matrix we have found is A^{-1}.

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Finding the Multiplicative Inverse of 2×2 Matrices Using a Formula

When we need to find the multiplicative inverse of a 2×2 matrix, we can use a special formula instead of using matrix multiplication or augmenting with the identity.

If A is a 2×2 matrix, such as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the multiplicative inverse of A is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $ad - bc \neq 0$. If $ad - bc = 0$, then A has no inverse.

EXAMPLE 4

Using the Formula to Find the Multiplicative Inverse of Matrix *A*

Use the formula to find the multiplicative inverse of

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Solution

Using the formula, we have

$$
\begin{aligned}
A^{-1} &= \frac{1}{(1)(-3)-(-2)(2)}\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \\
&= \frac{1}{-3+4}\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}
\end{aligned}
$$

Analysis

We can check that our formula works by using one of the other methods to calculate the inverse. Let's augment A with the identity.

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

Perform row operations with the goal of turning A into the identity.

1. Multiply row 1 by -2 and add to row 2.

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

2. Multiply row 1 by 2 and add to row 1.

$$\left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

So, we have verified our original solution.

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

TRY IT #2 Use the formula to find the inverse of matrix A. Verify your answer by augmenting with the identity matrix.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

EXAMPLE 5

Finding the Inverse of the Matrix, If It Exists

Find the inverse, if it exists, of the given matrix.

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

Solution

We will use the method of augmenting with the identity.

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

1. Switch row 1 and row 2.

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right]$$

2. Multiply row 1 by −3 and add it to row 2.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]$$

3. There is nothing further we can do. The zeros in row 2 indicate that this matrix has no inverse.

Finding the Multiplicative Inverse of 3×3 Matrices

Unfortunately, we do not have a formula similar to the one for a 2×2 matrix to find the inverse of a 3×3 matrix. Instead, we will augment the original matrix with the identity matrix and use row operations to obtain the inverse.

Given a 3×3 matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

augment A with the identity matrix

$$A \mid I = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

To begin, we write the augmented matrix with the identity on the right and A on the left. Performing elementary row operations so that the identity matrix appears on the left, we will obtain the inverse matrix on the right. We will find the inverse of this matrix in the next example.

HOW TO

Given a 3×3 matrix, find the inverse

1. Write the original matrix augmented with the identity matrix on the right.
2. Use elementary row operations so that the identity appears on the left.
3. What is obtained on the right is the inverse of the original matrix.
4. Use matrix multiplication to show that $AA^{-1} = I$ and $A^{-1}A = I$.

EXAMPLE 6

Finding the Inverse of a 3 × 3 Matrix

Given the 3×3 matrix A, find the inverse.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

Solution

Augment A with the identity matrix, and then begin row operations until the identity matrix replaces A. The matrix on the right will be the inverse of A.

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Interchange } R_2 \text{ and } R_1} \left[\begin{array}{ccc|ccc} 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-R_2 + R_1 = R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$R_3 \leftrightarrow R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$-2R_1 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array}\right]$$

$$-3R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

Thus,

$$A^{-1} = B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Analysis

To prove that $B = A^{-1}$, let's multiply the two matrices together to see if the product equals the identity, if $AA^{-1} = I$ and $A^{-1}A = I$.

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 2(-1)+3(-1)+1(6) & 2(1)+3(0)+1(-2) & 2(0)+3(1)+1(-3) \\ 3(-1)+3(-1)+1(6) & 3(1)+3(0)+1(-2) & 3(0)+3(1)+1(-3) \\ 2(-1)+4(-1)+1(6) & 2(1)+4(0)+1(-2) & 2(0)+4(1)+1(-3) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A^{-1}A &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1(2)+1(3)+0(2) & -1(3)+1(3)+0(4) & -1(1)+1(1)+0(1) \\ -1(2)+0(3)+1(2) & -1(3)+0(3)+1(4) & -1(1)+0(1)+1(1) \\ 6(2)+-2(3)+-3(2) & 6(3)+-2(3)+-3(4) & 6(1)+-2(1)+-3(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

TRY IT #3 Find the inverse of the 3×3 matrix.

$$A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$$

Solving a System of Linear Equations Using the Inverse of a Matrix

Solving a system of linear equations using the inverse of a matrix requires the definition of two new matrices: X is the matrix representing the variables of the system, and B is the matrix representing the constants. Using matrix multiplication, we may define a system of equations with the same number of equations as variables as

$$AX = B$$

To solve a system of linear equations using an inverse matrix, let A be the coefficient matrix, let X be the variable matrix, and let B be the constant matrix. Thus, we want to solve a system $AX = B$. For example, look at the following system of equations.

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}$$

From this system, the coefficient matrix is

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The variable matrix is

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

And the constant matrix is

$$B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then $AX = B$ looks like

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Recall the discussion earlier in this section regarding multiplying a real number by its inverse, $(2^{-1})\ 2 = \left(\frac{1}{2}\right)\ 2 = 1$. To solve a single linear equation $ax = b$ for x, we would simply multiply both sides of the equation by the multiplicative inverse (reciprocal) of a. Thus,

$$\begin{aligned} ax &= b \\ \left(\frac{1}{a}\right) ax &= \left(\frac{1}{a}\right) b \\ (a^{-1})ax &= (a^{-1})b \\ [(a^{-1})a]x &= (a^{-1})b \\ 1x &= (a^{-1})b \\ x &= (a^{-1})b \end{aligned}$$

The only difference between a solving a linear equation and a system of equations written in matrix form is that finding the inverse of a matrix is more complicated, and matrix multiplication is a longer process. However, the goal is the same—to isolate the variable.

We will investigate this idea in detail, but it is helpful to begin with a 2×2 system and then move on to a 3×3 system.

Solving a System of Equations Using the Inverse of a Matrix

Given a system of equations, write the coefficient matrix A, the variable matrix X, and the constant matrix B. Then

$$AX = B$$

Multiply both sides by the inverse of A to obtain the solution.

$$\begin{aligned} \left(A^{-1}\right) AX &= \left(A^{-1}\right) B \\ \left[\left(A^{-1}\right) A\right] X &= \left(A^{-1}\right) B \\ IX &= \left(A^{-1}\right) B \\ X &= \left(A^{-1}\right) B \end{aligned}$$

Q&A **If the coefficient matrix does not have an inverse, does that mean the system has no solution?**

No, if the coefficient matrix is not invertible, the system could be inconsistent and have no solution, or be dependent and have infinitely many solutions.

EXAMPLE 7

Solving a 2 × 2 System Using the Inverse of a Matrix

Solve the given system of equations using the inverse of a matrix.

$$\begin{aligned} 3x + 8y &= 5 \\ 4x + 11y &= 7 \end{aligned}$$

Solution

Write the system in terms of a coefficient matrix, a variable matrix, and a constant matrix.

$$A = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

First, we need to calculate A^{-1}. Using the formula to calculate the inverse of a 2 by 2 matrix, we have:

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3(11)-8(4)} \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \\ &= \frac{1}{1} \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \end{aligned}$$

So,

$$A^{-1} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

Now we are ready to solve. Multiply both sides of the equation by A^{-1}.

$$\begin{aligned} &(A^{-1})AX = (A^{-1})B \\ &\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} \\ &\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11(5) + (-8)7 \\ -4(5) + 3(7) \end{bmatrix} \\ &\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

The solution is $(-1, 1)$.

Q&A **Can we solve for X by finding the product BA^{-1}?**

No, recall that matrix multiplication is not commutative, so $A^{-1}B \neq BA^{-1}$. Consider our steps for solving the matrix equation.

$$\begin{aligned} \left(A^{-1}\right) AX &= \left(A^{-1}\right) B \\ \left[\left(A^{-1}\right) A\right] X &= \left(A^{-1}\right) B \\ IX &= \left(A^{-1}\right) B \\ X &= \left(A^{-1}\right) B \end{aligned}$$

Notice in the first step we multiplied both sides of the equation by A^{-1}, but the A^{-1} was to the left of A on the left side and to the left of B on the right side. Because matrix multiplication is not commutative, order matters.

EXAMPLE 8

Solving a 3 × 3 System Using the Inverse of a Matrix

Solve the following system using the inverse of a matrix.

$$\begin{aligned} 5x+15y+56z&=35\\ -4x-11y-41z&=-26\\ -x-3y-11z&=-7 \end{aligned}$$

Solution

Write the equation $AX=B$.

$$\begin{bmatrix} 5 & 15 & 56\\ -4 & -11 & -41\\ -1 & -3 & -11 \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 35\\ -26\\ -7 \end{bmatrix}$$

First, we will find the inverse of A by augmenting with the identity.

$$\left[\begin{array}{ccc|ccc} 5 & 15 & 56 & 1 & 0 & 0\\ -4 & -11 & -41 & 0 & 1 & 0\\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by $\frac{1}{5}$.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0\\ -4 & -11 & -41 & 0 & 1 & 0\\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by 4 and add to row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0\\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0\\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Add row 1 to row 3.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0\\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0\\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array}\right]$$

Multiply row 2 by −3 and add to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0\\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0\\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array}\right]$$

Multiply row 3 by 5.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0\\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

Multiply row 3 by $\frac{1}{5}$ and add to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1\\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0\\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

Multiply row 3 by $-\frac{19}{5}$ and add to row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & 1 & -19 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

So,

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix}$$

Multiply both sides of the equation by A^{-1}. We want $A^{-1}AX = A^{-1}B$:

$$\begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

Thus,

$$A^{-1}B = \begin{bmatrix} -70 + 78 - 7 \\ -105 - 26 + 133 \\ 35 + 0 - 35 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The solution is $(1, 2, 0)$.

TRY IT #4 Solve the system using the inverse of the coefficient matrix.

$$\begin{aligned} 2x - 17y + 11z &= 0 \\ -x + 11y - 7z &= 8 \\ 3y - 2z &= -2 \end{aligned}$$

HOW TO

Given a system of equations, solve with matrix inverses using a calculator.

1. Save the coefficient matrix and the constant matrix as matrix variables $[A]$ and $[B]$.
2. Enter the multiplication into the calculator, calling up each matrix variable as needed.
3. If the coefficient matrix is invertible, the calculator will present the solution matrix; if the coefficient matrix is not invertible, the calculator will present an error message.

EXAMPLE 9

Using a Calculator to Solve a System of Equations with Matrix Inverses

Solve the system of equations with matrix inverses using a calculator

$$\begin{aligned} 2x + 3y + z &= 32 \\ 3x + 3y + z &= -27 \\ 2x + 4y + z &= -2 \end{aligned}$$

Solution

On the matrix page of the calculator, enter the coefficient matrix as the matrix variable $[A]$, and enter the constant matrix as the matrix variable $[B]$.

$$[A] = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}, \quad [B] = \begin{bmatrix} 32 \\ -27 \\ -2 \end{bmatrix}$$

On the home screen of the calculator, type in the multiplication to solve for X, calling up each matrix variable as needed.

$$[A]^{-1} \times [B]$$

Evaluate the expression.

$$\begin{bmatrix} -59 \\ -34 \\ 252 \end{bmatrix}$$

MEDIA

Access these online resources for additional instruction and practice with solving systems with inverses.

[The Identity Matrix (http://openstax.org/l/identmatrix)](http://openstax.org/l/identmatrix)
[Determining Inverse Matrices (http://openstax.org/l/inversematrix)](http://openstax.org/l/inversematrix)
[Using a Matrix Equation to Solve a System of Equations (http://openstax.org/l/matrixsystem)](http://openstax.org/l/matrixsystem)

7.7 SECTION EXERCISES

Verbal

1. In a previous section, we showed that matrix multiplication is not commutative, that is, $AB \neq BA$ in most cases. Can you explain why matrix multiplication is commutative for matrix inverses, that is, $A^{-1}A = AA^{-1}$?

2. Does every 2×2 matrix have an inverse? Explain why or why not. Explain what condition is necessary for an inverse to exist.

3. Can you explain whether a 2×2 matrix with an entire row of zeros can have an inverse?

4. Can a matrix with an entire column of zeros have an inverse? Explain why or why not.

5. Can a matrix with zeros on the diagonal have an inverse? If so, find an example. If not, prove why not. For simplicity, assume a 2×2 matrix.

Algebraic

In the following exercises, show that matrix A is the inverse of matrix B.

6. $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

8. $A = \begin{bmatrix} 4 & 5 \\ 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{1}{5} & -\frac{4}{35} \end{bmatrix}$

9. $A = \begin{bmatrix} -2 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix}$

10. $A=\begin{bmatrix}1&0&1\\0&1&-1\\0&1&1\end{bmatrix},\ B=\frac{1}{2}\begin{bmatrix}2&1&-1\\0&1&1\\0&-1&1\end{bmatrix}$

11. $A=\begin{bmatrix}1&2&3\\4&0&2\\1&6&9\end{bmatrix},\ B=\frac{1}{4}\begin{bmatrix}6&0&-2\\17&-3&-5\\-12&2&4\end{bmatrix}$

12. $A=\begin{bmatrix}3&8&2\\1&1&1\\5&6&12\end{bmatrix},\ B=\frac{1}{36}\begin{bmatrix}-6&84&-6\\7&-26&1\\-1&-22&5\end{bmatrix}$

For the following exercises, find the multiplicative inverse of each matrix, if it exists.

13. $\begin{bmatrix}3&-2\\1&9\end{bmatrix}$

14. $\begin{bmatrix}-2&2\\3&1\end{bmatrix}$

15. $\begin{bmatrix}-3&7\\9&2\end{bmatrix}$

16. $\begin{bmatrix}-4&-3\\-5&8\end{bmatrix}$

17. $\begin{bmatrix}1&1\\2&2\end{bmatrix}$

18. $\begin{bmatrix}0&1\\1&0\end{bmatrix}$

19. $\begin{bmatrix}0.5&1.5\\1&-0.5\end{bmatrix}$

20. $\begin{bmatrix}1&0&6\\-2&1&7\\3&0&2\end{bmatrix}$

21. $\begin{bmatrix}0&1&-3\\4&1&0\\1&0&5\end{bmatrix}$

22. $\begin{bmatrix}1&2&-1\\-3&4&1\\-2&-4&-5\end{bmatrix}$

23. $\begin{bmatrix}1&9&-3\\2&5&6\\4&-2&7\end{bmatrix}$

24. $\begin{bmatrix}1&-2&3\\-4&8&-12\\1&4&2\end{bmatrix}$

25. $\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\\frac{1}{3}&\frac{1}{4}&\frac{1}{5}\\\frac{1}{6}&\frac{1}{7}&\frac{1}{8}\end{bmatrix}$

26. $\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$

For the following exercises, solve the system using the inverse of a 2×2 matrix.

27. $\begin{aligned}5x-6y&=-61\\4x+3y&=-2\end{aligned}$

28. $\begin{aligned}8x+4y&=-100\\3x-4y&=1\end{aligned}$

29. $\begin{aligned}3x-2y&=6\\-x+5y&=-2\end{aligned}$

30. $\begin{aligned}5x-4y&=-5\\4x+y&=2.3\end{aligned}$

31. $\begin{aligned}-3x-4y&=9\\12x+4y&=-6\end{aligned}$

32. $\begin{aligned}-2x+3y&=\tfrac{3}{10}\\-x+5y&=\tfrac{1}{2}\end{aligned}$

33. $\begin{aligned}\tfrac{8}{5}x-\tfrac{4}{5}y&=\tfrac{2}{5}\\-\tfrac{8}{5}x+\tfrac{1}{5}y&=\tfrac{7}{10}\end{aligned}$

34. $\begin{aligned}\tfrac{1}{2}x+\tfrac{1}{5}y&=-\tfrac{1}{4}\\\tfrac{1}{2}x-\tfrac{3}{5}y&=-\tfrac{9}{4}\end{aligned}$

For the following exercises, solve a system using the inverse of a 3×3 matrix.

35. $\begin{aligned}3x-2y+5z&=21\\5x+4y&=37\\x-2y-5z&=5\end{aligned}$

36. $\begin{aligned}4x+4y+4z&=40\\2x-3y+4z&=-12\\-x+3y+4z&=9\end{aligned}$

37. $\begin{aligned}6x-5y-z&=31\\-x+2y+z&=-6\\3x+3y+2z&=13\end{aligned}$

38. $$\begin{aligned} 6x - 5y + 2z &= -4 \\ 2x + 5y - z &= 12 \\ 2x + 5y + z &= 12 \end{aligned}$$

39. $$\begin{aligned} 4x - 2y + 3z &= -12 \\ 2x + 2y - 9z &= 33 \\ 6y - 4z &= 1 \end{aligned}$$

40. $$\begin{aligned} \tfrac{1}{10}x - \tfrac{1}{5}y + 4z &= \tfrac{-41}{2} \\ \tfrac{1}{5}x - 20y + \tfrac{2}{5}z &= -101 \\ \tfrac{3}{10}x + 4y - \tfrac{3}{10}z &= 23 \end{aligned}$$

41. $$\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{5}y + \tfrac{1}{5}z &= \tfrac{31}{100} \\ -\tfrac{3}{4}x - \tfrac{1}{4}y + \tfrac{1}{2}z &= \tfrac{7}{40} \\ -\tfrac{4}{5}x - \tfrac{1}{2}y + \tfrac{3}{2}z &= \tfrac{1}{4} \end{aligned}$$

42. $$\begin{aligned} 0.1x + 0.2y + 0.3z &= -1.4 \\ 0.1x - 0.2y + 0.3z &= 0.6 \\ 0.4y + 0.9z &= -2 \end{aligned}$$

Technology

For the following exercises, use a calculator to solve the system of equations with matrix inverses.

43. $$\begin{aligned} 2x - y &= -3 \\ -x + 2y &= 2.3 \end{aligned}$$

44. $$\begin{aligned} -\tfrac{1}{2}x - \tfrac{3}{2}y &= -\tfrac{43}{20} \\ \tfrac{5}{2}x + \tfrac{11}{5}y &= \tfrac{31}{4} \end{aligned}$$

45. $$\begin{aligned} 12.3x - 2y - 2.5z &= 2 \\ 36.9x + 7y - 7.5z &= -7 \\ 8y - 5z &= -10 \end{aligned}$$

46. $$\begin{aligned} 0.5x - 3y + 6z &= -0.8 \\ 0.7x - 2y &= -0.06 \\ 0.5x + 4y + 5z &= 0 \end{aligned}$$

Extensions

For the following exercises, find the inverse of the given matrix.

47. $$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

48. $$\begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & -1 & 0 \\ 1 & -3 & 0 & 1 \end{bmatrix}$$

49. $$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 4 & -2 & 3 \\ -5 & 0 & 1 & 1 \end{bmatrix}$$

50. $$\begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

51. $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Real-World Applications

For the following exercises, write a system of equations that represents the situation. Then, solve the system using the inverse of a matrix.

52. 2,400 tickets were sold for a basketball game. If the prices for floor 1 and floor 2 were different, and the total amount of money brought in is $64,000, how much was the price of each ticket?

53. In the previous exercise, if you were told there were 400 more tickets sold for floor 2 than floor 1, how much was the price of each ticket?

54. A food drive collected two different types of canned goods, green beans and kidney beans. The total number of collected cans was 350 and the total weight of all donated food was 348 lb, 12 oz. If the green bean cans weigh 2 oz less than the kidney bean cans, how many of each can was donated?

55. Students were asked to bring their favorite fruit to class. 95% of the fruits consisted of banana, apple, and oranges. If oranges were twice as popular as bananas, and apples were 5% less popular than bananas, what are the percentages of each individual fruit?

56. The nursing club held a bake sale to raise money and sold brownies and chocolate chip cookies. They priced the brownies at $1 and the chocolate chip cookies at $0.75. They raised $700 and sold 850 items. How many brownies and how many cookies were sold?

57. A clothing store needs to order new inventory. It has three different types of hats for sale: straw hats, beanies, and cowboy hats. The straw hat is priced at $13.99, the beanie at $7.99, and the cowboy hat at $14.49. If 100 hats were sold this past quarter, $1,119 was taken in by sales, and the amount of beanies sold was 10 more than cowboy hats, how many of each should the clothing store order to replace those already sold?

58. Anna, Percy, and Morgan weigh a combined 370 lb. If Morgan weighs 20 lb more than Percy, and Anna weighs 1.5 times as much as Percy, how much does each person weigh?

59. Three roommates shared a package of 12 ice cream bars, but no one remembers who ate how many. If Micah ate twice as many ice cream bars as Joe, and Albert ate three less than Micah, how many ice cream bars did each roommate eat?

60. A farmer constructed a chicken coop out of chicken wire, wood, and plywood. The chicken wire cost $2 per square foot, the wood $10 per square foot, and the plywood $5 per square foot. The farmer spent a total of $51, and the total amount of materials used was 14 ft^2. He used 3 ft^2 more chicken wire than plywood. How much of each material in did the farmer use?

61. Jay has lemon, orange, and pomegranate trees in his backyard. An orange weighs 8 oz, a lemon 5 oz, and a pomegranate 11 oz. Jay picked 142 pieces of fruit weighing a total of 70 lb, 10 oz. He picked 15.5 times more oranges than pomegranates. How many of each fruit did Jay pick?

7.8 Solving Systems with Cramer's Rule

Learning Objectives

In this section, you will:

- Evaluate 2×2 determinants.
- Use Cramer's Rule to solve a system of equations in two variables.
- Evaluate 3×3 determinants.
- Use Cramer's Rule to solve a system of three equations in three variables.
- Know the properties of determinants.

We have learned how to solve systems of equations in two variables and three variables, and by multiple methods: substitution, addition, Gaussian elimination, using the inverse of a matrix, and graphing. Some of these methods are easier to apply than others and are more appropriate in certain situations. In this section, we will study two more strategies for solving systems of equations.

Evaluating the Determinant of a 2×2 Matrix

A determinant is a real number that can be very useful in mathematics because it has multiple applications, such as calculating area, volume, and other quantities. Here, we will use determinants to reveal whether a matrix is invertible by using the entries of a square matrix to determine whether there is a solution to the system of equations. Perhaps one of the more interesting applications, however, is their use in cryptography. Secure signals or messages are sometimes sent encoded in a matrix. The data can only be decrypted with an invertible matrix and the determinant. For our purposes, we focus on the determinant as an indication of the invertibility of the matrix. Calculating the determinant of a matrix involves following the specific patterns that are outlined in this section.

Find the Determinant of a 2 × 2 Matrix

The **determinant** of a 2×2 matrix, given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Notice the change in notation. There are several ways to indicate the determinant, including $\det(A)$ and replacing the brackets in a matrix with straight lines, $|A|$.

EXAMPLE 1

Finding the Determinant of a 2 × 2 Matrix

Find the determinant of the given matrix.

$$A = \begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} \det(A) &= \begin{vmatrix} 5 & 2 \\ -6 & 3 \end{vmatrix} \\ &= 5(3) - (-6)(2) \\ &= 27 \end{aligned}$$

Using Cramer's Rule to Solve a System of Two Equations in Two Variables

We will now introduce a final method for solving systems of equations that uses determinants. Known as Cramer's Rule, this technique dates back to the middle of the 18th century and is named for its innovator, the Swiss mathematician Gabriel Cramer (1704-1752), who introduced it in 1750 in Introduction à l'Analyse des lignes Courbes algébriques. Cramer's Rule is a viable and efficient method for finding solutions to systems with an arbitrary number of unknowns, provided that we have the same number of equations as unknowns.

Cramer's Rule will give us the unique solution to a system of equations, if it exists. However, if the system has no solution or an infinite number of solutions, this will be indicated by a determinant of zero. To find out if the system is inconsistent or dependent, another method, such as elimination, will have to be used.

To understand Cramer's Rule, let's look closely at how we solve systems of linear equations using basic row operations. Consider a system of two equations in two variables.

$$\begin{aligned} a_1 x + b_1 y &= c_1 \quad (1) \\ a_2 x + b_2 y &= c_2 \quad (2) \end{aligned}$$

We eliminate one variable using row operations and solve for the other. Say that we wish to solve for x. If equation (2) is multiplied by the opposite of the coefficient of y in equation (1), equation (1) is multiplied by the coefficient of y in equation (2), and we add the two equations, the variable y will be eliminated.

$$\begin{array}{ll} b_2 a_1 x + b_2 b_1 y = b_2 c_1 & \text{Multiply } R_1 \text{ by } b_2 \\ \underline{-b_1 a_2 x - b_1 b_2 y = -b_1 c_2} & \underline{\text{Multiply } R_2 \text{ by} - b_1} \\ b_2 a_1 x - b_1 a_2 x = b_2 c_1 - b_1 c_2 & \end{array}$$

Now, solve for x.

$$\begin{aligned} b_2 a_1 x - b_1 a_2 x &= b_2 c_1 - b_1 c_2 \\ x(b_2 a_1 - b_1 a_2) &= b_2 c_1 - b_1 c_2 \\ x &= \frac{b_2 c_1 - b_1 c_2}{b_2 a_1 - b_1 a_2} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \end{aligned}$$

Similarly, to solve for y, we will eliminate x.

$$\begin{array}{ll} a_2 a_1 x + a_2 b_1 y = a_2 c_1 & \text{Multiply } R_1 \text{ by } a_2 \\ \underline{-a_1 a_2 x - a_1 b_2 y = -a_1 c_2} & \underline{\text{Multiply } R_2 \text{ by} - a_1} \\ a_2 b_1 y - a_1 b_2 y = a_2 c_1 - a_1 c_2 & \end{array}$$

Solving for y gives

$$\begin{aligned} a_2b_1y - a_1b_2y &= a_2c_1 - a_1c_2 \\ y(a_2b_1 - a_1b_2) &= a_2c_1 - a_1c_2 \end{aligned}$$

$$y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Notice that the denominator for both x and y is the determinant of the coefficient matrix.

We can use these formulas to solve for x and y, but Cramer's Rule also introduces new notation:

- D : determinant of the coefficient matrix
- D_x : determinant of the numerator in the solution of x
$$x = \frac{D_x}{D}$$
- D_y : determinant of the numerator in the solution of y
$$y = \frac{D_y}{D}$$

The key to Cramer's Rule is replacing the variable column of interest with the constant column and calculating the determinants. We can then express x and y as a quotient of two determinants.

Cramer's Rule for 2×2 Systems

Cramer's Rule is a method that uses determinants to solve systems of equations that have the same number of equations as variables.

Consider a system of two linear equations in two variables.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

The solution using Cramer's Rule is given as

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad D \neq 0; \qquad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad D \neq 0.$$

If we are solving for x, the x column is replaced with the constant column. If we are solving for y, the y column is replaced with the constant column.

EXAMPLE 2

Using Cramer's Rule to Solve a 2 × 2 System

Solve the following 2×2 system using Cramer's Rule.

$$\begin{aligned} 12x + 3y &= 15 \\ 2x - 3y &= 13 \end{aligned}$$

Solution

Solve for x.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{-45 - 39}{-36 - 6} = \frac{-84}{-42} = 2$$

Solve for y.

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{156 - 30}{-36 - 6} = -\frac{126}{42} = -3$$

The solution is $(2, -3)$.

> TRY IT #1 Use Cramer's Rule to solve the 2 × 2 system of equations.

$$\begin{aligned} x + 2y &= -11 \\ -2x + y &= -13 \end{aligned}$$

Evaluating the Determinant of a 3 × 3 Matrix

Finding the determinant of a 2×2 matrix is straightforward, but finding the determinant of a 3×3 matrix is more complicated. One method is to augment the 3×3 matrix with a repetition of the first two columns, giving a 3×5 matrix. Then we calculate the sum of the products of entries *down* each of the three diagonals (upper left to lower right), and subtract the products of entries *up* each of the three diagonals (lower left to upper right). This is more easily understood with a visual and an example.

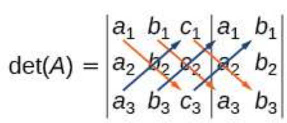
Find the determinant of the 3×3 matrix.

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

1. Augment A with the first two columns.

$$\det(A) = \left| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} \right| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} \right|$$

2. From upper left to lower right: Multiply the entries down the first diagonal. Add the result to the product of entries down the second diagonal. Add this result to the product of the entries down the third diagonal.
3. From lower left to upper right: Subtract the product of entries up the first diagonal. From this result subtract the product of entries up the second diagonal. From this result, subtract the product of entries up the third diagonal.



The algebra is as follows:

$$|A| = a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1$$

EXAMPLE 3

Finding the Determinant of a 3 × 3 Matrix

Find the determinant of the 3 × 3 matrix given

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

Solution

Augment the matrix with the first two columns and then follow the formula. Thus,

$$\begin{aligned}|A| &= \left|\begin{array}{ccc|cc} 0 & 2 & 1 & 0 & 2 \\ 3 & -1 & 1 & 3 & -1 \\ 4 & 0 & 1 & 4 & 0 \end{array}\right| \\ &= 0(-1)(1) + 2(1)(4) + 1(3)(0) - 4(-1)(1) - 0(1)(0) - 1(3)(2) \\ &= 0 + 8 + 0 + 4 - 0 - 6 \\ &= 6\end{aligned}$$

TRY IT #2 Find the determinant of the 3 × 3 matrix.

$$\det(A) = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

Q&A **Can we use the same method to find the determinant of a larger matrix?**

No, this method only works for 2×2 and 3×3 matrices. For larger matrices it is best to use a graphing utility or computer software.

Using Cramer's Rule to Solve a System of Three Equations in Three Variables

Now that we can find the determinant of a 3 × 3 matrix, we can apply Cramer's Rule to solve a system of three equations in three variables. Cramer's Rule is straightforward, following a pattern consistent with Cramer's Rule for 2 × 2 matrices. As the order of the matrix increases to 3 × 3, however, there are many more calculations required.

When we calculate the determinant to be zero, Cramer's Rule gives no indication as to whether the system has no solution or an infinite number of solutions. To find out, we have to perform elimination on the system.

Consider a 3 × 3 system of equations.

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}, D \neq 0$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

If we are writing the determinant D_x, we replace the x column with the constant column. If we are writing the determinant D_y, we replace the y column with the constant column. If we are writing the determinant D_z, we replace the z column with the constant column. Always check the answer.

EXAMPLE 4

Solving a 3 × 3 System Using Cramer's Rule

Find the solution to the given 3 × 3 system using Cramer's Rule.

$$\begin{aligned} x + y - z &= 6 \\ 3x - 2y + z &= -5 \\ x + 3y - 2z &= 14 \end{aligned}$$

Solution

Use Cramer's Rule.

$$D=\begin{vmatrix}1 & 1 & -1\\ 3 & -2 & 1\\ 1 & 3 & -2\end{vmatrix}, D_x=\begin{vmatrix}6 & 1 & -1\\ -5 & -2 & 1\\ 14 & 3 & -2\end{vmatrix}, D_y=\begin{vmatrix}1 & 6 & -1\\ 3 & -5 & 1\\ 1 & 14 & -2\end{vmatrix}, D_z=\begin{vmatrix}1 & 1 & 6\\ 3 & -2 & -5\\ 1 & 3 & 14\end{vmatrix}$$

Then,

$$\begin{aligned} x &= \tfrac{D_x}{D} = \tfrac{-3}{-3} = 1 \\ y &= \tfrac{D_y}{D} = \tfrac{-9}{-3} = 3 \\ z &= \tfrac{D_z}{D} = \tfrac{6}{-3} = -2 \end{aligned}$$

The solution is $(1, 3, -2)$.

TRY IT #3 Use Cramer's Rule to solve the 3 × 3 matrix.

$$\begin{aligned} x-3y+7z &= 13 \\ x+y+z &= 1 \\ x-2y+3z &= 4 \end{aligned}$$

EXAMPLE 5

Using Cramer's Rule to Solve an Inconsistent System

Solve the system of equations using Cramer's Rule.

$$\begin{aligned} 3x-2y &= 4 \quad (1) \\ 6x-4y &= 0 \quad (2) \end{aligned}$$

Solution

We begin by finding the determinants D, D_x, and D_y.

$$D=\begin{vmatrix}3 & -2\\ 6 & -4\end{vmatrix}=3(-4)-6(-2)=0$$

We know that a determinant of zero means that either the system has no solution or it has an infinite number of solutions. To see which one, we use the process of elimination. Our goal is to eliminate one of the variables.

1. Multiply equation (1) by -2.
2. Add the result to equation (2).

$$\begin{array}{l} -6x+4y \;\; = -8 \\ \underline{6x-4y=0 \qquad} \\ 0=-8 \end{array}$$

We obtain the equation $0=-8$, which is false. Therefore, the system has no solution. Graphing the system reveals two parallel lines. See Figure 1.

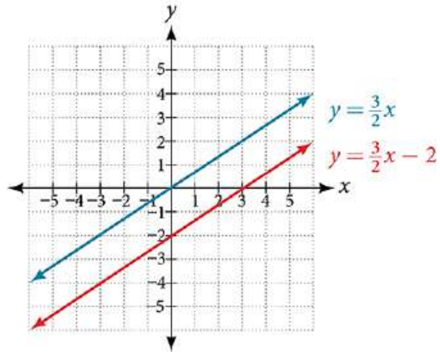


Figure 1

EXAMPLE 6

Use Cramer's Rule to Solve a Dependent System

Solve the system with an infinite number of solutions.

$$\begin{aligned} x - 2y + 3z &= 0 \quad (1) \\ 3x + y - 2z &= 0 \quad (2) \\ 2x - 4y + 6z &= 0 \quad (3) \end{aligned}$$

Solution

Let's find the determinant first. Set up a matrix augmented by the first two columns.

$$\left|\begin{array}{rrr|rr} 1 & -2 & 3 & 1 & -2 \\ 3 & 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & 2 & -4 \end{array}\right|$$

Then,

$$1\,(1)\,(6) + (-2)\,(-2)\,(2) + 3\,(3)\,(-4) - 2\,(1)\,(3) - (-4)\,(-2)\,(1) - 6\,(3)\,(-2) = 0$$

As the determinant equals zero, there is either no solution or an infinite number of solutions. We have to perform elimination to find out.

1. Multiply equation (1) by -2 and add the result to equation (3):

$$\begin{aligned} -2x + 4y - 6z &= 0 \\ 2x - 4y + 6z &= 0 \\ \hline 0 &= 0 \end{aligned}$$

2. Obtaining an answer of $0 = 0$, a statement that is always true, means that the system has an infinite number of solutions. Graphing the system, we can see that two of the planes are the same and they both intersect the third plane on a line. See Figure 2.

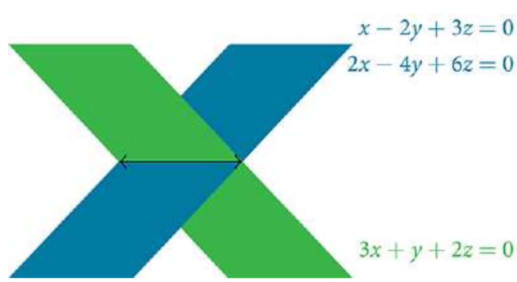


Figure 2

Understanding Properties of Determinants

There are many properties of determinants. Listed here are some properties that may be helpful in calculating the determinant of a matrix.

Properties of Determinants

1. If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.
2. When two rows are interchanged, the determinant changes sign.
3. If either two rows or two columns are identical, the determinant equals zero.
4. If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.
5. The determinant of an inverse matrix A^{-1} is the reciprocal of the determinant of the matrix A.
6. If any row or column is multiplied by a constant, the determinant is multiplied by the same factor.

EXAMPLE 7

Illustrating Properties of Determinants

Illustrate each of the properties of determinants.

Solution

Property 1 states that if the matrix is in upper triangular form, the determinant is the product of the entries down the main diagonal.

$$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 1\\ 0 & 0 & -1\end{bmatrix}$$

Augment A with the first two columns.

$$A=\left[\begin{array}{ccc|cc}1 & 2 & 3 & 1 & 2\\ 0 & 2 & 1 & 0 & 2\\ 0 & 0 & -1 & 0 & 0\end{array}\right]$$

Then

$$\begin{aligned}\det(A) &= 1(2)(-1)+2(1)(0)+3(0)(0)-0(2)(3)-0(1)(1)+1(0)(2)\\ &= -2\end{aligned}$$

Property 2 states that interchanging rows changes the sign. Given

$$A=\begin{bmatrix}-1 & 5\\ 4 & -3\end{bmatrix},\quad \det(A)=(-1)(-3)-(4)(5)=3-20=-17$$

$$B=\begin{bmatrix}4 & -3\\ -1 & 5\end{bmatrix},\quad \det(B)=(4)(5)-(-1)(-3)=20-3=17$$

Property 3 states that if two rows or two columns are identical, the determinant equals zero.

$$A=\left[\begin{array}{ccc|cc}1 & 2 & 2 & 1 & 2\\ 2 & 2 & 2 & 2 & 2\\ -1 & 2 & 2 & -1 & 2\end{array}\right]$$

$$\begin{aligned}\det(A) &= 1(2)(2)+2(2)(-1)+2(2)(2)+1(2)(2)-2(2)(1)-2(2)(2)\\ &= 4-4+8+4-4-8=0\end{aligned}$$

Property 4 states that if a row or column equals zero, the determinant equals zero. Thus,

$$A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix},\quad \det(A)=1\,(0)-2\,(0)=0$$

Property 5 states that the determinant of an inverse matrix A^{-1} is the reciprocal of the determinant A. Thus,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \det(A) = 1(4) - 3(2) = -2$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}, \det\left(A^{-1}\right) = -2\left(-\frac{1}{2}\right) - \left(\frac{3}{2}\right)(1) = -\frac{1}{2}$$

Property 6 states that if any row or column of a matrix is multiplied by a constant, the determinant is multiplied by the same factor. Thus,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \det(A) = 1(4) - 2(3) = -2$$

$$B = \begin{bmatrix} 2(1) & 2(2) \\ 3 & 4 \end{bmatrix}, \det(B) = 2(4) - 3(4) = -4$$

EXAMPLE 8

Using Cramer's Rule and Determinant Properties to Solve a System

Find the solution to the given 3 × 3 system.

$$\begin{aligned} 2x + 4y + 4z &= 2 && (1) \\ 3x + 7y + 7z &= -5 && (2) \\ x + 2y + 2z &= 4 && (3) \end{aligned}$$

Solution

Using Cramer's Rule, we have

$$D = \begin{vmatrix} 2 & 4 & 4 \\ 3 & 7 & 7 \\ 1 & 2 & 2 \end{vmatrix}$$

Notice that the second and third columns are identical. According to Property 3, the determinant will be zero, so there is either no solution or an infinite number of solutions. We have to perform elimination to find out.

1. Multiply equation (3) by –2 and add the result to equation (1).

$$\begin{array}{r} -2x - 4y - 4x = -8 \\ 2x + 4y + 4z = 2 \\ \hline 0 = -6 \end{array}$$

Obtaining a statement that is a contradiction means that the system has no solution.

MEDIA

Access these online resources for additional instruction and practice with Cramer's Rule.

[Solve a System of Two Equations Using Cramer's Rule (http://openstax.org/l/system2cramer)](http://openstax.org/l/system2cramer)
[Solve a Systems of Three Equations using Cramer's Rule (http://openstax.org/l/system3cramer)](http://openstax.org/l/system3cramer)

7.8 SECTION EXERCISES

Verbal

1. Explain why we can always evaluate the determinant of a square matrix.

2. Examining Cramer's Rule, explain why there is no unique solution to the system when the determinant of your matrix is 0. For simplicity, use a 2×2 matrix.

3. Explain what it means in terms of an inverse for a matrix to have a 0 determinant.

4. The determinant of 2×2 matrix A is 3. If you switch the rows and multiply the first row by 6 and the second row by 2, explain how to find the determinant and provide the answer.

Algebraic

For the following exercises, find the determinant.

5. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

6. $\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}$

7. $\begin{vmatrix} 2 & -5 \\ -1 & 6 \end{vmatrix}$

8. $\begin{vmatrix} -8 & 4 \\ -1 & 5 \end{vmatrix}$

9. $\begin{vmatrix} 1 & 0 \\ 3 & -4 \end{vmatrix}$

10. $\begin{vmatrix} 10 & 20 \\ 0 & -10 \end{vmatrix}$

11. $\begin{vmatrix} 10 & 0.2 \\ 5 & 0.1 \end{vmatrix}$

12. $\begin{vmatrix} 6 & -3 \\ 8 & 4 \end{vmatrix}$

13. $\begin{vmatrix} -2 & -3 \\ 3.1 & 4,000 \end{vmatrix}$

14. $\begin{vmatrix} -1.1 & 0.6 \\ 7.2 & -0.5 \end{vmatrix}$

15. $\begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix}$

16. $\begin{vmatrix} -1 & 4 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$

17. $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

18. $\begin{vmatrix} 2 & -3 & 1 \\ 3 & -4 & 1 \\ -5 & 6 & 1 \end{vmatrix}$

19. $\begin{vmatrix} -2 & 1 & 4 \\ -4 & 2 & -8 \\ 2 & -8 & -3 \end{vmatrix}$

20. $\begin{vmatrix} 6 & -1 & 2 \\ -4 & -3 & 5 \\ 1 & 9 & -1 \end{vmatrix}$

21. $\begin{vmatrix} 5 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -6 & -3 \end{vmatrix}$

22. $\begin{vmatrix} 1.1 & 2 & -1 \\ -4 & 0 & 0 \\ 4.1 & -0.4 & 2.5 \end{vmatrix}$

23. $\begin{vmatrix} 2 & -1.6 & 3.1 \\ 1.1 & 3 & -8 \\ -9.3 & 0 & 2 \end{vmatrix}$

24. $\begin{vmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & -\frac{1}{6} & \frac{1}{7} \\ 0 & 0 & \frac{1}{8} \end{vmatrix}$

For the following exercises, solve the system of linear equations using Cramer's Rule.

25. $$\begin{aligned} 2x - 3y &= -1 \\ 4x + 5y &= 9 \end{aligned}$$

26. $$\begin{aligned} 5x - 4y &= 2 \\ -4x + 7y &= 6 \end{aligned}$$

27. $$\begin{aligned} 6x - 3y &= 2 \\ -8x + 9y &= -1 \end{aligned}$$

28. $$\begin{aligned} 2x + 6y &= 12 \\ 5x - 2y &= 13 \end{aligned}$$

29. $$\begin{aligned} 4x + 3y &= 23 \\ 2x - y &= -1 \end{aligned}$$

30. $$\begin{aligned} 10x - 6y &= 2 \\ -5x + 8y &= -1 \end{aligned}$$

31. $$\begin{aligned} 4x - 3y &= -3 \\ 2x + 6y &= -4 \end{aligned}$$

32. $$\begin{aligned} 4x - 5y &= 7 \\ -3x + 9y &= 0 \end{aligned}$$

33. $$\begin{aligned} 4x + 10y &= 180 \\ -3x - 5y &= -105 \end{aligned}$$

34. $$\begin{aligned} 8x - 2y &= -3 \\ -4x + 6y &= 4 \end{aligned}$$

For the following exercises, solve the system of linear equations using Cramer's Rule.

35. $$\begin{aligned} x + 2y - 4z &= -1 \\ 7x + 3y + 5z &= 26 \\ -2x - 6y + 7z &= -6 \end{aligned}$$

36. $$\begin{aligned} -5x + 2y - 4z &= -47 \\ 4x - 3y - z &= -94 \\ 3x - 3y + 2z &= 94 \end{aligned}$$

37. $$\begin{aligned} 4x + 5y - z &= -7 \\ -2x - 9y + 2z &= 8 \\ 5y + 7z &= 21 \end{aligned}$$

38. $$\begin{aligned} 4x - 3y + 4z &= 10 \\ 5x - 2z &= -2 \\ 3x + 2y - 5z &= -9 \end{aligned}$$

39. $$\begin{aligned} 4x - 2y + 3z &= 6 \\ -6x + y &= -2 \\ 2x + 7y + 8z &= 24 \end{aligned}$$

40. $$\begin{aligned} 5x + 2y - z &= 1 \\ -7x - 8y + 3z &= 1.5 \\ 6x - 12y + z &= 7 \end{aligned}$$

41. $$\begin{aligned} 13x - 17y + 16z &= 73 \\ -11x + 15y + 17z &= 61 \\ 46x + 10y - 30z &= -18 \end{aligned}$$

42. $$\begin{aligned} -4x - 3y - 8z &= -7 \\ 2x - 9y + 5z &= 0.5 \\ 5x - 6y - 5z &= -2 \end{aligned}$$

43. $$\begin{aligned} 4x - 6y + 8z &= 10 \\ -2x + 3y - 4z &= -5 \\ x + y + z &= 1 \end{aligned}$$

44. $$\begin{aligned} 4x - 6y + 8z &= 10 \\ -2x + 3y - 4z &= -5 \\ 12x + 18y - 24z &= -30 \end{aligned}$$

Technology

For the following exercises, use the determinant function on a graphing utility.

45. $$\begin{vmatrix} 1 & 0 & 8 & 9 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 4 & 3 \end{vmatrix}$$

46. $$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & -9 & 1 & 3 \\ 3 & 0 & -2 & -1 \\ 0 & 1 & 1 & -2 \end{vmatrix}$$

47. $$\begin{vmatrix} \frac{1}{2} & 1 & 7 & 4 \\ 0 & \frac{1}{2} & 100 & 5 \\ 0 & 0 & 2 & 2{,}000 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

48. $$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{vmatrix}$$

Real-World Applications

For the following exercises, create a system of linear equations to describe the behavior. Then, calculate the determinant. Will there be a unique solution? If so, find the unique solution.

49. Two numbers add up to 56. One number is 20 less than the other.

50. Two numbers add up to 104. If you add two times the first number plus two times the second number, your total is 208

51. Three numbers add up to 106. The first number is 3 less than the second number. The third number is 4 more than the first number.

52. Three numbers add to 216. The sum of the first two numbers is 112. The third number is 8 less than the first two numbers combined.

For the following exercises, create a system of linear equations to describe the behavior. Then, solve the system for all solutions using Cramer's Rule.

53. You invest $10,000 into two accounts, which receive 8% interest and 5% interest. At the end of a year, you had $10,710 in your combined accounts. How much was invested in each account?

54. You invest $80,000 into two accounts, $22,000 in one account, and $58,000 in the other account. At the end of one year, assuming simple interest, you have earned $2,470 in interest. The second account receives half a percent less than twice the interest on the first account. What are the interest rates for your accounts?

55. A theater needs to know how many adult tickets and children tickets were sold out of the 1,200 total tickets. If children's tickets are $5.95, adult tickets are $11.15, and the total amount of revenue was $12,756, how many children's tickets and adult tickets were sold?

56. A concert venue sells single tickets for $40 each and couple's tickets for $65. If the total revenue was $18,090 and the 321 tickets were sold, how many single tickets and how many couple's tickets were sold?

57. You decide to paint your kitchen green. You create the color of paint by mixing yellow and blue paints. You cannot remember how many gallons of each color went into your mix, but you know there were 10 gal total. Additionally, you kept your receipt, and know the total amount spent was $29.50. If each gallon of yellow costs $2.59, and each gallon of blue costs $3.19, how many gallons of each color go into your green mix?

58. You sold two types of scarves at a farmers' market and would like to know which one was more popular. The total number of scarves sold was 56, the yellow scarf cost $10, and the purple scarf cost $11. If you had total revenue of $583, how many yellow scarves and how many purple scarves were sold?

59. Your garden produced two types of tomatoes, one green and one red. The red weigh 10 oz, and the green weigh 4 oz. You have 30 tomatoes, and a total weight of 13 lb, 14 oz. How many of each type of tomato do you have?

60. At a market, the three most popular vegetables make up 53% of vegetable sales. Corn has 4% higher sales than broccoli, which has 5% more sales than onions. What percentage does each vegetable have in the market share?

61. At the same market, the three most popular fruits make up 37% of the total fruit sold. Strawberries sell twice as much as oranges, and kiwis sell one more percentage point than oranges. For each fruit, find the percentage of total fruit sold.

62. Three artists performed at a concert venue. The first one charged $15 per ticket, the second artist charged $45 per ticket, and the final one charged $22 per ticket. There were 510 tickets sold, for a total of $12,700. If the first band had 40 more audience members than the second band, how many tickets were sold for each band?

63. A movie theatre sold tickets to three movies. The tickets to the first movie were $5, the tickets to the second movie were $11, and the third movie was $12. 100 tickets were sold to the first movie. The total number of tickets sold was 642, for a total revenue of $6,774. How many tickets for each movie were sold?

For the following exercises, use this scenario: A health-conscious company decides to make a trail mix out of almonds, dried cranberries, and chocolate-covered cashews. The nutritional information for these items is shown in Table 1.

	Fat (g)	Protein (g)	Carbohydrates (g)
Almonds (10)	6	2	3
Cranberries (10)	0.02	0	8
Cashews (10)	7	3.5	5.5

Table 1

64. For the special “low-carb”trail mix, there are 1,000 pieces of mix. The total number of carbohydrates is 425 g, and the total amount of fat is 570.2 g. If there are 200 more pieces of cashews than cranberries, how many of each item is in the trail mix?

65. For the “hiking” mix, there are 1,000 pieces in the mix, containing 390.8 g of fat, and 165 g of protein. If there is the same amount of almonds as cashews, how many of each item is in the trail mix?

66. For the “energy-booster” mix, there are 1,000 pieces in the mix, containing 145 g of protein and 625 g of carbohydrates. If the number of almonds and cashews summed together is equivalent to the amount of cranberries, how many of each item is in the trail mix?

Chapter Review

Key Terms

addition method an algebraic technique used to solve systems of linear equations in which the equations are added in a way that eliminates one variable, allowing the resulting equation to be solved for the remaining variable; substitution is then used to solve for the first variable

augmented matrix a coefficient matrix adjoined with the constant column separated by a vertical line within the matrix brackets

break-even point the point at which a cost function intersects a revenue function; where profit is zero

coefficient matrix a matrix that contains only the coefficients from a system of equations

column a set of numbers aligned vertically in a matrix

consistent system a system for which there is a single solution to all equations in the system and it is an independent system, or if there are an infinite number of solutions and it is a dependent system

cost function the function used to calculate the costs of doing business; it usually has two parts, fixed costs and variable costs

Cramer's Rule a method for solving systems of equations that have the same number of equations as variables using determinants

dependent system a system of linear equations in which the two equations represent the same line; there are an infinite number of solutions to a dependent system

determinant a number calculated using the entries of a square matrix that determines such information as whether there is a solution to a system of equations

entry an element, coefficient, or constant in a matrix

feasible region the solution to a system of nonlinear inequalities that is the region of the graph where the shaded regions of each inequality intersect

Gaussian elimination using elementary row operations to obtain a matrix in row-echelon form

identity matrix a square matrix containing ones down the main diagonal and zeros everywhere else; it acts as a 1 in matrix algebra

inconsistent system a system of linear equations with no common solution because they represent parallel lines, which have no point or line in common

independent system a system of linear equations with exactly one solution pair (x, y)

main diagonal entries from the upper left corner diagonally to the lower right corner of a square matrix

matrix a rectangular array of numbers

multiplicative inverse of a matrix a matrix that, when multiplied by the original, equals the identity matrix

nonlinear inequality an inequality containing a nonlinear expression

partial fraction decomposition the process of returning a simplified rational expression to its original form, a sum or difference of simpler rational expressions

partial fractions the individual fractions that make up the sum or difference of a rational expression before combining them into a simplified rational expression

profit function the profit function is written as $P(x) = R(x) - C(x)$, revenue minus cost

revenue function the function that is used to calculate revenue, simply written as $R = xp$, where $x =$ quantity and $p =$ price

row a set of numbers aligned horizontally in a matrix

row operations adding one row to another row, multiplying a row by a constant, interchanging rows, and so on, with the goal of achieving row-echelon form

row-echelon form after performing row operations, the matrix form that contains ones down the main diagonal and zeros at every space below the diagonal

row-equivalent two matrices A and B are row-equivalent if one can be obtained from the other by performing basic row operations

scalar multiple an entry of a matrix that has been multiplied by a scalar

solution set the set of all ordered pairs or triples that satisfy all equations in a system of equations

substitution method an algebraic technique used to solve systems of linear equations in which one of the two equations is solved for one variable and then substituted into the second equation to solve for the second variable

system of linear equations a set of two or more equations in two or more variables that must be considered simultaneously.

system of nonlinear equations a system of equations containing at least one equation that is of degree larger than one

system of nonlinear inequalities a system of two or more inequalities in two or more variables containing at least one inequality that is not linear

Key Equations

Identity matrix for a 2×2 matrix	$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Identity matrix for a 3×3 matrix	$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Multiplicative inverse of a 2×2 matrix	$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$

Key Concepts

7.1 Systems of Linear Equations: Two Variables

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See Example 1.
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See Example 2.
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See Example 3.
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See Example 4.
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See Example 5, Example 6, and Example 7.
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See Example 8.
- The solution to a system of dependent equations will always be true because both equations describe the same line. See Example 9.
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See Example 10 and Example 11.

7.2 Systems of Linear Equations: Three Variables

- A solution set is an ordered triple $\{(x, y, z)\}$ that represents the intersection of three planes in space. See Example 1.
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See Example 2.
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See Example 3.
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See Example 4.
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See Example 5.
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

7.3 Systems of Nonlinear Equations and Inequalities: Two Variables

- There are three possible types of solutions to a system of equations representing a line and a parabola: (1) no solution, the line does not intersect the parabola; (2) one solution, the line is tangent to the parabola; and (3) two

solutions, the line intersects the parabola in two points. See Example 1.
- There are three possible types of solutions to a system of equations representing a circle and a line: (1) no solution, the line does not intersect the circle; (2) one solution, the line is tangent to the circle; (3) two solutions, the line intersects the circle in two points. See Example 2.
- There are five possible types of solutions to the system of nonlinear equations representing an ellipse and a circle: (1) no solution, the circle and the ellipse do not intersect; (2) one solution, the circle and the ellipse are tangent to each other; (3) two solutions, the circle and the ellipse intersect in two points; (4) three solutions, the circle and ellipse intersect in three places; (5) four solutions, the circle and the ellipse intersect in four points. See Example 3.
- An inequality is graphed in much the same way as an equation, except for > or <, we draw a dashed line and shade the region containing the solution set. See Example 4.
- Inequalities are solved the same way as equalities, but solutions to systems of inequalities must satisfy both inequalities. See Example 5.

7.4 Partial Fractions

- Decompose $\frac{P(x)}{Q(x)}$ by writing the partial fractions as $\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}$. Solve by clearing the fractions, expanding the right side, collecting like terms, and setting corresponding coefficients equal to each other, then setting up and solving a system of equations. See Example 1.
- The decomposition of $\frac{P(x)}{Q(x)}$ with repeated linear factors must account for the factors of the denominator in increasing powers. See Example 2.
- The decomposition of $\frac{P(x)}{Q(x)}$ with a nonrepeated irreducible quadratic factor needs a linear numerator over the quadratic factor, as in $\frac{A}{x}+\frac{Bx+C}{\left(ax^2+bx+c\right)}$. See Example 3.
- In the decomposition of $\frac{P(x)}{Q(x)}$, where $Q\left(x\right)$ has a repeated irreducible quadratic factor, when the irreducible quadratic factors are repeated, powers of the denominator factors must be represented in increasing powers as

$$\frac{Ax+B}{\left(ax^2+bx+c\right)}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}.$$

  See Example 4.

7.5 Matrices and Matrix Operations

- A matrix is a rectangular array of numbers. Entries are arranged in rows and columns.
- The dimensions of a matrix refer to the number of rows and the number of columns. A 3×2 matrix has three rows and two columns. See Example 1.
- We add and subtract matrices of equal dimensions by adding and subtracting corresponding entries of each matrix. See Example 2, Example 3, Example 4, and Example 5.
- Scalar multiplication involves multiplying each entry in a matrix by a constant. See Example 6.
- Scalar multiplication is often required before addition or subtraction can occur. See Example 7.
- Multiplying matrices is possible when inner dimensions are the same—the number of columns in the first matrix must match the number of rows in the second.
- The product of two matrices, A and B, is obtained by multiplying each entry in row 1 of A by each entry in column 1 of B; then multiply each entry of row 1 of A by each entry in columns 2 of B, and so on. See Example 8 and Example 9.
- Many real-world problems can often be solved using matrices. See Example 10.
- We can use a calculator to perform matrix operations after saving each matrix as a matrix variable. See Example 11.

7.6 Solving Systems with Gaussian Elimination

- An augmented matrix is one that contains the coefficients and constants of a system of equations. See Example 1.
- A matrix augmented with the constant column can be represented as the original system of equations. See Example 2.
- Row operations include multiplying a row by a constant, adding one row to another row, and interchanging rows.
- We can use Gaussian elimination to solve a system of equations. See Example 3, Example 4, and Example 5.
- Row operations are performed on matrices to obtain row-echelon form. See Example 6.
- To solve a system of equations, write it in augmented matrix form. Perform row operations to obtain row-echelon form. Back-substitute to find the solutions. See Example 7 and Example 8.
- A calculator can be used to solve systems of equations using matrices. See Example 9.
- Many real-world problems can be solved using augmented matrices. See Example 10 and Example 11.

7.7 Solving Systems with Inverses

- An identity matrix has the property $AI = IA = A$. See Example 1.
- An invertible matrix has the property $AA^{-1} = A^{-1}A = I$. See Example 2.
- Use matrix multiplication and the identity to find the inverse of a 2×2 matrix. See Example 3.
- The multiplicative inverse can be found using a formula. See Example 4.
- Another method of finding the inverse is by augmenting with the identity. See Example 5.
- We can augment a 3×3 matrix with the identity on the right and use row operations to turn the original matrix into the identity, and the matrix on the right becomes the inverse. See Example 6.
- Write the system of equations as $AX = B$, and multiply both sides by the inverse of A : $A^{-1}AX = A^{-1}B$. See Example 7 and Example 8.
- We can also use a calculator to solve a system of equations with matrix inverses. See Example 9.

7.8 Solving Systems with Cramer's Rule

- The determinant for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. See Example 1.
- Cramer's Rule replaces a variable column with the constant column. Solutions are $x = \frac{D_x}{D}, y = \frac{D_y}{D}$. See Example 2.
- To find the determinant of a 3×3 matrix, augment with the first two columns. Add the three diagonal entries (upper left to lower right) and subtract the three diagonal entries (lower left to upper right). See Example 3.
- To solve a system of three equations in three variables using Cramer's Rule, replace a variable column with the constant column for each desired solution: $x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$. See Example 4.
- Cramer's Rule is also useful for finding the solution of a system of equations with no solution or infinite solutions. See Example 5 and Example 6.
- Certain properties of determinants are useful for solving problems. For example:
  - If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.
  - When two rows are interchanged, the determinant changes sign.
  - If either two rows or two columns are identical, the determinant equals zero.
  - If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.
  - The determinant of an inverse matrix A^{-1} is the reciprocal of the determinant of the matrix A.
  - If any row or column is multiplied by a constant, the determinant is multiplied by the same factor. See Example 7 and Example 8.

Exercises

Review Exercises

Systems of Linear Equations: Two Variables

For the following exercises, determine whether the ordered pair is a solution to the system of equations.

1. $\begin{aligned} 3x - y &= 4 \\ x + 4y &= -3 \end{aligned}$ and $(-1, 1)$

2. $\begin{aligned} 6x - 2y &= 24 \\ -3x + 3y &= 18 \end{aligned}$ and $(9, 15)$

For the following exercises, use substitution to solve the system of equations.

3. $\begin{aligned} 10x + 5y &= -5 \\ 3x - 2y &= -12 \end{aligned}$

4. $\begin{aligned} \tfrac{4}{7}x + \tfrac{1}{5}y &= \tfrac{43}{70} \\ \tfrac{5}{6}x - \tfrac{1}{3}y &= -\tfrac{2}{3} \end{aligned}$

5. $\begin{aligned} 5x + 6y &= 14 \\ 4x + 8y &= 8 \end{aligned}$

For the following exercises, use addition to solve the system of equations.

6. $\begin{aligned} 3x + 2y &= -7 \\ 2x + 4y &= 6 \end{aligned}$

7. $\begin{aligned} 3x + 4y &= 2 \\ 9x + 12y &= 3 \end{aligned}$

8. $\begin{aligned} 8x + 4y &= 2 \\ 6x - 5y &= 0.7 \end{aligned}$

For the following exercises, write a system of equations to solve each problem. Solve the system of equations.

9. A factory has a cost of production $C(x) = 150x + 15{,}000$ and a revenue function $R(x) = 200x$. What is the break-even point?

10. A performer charges $C(x) = 50x + 10{,}000$, where x is the total number of attendees at a show. The venue charges $75 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

Systems of Linear Equations: Three Variables

For the following exercises, solve the system of three equations using substitution or addition.

11. $$\begin{aligned} 0.5x - 0.5y &= 10 \\ -0.2y + 0.2x &= 4 \\ 0.1x + 0.1z &= 2 \end{aligned}$$

12. $$\begin{aligned} 5x + 3y - z &= 5 \\ 3x - 2y + 4z &= 13 \\ 4x + 3y + 5z &= 22 \end{aligned}$$

13. $$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + 2z &= 1 \\ 3x + 3y &= 2 \end{aligned}$$

14. $$\begin{aligned} 2x - 3y + z &= -1 \\ x + y + z &= -4 \\ 4x + 2y - 3z &= 33 \end{aligned}$$

15. $$\begin{aligned} 3x + 2y - z &= -10 \\ x - y + 2z &= 7 \\ -x + 3y + z &= -2 \end{aligned}$$

16. $$\begin{aligned} 3x + 4z &= -11 \\ x - 2y &= 5 \\ 4y - z &= -10 \end{aligned}$$

17. $$\begin{aligned} 2x - 3y + z &= 0 \\ 2x + 4y - 3z &= 0 \\ 6x - 2y - z &= 0 \end{aligned}$$

18. $$\begin{aligned} 6x - 4y - 2z &= 2 \\ 3x + 2y - 5z &= 4 \\ 6y - 7z &= 5 \end{aligned}$$

For the following exercises, write a system of equations to solve each problem. Solve the system of equations.

19. Three odd numbers sum up to 61. The smaller is one-third the larger and the middle number is 16 less than the larger. What are the three numbers?

20. A local theatre sells out for their show. They sell all 500 tickets for a total purse of $8,070.00. The tickets were priced at $15 for students, $12 for children, and $18 for adults. If the band sold three times as many adult tickets as children's tickets, how many of each type was sold?

Systems of Nonlinear Equations and Inequalities: Two Variables

For the following exercises, solve the system of nonlinear equations.

21. $$\begin{aligned} y &= x^2 - 7 \\ y &= 5x - 13 \end{aligned}$$

22. $$\begin{aligned} y &= x^2 - 4 \\ y &= 5x + 10 \end{aligned}$$

23. $$\begin{aligned} x^2 + y^2 &= 16 \\ y &= x - 8 \end{aligned}$$

24. $\begin{aligned} x^2 + y^2 &= 25 \\ y &= x^2 + 5 \end{aligned}$

25. $\begin{aligned} x^2 + y^2 &= 4 \\ y - x^2 &= 3 \end{aligned}$

For the following exercises, graph the inequality.

26. $y > x^2 - 1$

27. $\frac{1}{4}x^2 + y^2 < 4$

For the following exercises, graph the system of inequalities.

28. $\begin{aligned} x^2 + y^2 + 2x &< 3 \\ y &> -x^2 - 3 \end{aligned}$

29. $\begin{aligned} x^2 - 2x + y^2 - 4x &< 4 \\ y &< -x + 4 \end{aligned}$

30. $\begin{aligned} x^2 + y^2 &< 1 \\ y^2 &< x \end{aligned}$

Partial Fractions

For the following exercises, decompose into partial fractions.

31. $\frac{-2x+6}{x^2+3x+2}$

32. $\frac{10x+2}{4x^2+4x+1}$

33. $\frac{7x+20}{x^2+10x+25}$

34. $\frac{x-18}{x^2-12x+36}$

35. $\frac{-x^2+36x+70}{x^3-125}$

36. $\frac{-5x^2+6x-2}{x^3+27}$

37. $\frac{x^3-4x^2+3x+11}{(x^2-2)^2}$

38. $\frac{4x^4-2x^3+22x^2-6x+48}{x(x^2+4)^2}$

Matrices and Matrix Operations

For the following exercises, perform the requested operations on the given matrices.

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 6 & 7 & -3 \\ 11 & -2 & 4 \end{bmatrix}, C = \begin{bmatrix} 6 & 7 \\ 11 & -2 \\ 14 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & -4 & 9 \\ 10 & 5 & -7 \\ 2 & 8 & 5 \end{bmatrix}, E = \begin{bmatrix} 7 & -14 & 3 \\ 2 & -1 & 3 \\ 0 & 1 & 9 \end{bmatrix}$$

39. $-4A$

40. $10D - 6E$

41. $B + C$

42. AB

43. BA

44. BC

45. CB

46. DE

47. ED

48. EC

49. CE

50. A^3

Solving Systems with Gaussian Elimination

For the following exercises, write the system of linear equations from the augmented matrix. Indicate whether there will be a unique solution.

51. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 7 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

52. $\left[\begin{array}{ccc|c} 1 & 0 & 5 & -9 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right]$

For the following exercises, write the augmented matrix from the system of linear equations.

53. $$\begin{aligned} -2x + 2y + z &= 7 \\ 2x - 8y + 5z &= 0 \\ 19x - 10y + 22z &= 3 \end{aligned}$$

54. $$\begin{aligned} 4x + 2y - 3z &= 14 \\ -12x + 3y + z &= 100 \\ 9x - 6y + 2z &= 31 \end{aligned}$$

55. $$\begin{aligned} x + 3z &= 12 \\ -x + 4y &= 0 \\ y + 2z &= -7 \end{aligned}$$

For the following exercises, solve the system of linear equations using Gaussian elimination.

56. $$\begin{aligned} 3x - 4y &= -7 \\ -6x + 8y &= 14 \end{aligned}$$

57. $$\begin{aligned} 3x - 4y &= 1 \\ -6x + 8y &= 6 \end{aligned}$$

58. $$\begin{aligned} -1.1x - 2.3y &= 6.2 \\ -5.2x - 4.1y &= 4.3 \end{aligned}$$

59. $$\begin{aligned} 2x + 3y + 2z &= 1 \\ -4x - 6y - 4z &= -2 \\ 10x + 15y + 10z &= 0 \end{aligned}$$

60. $$\begin{aligned} -x + 2y - 4z &= 8 \\ 3y + 8z &= -4 \\ -7x + y + 2z &= 1 \end{aligned}$$

Solving Systems with Inverses

For the following exercises, find the inverse of the matrix.

61. $$\begin{bmatrix} -0.2 & 1.4 \\ 1.2 & -0.4 \end{bmatrix}$$

62. $$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

63. $$\begin{bmatrix} 12 & 9 & -6 \\ -1 & 3 & 2 \\ -4 & -3 & 2 \end{bmatrix}$$

64. $$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

For the following exercises, find the solutions by computing the inverse of the matrix.

65. $$\begin{aligned} 0.3x - 0.1y &= -10 \\ -0.1x + 0.3y &= 14 \end{aligned}$$

66. $$\begin{aligned} 0.4x - 0.2y &= -0.6 \\ -0.1x + 0.05y &= 0.3 \end{aligned}$$

67. $$\begin{aligned} 4x + 3y - 3z &= -4.3 \\ 5x - 4y - z &= -6.1 \\ x + z &= -0.7 \end{aligned}$$

68. $$\begin{aligned} -2x - 3y + 2z &= 3 \\ -x + 2y + 4z &= -5 \\ -2y + 5z &= -3 \end{aligned}$$

For the following exercises, write a system of equations to solve each problem. Solve the system of equations.

69. Students were asked to bring their favorite fruit to class. 90% of the fruits consisted of banana, apple, and oranges. If oranges were half as popular as bananas and apples were 5% more popular than bananas, what are the percentages of each individual fruit?

70. A school club held a bake sale to raise money and sold brownies and chocolate chip cookies. They priced the brownies at \$2 and the chocolate chip cookies at \$1. They raised \$250 and sold 175 items. How many brownies and how many cookies were sold?

Solving Systems with Cramer's Rule

For the following exercises, find the determinant.

71. $\begin{vmatrix} 100 & 0 \\ 0 & 0 \end{vmatrix}$

72. $\begin{vmatrix} 0.2 & -0.6 \\ 0.7 & -1.1 \end{vmatrix}$

73. $\begin{vmatrix} -1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$

74. $\begin{vmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{vmatrix}$

For the following exercises, use Cramer's Rule to solve the linear systems of equations.

75. $\begin{aligned} 4x - 2y &= 23 \\ -5x - 10y &= -35 \end{aligned}$

76. $\begin{aligned} 0.2x - 0.1y &= 0 \\ -0.3x + 0.3y &= 2.5 \end{aligned}$

77. $\begin{aligned} -0.5x + 0.1y &= 0.3 \\ -0.25x + 0.05y &= 0.15 \end{aligned}$

78. $\begin{aligned} x + 6y + 3z &= 4 \\ 2x + y + 2z &= 3 \\ 3x - 2y + z &= 0 \end{aligned}$

79. $\begin{aligned} 4x - 3y + 5z &= -\tfrac{5}{2} \\ 7x - 9y - 3z &= \tfrac{3}{2} \\ x - 5y - 5z &= \tfrac{5}{2} \end{aligned}$

80. $\begin{aligned} \tfrac{3}{10}x - \tfrac{1}{5}y - \tfrac{3}{10}z &= -\tfrac{1}{50} \\ \tfrac{1}{10}x - \tfrac{1}{10}y - \tfrac{1}{2}z &= -\tfrac{9}{50} \\ \tfrac{2}{5}x - \tfrac{1}{2}y - \tfrac{3}{5}z &= -\tfrac{1}{5} \end{aligned}$

Practice Test

Is the following ordered pair a solution to the system of equations?

1. $\begin{aligned} -5x - y &= 12 \\ x + 4y &= 9 \end{aligned}$ with $(-3, 3)$

For the following exercises, solve the systems of linear and nonlinear equations using substitution or elimination. Indicate if no solution exists.

2. $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 4 \\ \tfrac{3}{2}x - y &= 0 \end{aligned}$

3. $\begin{aligned} -\tfrac{1}{2}x - 4y &= 4 \\ 2x + 16y &= 2 \end{aligned}$

4. $\begin{aligned} 5x - y &= 1 \\ -10x + 2y &= -2 \end{aligned}$

5. $\begin{aligned} 4x - 6y - 2z &= \tfrac{1}{10} \\ x - 7y + 5z &= -\tfrac{1}{4} \\ 3x + 6y - 9z &= \tfrac{6}{5} \end{aligned}$

6. $\begin{aligned} x + z &= 20 \\ x + y + z &= 20 \\ x + 2y + z &= 10 \end{aligned}$

7. $\begin{aligned} 5x - 4y - 3z &= 0 \\ 2x + y + 2z &= 0 \\ x - 6y - 7z &= 0 \end{aligned}$

8. $\begin{aligned} y &= x^2 + 2x - 3 \\ y &= x - 1 \end{aligned}$

9. $\begin{aligned} y^2 + x^2 &= 25 \\ y^2 - 2x^2 &= 1 \end{aligned}$

For the following exercises, graph the following inequalities.

10. $y < x^2 + 9$

11. $\begin{aligned} x^2 + y^2 &> 4 \\ y &< x^2 + 1 \end{aligned}$

For the following exercises, write the partial fraction decomposition.

12. $\frac{-8x-30}{x^2+10x+25}$

13. $\frac{13x+2}{(3x+1)^2}$

14. $\frac{x^4-x^3+2x-1}{x(x^2+1)^2}$

For the following exercises, perform the given matrix operations.

15. $5\begin{bmatrix} 4 & 9 \\ -2 & 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -6 & 12 \\ 4 & -8 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 4 & -7 \\ -2 & 9 & 5 \\ 12 & 0 & -4 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 3 \\ 5 & 10 \end{bmatrix}$

17. $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}^{-1}$

18. $\det\begin{vmatrix} 0 & 0 \\ 400 & 4{,}000 \end{vmatrix}$

19. $\det\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{vmatrix}$

20. If $\det(A) = -6$, what would be the determinant if you switched rows 1 and 3, multiplied the second row by 12, and took the inverse?

21. Rewrite the system of linear equations as an augmented matrix.

$$\begin{aligned} 14x - 2y + 13z &= 140 \\ -2x + 3y - 6z &= -1 \\ x - 5y + 12z &= 11 \end{aligned}$$

22. Rewrite the augmented matrix as a system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 12 \\ -2 & 4 & 9 & -5 \\ -6 & 1 & 2 & 8 \end{array}\right]$$

For the following exercises, use Gaussian elimination to solve the systems of equations.

23. $$\begin{aligned} x - 6y &= 4 \\ 2x - 12y &= 0 \end{aligned}$$

24. $$\begin{aligned} 2x + y + z &= -3 \\ x - 2y + 3z &= 6 \\ x - y - z &= 6 \end{aligned}$$

For the following exercises, use the inverse of a matrix to solve the systems of equations.

25. $$\begin{aligned} 4x - 5y &= -50 \\ -x + 2y &= 80 \end{aligned}$$

26. $$\begin{aligned} \frac{1}{100}x - \frac{3}{100}y + \frac{1}{20}z &= -49 \\ \frac{3}{100}x - \frac{7}{100}y - \frac{1}{100}z &= 13 \\ \frac{9}{100}x - \frac{9}{100}y - \frac{9}{100}z &= 99 \end{aligned}$$

For the following exercises, use Cramer's Rule to solve the systems of equations.

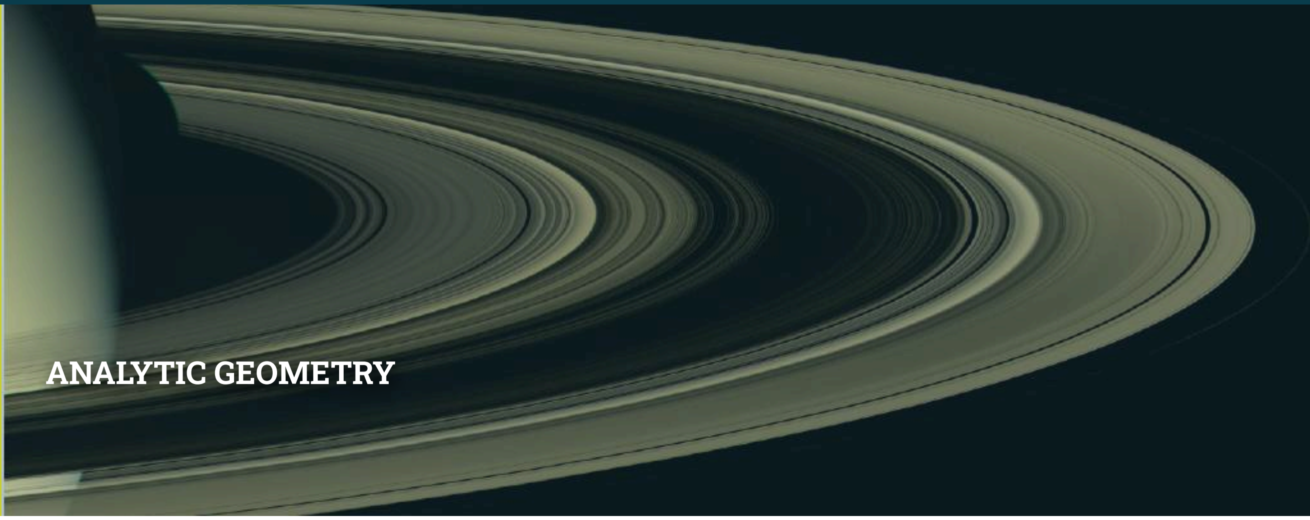
27. $$\begin{aligned} 200x - 300y &= 2 \\ 400x + 715y &= 4 \end{aligned}$$

28. $$\begin{aligned} 0.1x + 0.1y - 0.1z &= -1.2 \\ 0.1x - 0.2y + 0.4z &= -1.2 \\ 0.5x - 0.3y + 0.8z &= -5.9 \end{aligned}$$

For the following exercises, solve using a system of linear equations.

29. A factory producing cell phones has the following cost and revenue functions: $C(x) = x^2 + 75x + 2{,}688$ and $R(x) = x^2 + 160x$. What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.

30. A small fair charges $1.50 for students, $1 for children, and $2 for adults. In one day, three times as many children as adults attended. A total of 800 tickets were sold for a total revenue of $1,050. How many of each type of ticket was sold?



8 ANALYTIC GEOMETRY

The rings of Saturn have produced wonder, as well as misunderstanding, since Galileo first discovered them (he initially thought they were moons). Though they appear to be a series of solid discs even in this 2004 closeup from the Cassini probe, 19th century mathematicians proved that they are made up of billions of small objects clustered together. (credit: modificaion of "Saturn" by NASA/JPL-Caltech/SSI/Kevin M. Gill/flickr)

Chapter Outline

Introduction to Analytic Geometry

The Greek mathematician Menaechmus (c. 380–c. 320 BCE) is generally credited with discovering the shapes formed by the intersection of a plane and a right circular cone. Depending on how he tilted the plane when it intersected the cone, he formed different shapes at the intersection–beautiful shapes with near-perfect symmetry.

It was also said that Aristotle may have had an intuitive understanding of these shapes, as he observed the orbit of the planet to be circular. He presumed that the planets moved in circular orbits around Earth, and for nearly 2000 years this was the commonly held belief.

It was not until the Renaissance movement that Johannes Kepler noticed that the orbits of the planet were not circular in nature. His published law of planetary motion in the 1600s changed our view of the solar system forever. He claimed that the sun was at one end of the orbits, and the planets revolved around the sun in an oval-shaped path.

Other objects in the solar system (and perhaps other systems) follow a similar elliptical path, including the spectacular rings of Saturn. Using this understanding as a basis, 19th century mathematicians like James Clerk Maxwell and Sofya Kovalevskaya showed that despite their appearance through the telescopes of the day (and even in current telescopes), the rings are not solid and continuous, but are rather composed of small particles. Even after the Voyager and Cassini missions have provided close-up and detailed data regarding the ring structures, full understanding of their construction relies heavily on mathematical analysis. Of particular interest are the influences of Saturn's moons and moonlets, and the ways they both disrupt and preserve the ring structure.

In this chapter, we will investigate the two-dimensional figures that are formed when a right circular cone is intersected by a plane. We will begin by studying each of three figures created in this manner. We will develop defining equations for each figure and then learn how to use these equations to solve a variety of problems.

8.1 The Ellipse

Learning Objectives

In this section, you will:

- Write equations of ellipses in standard form.
- Graph ellipses centered at the origin.
- Graph ellipses not centered at the origin.
- Solve applied problems involving ellipses.

Figure 1 The National Statuary Hall in Washington, D.C. (credit: Greg Palmer, Flickr)

Can you imagine standing at one end of a large room and still being able to hear a whisper from a person standing at the other end? The National Statuary Hall in Washington, D.C., shown in Figure 1, is such a room.[1] It is an semi-circular room called a *whispering chamber* because the shape makes it possible for sound to travel along the walls and dome. In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.

Writing Equations of Ellipses in Standard Form

A conic section, or **conic**, is a shape resulting from intersecting a right circular cone with a plane. The angle at which the plane intersects the cone determines the shape, as shown in Figure 2.

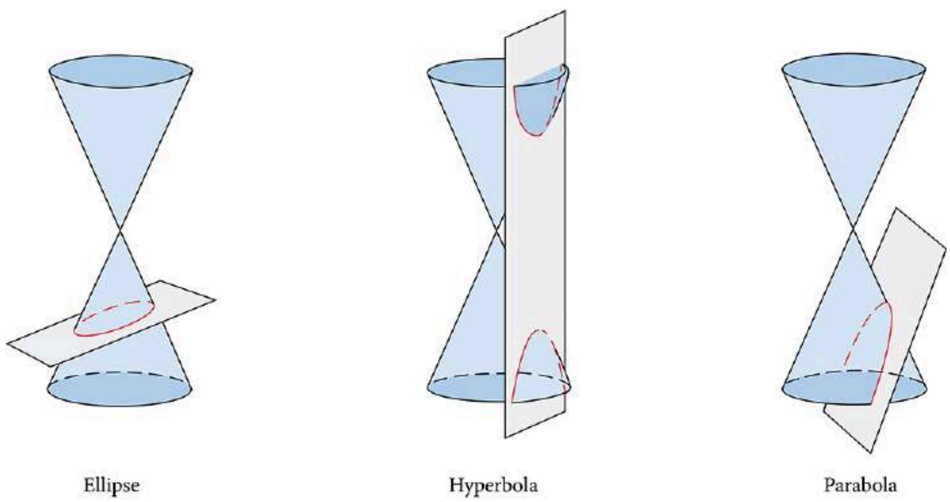


Figure 2

Conic sections can also be described by a set of points in the coordinate plane. Later in this chapter, we will see that the graph of any quadratic equation in two variables is a conic section. The signs of the equations and the coefficients of the variable terms determine the shape. This section focuses on the four variations of the standard form of the equation for the ellipse. An **ellipse** is the set of all points (x, y) in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a **focus** (plural: **foci**).

1 Architect of the Capitol. http://www.aoc.gov. Accessed April 15, 2014.

We can draw an ellipse using a piece of cardboard, two thumbtacks, a pencil, and string. Place the thumbtacks in the cardboard to form the foci of the ellipse. Cut a piece of string longer than the distance between the two thumbtacks (the length of the string represents the constant in the definition). Tack each end of the string to the cardboard, and trace a curve with a pencil held taut against the string. The result is an ellipse. See Figure 3.

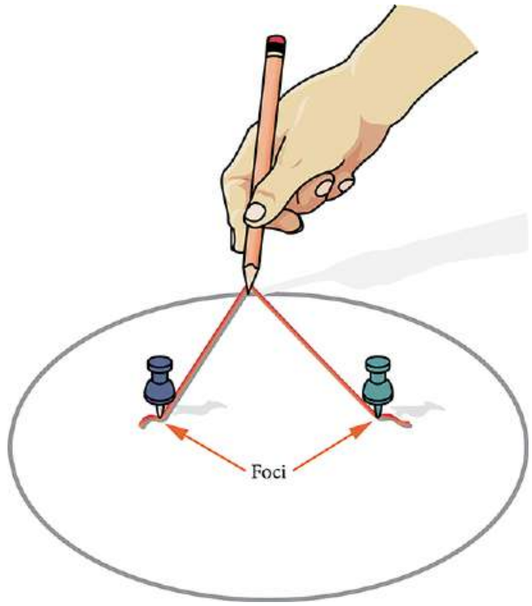


Figure 3

Every ellipse has two axes of symmetry. The longer axis is called the **major axis**, and the shorter axis is called the **minor axis**. Each endpoint of the major axis is the **vertex** of the ellipse (plural: **vertices**), and each endpoint of the minor axis is a **co-vertex** of the ellipse. The **center of an ellipse** is the midpoint of both the major and minor axes. The axes are perpendicular at the center. The foci always lie on the major axis, and the sum of the distances from the foci to any point on the ellipse (the constant sum) is greater than the distance between the foci. See Figure 4.

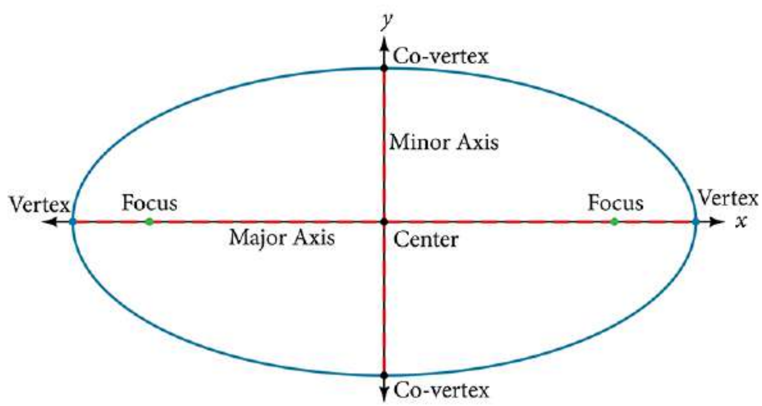


Figure 4

In this section, we restrict ellipses to those that are positioned vertically or horizontally in the coordinate plane. That is, the axes will either lie on or be parallel to the *x*- and *y*-axes. Later in the chapter, we will see ellipses that are rotated in the coordinate plane.

To work with horizontal and vertical ellipses in the coordinate plane, we consider two cases: those that are centered at the origin and those that are centered at a point other than the origin. First we will learn to derive the equations of ellipses, and then we will learn how to write the equations of ellipses in standard form. Later we will use what we learn to draw the graphs.

Deriving the Equation of an Ellipse Centered at the Origin

To derive the equation of an ellipse centered at the origin, we begin with the foci $(-c, 0)$ and $(c, 0)$. The ellipse is the set of all points (x, y) such that the sum of the distances from (x, y) to the foci is constant, as shown in Figure 5.

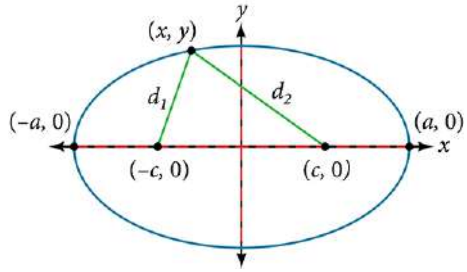


Figure 5

If $(a, 0)$ is a vertex of the ellipse, the distance from $(-c, 0)$ to $(a, 0)$ is $a - (-c) = a + c$. The distance from $(c, 0)$ to $(a, 0)$ is $a - c$. The sum of the distances from the foci to the vertex is

$$(a + c) + (a - c) = 2a$$

If (x, y) is a point on the ellipse, then we can define the following variables:

$$d_1 = \text{the distance from } (\text{-}c, 0) \text{ to } (x, y)$$
$$d_2 = \text{the distance from } (c, 0) \text{ to } (x, y)$$

By the definition of an ellipse, $d_1 + d_2$ is constant for any point (x, y) on the ellipse. We know that the sum of these distances is $2a$ for the vertex $(a, 0)$. It follows that $d_1 + d_2 = 2a$ for any point on the ellipse. We will begin the derivation by applying the distance formula. The rest of the derivation is algebraic.

$$\begin{array}{ll}
d_1 + d_2 = \sqrt{(x-(-c))^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a & \text{Distance formula} \\
\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a & \text{Simplify expressions.} \\
\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2} & \text{Move radical to opposite side.} \\
(x+c)^2 + y^2 = \left[2a - \sqrt{(x-c)^2 + y^2}\right]^2 & \text{Square both sides.} \\
x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 & \text{Expand the squares.} \\
x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2 & \text{Expand remaining squares.} \\
2cx = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} - 2cx & \text{Combine like terms.} \\
4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2} & \text{Isolate the radical.} \\
cx - a^2 = -a\sqrt{(x-c)^2 + y^2} & \text{Divide by 4.} \\
\left[cx - a^2\right]^2 = a^2\left[\sqrt{(x-c)^2 + y^2}\right]^2 & \text{Square both sides.} \\
c^2x^2 - 2a^2cx + a^4 = a^2\left(x^2 - 2cx + c^2 + y^2\right) & \text{Expand the squares.} \\
c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 & \text{Distribute } a^2. \\
a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2 & \text{Rewrite.} \\
x^2\left(a^2 - c^2\right) + a^2y^2 = a^2\left(a^2 - c^2\right) & \text{Factor common terms.} \\
x^2b^2 + a^2y^2 = a^2b^2 & \text{Set } b^2 = a^2 - c^2. \\
\frac{x^2b^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} & \text{Divide both sides by } a^2b^2. \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 & \text{Simplify.}
\end{array}$$

Thus, the standard equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This equation defines an ellipse centered at the origin. If $a > b$, the ellipse is stretched further in the horizontal direction, and if $b > a$, the ellipse is stretched further in the vertical direction.

Writing Equations of Ellipses Centered at the Origin in Standard Form

Standard forms of equations tell us about key features of graphs. Take a moment to recall some of the standard forms of equations we've worked with in the past: linear, quadratic, cubic, exponential, logarithmic, and so on. By learning to interpret standard forms of equations, we are bridging the relationship between algebraic and geometric representations of mathematical phenomena.

The key features of the ellipse are its center, vertices, co-vertices, foci, and lengths and positions of the major and minor axes. Just as with other equations, we can identify all of these features just by looking at the standard form of the equation. There are four variations of the standard form of the ellipse. These variations are categorized first by the location of the center (the origin or not the origin), and then by the position (horizontal or vertical). Each is presented along with a description of how the parts of the equation relate to the graph. Interpreting these parts allows us to form a mental picture of the ellipse.

Standard Forms of the Equation of an Ellipse with Center (0,0)

The standard form of the equation of an ellipse with center $(0, 0)$ and major axis on the *x-axis* is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = a^2 - b^2$. See Figure 6 **a**

The standard form of the equation of an ellipse with center $(0, 0)$ and major axis on the *y-axis* is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(\pm b, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = a^2 - b^2$. See Figure 6 **b**

Note that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 - b^2$. When we are given the coordinates of the foci and vertices of an ellipse, we can use this relationship to find the equation of the ellipse in standard form.

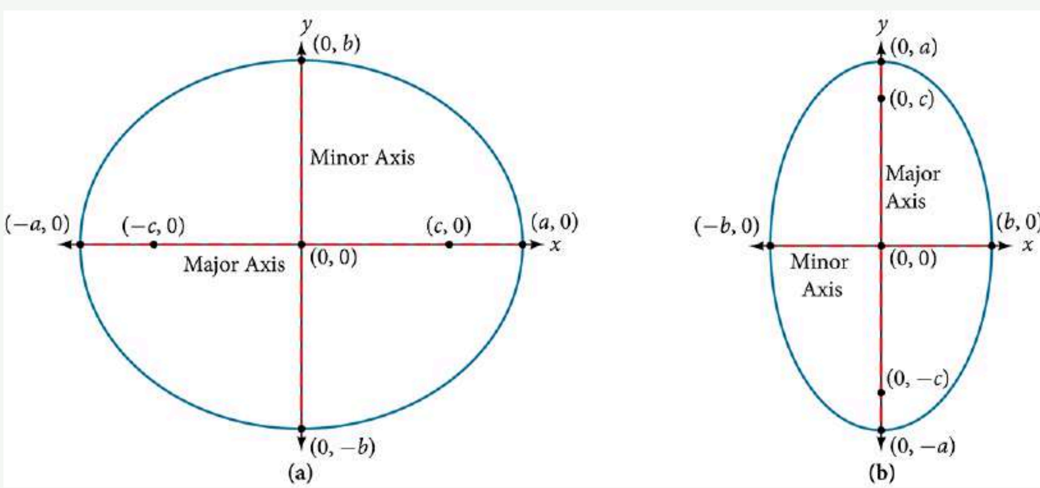


Figure 6 (a) Horizontal ellipse with center $(0, 0)$ (b) Vertical ellipse with center $(0, 0)$

HOW TO

Given the vertices and foci of an ellipse centered at the origin, write its equation in standard form.

1. Determine whether the major axis lies on the *x*- or *y*-axis.
   a. If the given coordinates of the vertices and foci have the form $(\pm a, 0)$ and $(\pm c, 0)$ respectively, then the major axis is the *x*-axis. Use the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
   b. If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(0, \pm c)$, respectively, then the major axis is the *y*-axis. Use the standard form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.
2. Use the equation $c^2 = a^2 - b^2$, along with the given coordinates of the vertices and foci, to solve for b^2.
3. Substitute the values for a^2 and b^2 into the standard form of the equation determined in Step 1.

EXAMPLE 1

Writing the Equation of an Ellipse Centered at the Origin in Standard Form
What is the standard form equation of the ellipse that has vertices $(\pm 8, 0)$ and foci $(\pm 5, 0)$?

Solution
The foci are on the *x*-axis, so the major axis is the *x*-axis. Thus, the equation will have the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The vertices are $(\pm 8, 0)$, so $a = 8$ and $a^2 = 64$.

The foci are $(\pm 5, 0)$, so $c = 5$ and $c^2 = 25$.

We know that the vertices and foci are related by the equation $c^2 = a^2 - b^2$. Solving for b^2, we have:

$$\begin{aligned} c^2 &= a^2 - b^2 & & \\ 25 &= 64 - b^2 & & \text{Substitute for } c^2 \text{ and } a^2. \\ b^2 &= 39 & & \text{Solve for } b^2. \end{aligned}$$

Now we need only substitute $a^2 = 64$ and $b^2 = 39$ into the standard form of the equation. The equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{39} = 1$.

TRY IT #1 What is the standard form equation of the ellipse that has vertices $(0, \pm 4)$ and foci $\left(0, \pm \sqrt{15}\right)$?

Q&A **Can we write the equation of an ellipse centered at the origin given coordinates of just one focus and vertex?**

Yes. Ellipses are symmetrical, so the coordinates of the vertices of an ellipse centered around the origin will always have the form $(\pm a, 0)$ or $(0, \pm a)$. Similarly, the coordinates of the foci will always have the form $(\pm c, 0)$ or $(0, \pm c)$. Knowing this, we can use a and c from the given points, along with the equation $c^2 = a^2 - b^2$, to find b^2.

Writing Equations of Ellipses Not Centered at the Origin

Like the graphs of other equations, the graph of an ellipse can be translated. If an ellipse is translated h units horizontally and k units vertically, the center of the ellipse will be (h, k). This translation results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and y replaced by $(y - k)$.

Standard Forms of the Equation of an Ellipse with Center (h, k)

The standard form of the equation of an ellipse with center $(h,\ \ k)$ and major axis parallel to the *x*-axis is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$. See Figure 7**a**

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the *y*-axis is

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = a^2 - b^2$. See Figure 7**b**

Just as with ellipses centered at the origin, ellipses that are centered at a point (h, k) have vertices, co-vertices, and foci that are related by the equation $c^2 = a^2 - b^2$. We can use this relationship along with the midpoint and distance formulas to find the equation of the ellipse in standard form when the vertices and foci are given.

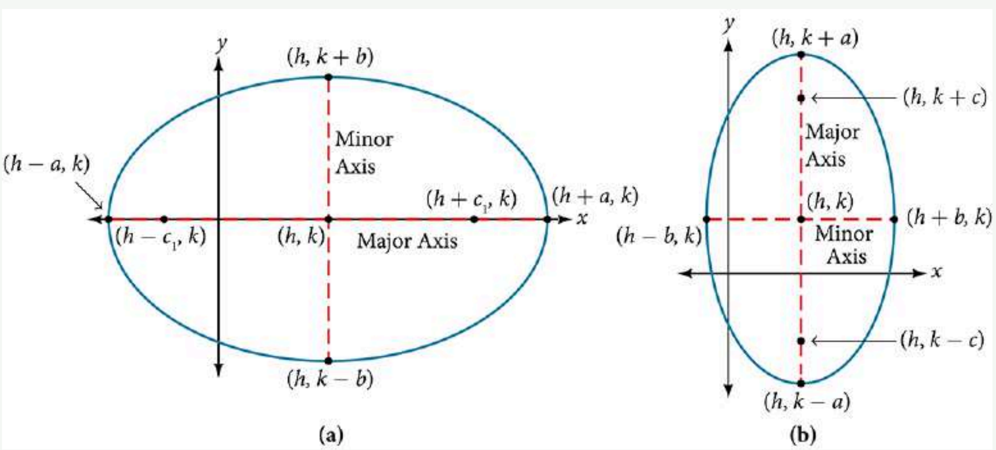


Figure 7 (a) Horizontal ellipse with center (h, k) (b) Vertical ellipse with center (h, k)

HOW TO

Given the vertices and foci of an ellipse not centered at the origin, write its equation in standard form.

1. Determine whether the major axis is parallel to the *x*- or *y*-axis.
    a. If the *y*-coordinates of the given vertices and foci are the same, then the major axis is parallel to the *x*-axis.

Use the standard form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$

b. If the *x*-coordinates of the given vertices and foci are the same, then the major axis is parallel to the *y*-axis. Use the standard form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$

2. Identify the center of the ellipse (h, k) using the midpoint formula and the given coordinates for the vertices.
3. Find a^2 by solving for the length of the major axis, $2a$, which is the distance between the given vertices.
4. Find c^2 using h and k, found in Step 2, along with the given coordinates for the foci.
5. Solve for b^2 using the equation $c^2 = a^2 - b^2$.
6. Substitute the values for h, k, a^2, and b^2 into the standard form of the equation determined in Step 1.

EXAMPLE 2

Writing the Equation of an Ellipse Centered at a Point Other Than the Origin

What is the standard form equation of the ellipse that has vertices $(-2, -8)$ and $(-2, 2)$

and foci $(-2, -7)$ and $(-2, 1)$?

Solution

The *x*-coordinates of the vertices and foci are the same, so the major axis is parallel to the *y*-axis. Thus, the equation of the ellipse will have the form

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

First, we identify the center, (h, k). The center is halfway between the vertices, $(-2, -8)$ and $(-2, 2)$. Applying the midpoint formula, we have:

$$\begin{aligned}(h, k) &= \left(\frac{-2+(-2)}{2}, \frac{-8+2}{2}\right) \\ &= (-2, -3)\end{aligned}$$

Next, we find a^2. The length of the major axis, $2a$, is bounded by the vertices. We solve for a by finding the distance between the *y*-coordinates of the vertices.

$$\begin{aligned}2a &= 2 - (-8) \\ 2a &= 10 \\ a &= 5\end{aligned}$$

So $a^2 = 25$.

Now we find c^2. The foci are given by $(h, k \pm c)$. So, $(h, k - c) = (-2, -7)$ and $(h, k + c) = (-2, 1)$. We substitute $k = -3$ using either of these points to solve for c.

$$\begin{aligned}k + c &= 1 \\ -3 + c &= 1 \\ c &= 4\end{aligned}$$

So $c^2 = 16$.

Next, we solve for b^2 using the equation $c^2 = a^2 - b^2$.

$$\begin{aligned}c^2 &= a^2 - b^2 \\ 16 &= 25 - b^2 \\ b^2 &= 9\end{aligned}$$

Finally, we substitute the values found for h, k, a^2, and b^2 into the standard form equation for an ellipse:

$$\frac{(x+2)^2}{9} + \frac{(y+3)^2}{25} = 1$$

TRY IT #2 What is the standard form equation of the ellipse that has vertices $(-3, 3)$ and $(5, 3)$ and foci $\left(1-2\sqrt{3}, 3\right)$ and $\left(1+2\sqrt{3}, 3\right)$?

Graphing Ellipses Centered at the Origin

Just as we can write the equation for an ellipse given its graph, we can graph an ellipse given its equation. To graph ellipses centered at the origin, we use the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b$ for horizontal ellipses and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a > b$ for vertical ellipses.

HOW TO

Given the standard form of an equation for an ellipse centered at $(0, 0)$, sketch the graph.

1. Use the standard forms of the equations of an ellipse to determine the major axis, vertices, co-vertices, and foci.
   a. If the equation is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, then
      - the major axis is the *x*-axis
      - the coordinates of the vertices are $(\pm a, 0)$
      - the coordinates of the co-vertices are $(0, \pm b)$
      - the coordinates of the foci are $(\pm c, 0)$

   b. If the equation is in the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a > b$, then
      - the major axis is the *y*-axis
      - the coordinates of the vertices are $(0, \pm a)$
      - the coordinates of the co-vertices are $(\pm b, 0)$
      - the coordinates of the foci are $(0, \pm c)$
2. Solve for c using the equation $c^2 = a^2 - b^2$.
3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

EXAMPLE 3

Graphing an Ellipse Centered at the Origin

Graph the ellipse given by the equation, $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Identify and label the center, vertices, co-vertices, and foci.

Solution

First, we determine the position of the major axis. Because $25 > 9$, the major axis is on the *y*-axis. Therefore, the equation is in the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $b^2 = 9$ and $a^2 = 25$. It follows that:

- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(0, \pm a) = \left(0, \pm\sqrt{25}\right) = (0, \pm 5)$
- the coordinates of the co-vertices are $(\pm b, 0) = \left(\pm\sqrt{9}, 0\right) = (\pm 3, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = a^2 - b^2$ Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{25-9} \\ &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned}$$

Therefore, the coordinates of the foci are $(0, \pm 4)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse. See Figure 8.

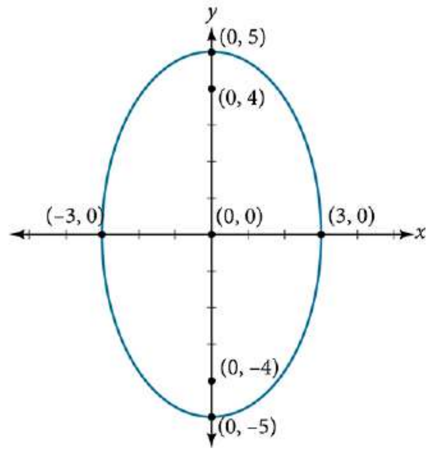


Figure 8

> TRY IT #3 Graph the ellipse given by the equation $\frac{x^2}{36} + \frac{y^2}{4} = 1$. Identify and label the center, vertices, co-vertices, and foci.

EXAMPLE 4

Graphing an Ellipse Centered at the Origin from an Equation Not in Standard Form

Graph the ellipse given by the equation $4x^2 + 25y^2 = 100$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Solution

First, use algebra to rewrite the equation in standard form.

$$\begin{aligned} 4x^2 + 25y^2 &= 100 \\ \frac{4x^2}{100} + \frac{25y^2}{100} &= \frac{100}{100} \\ \frac{x^2}{25} + \frac{y^2}{4} &= 1 \end{aligned}$$

Next, we determine the position of the major axis. Because $25 > 4$, the major axis is on the *x*-axis. Therefore, the equation is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 = 25$ and $b^2 = 4$. It follows that:

- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(\pm a, 0) = \left(\pm\sqrt{25}, 0\right) = (\pm 5, 0)$
- the coordinates of the co-vertices are $(0, \pm b) = \left(0, \pm\sqrt{4}\right) = (0, \pm 2)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = a^2 - b^2$. Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{25 - 4} \\ &= \pm\sqrt{21} \end{aligned}$$

Therefore the coordinates of the foci are $\left(\pm\sqrt{21}, 0\right)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

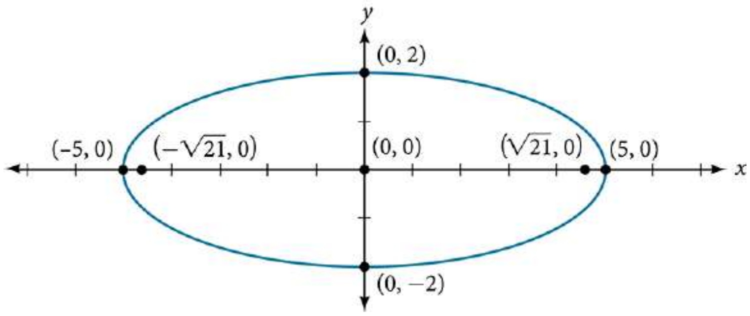


Figure 9

> **TRY IT** #4 Graph the ellipse given by the equation $49x^2 + 16y^2 = 784$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Graphing Ellipses Not Centered at the Origin

When an ellipse is not centered at the origin, we can still use the standard forms to find the key features of the graph. When the ellipse is centered at some point, (h, k), we use the standard forms $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a > b$ for horizontal ellipses and $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \quad a > b$ for vertical ellipses. From these standard equations, we can easily determine the center, vertices, co-vertices, foci, and positions of the major and minor axes.

HOW TO

Given the standard form of an equation for an ellipse centered at (h, k), sketch the graph.

1. Use the standard forms of the equations of an ellipse to determine the center, position of the major axis, vertices, co-vertices, and foci.
   a. If the equation is in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $a > b$, then
      - the center is (h, k)
      - the major axis is parallel to the *x*-axis
      - the coordinates of the vertices are $(h \pm a, k)$
      - the coordinates of the co-vertices are $(h, k \pm b)$
      - the coordinates of the foci are $(h \pm c, k)$
   b. If the equation is in the form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, where $a > b$, then
      - the center is (h, k)
      - the major axis is parallel to the *y*-axis
      - the coordinates of the vertices are $(h, k \pm a)$
      - the coordinates of the co-vertices are $(h \pm b, k)$
      - the coordinates of the foci are $(h, k \pm c)$
2. Solve for c using the equation $c^2 = a^2 - b^2$.
3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

EXAMPLE 5

Graphing an Ellipse Centered at (*h, k*)

Graph the ellipse given by the equation, $\frac{(x+2)^2}{4} + \frac{(y-5)^2}{9} = 1$. Identify and label the center, vertices, co-vertices, and foci.

Solution

First, we determine the position of the major axis. Because $9 > 4$, the major axis is parallel to the *y*-axis. Therefore, the equation is in the form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, where $b^2 = 4$ and $a^2 = 9$. It follows that:

- the center of the ellipse is $(h, k) = (-2, 5)$
- the coordinates of the vertices are $(h, k \pm a) = (-2, 5 \pm \sqrt{9}) = (-2, 5 \pm 3)$, or $(-2, 2)$ and $(-2, 8)$
- the coordinates of the co-vertices are $(h \pm b, k) = (-2 \pm \sqrt{4}, 5) = (-2 \pm 2, 5)$, or $(-4, 5)$ and $(0, 5)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = a^2 - b^2$. Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{9-4} \\ &= \pm\sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $\left(-2, 5 - \sqrt{5}\right)$ and $\left(-2, 5+\sqrt{5}\right)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

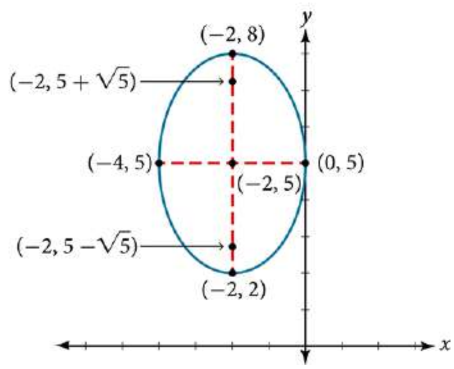


Figure 10

TRY IT #5 Graph the ellipse given by the equation $\frac{(x-4)^2}{36} + \frac{(y-2)^2}{20} = 1$. Identify and label the center, vertices, co-vertices, and foci.

HOW TO

Given the general form of an equation for an ellipse centered at (*h*, *k*), express the equation in standard form.

1. Recognize that an ellipse described by an equation in the form $ax^2 + by^2 + cx + dy + e = 0$ is in general form.
2. Rearrange the equation by grouping terms that contain the same variable. Move the constant term to the opposite side of the equation.
3. Factor out the coefficients of the x^2 and y^2 terms in preparation for completing the square.
4. Complete the square for each variable to rewrite the equation in the form of the sum of multiples of two binomials squared set equal to a constant, $m_1(x-h)^2 + m_2(y-k)^2 = m_3$, where m_1, m_2, and m_3 are constants.
5. Divide both sides of the equation by the constant term to express the equation in standard form.

EXAMPLE 6

Graphing an Ellipse Centered at (*h*, *k*) by First Writing It in Standard Form

Graph the ellipse given by the equation $4x^2 + 9y^2 - 40x + 36y + 100 = 0$. Identify and label the center, vertices, co-vertices, and foci.

Solution

We must begin by rewriting the equation in standard form.

$$4x^2 + 9y^2 - 40x + 36y + 100 = 0$$

Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$\left(4x^2 - 40x\right) + \left(9y^2 + 36y\right) = -100$$

Factor out the coefficients of the squared terms.

$$4\left(x^2 - 10x\right) + 9\left(y^2 + 4y\right) = -100$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$4\left(x^2 - 10x + 25\right) + 9\left(y^2 + 4y + 4\right) = -100 + 100 + 36$$

Rewrite as perfect squares.

$$4(x-5)^2 + 9(y+2)^2 = 36$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-5)^2}{9} + \frac{(y+2)^2}{4} = 1$$

Now that the equation is in standard form, we can determine the position of the major axis. Because $9 > 4$, the major axis is parallel to the *x*-axis. Therefore, the equation is in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $a^2 = 9$ and $b^2 = 4$. It follows that:

- the center of the ellipse is $(h, k) = (5, -2)$
- the coordinates of the vertices are $(h \pm a, k) = \left(5 \pm \sqrt{9}, -2\right) = (5 \pm 3, -2)$, or $(2, -2)$ and $(8, -2)$
- the coordinates of the co-vertices are $(h, k \pm b) = \left(5, -2 \pm \sqrt{4}\right) = (5, -2 \pm 2)$, or $(5, -4)$ and $(5, 0)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$. Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{9-4} \\ &= \pm\sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $\left(5 - \sqrt{5}, -2\right)$ and $\left(5+\sqrt{5}, -2\right)$.

Next we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse as shown in Figure 11.

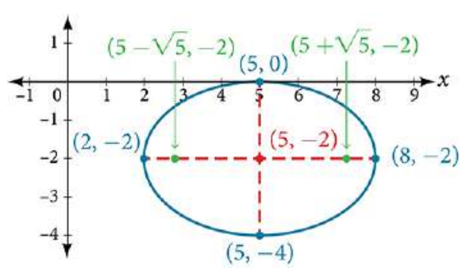


Figure 11

TRY IT #6 Express the equation of the ellipse given in standard form. Identify the center, vertices, co-vertices, and foci of the ellipse.

$$4x^2 + y^2 - 24x + 2y + 21 = 0$$

Solving Applied Problems Involving Ellipses

Many real-world situations can be represented by ellipses, including orbits of planets, satellites, moons and comets, and shapes of boat keels, rudders, and some airplane wings. A medical device called a lithotripter uses elliptical reflectors to break up kidney stones by generating sound waves. Some buildings, called whispering chambers, are designed with elliptical domes so that a person whispering at one focus can easily be heard by someone standing at the other focus. This occurs because of the acoustic properties of an ellipse. When a sound wave originates at one focus of a whispering chamber, the sound wave will be reflected off the elliptical dome and back to the other focus. See Figure 12. In the whisper chamber at the Museum of Science and Industry in Chicago, two people standing at the foci—about 43 feet apart—can hear each other whisper. When these chambers are placed in unexpected places, such as the ones inside Bush International Airport in Houston and Grand Central Terminal in New York City, they can induce surprised reactions among travelers.

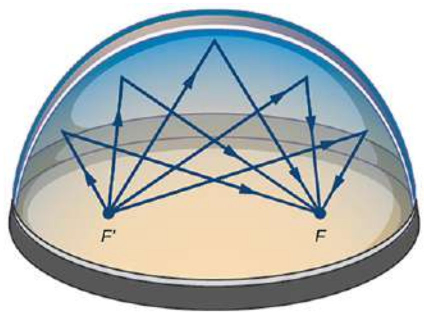


Figure 12 Sound waves are reflected between foci in an elliptical room, called a whispering chamber.

EXAMPLE 7

Locating the Foci of a Whispering Chamber

A large room in an art gallery is a whispering chamber. Its dimensions are 46 feet wide by 96 feet long as shown in Figure 13.

a. What is the standard form of the equation of the ellipse representing the outline of the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0, 0)$.
b. If two visitors standing at the foci of this room can hear each other whisper, how far apart are the two visitors? Round to the nearest foot.

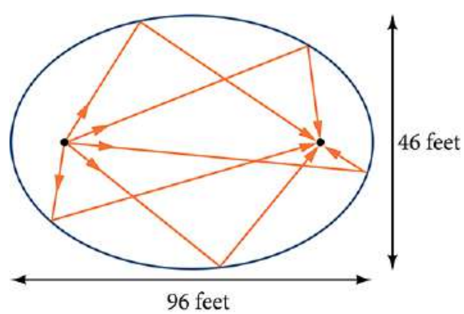


Figure 13

Solution

a. We are assuming a horizontal ellipse with center $(0, 0)$, so we need to find an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$ where $a > b$. We know that the length of the major axis, $2a$, is longer than the length of the minor axis, $2b$. So the length of the room, 96, is represented by the major axis, and the width of the room, 46, is represented by the minor axis.
   - Solving for a, we have $2a = 96$, so $a = 48$, and $a^2 = 2304$.
   - Solving for b, we have $2b = 46$, so $b = 23$, and $b^2 = 529$.

   Therefore, the equation of the ellipse is $\frac{x^2}{2304} + \frac{y^2}{529} = 1.$

b. To find the distance between the senators, we must find the distance between the foci, $(\pm c, 0)$, where $c^2 = a^2 - b^2$.

Solving for c, we have:

$$
\begin{array}{ll}
c^2 = a^2 - b^2 & \\
c^2 = 2304 - 529 & \text{Substitute using the values found in part (a).} \\
c = \pm\sqrt{2304 - 529} & \text{Take the square root of both sides.} \\
c = \pm\sqrt{1775} & \text{Subtract.} \\
c \approx \pm 42 & \text{Round to the nearest foot.}
\end{array}
$$

The points $(\pm 42, 0)$ represent the foci. Thus, the distance between the senators is $2\,(42) = 84$ feet.

TRY IT #7 Suppose a whispering chamber is 480 feet long and 320 feet wide.

ⓐ What is the standard form of the equation of the ellipse representing the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0, 0)$.

ⓑ If two people are standing at the foci of this room and can hear each other whisper, how far apart are the people? Round to the nearest foot.

MEDIA

Access these online resources for additional instruction and practice with ellipses.

[Conic Sections: The Ellipse (http://openstax.org/l/conicellipse)](http://openstax.org/l/conicellipse)
[Graph an Ellipse with Center at the Origin (http://openstax.org/l/grphellorigin)](http://openstax.org/l/grphellorigin)
[Graph an Ellipse with Center Not at the Origin (http://openstax.org/l/grphellnot)](http://openstax.org/l/grphellnot)

8.1 SECTION EXERCISES

Verbal

1. Define an ellipse in terms of its foci.

2. Where must the foci of an ellipse lie?

3. What special case of the ellipse do we have when the major and minor axis are of the same length?

4. For the special case mentioned in the previous question, what would be true about the foci of that ellipse?

5. What can be said about the symmetry of the graph of an ellipse with center at the origin and foci along the y-axis?

Algebraic

For the following exercises, determine whether the given equations represent ellipses. If yes, write in standard form.

6. $2x^2 + y = 4$

7. $4x^2 + 9y^2 = 36$

8. $4x^2 - y^2 = 4$

9. $4x^2 + 9y^2 = 1$

10. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$

For the following exercises, write the equation of an ellipse in standard form, and identify the end points of the major and minor axes as well as the foci.

11. $\frac{x^2}{4}+\frac{y^2}{49}=1$

12. $\frac{x^2}{100}+\frac{y^2}{64}=1$

13. $x^2+9y^2=1$

14. $4x^2+16y^2=1$

15. $\frac{(x-2)^2}{49}+\frac{(y-4)^2}{25}=1$

16. $\frac{(x-2)^2}{81}+\frac{(y+1)^2}{16}=1$

17. $\frac{(x+5)^2}{4}+\frac{(y-7)^2}{9}=1$

18. $\frac{(x-7)^2}{49}+\frac{(y-7)^2}{49}=1$

19. $4x^2-8x+9y^2-72y+112=0$

20. $9x^2-54x+9y^2-54y+81=0$

21. $4x^2-24x+36y^2-360y+864=0$

22. $4x^2+24x+16y^2-128y+228=0$

23. $4x^2+40x+25y^2-100y+100=0$

24. $x^2+2x+100y^2-1000y+2401=0$

25. $4x^2+24x+25y^2+200y+336=0$

26. $9x^2+72x+16y^2+16y+4=0$

For the following exercises, find the foci for the given ellipses.

27. $\frac{(x+3)^2}{25}+\frac{(y+1)^2}{36}=1$

28. $\frac{(x+1)^2}{100}+\frac{(y-2)^2}{4}=1$

29. $x^2+y^2=1$

30. $x^2+4y^2+4x+8y=1$

31. $10x^2+y^2+200x=0$

Graphical

For the following exercises, graph the given ellipses, noting center, vertices, and foci.

32. $\frac{x^2}{25}+\frac{y^2}{36}=1$

33. $\frac{x^2}{16}+\frac{y^2}{9}=1$

34. $4x^2+9y^2=1$

35. $81x^2+49y^2=1$

36. $\frac{(x-2)^2}{64}+\frac{(y-4)^2}{16}=1$

37. $\frac{(x+3)^2}{9}+\frac{(y-3)^2}{9}=1$

38. $\frac{x^2}{2}+\frac{(y+1)^2}{5}=1$

39. $4x^2-8x+16y^2-32y-44=0$

40. $x^2-8x+25y^2-100y+91=0$

41. $x^2+8x+4y^2-40y+112=0$

42. $64x^2+128x+9y^2-72y-368=0$

43. $16x^2+64x+4y^2-8y+4=0$

44. $100x^2+1000x+y^2-10y+2425=0$

45. $4x^2+16x+4y^2+16y+16=0$

For the following exercises, use the given information about the graph of each ellipse to determine its equation.

46. Center at the origin, symmetric with respect to the *x*- and *y*-axes, focus at $(4, 0)$, and point on graph $(0, 3)$.

47. Center at the origin, symmetric with respect to the *x*- and *y*-axes, focus at $(0, -2)$, and point on graph $(5, 0)$.

48. Center at the origin, symmetric with respect to the *x*- and *y*-axes, focus at $(3, 0)$, and major axis is twice as long as minor axis.

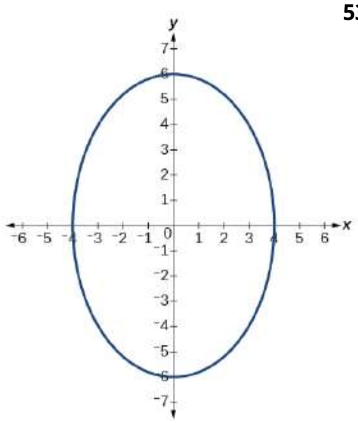
49. Center $(4, 2)$; vertex $(9, 2)$; one focus: $\left(4 + 2\sqrt{6}, 2\right)$.

50. Center $(3, 5)$; vertex $(3, 11)$; one focus: $\left(3,\ \ 5+4\sqrt{2}\right)$

51. Center $(-3, 4)$; vertex $(1, 4)$; one focus: $\left(-3 + 2\sqrt{3}, 4\right)$

For the following exercises, given the graph of the ellipse, determine its equation.

52.



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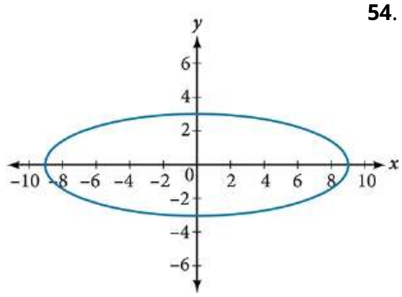

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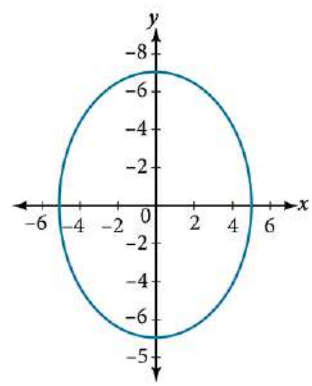

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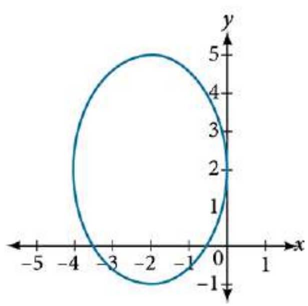
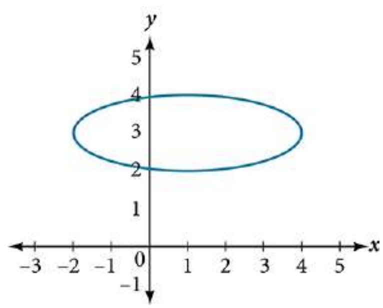


56.



Extensions

For the following exercises, find the area of the ellipse. The area of an ellipse is given by the formula $\text{Area} = a \cdot b \cdot \pi$.

57. $\frac{(x-3)^2}{9} + \frac{(y-3)^2}{16} = 1$

58. $\frac{(x+6)^2}{16} + \frac{(y-6)^2}{36} = 1$

59. $\frac{(x+1)^2}{4} + \frac{(y-2)^2}{5} = 1$

60. $4x^2 - 8x + 9y^2 - 72y + 112 = 0$

61. $9x^2 - 54x + 9y^2 - 54y + 81 = 0$

Real-World Applications

62. Find the equation of the ellipse that will just fit inside a box that is 8 units wide and 4 units high.

63. Find the equation of the ellipse that will just fit inside a box that is four times as wide as it is high. Express in terms of h, the height.

64. An arch has the shape of a semi-ellipse (the top half of an ellipse). The arch has a height of 8 feet and a span of 20 feet. Find an equation for the ellipse, and use that to find the height to the nearest 0.01 foot of the arch at a distance of 4 feet from the center.

65. An arch has the shape of a semi-ellipse. The arch has a height of 12 feet and a span of 40 feet. Find an equation for the ellipse, and use that to find the distance from the center to a point at which the height is 6 feet. Round to the nearest hundredth.

66. A bridge is to be built in the shape of a semi-elliptical arch and is to have a span of 120 feet. The height of the arch at a distance of 40 feet from the center is to be 8 feet. Find the height of the arch at its center.

67. A person in a whispering gallery standing at one focus of the ellipse can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet, and the foci are located 30 feet from the center, find the height of the ceiling at the center.

68. A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus, and the other focus is 80 feet away, what is the length and height at the center of the gallery?

8.2 The Hyperbola

Learning Objectives

In this section, you will:

- Locate a hyperbola’s vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Solve applied problems involving hyperbolas.

What do paths of comets, supersonic booms, ancient Grecian pillars, and natural draft cooling towers have in common? They can all be modeled by the same type of conic. For instance, when something moves faster than the speed of sound, a shock wave in the form of a cone is created. A portion of a conic is formed when the wave intersects the ground, resulting in a sonic boom. See Figure 1.

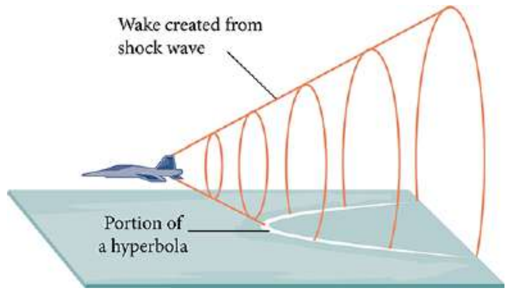


Figure 1 A shock wave intersecting the ground forms a portion of a conic and results in a sonic boom.

Most people are familiar with the sonic boom created by supersonic aircraft, but humans were breaking the sound barrier long before the first supersonic flight. The crack of a whip occurs because the tip is exceeding the speed of sound. The bullets shot from many firearms also break the sound barrier, although the bang of the gun usually supersedes the sound of the sonic boom.

Locating the Vertices and Foci of a Hyperbola

In analytic geometry, a **hyperbola** is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other. See Figure 2.

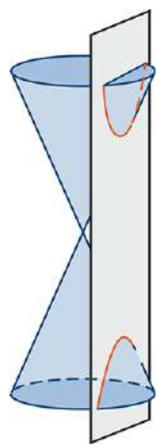


Figure 2 A hyperbola

Like the ellipse, the hyperbola can also be defined as a set of points in the coordinate plane. A hyperbola is the set of all points (x, y) in a plane such that the difference of the distances between (x, y) and the foci is a positive constant.

Notice that the definition of a hyperbola is very similar to that of an ellipse. The distinction is that the hyperbola is defined in terms of the *difference* of two distances, whereas the ellipse is defined in terms of the *sum* of two distances.

As with the ellipse, every hyperbola has two axes of symmetry. The **transverse axis** is a line segment that passes through the center of the hyperbola and has vertices as its endpoints. The foci lie on the line that contains the transverse axis. The **conjugate axis** is perpendicular to the transverse axis and has the co-vertices as its endpoints. The **center of a hyperbola** is the midpoint of both the transverse and conjugate axes, where they intersect. Every hyperbola also has two **asymptotes** that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes. The **central rectangle** of the hyperbola is centered at the origin with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle. See Figure 3.

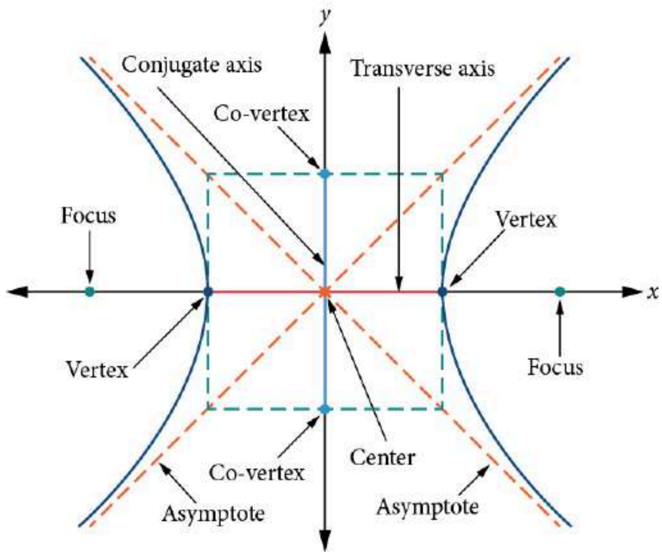


Figure 3 Key features of the hyperbola

In this section, we will limit our discussion to hyperbolas that are positioned vertically or horizontally in the coordinate plane; the axes will either lie on or be parallel to the *x*- and *y*-axes. We will consider two cases: those that are centered at the origin, and those that are centered at a point other than the origin.

Deriving the Equation of a Hyperbola Centered at the Origin

Let $(-c, 0)$ and $(c, 0)$ be the foci of a hyperbola centered at the origin. The hyperbola is the set of all points (x, y) such that the difference of the distances from (x, y) to the foci is constant. See Figure 4.

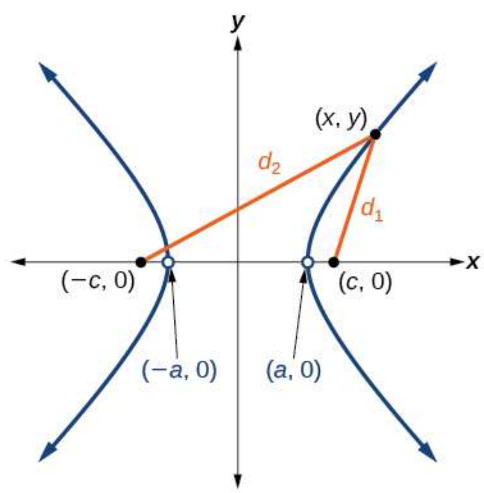


Figure 4

If $(a, 0)$ is a vertex of the hyperbola, the distance from $(-c, 0)$ to $(a, 0)$ is $a-(-c)=a+c$. The distance from $(c, 0)$ to $(a, 0)$ is $c-a$. The difference of the distances from the foci to the vertex is

$$(a+c)-(c-a)=2a$$

If (x, y) is a point on the hyperbola, we can define the following variables:

$$\begin{aligned} d_2 &= \text{the distance from } (-c, 0) \text{ to } (x, y) \\ d_1 &= \text{the distance from } (c, 0) \text{ to } (x, y) \end{aligned}$$

By definition of a hyperbola, d_2-d_1 is constant for any point (x, y) on the hyperbola. We know that the difference of these distances is $2a$ for the vertex $(a, 0)$. It follows that $d_2-d_1=2a$ for any point on the hyperbola. As with the derivation of the equation of an ellipse, we will begin by applying the distance formula. The rest of the derivation is algebraic. Compare this derivation with the one from the previous section for ellipses.

$$
\begin{aligned}
d_2 - d_1 &= \sqrt{(x-(-c))^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = 2a && \text{Distance Formula} \\
\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a && \text{Simplify expressions.} \\
\sqrt{(x+c)^2 + y^2} &= 2a + \sqrt{(x-c)^2 + y^2} && \text{Move radical to opposite side.} \\
(x+c)^2 + y^2 &= \left(2a + \sqrt{(x-c)^2 + y^2}\right)^2 && \text{Square both sides.} \\
x^2 + 2cx + c^2 + y^2 &= 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 && \text{Expand the squares.} \\
x^2 + 2cx + c^2 + y^2 &= 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2 && \text{Expand remaining square.} \\
2cx &= 4a^2 + 4a\sqrt{(x-c)^2 + y^2} - 2cx && \text{Combine like terms.} \\
4cx - 4a^2 &= 4a\sqrt{(x-c)^2 + y^2} && \text{Isolate the radical.} \\
cx - a^2 &= a\sqrt{(x-c)^2 + y^2} && \text{Divide by 4.} \\
\left(cx - a^2\right)^2 &= a^2\left(\sqrt{(x-c)^2 + y^2}\right)^2 && \text{Square both sides.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2\left(x^2 - 2cx + c^2 + y^2\right) && \text{Expand the squares.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 && \text{Distribute } a^2. \\
a^4 + c^2x^2 &= a^2x^2 + a^2c^2 + a^2y^2 && \text{Combine like terms.} \\
c^2x^2 - a^2x^2 - a^2y^2 &= a^2c^2 - a^4 && \text{Rearrange terms.} \\
x^2\left(c^2 - a^2\right) - a^2y^2 &= a^2\left(c^2 - a^2\right) && \text{Factor common terms.} \\
x^2b^2 - a^2y^2 &= a^2b^2 && \text{Set } b^2 = c^2 - a^2. \\
\frac{x^2b^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} && \text{Divide both sides by } a^2b^2 \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1
\end{aligned}
$$

This equation defines a hyperbola centered at the origin with vertices $(\pm a, 0)$ and co-vertices $(0 \pm b)$.

Standard Forms of the Equation of a Hyperbola with Center (0,0)

The standard form of the equation of a hyperbola with center $(0, 0)$ and transverse axis on the *x*-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(\pm c, 0)$
- the equations of the asymptotes are $y = \pm \frac{b}{a}x$

See Figure 5**a**.

The standard form of the equation of a hyperbola with center $(0, 0)$ and transverse axis on the *y*-axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(\pm b, 0)$

- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y = \pm \frac{a}{b} x$

See Figure 5**b**.

Note that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. When we are given the equation of a hyperbola, we can use this relationship to identify its vertices and foci.

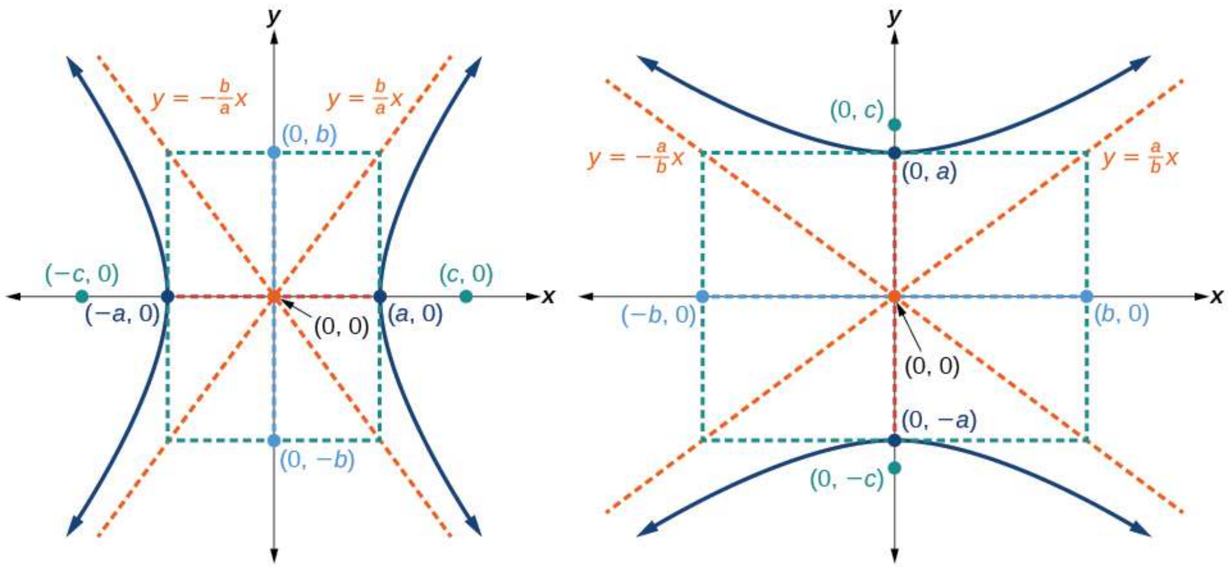


Figure 5 (a) Horizontal hyperbola with center $(0, 0)$ (b) Vertical hyperbola with center $(0, 0)$

HOW TO

Given the equation of a hyperbola in standard form, locate its vertices and foci.

1. Determine whether the transverse axis lies on the *x*- or *y*-axis. Notice that a^2 is always under the variable with the positive coefficient. So, if you set the other variable equal to zero, you can easily find the intercepts. In the case where the hyperbola is centered at the origin, the intercepts coincide with the vertices.
   a. If the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the transverse axis lies on the *x*-axis. The vertices are located at $(\pm a, 0)$, and the foci are located at $(\pm c, 0)$.
   b. If the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then the transverse axis lies on the *y*-axis. The vertices are located at $(0, \pm a)$, and the foci are located at $(0, \pm c)$.
2. Solve for a using the equation $a = \sqrt{a^2}$.
3. Solve for c using the equation $c = \sqrt{a^2 + b^2}$.

EXAMPLE 1

Locating a Hyperbola's Vertices and Foci

Identify the vertices and foci of the hyperbola with equation $\frac{y^2}{49} - \frac{x^2}{32} = 1$.

Solution

The equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, so the transverse axis lies on the *y*-axis. The hyperbola is centered at the origin, so the vertices serve as the y-intercepts of the graph. To find the vertices, set $x = 0$, and solve for y.

$$\begin{aligned}1 &= \frac{y^2}{49} - \frac{x^2}{32}\\ 1 &= \frac{y^2}{49} - \frac{0^2}{32}\\ 1 &= \frac{y^2}{49}\\ y^2 &= 49\\ y &= \pm\sqrt{49} = \pm 7\end{aligned}$$

The foci are located at $(0, \pm c)$. Solving for c,

$$c = \sqrt{a^2 + b^2} = \sqrt{49 + 32} = \sqrt{81} = 9$$

Therefore, the vertices are located at $(0, \pm 7)$, and the foci are located at $(0, 9)$.

TRY IT #1 Identify the vertices and foci of the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{25} = 1$.

Writing Equations of Hyperbolas in Standard Form

Just as with ellipses, writing the equation for a hyperbola in standard form allows us to calculate the key features: its center, vertices, co-vertices, foci, asymptotes, and the lengths and positions of the transverse and conjugate axes. Conversely, an equation for a hyperbola can be found given its key features. We begin by finding standard equations for hyperbolas centered at the origin. Then we will turn our attention to finding standard equations for hyperbolas centered at some point other than the origin.

Hyperbolas Centered at the Origin

Reviewing the standard forms given for hyperbolas centered at $(0, 0)$, we see that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. Note that this equation can also be rewritten as $b^2 = c^2 - a^2$. This relationship is used to write the equation for a hyperbola when given the coordinates of its foci and vertices.

HOW TO

Given the vertices and foci of a hyperbola centered at $(0, \ 0)$, write its equation in standard form.

1. Determine whether the transverse axis lies on the x- or y-axis.
   a. If the given coordinates of the vertices and foci have the form $(\pm a, 0)$ and $(\pm c, 0)$, respectively, then the transverse axis is the x-axis. Use the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
   b. If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(0, \pm c)$, respectively, then the transverse axis is the y-axis. Use the standard form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.
2. Find b^2 using the equation $b^2 = c^2 - a^2$.
3. Substitute the values for a^2 and b^2 into the standard form of the equation determined in Step 1.

EXAMPLE 2

Finding the Equation of a Hyperbola Centered at (0,0) Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices $(\pm 6, 0)$ and foci $\left(\pm 2\sqrt{10}, 0\right)$?

Solution

The vertices and foci are on the x-axis. Thus, the equation for the hyperbola will have the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The vertices are $(\pm 6, 0)$, so $a = 6$ and $a^2 = 36$.

The foci are $\left(\pm 2\sqrt{10}, 0\right)$, so $c = 2\sqrt{10}$ and $c^2 = 40.$

Solving for b^2, we have

$$\begin{aligned} b^2 &= c^2 - a^2 & \\ b^2 &= 40 - 36 & \text{Substitute for } c^2 \text{ and } a^2. \\ b^2 &= 4 & \text{Subtract.} \end{aligned}$$

Finally, we substitute $a^2 = 36$ and $b^2 = 4$ into the standard form of the equation, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The equation of the hyperbola is $\frac{x^2}{36} - \frac{y^2}{4} = 1$, as shown in Figure 6.

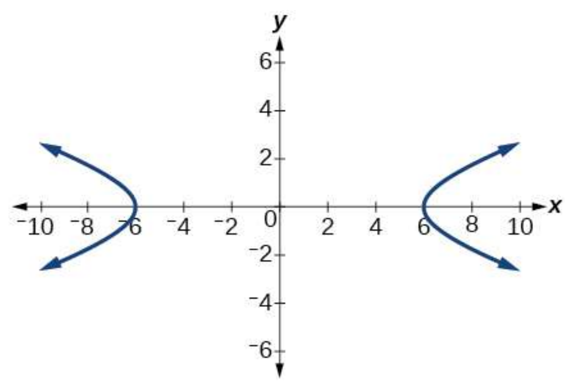


Figure 6

TRY IT #2 What is the standard form equation of the hyperbola that has vertices $(0, \pm 2)$ and foci $\left(0, \pm 2\sqrt{5}\right)$?

Hyperbolas Not Centered at the Origin

Like the graphs for other equations, the graph of a hyperbola can be translated. If a hyperbola is translated h units horizontally and k units vertically, the center of the hyperbola will be (h, k). This translation results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and y replaced by $(y - k)$.

Standard Forms of the Equation of a Hyperbola with Center (*h*, *k*)

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the *x*-axis is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h \pm c, k)$

The asymptotes of the hyperbola coincide with the diagonals of the central rectangle. The length of the rectangle is $2a$ and its width is $2b$. The slopes of the diagonals are $\pm\frac{b}{a}$, and each diagonal passes through the center (h, k). Using the **point-slope formula**, it is simple to show that the equations of the asymptotes are $y = \pm\frac{b}{a}(x - h) + k$. See Figure 7**a**

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the *y*-axis is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h, k \pm c)$

Using the reasoning above, the equations of the asymptotes are $y = \pm\frac{a}{b}(x-h) + k$. See Figure 7**b**.

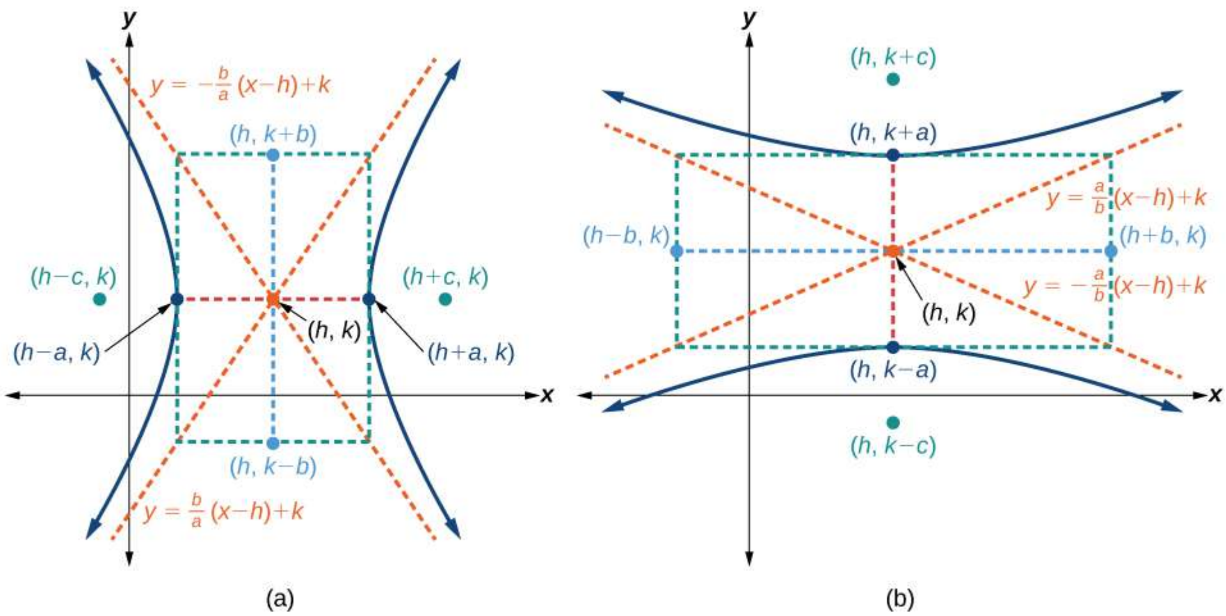


Figure 7 (a) Horizontal hyperbola with center (h, k) (b) Vertical hyperbola with center (h, k)

Like hyperbolas centered at the origin, hyperbolas centered at a point (h, k) have vertices, co-vertices, and foci that are related by the equation $c^2 = a^2 + b^2$. We can use this relationship along with the midpoint and distance formulas to find the standard equation of a hyperbola when the vertices and foci are given.

HOW TO

Given the vertices and foci of a hyperbola centered at (h, k), write its equation in standard form.

1. Determine whether the transverse axis is parallel to the *x*- or *y*-axis.
   a. If the *y*-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the *x*-axis. Use the standard form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.
   b. If the *x*-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the *y*-axis. Use the standard form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.
2. Identify the center of the hyperbola, (h, k), using the midpoint formula and the given coordinates for the vertices.
3. Find a^2 by solving for the length of the transverse axis, $2a$, which is the distance between the given vertices.

4. Find c^2 using h and k found in Step 2 along with the given coordinates for the foci.
5. Solve for b^2 using the equation $b^2 = c^2 - a^2$.
6. Substitute the values for h, k, a^2, and b^2 into the standard form of the equation determined in Step 1.

EXAMPLE 3

Finding the Equation of a Hyperbola Centered at (*h*, *k*) Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices at $(0, -2)$ and $(6, -2)$ and foci at $(-2, -2)$ and $(8, -2)$?

Solution

The *y*-coordinates of the vertices and foci are the same, so the transverse axis is parallel to the *x*-axis. Thus, the equation of the hyperbola will have the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

First, we identify the center, (h, k). The center is halfway between the vertices $(0, -2)$ and $(6, -2)$. Applying the midpoint formula, we have

$$(h, k) = \left(\frac{0+6}{2}, \frac{-2+(-2)}{2}\right) = (3, -2)$$

Next, we find a^2. The length of the transverse axis, $2a$, is bounded by the vertices. So, we can find a^2 by finding the distance between the *x*-coordinates of the vertices.

$$\begin{aligned} 2a &= |0-6| \\ 2a &= 6 \\ a &= 3 \\ a^2 &= 9 \end{aligned}$$

Now we need to find c^2. The coordinates of the foci are $(h \pm c, k)$. So $(h - c, k) = (-2, -2)$ and $(h + c, k) = (8, -2)$. We can use the *x*-coordinate from either of these points to solve for c. Using the point $(8, -2)$, and substituting $h = 3$,

$$\begin{aligned} h + c &= 8 \\ 3 + c &= 8 \\ c &= 5 \\ c^2 &= 25 \end{aligned}$$

Next, solve for b^2 using the equation $b^2 = c^2 - a^2$:

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 25 - 9 \\ &= 16 \end{aligned}$$

Finally, substitute the values found for h, k, a^2, and b^2 into the standard form of the equation.

$$\frac{(x-3)^2}{9} - \frac{(y+2)^2}{16} = 1$$

TRY IT #3 What is the standard form equation of the hyperbola that has vertices $(1, -2)$ and $(1, 8)$ and foci $(1, -10)$ and $(1, 16)$?

Graphing Hyperbolas Centered at the Origin

When we have an equation in standard form for a hyperbola centered at the origin, we can interpret its parts to identify the key features of its graph: the center, vertices, co-vertices, asymptotes, foci, and lengths and positions of the transverse and conjugate axes. To graph hyperbolas centered at the origin, we use the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for horizontal hyperbolas and the standard form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ for vertical hyperbolas.

HOW TO

Given a standard form equation for a hyperbola centered at $(0, 0)$, sketch the graph.

1. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the vertices, co-vertices, and foci; and the equations for the asymptotes.
   a. If the equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then
      - the transverse axis is on the *x*-axis
      - the coordinates of the vertices are $(\pm a, 0)$
      - the coordinates of the co-vertices are $(0, \pm b)$
      - the coordinates of the foci are $(\pm c, 0)$
      - the equations of the asymptotes are $y = \pm \frac{b}{a} x$
   b. If the equation is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then
      - the transverse axis is on the *y*-axis
      - the coordinates of the vertices are $(0, \pm a)$
      - the coordinates of the co-vertices are $(\pm b, 0)$
      - the coordinates of the foci are $(0, \pm c)$
      - the equations of the asymptotes are $y = \pm \frac{a}{b} x$
3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2 + b^2}$.
4. Plot the vertices, co-vertices, foci, and asymptotes in the coordinate plane, and draw a smooth curve to form the hyperbola.

EXAMPLE 4

Graphing a Hyperbola Centered at (0, 0) Given an Equation in Standard Form

Graph the hyperbola given by the equation $\frac{y^2}{64} - \frac{x^2}{36} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

Solution

The standard form that applies to the given equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Thus, the transverse axis is on the *y*-axis

The coordinates of the vertices are $(0, \pm a) = \left(0, \pm\sqrt{64}\right) = (0, \pm 8)$

The coordinates of the co-vertices are $(\pm b, 0) = \left(\pm\sqrt{36},\ 0\right) = (\pm 6, 0)$

The coordinates of the foci are $(0, \pm c)$, where $c = \pm\sqrt{a^2 + b^2}$. Solving for c, we have

$$c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{64 + 36} = \pm\sqrt{100} = \pm 10$$

Therefore, the coordinates of the foci are $(0, \pm 10)$

The equations of the asymptotes are $y = \pm \frac{a}{b} x = \pm \frac{8}{6} x = \pm \frac{4}{3} x$

Plot and label the vertices and co-vertices, and then sketch the central rectangle. Sides of the rectangle are parallel to the axes and pass through the vertices and co-vertices. Sketch and extend the diagonals of the central rectangle to show the asymptotes. The central rectangle and asymptotes provide the framework needed to sketch an accurate graph of the hyperbola. Label the foci and asymptotes, and draw a smooth curve to form the hyperbola, as shown in Figure 8.

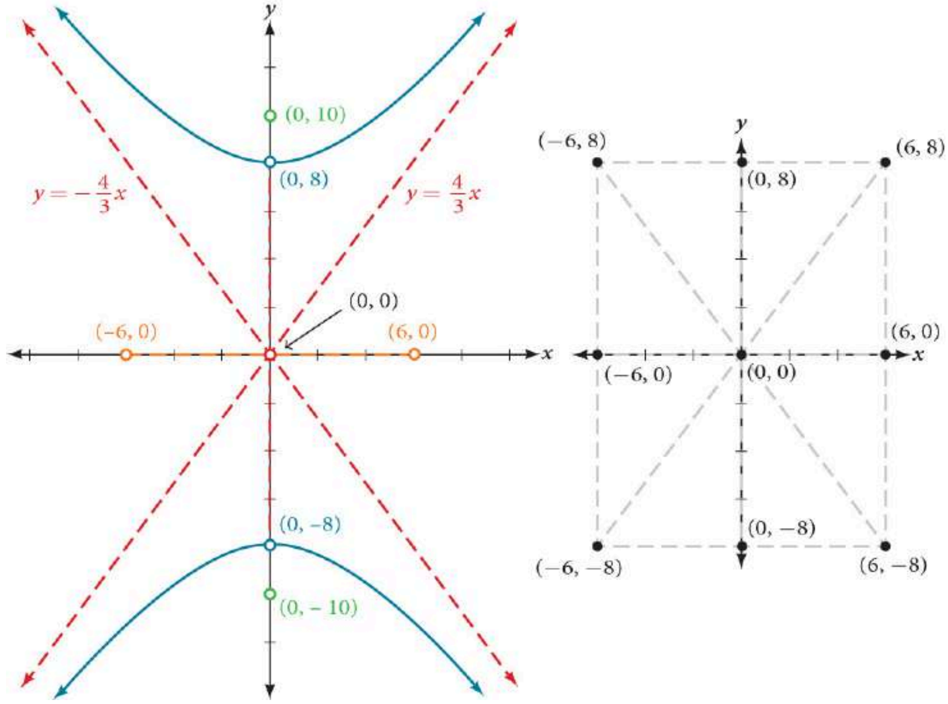


Figure 8

TRY IT #4 Graph the hyperbola given by the equation $\frac{x^2}{144} - \frac{y^2}{81} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

Graphing Hyperbolas Not Centered at the Origin

Graphing hyperbolas centered at a point (h, k) other than the origin is similar to graphing ellipses centered at a point other than the origin. We use the standard forms $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ for horizontal hyperbolas, and $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ for vertical hyperbolas. From these standard form equations we can easily calculate and plot key features of the graph: the coordinates of its center, vertices, co-vertices, and foci; the equations of its asymptotes; and the positions of the transverse and conjugate axes.

HOW TO

Given a general form for a hyperbola centered at (h, k) , sketch the graph.

1. Convert the general form to that standard form. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the center, vertices, co-vertices, foci; and equations for the asymptotes.
   a. If the equation is in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, then
      - the transverse axis is parallel to the *x*-axis
      - the center is (h, k)
      - the coordinates of the vertices are $(h \pm a, k)$
      - the coordinates of the co-vertices are $(h, k \pm b)$
      - the coordinates of the foci are $(h \pm c, k)$
      - the equations of the asymptotes are $y = \pm\frac{b}{a}(x - h) + k$

b. If the equation is in the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, then
   - the transverse axis is parallel to the *y*-axis
   - the center is (h, k)
   - the coordinates of the vertices are $(h, k \pm a)$
   - the coordinates of the co-vertices are $(h \pm b, k)$
   - the coordinates of the foci are $(h, k \pm c)$
   - the equations of the asymptotes are $y = \pm\frac{a}{b}(x-h) + k$

3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2 + b^2}$.
4. Plot the center, vertices, co-vertices, foci, and asymptotes in the coordinate plane and draw a smooth curve to form the hyperbola.

EXAMPLE 5

Graphing a Hyperbola Centered at (*h*, *k*) Given an Equation in General Form

Graph the hyperbola given by the equation $9x^2 - 4y^2 - 36x - 40y - 388 = 0$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Solution

Start by expressing the equation in standard form. Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$\left(9x^2 - 36x\right) - \left(4y^2 + 40y\right) = 388$$

Factor the leading coefficient of each expression.

$$9\left(x^2 - 4x\right) - 4\left(y^2 + 10y\right) = 388$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$9\left(x^2 - 4x + 4\right) - 4\left(y^2 + 10y + 25\right) = 388 + 36 - 100$$

Rewrite as perfect squares.

$$9(x-2)^2 - 4(y+5)^2 = 324$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-2)^2}{36} - \frac{(y+5)^2}{81} = 1$$

The standard form that applies to the given equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where $a^2 = 36$ and $b^2 = 81$, or $a = 6$ and $b = 9$. Thus, the transverse axis is parallel to the *x*-axis. It follows that:

- the center of the ellipse is $(h, k) = (2, -5)$
- the coordinates of the vertices are $(h \pm a, k) = (2 \pm 6, -5)$, or $(-4, -5)$ and $(8, -5)$
- the coordinates of the co-vertices are $(h, k \pm b) = (2, -5 \pm 9)$, or $(2, -14)$ and $(2, 4)$
- the coordinates of the foci are $(h \pm c, k)$, where $c = \pm\sqrt{a^2 + b^2}$. Solving for c, we have

$$c = \pm\sqrt{36 + 81} = \pm\sqrt{117} = \pm 3\sqrt{13}$$

Therefore, the coordinates of the foci are $\left(2 - 3\sqrt{13}, -5\right)$ and $\left(2 + 3\sqrt{13}, -5\right)$.

The equations of the asymptotes are $y = \pm\frac{b}{a}(x-h) + k = \pm\frac{3}{2}(x-2) - 5$.

Next, we plot and label the center, vertices, co-vertices, foci, and asymptotes and draw smooth curves to form the hyperbola, as shown in Figure 9.

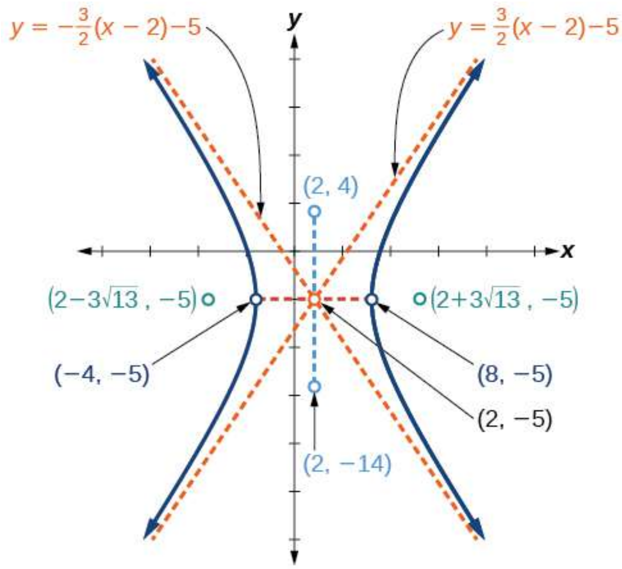


Figure 9

TRY IT #5 Graph the hyperbola given by the standard form of an equation $\frac{(y+4)^2}{100} - \frac{(x-3)^2}{64} = 1$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Solving Applied Problems Involving Hyperbolas

As we discussed at the beginning of this section, hyperbolas have real-world applications in many fields, such as astronomy, physics, engineering, and architecture. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere and are often touted for their ability to generate power efficiently. Because of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength. See Figure 10. For example, a 500-foot tower can be made of a reinforced concrete shell only 6 or 8 inches wide!

Figure 10 Cooling towers at the Drax power station in North Yorkshire, United Kingdom (credit: Les Haines, Flickr)

The first hyperbolic towers were designed in 1914 and were 35 meters high. Today, the tallest cooling towers are in France, standing a remarkable 170 meters tall. In Example 6 we will use the design layout of a cooling tower to find a hyperbolic equation that models its sides.

EXAMPLE 6

Solving Applied Problems Involving Hyperbolas

The design layout of a cooling tower is shown in Figure 11. The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart.

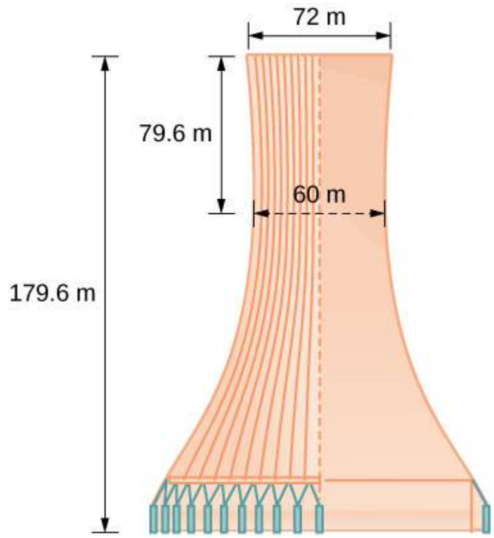


Figure 11 Project design for a natural draft cooling tower

Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

Solution

We are assuming the center of the tower is at the origin, so we can use the standard form of a horizontal hyperbola centered at the origin: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the branches of the hyperbola form the sides of the cooling tower. We must find the values of a^2 and b^2 to complete the model.

First, we find a^2. Recall that the length of the transverse axis of a hyperbola is $2a$. This length is represented by the distance where the sides are closest, which is given as 60 meters. So, $2a = 60$. Therefore, $a = 30$ and $a^2 = 900$.

To solve for b^2, we need to substitute for x and y in our equation using a known point. To do this, we can use the dimensions of the tower to find some point (x, y) that lies on the hyperbola. We will use the top right corner of the tower to represent that point. Since the *y*-axis bisects the tower, our *x*-value can be represented by the radius of the top, or 36 meters. The *y*-value is represented by the distance from the origin to the top, which is given as 79.6 meters. Therefore,

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 && \text{Standard form of horizontal hyperbola.} \\ b^2 &= \frac{y^2}{\frac{x^2}{a^2} - 1} && \text{Isolate } b^2 \\ &= \frac{(79.6)^2}{\frac{(36)^2}{900} - 1} && \text{Substitute for } a^2, x, \text{ and } y \\ &\approx 14400.3636 && \text{Round to four decimal places} \end{aligned}$$

The sides of the tower can be modeled by the hyperbolic equation

$$\frac{x^2}{900} - \frac{y^2}{14400.3636} = 1, \text{or } \frac{x^2}{30^2} - \frac{y^2}{120.0015^2} = 1$$

TRY IT #6 A design for a cooling tower project is shown in Figure 12. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

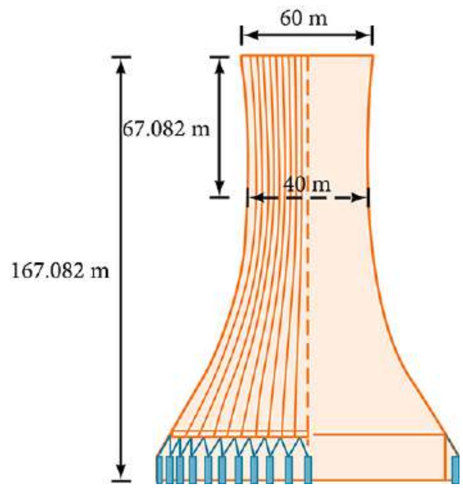


Figure 12

MEDIA

Access these online resources for additional instruction and practice with hyperbolas.

[Conic Sections: The Hyperbola Part 1 of 2 (http://openstax.org/l/hyperbola1)](http://openstax.org/l/hyperbola1)
[Conic Sections: The Hyperbola Part 2 of 2 (http://openstax.org/l/hyperbola2)](http://openstax.org/l/hyperbola2)
[Graph a Hyperbola with Center at Origin (http://openstax.org/l/hyperbolaorigin)](http://openstax.org/l/hyperbolaorigin)
[Graph a Hyperbola with Center not at Origin (http://openstax.org/l/hbnotorigin)](http://openstax.org/l/hbnotorigin)

8.2 SECTION EXERCISES

Verbal

1. Define a hyperbola in terms of its foci.

2. What can we conclude about a hyperbola if its asymptotes intersect at the origin?

3. What must be true of the foci of a hyperbola?

4. If the transverse axis of a hyperbola is vertical, what do we know about the graph?

5. Where must the center of hyperbola be relative to its foci?

Algebraic

For the following exercises, determine whether the following equations represent hyperbolas. If so, write in standard form.

6. $3y^2 + 2x = 6$

7. $\frac{x^2}{36} - \frac{y^2}{9} = 1$

8. $5y^2 + 4x^2 = 6x$

9. $25x^2 - 16y^2 = 400$

10. $-9x^2 + 18x + y^2 + 4y - 14 = 0$

For the following exercises, write the equation for the hyperbola in standard form if it is not already, and identify the vertices and foci, and write equations of asymptotes.

11. $\frac{x^2}{25} - \frac{y^2}{36} = 1$

12. $\frac{x^2}{100} - \frac{y^2}{9} = 1$

13. $\frac{y^2}{4} - \frac{x^2}{81} = 1$

14. $9y^2 - 4x^2 = 1$

15. $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{16} = 1$

16. $\frac{(y-6)^2}{36} - \frac{(x+1)^2}{16} = 1$

17. $\frac{(x-2)^2}{49} - \frac{(y+7)^2}{49} = 1$

18. $4x^2 - 8x - 9y^2 - 72y + 112 = 0$

19. $-9x^2 - 54x + 9y^2 - 54y + 81 = 0$

20. $4x^2 - 24x - 36y^2 - 360y + 864 = 0$

21. $-4x^2 + 24x + 16y^2 - 128y + 156 = 0$

22. $-4x^2 + 40x + 25y^2 - 100y + 100 = 0$

23. $x^2 + 2x - 100y^2 - 1000y + 2401 = 0$

24. $-9x^2 + 72x + 16y^2 + 16y + 4 = 0$

25. $4x^2 + 24x - 25y^2 + 200y - 464 = 0$

For the following exercises, find the equations of the asymptotes for each hyperbola.

26. $\frac{y^2}{3^2} - \frac{x^2}{3^2} = 1$

27. $\frac{(x-3)^2}{5^2} - \frac{(y+4)^2}{2^2} = 1$

28. $\frac{(y-3)^2}{3^2} - \frac{(x+5)^2}{6^2} = 1$

29. $9x^2 - 18x - 16y^2 + 32y - 151 = 0$

30. $16y^2 + 96y - 4x^2 + 16x + 112 = 0$

Graphical

For the following exercises, sketch a graph of the hyperbola, labeling vertices and foci.

31. $\frac{x^2}{49} - \frac{y^2}{16} = 1$

32. $\frac{x^2}{64} - \frac{y^2}{4} = 1$

33. $\frac{y^2}{9} - \frac{x^2}{25} = 1$

34. $81x^2 - 9y^2 = 1$

35. $\frac{(y+5)^2}{9} - \frac{(x-4)^2}{25} = 1$

36. $\frac{(x-2)^2}{8} - \frac{(y+3)^2}{27} = 1$

37. $\frac{(y-3)^2}{9} - \frac{(x-3)^2}{9} = 1$

38. $-4x^2 - 8x + 16y^2 - 32y - 52 = 0$

39. $x^2 - 8x - 25y^2 - 100y - 109 = 0$

40. $-x^2 + 8x + 4y^2 - 40y + 88 = 0$

41. $64x^2 + 128x - 9y^2 - 72y - 656 = 0$

42. $16x^2 + 64x - 4y^2 - 8y - 4 = 0$

43. $-100x^2 + 1000x + y^2 - 10y - 2575 = 0$

44. $4x^2 + 16x - 4y^2 + 16y + 16 = 0$

For the following exercises, given information about the graph of the hyperbola, find its equation.

45. Vertices at $(3, 0)$ and $(-3, 0)$ and one focus at $(5, 0)$.

46. Vertices at $(0, 6)$ and $(0, -6)$ and one focus at $(0, -8)$.

47. Vertices at $(1, 1)$ and $(11, 1)$ and one focus at $(12, 1)$.

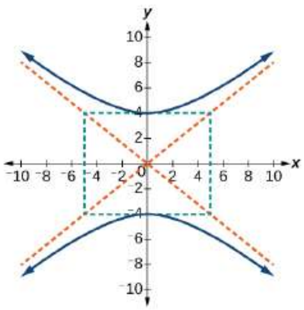
48. Center: $(0, 0)$; vertex: $(0, -13)$; one focus: $\left(0, \sqrt{313}\right)$.

49. Center: $(4, 2)$; vertex: $(9, 2)$; one focus: $\left(4 + \sqrt{26}, 2\right)$.

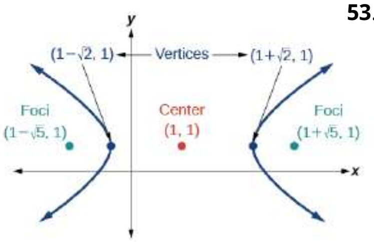
50. Center: $(3, 5)$; vertex: $(3, 11)$; one focus: $\left(3, 5 + 2\sqrt{10}\right)$.

For the following exercises, given the graph of the hyperbola, find its equation.

51.



52.



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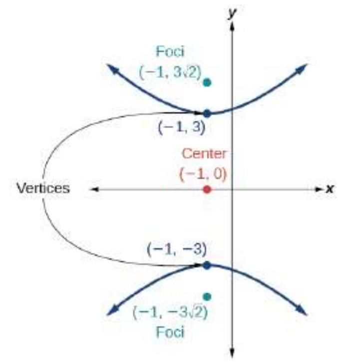
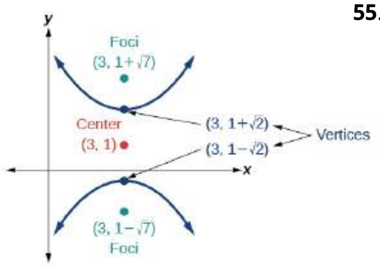


54.



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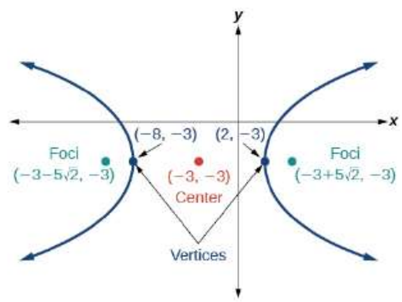


Extensions

For the following exercises, express the equation for the hyperbola as two functions, with y as a function of x. Express as simply as possible. Use a graphing calculator to sketch the graph of the two functions on the same axes.

56. $\frac{x^2}{4} - \frac{y^2}{9} = 1$

57. $\frac{y^2}{9} - \frac{x^2}{1} = 1$

58. $\frac{(x-2)^2}{16} - \frac{(y+3)^2}{25} = 1$

59. $-4x^2 - 16x + y^2 - 2y - 19 = 0$

60. $4x^2 - 24x - y^2 - 4y + 16 = 0$

Real-World Applications

For the following exercises, a hedge is to be constructed in the shape of a hyperbola near a fountain at the center of the yard. Find the equation of the hyperbola and sketch the graph.

61. The hedge will follow the asymptotes $y = x$ and $y = -x$, and its closest distance to the center fountain is 5 yards.

62. The hedge will follow the asymptotes $y = 2x$ and $y = -2x$, and its closest distance to the center fountain is 6 yards.

63. The hedge will follow the asymptotes $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$, and its closest distance to the center fountain is 10 yards.

64. The hedge will follow the asymptotes $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$, and its closest distance to the center fountain is 12 yards.

65. The hedge will follow the asymptotes $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$, and its closest distance to the center fountain is 20 yards.

For the following exercises, assume an object enters our solar system and we want to graph its path on a coordinate system with the sun at the origin and the x-axis as the axis of symmetry for the object's path. Give the equation of the flight path of each object using the given information.

66. The object enters along a path approximated by the line $y = x - 2$ and passes within 1 au (astronomical unit) of the sun at its closest approach, so that the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -x + 2$.

67. The object enters along a path approximated by the line $y = 2x - 2$ and passes within 0.5 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -2x + 2$.

68. The object enters along a path approximated by the line $y = 0.5x + 2$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -0.5x - 2$.

69. The object enters along a path approximated by the line $y = \frac{1}{3}x - 1$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -\frac{1}{3}x + 1$.

70. The object enters along a path approximated by the line $y = 3x - 9$ and passes within 1 au of the sun at its closest approach, so the sun is one focus of the hyperbola. It then departs the solar system along a path approximated by the line $y = -3x + 9$.

8.3 The Parabola

Learning Objectives

In this section, you will:

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Solve applied problems involving parabolas.

Figure 1 Katherine Johnson's pioneering mathematical work in the area of parabolic and other orbital calculations played a significant role in the development of U.S space flight. (credit: NASA)

Katherine Johnson is the pioneering NASA mathematician who was integral to the successful and safe flight and return of many human missions as well as satellites. Prior to the work featured in the movie *Hidden Figures*, she had already made major contributions to the space program. She provided trajectory analysis for the Mercury mission, in which Alan Shepard became the first American to reach space, and she and engineer Ted Sopinski authored a monumental paper regarding placing an object in a precise orbital position and having it return safely to Earth. Many of the orbits she determined were made up of parabolas, and her ability to combine different types of math enabled an unprecedented level of precision. Johnson said, "You tell me when you want it and where you want it to land, and I'll do it backwards and tell you when to take off."

Johnson's work on parabolic orbits and other complex mathematics resulted in successful orbits, Moon landings, and the development of the Space Shuttle program. Applications of parabolas are also critical to other areas of science. Parabolic mirrors (or reflectors) are able to capture energy and focus it to a single point. The advantages of this property are evidenced by the vast list of parabolic objects we use every day: satellite dishes, suspension bridges, telescopes, microphones, spotlights, and car headlights, to name a few. Parabolic reflectors are also used in alternative energy devices, such as solar cookers and water heaters, because they are inexpensive to manufacture and need little maintenance. In this section we will explore the parabola and its uses, including low-cost, energy-efficient solar designs.

Graphing Parabolas with Vertices at the Origin

In The Ellipse, we saw that an ellipse is formed when a plane cuts through a right circular cone. If the plane is parallel to the edge of the cone, an unbounded curve is formed. This curve is a **parabola**. See Figure 2.

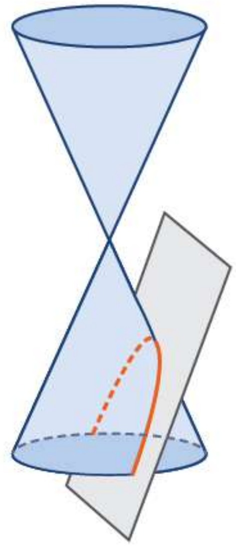


Figure 2 Parabola

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points (x, y) in a plane that are the same distance from a fixed line, called the **directrix**, and a fixed point (the **focus**) not on the directrix.

In Quadratic Functions (http://openstax.org/books/precalculus-2e/pages/3-3-power-functions-and-polynomial-functions), we learned about a parabola's vertex and axis of symmetry. Now we extend the discussion to include other key features of the parabola. See Figure 3. Notice that the axis of symmetry passes through the focus and vertex and is perpendicular to the directrix. The vertex is the midpoint between the directrix and the focus.

The line segment that passes through the focus and is parallel to the directrix is called the **latus rectum**. The endpoints of the latus rectum lie on the curve. By definition, the distance d from the focus to any point P on the parabola is equal to the distance from P to the directrix.

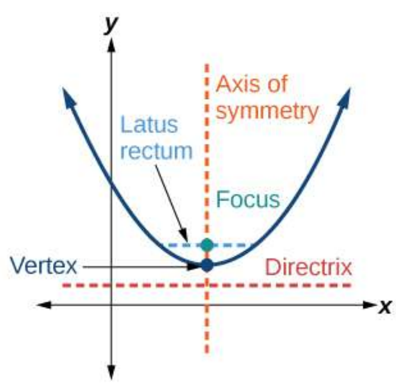


Figure 3 Key features of the parabola

To work with parabolas in the coordinate plane, we consider two cases: those with a vertex at the origin and those with a vertex at a point other than the origin. We begin with the former.

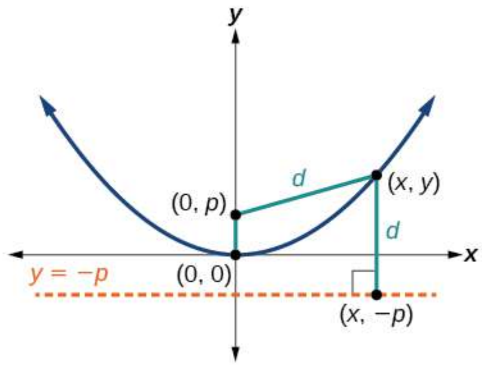


Figure 4

Let (x, y) be a point on the parabola with vertex $(0, 0)$, focus $(0, p)$, and directrix $y = - p$ as shown in Figure 4. The distance d from point (x, y) to point $(x, -p)$ on the directrix is the difference of the *y*-values: $d = y + p$. The distance from the focus $(0, p)$ to the point (x, y) is also equal to d and can be expressed using the distance formula.

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-p)^2} \\ &= \sqrt{x^2 + (y-p)^2} \end{aligned}$$

Set the two expressions for d equal to each other and solve for y to derive the equation of the parabola. We do this because the distance from (x, y) to $(0, p)$ equals the distance from (x, y) to $(x\ , -\ p)$.

$$\sqrt{x^2 + (y-p)^2} = y + p$$

We then square both sides of the equation, expand the squared terms, and simplify by combining like terms.

$$\begin{aligned} x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 - 2py &= 2py \\ x^2 &= 4py \end{aligned}$$

The equations of parabolas with vertex $(0, 0)$ are $y^2 = 4px$ when the *x*-axis is the axis of symmetry and $x^2 = 4py$ when the *y*-axis is the axis of symmetry. These standard forms are given below, along with their general graphs and key features.

Standard Forms of Parabolas with Vertex (0, 0)

Table 1 and Figure 5 summarize the standard features of parabolas with a vertex at the origin.

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
x-axis	$y^2 = 4px$	$(p,\ \ 0)$	$x = -p$	$(p,\ \ \pm 2p)$
y-axis	$x^2 = 4py$	$(0,\ \ p)$	$y = -p$	$(\pm 2p,\ \ p)$

Table 1

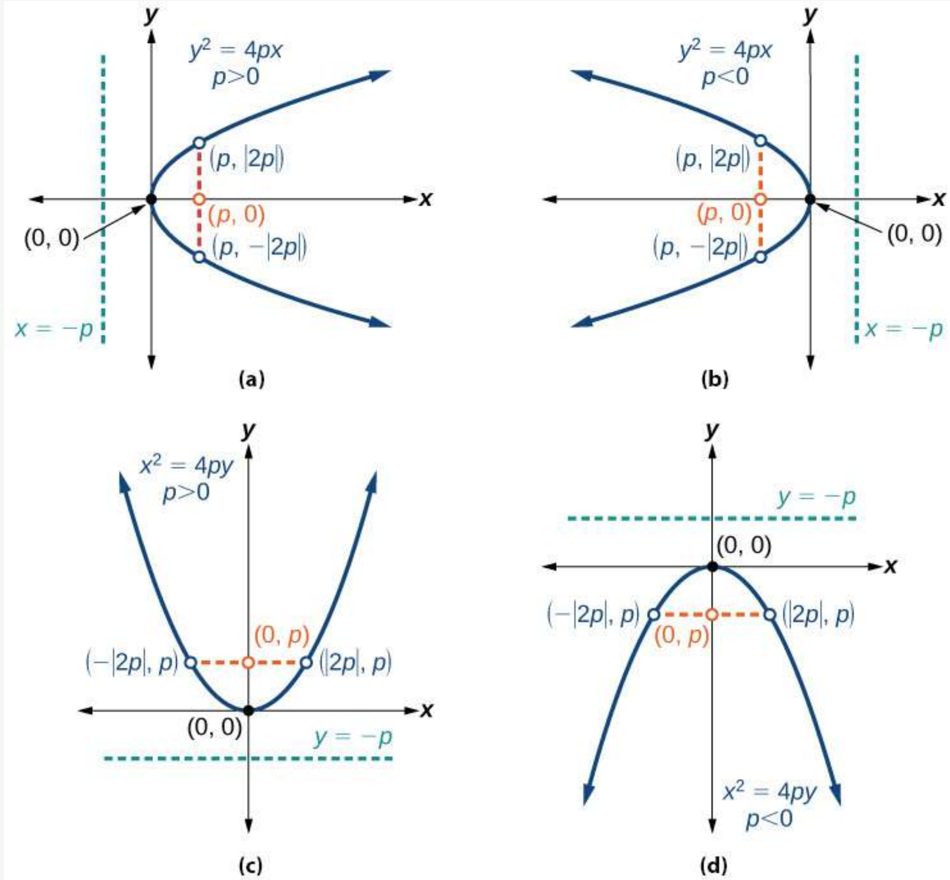


Figure 5 (a) When $p > 0$ and the axis of symmetry is the *x*-axis, the parabola opens right. (b) When $p < 0$ and the axis of symmetry is the *x*-axis, the parabola opens left. (c) When $p > 0$ and the axis of symmetry is the *y*-axis, the parabola opens up. (d) When $p < 0$ and the axis of symmetry is the *y*-axis, the parabola opens down.

The key features of a parabola are its vertex, axis of symmetry, focus, directrix, and latus rectum. See Figure 5. When given a standard equation for a parabola centered at the origin, we can easily identify the key features to graph the parabola.

A line is said to be tangent to a curve if it intersects the curve at exactly one point. If we sketch lines tangent to the parabola at the endpoints of the latus rectum, these lines intersect on the axis of symmetry, as shown in Figure 6.

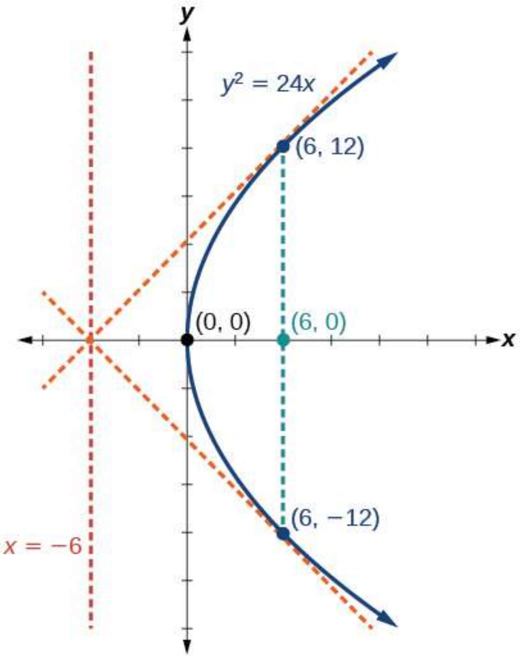


Figure 6

HOW TO

Given a standard form equation for a parabola centered at (0, 0), sketch the graph.

1. Determine which of the standard forms applies to the given equation: $y^2 = 4px$ or $x^2 = 4py$.
2. Use the standard form identified in Step 1 to determine the axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
    a. If the equation is in the form $y^2 = 4px$, then
        - the axis of symmetry is the x-axis, $y = 0$
        - set $4p$ equal to the coefficient of x in the given equation to solve for p. If $p > 0$, the parabola opens right. If $p < 0$, the parabola opens left.
        - use p to find the coordinates of the focus, $(p, 0)$
        - use p to find the equation of the directrix, $x = -p$
        - use p to find the endpoints of the latus rectum, $(p, \pm 2p)$. Alternately, substitute $x = p$ into the original equation.

    b. If the equation is in the form $x^2 = 4py$, then
        - the axis of symmetry is the y-axis, $x = 0$
        - set $4p$ equal to the coefficient of y in the given equation to solve for p. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down.
        - use p to find the coordinates of the focus, $(0, p)$
        - use p to find equation of the directrix, $y = -p$
        - use p to find the endpoints of the latus rectum, $(\pm 2p, p)$
3. Plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

EXAMPLE 1

Graphing a Parabola with Vertex (0, 0) and the *x*-axis as the Axis of Symmetry

Graph $y^2 = 24x$. Identify and label the focus, directrix, and endpoints of the latus rectum.

Solution

The standard form that applies to the given equation is $y^2 = 4px$. Thus, the axis of symmetry is the *x*-axis. It follows that:

- $24 = 4p$, so $p = 6$. Since $p > 0$, the parabola opens right
- the coordinates of the focus are $(p, 0) = (6, 0)$
- the equation of the directrix is $x = -p = -6$
- the endpoints of the latus rectum have the same *x*-coordinate at the focus. To find the endpoints, substitute $x = 6$ into the original equation: $(6, \pm 12)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola. Figure 7

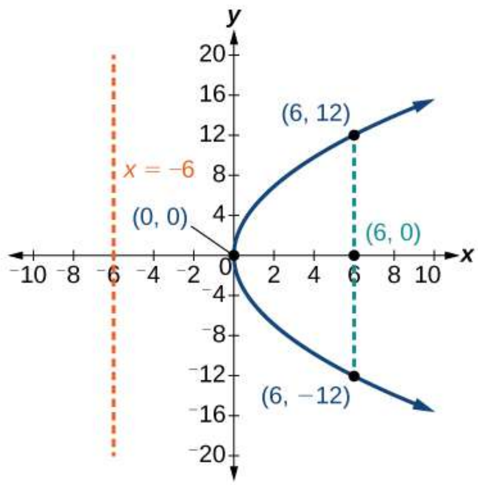


Figure 7

TRY IT #1 Graph $y^2 = -16x$. Identify and label the focus, directrix, and endpoints of the latus rectum.

EXAMPLE 2

Graphing a Parabola with Vertex (0, 0) and the *y*-axis as the Axis of Symmetry

Graph $x^2 = -6y$. Identify and label the focus, directrix, and endpoints of the latus rectum.

Solution

The standard form that applies to the given equation is $x^2 = 4py$. Thus, the axis of symmetry is the *y*-axis. It follows that:

- $-6 = 4p$, so $p = -\frac{3}{2}$. Since $p < 0$, the parabola opens down.
- the coordinates of the focus are $(0, p) = \left(0, -\frac{3}{2}\right)$
- the equation of the directrix is $y = -p = \frac{3}{2}$
- the endpoints of the latus rectum can be found by substituting $y = \frac{3}{2}$ into the original equation, $\left(\pm 3, -\frac{3}{2}\right)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

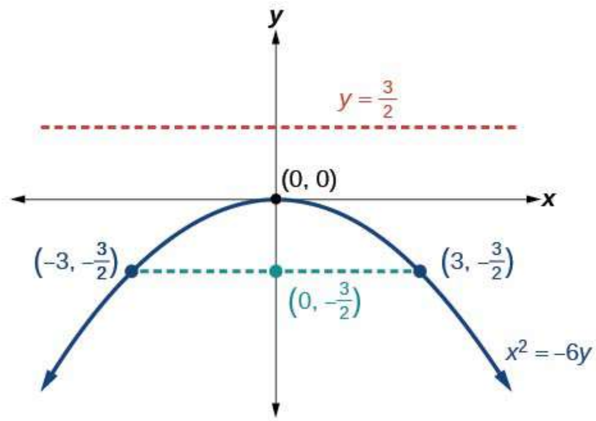


Figure 8

TRY IT #2 Graph $x^2 = 8y$. Identify and label the focus, directrix, and endpoints of the latus rectum.

Writing Equations of Parabolas in Standard Form

In the previous examples, we used the standard form equation of a parabola to calculate the locations of its key features. We can also use the calculations in reverse to write an equation for a parabola when given its key features.

HOW TO

Given its focus and directrix, write the equation for a parabola in standard form.

1. Determine whether the axis of symmetry is the *x*- or *y*-axis.
    a. If the given coordinates of the focus have the form $(p, 0)$, then the axis of symmetry is the *x*-axis. Use the standard form $y^2 = 4px$.
    b. If the given coordinates of the focus have the form $(0, p)$, then the axis of symmetry is the *y*-axis. Use the standard form $x^2 = 4py$.
2. Multiply $4p$.
3. Substitute the value from Step 2 into the equation determined in Step 1.

EXAMPLE 3

Writing the Equation of a Parabola in Standard Form Given its Focus and Directrix

What is the equation for the parabola with focus $\left(-\frac{1}{2}, 0\right)$ and directrix $x = \frac{1}{2}$?

Solution

The focus has the form $(p, 0)$, so the equation will have the form $y^2 = 4px$.

- Multiplying $4p$, we have $4p = 4\left(-\frac{1}{2}\right) = -2$.
- Substituting for $4p$, we have $y^2 = 4px = -2x$.

Therefore, the equation for the parabola is $y^2 = -2x$.

TRY IT #3 What is the equation for the parabola with focus $\left(0, \frac{7}{2}\right)$ and directrix $y = -\frac{7}{2}$?

Graphing Parabolas with Vertices Not at the Origin

Like other graphs we've worked with, the graph of a parabola can be translated. If a parabola is translated h units horizontally and k units vertically, the vertex will be (h, k) . This translation results in the standard form of the equation we saw previously with x replaced by $(x-h)$ and y replaced by $(y-k)$.

To graph parabolas with a vertex (h, k) other than the origin, we use the standard form $(y-k)^2 = 4p(x-h)$ for parabolas that have an axis of symmetry parallel to the *x*-axis, and $(x-h)^2 = 4p(y-k)$ for parabolas that have an axis of symmetry parallel to the *y*-axis. These standard forms are given below, along with their general graphs and key features.

Standard Forms of Parabolas with Vertex (h, k)

Table 2 and Figure 9 summarize the standard features of parabolas with a vertex at a point (h, k).

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
$y=k$	$(y-k)^2=4p(x-h)$	$(h+p,\ \ k)$	$x=h-p$	$(h+p,\ \ k\pm 2p)$
$x=h$	$(x-h)^2=4p(y-k)$	$(h,\ \ k+p)$	$y=k-p$	$(h\pm 2p,\ \ k+p)$

Table 2

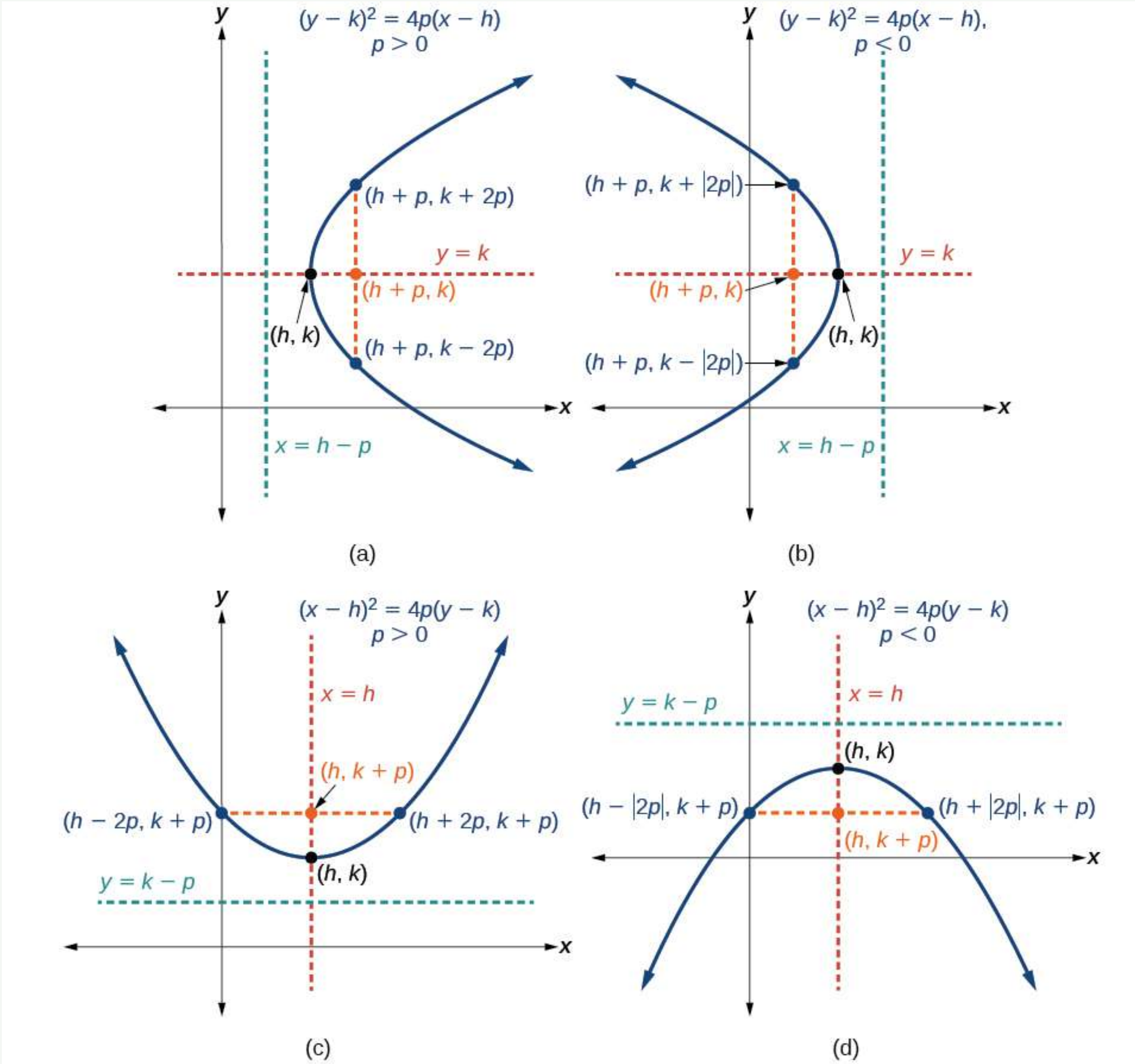


Figure 9 (a) When $p > 0$, the parabola opens right. (b) When $p < 0$, the parabola opens left. (c) When $p > 0$, the parabola opens up. (d) When $p < 0$, the parabola opens down.

HOW TO

Given a standard form equation for a parabola centered at (*h*, *k*), sketch the graph.

1. Determine which of the standard forms applies to the given equation: $(y-k)^2 = 4p(x-h)$ or $(x-h)^2 = 4p(y-k)$.
2. Use the standard form identified in Step 1 to determine the vertex, axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   a. If the equation is in the form $(y-k)^2 = 4p(x-h)$, then:
      - use the given equation to identify h and k for the vertex, (h, k)
      - use the value of k to determine the axis of symmetry, $y = k$
      - set $4p$ equal to the coefficient of $(x-h)$ in the given equation to solve for p. If $p > 0$, the parabola opens right. If $p < 0$, the parabola opens left.

- use $h, k,$ and p to find the coordinates of the focus, $(h+p, \quad k)$
- use h and p to find the equation of the directrix, $x = h - p$
- use $h, k,$ and p to find the endpoints of the latus rectum, $(h+p, k \pm 2p)$

b. If the equation is in the form $(x-h)^2 = 4p(y-k)$, then:
- use the given equation to identify h and k for the vertex, (h, k)
- use the value of h to determine the axis of symmetry, $x = h$
- set $4p$ equal to the coefficient of $(y-k)$ in the given equation to solve for p. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down.
- use $h, k,$ and p to find the coordinates of the focus, $(h, \quad k+p)$
- use k and p to find the equation of the directrix, $y = k - p$
- use $h, k,$ and p to find the endpoints of the latus rectum, $(h \pm 2p, \quad k+p)$

3. Plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

EXAMPLE 4

Graphing a Parabola with Vertex (*h*, *k*) and Axis of Symmetry Parallel to the *x*-axis

Graph $(y-1)^2 = -16(x+3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solution

The standard form that applies to the given equation is $(y-k)^2 = 4p(x-h)$. Thus, the axis of symmetry is parallel to the *x*-axis. It follows that:

- the vertex is $(h, k) = (-3, 1)$
- the axis of symmetry is $y = k = 1$
- $-16 = 4p$, so $p = -4$. Since $p < 0$, the parabola opens left.
- the coordinates of the focus are $(h+p, k) = (-3+(-4), 1) = (-7, 1)$
- the equation of the directrix is $x = h - p = -3 - (-4) = 1$
- the endpoints of the latus rectum are $(h+p, k \pm 2p) = (-3+(-4), 1 \pm 2(-4))$, or $(-7, -7)$ and $(-7, 9)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola. See Figure 10.

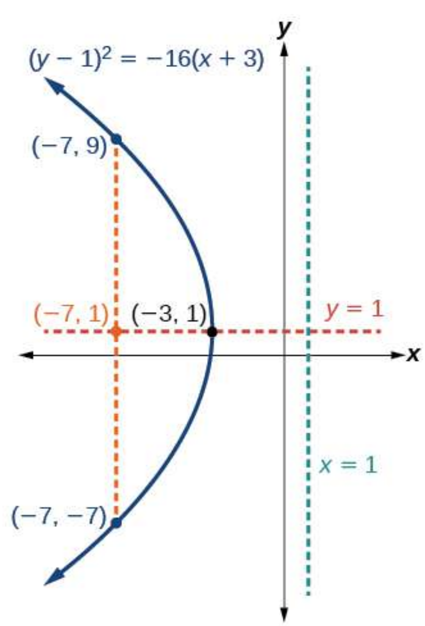


Figure 10

TRY IT #4 Graph $(y+1)^2 = 4(x-8)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

EXAMPLE 5

Graphing a Parabola from an Equation Given in General Form

Graph $x^2 - 8x - 28y - 208 = 0$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solution

Start by writing the equation of the parabola in standard form. The standard form that applies to the given equation is $(x-h)^2 = 4p(y-k)$. Thus, the axis of symmetry is parallel to the y-axis. To express the equation of the parabola in this form, we begin by isolating the terms that contain the variable x in order to complete the square.

$$\begin{aligned} x^2 - 8x - 28y - 208 &= 0 \\ x^2 - 8x &= 28y + 208 \\ x^2 - 8x + 16 &= 28y + 208 + 16 \\ (x-4)^2 &= 28y + 224 \\ (x-4)^2 &= 28(y+8) \\ (x-4)^2 &= 4 \cdot 7 \cdot (y+8) \end{aligned}$$

It follows that:

- the vertex is $(h, k) = (4, -8)$
- the axis of symmetry is $x = h = 4$
- since $p = 7$, $p > 0$ and so the parabola opens up
- the coordinates of the focus are $(h, k+p) = (4, -8+7) = (4, -1)$
- the equation of the directrix is $y = k - p = -8 - 7 = -15$
- the endpoints of the latus rectum are $(h \pm 2p, k+p) = (4 \pm 2(7), -8+7)$, or $(-10, -1)$ and $(18, -1)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola. See Figure 11.

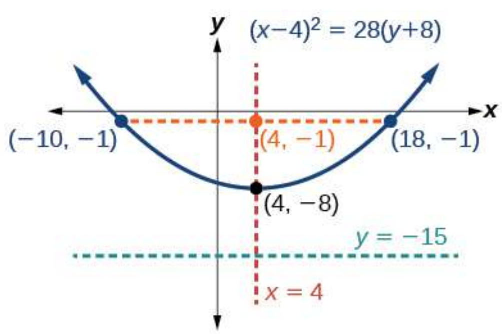


Figure 11

TRY IT #5 Graph $(x+2)^2 = -20(y-3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solving Applied Problems Involving Parabolas

As we mentioned at the beginning of the section, parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones, and radar equipment. Parabolic mirrors, such as the one used to light the Olympic torch, have a very unique reflecting property. When rays of light parallel to the parabola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus. See Figure 12. This is why the Olympic torch is ignited when it is held at the focus of the parabolic mirror.

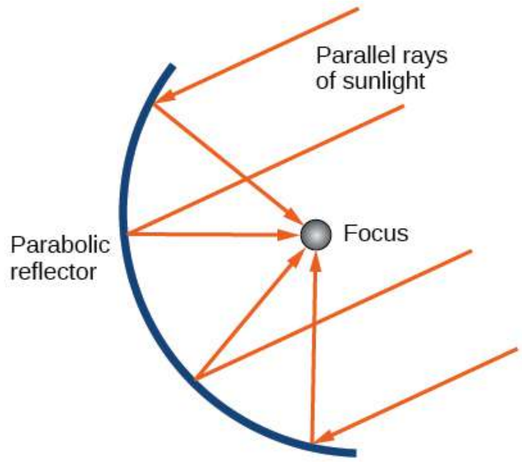


Figure 12 Reflecting property of parabolas

Parabolic mirrors have the ability to focus the sun’s energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus, parabolic mirrors are featured in many low-cost, energy efficient solar products, such as solar cookers, solar heaters, and even travel-sized fire starters.

EXAMPLE 6

Solving Applied Problems Involving Parabolas

A cross-section of a design for a travel-sized solar fire starter is shown in Figure 13. The sun’s rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds.

ⓐ Find the equation of the parabola that models the fire starter. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane.

ⓑ Use the equation found in part ⓐ to find the depth of the fire starter.

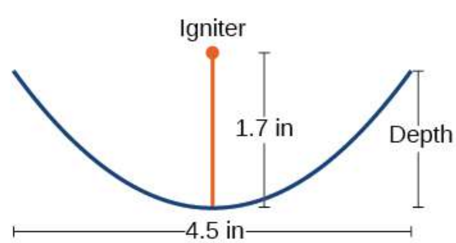


Figure 13 Cross-section of a travel-sized solar fire starter

Solution

ⓐ The vertex of the dish is the origin of the coordinate plane, so the parabola will take the standard form $x^2 = 4py$, where $p > 0$. The igniter, which is the focus, is 1.7 inches above the vertex of the dish. Thus we have $p = 1.7$.

$$\begin{aligned} x^2 &= 4py && \text{Standard form of upward-facing parabola with vertex (0,0)} \\ x^2 &= 4(1.7)y && \text{Substitute 1.7 for } p. \\ x^2 &= 6.8y && \text{Multiply.} \end{aligned}$$

ⓑ The dish extends $\frac{4.5}{2} = 2.25$ inches on either side of the origin. We can substitute 2.25 for x in the equation from part (a) to find the depth of the dish.

$$\begin{aligned} x^2 &= 6.8y && \text{Equation found in part (a).} \\ (2.25)^2 &= 6.8y && \text{Substitute 2.25 for } x. \\ y &\approx 0.74 && \text{Solve for } y. \end{aligned}$$

The dish is about 0.74 inches deep.

TRY IT #6 Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of 1600 mm. The sun's rays reflect off the parabolic mirror toward the "cooker," which is placed 320 mm from the base.

ⓐ Find an equation that models a cross-section of the solar cooker. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane, and that the parabola opens to the right (i.e., has the *x*-axis as its axis of symmetry).

ⓑ Use the equation found in part ⓐ to find the depth of the cooker.

MEDIA

Access these online resources for additional instruction and practice with parabolas.

[Conic Sections: The Parabola Part 1 of 2 (http://openstax.org/l/parabola1)](http://openstax.org/l/parabola1)
[Conic Sections: The Parabola Part 2 of 2 (http://openstax.org/l/parabola2)](http://openstax.org/l/parabola2)
[Parabola with Vertical Axis (http://openstax.org/l/parabolavertcal)](http://openstax.org/l/parabolavertcal)
[Parabola with Horizontal Axis (http://openstax.org/l/parabolahoriz)](http://openstax.org/l/parabolahoriz)

8.3 SECTION EXERCISES

Verbal

1. Define a parabola in terms of its focus and directrix.

2. If the equation of a parabola is written in standard form and p is positive and the directrix is a vertical line, then what can we conclude about its graph?

3. If the equation of a parabola is written in standard form and p is negative and the directrix is a horizontal line, then what can we conclude about its graph?

4. What is the effect on the graph of a parabola if its equation in standard form has increasing values of p?

5. As the graph of a parabola becomes wider, what will happen to the distance between the focus and directrix?

Algebraic

For the following exercises, determine whether the given equation is a parabola. If so, rewrite the equation in standard form.

6. $y^2 = 4 - x^2$

7. $y = 4x^2$

8. $3x^2 - 6y^2 = 12$

9. $(y - 3)^2 = 8(x - 2)$

10. $y^2 + 12x - 6y - 51 = 0$

For the following exercises, rewrite the given equation in standard form, and then determine the vertex (V), focus (F), and directrix (d) of the parabola.

11. $x = 8y^2$

12. $y = \frac{1}{4}x^2$

13. $y = -4x^2$

14. $x = \frac{1}{8}y^2$

15. $x = 36y^2$

16. $x = \frac{1}{36}y^2$

17. $(x-1)^2 = 4(y-1)$

18. $(y-2)^2 = \frac{4}{5}(x+4)$

19. $(y-4)^2 = 2(x+3)$

20. $(x+1)^2 = 2(y+4)$

21. $(x+4)^2 = 24(y+1)$

22. $(y+4)^2 = 16(x+4)$

23. $y^2 + 12x - 6y + 21 = 0$

24. $x^2 - 4x - 24y + 28 = 0$

25. $5x^2 - 50x - 4y + 113 = 0$

26. $y^2 - 24x + 4y - 68 = 0$

27. $x^2 - 4x + 2y - 6 = 0$

28. $y^2 - 6y + 12x - 3 = 0$

29. $3y^2 - 4x - 6y + 23 = 0$

30. $x^2 + 4x + 8y - 4 = 0$

Graphical

For the following exercises, graph the parabola, labeling the focus and the directrix.

31. $x = \frac{1}{8}y^2$

32. $y = 36x^2$

33. $y = \frac{1}{36}x^2$

34. $y = -9x^2$

35. $(y-2)^2 = -\frac{4}{3}(x+2)$

36. $-5(x+5)^2 = 4(y+5)$

37. $-6(y+5)^2 = 4(x-4)$

38. $y^2 - 6y - 8x + 1 = 0$

39. $x^2 + 8x + 4y + 20 = 0$

40. $3x^2 + 30x - 4y + 95 = 0$

41. $y^2 - 8x + 10y + 9 = 0$

42. $x^2 + 4x + 2y + 2 = 0$

43. $y^2 + 2y - 12x + 61 = 0$

44. $-2x^2 + 8x - 4y - 24 = 0$

For the following exercises, find the equation of the parabola given information about its graph.

45. Vertex is $(0,0)$; directrix is $y = 4$, focus is $(0,-4)$.

46. Vertex is $(0,0)$; directrix is $x = 4$, focus is $(-4,0)$.

47. Vertex is $(2,2)$; directrix is $x = 2-\sqrt{2}$, focus is $\left(2+\sqrt{2}, 2\right)$.

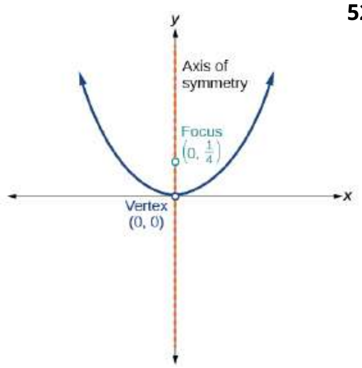
48. Vertex is $(-2,3)$; directrix is $x = -\frac{7}{2}$, focus is $\left(-\frac{1}{2}, 3\right)$.

49. Vertex is $\left(\sqrt{2}, -\sqrt{3}\right)$; directrix is $x = 2\sqrt{2}$, focus is $\left(0, -\sqrt{3}\right)$.

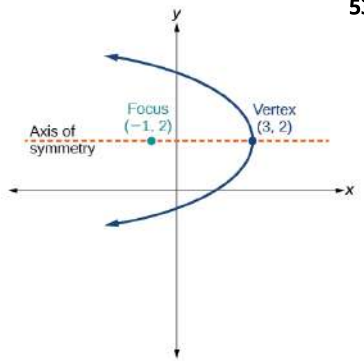
50. Vertex is $(1,2)$; directrix is $y = \frac{11}{3}$, focus is $\left(1, \frac{1}{3}\right)$.

For the following exercises, determine the equation for the parabola from its graph.

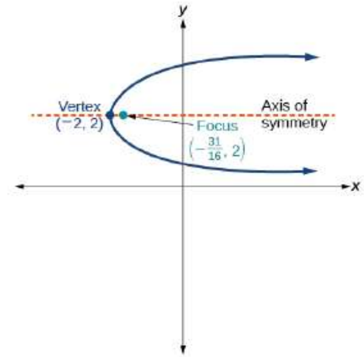
51.



52.



53.



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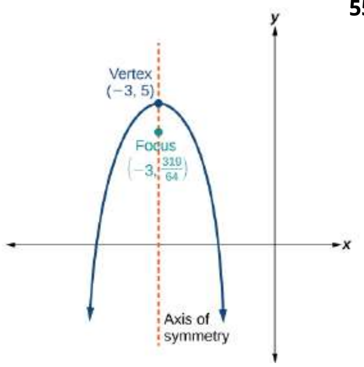

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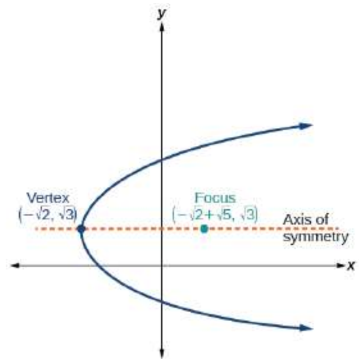


Extensions

For the following exercises, the vertex and endpoints of the latus rectum of a parabola are given. Find the equation.

56. $V\,(0,0)$, Endpoints $(2,1)$, $(-2,1)$

57. $V\,(0,0)$, Endpoints $(-2,4)$, $(-2,-4)$

58. $V\,(1,2)$, Endpoints $(-5,5)$, $(7,5)$

59. $V\,(-3,-1)$, Endpoints $(0,5)$, $(0,-7)$

60. $V\,(4,-3)$, Endpoints $\left(5,-\frac{7}{2}\right)$, $\left(3,-\frac{7}{2}\right)$

Real-World Applications

61. The mirror in an automobile headlight has a parabolic cross-section with the light bulb at the focus. On a schematic, the equation of the parabola is given as $x^2 = 4y$. At what coordinates should you place the light bulb?

62. If we want to construct the mirror from the previous exercise such that the focus is located at $(0, 0.25)$, what should the equation of the parabola be?

63. A satellite dish is shaped like a paraboloid of revolution. This means that it can be formed by rotating a parabola around its axis of symmetry. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

64. Consider the satellite dish from the previous exercise. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?

65. The reflector in a searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

66. If the reflector in the searchlight from the previous exercise has the light source located 6 inches from the base along the axis of symmetry and the opening is 4 feet, find the depth.

67. An arch is in the shape of a parabola. It has a span of 100 feet and a maximum height of 20 feet. Find the equation of the parabola, and determine the height of the arch 40 feet from the center.

68. If the arch from the previous exercise has a span of 160 feet and a maximum height of 40 feet, find the equation of the parabola, and determine the distance from the center at which the height is 20 feet.

69. An object is projected so as to follow a parabolic path given by $y = -x^2 + 96x$, where x is the horizontal distance traveled in feet and y is the height. Determine the maximum height the object reaches.

70. For the object from the previous exercise, assume the path followed is given by $y = -0.5x^2 + 80x$. Determine how far along the horizontal the object traveled to reach maximum height.

8.4 Rotation of Axes

Learning Objectives

In this section, you will:

- Identify nondegenerate conic sections given their general form equations.
- Use rotation of axes formulas.
- Write equations of rotated conics in standard form.
- Identify conics without rotating axes.

As we have seen, conic sections are formed when a plane intersects two right circular cones aligned tip to tip and extending infinitely far in opposite directions, which we also call a *cone*. The way in which we slice the cone will determine the type of conic section formed at the intersection. A circle is formed by slicing a cone with a plane perpendicular to the axis of symmetry of the cone. An ellipse is formed by slicing a single cone with a slanted plane not perpendicular to the axis of symmetry. A parabola is formed by slicing the plane through the top or bottom of the double-cone, whereas a hyperbola is formed when the plane slices both the top and bottom of the cone. See Figure 1.

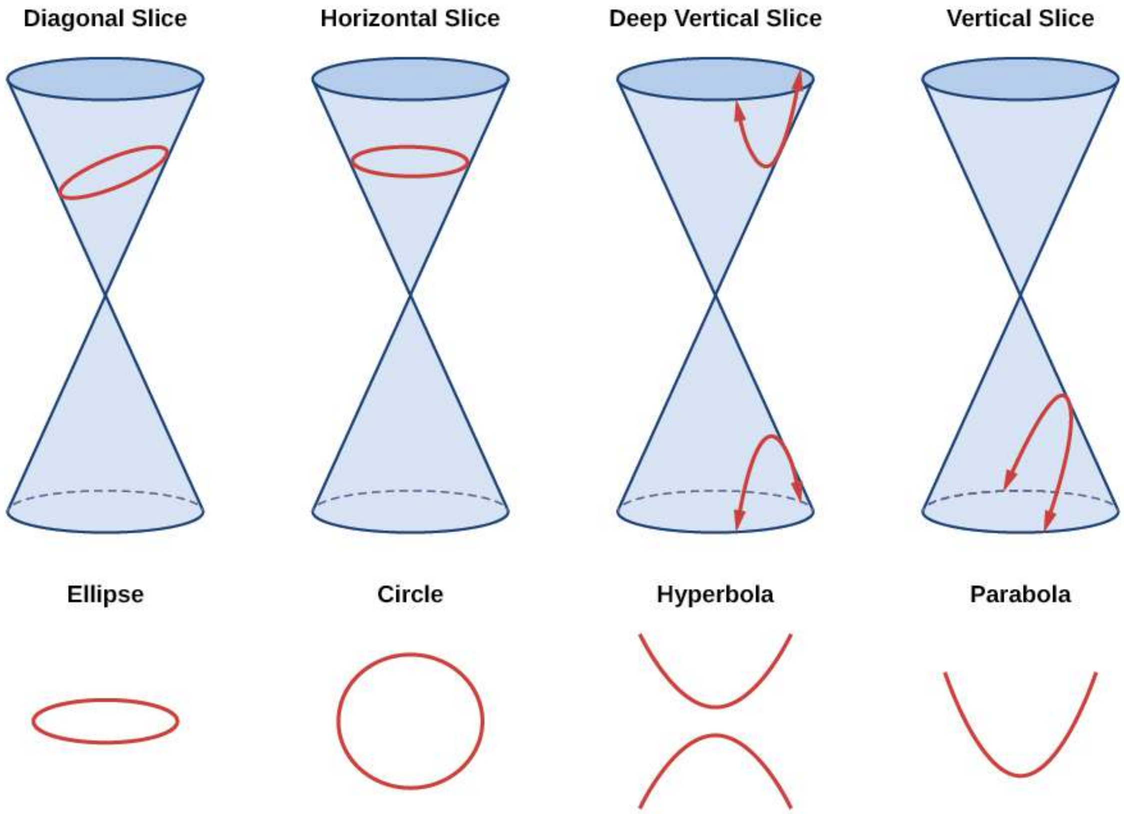


Figure 1 The nondegenerate conic sections

Ellipses, circles, hyperbolas, and parabolas are sometimes called the **nondegenerate conic sections**, in contrast to the **degenerate conic sections**, which are shown in Figure 2. A degenerate conic results when a plane intersects the double cone and passes through the apex. Depending on the angle of the plane, three types of degenerate conic sections are possible: a point, a line, or two intersecting lines.

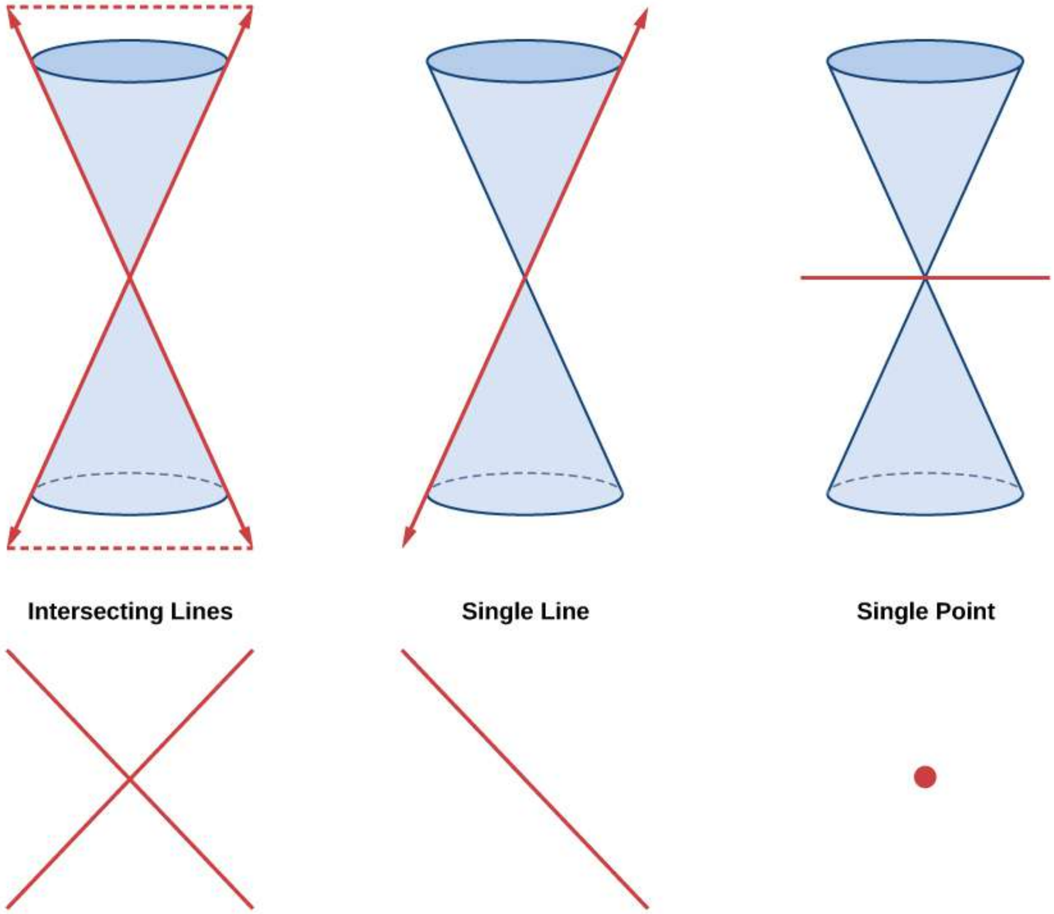


Figure 2 Degenerate conic sections

Identifying Nondegenerate Conics in General Form

In previous sections of this chapter, we have focused on the standard form equations for nondegenerate conic sections. In this section, we will shift our focus to the general form equation, which can be used for any conic. The general form is set equal to zero, and the terms and coefficients are given in a particular order, as shown below.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, and C are not all zero. We can use the values of the coefficients to identify which type conic is represented by a given equation.

You may notice that the general form equation has an xy term that we have not seen in any of the standard form equations. As we will discuss later, the xy term rotates the conic whenever B is not equal to zero.

Conic Sections	Example
ellipse	$4x^2 + 9y^2 = 1$
circle	$4x^2 + 4y^2 = 1$
hyperbola	$4x^2 - 9y^2 = 1$
parabola	$4x^2 = 9y \text{ or } 4y^2 = 9x$
one line	$4x + 9y = 1$

Table 1

Conic Sections	Example
intersecting lines	$(x-4)(y+4)=0$
parallel lines	$(x-4)(x-9)=0$
a point	$4x^2+4y^2=0$
no graph	$4x^2+4y^2=-1$

Table 1

General Form of Conic Sections

A **conic section** has the general form

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

where A, B, and C are not all zero.

Table 2 summarizes the different conic sections where $B=0$, and A and C are nonzero real numbers. This indicates that the conic has not been rotated.

ellipse	$Ax^2+Cy^2+Dx+Ey+F=0,\quad A\neq C\ \text{ and } AC>0$
circle	$Ax^2+Cy^2+Dx+Ey+F=0,\quad A=C$
hyperbola	$Ax^2-Cy^2+Dx+Ey+F=0\ \text{ or }\ -Ax^2+Cy^2+Dx+Ey+F=0$, where A and C are positive
parabola	$Ax^2+Dx+Ey+F=0\ \text{ or } Cy^2+Dx+Ey+F=0$

Table 2

HOW TO

Given the equation of a conic, identify the type of conic.

1. Rewrite the equation in the general form, $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.
2. Identify the values of A and C from the general form.
   a. If A and C are nonzero, have the same sign, and are not equal to each other, then the graph may be an ellipse.
   b. If A and C are equal and nonzero and have the same sign, then the graph may be a circle.
   c. If A and C are nonzero and have opposite signs, then the graph may be a hyperbola.
   d. If either A or C is zero, then the graph may be a parabola.

   If *B* = 0, the conic section will have a vertical and/or horizontal axes. If *B* does not equal 0, as shown below, the conic section is rotated. Notice the phrase “may be” in the definitions. That is because the equation may not represent a conic section at all, depending on the values of *A*, *B*, *C*, *D*, *E*, and *F*. For example, the degenerate case of a circle or an ellipse is a point:
   $Ax^2+By^2=0$, when *A* and *B* have the same sign.
   The degenerate case of a hyperbola is two intersecting straight lines: $Ax^2+By^2=0$, when A and B have opposite signs.
   On the other hand, the equation, $Ax^2+By^2+1=0$, when A and B are positive does not represent a graph at

all, since there are no real ordered pairs which satisfy it.

EXAMPLE 1

Identifying a Conic from Its General Form

Identify the graph of each of the following nondegenerate conic sections.

ⓐ $4x^2 - 9y^2 + 36x + 36y - 125 = 0$ ⓑ $9y^2 + 16x + 36y - 10 = 0$ ⓒ $3x^2 + 3y^2 - 2x - 6y - 4 = 0$

ⓓ $-25x^2 - 4y^2 + 100x + 16y + 20 = 0$

Solution

ⓐ Rewriting the general form, we have
$$\begin{gathered} Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \\ 4x^2 + 0xy + (-9)y^2 + 36x + 36y + (-125) = 0 \end{gathered}$$
$A = 4$ and $C = -9$, so we observe that A and C have opposite signs. The graph of this equation is a hyperbola.

ⓑ Rewriting the general form, we have
$$\begin{gathered} Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \\ 0x^2 + 0xy + 9y^2 + 16x + 36y + (-10) = 0 \end{gathered}$$
$A = 0$ and $C = 9$. We can determine that the equation is a parabola, since A is zero.

ⓒ Rewriting the general form, we have
$$\begin{gathered} Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \\ 3x^2 + 0xy + 3y^2 + (-2)x + (-6)y + (-4) = 0 \end{gathered}$$
$A = 3$ and $C = 3$. Because $A = C$, the graph of this equation is a circle.

ⓓ Rewriting the general form, we have
$$\begin{gathered} Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \\ (-25)x^2 + 0xy + (-4)y^2 + 100x + 16y + 20 = 0 \end{gathered}$$
$A = -25$ and $C = -4$. Because $AC > 0$ and $A \neq C$, the graph of this equation is an ellipse.

TRY IT #1 Identify the graph of each of the following nondegenerate conic sections.

ⓐ $16y^2 - x^2 + x - 4y - 9 = 0$ ⓑ $16x^2 + 4y^2 + 16x + 49y - 81 = 0$

Finding a New Representation of the Given Equation after Rotating through a Given Angle

Until now, we have looked at equations of conic sections without an xy term, which aligns the graphs with the *x*- and *y*-axes. When we add an xy term, we are rotating the conic about the origin. If the *x*- and *y*-axes are rotated through an angle, say θ, then every point on the plane may be thought of as having two representations: (x, y) on the Cartesian plane with the original *x*-axis and *y*-axis, and $\left(x', y'\right)$ on the new plane defined by the new, rotated axes, called the *x'*-axis and *y'*-axis. See Figure 3.

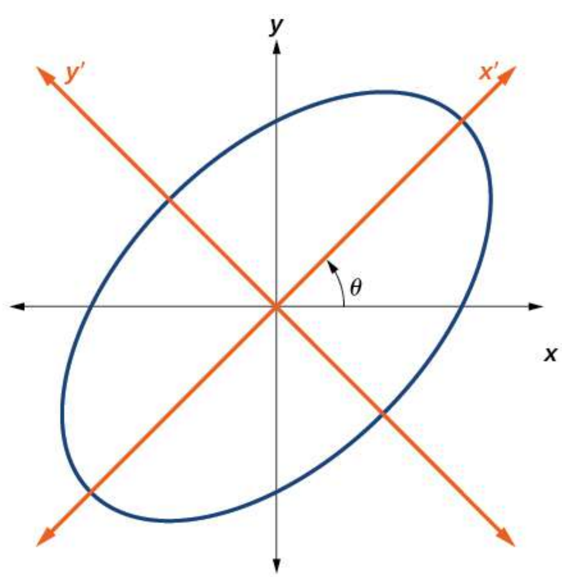


Figure 3 The graph of the rotated ellipse $x^2 + y^2 - xy - 15 = 0$

We will find the relationships between x and y on the Cartesian plane with x' and y' on the new rotated plane. See Figure 4.

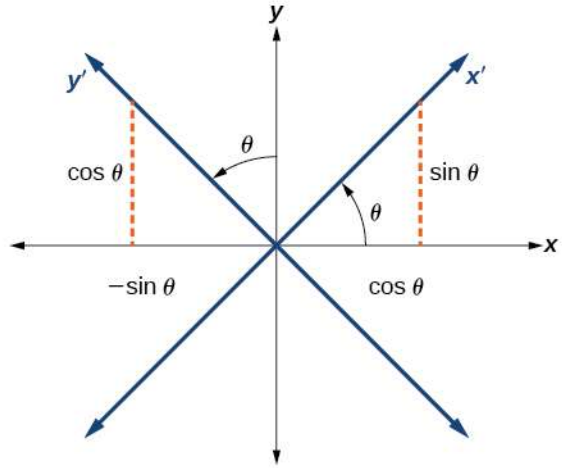


Figure 4 The Cartesian plane with x- and y-axes and the resulting x′− and y′−axes formed by a rotation by an angle θ.

The original coordinate x- and y-axes have unit vectors i and j. The rotated coordinate axes have unit vectors i' and j'. The angle θ is known as the **angle of rotation**. See Figure 5. We may write the new unit vectors in terms of the original ones.

$$\begin{aligned} i' &= \cos\ \theta i + \sin\ \theta j \\ j' &= -\sin\ \theta i + \cos\ \theta j \end{aligned}$$

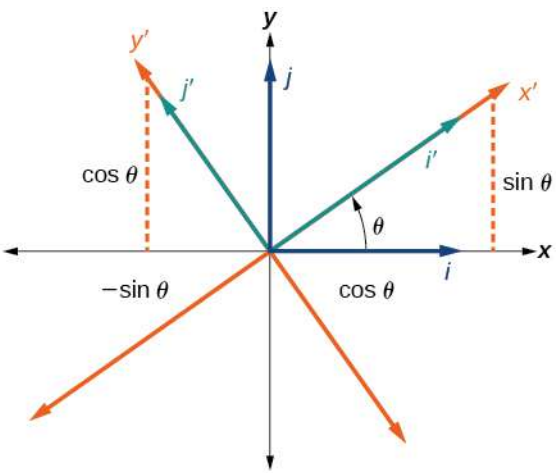


Figure 5 Relationship between the old and new coordinate planes.

Consider a vector u in the new coordinate plane. It may be represented in terms of its coordinate axes.

$$\begin{array}{ll} u = x' i' + y' j' & \\ u = x'(i\ \cos\ \theta + j\ \sin\ \theta) + y'(-i\ \sin\ \theta + j\ \cos\ \theta) & \text{Substitute.} \\ u = ix'\ \cos\ \theta + jx'\ \sin\ \theta - iy'\ \sin\ \theta + jy'\ \cos\ \theta & \text{Distribute.} \\ u = ix'\ \cos\ \theta - iy'\ \sin\ \theta + jx'\ \sin\ \theta + jy'\ \cos\ \theta & \text{Apply commutative property.} \\ u = (x'\ \cos\ \theta - y'\ \sin\ \theta)i + (x'\ \sin\ \theta + y'\ \cos\ \theta)j & \text{Factor by grouping.} \end{array}$$

Because $u = x' i' + y' j'$, we have representations of x and y in terms of the new coordinate system.

$$x = x' \cos\ \theta - y' \sin\ \theta$$

and

$$y = x' \sin\ \theta + y' \cos\ \theta$$

Equations of Rotation

If a point (x, y) on the Cartesian plane is represented on a new coordinate plane where the axes of rotation are formed by rotating an angle θ from the positive x-axis, then the coordinates of the point with respect to the new axes are $\left(x', y'\right)$. We can use the following equations of rotation to define the relationship between (x, y) and $\left(x', y'\right)$:

$$x = x' \cos\ \theta - y' \sin\ \theta$$

and

$$y = x' \sin\ \theta + y' \cos\ \theta$$

HOW TO

Given the equation of a conic, find a new representation after rotating through an angle.

1. Find x and y where $x = x' \cos\ \theta - y' \sin\ \theta$ and $y = x' \sin\ \theta + y' \cos\ \theta$.
2. Substitute the expression for x and y into in the given equation, then simplify.
3. Write the equations with x' and y' in standard form.

EXAMPLE 2

Finding a New Representation of an Equation after Rotating through a Given Angle

Find a new representation of the equation $2x^2 - xy + 2y^2 - 30 = 0$ after rotating through an angle of $\theta = 45°$.

Solution

Find x and y, where $x = x' \cos\ \theta - y' \sin\ \theta$ and $y = x' \sin\ \theta + y' \cos\ \theta$.

Because $\theta = 45°$,

$$\begin{aligned} x &= x' \cos(45°) - y' \sin(45°) \\ x &= x' \left(\frac{1}{\sqrt{2}}\right) - y' \left(\frac{1}{\sqrt{2}}\right) \\ x &= \frac{x' - y'}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} y &= x' \sin(45°) + y' \cos(45°) \\ y &= x' \left(\frac{1}{\sqrt{2}}\right) + y' \left(\frac{1}{\sqrt{2}}\right) \\ y &= \frac{x' + y'}{\sqrt{2}} \end{aligned}$$

Substitute $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\ \theta + y' \cos\ \theta$ into $2x^2 - xy + 2y^2 - 30 = 0$.

$$2\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - \left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 2\left(\frac{x' + y'}{\sqrt{2}}\right)^2 - 30 = 0$$

Simplify.

$$\begin{aligned}
\cancel{2}\frac{(x'-y')(x'-y')}{\cancel{2}} - \frac{(x'-y')(x'+y')}{2} + \cancel{2}\frac{(x'+y')(x'+y')}{\cancel{2}} - 30 &= 0 && \text{FOIL method} \\
x'^2 \cancel{-2x'y'} + y'^2 - \frac{(x'^2-y'^2)}{2} + x'^2 \cancel{+2x'y'} + y'^2 - 30 &= 0 && \text{Combine like terms.} \\
2x'^2 + 2y'^2 - \frac{(x'^2-y'^2)}{2} &= 30 && \text{Combine like terms.} \\
2\left(2x'^2 + 2y'^2 - \frac{(x'^2-y'^2)}{2}\right) &= 2(30) && \text{Multiply both sides by 2.} \\
4x'^2 + 4y'^2 - (x'^2 - y'^2) &= 60 && \text{Simplify.} \\
4x'^2 + 4y'^2 - x'^2 + y'^2 &= 60 && \text{Distribute.} \\
\frac{3x'^2}{60} + \frac{5y'^2}{60} &= \frac{60}{60} && \text{Set equal to 1.}
\end{aligned}$$

Write the equations with x' and y' in the standard form.

$$\frac{x'^2}{20} + \frac{y'^2}{12} = 1$$

This equation is an ellipse. Figure 6 shows the graph.

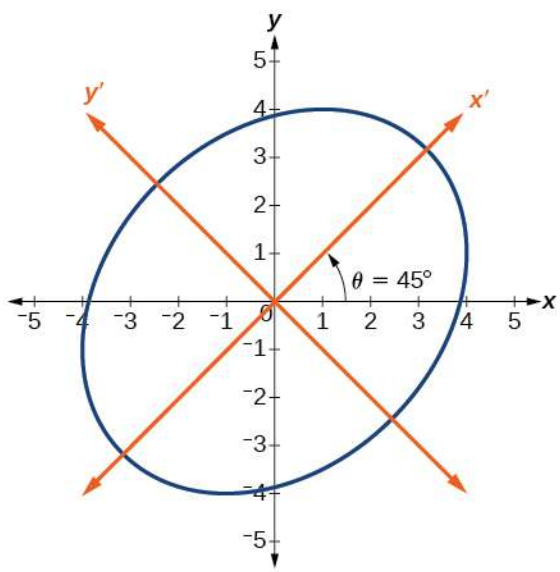


Figure 6

Writing Equations of Rotated Conics in Standard Form

Now that we can find the standard form of a conic when we are given an angle of rotation, we will learn how to transform the equation of a conic given in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ into standard form by rotating the axes. To do so, we will rewrite the general form as an equation in the x' and y' coordinate system without the $x'y'$ term, by rotating the axes by a measure of θ that satisfies

$$\cot(2\theta) = \frac{A-C}{B}$$

We have learned already that any conic may be represented by the second degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, and C are not all zero. However, if $B \neq 0$, then we have an xy term that prevents us from rewriting the equation in standard form. To eliminate it, we can rotate the axes by an acute angle θ where $\cot(2\theta) = \frac{A-C}{B}$.

- If $\cot(2\theta) > 0$, then 2θ is in the first quadrant, and θ is between $(0°, 45°)$.
- If $\cot(2\theta) < 0$, then 2θ is in the second quadrant, and θ is between $(45°, 90°)$.
- If $A = C$, then $\theta = 45°$.

HOW TO

Given an equation for a conic in the $x' y'$ system, rewrite the equation without the $x' y'$ term in terms of x' and y', where the x' and y' axes are rotations of the standard axes by θ degrees.

1. Find $\cot(2\theta)$.
2. Find $\sin\ \theta$ and $\cos\ \theta$.
3. Substitute $\sin\ \theta$ and $\cos\ \theta$ into $x = x' \cos\ \theta - y' \sin\ \theta$ and $y = x' \sin\ \theta + y' \cos\ \theta$.
4. Substitute the expression for x and y into in the given equation, and then simplify.
5. Write the equations with x' and y' in the standard form with respect to the rotated axes.

EXAMPLE 3

Rewriting an Equation with respect to the *x′* and *y′* axes without the *x′y′* Term

Rewrite the equation $8x^2 - 12xy + 17y^2 = 20$ in the $x' y'$ system without an $x' y'$ term.

Solution

First, we find $\cot(2\theta)$. See Figure 7.

$$\begin{aligned} 8x^2 - 12xy + 17y^2 = 20 &\Rightarrow A = 8,\quad B = -12 \text{ and } C = 17 \\ \cot(2\theta) &= \tfrac{A-C}{B} = \tfrac{8-17}{-12} \\ \cot(2\theta) &= \tfrac{-9}{-12} = \tfrac{3}{4} \end{aligned}$$

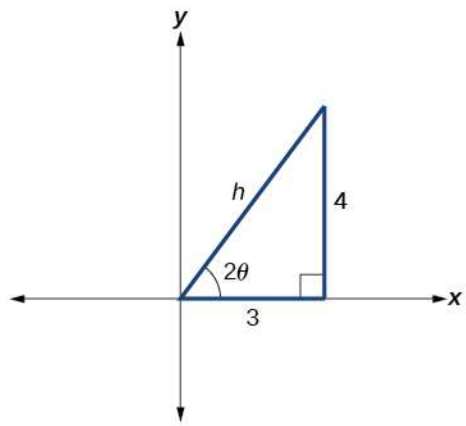


Figure 7

$$\cot(2\theta) = \frac{3}{4} = \frac{\text{adjacent}}{\text{opposite}}$$

So the hypotenuse is

$$\begin{aligned} 3^2 + 4^2 &= h^2 \\ 9 + 16 &= h^2 \\ 25 &= h^2 \\ h &= 5 \end{aligned}$$

Next, we find $\sin\ \theta$ and $\cos\ \theta$.

$$\sin\ \theta = \sqrt{\frac{1-\cos(2\theta)}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{\frac{5}{5}-\frac{3}{5}}{2}} = \sqrt{\frac{5-3}{5}\cdot\frac{1}{2}} = \sqrt{\frac{2}{10}} = \sqrt{\frac{1}{5}}$$
$$\sin\ \theta = \frac{1}{\sqrt{5}}$$
$$\cos\ \theta = \sqrt{\frac{1+\cos(2\theta)}{2}} = \sqrt{\frac{1+\frac{3}{5}}{2}} = \sqrt{\frac{\frac{5}{5}+\frac{3}{5}}{2}} = \sqrt{\frac{5+3}{5}\cdot\frac{1}{2}} = \sqrt{\frac{8}{10}} = \sqrt{\frac{4}{5}}$$
$$\cos\ \theta = \frac{2}{\sqrt{5}}$$

Substitute the values of $\sin\ \theta$ and $\cos\ \theta$ into $x = x'\cos\ \theta - y'\sin\ \theta$ and $y = x'\sin\ \theta + y'\cos\ \theta$.

$$x = x'\cos\ \theta - y'\sin\ \theta$$
$$x = x'\left(\frac{2}{\sqrt{5}}\right) - y'\left(\frac{1}{\sqrt{5}}\right)$$
$$x = \frac{2x'-y'}{\sqrt{5}}$$

and

$$y = x'\sin\ \theta + y'\cos\ \theta$$
$$y = x'\left(\frac{1}{\sqrt{5}}\right) + y'\left(\frac{2}{\sqrt{5}}\right)$$
$$y = \frac{x'+2y'}{\sqrt{5}}$$

Substitute the expressions for x and y into in the given equation, and then simplify.

$$8\left(\frac{2x'-y'}{\sqrt{5}}\right)^2 - 12\left(\frac{2x'-y'}{\sqrt{5}}\right)\left(\frac{x'+2y'}{\sqrt{5}}\right) + 17\left(\frac{x'+2y'}{\sqrt{5}}\right)^2 = 20$$
$$8\left(\frac{(2x'-y')(2x'-y')}{5}\right) - 12\left(\frac{(2x'-y')(x'+2y')}{5}\right) + 17\left(\frac{(x'+2y')(x'+2y')}{5}\right) = 20$$
$$8\left(4x'^2 - 4x'y' + y'^2\right) - 12\left(2x'^2 + 3x'y' - 2y'^2\right) + 17\left(x'^2 + 4x'y' + 4y'^2\right) = 100$$
$$32x'^2 - 32x'y' + 8y'^2 - 24x'^2 - 36x'y' + 24y'^2 + 17x'^2 + 68x'y' + 68y'^2 = 100$$
$$25x'^2 + 100y'^2 = 100$$
$$\frac{25}{100}x'^2 + \frac{100}{100}y'^2 = \frac{100}{100}$$

Write the equations with x' and y' in the standard form with respect to the new coordinate system.

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1$$

Figure 8 shows the graph of the ellipse.

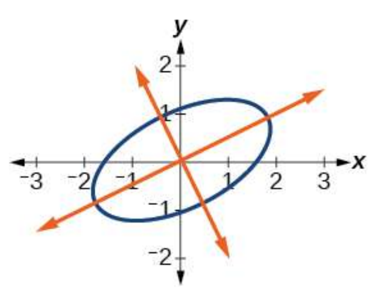


Figure 8

TRY IT #2 Rewrite the $13x^2 - 6\sqrt{3}xy + 7y^2 = 16$ in the $x'\,y'$ system without the $x'\,y'$ term.

EXAMPLE 4

Graphing an Equation That Has No *x′y′* Terms

Graph the following equation relative to the $x'\,y'$ system:

$$x^2 + 12xy - 4y^2 = 30$$

Solution

First, we find $\cot(2\theta)$.

$$x^2 + 12xy - 4y^2 = 20 \Rightarrow A = 1, \quad B = 12, \text{and } C = -4$$

$$\begin{aligned} \cot(2\theta) &= \tfrac{A-C}{B} \\ \cot(2\theta) &= \tfrac{1-(-4)}{12} \\ \cot(2\theta) &= \tfrac{5}{12} \end{aligned}$$

Because $\cot(2\theta) = \frac{5}{12}$, we can draw a reference triangle as in Figure 9.

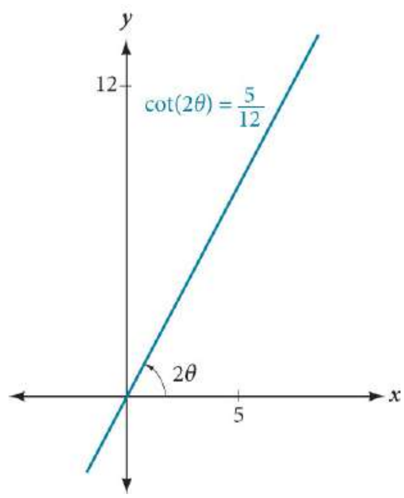


Figure 9

$$\cot(2\theta) = \frac{5}{12} = \frac{\text{adjacent}}{\text{opposite}}$$

Thus, the hypotenuse is

$$\begin{aligned} 5^2 + 12^2 &= h^2 \\ 25 + 144 &= h^2 \\ 169 &= h^2 \\ h &= 13 \end{aligned}$$

Next, we find $\sin\ \theta$ and $\cos\ \theta$. We will use half-angle identities.

$$\sin\ \theta = \sqrt{\frac{1-\cos(2\theta)}{2}} = \sqrt{\frac{1-\frac{5}{13}}{2}} = \sqrt{\frac{\frac{13}{13}-\frac{5}{13}}{2}} = \sqrt{\frac{8}{13}\cdot\frac{1}{2}} = \frac{2}{\sqrt{13}}$$

$$\cos\ \theta = \sqrt{\frac{1+\cos(2\theta)}{2}} = \sqrt{\frac{1+\frac{5}{13}}{2}} = \sqrt{\frac{\frac{13}{13}+\frac{5}{13}}{2}} = \sqrt{\frac{18}{13}\cdot\frac{1}{2}} = \frac{3}{\sqrt{13}}$$

Now we find x and y.

$$\begin{aligned}
x &= x' \cos\ \theta - y' \sin\ \theta \\
x &= x'\left(\frac{3}{\sqrt{13}}\right) - y'\left(\frac{2}{\sqrt{13}}\right) \\
x &= \frac{3x' - 2y'}{\sqrt{13}}
\end{aligned}$$

and

$$\begin{aligned}
y &= x' \sin\ \theta + y' \cos\ \theta \\
y &= x'\left(\frac{2}{\sqrt{13}}\right) + y'\left(\frac{3}{\sqrt{13}}\right) \\
y &= \frac{2x' + 3y'}{\sqrt{13}}
\end{aligned}$$

Now we substitute $x = \frac{3x' - 2y'}{\sqrt{13}}$ and $y = \frac{2x' + 3y'}{\sqrt{13}}$ into $x^2 + 12xy - 4y^2 = 30$.

$$\begin{aligned}
\left(\frac{3x'-2y'}{\sqrt{13}}\right)^2 + 12\left(\frac{3x'-2y'}{\sqrt{13}}\right)\left(\frac{2x'+3y'}{\sqrt{13}}\right) - 4\left(\frac{2x'+3y'}{\sqrt{13}}\right)^2 &= 30 \\
\left(\frac{1}{13}\right)\left[(3x' - 2y')^2 + 12(3x' - 2y')(2x' + 3y') - 4(2x' + 3y')^2\right] &= 30 && \text{Factor.} \\
\left(\frac{1}{13}\right)\left[9x'^2 - 12x'y' + 4y'^2 + 12\left(6x'^2 + 5x'y' - 6y'^2\right) - 4\left(4x'^2 + 12x'y' + 9y'^2\right)\right] &= 30 && \text{Multiply.} \\
\left(\frac{1}{13}\right)\left[9x'^2 - 12x'y' + 4y'^2 + 72x'^2 + 60x'y' - 72y'^2 - 16x'^2 - 48x'y' - 36y'^2\right] &= 30 && \text{Distribute.} \\
\left(\frac{1}{13}\right)\left[65x'^2 - 104y'^2\right] &= 30 && \text{Combine like terms.} \\
65x'^2 - 104y'^2 &= 390 && \text{Multiply.} \\
\frac{x'^2}{6} - \frac{4y'^2}{15} &= 1 && \text{Divide by 390.}
\end{aligned}$$

Figure 10 shows the graph of the hyperbola $\frac{x'^2}{6} - \frac{4y'^2}{15} = 1$.

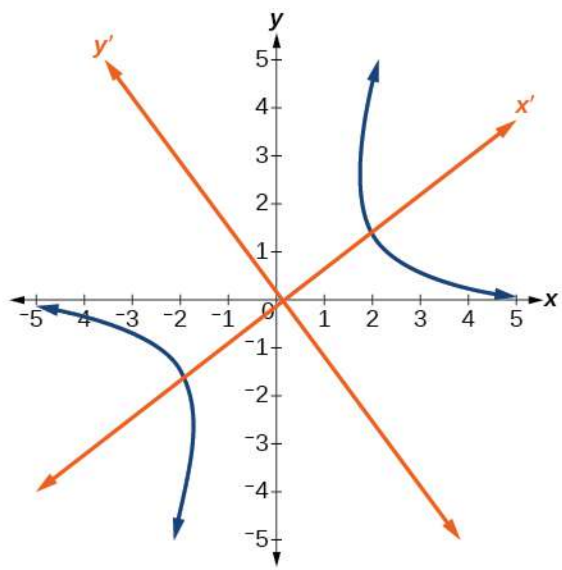


Figure 10

Identifying Conics without Rotating Axes

Now we have come full circle. How do we identify the type of conic described by an equation? What happens when the axes are rotated? Recall, the general form of a conic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

If we apply the rotation formulas to this equation we get the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

It may be shown that $B^2 - 4AC = B'^2 - 4A'C'$. The expression does not vary after rotation, so we call the expression invariant**.** The discriminant, $B^2 - 4AC$, is invariant and remains unchanged after rotation. Because the discriminant remains unchanged, observing the discriminant enables us to identify the conic section.

Using the Discriminant to Identify a Conic

If the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is transformed by rotating axes into the equation $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$, then $B^2 - 4AC = B'^2 - 4A'C'$.

The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is an ellipse, a parabola, or a hyperbola, or a degenerate case of one of these.

If the discriminant, $B^2 - 4AC$, is

- < 0, the conic section is an ellipse
- $= 0$, the conic section is a parabola
- > 0, the conic section is a hyperbola

EXAMPLE 5

Identifying the Conic without Rotating Axes

Identify the conic for each of the following without rotating axes.

ⓐ $5x^2 + 2\sqrt{3}xy + 2y^2 - 5 = 0$ ⓑ $5x^2 + 2\sqrt{3}xy + 12y^2 - 5 = 0$

Solution

ⓐ Let's begin by determining A, B, and C.

$$\underbrace{5}_{A}x^2 + \underbrace{2\sqrt{3}}_{B}xy + \underbrace{2}_{C}y^2 - 5 = 0$$

Now, we find the discriminant.

$$\begin{aligned} B^2 - 4AC &= \left(2\sqrt{3}\right)^2 - 4(5)(2) \\ &= 4(3) - 40 \\ &= 12 - 40 \\ &= -28 < 0 \end{aligned}$$

Therefore, $5x^2 + 2\sqrt{3}xy + 2y^2 - 5 = 0$ represents an ellipse.

ⓑ Again, let's begin by determining A, B, and C.

$$\underbrace{5}_{A}x^2 + \underbrace{2\sqrt{3}}_{B}xy + \underbrace{12}_{C}y^2 - 5 = 0$$

Now, we find the discriminant.

$$\begin{aligned} B^2 - 4AC &= \left(2\sqrt{3}\right)^2 - 4(5)(12) \\ &= 4(3) - 240 \\ &= 12 - 240 \\ &= -228 < 0 \end{aligned}$$

Therefore, $5x^2 + 2\sqrt{3}xy + 12y^2 - 5 = 0$ represents an ellipse.

TRY IT #3 Identify the conic for each of the following without rotating axes.

ⓐ $x^2 - 9xy + 3y^2 - 12 = 0$ ⓑ $10x^2 - 9xy + 4y^2 - 4 = 0$

MEDIA

Access this online resource for additional instruction and practice with conic sections and rotation of axes.

Introduction to Conic Sections (http://openstax.org/l/introconic)

8.4 SECTION EXERCISES

Verbal

1. What effect does the xy term have on the graph of a conic section?

2. If the equation of a conic section is written in the form $Ax^2 + By^2 + Cx + Dy + E = 0$ and $AB = 0$, what can we conclude?

3. If the equation of a conic section is written in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, and $B^2 - 4AC > 0$, what can we conclude?

4. Given the equation $ax^2 + 4x + 3y^2 - 12 = 0$, what can we conclude if $a > 0$?

5. For the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, the value of θ that satisfies $\cot(2\theta) = \frac{A-C}{B}$ gives us what information?

Algebraic

For the following exercises, determine which conic section is represented based on the given equation.

6. $9x^2 + 4y^2 + 72x + 36y - 500 = 0$

7. $x^2 - 10x + 4y - 10 = 0$

8. $2x^2 - 2y^2 + 4x - 6y - 2 = 0$

9. $4x^2 - y^2 + 8x - 1 = 0$

10. $4y^2 - 5x + 9y + 1 = 0$

11. $2x^2 + 3y^2 - 8x - 12y + 2 = 0$

12. $4x^2 + 9xy + 4y^2 - 36y - 125 = 0$

13. $3x^2 + 6xy + 3y^2 - 36y - 125 = 0$

14. $-3x^2 + 3\sqrt{3}xy - 4y^2 + 9 = 0$

15. $2x^2 + 4\sqrt{3}xy + 6y^2 - 6x - 3 = 0$

16. $-x^2 + 4\sqrt{2}xy + 2y^2 - 2y + 1 = 0$

17. $8x^2 + 4\sqrt{2}xy + 4y^2 - 10x + 1 = 0$

For the following exercises, find a new representation of the given equation after rotating through the given angle.

18. $3x^2 + xy + 3y^2 - 5 = 0, \theta = 45°$ **19.** $4x^2 - xy + 4y^2 - 2 = 0, \theta = 45°$ **20.** $2x^2 + 8xy - 1 = 0, \theta = 30°$

21. $-2x^2 + 8xy + 1 = 0, \theta = 45°$ **22.** $4x^2 + \sqrt{2}xy + 4y^2 + y + 2 = 0, \theta = 45°$

For the following exercises, determine the angle θ that will eliminate the xy term and write the corresponding equation without the xy term.

23. $x^2 + 3\sqrt{3}xy + 4y^2 + y - 2 = 0$ **24.** $4x^2 + 2\sqrt{3}xy + 6y^2 + y - 2 = 0$

25. $9x^2 - 3\sqrt{3}xy + 6y^2 + 4y - 3 = 0$ **26.** $-3x^2 - \sqrt{3}xy - 2y^2 - x = 0$

27. $16x^2 + 24xy + 9y^2 + 6x - 6y + 2 = 0$ **28.** $x^2 + 4xy + 4y^2 + 3x - 2 = 0$

29. $x^2 + 4xy + y^2 - 2x + 1 = 0$ **30.** $4x^2 - 2\sqrt{3}xy + 6y^2 - 1 = 0$

Graphical

For the following exercises, rotate through the given angle based on the given equation. Give the new equation and graph the original and rotated equation.

31. $y = -x^2, \theta = -45°$ **32.** $x = y^2, \theta = 45°$ **33.** $\frac{x^2}{4} + \frac{y^2}{1} = 1, \theta = 45°$

34. $\frac{y^2}{16} + \frac{x^2}{9} = 1, \theta = 45°$ **35.** $y^2 - x^2 = 1, \theta = 45°$ **36.** $y = \frac{x^2}{2}, \theta = 30°$

37. $x = (y-1)^2, \theta = 30°$ **38.** $\frac{x^2}{9} + \frac{y^2}{4} = 1, \theta = 30°$

For the following exercises, graph the equation relative to the $x'y'$ system in which the equation has no $x'y'$ term.

39. $xy = 9$ **40.** $x^2 + 10xy + y^2 - 6 = 0$ **41.** $x^2 - 10xy + y^2 - 24 = 0$

42. $4x^2 - 3\sqrt{3}xy + y^2 - 22 = 0$ **43.** $6x^2 + 2\sqrt{3}xy + 4y^2 - 21 = 0$ **44.** $11x^2 + 10\sqrt{3}xy + y^2 - 64 = 0$

45. $21x^2 + 2\sqrt{3}xy + 19y^2 - 18 = 0$ **46.** $16x^2 + 24xy + 9y^2 - 130x + 90y = 0$

47. $16x^2 + 24xy + 9y^2 - 60x + 80y = 0$ **48.** $13x^2 - 6\sqrt{3}xy + 7y^2 - 16 = 0$

49. $4x^2 - 4xy + y^2 - 8\sqrt{5}x - 16\sqrt{5}y = 0$

For the following exercises, determine the angle of rotation in order to eliminate the xy term. Then graph the new set of axes.

50. $6x^2 - 5\sqrt{3}xy + y^2 + 10x - 12y = 0$ **51.** $6x^2 - 5xy + 6y^2 + 20x - y = 0$

52. $6x^2 - 8\sqrt{3}xy + 14y^2 + 10x - 3y = 0$

53. $4x^2 + 6\sqrt{3}xy + 10y^2 + 20x - 40y = 0$

54. $8x^2 + 3xy + 4y^2 + 2x - 4 = 0$

55. $16x^2 + 24xy + 9y^2 + 20x - 44y = 0$

For the following exercises, determine the value of k based on the given equation.

56. Given $4x^2 + kxy + 16y^2 + 8x + 24y - 48 = 0$, find k for the graph to be a parabola.

57. Given $2x^2 + kxy + 12y^2 + 10x - 16y + 28 = 0$, find k for the graph to be an ellipse.

58. Given $3x^2 + kxy + 4y^2 - 6x + 20y + 128 = 0$, find k for the graph to be a hyperbola.

59. Given $kx^2 + 8xy + 8y^2 - 12x + 16y + 18 = 0$, find k for the graph to be a parabola.

60. Given $6x^2 + 12xy + ky^2 + 16x + 10y + 4 = 0$, find k for the graph to be an ellipse.

8.5 Conic Sections in Polar Coordinates

Learning Objectives

In this section, you will:

- Identify a conic in polar form.
- Graph the polar equations of conics.
- Define conics in terms of a focus and a directrix.

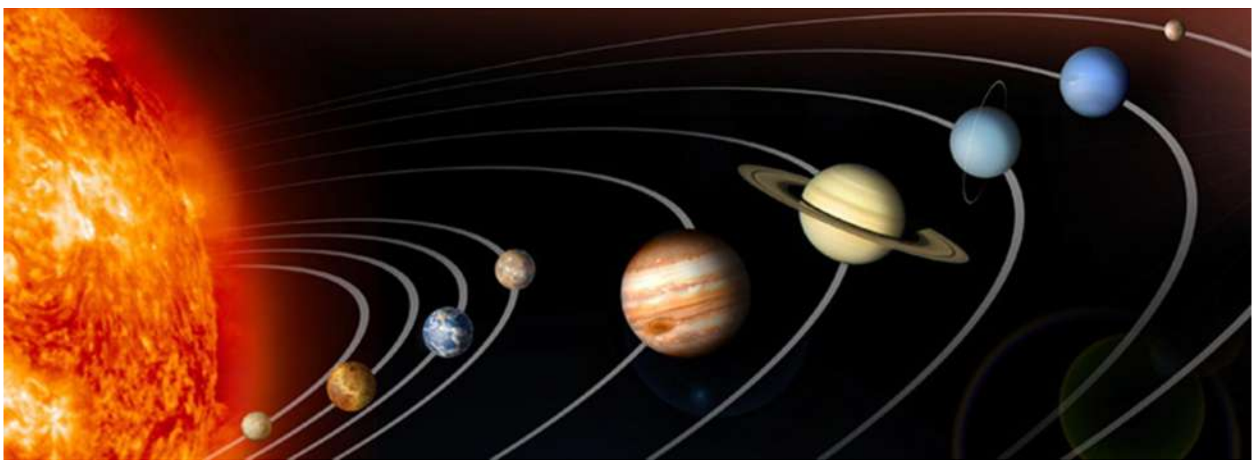


Figure 1 Planets orbiting the sun follow elliptical paths. (credit: NASA Blueshift, Flickr)

Most of us are familiar with orbital motion, such as the motion of a planet around the sun or an electron around an atomic nucleus. Within the planetary system, orbits of planets, asteroids, and comets around a larger celestial body are often elliptical. Comets, however, may take on a parabolic or hyperbolic orbit instead. And, in reality, the characteristics of the planets' orbits may vary over time. Each orbit is tied to the location of the celestial body being orbited and the distance and direction of the planet or other object from that body. As a result, we tend to use polar coordinates to represent these orbits.

In an elliptical orbit, the periapsis is the point at which the two objects are closest, and the apoapsis is the point at which they are farthest apart. Generally, the velocity of the orbiting body tends to increase as it approaches the periapsis and decrease as it approaches the apoapsis. Some objects reach an escape velocity, which results in an infinite orbit. These bodies exhibit either a parabolic or a hyperbolic orbit about a body; the orbiting body breaks free of the celestial body's gravitational pull and fires off into space. Each of these orbits can be modeled by a conic section in the polar coordinate system.

Identifying a Conic in Polar Form

Any conic may be determined by three characteristics: a single focus, a fixed line called the directrix, and the ratio of the distances of each to a point on the graph. Consider the parabola $x = 2 + y^2$ shown in Figure 2.

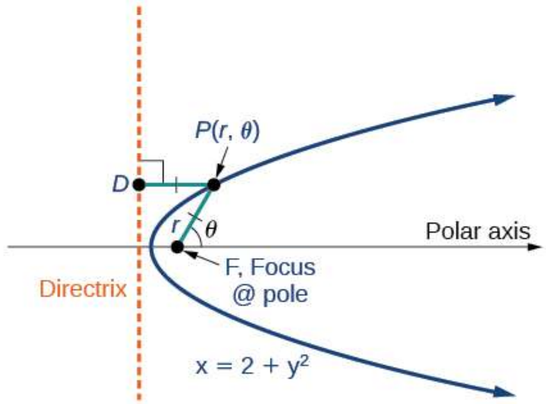


Figure 2

In The Parabola, we learned how a parabola is defined by the focus (a fixed point) and the directrix (a fixed line). In this section, we will learn how to define any conic in the polar coordinate system in terms of a fixed point, the focus $P(r, \theta)$ at the pole, and a line, the directrix, which is perpendicular to the polar axis.

If F is a fixed point, the focus, and D is a fixed line, the directrix, then we can let e be a fixed positive number, called the **eccentricity**, which we can define as the ratio of the distances from a point on the graph to the focus and the point on the graph to the directrix. Then the set of all points P such that $e = \frac{PF}{PD}$ is a conic. In other words, we can define a conic as the set of all points P with the property that the ratio of the distance from P to F to the distance from P to D is equal to the constant e.

For a conic with eccentricity e,

- if $0 \leq e < 1$, the conic is an ellipse
- if $e = 1$, the conic is a parabola
- if $e > 1$, the conic is an hyperbola

With this definition, we may now define a conic in terms of the directrix, $x = \pm p$, the eccentricity e, and the angle θ. Thus, each conic may be written as a **polar equation**, an equation written in terms of r and θ.

The Polar Equation for a Conic

For a conic with a focus at the origin, if the directrix is $x = \pm p$, where p is a positive real number, and the **eccentricity** is a positive real number e, the conic has a **polar equation**

$$r = \frac{ep}{1 \pm e \quad \cos \quad \theta}$$

For a conic with a focus at the origin, if the directrix is $y = \pm p$, where p is a positive real number, and the eccentricity is a positive real number e, the conic has a polar equation

$$r = \frac{ep}{1 \pm e \quad \sin \quad \theta}$$

HOW TO

Given the polar equation for a conic, identify the type of conic, the directrix, and the eccentricity.

1. Multiply the numerator and denominator by the reciprocal of the constant in the denominator to rewrite the equation in standard form.
2. Identify the eccentricity e as the coefficient of the trigonometric function in the denominator.
3. Compare e with 1 to determine the shape of the conic.
4. Determine the directrix as $x = p$ if cosine is in the denominator and $y = p$ if sine is in the denominator. Set ep

equal to the numerator in standard form to solve for x or y.

EXAMPLE 1

Identifying a Conic Given the Polar Form

For each of the following equations, identify the conic with focus at the origin, the directrix, and the eccentricity.

a. $r = \frac{6}{3+2 \sin \theta}$
b. $r = \frac{12}{4+5 \cos \theta}$
c. $r = \frac{7}{2-2 \sin \theta}$

Solution

For each of the three conics, we will rewrite the equation in standard form. Standard form has a 1 as the constant in the denominator. Therefore, in all three parts, the first step will be to multiply the numerator and denominator by the reciprocal of the constant of the original equation, $\frac{1}{c}$, where c is that constant.

a. Multiply the numerator and denominator by $\frac{1}{3}$.

$$r = \frac{6}{3+2\sin \theta} \cdot \frac{\left(\frac{1}{3}\right)}{\left(\frac{1}{3}\right)} = \frac{6\left(\frac{1}{3}\right)}{3\left(\frac{1}{3}\right)+2\left(\frac{1}{3}\right)\sin \theta} = \frac{2}{1+\frac{2}{3} \sin \theta}$$

Because $\sin \theta$ is in the denominator, the directrix is $y = p$. Comparing to standard form, note that $e = \frac{2}{3}$. Therefore, from the numerator,

$$\begin{aligned} 2 &= ep \\ 2 &= \tfrac{2}{3}p \\ \left(\tfrac{3}{2}\right)2 &= \left(\tfrac{3}{2}\right)\tfrac{2}{3}p \\ 3 &= p \end{aligned}$$

Since $e < 1$, the conic is an ellipse. The eccentricity is $e = \frac{2}{3}$ and the directrix is $y = 3$.

b. Multiply the numerator and denominator by $\frac{1}{4}$.

$$\begin{aligned} r &= \frac{12}{4+5 \cos \theta} \cdot \frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{4}\right)} \\ r &= \frac{12\left(\frac{1}{4}\right)}{4\left(\frac{1}{4}\right)+5\left(\frac{1}{4}\right)\cos \theta} \\ r &= \frac{3}{1+\frac{5}{4} \cos \theta} \end{aligned}$$

Because $\cos \theta$ is in the denominator, the directrix is $x = p$. Comparing to standard form, $e = \frac{5}{4}$. Therefore, from the numerator,

$$\begin{aligned} 3 &= ep \\ 3 &= \tfrac{5}{4}p \\ \left(\tfrac{4}{5}\right)3 &= \left(\tfrac{4}{5}\right)\tfrac{5}{4}p \\ \tfrac{12}{5} &= p \end{aligned}$$

Since $e > 1$, the conic is a hyperbola. The eccentricity is $e = \frac{5}{4}$ and the directrix is $x = \frac{12}{5} = 2.4$.

c. Multiply the numerator and denominator by $\frac{1}{2}$.

$$r = \frac{7}{2-2\sin\theta} \cdot \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}$$

$$r = \frac{7\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}\right)-2\left(\frac{1}{2}\right)\sin\theta}$$

$$r = \frac{\frac{7}{2}}{1-\sin\theta}$$

Because sine is in the denominator, the directrix is $y = -p$. Comparing to standard form, $e = 1$. Therefore, from the numerator,

$$\frac{7}{2} = ep$$
$$\frac{7}{2} = (1)\,p$$
$$\frac{7}{2} = p$$

Because $e = 1$, the conic is a parabola. The eccentricity is $e = 1$ and the directrix is $y = -\frac{7}{2} = -3.5$.

TRY IT #1 Identify the conic with focus at the origin, the directrix, and the eccentricity for $r = \frac{2}{3-\cos\theta}$.

Graphing the Polar Equations of Conics

When graphing in Cartesian coordinates, each conic section has a unique equation. This is not the case when graphing in polar coordinates. We must use the eccentricity of a conic section to determine which type of curve to graph, and then determine its specific characteristics. The first step is to rewrite the conic in standard form as we have done in the previous example. In other words, we need to rewrite the equation so that the denominator begins with 1. This enables us to determine e and, therefore, the shape of the curve. The next step is to substitute values for θ and solve for r to plot a few key points. Setting θ equal to $0, \frac{\pi}{2}, \pi,$ and $\frac{3\pi}{2}$ provides the vertices so we can create a rough sketch of the graph.

EXAMPLE 2

Graphing a Parabola in Polar Form

Graph $r = \frac{5}{3+3\cos\theta}$.

Solution

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 3, which is $\frac{1}{3}$.

$$r = \frac{5}{3+3\cos\theta} = \frac{5\left(\frac{1}{3}\right)}{3\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)\cos\theta}$$

$$r = \frac{\frac{5}{3}}{1+\cos\theta}$$

Because $e = 1$, we will graph a parabola with a focus at the origin. The function has a $\cos\theta$, and there is an addition sign in the denominator, so the directrix is $x = p$.

$$\frac{5}{3} = ep$$
$$\frac{5}{3} = (1)p$$
$$\frac{5}{3} = p$$

The directrix is $x = \frac{5}{3}$.

Plotting a few key points as in Table 1 will enable us to see the vertices. See Figure 3.

	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{5}{3+3\cos\theta}$	$\frac{5}{6} \approx 0.83$	$\frac{5}{3} \approx 1.67$	undefined	$\frac{5}{3} \approx 1.67$

Table 1

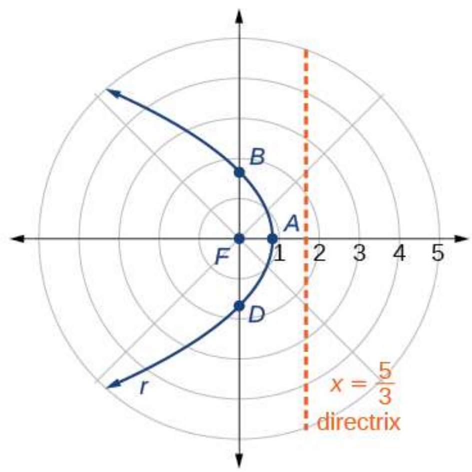


Figure 3

Analysis

We can check our result with a graphing utility. See Figure 4.

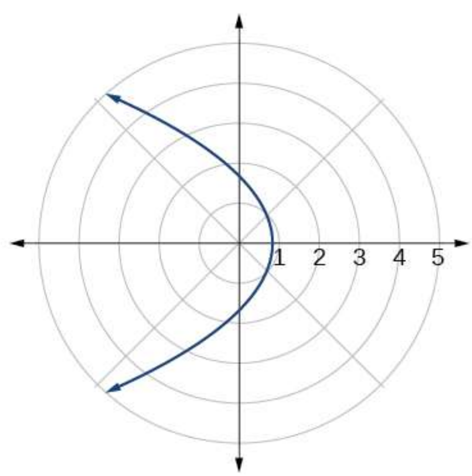


Figure 4

EXAMPLE 3

Graphing a Hyperbola in Polar Form

Graph $r = \frac{8}{2-3\sin\theta}$.

Solution

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 2, which is $\frac{1}{2}$.

$$r=\frac{8}{2-3\sin\ \theta}=\frac{8\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}\right)-3\left(\frac{1}{2}\right)\sin\ \theta}$$

$$r=\frac{4}{1-\frac{3}{2}\ \sin\ \theta}$$

Because $e=\frac{3}{2}, e>1,$ so we will graph a hyperbola with a focus at the origin. The function has a $\sin\ \theta$ term and there is a subtraction sign in the denominator, so the directrix is $y = -p$.

$$\begin{aligned}4&=ep\\4&=\left(\tfrac{3}{2}\right)p\\4\left(\tfrac{2}{3}\right)&=p\\\tfrac{8}{3}&=p\end{aligned}$$

The directrix is $y=-\frac{8}{3}$.

Plotting a few key points as in Table 2 will enable us to see the vertices. See Figure 5.

	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r=\dfrac{8}{2-3\sin\ \theta}$	4	-8	4	$\frac{8}{5}=1.6$

Table 2

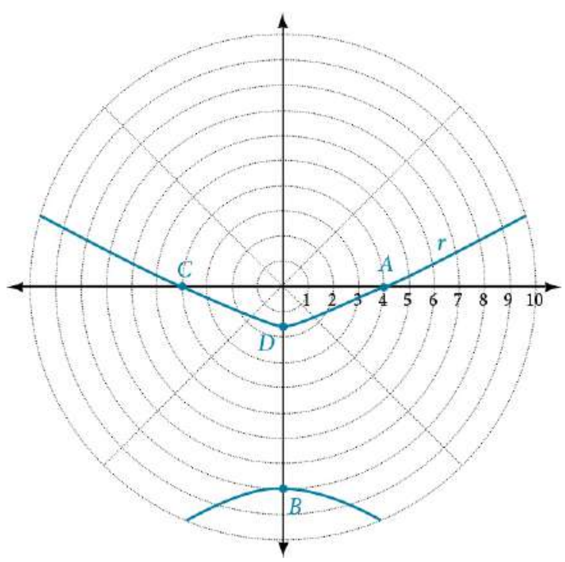


Figure 5

EXAMPLE 4

Graphing an Ellipse in Polar Form

Graph $r = \frac{10}{5-4 \cos \theta}$.

Solution

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 5, which is $\frac{1}{5}$.

$$r = \frac{10}{5-4\cos \theta} = \frac{10\left(\frac{1}{5}\right)}{5\left(\frac{1}{5}\right)-4\left(\frac{1}{5}\right)\cos \theta}$$
$$r = \frac{2}{1-\frac{4}{5} \cos \theta}$$

Because $e = \frac{4}{5}, e < 1$, so we will graph an ellipse with a focus at the origin. The function has a $\cos \theta$, and there is a subtraction sign in the denominator, so the directrix is $x = -p$.

$$2 = ep$$
$$2 = \left(\frac{4}{5}\right) p$$
$$2\left(\frac{5}{4}\right) = p$$
$$\frac{5}{2} = p$$

The directrix is $x = -\frac{5}{2}$.

Plotting a few key points as in Table 3 will enable us to see the vertices. See Figure 6.

	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{10}{5-4 \cos \theta}$	10	2	$\frac{10}{9} \approx 1.1$	2

Table 3

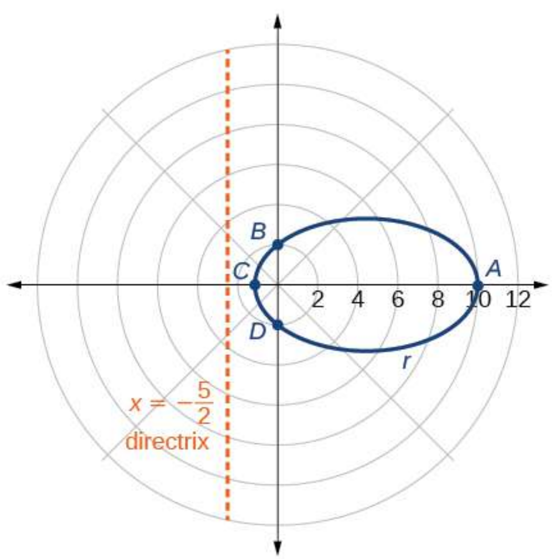


Figure 6

Analysis

We can check our result using a graphing utility. See Figure 7.

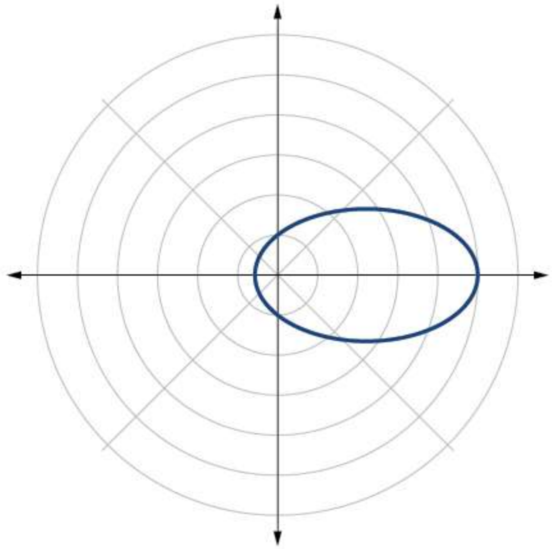


Figure 7 $r = \frac{10}{5-4\cos\theta}$ graphed on a viewing window of $[-3, 12, 1]$ by $[-4, 4, 1]$, θ min $= 0$ and θ max $= 2\pi$.

TRY IT #2 Graph $r = \frac{2}{4-\cos\theta}$.

Defining Conics in Terms of a Focus and a Directrix

So far we have been using polar equations of conics to describe and graph the curve. Now we will work in reverse; we will use information about the origin, eccentricity, and directrix to determine the polar equation.

HOW TO

Given the focus, eccentricity, and directrix of a conic, determine the polar equation.

1. Determine whether the directrix is horizontal or vertical. If the directrix is given in terms of y, we use the general polar form in terms of sine. If the directrix is given in terms of x, we use the general polar form in terms of cosine.
2. Determine the sign in the denominator. If $p < 0$, use subtraction. If $p > 0$, use addition.
3. Write the coefficient of the trigonometric function as the given eccentricity.
4. Write the absolute value of p in the numerator, and simplify the equation.

EXAMPLE 5

Finding the Polar Form of a Vertical Conic Given a Focus at the Origin and the Eccentricity and Directrix

Find the polar form of the conic given a focus at the origin, $e = 3$ and directrix $y = -2$.

Solution

The directrix is $y = -p$, so we know the trigonometric function in the denominator is sine.

Because $y = -2, -2 < 0$, so we know there is a subtraction sign in the denominator. We use the standard form of

$$r = \frac{ep}{1 - e\sin\theta}$$

and $e = 3$ and $|-2| = 2 = p$.

Therefore,

$$r = \frac{(3)(2)}{1-3 \sin \theta}$$
$$r = \frac{6}{1-3 \sin \theta}$$

EXAMPLE 6

Finding the Polar Form of a Horizontal Conic Given a Focus at the Origin and the Eccentricity and Directrix

Find the polar form of a conic given a focus at the origin, $e = \frac{3}{5}$, and directrix $x = 4$.

Solution

Because the directrix is $x = p$, we know the function in the denominator is cosine. Because $x = 4, 4 > 0$, so we know there is an addition sign in the denominator. We use the standard form of

$$r = \frac{ep}{1 + e \cos \theta}$$

and $e = \frac{3}{5}$ and $|4| = 4 = p$.

Therefore,

$$r = \frac{\left(\frac{3}{5}\right)(4)}{1+\frac{3}{5} \cos \theta}$$
$$r = \frac{\frac{12}{5}}{1+\frac{3}{5} \cos \theta}$$
$$r = \frac{\frac{12}{5}}{1\left(\frac{5}{5}\right)+\frac{3}{5} \cos \theta}$$
$$r = \frac{\frac{12}{5}}{\frac{5}{5}+\frac{3}{5} \cos \theta}$$
$$r = \frac{12}{5} \cdot \frac{5}{5+3 \cos \theta}$$
$$r = \frac{12}{5+3 \cos \theta}$$

TRY IT #3 Find the polar form of the conic given a focus at the origin, $e = 1$, and directrix $x = -1$.

EXAMPLE 7

Converting a Conic in Polar Form to Rectangular Form

Convert the conic $r = \frac{1}{5-5 \sin \theta}$ to rectangular form.

Solution

We will rearrange the formula to use the identities $r = \sqrt{x^2 + y^2}, x = r \cos \theta,$ and $y = r \sin \theta$.

$$\begin{aligned} r &= \frac{1}{5-5\sin\theta} & \\ r\cdot(5-5\sin\theta) &= \frac{1}{5-5\sin\theta}\cdot(5-5\sin\theta) & \text{Eliminate the fraction.} \\ 5r-5r\sin\theta &= 1 & \text{Distribute.} \\ 5r &= 1+5r\sin\theta & \text{Isolate } 5r. \\ 25r^2 &= (1+5r\sin\theta)^2 & \text{Square both sides.} \\ 25(x^2+y^2) &= (1+5y)^2 & \text{Substitute } r=\sqrt{x^2+y^2} \text{ and } y=r\sin\theta. \\ 25x^2+25y^2 &= 1+10y+25y^2 & \text{Distribute and use FOIL.} \\ 25x^2-10y &= 1 & \text{Rearrange terms and set equal to 1.} \end{aligned}$$

TRY IT #4 Convert the conic $r=\frac{2}{1+2\cos\theta}$ to rectangular form.

MEDIA

Access these online resources for additional instruction and practice with conics in polar coordinates.

[Polar Equations of Conic Sections (http://openstax.org/l/determineconic)](http://openstax.org/l/determineconic)
[Graphing Polar Equations of Conics - 1 (http://openstax.org/l/graphconic1)](http://openstax.org/l/graphconic1)
[Graphing Polar Equations of Conics - 2 (http://openstax.org/l/graphconic2)](http://openstax.org/l/graphconic2)

8.5 SECTION EXERCISES

Verbal

1. Explain how eccentricity determines which conic section is given.

2. If a conic section is written as a polar equation, what must be true of the denominator?

3. If a conic section is written as a polar equation, and the denominator involves $\sin\theta$, what conclusion can be drawn about the directrix?

4. If the directrix of a conic section is perpendicular to the polar axis, what do we know about the equation of the graph?

5. What do we know about the focus/foci of a conic section if it is written as a polar equation?

Algebraic

For the following exercises, identify the conic with a focus at the origin, and then give the directrix and eccentricity.

6. $r=\frac{6}{1-2\cos\theta}$

7. $r=\frac{3}{4-4\sin\theta}$

8. $r=\frac{8}{4-3\cos\theta}$

9. $r=\frac{5}{1+2\sin\theta}$

10. $r=\frac{16}{4+3\cos\theta}$

11. $r=\frac{3}{10+10\cos\theta}$

12. $r=\frac{2}{1-\cos\theta}$

13. $r=\frac{4}{7+2\cos\theta}$

14. $r(1-\cos\theta)=3$

15. $r(3+5\sin\theta)=11$

16. $r(4-5\sin\theta)=1$

17. $r(7+8\cos\theta)=7$

For the following exercises, convert the polar equation of a conic section to a rectangular equation.

18. $r = \frac{4}{1+3 \sin \theta}$

19. $r = \frac{2}{5-3 \sin \theta}$

20. $r = \frac{8}{3-2 \cos \theta}$

21. $r = \frac{3}{2+5 \cos \theta}$

22. $r = \frac{4}{2+2 \sin \theta}$

23. $r = \frac{3}{8-8 \cos \theta}$

24. $r = \frac{2}{6+7 \cos \theta}$

25. $r = \frac{5}{5-11 \sin \theta}$

26. $r(5 + 2 \cos \theta) = 6$

27. $r(2 - \cos \theta) = 1$

28. $r(2.5 - 2.5 \sin \theta) = 5$

29. $r = \frac{6 \sec \theta}{-2+3 \sec \theta}$

30. $r = \frac{6 \csc \theta}{3+2 \csc \theta}$

For the following exercises, graph the given conic section. If it is a parabola, label the vertex, focus, and directrix. If it is an ellipse, label the vertices and foci. If it is a hyperbola, label the vertices and foci.

31. $r = \frac{5}{2+\cos \theta}$

32. $r = \frac{2}{3+3 \sin \theta}$

33. $r = \frac{10}{5-4 \sin \theta}$

34. $r = \frac{3}{1+2 \cos \theta}$

35. $r = \frac{8}{4-5 \cos \theta}$

36. $r = \frac{3}{4-4 \cos \theta}$

37. $r = \frac{2}{1-\sin \theta}$

38. $r = \frac{6}{3+2 \sin \theta}$

39. $r(1 + \cos \theta) = 5$

40. $r(3 - 4 \sin \theta) = 9$

41. $r(3 - 2 \sin \theta) = 6$

42. $r(6 - 4 \cos \theta) = 5$

For the following exercises, find the polar equation of the conic with focus at the origin and the given eccentricity and directrix.

43. Directrix: $x = 4$; $e = \frac{1}{5}$

44. Directrix: $x = -4$; $e = 5$

45. Directrix: $y = 2$; $e = 2$

46. Directrix: $y = -2$; $e = \frac{1}{2}$

47. Directrix: $x = 1$; $e = 1$

48. Directrix: $x = -1$; $e = 1$

49. Directrix: $x = -\frac{1}{4}$; $e = \frac{7}{2}$

50. Directrix: $y = \frac{2}{5}$; $e = \frac{7}{2}$

51. Directrix: $y = 4$; $e = \frac{3}{2}$

52. Directrix: $x = -2$; $e = \frac{8}{3}$

53. Directrix: $x = -5$; $e = \frac{3}{4}$

54. Directrix: $y = 2$; $e = 2.5$

55. Directrix: $x = -3$; $e = \frac{1}{3}$

Extensions

Recall from Rotation of Axes that equations of conics with an xy term have rotated graphs. For the following exercises, express each equation in polar form with r as a function of θ.

56. $xy = 2$

57. $x^2 + xy + y^2 = 4$

58. $2x^2 + 4xy + 2y^2 = 9$

59. $16x^2 + 24xy + 9y^2 = 4$

60. $2xy + y = 1$

Chapter Review

Key Terms

angle of rotation an acute angle formed by a set of axes rotated from the Cartesian plane where, if $\cot(2\theta) > 0$, then θ is between $(0°, 45°)$; if $\cot(2\theta) < 0$, then θ is between $(45°, 90°)$; and if $\cot(2\theta) = 0$, then $\theta = 45°$

center of a hyperbola the midpoint of both the transverse and conjugate axes of a hyperbola

center of an ellipse the midpoint of both the major and minor axes

conic section any shape resulting from the intersection of a right circular cone with a plane

conjugate axis the axis of a hyperbola that is perpendicular to the transverse axis and has the co-vertices as its endpoints

degenerate conic sections any of the possible shapes formed when a plane intersects a double cone through the apex. Types of degenerate conic sections include a point, a line, and intersecting lines.

directrix a line perpendicular to the axis of symmetry of a parabola; a line such that the ratio of the distance between the points on the conic and the focus to the distance to the directrix is constant

eccentricity the ratio of the distances from a point P on the graph to the focus F and to the directrix D represented by $e = \frac{PF}{PD}$, where e is a positive real number

ellipse the set of all points (x, y) in a plane such that the sum of their distances from two fixed points is a constant

foci plural of focus

focus (of a parabola) a fixed point in the interior of a parabola that lies on the axis of symmetry

focus (of an ellipse) one of the two fixed points on the major axis of an ellipse such that the sum of the distances from these points to any point (x, y) on the ellipse is a constant

hyperbola the set of all points (x, y) in a plane such that the difference of the distances between (x, y) and the foci is a positive constant

latus rectum the line segment that passes through the focus of a parabola parallel to the directrix, with endpoints on the parabola

major axis the longer of the two axes of an ellipse

minor axis the shorter of the two axes of an ellipse

nondegenerate conic section a shape formed by the intersection of a plane with a double right cone such that the plane does not pass through the apex; nondegenerate conics include circles, ellipses, hyperbolas, and parabolas

parabola the set of all points (x, y) in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix

polar equation an equation of a curve in polar coordinates r and θ

transverse axis the axis of a hyperbola that includes the foci and has the vertices as its endpoints

Key Equations

Horizontal ellipse, center at origin	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b$
Vertical ellipse, center at origin	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a > b$
Horizontal ellipse, center (h, k)	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a > b$
Vertical ellipse, center (h, k)	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \quad a > b$

Hyperbola, center at origin, transverse axis on *x*-axis	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Hyperbola, center at origin, transverse axis on *y*-axis	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Hyperbola, center at (h, k), transverse axis parallel to *x*-axis	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Hyperbola, center at (h, k), transverse axis parallel to y-axis	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Parabola, vertex at origin, axis of symmetry on x-axis	$y^2 = 4px$
Parabola, vertex at origin, axis of symmetry on y-axis	$x^2 = 4py$
Parabola, vertex at (h, k), axis of symmetry on x-axis	$(y-k)^2 = 4p(x-h)$
Parabola, vertex at (h, k), axis of symmetry on y-axis	$(x-h)^2 = 4p(y-k)$

General Form equation of a conic section	$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
Rotation of a conic section	$x = x' \cos \theta - y' \sin \theta$ $y = x' \sin \theta + y' \cos \theta$
Angle of rotation	θ, where $\cot(2\theta) = \frac{A-C}{B}$

Key Concepts

8.1 The Ellipse

- An ellipse is the set of all points (x, y) in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus (plural: foci).
- When given the coordinates of the foci and vertices of an ellipse, we can write the equation of the ellipse in standard form. See Example 1 and Example 2.
- When given an equation for an ellipse centered at the origin in standard form, we can identify its vertices, co-vertices, foci, and the lengths and positions of the major and minor axes in order to graph the ellipse. See Example 3 and Example 4.
- When given the equation for an ellipse centered at some point other than the origin, we can identify its key features and graph the ellipse. See Example 5 and Example 6.
- Real-world situations can be modeled using the standard equations of ellipses and then evaluated to find key features, such as lengths of axes and distance between foci. See Example 7.

8.2 The Hyperbola

- A hyperbola is the set of all points (x, y) in a plane such that the difference of the distances between (x, y) and the foci is a positive constant.
- The standard form of a hyperbola can be used to locate its vertices and foci. See Example 1.
- When given the coordinates of the foci and vertices of a hyperbola, we can write the equation of the hyperbola in standard form. See Example 2 and Example 3.
- When given an equation for a hyperbola, we can identify its vertices, co-vertices, foci, asymptotes, and lengths and positions of the transverse and conjugate axes in order to graph the hyperbola. See Example 4 and Example 5.
- Real-world situations can be modeled using the standard equations of hyperbolas. For instance, given the dimensions of a natural draft cooling tower, we can find a hyperbolic equation that models its sides. See Example 6.

8.3 The Parabola

- A parabola is the set of all points (x, y) in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix.
- The standard form of a parabola with vertex $(0, 0)$ and the x-axis as its axis of symmetry can be used to graph the parabola. If $p > 0$, the parabola opens right. If $p < 0$, the parabola opens left. See Example 1.
- The standard form of a parabola with vertex $(0, 0)$ and the y-axis as its axis of symmetry can be used to graph the parabola. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down. See Example 2.
- When given the focus and directrix of a parabola, we can write its equation in standard form. See Example 3.
- The standard form of a parabola with vertex (h, k) and axis of symmetry parallel to the x-axis can be used to graph

the parabola. If $p > 0$, the parabola opens right. If $p < 0$, the parabola opens left. See Example 4.

- The standard form of a parabola with vertex (h, k) and axis of symmetry parallel to the y-axis can be used to graph the parabola. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down. See Example 5.
- Real-world situations can be modeled using the standard equations of parabolas. For instance, given the diameter and focus of a cross-section of a parabolic reflector, we can find an equation that models its sides. See Example 6.

8.4 Rotation of Axes

- Four basic shapes can result from the intersection of a plane with a pair of right circular cones connected tail to tail. They include an ellipse, a circle, a hyperbola, and a parabola.
- A nondegenerate conic section has the general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ where A, B and C are not all zero. The values of A, B, and C determine the type of conic. See Example 1.
- Equations of conic sections with an xy term have been rotated about the origin. See Example 2.
- The general form can be transformed into an equation in the x' and y' coordinate system without the $x'y'$ term. See Example 3 and Example 4.
- An expression is described as invariant if it remains unchanged after rotating. Because the discriminant is invariant, observing it enables us to identify the conic section. See Example 5.

8.5 Conic Sections in Polar Coordinates

- Any conic may be determined by a single focus, the corresponding eccentricity, and the directrix. We can also define a conic in terms of a fixed point, the focus $P(r, \theta)$ at the pole, and a line, the directrix, which is perpendicular to the polar axis.
- A conic is the set of all points $e = \frac{PF}{PD}$, where eccentricity e is a positive real number. Each conic may be written in terms of its polar equation. See Example 1.
- The polar equations of conics can be graphed. See Example 2, Example 3, and Example 4.
- Conics can be defined in terms of a focus, a directrix, and eccentricity. See Example 5 and Example 6.
- We can use the identities $r = \sqrt{x^2 + y^2}$, $x = r\ \cos\ \theta$, and $y = r\ \sin\ \theta$ to convert the equation for a conic from polar to rectangular form. See Example 7.

Exercises

Review Exercises

The Ellipse

For the following exercises, write the equation of the ellipse in standard form. Then identify the center, vertices, and foci.

1. $\frac{x^2}{25} + \frac{y^2}{64} = 1$

2. $\frac{(x-2)^2}{100} + \frac{(y+3)^2}{36} = 1$

3. $9x^2 + y^2 + 54x - 4y + 76 = 0$

4. $9x^2 + 36y^2 - 36x + 72y + 36 = 0$

For the following exercises, graph the ellipse, noting center, vertices, and foci.

5. $\frac{x^2}{36} + \frac{y^2}{9} = 1$

6. $\frac{(x-4)^2}{25} + \frac{(y+3)^2}{49} = 1$

7. $4x^2 + y^2 + 16x + 4y - 44 = 0$

8. $2x^2 + 3y^2 - 20x + 12y + 38 = 0$

For the following exercises, use the given information to find the equation for the ellipse.

9. Center at $(0, 0)$, focus at $(3, 0)$, vertex at $(-5, 0)$

10. Center at $(2, -2)$, vertex at $(7, -2)$, focus at $(4, -2)$

11. A whispering gallery is to be constructed such that the foci are located 35 feet from the center. If the length of the gallery is to be 100 feet, what should the height of the ceiling be?

The Hyperbola

For the following exercises, write the equation of the hyperbola in standard form. Then give the center, vertices, and foci.

12. $\frac{x^2}{81} - \frac{y^2}{9} = 1$

13. $\frac{(y+1)^2}{16} - \frac{(x-4)^2}{36} = 1$

14. $9y^2 - 4x^2 + 54y - 16x + 29 = 0$

15. $3x^2 - y^2 - 12x - 6y - 9 = 0$

For the following exercises, graph the hyperbola, labeling vertices and foci.

16. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

17. $\frac{(y-1)^2}{49} - \frac{(x+1)^2}{4} = 1$

18. $x^2 - 4y^2 + 6x + 32y - 91 = 0$

19. $2y^2 - x^2 - 12y - 6 = 0$

For the following exercises, find the equation of the hyperbola.

20. Center at $(0, 0)$, vertex at $(0, 4)$, focus at $(0, -6)$

21. Foci at $(3, 7)$ and $(7, 7)$, vertex at $(6, 7)$

The Parabola

For the following exercises, write the equation of the parabola in standard form. Then give the vertex, focus, and directrix.

22. $y^2 = 12x$

23. $(x + 2)^2 = \frac{1}{2}(y - 1)$

24. $y^2 - 6y - 6x - 3 = 0$

25. $x^2 + 10x - y + 23 = 0$

For the following exercises, graph the parabola, labeling vertex, focus, and directrix.

26. $x^2 + 4y = 0$

27. $(y - 1)^2 = \frac{1}{2}(x + 3)$

28. $x^2 - 8x - 10y + 46 = 0$

29. $2y^2 + 12y + 6x + 15 = 0$

For the following exercises, write the equation of the parabola using the given information.

30. Focus at $(-4, 0)$; directrix is $x = 4$

31. Focus at $\left(2, \frac{9}{8}\right)$; directrix is $y = \frac{7}{8}$

32. A cable TV receiving dish is the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is 5 feet across at its opening and 1.5 feet deep.

Rotation of Axes

For the following exercises, determine which of the conic sections is represented.

33. $16x^2 + 24xy + 9y^2 + 24x - 60y - 60 = 0$

34. $4x^2 + 14xy + 5y^2 + 18x - 6y + 30 = 0$

35. $4x^2 + xy + 2y^2 + 8x - 26y + 9 = 0$

For the following exercises, determine the angle θ that will eliminate the xy term, and write the corresponding equation without the xy term.

36. $x^2 + 4xy - 2y^2 - 6 = 0$

37. $x^2 - xy + y^2 - 6 = 0$

For the following exercises, graph the equation relative to the $x'y'$ system in which the equation has no $x'y'$ term.

38. $9x^2 - 24xy + 16y^2 - 80x - 60y + 100 = 0$

39. $x^2 - xy + y^2 - 2 = 0$

40. $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$

Conic Sections in Polar Coordinates

For the following exercises, given the polar equation of the conic with focus at the origin, identify the eccentricity and directrix.

41. $r = \frac{10}{1-5\cos\theta}$

42. $r = \frac{6}{3+2\cos\theta}$

43. $r = \frac{1}{4+3\sin\theta}$

44. $r = \frac{3}{5-5\sin\theta}$

For the following exercises, graph the conic given in polar form. If it is a parabola, label the vertex, focus, and directrix. If it is an ellipse or a hyperbola, label the vertices and foci.

45. $r = \frac{3}{1-\sin\theta}$

46. $r = \frac{8}{4+3\sin\theta}$

47. $r = \frac{10}{4+5\cos\theta}$

48. $r = \frac{9}{3-6\cos\theta}$

For the following exercises, given information about the graph of a conic with focus at the origin, find the equation in polar form.

49. Directrix is $x = 3$ and eccentricity $e = 1$

50. Directrix is $y = -2$ and eccentricity $e = 4$

Practice Test

For the following exercises, write the equation in standard form and state the center, vertices, and foci.

1. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

2. $9y^2 + 16x^2 - 36y + 32x - 92 = 0$

For the following exercises, sketch the graph, identifying the center, vertices, and foci.

3. $\frac{(x-3)^2}{64} + \frac{(y-2)^2}{36} = 1$

4. $2x^2 + y^2 + 8x - 6y - 7 = 0$

5. Write the standard form equation of an ellipse with a center at $(1, 2)$, vertex at $(7, 2)$, and focus at $(4, 2\ )$.

6. A whispering gallery is to be constructed with a length of 150 feet. If the foci are to be located 20 feet away from the wall, how high should the ceiling be?

For the following exercises, write the equation of the hyperbola in standard form, and give the center, vertices, foci, and asymptotes.

7. $\frac{x^2}{49} - \frac{y^2}{81} = 1$

8. $16y^2 - 9x^2 + 128y + 112 = 0$

For the following exercises, graph the hyperbola, noting its center, vertices, and foci. State the equations of the asymptotes.

9. $\frac{(x-3)^2}{25} - \frac{(y+3)^2}{1} = 1$

10. $y^2 - x^2 + 4y - 4x - 18 = 0$

11. Write the standard form equation of a hyperbola with foci at $(1, 0)$ and $(1, 6)$, and a vertex at $(1, 2)$.

For the following exercises, write the equation of the parabola in standard form, and give the vertex, focus, and equation of the directrix.

12. $y^2 + 10x = 0$

13. $3x^2 - 12x - y + 11 = 0$

For the following exercises, graph the parabola, labeling the vertex, focus, and directrix.

14. $(x - 1)^2 = -4\,(y + 3)$

15. $y^2 + 8x - 8y + 40 = 0$

16. Write the equation of a parabola with a focus at $(2, 3)$ and directrix $y = -1$.

17. A searchlight is shaped like a paraboloid of revolution. If the light source is located 1.5 feet from the base along the axis of symmetry, and the depth of the searchlight is 3 feet, what should the width of the opening be?

For the following exercises, determine which conic section is represented by the given equation, and then determine the angle θ that will eliminate the xy term.

18. $3x^2 - 2xy + 3y^2 = 4$

19. $x^2 + 4xy + 4y^2 + 6x - 8y = 0$

For the following exercises, rewrite in the $x'y'$ system without the $x'y'$ term, and graph the rotated graph.

20. $11x^2 + 10\sqrt{3}xy + y^2 = 4$

21. $16x^2 + 24xy + 9y^2 - 125x = 0$

For the following exercises, identify the conic with focus at the origin, and then give the directrix and eccentricity.

22. $r = \frac{3}{2 - \sin\ \theta}$

23. $r = \frac{5}{4 + 6\ \cos\ \theta}$

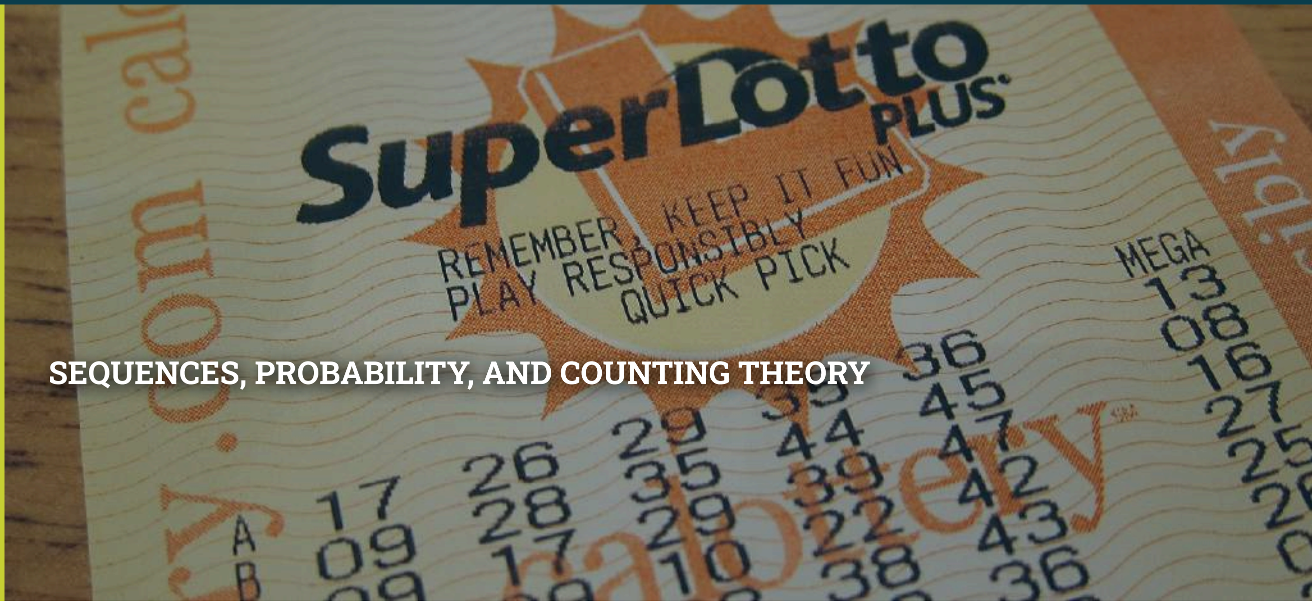
For the following exercises, graph the given conic section. If it is a parabola, label vertex, focus, and directrix. If it is an ellipse or a hyperbola, label vertices and foci.

24. $r = \frac{12}{4 - 8\ \sin\ \theta}$

25. $r = \frac{2}{4 + 4\ \sin\ \theta}$

26. Find a polar equation of the conic with focus at the origin, eccentricity of $e = 2$, and directrix: $x = 3$.

9 SEQUENCES, PROBABILITY, AND COUNTING THEORY



(credit: Robert Couse-Baker, Flickr.)

Chapter Outline

Introduction to Sequences, Probability and Counting Theory

A lottery winner has some big decisions to make regarding what to do with the winnings. Buy a new home? A luxury convertible? A cruise around the world?

The likelihood of winning the lottery is slim, but we all love to fantasize about what we could buy with the winnings. One of the first things a lottery winner has to decide is whether to take the winnings in the form of a lump sum or as a series of regular payments, called an annuity, over an extended period of time.

This decision is often based on many factors, such as tax implications, interest rates, and investment strategies. There are also personal reasons to consider when making the choice, and one can make many arguments for either decision. However, most lottery winners opt for the lump sum.

In this chapter, we will explore the mathematics behind situations such as these. We will take an in-depth look at annuities. We will also look at the branch of mathematics that would allow us to calculate the number of ways to choose lottery numbers and the probability of winning.

9.1 Sequences and Their Notations

Learning Objectives

In this section, you will:

- Write the terms of a sequence defined by an explicit formula.
- Write the terms of a sequence defined by a recursive formula.
- Use factorial notation.

A video game company launches an exciting new advertising campaign. They predict the number of online visits to their website, or hits, will double each day. The model they are using shows 2 hits the first day, 4 hits the second day, 8 hits the third day, and so on. See Table 1.

Day	1	2	3	4	5	…
Hits	2	4	8	16	32	…

Table 1

If their model continues, how many hits will there be at the end of the month? To answer this question, we'll first need to know how to determine a list of numbers written in a specific order. In this section, we will explore these kinds of ordered lists.

Writing the Terms of a Sequence Defined by an Explicit Formula

One way to describe an ordered list of numbers is as a **sequence**. A sequence is a function whose domain is a subset of the counting numbers. The sequence established by the number of hits on the website is

$$\{2, 4, 8, 16, 32, \ldots\}.$$

The ellipsis (…) indicates that the sequence continues indefinitely. Each number in the sequence is called a **term**. The first five terms of this sequence are 2, 4, 8, 16, and 32.

Listing all of the terms for a sequence can be cumbersome. For example, finding the number of hits on the website at the end of the month would require listing out as many as 31 terms. A more efficient way to determine a specific term is by writing a formula to define the sequence.

One type of formula is an explicit formula, which defines the terms of a sequence using their position in the sequence. Explicit formulas are helpful if we want to find a specific term of a sequence without finding all of the previous terms. We can use the formula to find the nth term of the sequence, where n is any positive number. In our example, each number in the sequence is double the previous number, so we can use powers of 2 to write a formula for the nth term.

$$\begin{array}{cccccccc} \{2, & 4, & 8, & 16, & 32, & \ldots, & ?, & \ldots\} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ \{2^1, & 2^2, & 2^3, & 2^4, & 2^5, & \ldots, & 2^n, & \ldots\} \end{array}$$

The first term of the sequence is $2^1 = 2$, the second term is $2^2 = 4$, the third term is $2^3 = 8$, and so on. The nth term of the sequence can be found by raising 2 to the nth power. An explicit formula for a sequence is named by a lower case letter a, b, c... with the subscript n. The explicit formula for this sequence is

$$a_n = 2^n.$$

Now that we have a formula for the nth term of the sequence, we can answer the question posed at the beginning of this section. We were asked to find the number of hits at the end of the month, which we will take to be 31 days. To find the number of hits on the last day of the month, we need to find the 31st term of the sequence. We will substitute 31 for n in the formula.

$$\begin{aligned} a_{31} &= 2^{31} \\ &= 2{,}147{,}483{,}648 \end{aligned}$$

If the doubling trend continues, the company will get $2{,}147{,}483{,}648$ hits on the last day of the month. That is over 2.1 billion hits! The huge number is probably a little unrealistic because it does not take consumer interest and competition into account. It does, however, give the company a starting point from which to consider business decisions.

Another way to represent the sequence is by using a table. The first five terms of the sequence and the nth term of the sequence are shown in Table 2.

n	1	2	3	4	5	n
nth **term of the sequence,** a_n	2	4	8	16	32	2^n

Table 2

Graphing provides a visual representation of the sequence as a set of distinct points. We can see from the graph in Figure 1 that the number of hits is rising at an exponential rate. This particular sequence forms an exponential function.

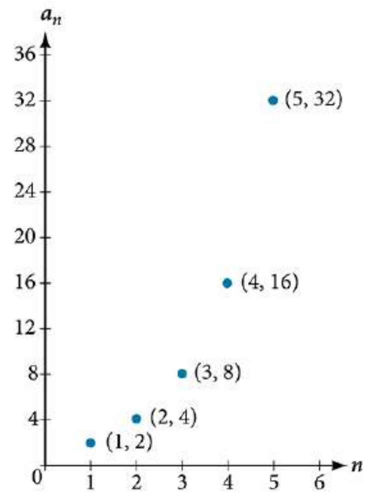


Figure 1

Lastly, we can write this particular sequence as

$$\{2, 4, 8, 16, 32, \ldots, 2^n, \ldots\}.$$

A sequence that continues indefinitely is called an **infinite sequence**. The domain of an infinite sequence is the set of counting numbers. If we consider only the first 10 terms of the sequence, we could write

$$\{2, 4, 8, 16, 32, \ldots, 2^n, \ldots, 1024\}.$$

This sequence is called a **finite sequence** because it does not continue indefinitely.

Sequence

A **sequence** is a function whose domain is the set of positive integers. A **finite sequence** is a sequence whose domain consists of only the first n positive integers. The numbers in a sequence are called **terms**. The variable a with a number subscript is used to represent the terms in a sequence and to indicate the position of the term in the sequence.

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

We call a_1 the first term of the sequence, a_2 the second term of the sequence, a_3 the third term of the sequence, and so on. The term a_n is called the **nth term of the sequence**, or the general term of the sequence. An **explicit formula** defines the nth term of a sequence using the position of the term. A sequence that continues indefinitely is an **infinite sequence**.

Q&A **Does a sequence always have to begin with a_1?**

No. In certain problems, it may be useful to define the initial term as a_0 instead of a_1. In these problems, the domain of the function includes 0.

HOW TO

Given an explicit formula, write the first n terms of a sequence.

1. Substitute each value of n into the formula. Begin with $n = 1$ to find the first term, a_1.
2. To find the second term, a_2, use $n = 2$.
3. Continue in the same manner until you have identified all n terms.

EXAMPLE 1

Writing the Terms of a Sequence Defined by an Explicit Formula

Write the first five terms of the sequence defined by the explicit formula $a_n = -3n + 8$.

Solution

Substitute $n = 1$ into the formula. Repeat with values 2 through 5 for n.

$$\begin{array}{ll} n = 1 & a_1 = -3(1) + 8 = 5 \\ n = 2 & a_2 = -3(2) + 8 = 2 \\ n = 3 & a_3 = -3(3) + 8 = -1 \\ n = 4 & a_4 = -3(4) + 8 = -4 \\ n = 5 & a_5 = -3(5) + 8 = -7 \end{array}$$

The first five terms are $\{5, \ 2, \ -1, \ -4, \ -7\}$.

Analysis

The sequence values can be listed in a table. A table, such as Table 3, is a convenient way to input the function into a graphing utility.

n	1	2	3	4	5
a_n	5	2	–1	–4	–7

Table 3

A graph can be made from this table of values. From the graph in Figure 2, we can see that this sequence represents a linear function, but notice the graph is not continuous because the domain is over the positive integers only.

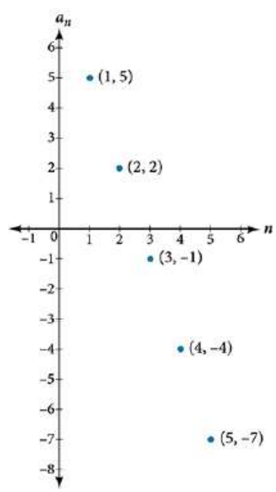


Figure 2

TRY IT #1 Write the first five terms of the sequence defined by the explicit formula $t_n = 5n - 4$.

Investigating Alternating Sequences

Sometimes sequences have terms that are alternate. In fact, the terms may actually alternate in sign. The steps to finding terms of the sequence are the same as if the signs did not alternate. However, the resulting terms will not show increase or decrease as n increases. Let's take a look at the following sequence.

$$\{2, -4, 6, -8\}$$

Notice the first term is greater than the second term, the second term is less than the third term, and the third term is greater than the fourth term. This trend continues forever. Do not rearrange the terms in numerical order to interpret the sequence.

HOW TO

Given an explicit formula with alternating terms, write the first n terms of a sequence.

1. Substitute each value of n into the formula. Begin with $n = 1$ to find the first term, a_1. The sign of the term is given by the $(-1)^n$ in the explicit formula.
2. To find the second term, a_2, use $n = 2$.
3. Continue in the same manner until you have identified all n terms.

EXAMPLE 2

Writing the Terms of an Alternating Sequence Defined by an Explicit Formula

Write the first five terms of the sequence.

$$a_n = \frac{(-1)^n n^2}{n+1}$$

Solution

Substitute $n = 1$, $n = 2$, and so on in the formula.

$$\begin{array}{ll} n = 1 & a_1 = \frac{(-1)^1 1^2}{1+1} = -\frac{1}{2} \\ n = 2 & a_2 = \frac{(-1)^2 2^2}{2+1} = \frac{4}{3} \\ n = 3 & a_3 = \frac{(-1)^3 3^2}{3+1} = -\frac{9}{4} \\ n = 4 & a_4 = \frac{(-1)^4 4^2}{4+1} = \frac{16}{5} \\ n = 5 & a_5 = \frac{(-1)^5 5^2}{5+1} = -\frac{25}{6} \end{array}$$

The first five terms are $\left\{-\frac{1}{2}, \frac{4}{3}, -\frac{9}{4}, \frac{16}{5}, -\frac{25}{6}\right\}$.

Analysis

The graph of this function, shown in Figure 3, looks different from the ones we have seen previously in this section because the terms of the sequence alternate between positive and negative values.

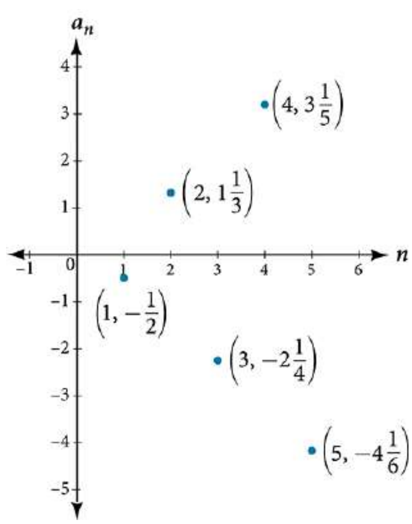


Figure 3

Q&A In **Example 2**, does the (–1) to the power of n account for the oscillations of signs?

Yes, the power might be $n, n+1, n-1$, and so on, but any odd powers will result in a negative term, and any even power will result in a positive term.

TRY IT #2 Write the first five terms of the sequence.

$$a_n = \frac{4n}{(-2)^n}$$

Investigating Piecewise Explicit Formulas

We've learned that sequences are functions whose domain is over the positive integers. This is true for other types of functions, including some piecewise functions. Recall that a piecewise function is a function defined by multiple subsections. A different formula might represent each individual subsection.

HOW TO

Given an explicit formula for a piecewise function, write the first n terms of a sequence

1. Identify the formula to which $n = 1$ applies.
2. To find the first term, a_1, use $n = 1$ in the appropriate formula.
3. Identify the formula to which $n = 2$ applies.
4. To find the second term, a_2, use $n = 2$ in the appropriate formula.
5. Continue in the same manner until you have identified all n terms.

EXAMPLE 3

Writing the Terms of a Sequence Defined by a Piecewise Explicit Formula

Write the first six terms of the sequence.

$$a_n = \begin{cases} n^2 & \text{if } n \text{ is not divisible by 3} \\ \frac{n}{3} & \text{if } n \text{ is divisible by 3} \end{cases}$$

Solution

Substitute $n = 1, n = 2$, and so on in the appropriate formula. Use n^2 when n is not a multiple of 3. Use $\frac{n}{3}$ when n is a multiple of 3.

$$\begin{array}{ll} a_1 = 1^2 = 1 & \text{1 is not a multiple of 3. Use } n^2. \\ a_2 = 2^2 = 4 & \text{2 is not a multiple of 3. Use } n^2. \\ a_3 = \frac{3}{3} = 1 & \text{3 is a multiple of 3. Use } \frac{n}{3}. \\ a_4 = 4^2 = 16 & \text{4 is not a multiple of 3. Use } n^2. \\ a_5 = 5^2 = 25 & \text{5 is not a multiple of 3. Use } n^2. \\ a_6 = \frac{6}{3} = 2 & \text{6 is a multiple of 3. Use } \frac{n}{3}. \end{array}$$

The first six terms are $\{1, 4, 1, 16, 25, 2\}$.

Analysis

Every third point on the graph shown in Figure 4 stands out from the two nearby points. This occurs because the sequence was defined by a piecewise function.

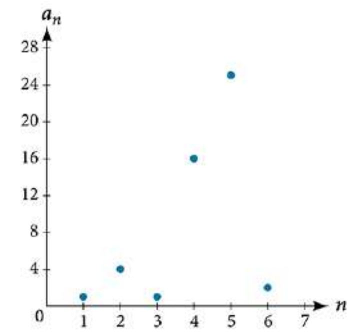


Figure 4

TRY IT #3 Write the first six terms of the sequence.

$$a_n = \begin{cases} 2n^3 & \text{if } n \text{ is odd} \\ \frac{5n}{2} & \text{if } n \text{ is even} \end{cases}$$

Finding an Explicit Formula

Thus far, we have been given the explicit formula and asked to find a number of terms of the sequence. Sometimes, the explicit formula for the nth term of a sequence is not given. Instead, we are given several terms from the sequence. When this happens, we can work in reverse to find an explicit formula from the first few terms of a sequence. The key to finding an explicit formula is to look for a pattern in the terms. Keep in mind that the pattern may involve alternating terms, formulas for numerators, formulas for denominators, exponents, or bases.

HOW TO

Given the first few terms of a sequence, find an explicit formula for the sequence.

1. Look for a pattern among the terms.
2. If the terms are fractions, look for a separate pattern among the numerators and denominators.
3. Look for a pattern among the signs of the terms.
4. Write a formula for a_n in terms of n. Test your formula for $n = 1,\ \ n = 2$, and $n = 3$.

EXAMPLE 4

Writing an Explicit Formula for the nth Term of a Sequence

Write an explicit formula for the nth term of each sequence.

ⓐ $\left\{-\frac{2}{11}, \frac{3}{13}, -\frac{4}{15}, \frac{5}{17}, -\frac{6}{19}, \ldots\right\}$ ⓑ $\left\{-\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, -\frac{2}{3{,}125}, -\frac{2}{15{,}625}, \ldots\right\}$ ⓒ $\{e^4, e^5, e^6, e^7, e^8, \ldots\}$

Solution

Look for the pattern in each sequence.

ⓐ The terms alternate between positive and negative. We can use $(-1)^n$ to make the terms alternate. The numerator can be represented by $n + 1$. The denominator can be represented by $2n + 9$.

$a_n = \frac{(-1)^n(n+1)}{2n+9}$

ⓑ

The terms are all negative.

$\left\{-\frac{2}{25}, -\frac{2}{125}, -\frac{2}{625}, -\frac{2}{3{,}125}, -\frac{2}{15{,}125}, \cdots\right\}$ The numerator is 2.

$\left\{-\frac{2}{5^2}, -\frac{2}{5^3}, -\frac{2}{5^4}, -\frac{2}{5^6}, -\frac{2}{5^7}, \cdots, -\frac{2}{5^n}\right\}$ The denominators are increasing powers of 5.

So we know that the fraction is negative, the numerator is 2, and the denominator can be represented by 5^{n+1}.

$$a_n = -\frac{2}{5^{n+1}}$$

ⓒ

The terms are powers of e. For $n = 1$, the first term is e^4 so the exponent must be $n + 3$.

$$a_n = e^{n+3}$$

> TRY IT #4 Write an explicit formula for the nth term of the sequence.

$$\{9, \ -81, \ 729, \ -6{,}561, \ 59{,}049, \ \ldots\}$$

> TRY IT #5 Write an explicit formula for the nth term of the sequence.

$$\left\{-\frac{3}{4}, -\frac{9}{8}, -\frac{27}{12}, -\frac{81}{16}, -\frac{243}{20}, \ldots\right\}$$

> TRY IT #6 Write an explicit formula for the nth term of the sequence.

$$\left\{\frac{1}{e^2}, \frac{1}{e}, 1, e, e^2, \ldots\right\}$$

Writing the Terms of a Sequence Defined by a Recursive Formula

Sequences occur naturally in the growth patterns of nautilus shells, pinecones, tree branches, and many other natural structures. We may see the sequence in the leaf or branch arrangement, the number of petals of a flower, or the pattern of the chambers in a nautilus shell. Their growth follows the Fibonacci sequence, a famous sequence in which each term can be found by adding the preceding two terms. The numbers in the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34,…. Other examples from the natural world that exhibit the Fibonacci sequence are the Calla Lily, which has just one petal, the Black-Eyed Susan with 13 petals, and different varieties of daisies that may have 21 or 34 petals.

Each term of the Fibonacci sequence depends on the terms that come before it. The Fibonacci sequence cannot easily be written using an explicit formula. Instead, we describe the sequence using a **recursive formula**, a formula that defines the terms of a sequence using previous terms.

A recursive formula always has two parts: the value of an initial term (or terms), and an equation defining a_n in terms of preceding terms. For example, suppose we know the following:

$$\begin{aligned} a_1 &= 3 \\ a_n &= 2a_{n-1} - 1 \text{ for } n \geq 2 \end{aligned}$$

We can find the subsequent terms of the sequence using the first term.

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 2a_1 - 1 = 2(3) - 1 = 5 \\ a_3 &= 2a_2 - 1 = 2(5) - 1 = 9 \\ a_4 &= 2a_3 - 1 = 2(9) - 1 = 17 \end{aligned}$$

So the first four terms of the sequence are $\{3, \ 5, \ 9, \ 17\}$.

The recursive formula for the Fibonacci sequence states the first two terms and defines each successive term as the sum of the preceding two terms.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-1} + a_{n-2} \text{ for } \ n \geq 3 \end{aligned}$$

To find the tenth term of the sequence, for example, we would need to add the eighth and ninth terms. We were told previously that the eighth and ninth terms are 21 and 34, so

$$a_{10} = a_9 + a_8 = 34 + 21 = 55$$

Recursive Formula

A **recursive formula** is a formula that defines each term of a sequence using preceding term(s). Recursive formulas must always state the initial term, or terms, of the sequence.

Q&A **Must the first two terms always be given in a recursive formula?**

No. The Fibonacci sequence defines each term using the two preceding terms, but many recursive formulas define each term using only one preceding term. These sequences need only the first term to be defined.

HOW TO

Given a recursive formula with only the first term provided, write the first n terms of a sequence.

1. Identify the initial term, a_1, which is given as part of the formula. This is the first term.
2. To find the second term, a_2, substitute the initial term into the formula for a_{n-1}. Solve.
3. To find the third term, a_3, substitute the second term into the formula. Solve.
4. Repeat until you have solved for the nth term.

EXAMPLE 5

Writing the Terms of a Sequence Defined by a Recursive Formula

Write the first five terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 9 \\ a_n &= 3a_{n-1} - 20, \text{ for } n \geq 2 \end{aligned}$$

Solution

The first term is given in the formula. For each subsequent term, we replace a_{n-1} with the value of the preceding term.

$$\begin{array}{ll} n = 1 & a_1 = 9 \\ n = 2 & a_2 = 3a_1 - 20 = 3(9) - 20 = 27 - 20 = 7 \\ n = 3 & a_3 = 3a_2 - 20 = 3(7) - 20 = 21 - 20 = 1 \\ n = 4 & a_4 = 3a_3 - 20 = 3(1) - 20 = 3 - 20 = -17 \\ n = 5 & a_5 = 3a_4 - 20 = 3(-17) - 20 = -51 - 20 = -71 \end{array}$$

The first five terms are $\{9, \ 7, \ 1, \ -17, \ -71\}$. See Figure 5.

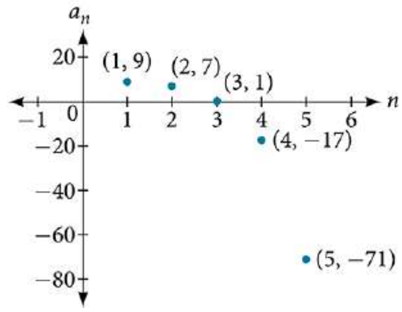


Figure 5

TRY IT #7 Write the first five terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 2 \\ a_n &= 2a_{n-1} + 1, \text{ for } n \geq 2 \end{aligned}$$

HOW TO

Given a recursive formula with two initial terms, write the first n terms of a sequence.

1. Identify the initial term, a_1, which is given as part of the formula.
2. Identify the second term, a_2, which is given as part of the formula.
3. To find the third term, substitute the initial term and the second term into the formula. Evaluate.
4. Repeat until you have evaluated the nth term.

EXAMPLE 6

Writing the Terms of a Sequence Defined by a Recursive Formula

Write the first six terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \\ a_n &= 3a_{n-1} + 4a_{n-2}, \text{ for } n \geq 3 \end{aligned}$$

Solution

The first two terms are given. For each subsequent term, we replace a_{n-1} and a_{n-2} with the values of the two preceding terms.

$$\begin{array}{ll} n = 3 & a_3 = 3a_2 + 4a_1 = 3(2) + 4(1) = 10 \\ n = 4 & a_4 = 3a_3 + 4a_2 = 3(10) + 4(2) = 38 \\ n = 5 & a_5 = 3a_4 + 4a_3 = 3(38) + 4(10) = 154 \\ n = 6 & a_6 = 3a_5 + 4a_4 = 3(154) + 4(38) = 614 \end{array}$$

The first six terms are $\{1, 2, 10, 38, 154, 614\}$. See Figure 6.

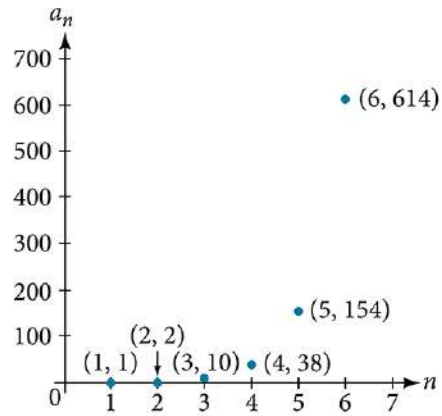


Figure 6

> **TRY IT** #8 Write the first 8 terms of the sequence defined by the recursive formula.

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 1 \\ a_3 &= 1 \\ a_n &= \frac{a_{n-1}}{a_{n-2}} + a_{n-3}, \text{ for } n \geq 4 \end{aligned}$$

Using Factorial Notation

The formulas for some sequences include products of consecutive positive integers. n **factorial**, written as $n!$, is the product of the positive integers from 1 to n. For example,

$$\begin{aligned} 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \end{aligned}$$

An example of formula containing a factorial is $a_n = (n+1)!$. The sixth term of the sequence can be found by substituting 6 for n.

$$a_6 = (6+1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

The factorial of any whole number n is $n(n-1)!$ We can therefore also think of $5!$ as $5 \cdot 4!$.

n Factorial

***n* factorial** is a mathematical operation that can be defined using a recursive formula. The factorial of n, denoted $n!$, is defined for a positive integer n as:

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ n! &= n(n-1)(n-2)\cdots(2)(1), \text{ for } n \geq 2 \end{aligned}$$

The special case $0!$ is defined as $0! = 1$.

> **Q&A** **Can factorials always be found using a calculator?**
>
> *No. Factorials get large very quickly—faster than even exponential functions! When the output gets too large for the calculator, it will not be able to calculate the factorial.*

EXAMPLE 7

Writing the Terms of a Sequence Using Factorials

Write the first five terms of the sequence defined by the explicit formula $a_n = \frac{5n}{(n+2)!}$.

Solution

Substitute $n = 1, n = 2$, and so on in the formula.

$$\begin{array}{ll}
n = 1 & a_1 = \frac{5(1)}{(1+2)!} = \frac{5}{3!} = \frac{5}{3 \cdot 2 \cdot 1} = \frac{5}{6} \\
n = 2 & a_2 = \frac{5(2)}{(2+2)!} = \frac{10}{4!} = \frac{10}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{5}{12} \\
n = 3 & a_3 = \frac{5(3)}{(3+2)!} = \frac{15}{5!} = \frac{15}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{8} \\
n = 4 & a_4 = \frac{5(4)}{(4+2)!} = \frac{20}{6!} = \frac{20}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{36} \\
n = 5 & a_5 = \frac{5(5)}{(5+2)!} = \frac{25}{7!} = \frac{25}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{5}{1{,}008}
\end{array}$$

The first five terms are $\left\{\frac{5}{6}, \frac{5}{12}, \frac{1}{8}, \frac{1}{36}, \frac{5}{1{,}008}\right\}$.

Analysis

Figure 7 shows the graph of the sequence. Notice that, since factorials grow very quickly, the presence of the factorial term in the denominator results in the denominator becoming much larger than the numerator as n increases. This means the quotient gets smaller and, as the plot of the terms shows, the terms are decreasing and nearing zero.

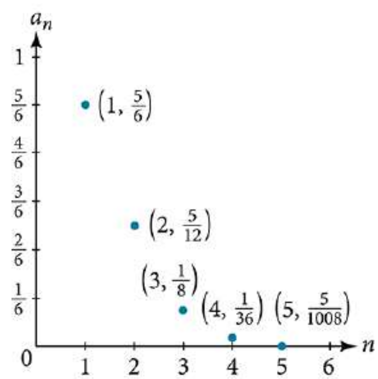


Figure 7

TRY IT #9 Write the first five terms of the sequence defined by the explicit formula $a_n = \frac{(n+1)!}{2n}$.

MEDIA

Access this online resource for additional instruction and practice with sequences.

Finding Terms in a Sequence (http://openstax.org/l/findingterms)

9.1 SECTION EXERCISES

Verbal

1. Discuss the meaning of a sequence. If a finite sequence is defined by a formula, what is its domain? What about an infinite sequence?

2. Describe three ways that a sequence can be defined.

3. Is the ordered set of even numbers an infinite sequence? What about the ordered set of odd numbers? Explain why or why not.

4. What happens to the terms a_n of a sequence when there is a negative factor in the formula that is raised to a power that includes n? What is the term used to describe this phenomenon?

5. What is a factorial, and how is it denoted? Use an example to illustrate how factorial notation can be beneficial.

Algebraic

For the following exercises, write the first four terms of the sequence.

6. $a_n = 2^n - 2$

7. $a_n = -\frac{16}{n+1}$

8. $a_n = -(-5)^{n-1}$

9. $a_n = \frac{2^n}{n^3}$

10. $a_n = \frac{2n+1}{n^3}$

11. $a_n = 1.25 \cdot (-4)^{n-1}$

12. $a_n = -4 \cdot (-6)^{n-1}$

13. $a_n = \frac{n^2}{2n+1}$

14. $a_n = (-10)^n + 1$

15. $a_n = -\left(\frac{4\cdot(-5)^{n-1}}{5}\right)$

For the following exercises, write the first eight terms of the piecewise sequence.

16. $a_n = \begin{cases} (-2)^n - 2 & \text{if } n \text{ is even} \\ (3)^{n-1} & \text{if } n \text{ is odd} \end{cases}$

17. $a_n = \begin{cases} \frac{n^2}{2n+1} & \text{if } n \le 5 \\ n^2 - 5 & \text{if } n > 5 \end{cases}$

18. $a_n = \begin{cases} (2n+1)^2 & \text{if } n \text{ is divisible by } 4 \\ \frac{2}{n} & \text{if } n \text{ is not divisible by } 4 \end{cases}$

19. $a_n = \begin{cases} -0.6 \cdot 5^{n-1} & \text{if } n \text{ is prime or } 1 \\ 2.5 \cdot (-2)^{n-1} & \text{if } n \text{ is composite} \end{cases}$

20. $a_n = \begin{cases} 4(n^2 - 2) & \text{if } n \le 3 \text{ or } n > 6 \\ \frac{n^2-2}{4} & \text{if } 3 < n \le 6 \end{cases}$

For the following exercises, write an explicit formula for each sequence.

21. $4, 7, 12, 19, 28, \ldots$

22. $-4, 2, -10, 14, -34, \ldots$

23. $1, 1, \frac{4}{3}, 2, \frac{16}{5}, \ldots$

24. $0, \frac{1-e^1}{1+e^2}, \frac{1-e^2}{1+e^3}, \frac{1-e^3}{1+e^4}, \frac{1-e^4}{1+e^5}, \ldots$

25. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

For the following exercises, write the first five terms of the sequence.

26. $a_1 = 9, \quad a_n = a_{n-1} + n$

27. $a_1 = 3, \quad a_n = (-3)\, a_{n-1}$

28. $a_1 = -4, \quad a_n = \frac{a_{n-1}+2n}{a_{n-1}-1}$

29. $a_1 = -1, \quad a_n = \frac{(-3)^{n-1}}{a_{n-1}-2}$

30. $a_1 = -30, \quad a_n = \left(2 + a_{n-1}\right)\left(\frac{1}{2}\right)^n$

For the following exercises, write the first eight terms of the sequence.

31. $a_1 = \frac{1}{24}, \quad \mathrm{a}_2 = 1, \quad a_n = \left(2a_{n-2}\right)\left(3a_{n-1}\right)$

32. $a_1 = -1, \quad \mathrm{a}_2 = 5, \quad a_n = a_{n-2}\left(3 - a_{n-1}\right)$

33. $a_1 = 2, \quad \mathrm{a}_2 = 10, \quad a_n = \frac{2\left(a_{n-1}+2\right)}{a_{n-2}}$

For the following exercises, write a recursive formula for each sequence.

34. $-2.5, -5, -10, -20, -40, \ldots$

35. $-8, -6, -3, 1, 6, \ldots$

36. $2, \ 4, \ 12, \ 48, \ 240, \ \ldots$

37. $35, \ 38, \ 41, \ 44, \ 47, \ \ldots$

38. $15, 3, \frac{3}{5}, \frac{3}{25}, \frac{3}{125}, \cdots$

For the following exercises, evaluate the factorial.

39. $6!$

40. $\left(\frac{12}{6}\right)!$

41. $\frac{12!}{6!}$

42. $\frac{100!}{99!}$

For the following exercises, write the first four terms of the sequence.

43. $a_n = \frac{n!}{n^2}$

44. $a_n = \frac{3 \cdot n!}{4 \cdot n!}$

45. $a_n = \frac{n!}{n^2-n-1}$

46. $a_n = \frac{100 \cdot n}{n(n-1)!}$

Graphical

For the following exercises, graph the first five terms of the indicated sequence

47. $a_n = \frac{(-1)^n}{n} + n$

48. $a_n = \begin{cases} \frac{4+n}{2n} & \text{if } n \text{ is even} \\ 3 + n & \text{if } n \text{ is odd} \end{cases}$

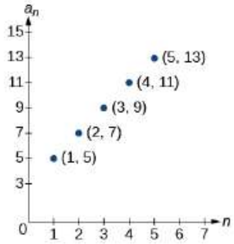
49. $a_1 = 2, \quad a_n = \left(-a_{n-1} + 1\right)^2$

50. $a_1 = 1, \quad a_n = a_{n-1} + 8$

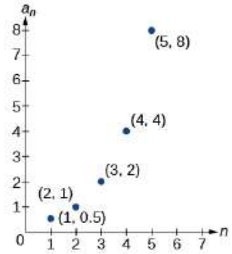
51. $a_n = \frac{(n+1)!}{(n-1)!}$

For the following exercises, write an explicit formula for the sequence using the first five points shown on the graph.

52.



53.



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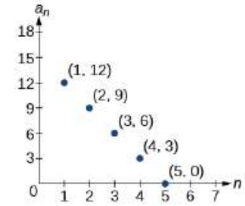
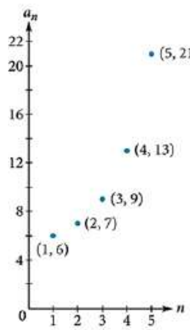
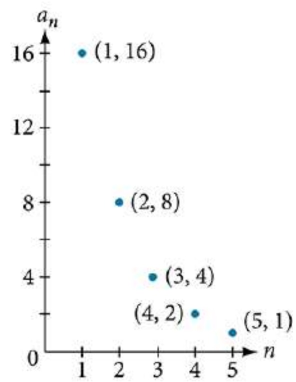


For the following exercises, write a recursive formula for the sequence using the first five points shown on the graph.

55.



56.



Technology

Follow these steps to evaluate a sequence defined recursively using a graphing calculator:

- *On the home screen, key in the value for the initial term a_1 and press **[ENTER]**.*
- *Enter the recursive formula by keying in all numerical values given in the formula, along with the key strokes **[2ND] ANS** for the previous term a_{n-1}. Press **[ENTER]**.*
- *Continue pressing **[ENTER]** to calculate the values for each successive term.*

For the following exercises, use the steps above to find the indicated term or terms for the sequence.

57. Find the first five terms of the sequence $a_1 = \frac{87}{111}$, $a_n = \frac{4}{3}a_{n-1} + \frac{12}{37}$. Use the >**Frac** feature to give fractional results.

58. Find the 15th term of the sequence $a_1 = 625$, $a_n = 0.8a_{n-1} + 18$.

59. Find the first five terms of the sequence $a_1 = 2$, $a_n = 2^{[(a_n - 1) - 1]} + 1$.

60. Find the first ten terms of the sequence $a_1 = 8$, $a_n = \frac{(a_{n-1}+1)!}{a_{n-1}!}$.

61. Find the tenth term of the sequence $a_1 = 2$, $a_n = na_{n-1}$

Follow these steps to evaluate a finite sequence defined by an explicit formula. Using a TI-84, do the following.

- *In the home screen, press **[2ND] LIST**.*
- *Scroll over to **OPS** and choose **"seq("** from the dropdown list. Press **[ENTER]**.*
- *In the line headed **"Expr:"** type in the explicit formula, using the $[\mathrm{X,T},\theta,n]$ button for n*

- *In the line headed* ***"Variable:"*** *type in the variable used on the previous step.*
- *In the line headed* ***"start:"*** *key in the value of n that begins the sequence.*
- *In the line headed* ***"end:"*** *key in the value of n that ends the sequence.*
- *Press* ***[ENTER]*** *3 times to return to the home screen. You will see the sequence syntax on the screen. Press* ***[ENTER]*** *to see the list of terms for the finite sequence defined. Use the right arrow key to scroll through the list of terms.*

Using a TI-83, do the following.

- *In the home screen, press* ***[2ND] LIST****.*
- *Scroll over to* ***OPS*** *and choose* ***"seq("*** *from the dropdown list. Press* ***[ENTER]****.*
- *Enter the items in the order* ***"Expr"****,* ***"Variable"****,* ***"start"****,* ***"end"*** *separated by commas. See the instructions above for the description of each item.*
- *Press* ***[ENTER]*** *to see the list of terms for the finite sequence defined. Use the right arrow key to scroll through the list of terms.*

For the following exercises, use the steps above to find the indicated terms for the sequence. Round to the nearest thousandth when necessary.

62. List the first five terms of the sequence $a_n = -\frac{28}{9}n + \frac{5}{3}$.

63. List the first six terms of the sequence $a_n = \frac{n^3 - 3.5n^2 + 4.1n - 1.5}{2.4n}$.

64. List the first five terms of the sequence $a_n = \frac{15n \cdot (-2)^{n-1}}{47}$

65. List the first four terms of the sequence $a_n = 5.7^n + 0.275\,(n-1)!$

66. List the first six terms of the sequence $a_n = \frac{n!}{n}$.

Extensions

67. Consider the sequence defined by $a_n = -6 - 8n$. Is $a_n = -421$ a term in the sequence? Verify the result.

68. What term in the sequence $a_n = \frac{n^2+4n+4}{2(n+2)}$ has the value 41? Verify the result.

69. Find a recursive formula for the sequence 1, 0, −1, −1, 0, 1, 1, 0, −1, −1, 0, 1, 1, ... (*Hint*: find a pattern for a_n based on the first two terms.)

70. Calculate the first eight terms of the sequences $a_n = \frac{(n+2)!}{(n-1)!}$ and $b_n = n^3 + 3n^2 + 2n$, and then make a conjecture about the relationship between these two sequences.

71. Prove the conjecture made in the preceding exercise.

9.2 Arithmetic Sequences

Learning Objectives

In this section, you will:

- Find the common difference for an arithmetic sequence.
- Write terms of an arithmetic sequence.
- Use a recursive formula for an arithmetic sequence.
- Use an explicit formula for an arithmetic sequence.

Companies often make large purchases, such as computers and vehicles, for business use. The book-value of these supplies decreases each year for tax purposes. This decrease in value is called depreciation. One method of calculating depreciation is straight-line depreciation, in which the value of the asset decreases by the same amount each year.

As an example, consider a woman who starts a small contracting business. She purchases a new truck for $25,000. After five years, she estimates that she will be able to sell the truck for $8,000. The loss in value of the truck will therefore be $17,000, which is $3,400 per year for five years. The truck will be worth $21,600 after the first year; $18,200 after two years; $14,800 after three years; $11,400 after four years; and $8,000 at the end of five years. In this section, we will consider specific kinds of sequences that will allow us to calculate depreciation, such as the truck's value.

Finding Common Differences

The values of the truck in the example are said to form an **arithmetic sequence** because they change by a constant amount each year. Each term increases or decreases by the same constant value called the **common difference** of the sequence. For this sequence, the common difference is –3,400.

$$\{25000, \xrightarrow{-3{,}400} 21600, \xrightarrow{-3{,}400} 18200, \xrightarrow{-3{,}400} 14800, \xrightarrow{-3{,}400} 11400, \xrightarrow{-3{,}400} 8000\}$$

The sequence below is another example of an arithmetic sequence. In this case, the constant difference is 3. You can choose any term of the sequence, and add 3 to find the subsequent term.

$$\{3, \xrightarrow{+3} 6, \xrightarrow{+3} 9, \xrightarrow{+3} 12, \xrightarrow{+3} 15, \ldots\}$$

Arithmetic Sequence

An **arithmetic sequence** is a sequence that has the property that the difference between any two consecutive terms is a constant. This constant is called the **common difference**. If a_1 is the first term of an arithmetic sequence and d is the common difference, the sequence will be:

$$\{a_n\} = \{a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots\}$$

EXAMPLE 1

Finding Common Differences

Is each sequence arithmetic? If so, find the common difference.

ⓐ $\{1, 2, 4, 8, 16, \ldots\}$ ⓑ $\{-3, 1, 5, 9, 13, \ldots\}$

Solution

Subtract each term from the subsequent term to determine whether a common difference exists.

ⓐ The sequence is not arithmetic because there is no common difference.

2 – 1 = 1   4 – 2 = 2   8 – 4 = 4   16 – 8 = 8

ⓑ The sequence is arithmetic because there is a common difference. The common difference is 4.

1 – (–3) = 4   5 – 1 = 4   9 – 5 = 4   13 – 9 = 4

Analysis

The graph of each of these sequences is shown in Figure 1. We can see from the graphs that, although both sequences show growth, a is not linear whereas b is linear. Arithmetic sequences have a constant rate of change so their graphs will always be points on a line.

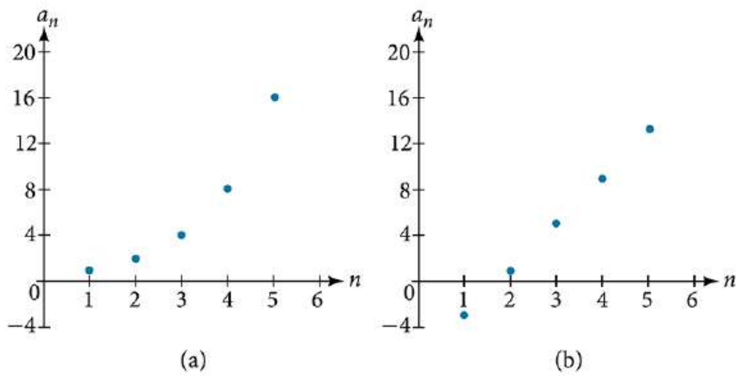


Figure 1

Q&A **If we are told that a sequence is arithmetic, do we have to subtract every term from the following term to find the common difference?**

No. If we know that the sequence is arithmetic, we can choose any one term in the sequence, and subtract it from the subsequent term to find the common difference.

TRY IT #1 Is the given sequence arithmetic? If so, find the common difference.

$$\{18,\ \ 16,\ \ 14,\ \ 12,\ \ 10, \ldots\}$$

TRY IT #2 Is the given sequence arithmetic? If so, find the common difference.

$$\{1,\ \ 3,\ \ 6,\ \ 10,\ \ 15, \ldots\}$$

Writing Terms of Arithmetic Sequences

Now that we can recognize an arithmetic sequence, we will find the terms if we are given the first term and the common difference. The terms can be found by beginning with the first term and adding the common difference repeatedly. In addition, any term can also be found by plugging in the values of n and d into formula below.

$$a_n = a_1 + (n-1)d$$

HOW TO

Given the first term and the common difference of an arithmetic sequence, find the first several terms.

1. Add the common difference to the first term to find the second term.
2. Add the common difference to the second term to find the third term.
3. Continue until all of the desired terms are identified.
4. Write the terms separated by commas within brackets.

EXAMPLE 2

Writing Terms of Arithmetic Sequences

Write the first five terms of the arithmetic sequence with $a_1 = 17$ and $d = -3$.

Solution

Adding -3 is the same as subtracting 3. Beginning with the first term, subtract 3 from each term to find the next term.

The first five terms are $\{17,\ \ 14,\ \ 11,\ \ 8,\ \ 5\}$

Analysis

As expected, the graph of the sequence consists of points on a line as shown in Figure 2.

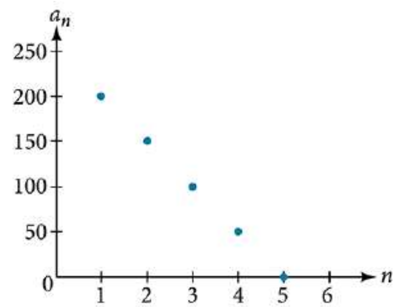


Figure 2

TRY IT #3 List the first five terms of the arithmetic sequence with $a_1 = 1$ and $d = 5$.

HOW TO

Given any first term and any other term in an arithmetic sequence, find a given term.

1. Substitute the values given for a_1 , a_n, n into the formula $a_n = a_1 + (n-1)d$ to solve for d.
2. Find a given term by substituting the appropriate values for a_1 , n, and d into the formula $a_n = a_1 + (n-1)d$.

EXAMPLE 3

Writing Terms of Arithmetic Sequences

Given $a_1 = 8$ and $a_4 = 14$, find a_5 .

Solution

The sequence can be written in terms of the initial term 8 and the common difference d .

$$\{8, 8+d, 8+2d, 8+3d\}$$

We know the fourth term equals 14; we know the fourth term has the form $a_1 + 3d = 8 + 3d$.

We can find the common difference d .

$$\begin{aligned}
a_n &= a_1 + (n-1)d & & \\
a_4 &= a_1 + 3d & & \\
a_4 &= 8 + 3d & & \text{Write the fourth term of the sequence in terms of } a_1 \text{ and } d. \\
14 &= 8 + 3d & & \text{Substitute 14 for } a_4. \\
d &= 2 & & \text{Solve for the common difference.}
\end{aligned}$$

Find the fifth term by adding the common difference to the fourth term.

$$a_5 = a_4 + 2 = 16$$

Analysis

Notice that the common difference is added to the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by adding the common difference to the first term nine times or by using the equation $a_n = a_1 + (n-1)\,d$.

TRY IT #4 Given $a_3 = 7$ and $a_5 = 17$, find a_2 .

Using Recursive Formulas for Arithmetic Sequences

Some arithmetic sequences are defined in terms of the previous term using a recursive formula. The formula provides an algebraic rule for determining the terms of the sequence. A recursive formula allows us to find any term of an arithmetic sequence using a function of the preceding term. Each term is the sum of the previous term and the common difference. For example, if the common difference is 5, then each term is the previous term plus 5. As with any recursive formula, the first term must be given.

$$a_n = a_{n-1} + d \qquad n \geq 2$$

Recursive Formula for an Arithmetic Sequence

The recursive formula for an arithmetic sequence with common difference d is:

$$a_n = a_{n-1} + d \qquad n \geq 2$$

HOW TO

Given an arithmetic sequence, write its recursive formula.

1. Subtract any term from the subsequent term to find the common difference.
2. State the initial term and substitute the common difference into the recursive formula for arithmetic sequences.

EXAMPLE 4

Writing a Recursive Formula for an Arithmetic Sequence

Write a recursive formula for the arithmetic sequence.

$$\{-18, \ -7, 4, 15, 26, \ldots\}$$

Solution

The first term is given as -18 . The common difference can be found by subtracting the first term from the second term.

$$d = -7 - (-18) = 11$$

Substitute the initial term and the common difference into the recursive formula for arithmetic sequences.

$$\begin{aligned} a_1 &= -18 \\ a_n &= a_{n-1} + 11, \quad \text{for } n \geq 2 \end{aligned}$$

Analysis

We see that the common difference is the slope of the line formed when we graph the terms of the sequence, as shown in Figure 3. The growth pattern of the sequence shows the constant difference of 11 units.

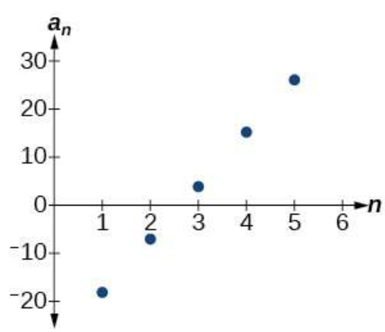


Figure 3

Q&A **Do we have to subtract the first term from the second term to find the common difference?**

No. We can subtract any term in the sequence from the subsequent term. It is, however, most common to subtract the first term from the second term because it is often the easiest method of finding the common difference.

TRY IT #5 Write a recursive formula for the arithmetic sequence.

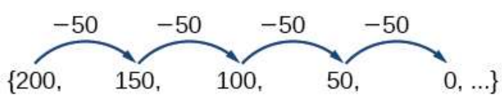
$$\{25,\ 37,\ 49,\ 61,\ \ldots\}$$

Using Explicit Formulas for Arithmetic Sequences

We can think of an arithmetic sequence as a function on the domain of the natural numbers; it is a linear function because it has a constant rate of change. The common difference is the constant rate of change, or the slope of the function. We can construct the linear function if we know the slope and the vertical intercept.

$$a_n = a_1 + d(n-1)$$

To find the y-intercept of the function, we can subtract the common difference from the first term of the sequence. Consider the following sequence.



The common difference is -50 , so the sequence represents a linear function with a slope of -50 . To find the y -intercept, we subtract -50 from 200 : $200-(-50)=200+50=250$. You can also find the y -intercept by graphing the function and determining where a line that connects the points would intersect the vertical axis. The graph is shown in Figure 4.

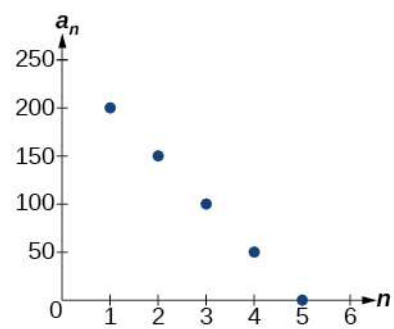


Figure 4

Recall the slope-intercept form of a line is $y = mx + b$. When dealing with sequences, we use a_n in place of y and n in place of x. If we know the slope and vertical intercept of the function, we can substitute them for m and b in the slope-intercept form of a line. Substituting -50 for the slope and 250 for the vertical intercept, we get the following equation:

$$a_n = -50n + 250$$

We do not need to find the vertical intercept to write an explicit formula for an arithmetic sequence. Another explicit formula for this sequence is $a_n = 200 - 50(n-1)$, which simplifies to $a_n = -50n + 250$.

Explicit Formula for an Arithmetic Sequence

An explicit formula for the nth term of an arithmetic sequence is given by

$$a_n = a_1 + d(n-1)$$

HOW TO

Given the first several terms for an arithmetic sequence, write an explicit formula.

1. Find the common difference, $a_2 - a_1$.

2. Substitute the common difference and the first term into $a_n = a_1 + d(n-1)$.

EXAMPLE 5

Writing the *n*th Term Explicit Formula for an Arithmetic Sequence

Write an explicit formula for the arithmetic sequence.

$$\{2, 12, 22, 32, 42, \ldots\}$$

Solution

The common difference can be found by subtracting the first term from the second term.

$$\begin{aligned} d &= a_2 - a_1 \\ &= 12 - 2 \\ &= 10 \end{aligned}$$

The common difference is 10. Substitute the common difference and the first term of the sequence into the formula and simplify.

$$\begin{aligned} a_n &= 2 + 10(n-1) \\ a_n &= 10n - 8 \end{aligned}$$

Analysis

The graph of this sequence, represented in Figure 5, shows a slope of 10 and a vertical intercept of -8 .

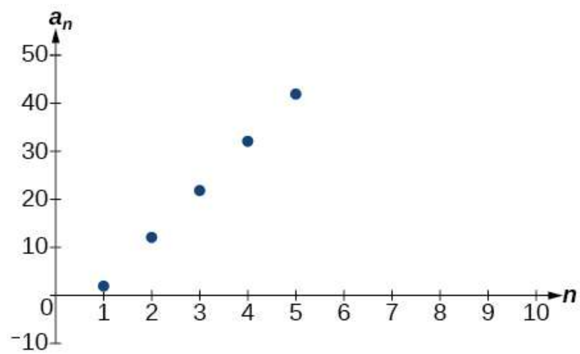


Figure 5

TRY IT #6 Write an explicit formula for the following arithmetic sequence.

$$\{50, 47, 44, 41, \ldots\}$$

Finding the Number of Terms in a Finite Arithmetic Sequence

Explicit formulas can be used to determine the number of terms in a finite arithmetic sequence. We need to find the common difference, and then determine how many times the common difference must be added to the first term to obtain the final term of the sequence.

HOW TO

Given the first three terms and the last term of a finite arithmetic sequence, find the total number of terms.

1. Find the common difference d.
2. Substitute the common difference and the first term into $a_n = a_1 + d(n-1)$.
3. Substitute the last term for a_n and solve for n.

EXAMPLE 6

Finding the Number of Terms in a Finite Arithmetic Sequence

Find the number of terms in the finite arithmetic sequence.

$$\{8, 1, -6, ..., -41\}$$

Solution

The common difference can be found by subtracting the first term from the second term.

$$1 - 8 = -7$$

The common difference is -7 . Substitute the common difference and the initial term of the sequence into the nth term formula and simplify.

$$\begin{aligned} a_n &= a_1 + d(n-1) \\ a_n &= 8 + (-7)(n-1) \\ a_n &= 15 - 7n \end{aligned}$$

Substitute -41 for a_n and solve for n

$$\begin{aligned} -41 &= 15 - 7n \\ 8 &= n \end{aligned}$$

There are eight terms in the sequence.

TRY IT #7 Find the number of terms in the finite arithmetic sequence.

$$\{6, 11, 16, ..., 56\}$$

Solving Application Problems with Arithmetic Sequences

In many application problems, it often makes sense to use an initial term of a_0 instead of a_1 . In these problems, we alter the explicit formula slightly to account for the difference in initial terms. We use the following formula:

$$a_n = a_0 + dn$$

EXAMPLE 7

Solving Application Problems with Arithmetic Sequences

A five-year old child receives an allowance of $1 each week. His parents promise him an annual increase of $2 per week.

ⓐ Write a formula for the child's weekly allowance in a given year.

ⓑ What will the child's allowance be when he is 16 years old?

Solution

ⓐ

The situation can be modeled by an arithmetic sequence with an initial term of 1 and a common difference of 2.

Let A be the amount of the allowance and n be the number of years after age 5. Using the altered explicit formula for an arithmetic sequence we get:

$$A_n = 1 + 2n$$

ⓑ

We can find the number of years since age 5 by subtracting.

$$16 - 5 = 11$$

We are looking for the child's allowance after 11 years. Substitute 11 into the formula to find the child's allowance at age 16.

$$A_{11} = 1 + 2(11) = 23$$

The child's allowance at age 16 will be $23 per week.

TRY IT #8 A woman decides to go for a 10-minute run every day this week and plans to increase the time of her daily run by 4 minutes each week. Write a formula for the time of her run after n weeks. How long will her daily run be 8 weeks from today?

MEDIA

Access this online resource for additional instruction and practice with arithmetic sequences.

[Arithmetic Sequences (http://openstax.org/l/arithmeticseq)](http://openstax.org/l/arithmeticseq)

9.2 SECTION EXERCISES

Verbal

1. What is an arithmetic sequence?

2. How is the common difference of an arithmetic sequence found?

3. How do we determine whether a sequence is arithmetic?

4. What are the main differences between using a recursive formula and using an explicit formula to describe an arithmetic sequence?

5. Describe how linear functions and arithmetic sequences are similar. How are they different?

Algebraic

For the following exercises, find the common difference for the arithmetic sequence provided.

6. $\{5, 11, 17, 23, 29, ...\}$

7. $\left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...\right\}$

For the following exercises, determine whether the sequence is arithmetic. If so find the common difference.

8. $\{11.4, 9.3, 7.2, 5.1, 3, ...\}$

9. $\{4, 16, 64, 256, 1024, ...\}$

For the following exercises, write the first five terms of the arithmetic sequence given the first term and common difference.

10. $a_1 = -25$, $d = -9$

11. $a_1 = 0$, $d = \frac{2}{3}$

For the following exercises, write the first five terms of the arithmetic series given two terms.

12. $a_1 = 17, \quad a_7 = -31$

13. $a_{13} = -60, \quad a_{33} = -160$

For the following exercises, find the specified term for the arithmetic sequence given the first term and common difference.

14. First term is 3, common difference is 4, find the 5th term.

15. First term is 4, common difference is 5, find the 4th term.

16. First term is 5, common difference is 6, find the 8th term.

17. First term is 6, common difference is 7, find the 6th term.

18. First term is 7, common difference is 8, find the 7th term.

For the following exercises, find the first term given two terms from an arithmetic sequence.

19. Find the first term or a_1 of an arithmetic sequence if $a_6 = 12$ and $a_{14} = 28$.

20. Find the first term or a_1 of an arithmetic sequence if $a_7 = 21$ and $a_{15} = 42$.

21. Find the first term or a_1 of an arithmetic sequence if $a_8 = 40$ and $a_{23} = 115$.

22. Find the first term or a_1 of an arithmetic sequence if $a_9 = 54$ and $a_{17} = 102$.

23. Find the first term or a_1 of an arithmetic sequence if $a_{11} = 11$ and $a_{21} = 16$.

For the following exercises, find the specified term given two terms from an arithmetic sequence.

24. $a_1 = 33$ and $a_7 = -15$. Find a_4.

25. $a_3 = -17.1$ and $a_{10} = -15.7$. Find a_{21}.

For the following exercises, use the recursive formula to write the first five terms of the arithmetic sequence.

26. $a_1 = 39; \quad a_n = a_{n-1} - 3$

27. $a_1 = -19; \quad a_n = a_{n-1} - 1.4$

For the following exercises, write a recursive formula for each arithmetic sequence.

28. $a = \{40, 60, 80, ...\}$

29. $a = \{17, 26, 35, ...\}$

30. $a = \{-1, 2, 5, ...\}$

31. $a = \{12, 17, 22, ...\}$

32. $a = \{-15, -7, 1, ...\}$

33. $a = \{8.9, 10.3, 11.7, ...\}$

34. $a = \{-0.52, -1.02, -1.52, ...\}$

35. $a = \left\{\frac{1}{5}, \frac{9}{20}, \frac{7}{10}, ...\right\}$

36. $a = \left\{-\frac{1}{2}, -\frac{5}{4}, -2, ...\right\}$

37. $a = \left\{\frac{1}{6}, -\frac{11}{12}, -2, ...\right\}$

For the following exercises, write a recursive formula for the given arithmetic sequence, and then find the specified term.

38. $a = \{7, 4, 1, ...\}$; Find the 17th term.

39. $a = \{4, 11, 18, ...\}$; Find the 14th term.

40. $a = \{2, 6, 10, ...\}$; Find the 12th term.

For the following exercises, use the explicit formula to write the first five terms of the arithmetic sequence.

41. $a_n = 24 - 4n$

42. $a_n = \frac{1}{2}n - \frac{1}{2}$

For the following exercises, write an explicit formula for each arithmetic sequence.

43. $a = \{3, 5, 7, ...\}$

44. $a = \{32, 24, 16, ...\}$

45. $a = \{-5, 95, 195, ...\}$

46. $a = \{-17, -217, -417, ...\}$

47. $a = \{1.8, 3.6, 5.4, ...\}$

48. $a = \{-18.1, -16.2, -14.3, ...\}$

49. $a = \{15.8, 18.5, 21.2, ...\}$

50. $a = \left\{\frac{1}{3}, -\frac{4}{3}, -3, ...\right\}$

51. $a = \left\{0, \frac{1}{3}, \frac{2}{3}, ...\right\}$

52. $a = \left\{-5, -\frac{10}{3}, -\frac{5}{3}, \dots\right\}$

For the following exercises, find the number of terms in the given finite arithmetic sequence.

53. $a = \{3, -4, -11, ..., -60\}$

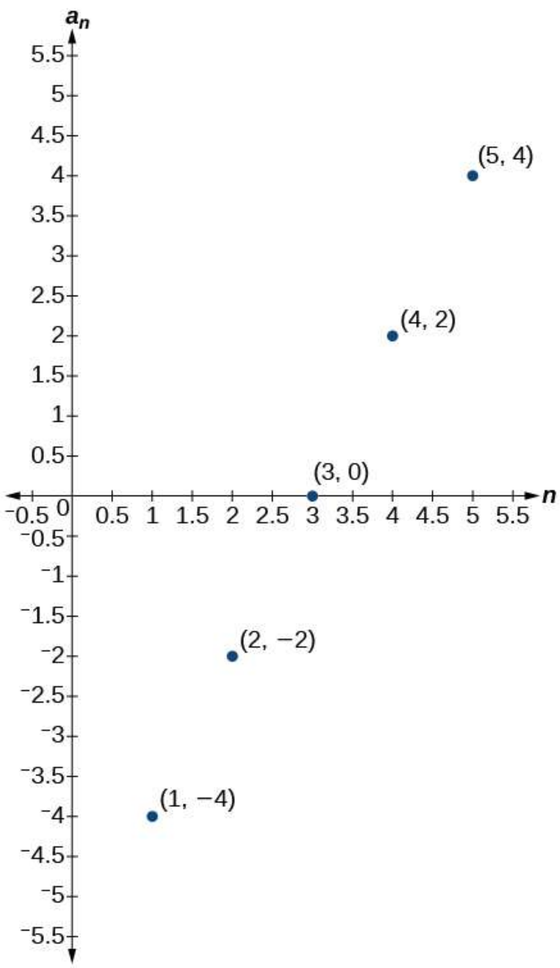
54. $a = \{1.2, 1.4, 1.6, ..., 3.8\}$

55. $a = \left\{\frac{1}{2}, 2, \frac{7}{2}, ..., 8\right\}$

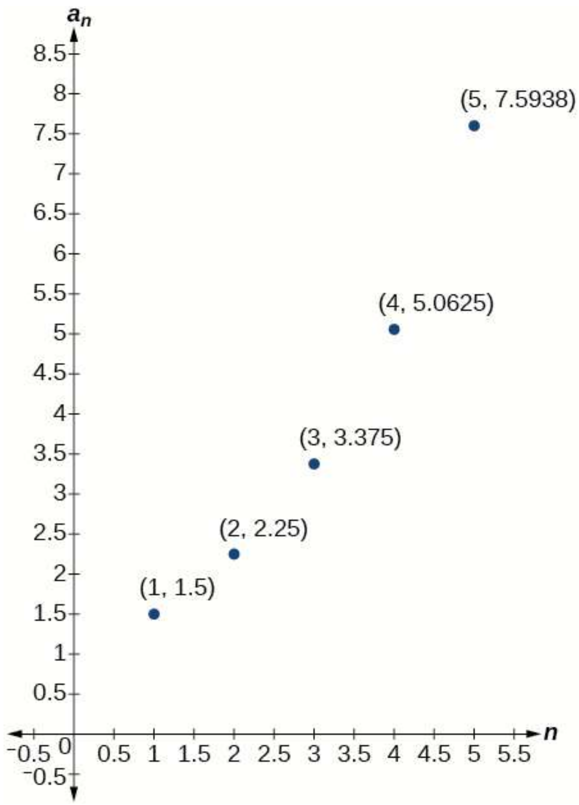
Graphical

For the following exercises, determine whether the graph shown represents an arithmetic sequence.

56.



57.



For the following exercises, use the information provided to graph the first 5 terms of the arithmetic sequence.

58. $a_1 = 0, d = 4$

59. $a_1 = 9; a_n = a_{n-1} - 10$

60. $a_n = -12 + 5n$

Technology

For the following exercises, follow the steps to work with the arithmetic sequence $a_n = 3n - 2$ using a graphing calculator:

- *Press **[MODE]***
  - *Select SEQ in the fourth line*
  - *Select DOT in the fifth line*
  - *Press **[ENTER]***
- *Press **[Y=]***
  - $n\text{Min}$ *is the first counting number for the sequence. Set* $n\text{Min} = 1$
  - $u(n)$ *is the pattern for the sequence. Set* $u(n) = 3n - 2$
  - $u(n\text{Min})$ *is the first number in the sequence. Set* $u(n\text{Min}) = 1$
- *Press **[2ND]** then **[WINDOW]** to go to **TBLSET***
  - *Set* $\text{TblStart} = 1$
  - *Set* $\Delta\text{Tbl} = 1$
  - *Set Indpnt: Auto and Depend: Auto*
- *Press **[2ND]** then **[GRAPH]** to go to the **TABLE***

61. What are the first seven terms shown in the column with the heading $u(n)$?

62. Use the scroll-down arrow to scroll to $n = 50$. What value is given for $u(n)$?

63. Press **[WINDOW]**. Set $n\text{Min} = 1$, $n\text{Max} = 5$, $x\text{Min} = 0$, $x\text{Max} = 6$, $y\text{Min} = -1$, and $y\text{Max} = 14$. Then press **[GRAPH]**. Graph the sequence as it appears on the graphing calculator.

For the following exercises, follow the steps given above to work with the arithmetic sequence $a_n = \frac{1}{2}n + 5$ using a graphing calculator.

64. What are the first seven terms shown in the column with the heading $u(n)$ in the TABLE feature?

65. Graph the sequence as it appears on the graphing calculator. Be sure to adjust the WINDOW settings as needed.

Extensions

66. Give two examples of arithmetic sequences whose 4th terms are 9.

67. Give two examples of arithmetic sequences whose 10th terms are 206.

68. Find the 5th term of the arithmetic sequence $\{9b, 5b, b, \ldots\}$.

69. Find the 11th term of the arithmetic sequence $\{3a - 2b, a + 2b, -a + 6b \ldots\}$.

70. At which term does the sequence $\{5.4, 14.5, 23.6, ...\}$ exceed 151?

71. At which term does the sequence $\left\{\frac{17}{3}, \frac{31}{6}, \frac{14}{3}, ...\right\}$ begin to have negative values?

72. For which terms does the finite arithmetic sequence $\left\{\frac{5}{2}, \frac{19}{8}, \frac{9}{4}, ..., \frac{1}{8}\right\}$ have integer values?

73. Write an arithmetic sequence using a recursive formula. Show the first 4 terms, and then find the 31st term.

74. Write an arithmetic sequence using an explicit formula. Show the first 4 terms, and then find the 28th term.

9.3 Geometric Sequences

Learning Objectives

In this section, you will:

- Find the common ratio for a geometric sequence.
- List the terms of a geometric sequence.
- Use a recursive formula for a geometric sequence.
- Use an explicit formula for a geometric sequence.

Many jobs offer an annual cost-of-living increase to keep salaries consistent with inflation. Suppose, for example, a recent college graduate finds a position as a sales manager earning an annual salary of $26,000. He is promised a 2% cost of living increase each year. His annual salary in any given year can be found by multiplying his salary from the previous year by 102%. His salary will be $26,520 after one year; $27,050.40 after two years; $27,591.41 after three years; and so on. When a salary increases by a constant rate each year, the salary grows by a constant factor. In this section, we will review sequences that grow in this way.

Finding Common Ratios

The yearly salary values described form a **geometric sequence** because they change by a constant factor each year. Each term of a geometric sequence increases or decreases by a constant factor called the **common ratio**. The sequence below is an example of a geometric sequence because each term increases by a constant factor of 6. Multiplying any term of the sequence by the common ratio 6 generates the subsequent term.

$$\{1, \xrightarrow{\times 6} 6, \xrightarrow{\times 6} 36, \xrightarrow{\times 6} 216, \xrightarrow{\times 6} 1{,}296, \ldots\}$$

Definition of a Geometric Sequence

A **geometric sequence** is one in which any term divided by the previous term is a constant. This constant is called the **common ratio** of the sequence. The common ratio can be found by dividing any term in the sequence by the previous term. If a_1 is the initial term of a geometric sequence and r is the common ratio, the sequence will be

$$\{a_1, \ a_1 r, \ a_1 r^2, \ a_1 r^3, \ \ldots\}.$$

HOW TO

Given a set of numbers, determine if they represent a geometric sequence.

1. Divide each term by the previous term.
2. Compare the quotients. If they are the same, a common ratio exists and the sequence is geometric.

EXAMPLE 1

Finding Common Ratios

Is the sequence geometric? If so, find the common ratio.

ⓐ $1, \ 2, \ 4, \ 8, \ 16, \ \ldots$ ⓑ $48, \ 12, \ 4, 2, \ \ldots$

Solution

Divide each term by the previous term to determine whether a common ratio exists.

ⓐ $\frac{2}{1} = 2 \qquad \frac{4}{2} = 2 \qquad \frac{8}{4} = 2 \qquad \frac{16}{8} = 2$

The sequence is geometric because there is a common ratio. The common ratio is 2.

ⓑ $\frac{12}{48} = \frac{1}{4} \qquad \frac{4}{12} = \frac{1}{3} \qquad \frac{2}{4} = \frac{1}{2}$

The sequence is not geometric because there is not a common ratio.

Analysis

The graph of each sequence is shown in Figure 1. It seems from the graphs that both (a) and (b) appear have the form of the graph of an exponential function in this viewing window. However, we know that (a) is geometric and so this interpretation holds, but (b) is not.

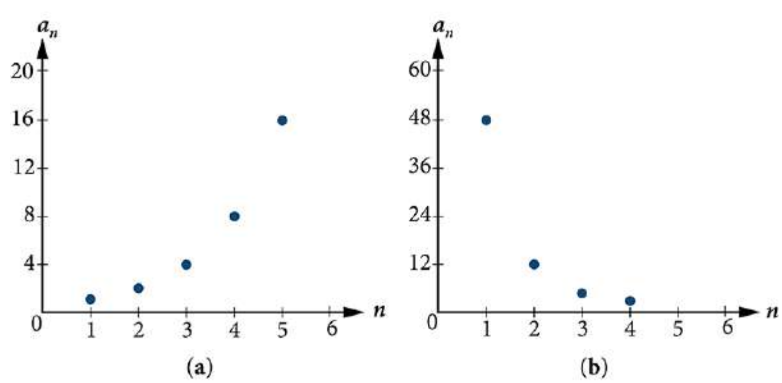


Figure 1

Q&A **If you are told that a sequence is geometric, do you have to divide every term by the previous term to find the common ratio?**

No. If you know that the sequence is geometric, you can choose any one term in the sequence and divide it by the previous term to find the common ratio.

TRY IT #1 Is the sequence geometric? If so, find the common ratio.

$$5, 10, 15, 20, \ldots$$

TRY IT #2 Is the sequence geometric? If so, find the common ratio.

$$100, 20, 4, \frac{4}{5}, \ldots$$

Writing Terms of Geometric Sequences

Now that we can identify a geometric sequence, we will learn how to find the terms of a geometric sequence if we are given the first term and the common ratio. The terms of a geometric sequence can be found by beginning with the first term and multiplying by the common ratio repeatedly. For instance, if the first term of a geometric sequence is $a_1 = -2$ and the common ratio is $r = 4,$ we can find subsequent terms by multiplying $-2 \cdot 4$ to get -8 then multiplying the result $-8 \cdot 4$ to get -32 and so on.

$$\begin{aligned} a_1 &= -2 \\ a_2 &= (-2 \cdot 4) = -8 \\ a_3 &= (-8 \cdot 4) = -32 \\ a_4 &= (-32 \cdot 4) = -128 \end{aligned}$$

The first four terms are $\{-2, -8, -32, -128\}$.

HOW TO

Given the first term and the common factor, find the first four terms of a geometric sequence.

1. Multiply the initial term, a_1, by the common ratio to find the next term, a_2.
2. Repeat the process, using $a_n = a_2$ to find a_3 and then a_3 to find $a_{4,}$ until all four terms have been identified.
3. Write the terms separated by commons within brackets.

EXAMPLE 2

Writing the Terms of a Geometric Sequence

List the first four terms of the geometric sequence with $a_1 = 5$ and $r = -2$.

Solution

Multiply a_1 by -2 to find a_2. Repeat the process, using a_2 to find a_3, and so on.

$$\begin{aligned} a_1 &= 5 \\ a_2 &= -2a_1 = -10 \\ a_3 &= -2a_2 = 20 \\ a_4 &= -2a_3 = -40 \end{aligned}$$

The first four terms are $\{5, -10, 20, -40\}$.

TRY IT #3 List the first five terms of the geometric sequence with $a_1 = 18$ and $r = \frac{1}{3}$.

Using Recursive Formulas for Geometric Sequences

A recursive formula allows us to find any term of a geometric sequence by using the previous term. Each term is the product of the common ratio and the previous term. For example, suppose the common ratio is 9. Then each term is nine times the previous term. As with any recursive formula, the initial term must be given.

Recursive Formula for a Geometric Sequence

The recursive formula for a geometric sequence with common ratio r and first term a_1 is

$$a_n = ra_{n-1}, n \geq 2$$

HOW TO

Given the first several terms of a geometric sequence, write its recursive formula.

1. State the initial term.
2. Find the common ratio by dividing any term by the preceding term.
3. Substitute the common ratio into the recursive formula for a geometric sequence.

EXAMPLE 3

Using Recursive Formulas for Geometric Sequences

Write a recursive formula for the following geometric sequence.

$$\{6, 9, 13.5, 20.25, \ldots\}$$

Solution

The first term is given as 6. The common ratio can be found by dividing the second term by the first term.

$$r = \frac{9}{6} = 1.5$$

Substitute the common ratio into the recursive formula for geometric sequences and define a_1.

$$\begin{aligned} a_n &= ra_{n-1} \\ a_n &= 1.5a_{n-1} \quad \text{for } n \geq 2 \\ a_1 &= 6 \end{aligned}$$

Analysis

The sequence of data points follows an exponential pattern. The common ratio is also the base of an exponential function as shown in Figure 2

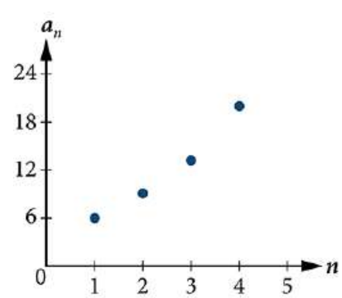


Figure 2

Q&A **Do we have to divide the second term by the first term to find the common ratio?**

No. We can divide any term in the sequence by the previous term. It is, however, most common to divide *the second term by the first term because it is often the easiest method of finding the common ratio.*

TRY IT #4 Write a recursive formula for the following geometric sequence.

$$\left\{2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots\right\}$$

Using Explicit Formulas for Geometric Sequences

Because a geometric sequence is an exponential function whose domain is the set of positive integers, and the common ratio is the base of the function, we can write explicit formulas that allow us to find particular terms.

$$a_n = a_1 r^{n-1}$$

Let's take a look at the sequence $\{18, 36, 72, 144, 288, \ldots\}$. This is a geometric sequence with a common ratio of 2 and an exponential function with a base of 2. An explicit formula for this sequence is

$$a_n = 18 \cdot 2^{n-1}$$

The graph of the sequence is shown in Figure 3.

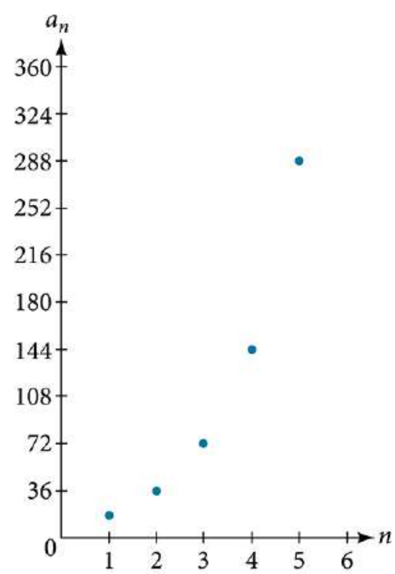


Figure 3

Explicit Formula for a Geometric Sequence

The n th term of a geometric sequence is given by the explicit formula:

$$a_n = a_1 r^{n-1}$$

EXAMPLE 4

Writing Terms of Geometric Sequences Using the Explicit Formula

Given a geometric sequence with $a_1 = 3$ and $a_4 = 24$, find a_2.

Solution

The sequence can be written in terms of the initial term and the common ratio r.

$$3, 3r, 3r^2, 3r^3, \ldots$$

Find the common ratio using the given fourth term.

$$\begin{aligned} a_n &= a_1 r^{n-1} & \\ a_4 &= 3r^3 & \text{Write the fourth term of sequence in terms of } \alpha_1 \text{ and } r \\ 24 &= 3r^3 & \text{Substitute 24 for } a_4 \\ 8 &= r^3 & \text{Divide} \\ r &= 2 & \text{Solve for the common ratio} \end{aligned}$$

Find the second term by multiplying the first term by the common ratio.

$$\begin{aligned} a_2 &= 2a_1 \\ &= 2(3) \\ &= 6 \end{aligned}$$

Analysis

The common ratio is multiplied by the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by multiplying the first term by the common ratio nine times or by multiplying by the common ratio raised to the ninth power.

TRY IT #5 Given a geometric sequence with $a_2 = 4$ and $a_3 = 32$, find a_6.

EXAMPLE 5

Writing an Explicit Formula for the n th Term of a Geometric Sequence

Write an explicit formula for the nth term of the following geometric sequence.

$$\{2, 10, 50, 250, \ldots\}$$

Solution

The first term is 2. The common ratio can be found by dividing the second term by the first term.

$$\frac{10}{2} = 5$$

The common ratio is 5. Substitute the common ratio and the first term of the sequence into the formula.

$$\begin{aligned} a_n &= a_1 r^{(n-1)} \\ a_n &= 2 \cdot 5^{n-1} \end{aligned}$$

The graph of this sequence in Figure 4 shows an exponential pattern.

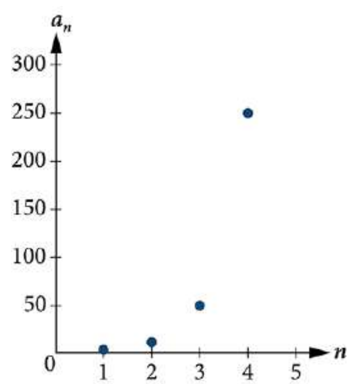


Figure 4

TRY IT #6 Write an explicit formula for the following geometric sequence.

$$\{-1, 3, -9, 27, \ldots\}$$

Solving Application Problems with Geometric Sequences

In real-world scenarios involving geometric sequences, we may need to use an initial term of a_0 instead of a_1. In these problems, we can alter the explicit formula slightly by using the following formula:

$$a_n = a_0 r^n$$

EXAMPLE 6

Solving Application Problems with Geometric Sequences

In 2013, the number of students in a small school is 284. It is estimated that the student population will increase by 4% each year.

ⓐ Write a formula for the student population. ⓑ Estimate the student population in 2020.

Solution

ⓐ

The situation can be modeled by a geometric sequence with an initial term of 284. The student population will be 104% of the prior year, so the common ratio is 1.04.

Let P be the student population and n be the number of years after 2013. Using the explicit formula for a geometric sequence we get

$$P_n = 284 \cdot 1.04^n$$

ⓑ

We can find the number of years since 2013 by subtracting.

$$2020 - 2013 = 7$$

We are looking for the population after 7 years. We can substitute 7 for n to estimate the population in 2020.

$$P_7 = 284 \cdot 1.04^7 \approx 374$$

The student population will be about 374 in 2020.

TRY IT #7 A business starts a new website. Initially the number of hits is 293 due to the curiosity factor. The business estimates the number of hits will increase by 2.6% per week.

ⓐ Write a formula for the number of hits. ⓑ Estimate the number of hits in 5 weeks.

MEDIA

Access these online resources for additional instruction and practice with geometric sequences.

[Geometric Sequences (http://openstax.org/l/geometricseq)](http://openstax.org/l/geometricseq)
[Determine the Type of Sequence (http://openstax.org/l/sequencetype)](http://openstax.org/l/sequencetype)
[Find the Formula for a Sequence (http://openstax.org/l/sequenceformula)](http://openstax.org/l/sequenceformula)

9.3 SECTION EXERCISES

Verbal

1. What is a geometric sequence?
2. How is the common ratio of a geometric sequence found?
3. What is the procedure for determining whether a sequence is geometric?
4. What is the difference between an arithmetic sequence and a geometric sequence?
5. Describe how exponential functions and geometric sequences are similar. How are they different?

Algebraic

For the following exercises, find the common ratio for the geometric sequence.

6. $1, 3, 9, 27, 81, \ldots$

7. $-0.125, 0.25, -0.5, 1, -2, \ldots$

8. $-2, -\frac{1}{2}, -\frac{1}{8}, -\frac{1}{32}, -\frac{1}{128}, \ldots$

For the following exercises, determine whether the sequence is geometric. If so, find the common ratio.

9. $-6, -12, -24, -48, -96, \ldots$

10. $5, 5.2, 5.4, 5.6, 5.8, \ldots$

11. $-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$

12. $6, 8, 11, 15, 20, \ldots$

13. $0.8, 4, 20, 100, 500, \ldots$

For the following exercises, write the first five terms of the geometric sequence, given the first term and common ratio.

14. $a_1 = 8, \quad r = 0.3$

15. $a_1 = 5, \quad r = \frac{1}{5}$

For the following exercises, write the first five terms of the geometric sequence, given any two terms.

16. $a_7 = 64, \quad a_{10} = 512$

17. $a_6 = 25, \quad a_8 = 6.25$

For the following exercises, find the specified term for the geometric sequence, given the first term and common ratio.

18. The first term is 2, and the common ratio is 3. Find the 5th term.

19. The first term is 16 and the common ratio is $-\frac{1}{3}$. Find the 4th term.

For the following exercises, find the specified term for the geometric sequence, given the first four terms.

20. $a_n = \{-1, 2, -4, 8, \ldots\}$. Find a_{12}.

21. $a_n = \left\{-2, \frac{2}{3}, -\frac{2}{9}, \frac{2}{27}, \ldots\right\}$. Find a_7.

For the following exercises, write the first five terms of the geometric sequence.

22. $a_1 = -486, \quad a_n = -\frac{1}{3} a_{n-1}$

23. $a_1 = 7, \quad a_n = 0.2a_{n-1}$

For the following exercises, write a recursive formula for each geometric sequence.

24. $a_n = \{-1, 5, -25, 125, \ldots\}$

25. $a_n = \{-32, -16, -8, -4, \ldots\}$

26. $a_n = \{14, 56, 224, 896, \ldots\}$

27. $a_n = \{10, -3, 0.9, -0.27, \ldots\}$

28. $a_n = \{0.61, 1.83, 5.49, 16.47, \ldots\}$

29. $a_n = \left\{\frac{3}{5}, \frac{1}{10}, \frac{1}{60}, \frac{1}{360}, \ldots\right\}$

30. $a_n = \left\{-2, \frac{4}{3}, -\frac{8}{9}, \frac{16}{27}, \ldots\right\}$

31. $a_n = \left\{\frac{1}{512}, -\frac{1}{128}, \frac{1}{32}, -\frac{1}{8}, \ldots\right\}$

For the following exercises, write the first five terms of the geometric sequence.

32. $a_n = -4 \cdot 5^{n-1}$

33. $a_n = 12 \cdot \left(-\frac{1}{2}\right)^{n-1}$

For the following exercises, write an explicit formula for each geometric sequence.

34. $a_n = \{-2, -4, -8, -16, ...\}$

35. $a_n = \{1, 3, 9, 27, ...\}$

36. $a_n = \{-4, -12, -36, -108, ...\}$

37. $a_n = \{0.8, -4, 20, -100, ...\}$

38. $a_n = \{-1.25, -5, -20, -80, ...\}$

39. $a_n = \left\{-1, -\frac{4}{5}, -\frac{16}{25}, -\frac{64}{125}, ...\right\}$

40. $a_n = \left\{2, \frac{1}{3}, \frac{1}{18}, \frac{1}{108}, ...\right\}$

41. $a_n = \left\{3, -1, \frac{1}{3}, -\frac{1}{9}, ...\right\}$

For the following exercises, find the specified term for the geometric sequence given.

42. Let $a_1 = 4$, $a_n = -3a_{n-1}$. Find a_8.

43. Let $a_n = -\left(-\frac{1}{3}\right)^{n-1}$. Find a_{12}.

For the following exercises, find the number of terms in the given finite geometric sequence.

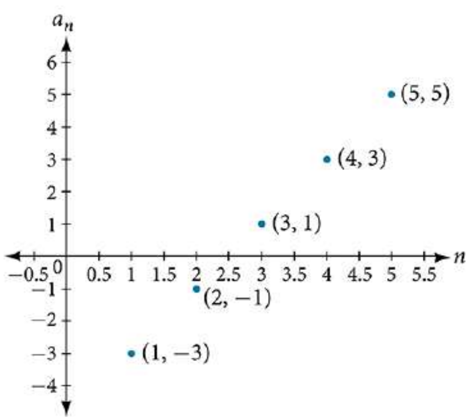
44. $a_n = \{-1, 3, -9, ..., 2187\}$

45. $a_n = \left\{2, 1, \frac{1}{2}, ..., \frac{1}{1024}\right\}$

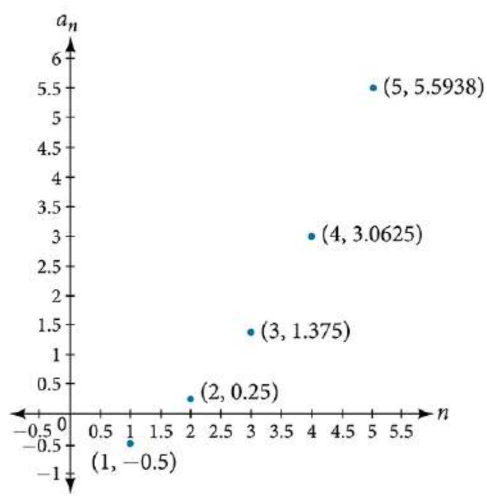
Graphical

For the following exercises, determine whether the graph shown represents a geometric sequence.

46.



47.



For the following exercises, use the information provided to graph the first five terms of the geometric sequence.

48. $a_1 = 1, \quad r = \frac{1}{2}$

49. $a_1 = 3, \quad a_n = 2a_{n-1}$

50. $a_n = 27 \cdot 0.3^{n-1}$

Extensions

51. Use recursive formulas to give two examples of geometric sequences whose 3rd terms are 200.

52. Use explicit formulas to give two examples of geometric sequences whose 7th terms are 1024.

53. Find the 5th term of the geometric sequence $\{b, 4b, 16b, ...\}$.

54. Find the 7th term of the geometric sequence $\{64a(-b), 32a(-3b), 16a(-9b), ...\}$.

55. At which term does the sequence $\{10, 12, 14.4, 17.28, \ \ ...\}$ exceed 100?

56. At which term does the sequence $\left\{\frac{1}{2187}, \frac{1}{729}, \frac{1}{243}, \frac{1}{81} \ \ ...\right\}$ begin to have integer values?

57. For which term does the geometric sequence $a_{\mathrm{n}} = -36\left(\frac{2}{3}\right)^{n-1}$ first have a non-integer value?

58. Use the recursive formula to write a geometric sequence whose common ratio is an integer. Show the first four terms, and then find the 10th term.

59. Use the explicit formula to write a geometric sequence whose common ratio is a decimal number between 0 and 1. Show the first 4 terms, and then find the 8th term.

60. Is it possible for a sequence to be both arithmetic and geometric? If so, give an example.

9.4 Series and Their Notations

Learning Objectives

In this section, you will:

- Use summation notation.
- Use the formula for the sum of the first n terms of an arithmetic series.
- Use the formula for the sum of the first n terms of a geometric series.
- Use the formula for the sum of an infinite geometric series.
- Solve annuity problems.

A parent decides to start a college fund for their daughter. They plan to invest \$50 in the fund each month. The fund pays 6% annual interest, compounded monthly. How much money will they have saved when their daughter is ready to start college in 6 years? In this section, we will learn how to answer this question. To do so, we need to consider the amount of money invested and the amount of interest earned.

Using Summation Notation

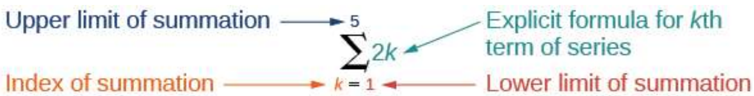
To find the total amount of money in the college fund and the sum of the amounts deposited, we need to add the amounts deposited each month and the amounts earned monthly. The sum of the terms of a sequence is called a **series**. Consider, for example, the following series.

$$3 + 7 + 11 + 15 + 19 + ...$$

The ***n*th partial sum** of a series is the sum of a finite number of consecutive terms beginning with the first term. The notation S_n represents the partial sum.

$$\begin{aligned} S_1 &= 3 \\ S_2 &= 3 + 7 = 10 \\ S_3 &= 3 + 7 + 11 = 21 \\ S_4 &= 3 + 7 + 11 + 15 = 36 \end{aligned}$$

Summation notation is used to represent series. Summation notation is often known as sigma notation because it uses the Greek capital letter sigma, Σ, to represent the sum. Summation notation includes an explicit formula and specifies the first and last terms in the series. An explicit formula for each term of the series is given to the right of the sigma. A variable called the **index of summation** is written below the sigma. The index of summation is set equal to the **lower limit of summation**, which is the number used to generate the first term in the series. The number above the sigma, called the **upper limit of summation**, is the number used to generate the last term in a series.



If we interpret the given notation, we see that it asks us to find the sum of the terms in the series $a_k = 2k$ for $k = 1$ through $k = 5$. We can begin by substituting the terms for k and listing out the terms of this series.

$$\begin{aligned} a_1 &= 2(1) = 2 \\ a_2 &= 2(2) = 4 \\ a_3 &= 2(3) = 6 \\ a_4 &= 2(4) = 8 \\ a_5 &= 2(5) = 10 \end{aligned}$$

We can find the sum of the series by adding the terms:

$$\sum_{k=1}^{5} 2k = 2 + 4 + 6 + 8 + 10 = 30$$

Summation Notation

The sum of the first n terms of a **series** can be expressed in **summation notation** as follows:

$$\sum_{k=1}^{n} a_k$$

This notation tells us to find the sum of a_k from $k = 1$ to $k = n$.

k is called the **index of summation**, 1 is the **lower limit of summation**, and n is the **upper limit of summation**.

Q&A **Does the lower limit of summation have to be 1?**

No. The lower limit of summation can be any number, but 1 is frequently used. We will look at examples with lower limits of summation other than 1.

HOW TO

Given summation notation for a series, evaluate the value.

1. Identify the lower limit of summation.
2. Identify the upper limit of summation.
3. Substitute each value of k from the lower limit to the upper limit into the formula.
4. Add to find the sum.

EXAMPLE 1

Using Summation Notation

Evaluate $\sum_{k=3}^{7} k^2$.

Solution

According to the notation, the lower limit of summation is 3 and the upper limit is 7. So we need to find the sum of k^2 from $k = 3$ to $k = 7$. We find the terms of the series by substituting $k = 3,4,5,6,$ and 7 into the function k^2. We add the terms to find the sum.

$$\begin{aligned}\sum_{k=3}^{7} k^2 &= 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \\ &= 9 + 16 + 25 + 36 + 49 \\ &= 135\end{aligned}$$

TRY IT #1 Evaluate $\sum_{k=2}^{5} (3k-1)$.

Using the Formula for Arithmetic Series

Just as we studied special types of sequences, we will look at special types of series. Recall that an arithmetic sequence is a sequence in which the difference between any two consecutive terms is the common difference, d. The sum of the terms of an arithmetic sequence is called an **arithmetic series**. We can write the sum of the first n terms of an arithmetic series as:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_n - d) + a_n.$$

We can also reverse the order of the terms and write the sum as

$$S_n = a_n + (a_n - d) + (a_n - 2d) + ... + (a_1 + d) + a_1.$$

If we add these two expressions for the sum of the first n terms of an arithmetic series, we can derive a formula for the sum of the first n terms of any arithmetic series.

$$\begin{array}{rl} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_n - d) + a_n \\ + \quad S_n &= a_n + (a_n - d) + (a_n - 2d) + ... + (a_1 + d) + a_1 \\ \hline 2S_n &= (a_1 + a_n) + (a_1 + a_n) + ... + (a_1 + a_n) \end{array}$$

Because there are n terms in the series, we can simplify this sum to

$$2S_n = n(a_1 + a_n).$$

We divide by 2 to find the formula for the sum of the first n terms of an arithmetic series.

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Formula for the Sum of the First n Terms of an Arithmetic Series

An **arithmetic series** is the sum of the terms of an arithmetic sequence. The formula for the sum of the first n terms of an arithmetic sequence is

$$S_n = \frac{n(a_1 + a_n)}{2}$$

HOW TO

Given terms of an arithmetic series, find the sum of the first n terms.

1. Identify a_1 and a_n.
2. Determine n.
3. Substitute values for a_1, a_n, and n into the formula $S_n = \frac{n(a_1+a_n)}{2}$.
4. Simplify to find S_n.

EXAMPLE 2

Finding the First *n* Terms of an Arithmetic Series

Find the sum of each arithmetic series.

ⓐ $5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32$ ⓑ $20 + 15 + 10 + \ldots + -50$ ⓒ $\sum_{k=1}^{12} 3k - 8$

Solution

ⓐ

We are given $a_1 = 5$ and $a_n = 32$.

Count the number of terms in the sequence to find $n = 10$.

Substitute values for $a_1, a_n,$ and n into the formula and simplify.

$$\begin{aligned} S_n &= \frac{n(a_1+a_n)}{2} \\ S_{10} &= \frac{10(5+32)}{2} = 185 \end{aligned}$$

ⓑ

We are given $a_1 = 20$ and $a_n = -50$.

Use the formula for the general term of an arithmetic sequence to find n.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ -50 &= 20 + (n-1)(-5) \\ -70 &= (n-1)(-5) \\ 14 &= n - 1 \\ 15 &= n \end{aligned}$$

Substitute values for $a_1, a_n,\ n$ into the formula and simplify.

$$\begin{aligned} S_n &= \frac{n(a_1+a_n)}{2} \\ S_{15} &= \frac{15(20-50)}{2} = -225 \end{aligned}$$

ⓒ

To find a_1, substitute $k = 1$ into the given explicit formula.

$$\begin{aligned} a_k &= 3k - 8 \\ a_1 &= 3(1) - 8 = -5 \end{aligned}$$

We are given that $n = 12$. To find a_{12}, substitute $k = 12$ into the given explicit formula.

$$\begin{aligned} a_k &= 3k - 8 \\ a_{12} &= 3(12) - 8 = 28 \end{aligned}$$

Substitute values for $a_1, a_n,$ and n into the formula and simplify.

$$\begin{aligned} S_n &= \frac{n(a_1+a_n)}{2} \\ S_{12} &= \frac{12(-5+28)}{2} = 138 \end{aligned}$$

Use the formula to find the sum of each arithmetic series.

TRY IT #2 $1.4 + 1.6 + 1.8 + 2.0 + 2.2 + 2.4 + 2.6 + 2.8 + 3.0 + 3.2 + 3.4$

TRY IT #3 $13 + 21 + 29 + \ \ldots + 69$

TRY IT #4 $\sum_{k=1}^{10} 5 - 6k$

EXAMPLE 3

Solving Application Problems with Arithmetic Series

On the Sunday after a minor surgery, a woman is able to walk a half-mile. Each Sunday, she walks an additional quarter-mile. After 8 weeks, what will be the total number of miles she has walked?

Solution

This problem can be modeled by an arithmetic series with $a_1 = \frac{1}{2}$ and $d = \frac{1}{4}$. We are looking for the total number of miles walked after 8 weeks, so we know that $n = 8$, and we are looking for S_8. To find a_8, we can use the explicit formula for an arithmetic sequence.

$$\begin{aligned} a_n &= a_1 + d(n-1) \\ a_8 &= \tfrac{1}{2} + \tfrac{1}{4}(8-1) = \tfrac{9}{4} \end{aligned}$$

We can now use the formula for arithmetic series.

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} \\ S_8 &= \frac{8(\frac{1}{2} + \frac{9}{4})}{2} = 11 \end{aligned}$$

She will have walked a total of 11 miles.

TRY IT #5 A man earns \$100 in the first week of June. Each week, he earns \$12.50 more than the previous week. After 12 weeks, how much has he earned?

Using the Formula for Geometric Series

Just as the sum of the terms of an arithmetic sequence is called an arithmetic series, the sum of the terms in a geometric sequence is called a **geometric series**. Recall that a geometric sequence is a sequence in which the ratio of any two consecutive terms is the common ratio, r. We can write the sum of the first n terms of a geometric series as

$$S_n = a_1 + ra_1 + r^2 a_1 + \ldots + r^{n-1} a_1.$$

Just as with arithmetic series, we can do some algebraic manipulation to derive a formula for the sum of the first n terms of a geometric series. We will begin by multiplying both sides of the equation by r.

$$rS_n = ra_1 + r^2 a_1 + r^3 a_1 + \ldots + r^n a_1$$

Next, we subtract this equation from the original equation.

$$\begin{array}{r} S_n = a_1 + ra_1 + r^2 a_1 + \ldots + r^{n-1} a_1 \\ \underline{-rS_n = -(ra_1 + r^2 a_1 + r^3 a_1 + \ldots + r^n a_1)} \\ (1-r)S_n = a_1 - r^n a_1 \end{array}$$

Notice that when we subtract, all but the first term of the top equation and the last term of the bottom equation cancel out. To obtain a formula for S_n, divide both sides by $(1 - r)$.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad r \neq 1$$

Formula for the Sum of the First n Terms of a Geometric Series

A **geometric series** is the sum of the terms in a geometric sequence. The formula for the sum of the first n terms of a geometric sequence is represented as

$$S_n = \frac{a_1(1-r^n)}{1-r} \quad r \neq 1$$

HOW TO

Given a geometric series, find the sum of the first *n* terms.

1. Identify $a_1, \ r, \ \text{and} \ n.$
2. Substitute values for $a_1, \ r$, and n into the formula $S_n = \frac{a_1(1-r^n)}{1-r}$.
3. Simplify to find S_n.

EXAMPLE 4

Finding the First *n* Terms of a Geometric Series

Use the formula to find the indicated partial sum of each geometric series.

ⓐ S_{11} for the series $8 + \text{-}4 + 2 + \ \ldots$ ⓑ $\sum_{k=1}^{6} 3 \cdot 2^k$

Solution

ⓐ

$a_1 = 8$, and we are given that $n = 11$.

We can find r by dividing the second term of the series by the first.

$$r = \frac{-4}{8} = -\frac{1}{2}$$

Substitute values for $a_1, \ r, \ \text{and} \ n$ into the formula and simplify.

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$

$$S_{11} = \frac{8\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)} \approx 5.336$$

ⓑ

Find a_1 by substituting $k = 1$ into the given explicit formula.

$$a_1 = 3 \cdot 2^1 = 6$$

We can see from the given explicit formula that $r = 2$. The upper limit of summation is 6, so $n = 6$.

Substitute values for $a_1, \ r$, and n into the formula, and simplify.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_6 = \frac{6(1-2^6)}{1-2} = 378$$

Use the formula to find the indicated partial sum of each geometric series.

TRY IT #6 S_{20} for the series $1{,}000 + 500 + 250 + \ \ldots$

TRY IT #7 $\sum_{k=1}^{8} 3^k$

EXAMPLE 5

Solving an Application Problem with a Geometric Series

At a new job, an employee's starting salary is $26,750. He receives a 1.6% annual raise. Find his total earnings at the end of 5 years.

Solution

The problem can be represented by a geometric series with $a_1 = 26,750$; $n = 5$; and $r = 1.016$. Substitute values for a_1, r, and n into the formula and simplify to find the total amount earned at the end of 5 years.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_5 = \frac{26{,}750(1-1.016^5)}{1-1.016} \approx 138{,}099.03$$

He will have earned a total of $138,099.03 by the end of 5 years.

TRY IT #8 At a new job, an employee's starting salary is $32,100. She receives a 2% annual raise. How much will she have earned by the end of 8 years?

Using the Formula for the Sum of an Infinite Geometric Series

Thus far, we have looked only at finite series. Sometimes, however, we are interested in the sum of the terms of an infinite sequence rather than the sum of only the first n terms. An **infinite series** is the sum of the terms of an infinite sequence. An example of an infinite series is $2 + 4 + 6 + 8 + \ldots$

This series can also be written in summation notation as $\sum_{k=1}^{\infty} 2k$, where the upper limit of summation is infinity. Because the terms are not tending to zero, the sum of the series increases without bound as we add more terms. Therefore, the sum of this infinite series is not defined. When the sum is not a real number, we say the series **diverges**.

Determining Whether the Sum of an Infinite Geometric Series is Defined

If the terms of an infinite geometric sequence approach 0, the sum of an infinite geometric series can be defined. The terms in this series approach 0:

$$1 + 0.2 + 0.04 + 0.008 + 0.0016 + \ldots$$

The common ratio $r = 0.2$. As n gets very large, the values of r^n get very small and approach 0. Each successive term affects the sum less than the preceding term. As each succeeding term gets closer to 0, the sum of the terms approaches a finite value. The terms of any infinite geometric series with $-1 < r < 1$ approach 0; the sum of a geometric series is defined when $-1 < r < 1$.

Determining Whether the Sum of an Infinite Geometric Series is Defined

The sum of an infinite series is defined if the series is geometric and $-1 < r < 1$.

HOW TO

Given the first several terms of an infinite series, determine if the sum of the series exists.

1. Find the ratio of the second term to the first term.
2. Find the ratio of the third term to the second term.
3. Continue this process to ensure the ratio of a term to the preceding term is constant throughout. If so, the series is geometric.
4. If a common ratio, r, was found in step 3, check to see if $-1 < r < 1$. If so, the sum is defined. If not, the sum is not defined.

EXAMPLE 6

Determining Whether the Sum of an Infinite Series is Defined

Determine whether the sum of each infinite series is defined.

ⓐ $12 + 8 + 4 + \ldots$ ⓑ $\frac{3}{4} + \frac{1}{2} + \frac{1}{3} + \ldots$ ⓒ $\sum_{k=1}^{\infty} 27 \cdot \left(\frac{1}{3}\right)^k$ ⓓ $\sum_{k=1}^{\infty} 5k$

Solution

ⓐ The ratio of the second term to the first is $\frac{2}{3}$, which is not the same as the ratio of the third term to the second, $\frac{1}{2}$. The series is not geometric.

ⓑ The ratio of the second term to the first is the same as the ratio of the third term to the second. The series is geometric with a common ratio of $\frac{2}{3}$. The sum of the infinite series is defined.

ⓒ The given formula is exponential with a base of $\frac{1}{3}$; the series is geometric with a common ratio of $\frac{1}{3}$. The sum of the infinite series is defined.

ⓓ The given formula is not exponential; the series is not geometric because the terms are increasing, and so cannot yield a finite sum.

Determine whether the sum of the infinite series is defined.

TRY IT #9 $\frac{1}{3} + \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \ldots$

TRY IT #10 $24 + (-12) + 6 + (-3) + \ldots$

TRY IT #11 $\sum_{k=1}^{\infty} 15 \cdot (-0.3)^k$

Finding Sums of Infinite Series

When the sum of an infinite geometric series exists, we can calculate the sum. The formula for the sum of an infinite series is related to the formula for the sum of the first n terms of a geometric series.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

We will examine an infinite series with $r = \frac{1}{2}$. What happens to r^n as n increases?

$$\begin{aligned} \left(\tfrac{1}{2}\right)^2 &= \tfrac{1}{4} \\ \left(\tfrac{1}{2}\right)^3 &= \tfrac{1}{8} \\ \left(\tfrac{1}{2}\right)^4 &= \tfrac{1}{16} \end{aligned}$$

The value of r^n decreases rapidly. What happens for greater values of n?

$$\begin{aligned} \left(\tfrac{1}{2}\right)^{10} &= \tfrac{1}{1{,}024} \\ \left(\tfrac{1}{2}\right)^{20} &= \tfrac{1}{1{,}048{,}576} \\ \left(\tfrac{1}{2}\right)^{30} &= \tfrac{1}{1{,}073{,}741{,}824} \end{aligned}$$

As n gets very large, r^n gets very small. We say that, as n increases without bound, r^n approaches 0. As r^n approaches 0, $1 - r^n$ approaches 1. When this happens, the numerator approaches a_1. This give us a formula for the sum of an infinite geometric series.

Formula for the Sum of an Infinite Geometric Series

The formula for the sum of an infinite geometric series with $-1 < r < 1$ is

$$S = \frac{a_1}{1 - r}$$

HOW TO

Given an infinite geometric series, find its sum.

1. Identify a_1 and r.
2. Confirm that $-1 < r < 1$.
3. Substitute values for a_1 and r into the formula, $S = \frac{a_1}{1-r}$.
4. Simplify to find S.

EXAMPLE 7

Finding the Sum of an Infinite Geometric Series

Find the sum, if it exists, for the following:

ⓐ $10 + 9 + 8 + 7 + \ldots$ ⓑ $248.6 + 99.44 + 39.776 + \ \ldots$ ⓒ $\sum_{k=1}^{\infty} 4{,}374 \cdot (-\frac{1}{3})^{k-1}$

ⓓ $\sum_{k=1}^{\infty} \frac{1}{9} \cdot (\frac{4}{3})^{k}$

Solution

ⓐ There is not a constant ratio; the series is not geometric.

ⓑ

There is a constant ratio; the series is geometric. $a_1 = 248.6$ and $r = \frac{99.44}{248.6} = 0.4$, so the sum exists. Substitute $a_1 = 248.6$ and $r = 0.4$ into the formula and simplify to find the sum:

$$S = \frac{a_1}{1-r}$$
$$S = \frac{248.6}{1-0.4} = 414.\overline{3}$$

ⓒ

The formula is exponential, so the series is geometric with $r =-\frac{1}{3}$. Find a_1 by substituting $k = 1$ into the given explicit formula:

$$a_1 = 4{,}374 \cdot (-\frac{1}{3})^{1-1} = 4{,}374$$

Substitute $a_1 = 4{,}374$ and $r = -\frac{1}{3}$ into the formula, and simplify to find the sum:

$$S = \frac{a_1}{1-r}$$
$$S = \frac{4{,}374}{1-(-\frac{1}{3})} = 3{,}280.5$$

ⓓ The formula is exponential, so the series is geometric, but $r > 1$. The sum does not exist.

EXAMPLE 8

Finding an Equivalent Fraction for a Repeating Decimal

Find an equivalent fraction for the repeating decimal $0.\overline{3}$

Solution

We notice the repeating decimal $0.\overline{3} = 0.333...$ so we can rewrite the repeating decimal as a sum of terms.

$$0.\overline{3} = 0.3 + 0.03 + 0.003 + ...$$

Looking for a pattern, we rewrite the sum, noticing that we see the first term multiplied to 0.1 in the second term, and the second term multiplied to 0.1 in the third term.

$$0.\overline{3} = 0.3 + (0.1)\underbrace{(0.3)}_{\text{First Term}} + (0.1)\underbrace{(0.1)(0.3)}_{\text{Second Term}}$$

Notice the pattern; we multiply each consecutive term by a common ratio of 0.1 starting with the first term of 0.3. So, substituting into our formula for an infinite geometric sum, we have

$$S_n = \frac{a_1}{1-r} = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}.$$

Find the sum, if it exists.

TRY IT #12 $2 + \frac{2}{3} + \frac{2}{9} + ...$

TRY IT #13 $\sum_{k=1}^{\infty} 0.76k + 1$

TRY IT #14 $\sum_{k=1}^{\infty} \left(-\frac{3}{8}\right)^k$

Solving Annuity Problems

At the beginning of the section, we looked at a problem in which a parent invested a set amount of money each month into a college fund for six years. An **annuity** is an investment in which the purchaser makes a sequence of periodic, equal payments. To find the amount of an annuity, we need to find the sum of all the payments and the interest earned. In the example, the parent invests $50 each month. This is the value of the initial deposit. The account paid 6% annual interest, compounded monthly. To find the interest rate per payment period, we need to divide the 6% annual percentage interest (APR) rate by 12. So the monthly interest rate is 0.5%. We can multiply the amount in the account each month by 100.5% to find the value of the account after interest has been added.

We can find the value of the annuity right after the last deposit by using a geometric series with $a_1 = 50$ and $r = 100.5\% = 1.005$. After the first deposit, the value of the annuity will be $50. Let us see if we can determine the amount in the college fund and the interest earned.

We can find the value of the annuity after n deposits using the formula for the sum of the first n terms of a geometric series. In 6 years, there are 72 months, so $n = 72$. We can substitute $a_1 = 50,\ r = 1.005,$ and $n = 72$ into the formula, and simplify to find the value of the annuity after 6 years.

$$S_{72} = \frac{50(1 - 1.005^{72})}{1 - 1.005} \approx 4{,}320.44$$

After the last deposit, the parent will have a total of $4,320.44 in the account. Notice, the parent made 72 payments of $50 each for a total of $72(50) = \$3{,}600$. This means that because of the annuity, the parent earned $720.44 interest in their college fund.

HOW TO

Given an initial deposit and an interest rate, find the value of an annuity.

1. Determine a_1, the value of the initial deposit.
2. Determine n, the number of deposits.
3. Determine r.
   a. Divide the annual interest rate by the number of times per year that interest is compounded.
   b. Add 1 to this amount to find r.
4. Substitute values for a_1, r, and n into the formula for the sum of the first n terms of a geometric series, $S_n = \frac{a_1(1-r^n)}{1-r}$.
5. Simplify to find S_n, the value of the annuity after n deposits.

EXAMPLE 9

Solving an Annuity Problem

A deposit of $100 is placed into a college fund at the beginning of every month for 10 years. The fund earns 9% annual interest, compounded monthly, and paid at the end of the month. How much is in the account right after the last deposit?

Solution

The value of the initial deposit is $100, so $a_1 = 100$. A total of 120 monthly deposits are made in the 10 years, so $n = 120$. To find r, divide the annual interest rate by 12 to find the monthly interest rate and add 1 to represent the new monthly deposit.

$$r = 1 + \frac{0.09}{12} = 1.0075$$

Substitute $a_1 = 100,\ r = 1.0075,\ \text{and}\ n = 120$ into the formula for the sum of the first n terms of a geometric series, and simplify to find the value of the annuity.

$$S_{120} = \frac{100(1 - 1.0075^{120})}{1 - 1.0075} \approx 19{,}351.43$$

So the account has $19,351.43 after the last deposit is made.

TRY IT #15 At the beginning of each month, $200 is deposited into a retirement fund. The fund earns 6% annual interest, compounded monthly, and paid into the account at the end of the month. How much is in the account if deposits are made for 10 years?

MEDIA

Access these online resources for additional instruction and practice with series.

[Arithmetic Series (http://openstax.org/l/arithmeticser)](http://openstax.org/l/arithmeticser)
[Geometric Series (http://openstax.org/l/geometricser)](http://openstax.org/l/geometricser)
[Summation Notation (http://openstax.org/l/sumnotation)](http://openstax.org/l/sumnotation)

9.4 SECTION EXERCISES

Verbal

1. What is an nth partial sum?

2. What is the difference between an arithmetic sequence and an arithmetic series?

3. What is a geometric series?

4. How is finding the sum of an infinite geometric series different from finding the nth partial sum?

5. What is an annuity?

Algebraic

For the following exercises, express each description of a sum using summation notation.

6. The sum of terms $m^2 + 3m$ from $m = 1$ to $m = 5$

7. The sum from of $n = 0$ to $n = 4$ of $5n$

8. The sum of $6k - 5$ from $k = -2$ to $k = 1$

9. The sum that results from adding the number 4 five times

For the following exercises, express each arithmetic sum using summation notation.

10. $5 + 10 + 15 + 20 + 25 + 30 + 35 + 40 + 45 + 50$

11. $10 + 18 + 26 + \ldots + 162$

12. $\frac{1}{2} + 1 + \frac{3}{2} + 2 + \ldots + 4$

For the following exercises, use the formula for the sum of the first n terms of each arithmetic sequence.

13. $\frac{3}{2} + 2 + \frac{5}{2} + 3 + \frac{7}{2}$

14. $19 + 25 + 31 + \ldots + 73$

15. $3.2 + 3.4 + 3.6 + \ldots + 5.6$

For the following exercises, express each geometric sum using summation notation.

16. $1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$

17. $8 + 4 + 2 + \ldots + 0.125$

18. $-\frac{1}{6} + \frac{1}{12} - \frac{1}{24} + \ldots + \frac{1}{768}$

For the following exercises, use the formula for the sum of the first n terms of each geometric sequence, and then state the indicated sum.

19. $9 + 3 + 1 + \frac{1}{3} + \frac{1}{9}$

20. $\sum_{n=1}^{9} 5 \cdot 2^{n-1}$

21. $\sum_{a=1}^{11} 64 \cdot 0.2^{a-1}$

For the following exercises, determine whether the infinite series has a sum. If so, write the formula for the sum. If not, state the reason.

22. $12 + 18 + 24 + 30 + \ldots$

23. $2 + 1.6 + 1.28 + 1.024 + \ldots$

24. $\sum_{m=1}^{\infty} 4^{m-1}$

25. $\sum_{\infty}^{k=1} -\left(-\frac{1}{2}\right)^{k-1}$

Graphical

For the following exercises, use the following scenario. Javier makes monthly deposits into a savings account. He opened the account with an initial deposit of $50. Each month thereafter he increased the previous deposit amount by $20.

26. Graph the arithmetic sequence showing one year of Javier's deposits.

27. Graph the arithmetic series showing the monthly sums of one year of Javier's deposits.

For the following exercises, use the geometric series $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$.

28. Graph the first 7 partial sums of the series.

29. What number does S_n seem to be approaching in the graph? Find the sum to explain why this makes sense.

Numeric

For the following exercises, find the indicated sum.

30. $\sum_{a=1}^{14} a$

31. $\sum_{n=1}^{6} n(n-2)$

32. $\sum_{k=1}^{17} k^2$

33. $\sum_{k=1}^{7} 2^k$

For the following exercises, use the formula for the sum of the first n terms of an arithmetic series to find the sum.

34. $-1.7 + -0.4 + 0.9 + 2.2 + 3.5 + 4.8$

35. $6 + \frac{15}{2} + 9 + \frac{21}{2} + 12 + \frac{27}{2} + 15$

36. $-1 + 3 + 7 + ... + 31$

37. $\sum_{k=1}^{11}\left(\frac{k}{2} - \frac{1}{2}\right)$

For the following exercises, use the formula for the sum of the first n terms of a geometric series to find the partial sum.

38. S_6 for the series $-2 - 10 - 50 - 250...$

39. S_7 for the series $0.4 - 2 + 10 - 50...$

40. $\sum_{k=1}^{9} 2^{k-1}$

41. $\sum_{n=1}^{10} -2 \cdot \left(\frac{1}{2}\right)^{n-1}$

For the following exercises, find the sum of the infinite geometric series.

42. $4 + 2 + 1 + \frac{1}{2} \ldots$

43. $-1 - \frac{1}{4} - \frac{1}{16} - \frac{1}{64} \ldots$

44. $\sum_{\infty}^{k=1} 3 \cdot \left(\frac{1}{4}\right)^{k-1}$

45. $\sum_{n=1}^{\infty} 4.6 \cdot 0.5^{n-1}$

For the following exercises, determine the value of the annuity for the indicated monthly deposit amount, the number of deposits, and the interest rate.

46. Deposit amount: $\$50$; total deposits: 60; interest rate: 5%, compounded monthly

47. Deposit amount: $\$150$; total deposits: 24; interest rate: 3%, compounded monthly

48. Deposit amount: $\$450$; total deposits: 60; interest rate: 4.5%, compounded quarterly

49. Deposit amount: $\$100$; total deposits: 120; interest rate: 10%, compounded semi-annually

Extensions

50. The sum of terms $50 - k^2$ from $k = x$ through 7 is 115. What is *x*?

51. Write an explicit formula for a_k such that $\sum_{k=0}^{6} a_k = 189$. Assume this is an arithmetic series.

52. Find the smallest value of *n* such that $\sum_{k=1}^{n} (3k - 5) > 100$.

53. How many terms must be added before the series $-1 - 3 - 5 - 7....$ has a sum less than -75?

54. Write $0.\overline{65}$ as an infinite geometric series using summation notation. Then use the formula for finding the sum of an infinite geometric series to convert $0.\overline{65}$ to a fraction.

55. The sum of an infinite geometric series is five times the value of the first term. What is the common ratio of the series?

56. To get the best loan rates available, the Coleman family want to save enough money to place 20% down on a $160,000 home. They plan to make monthly deposits of $125 in an investment account that offers 8.5% annual interest compounded semi-annually. Will the Colemans have enough for a 20% down payment after five years of saving? How much money will they have saved?

57. Karl has two years to save $\$10,000$ to buy a used car when he graduates. To the nearest dollar, what would his monthly deposits need to be if he invests in an account offering a 4.2% annual interest rate that compounds monthly?

Real-World Applications

58. Keisha devised a week-long study plan to prepare for finals. On the first day, she plans to study for 1 hour, and each successive day she will increase her study time by 30 minutes. How many hours will Keisha have studied after one week?

59. A boulder rolled down a mountain, traveling 6 feet in the first second. Each successive second, its distance increased by 8 feet. How far did the boulder travel after 10 seconds?

60. A scientist places 50 cells in a petri dish. Every hour, the population increases by 1.5%. What will the cell count be after 1 day?

61. A pendulum travels a distance of 3 feet on its first swing. On each successive swing, it travels $\frac{3}{4}$ the distance of the previous swing. What is the total distance traveled by the pendulum when it stops swinging?

62. Rachael deposits $1,500 into a retirement fund each year. The fund earns 8.2% annual interest, compounded monthly. If she opened her account when she was 19 years old, how much will she have by the time she is 55? How much of that amount will be interest earned?

9.5 Counting Principles

Learning Objectives

In this section, you will:

- Solve counting problems using the Addition Principle.
- Solve counting problems using the Multiplication Principle.
- Solve counting problems using permutations involving n distinct objects.
- Solve counting problems using combinations.
- Find the number of subsets of a given set.
- Solve counting problems using permutations involving n non-distinct objects.

A new company sells customizable cases for tablets and smartphones. Each case comes in a variety of colors and can be personalized for an additional fee with images or a monogram. A customer can choose not to personalize or could choose to have one, two, or three images or a monogram. The customer can choose the order of the images and the letters in the monogram. The company is working with an agency to develop a marketing campaign with a focus on the huge number of options they offer. Counting the possibilities is challenging!

We encounter a wide variety of counting problems every day. There is a branch of mathematics devoted to the study of counting problems such as this one. Other applications of counting include secure passwords, horse racing outcomes, and college scheduling choices. We will examine this type of mathematics in this section.

Using the Addition Principle

The company that sells customizable cases offers cases for tablets and smartphones. There are 3 supported tablet models and 5 supported smartphone models. The **Addition Principle** tells us that we can add the number of tablet options to the number of smartphone options to find the total number of options. By the Addition Principle, there are 8 total options, as we can see in Figure 1.

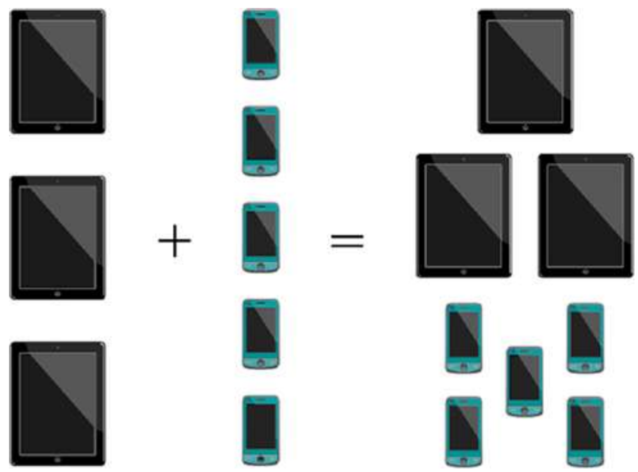


Figure 1

The Addition Principle

According to the **Addition Principle**, if one event can occur in m ways and a second event with no common outcomes can occur in n ways, then the first *or* second event can occur in $m + n$ ways.

EXAMPLE 1

Using the Addition Principle

There are 2 vegetarian entrée options and 5 meat entrée options on a dinner menu. What is the total number of entrée options?

Solution

We can add the number of vegetarian options to the number of meat options to find the total number of entrée options.

$$\begin{array}{ccccccccccccc} \text{Vegetarian} & + & \text{Vegetarian} & + & \text{Meat} & + & \text{Meat} & + & \text{Meat} & + & \text{Meat} & + & \text{Meat} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Option 1} & + & \text{Option 2} & + & \text{Option 3} & + & \text{Option 4} & + & \text{Option 5} & + & \text{Option 6} & + & \text{Option 7} \end{array}$$

There are 7 total options.

TRY IT #1 A student is shopping for a new computer. He is deciding among 3 desktop computers and 4 laptop computers. What is the total number of computer options?

Using the Multiplication Principle

The **Multiplication Principle** applies when we are making more than one selection. Suppose we are choosing an appetizer, an entrée, and a dessert. If there are 2 appetizer options, 3 entrée options, and 2 dessert options on a fixed-price dinner menu, there are a total of 12 possible choices of one each as shown in the tree diagram in Figure 2.

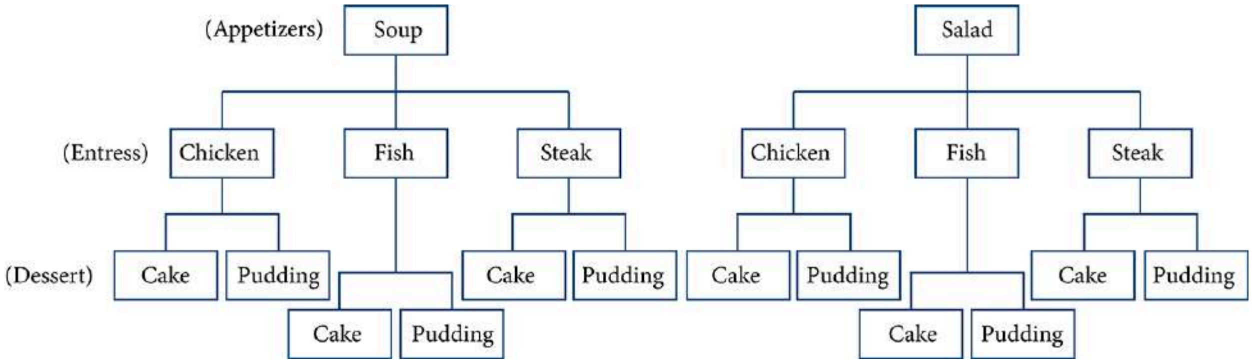


Figure 2

The possible choices are:

1. soup, chicken, cake
2. soup, chicken, pudding
3. soup, fish, cake
4. soup, fish, pudding
5. soup, steak, cake
6. soup, steak, pudding
7. salad, chicken, cake
8. salad, chicken, pudding
9. salad, fish, cake
10. salad, fish, pudding
11. salad, steak, cake
12. salad, steak, pudding

We can also find the total number of possible dinners by multiplying.

We could also conclude that there are 12 possible dinner choices simply by applying the Multiplication Principle.

$$\begin{array}{ccccccc} \#\text{ of appetizer options} & \times & \#\text{ of entree options} & \times & \#\text{ of dessert options} & & \\ 2 & \times & 3 & \times & 2 & = 12 \end{array}$$

The Multiplication Principle

According to the **Multiplication Principle**, if one event can occur in m ways and a second event can occur in n ways after the first event has occurred, then the two events can occur in $m \times n$ ways. This is also known as the **Fundamental Counting Principle**.

EXAMPLE 2

Using the Multiplication Principle

Diane packed 2 skirts, 4 blouses, and a sweater for her business trip. She will need to choose a skirt and a blouse for each outfit and decide whether to wear the sweater. Use the Multiplication Principle to find the total number of possible outfits.

Solution

To find the total number of outfits, find the product of the number of skirt options, the number of blouse options, and the number of sweater options.

$$\begin{array}{ccccccc} \begin{array}{c}\#\text{ of skirt}\\\text{options}\end{array} & \times & \begin{array}{c}\#\text{ of blouse}\\\text{options}\end{array} & \times & \begin{array}{c}\#\text{ of sweater}\\\text{options}\end{array} & & \\ 2 & \times & 4 & \times & 2 & = 16 \end{array}$$

There are 16 possible outfits.

TRY IT #2 A restaurant offers a breakfast special that includes a breakfast sandwich, a side dish, and a beverage. There are 3 types of breakfast sandwiches, 4 side dish options, and 5 beverage choices. Find the total number of possible breakfast specials.

Finding the Number of Permutations of *n* Distinct Objects

The Multiplication Principle can be used to solve a variety of problem types. One type of problem involves placing objects in order. We arrange letters into words and digits into numbers, line up for photographs, decorate rooms, and more. An ordering of objects is called a **permutation**.

Finding the Number of Permutations of *n* Distinct Objects Using the Multiplication Principle

To solve permutation problems, it is often helpful to draw line segments for each option. That enables us to determine the number of each option so we can multiply. For instance, suppose we have four paintings, and we want to find the number of ways we can hang three of the paintings in order on the wall. We can draw three lines to represent the three places on the wall.

$$\underline{\qquad} \times \underline{\qquad} \times \underline{\qquad}$$

There are four options for the first place, so we write a 4 on the first line.

$$\underline{\quad 4 \quad} \times \underline{\qquad} \times \underline{\qquad}$$

After the first place has been filled, there are three options for the second place so we write a 3 on the second line.

$$\underline{\quad 4 \quad} \times \underline{\quad 3 \quad} \times \underline{\qquad}$$

After the second place has been filled, there are two options for the third place so we write a 2 on the third line. Finally, we find the product.

$$\underline{\quad 4 \quad} \times \underline{\quad 3 \quad} \times \underline{\quad 2 \quad} = 24$$

There are 24 possible permutations of the paintings.

HOW TO

Given n distinct options, determine how many permutations there are.

1. Determine how many options there are for the first situation.
2. Determine how many options are left for the second situation.
3. Continue until all of the spots are filled.
4. Multiply the numbers together.

EXAMPLE 3

Finding the Number of Permutations Using the Multiplication Principle

At a swimming competition, nine swimmers compete in a race.

ⓐ How many ways can they place first, second, and third?

ⓑ How many ways can they place first, second, and third if a swimmer named Ariel wins first place? (Assume there is only one contestant named Ariel.)

ⓒ How many ways can all nine swimmers line up for a photo?

Solution

ⓐ Draw lines for each place.

options for 1st place × options for 2nd place × options for 3rd place

There are 9 options for first place. Once someone has won first place, there are 8 remaining options for second place. Once first and second place have been won, there are 7 remaining options for third place.

9 × 8 × 7 = 504

Multiply to find that there are 504 ways for the swimmers to place.

ⓑ Draw lines for describing each place.

options for 1st place × options for 2nd place × options for 3rd place

We know Ariel must win first place, so there is only 1 option for first place. There are 8 remaining options for second place, and then 7 remaining options for third place.

1 × 8 × 7 = 56

Multiply to find that there are 56 ways for the swimmers to place if Ariel wins first.

ⓒ

Draw lines for describing each place in the photo.

___ × ___ × ___ × ___ × ___ × ___ × ___ × ___ × ___

There are 9 choices for the first spot, then 8 for the second, 7 for the third, 6 for the fourth, and so on until only 1 person remains for the last spot.

9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 362,880

There are 362,880 possible permutations for the swimmers to line up.

Analysis

Note that in part c, we found there were 9! ways for 9 people to line up. The number of permutations of n distinct objects can always be found by $n!$.

A family of five is having portraits taken. Use the Multiplication Principle to find the following.

TRY IT #3 How many ways can the family line up for the portrait?

TRY IT #4 How many ways can the photographer line up 3 family members?

TRY IT #5 How many ways can the family line up for the portrait if the parents are required to stand on each end?

Finding the Number of Permutations of n Distinct Objects Using a Formula

For some permutation problems, it is inconvenient to use the Multiplication Principle because there are so many numbers to multiply. Fortunately, we can solve these problems using a formula. Before we learn the formula, let's look at two common notations for permutations. If we have a set of n objects and we want to choose r objects from the set in order, we write $P(n, r)$. Another way to write this is nP_r, a notation commonly seen on computers and calculators. To calculate $P(n, r)$, we begin by finding $n!$, the number of ways to line up all n objects. We then divide by $(n-r)!$ to cancel out the $(n-r)$ items that we do not wish to line up.

Let's see how this works with a simple example. Imagine a club of six people. They need to elect a president, a vice president, and a treasurer. Six people can be elected president, any one of the five remaining people can be elected vice president, and any of the remaining four people could be elected treasurer. The number of ways this may be done is $6 \times 5 \times 4 = 120$. Using factorials, we get the same result.

$$\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 6 \cdot 5 \cdot 4 = 120$$

There are 120 ways to select 3 officers in order from a club with 6 members. We refer to this as a permutation of 6 taken 3 at a time. The general formula is as follows.

$$P(n, r) = \frac{n!}{(n-r)!}$$

Note that the formula stills works if we are choosing all n objects and placing them in order. In that case we would be dividing by $(n-n)!$ or $0!$, which we said earlier is equal to 1. So the number of permutations of n objects taken n at a time is $\frac{n!}{1}$ or just $n!$.

Formula for Permutations of n Distinct Objects

Given n distinct objects, the number of ways to select r objects from the set in order is

$$P(n, r) = \frac{n!}{(n-r)!}$$

HOW TO

Given a word problem, evaluate the possible permutations.

1. Identify n from the given information.
2. Identify r from the given information.
3. Replace n and r in the formula with the given values.
4. Evaluate.

EXAMPLE 4

Finding the Number of Permutations Using the Formula

A professor is creating an exam of 9 questions from a test bank of 12 questions. How many ways can she select and arrange the questions?

Solution

Substitute $n = 12$ and $r = 9$ into the permutation formula and simplify.

$$P(n, r) = \frac{n!}{(n-r)!}$$
$$P(12, 9) = \frac{12!}{(12-9)!} = \frac{12!}{3!} = 79{,}833{,}600$$

There are 79,833,600 possible permutations of exam questions!

Analysis

We can also use a calculator to find permutations. For this problem, we would enter 12, press the ${}_nP_r$ function, enter 9, and then press the equal sign. The ${}_nP_r$ function may be located under the MATH menu with probability commands.

Q&A **Could we have solved Example 4 using the Multiplication Principle?**

Yes. We could have multiplied $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ *to find the same answer.*

A play has a cast of 7 actors preparing to make their curtain call. Use the permutation formula to find the following.

TRY IT #6 How many ways can the 7 actors line up?

TRY IT #7 How many ways can 5 of the 7 actors be chosen to line up?

Find the Number of Combinations Using the Formula

So far, we have looked at problems asking us to put objects in order. There are many problems in which we want to select a few objects from a group of objects, but we do not care about the order. When we are selecting objects and the order does not matter, we are dealing with **combinations**. A selection of r objects from a set of n objects where the order does not matter can be written as $C(n, r)$. Just as with permutations, $\mathrm{C}(n, r)$ can also be written as $_nC_r$. In this case, the general formula is as follows.

$$\mathrm{C}(n, r) = \frac{n!}{r!(n-r)!}$$

An earlier problem considered choosing 3 of 4 possible paintings to hang on a wall. We found that there were 24 ways to select 3 of the 4 paintings in order. But what if we did not care about the order? We would expect a smaller number because selecting paintings 1, 2, 3 would be the same as selecting paintings 2, 3, 1. To find the number of ways to select 3 of the 4 paintings, disregarding the order of the paintings, divide the number of permutations by the number of ways to order 3 paintings. There are $3! = 3 \cdot 2 \cdot 1 = 6$ ways to order 3 paintings. There are $\frac{24}{6}$, or 4 ways to select 3 of the 4 paintings. This number makes sense because every time we are selecting 3 paintings, we are *not* selecting 1 painting. There are 4 paintings we could choose *not* to select, so there are 4 ways to select 3 of the 4 paintings.

Formula for Combinations of n Distinct Objects

Given n distinct objects, the number of ways to select r objects from the set is

$$\mathrm{C}(n, r) = \frac{n!}{r!(n-r)!}$$

HOW TO

Given a number of options, determine the possible number of combinations.

1. Identify n from the given information.
2. Identify r from the given information.
3. Replace n and r in the formula with the given values.
4. Evaluate.

EXAMPLE 5

Finding the Number of Combinations Using the Formula

A fast food restaurant offers five side dish options. Your meal comes with two side dishes.

ⓐ How many ways can you select your side dishes?

ⓑ How many ways can you select 3 side dishes?

Solution

ⓐ We want to choose 2 side dishes from 5 options.

$$\mathrm{C}(5, 2) = \frac{5!}{2!(5-2)!} = 10$$

ⓑ We want to choose 3 side dishes from 5 options.

$$\mathrm{C}(5, 3) = \frac{5!}{3!(5-3)!} = 10$$

Analysis

We can also use a graphing calculator to find combinations. Enter 5, then press $_nC_r$, enter 3, and then press the equal sign. The $_nC_r$, function may be located under the MATH menu with probability commands.

Q&A **Is it a coincidence that parts (a) and (b) in Example 5 have the same answers?**

*No. When we choose r objects from n objects, we are **not** choosing $(n-r)$ objects. Therefore, $C(n, r) = C(n, n-r)$.*

TRY IT #8 An ice cream shop offers 10 flavors of ice cream. How many ways are there to choose 3 flavors for a banana split?

Finding the Number of Subsets of a Set

We have looked only at combination problems in which we chose exactly r objects. In some problems, we want to consider choosing every possible number of objects. Consider, for example, a pizza restaurant that offers 5 toppings. Any number of toppings can be ordered. How many different pizzas are possible?

To answer this question, we need to consider pizzas with any number of toppings. There is $C(5,0)=1$ way to order a pizza with no toppings. There are $C(5,1)=5$ ways to order a pizza with exactly one topping. If we continue this process, we get

$$C(5,0)+C(5,1)+C(5,2)+C(5,3)+C(5,4)+C(5,5)=32$$

There are 32 possible pizzas. This result is equal to 2^5.

We are presented with a sequence of choices. For each of the n objects we have two choices: include it in the subset or not. So for the whole subset we have made n choices, each with two options. So there are a total of $2\cdot 2\cdot 2\cdot \ldots \cdot 2$ possible resulting subsets, all the way from the empty subset, which we obtain when we say “no” each time, to the original set itself, which we obtain when we say “yes” each time.

Formula for the Number of Subsets of a Set

A set containing n distinct objects has 2^n subsets.

EXAMPLE 6

Finding the Number of Subsets of a Set

A restaurant offers butter, cheese, chives, and sour cream as toppings for a baked potato. How many different ways are there to order a potato?

Solution

We are looking for the number of subsets of a set with 4 objects. Substitute $n=4$ into the formula.

$$\begin{aligned}2^n &= 2^4\\ &= 16\end{aligned}$$

There are 16 possible ways to order a potato.

TRY IT #9 A sundae bar at a wedding has 6 toppings to choose from. Any number of toppings can be chosen. How many different sundaes are possible?

Finding the Number of Permutations of n Non-Distinct Objects

We have studied permutations where all of the objects involved were distinct. What happens if some of the objects are indistinguishable? For example, suppose there is a sheet of 12 stickers. If all of the stickers were distinct, there would be $12!$ ways to order the stickers. However, 4 of the stickers are identical stars, and 3 are identical moons. Because all of the objects are not distinct, many of the $12!$ permutations we counted are duplicates. The general formula for this situation is as follows.

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

In this example, we need to divide by the number of ways to order the 4 stars and the ways to order the 3 moons to find the number of unique permutations of the stickers. There are $4!$ ways to order the stars and $3!$ ways to order the moon.

$$\frac{12!}{4!3!}=3{,}326{,}400$$

There are 3,326,400 ways to order the sheet of stickers.

Formula for Finding the Number of Permutations of n Non-Distinct Objects

If there are n elements in a set and r_1 are alike, r_2 are alike, r_3 are alike, and so on through r_k, the number of permutations can be found by

$$\frac{n!}{r_1!r_2!\ldots r_k!}$$

EXAMPLE 7

Finding the Number of Permutations of n Non-Distinct Objects

Find the number of rearrangements of the letters in the word DISTINCT.

Solution

There are 8 letters. Both I and T are repeated 2 times. Substitute $n = 8,\ r_1 = 2,$ and $r_2 = 2$ into the formula.

$$\frac{8!}{2!2!} = 10{,}080$$

There are 10,080 arrangements.

TRY IT #10 Find the number of rearrangements of the letters in the word CARRIER.

MEDIA

Access these online resources for additional instruction and practice with combinations and permutations.

[Combinations (http://openstax.org/l/combinations)](http://openstax.org/l/combinations)
[Permutations (http://openstax.org/l/permutations)](http://openstax.org/l/permutations)

9.5 SECTION EXERCISES

Verbal

For the following exercises, assume that there are n ways an event A can happen, m ways an event B can happen, and that A and B are non-overlapping.

1. Use the Addition Principle of counting to explain how many ways event A or B can occur.

2. Use the Multiplication Principle of counting to explain how many ways event A and B can occur.

Answer the following questions.

3. When given two separate events, how do we know whether to apply the Addition Principle or the Multiplication Principle when calculating possible outcomes? What conjunctions may help to determine which operations to use?

4. Describe how the permutation of n objects differs from the permutation of choosing r objects from a set of n objects. Include how each is calculated.

5. What is the term for the arrangement that selects r objects from a set of n objects when the order of the r objects is not important? What is the formula for calculating the number of possible outcomes for this type of arrangement?

Numeric

For the following exercises, determine whether to use the Addition Principle or the Multiplication Principle. Then perform the calculations.

6. Let the set $A = \{-5, -3, -1, 2, 3, 4, 5, 6\}$. How many ways are there to choose a negative or an even number from A?

7. Let the set $B = \{-23, -16, -7, -2, 20, 36, 48, 72\}$. How many ways are there to choose a positive or an odd number from A?

8. How many ways are there to pick a red ace or a club from a standard card playing deck?

9. How many ways are there to pick a paint color from 5 shades of green, 4 shades of blue, or 7 shades of yellow?

10. How many outcomes are possible from tossing a pair of coins?

11. How many outcomes are possible from tossing a coin and rolling a 6-sided die?

12. How many two-letter strings—the first letter from A and the second letter from B— can be formed from the sets $A = \{b, c, d\}$ and $B = \{a, e, i, o, u\}$?

13. How many ways are there to construct a string of 3 digits if numbers can be repeated?

14. How many ways are there to construct a string of 3 digits if numbers cannot be repeated?

For the following exercises, compute the value of the expression.

15. $P(5, 2)$

16. $P(8, 4)$

17. $P(3, 3)$

18. $P(9, 6)$

19. $P(11, 5)$

20. $C(8, 5)$

21. $C(12, 4)$

22. $C(26, 3)$

23. $C(7, 6)$

24. $C(10, 3)$

For the following exercises, find the number of subsets in each given set.

25. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

26. $\{a, b, c, \ldots, z\}$

27. A set containing 5 distinct numbers, 4 distinct letters, and 3 distinct symbols

28. The set of even numbers from 2 to 28

29. The set of two-digit numbers between 1 and 100 containing the digit 0

For the following exercises, find the distinct number of arrangements.

30. The letters in the word "juggernaut"

31. The letters in the word "academia"

32. The letters in the word "academia" that begin and end in "a"

33. The symbols in the string #,#,#,@,@,$,$,$,%,%,%,%

34. The symbols in the string #,#,#,@,@,$,$,$,%,%,%,% that begin and end with "%"

Extensions

35. The set, S consists of $900{,}000{,}000$ whole numbers, each being the same number of digits long. How many digits long is a number from S? (*Hint:* use the fact that a whole number cannot start with the digit 0.)

36. The number of 5-element subsets from a set containing n elements is equal to the number of 6-element subsets from the same set. What is the value of n? (*Hint:* the order in which the elements for the subsets are chosen is not important.)

37. Can $C(n, r)$ ever equal $P(n, r)$? Explain.

38. Suppose a set A has 2,048 subsets. How many distinct objects are contained in A?

39. How many arrangements can be made from the letters of the word "mountains" if all the vowels must form a string?

Real-World Applications

40. A family consisting of 2 parents and 3 children is to pose for a picture with 2 family members in the front and 3 in the back.

ⓐ How many arrangements are possible with no restrictions?
ⓑ How many arrangements are possible if the parents must sit in the front?
ⓒ How many arrangements are possible if the parents must be next to each other?

41. A cell phone company offers 6 different voice packages and 8 different data packages. Of those, 3 packages include both voice and data. How many ways are there to choose either voice or data, but not both?

42. In horse racing, a "trifecta" occurs when a bettor wins by selecting the first three finishers in the exact order (1st place, 2nd place, and 3rd place). How many different trifectas are possible if there are 14 horses in a race?

43. A wholesale T-shirt company offers sizes small, medium, large, and extra-large in organic or non-organic cotton and colors white, black, gray, blue, and red. How many different T-shirts are there to choose from?

44. Hector wants to place billboard advertisements throughout the county for his new business. How many ways can Hector choose 15 neighborhoods to advertise in if there are 30 neighborhoods in the county?

45. An art store has 4 brands of paint pens in 12 different colors and 3 types of ink. How many paint pens are there to choose from?

46. How many ways can a committee of 3 freshmen and 4 juniors be formed from a group of 8 freshmen and 11 juniors?

47. How many ways can a baseball coach arrange the order of 9 batters if there are 15 players on the team?

48. A conductor needs 5 cellists and 5 violinists to play at a diplomatic event. To do this, he ranks the orchestra's 10 cellists and 16 violinists in order of musical proficiency. What is the ratio of the total cellist rankings possible to the total violinist rankings possible?

49. A motorcycle shop has 10 choppers, 6 bobbers, and 5 café racers—different types of vintage motorcycles. How many ways can the shop choose 3 choppers, 5 bobbers, and 2 café racers for a weekend showcase?

50. A skateboard shop stocks 10 types of board decks, 3 types of trucks, and 4 types of wheels. How many different skateboards can be constructed?

51. Just-For-Kicks Sneaker Company offers an online customizing service. How many ways are there to design a custom pair of Just-For-Kicks sneakers if a customer can choose from a basic shoe up to 11 customizable options?

52. A car wash offers the following optional services to the basic wash: clear coat wax, triple foam polish, undercarriage wash, rust inhibitor, wheel brightener, air freshener, and interior shampoo. How many washes are possible if any number of options can be added to the basic wash?

53. Suni bought 20 plants to arrange along the border of her garden. How many distinct arrangements can she make if the plants are comprised of 6 tulips, 6 roses, and 8 daisies?

54. How many unique ways can a string of Christmas lights be arranged from 9 red, 10 green, 6 white, and 12 gold color bulbs?

9.6 Binomial Theorem

Learning Objectives

In this section, you will:

- Apply the Binomial Theorem.

A polynomial with two terms is called a binomial. We have already learned to multiply binomials and to raise binomials to powers, but raising a binomial to a high power can be tedious and time-consuming. In this section, we will discuss a shortcut that will allow us to find $(x + y)^n$ without multiplying the binomial by itself n times.

Identifying Binomial Coefficients

In Counting Principles, we studied combinations. In the shortcut to finding $(x+y)^n$, we will need to use combinations to find the coefficients that will appear in the expansion of the binomial. In this case, we use the notation $\binom{n}{r}$ instead of $C(n,r)$, but it can be calculated in the same way. So

$$\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!}$$

The combination $\binom{n}{r}$ is called a **binomial coefficient**. An example of a binomial coefficient is $\binom{5}{2} = C(5,2) = 10.$

Binomial Coefficients

If n and r are integers greater than or equal to 0 with $n \geq r$, then the **binomial coefficient** is

$$\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!}$$

Q&A **Is a binomial coefficient always a whole number?**

Yes. Just as the number of combinations must always be a whole number, a binomial coefficient will always be a whole number.

EXAMPLE 1

Finding Binomial Coefficients

Find each binomial coefficient.

ⓐ $\binom{5}{3}$ ⓑ $\binom{9}{2}$ ⓒ $\binom{9}{7}$

Solution

Use the formula to calculate each binomial coefficient. You can also use the nC_r function on your calculator.

$$\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!}$$

ⓐ $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot 3!}{3!2!} = 10$ ⓑ $\binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{9 \cdot 8 \cdot 7!}{2!7!} = 36$ ⓒ $\binom{9}{7} = \frac{9!}{7!(9-7)!} = \frac{9 \cdot 8 \cdot 7!}{7!2!} = 36$

Analysis

Notice that we obtained the same result for parts (b) and (c). If you look closely at the solution for these two parts, you will see that you end up with the same two factorials in the denominator, but the order is reversed, just as with combinations.

$$\binom{n}{r} = \binom{n}{n-r}$$

TRY IT #1 Find each binomial coefficient.

ⓐ $\binom{7}{3}$ ⓑ $\binom{11}{4}$

Using the Binomial Theorem

When we expand $(x+y)^n$ by multiplying, the result is called a **binomial expansion**, and it includes binomial coefficients. If we wanted to expand $(x+y)^{52}$, we might multiply $(x+y)$ by itself fifty-two times. This could take hours! If we examine some simple binomial expansions, we can find patterns that will lead us to a shortcut for finding more complicated binomial expansions.

$$\begin{aligned}(x+y)^2 &= x^2+2xy+y^2\\(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\end{aligned}$$

First, let’s examine the exponents. With each successive term, the exponent for x decreases and the exponent for y increases. The sum of the two exponents is n for each term.

Next, let’s examine the coefficients. Notice that the coefficients increase and then decrease in a symmetrical pattern. The coefficients follow a pattern:

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}.$$

These patterns lead us to the **Binomial Theorem**, which can be used to expand any binomial.

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\\ &= x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\ldots+\binom{n}{n-1}xy^{n-1}+y^n\end{aligned}$$

Another way to see the coefficients is to examine the expansion of a binomial in general form, $x+y$, to successive powers 1, 2, 3, and 4.

$$\begin{aligned}(x+y)^1 &= x+y\\(x+y)^2 &= x^2+2xy+y^2\\(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\end{aligned}$$

Can you guess the next expansion for the binomial $(x+y)^5$?

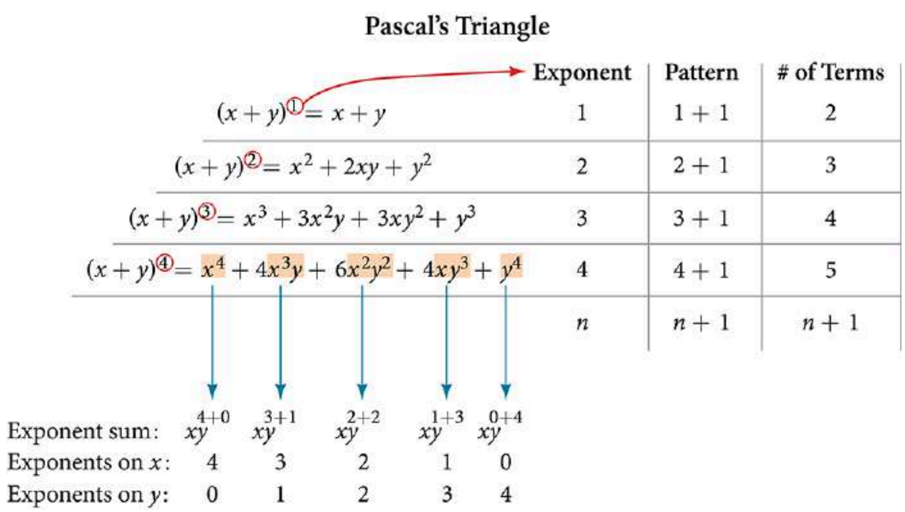


Figure 1

See Figure 1, which illustrates the following:

- There are $n+1$ terms in the expansion of $(x+y)^n$.
- The degree (or sum of the exponents) for each term is n.
- The powers on x begin with n and decrease to 0.
- The powers on y begin with 0 and increase to n.
- The coefficients are symmetric.

To determine the expansion on $(x+y)^5$, we see $n=5$, thus, there will be 5+1 = 6 terms. Each term has a combined degree of 5. In descending order for powers of x, the pattern is as follows:

- Introduce x^5, and then for each successive term reduce the exponent on x by 1 until $x^0=1$ is reached.
- Introduce $y^0=1$, and then increase the exponent on y by 1 until y^5 is reached.

$$x^5, \quad x^4y, \quad x^3y^2, \quad x^2y^3, \quad xy^4, \quad y^5$$

The next expansion would be

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

But where do those coefficients come from? The binomial coefficients are symmetric. We can see these coefficients in an array known as Pascal's Triangle, shown in Figure 2. Pascal didn't invent the triangle. The underlying principles had been developed and written about for over 1500 years, first by the Indian mathematician (and poet) Pingala in the second century BCE. Others throughout Asia and Europe worked with the concepts throughout, and the triangle was first published in its graphical form by Omar Khayyam, an Iranian mathematician and astronomer, for whom the triangle is named in Iran. French mathematician Blaise Pascal repopularized it when he republished it and used it to solve a number of probability problems.

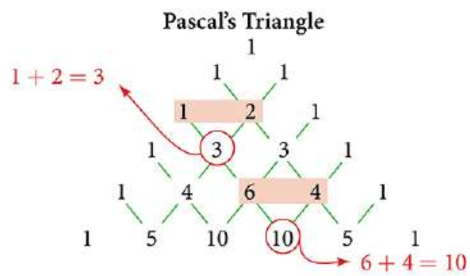


Figure 2

To generate Pascal's Triangle, we start by writing a 1. In the row below, row 2, we write two 1's. In the 3rd row, flank the ends of the rows with 1's, and add $1+1$ to find the middle number, 2. In the nth row, flank the ends of the row with 1's. Each element in the triangle is the sum of the two elements immediately above it.

To see the connection between Pascal's Triangle and binomial coefficients, let us revisit the expansion of the binomials in general form.

$$1 \longrightarrow (x+y)^0 = 1$$
$$1 \quad 1 \longrightarrow (x+y)^1 = x + y$$
$$1 \quad 2 \quad 1 \longrightarrow (x+y)^2 = x^2 + 2xy + y^2$$
$$1 \quad 3 \quad 3 \quad 1 \longrightarrow (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1 \longrightarrow (x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \longrightarrow (x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

The Binomial Theorem

The **Binomial Theorem** is a formula that can be used to expand any binomial.

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\ &= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + y^n\end{aligned}$$

HOW TO

Given a binomial, write it in expanded form.

1. Determine the value of n according to the exponent.
2. Evaluate the $k=0$ through $k=n$ using the Binomial Theorem formula.
3. Simplify.

EXAMPLE 2

Expanding a Binomial

Write in expanded form.

ⓐ $(x+y)^5$ ⓑ $(3x-y)^4$

Solution

ⓐ Substitute $n=5$ into the formula. Evaluate the $k=0$ through $k=5$ terms. Simplify.

$$\begin{aligned}(x+y)^5 &= \binom{5}{0}x^5y^0+\binom{5}{1}x^4y^1+\binom{5}{2}x^3y^2+\binom{5}{3}x^2y^3+\binom{5}{4}x^1y^4+\binom{5}{5}x^0y^5\\(x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5\end{aligned}$$

ⓑ Substitute $n=4$ into the formula. Evaluate the $k=0$ through $k=4$ terms. Notice that $3x$ is in the place that was occupied by x and that $-y$ is in the place that was occupied by y. So we substitute them. Simplify.

$$\begin{aligned}(3x-y)^4 &= \binom{4}{0}(3x)^4(-y)^0+\binom{4}{1}(3x)^3(-y)^1+\binom{4}{2}(3x)^2(-y)^2+\binom{4}{3}(3x)^1(-y)^3+\binom{4}{4}(3x)^0(-y)^4\\(3x-y)^4 &= 81x^4-108x^3y+54x^2y^2-12xy^3+y^4\end{aligned}$$

Analysis

Notice the alternating signs in part b. This happens because $(-y)$ raised to odd powers is negative, but $(-y)$ raised to even powers is positive. This will occur whenever the binomial contains a subtraction sign.

TRY IT #2 Write in expanded form.

ⓐ $(x-y)^5$ ⓑ $(2x+5y)^3$

Using the Binomial Theorem to Find a Single Term

Expanding a binomial with a high exponent such as $(x+2y)^{16}$ can be a lengthy process.

Sometimes we are interested only in a certain term of a binomial expansion. We do not need to fully expand a binomial to find a single specific term.

Note the pattern of coefficients in the expansion of $(x+y)^5$.

$$(x+y)^5 = x^5+\binom{5}{1}x^4y+\binom{5}{2}x^3y^2+\binom{5}{3}x^2y^3+\binom{5}{4}xy^4+y^5$$

The second term is $\binom{5}{1}x^4y$. The third term is $\binom{5}{2}x^3y^2$. We can generalize this result.

$$\binom{n}{r}x^{n-r}y^r$$

The (r+1)th Term of a Binomial Expansion

The $(r+1)$th term of the binomial expansion of $(x+y)^n$ is:

$$\binom{n}{r}x^{n-r}y^r$$

HOW TO

Given a binomial, write a specific term without fully expanding.

1. Determine the value of n according to the exponent.

2. Determine $(r+1)$.
3. Determine r.
4. Replace r in the formula for the $(r+1)$th term of the binomial expansion.

EXAMPLE 3

Writing a Given Term of a Binomial Expansion

Find the tenth term of $(x+2y)^{16}$ without fully expanding the binomial.

Solution

Because we are looking for the tenth term, $r+1=10$, we will use $r=9$ in our calculations.

$$\binom{n}{r} x^{n-r} y^r$$

$$\binom{16}{9} x^{16-9} (2y)^9 = 5{,}857{,}280 x^7 y^9$$

TRY IT #3 Find the sixth term of $(3x-y)^9$ without fully expanding the binomial.

MEDIA

Access these online resources for additional instruction and practice with binomial expansion.

[The Binomial Theorem (http://openstax.org/l/binomialtheorem)](http://openstax.org/l/binomialtheorem)
[Binomial Theorem Example (http://openstax.org/l/btexample)](http://openstax.org/l/btexample)

9.6 SECTION EXERCISES

Verbal

1. What is a binomial coefficient, and how it is calculated?

2. What role do binomial coefficients play in a binomial expansion? Are they restricted to any type of number?

3. What is the Binomial Theorem and what is its use?

4. When is it an advantage to use the Binomial Theorem? Explain.

Algebraic

For the following exercises, evaluate the binomial coefficient.

5. $\binom{6}{2}$

6. $\binom{5}{3}$

7. $\binom{7}{4}$

8. $\binom{9}{7}$

9. $\binom{10}{9}$

10. $\binom{25}{11}$

11. $\binom{17}{6}$

12. $\binom{200}{199}$

For the following exercises, use the Binomial Theorem to expand each binomial.

13. $(4a-b)^3$

14. $(5a+2)^3$

15. $(3a+2b)^3$

16. $(2x+3y)^4$

17. $(4x+2y)^5$

18. $(3x-2y)^4$

19. $(4x-3y)^5$

20. $\left(\frac{1}{x}+3y\right)^5$

21. $(x^{-1}+2y^{-1})^4$

22. $(\sqrt{x}-\sqrt{y})^5$

For the following exercises, use the Binomial Theorem to write the first three terms of each binomial.

23. $(a+b)^{17}$

24. $(x-1)^{18}$

25. $(a-2b)^{15}$

26. $(x-2y)^8$

27. $(3a+b)^{20}$

28. $(2a+4b)^7$

29. $(x^3-\sqrt{y})^8$

For the following exercises, find the indicated term of each binomial without fully expanding the binomial.

30. The fourth term of $(2x-3y)^4$

31. The fourth term of $(3x-2y)^5$

32. The third term of $(6x-3y)^7$

33. The eighth term of $(7+5y)^{14}$

34. The seventh term of $(a+b)^{11}$

35. The fifth term of $(x-y)^7$

36. The tenth term of $(x-1)^{12}$

37. The ninth term of $(a-3b^2)^{11}$

38. The fourth term of $\left(x^3-\frac{1}{2}\right)^{10}$

39. The eighth term of $\left(\frac{y}{2}+\frac{2}{x}\right)^9$

Graphical

For the following exercises, use the Binomial Theorem to expand the binomial $f(x)=(x+3)^4$. Then find and graph each indicated sum on one set of axes.

40. Find and graph $f_1(x)$, such that $f_1(x)$ is the first term of the expansion.

41. Find and graph $f_2(x)$, such that $f_2(x)$ is the sum of the first two terms of the expansion.

42. Find and graph $f_3(x)$, such that $f_3(x)$ is the sum of the first three terms of the expansion.

43. Find and graph $f_4(x)$, such that $f_4(x)$ is the sum of the first four terms of the expansion.

44. Find and graph $f_5(x)$, such that $f_5(x)$ is the sum of the first five terms of the expansion.

Extensions

45. In the expansion of $(5x+3y)^n$, each term has the form $\binom{n}{k} a^{n-k} b^k$, where k successively takes on the value $0, 1, 2, \ldots, n$. If $\binom{n}{k} = \binom{7}{2}$, what is the corresponding term?

46. In the expansion of $(a+b)^n$, the coefficient of $a^{n-k}b^k$ is the same as the coefficient of which other term?

47. Consider the expansion of $(x+b)^{40}$. What is the exponent of b in the kth term?

48. Find $\binom{n}{k-1} + \binom{n}{k}$ and write the answer as a binomial coefficient in the form $\binom{n}{k}$. Prove it.
Hint: Use the fact that, for any integer p, such that $p \geq 1$, $p! = p(p-1)!$.

49. Which expression cannot be expanded using the Binomial Theorem? Explain.

- $(x^2 - 2x + 1)$
- $(\sqrt{a} + 4\sqrt{a} - 5)^8$
- $(x^3 + 2y^2 - z)^5$
- $(3x^2 - \sqrt{2y^3})^{12}$

9.7 Probability

Learning Objectives

In this section, you will:

- Construct probability models.
- Compute probabilities of equally likely outcomes.
- Compute probabilities of the union of two events.
- Use the complement rule to find probabilities.
- Compute probability using counting theory.

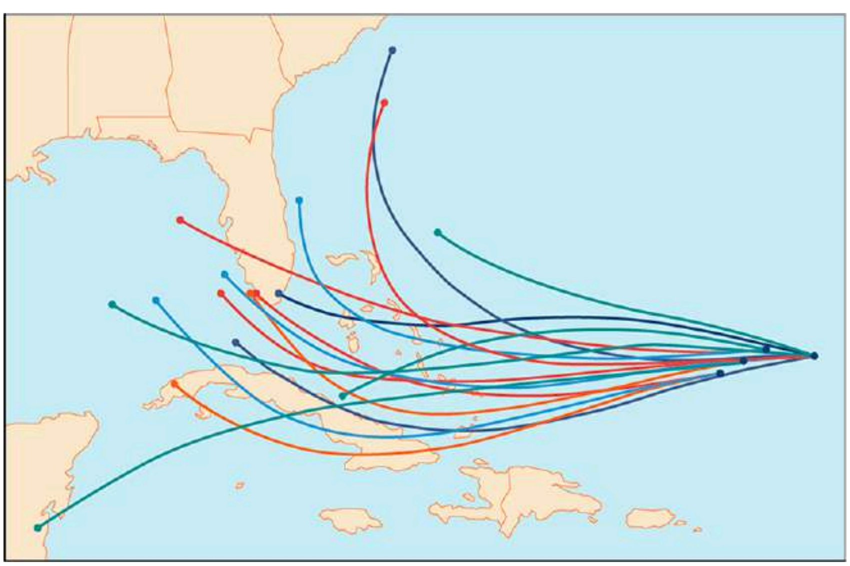


Figure 1 An example of a “spaghetti model,” which can be used to predict possible paths of a tropical storm.[1]

Residents of the Southeastern United States are all too familiar with charts, known as spaghetti models, such as the one in Figure 1. They combine a collection of weather data to predict the most likely path of a hurricane. Each colored line represents one possible path. The group of squiggly lines can begin to resemble strands of spaghetti, hence the name. In this section, we will investigate methods for making these types of predictions.

Constructing Probability Models

Suppose we roll a six-sided number cube. Rolling a number cube is an example of an **experiment**, or an activity with an observable result. The numbers on the cube are possible results, or **outcomes**, of this experiment. The set of all possible outcomes of an experiment is called the **sample space** of the experiment. The sample space for this experiment is $\{1, 2, 3, 4, 5, 6\}$. An **event** is any subset of a sample space.

The likelihood of an event is known as **probability**. The probability of an event p is a number that always satisfies $0 \leq p \leq 1$, where 0 indicates an impossible event and 1 indicates a certain event. A **probability model** is a mathematical description of an experiment listing all possible outcomes and their associated probabilities. For instance, if there is a 1% chance of winning a raffle and a 99% chance of losing the raffle, a probability model would look much like Table 1.

Outcome	Probability
Winning the raffle	1%
Losing the raffle	99%

Table 1

The sum of the probabilities listed in a probability model must equal 1, or 100%.

HOW TO

Given a probability event where each event is equally likely, construct a probability model.

1. Identify every outcome.
2. Determine the total number of possible outcomes.
3. Compare each outcome to the total number of possible outcomes.

EXAMPLE 1

Constructing a Probability Model

Construct a probability model for rolling a single, fair die, with the event being the number shown on the die.

Solution

Begin by making a list of all possible outcomes for the experiment. The possible outcomes are the numbers that can be rolled: 1, 2, 3, 4, 5, and 6. There are six possible outcomes that make up the sample space.

Assign probabilities to each outcome in the sample space by determining a ratio of the outcome to the number of possible outcomes. There is one of each of the six numbers on the cube, and there is no reason to think that any particular face is more likely to show up than any other one, so the probability of rolling any number is $\frac{1}{6}$.

Outcome	Roll of 1	Roll of 2	Roll of 3	Roll of 4	Roll of 5	Roll of 6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Table 2

1 The figure is for illustrative purposes only and does not model any particular storm.

Q&A **Do probabilities always have to be expressed as fractions?**

No. Probabilities can be expressed as fractions, decimals, or percents. Probability must always be a number between 0 and 1, inclusive of 0 and 1.

TRY IT #1 Construct a probability model for tossing a fair coin.

Computing Probabilities of Equally Likely Outcomes

Let S be a sample space for an experiment. When investigating probability, an event is any subset of S. When the outcomes of an experiment are all equally likely, we can find the probability of an event by dividing the number of outcomes in the event by the total number of outcomes in S. Suppose a number cube is rolled, and we are interested in finding the probability of the event "rolling a number less than or equal to 4." There are 4 possible outcomes in the event and 6 possible outcomes in S, so the probability of the event is $\frac{4}{6} = \frac{2}{3}$.

Computing the Probability of an Event with Equally Likely Outcomes

The probability of an event E in an experiment with sample space S with equally likely outcomes is given by

$$P(E) = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{n(E)}{n(S)}$$

E is a subset of S, so it is always true that $0 \leq P(E) \leq 1$.

EXAMPLE 2

Computing the Probability of an Event with Equally Likely Outcomes

A six-sided number cube is rolled. Find the probability of rolling an odd number.

Solution

The event "rolling an odd number" contains three outcomes. There are 6 equally likely outcomes in the sample space. Divide to find the probability of the event.

$$P(E) = \frac{3}{6} = \frac{1}{2}$$

TRY IT #2 A number cube is rolled. Find the probability of rolling a number greater than 2.

Computing the Probability of the Union of Two Events

We are often interested in finding the probability that one of multiple events occurs. Suppose we are playing a card game, and we will win if the next card drawn is either a heart or a king. We would be interested in finding the probability of the next card being a heart or a king. The **union of two events** E and F, written $E \cup F$, is the event that occurs if either or both events occur.

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Suppose the spinner in Figure 2 is spun. We want to find the probability of spinning orange or spinning a b.

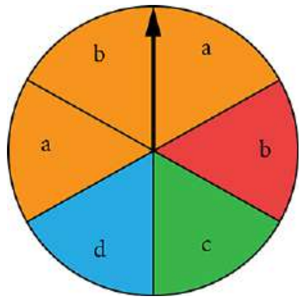


Figure 2

There are a total of 6 sections, and 3 of them are orange. So the probability of spinning orange is $\frac{3}{6}=\frac{1}{2}$. There are a total of 6 sections, and 2 of them have a b. So the probability of spinning a b is $\frac{2}{6}=\frac{1}{3}$. If we added these two probabilities, we would be counting the sector that is both orange and a b twice. To find the probability of spinning an orange or a b, we need to subtract the probability that the sector is both orange and has a b.

$$\frac{1}{2}+\frac{1}{3}-\frac{1}{6}=\frac{2}{3}$$

The probability of spinning orange or a b is $\frac{2}{3}$.

Probability of the Union of Two Events

The probability of the union of two events E and F (written $E \cup F$) equals the sum of the probability of E and the probability of F minus the probability of E and F occurring together (which is called the intersection of E and F and is written as $E \cap F$).

$$P(E \cup F)=P(E)+P(F)-P(E \cap F)$$

EXAMPLE 3

Computing the Probability of the Union of Two Events

A card is drawn from a standard deck. Find the probability of drawing a heart or a 7.

Solution

A standard deck contains an equal number of hearts, diamonds, clubs, and spades. So the probability of drawing a heart is $\frac{1}{4}$. There are four 7s in a standard deck, and there are a total of 52 cards. So the probability of drawing a 7 is $\frac{1}{13}$.

The only card in the deck that is both a heart and a 7 is the 7 of hearts, so the probability of drawing both a heart and a 7 is $\frac{1}{52}$. Substitute $P(H)=\frac{1}{4},\ P(7)=\frac{1}{13}$, and $P(H \cap 7)=\frac{1}{52}$ into the formula.

$$\begin{aligned} P(E \cup F) &= P(E)+P(F)-P(E \cap F) \\ &= \frac{1}{4}+\frac{1}{13}-\frac{1}{52} \\ &= \frac{4}{13} \end{aligned}$$

The probability of drawing a heart or a 7 is $\frac{4}{13}$.

TRY IT #3 A card is drawn from a standard deck. Find the probability of drawing a red card or an ace.

Computing the Probability of Mutually Exclusive Events

Suppose the spinner in Figure 2 is spun again, but this time we are interested in the probability of spinning an orange or a d. There are no sectors that are both orange and contain a d, so these two events have no outcomes in common. Events are said to be **mutually exclusive events** when they have no outcomes in common. Because there is no overlap, there is nothing to subtract, so the general formula is

$$P(E \cup F)=P(E)+P(F)$$

Notice that with mutually exclusive events, the intersection of E and F is the empty set. The probability of spinning an orange is $\frac{3}{6} = \frac{1}{2}$ and the probability of spinning a d is $\frac{1}{6}$. We can find the probability of spinning an orange or a d simply by adding the two probabilities.

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) \\ &= \frac{1}{2} + \frac{1}{6} \\ &= \frac{2}{3} \end{aligned}$$

The probability of spinning an orange or a d is $\frac{2}{3}$.

Probability of the Union of Mutually Exclusive Events

The probability of the union of two *mutually exclusive* events E and F is given by

$$P(E \cup F) = P(E) + P(F)$$

HOW TO

Given a set of events, compute the probability of the union of mutually exclusive events.

1. Determine the total number of outcomes for the first event.
2. Find the probability of the first event.
3. Determine the total number of outcomes for the second event.
4. Find the probability of the second event.
5. Add the probabilities.

EXAMPLE 4

Computing the Probability of the Union of Mutually Exclusive Events

A card is drawn from a standard deck. Find the probability of drawing a heart or a spade.

Solution

The events “drawing a heart” and “drawing a spade” are mutually exclusive because they cannot occur at the same time. The probability of drawing a heart is $\frac{1}{4}$, and the probability of drawing a spade is also $\frac{1}{4}$, so the probability of drawing a heart or a spade is

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

TRY IT #4 A card is drawn from a standard deck. Find the probability of drawing an ace or a king.

Using the Complement Rule to Compute Probabilities

We have discussed how to calculate the probability that an event will happen. Sometimes, we are interested in finding the probability that an event will *not* happen. The **complement of an event** E, denoted E', is the set of outcomes in the sample space that are not in E. For example, suppose we are interested in the probability that a horse will lose a race. If event W is the horse winning the race, then the complement of event W is the horse losing the race.

To find the probability that the horse loses the race, we need to use the fact that the sum of all probabilities in a probability model must be 1.

$$P(E') = 1 - P(E)$$

The probability of the horse winning added to the probability of the horse losing must be equal to 1. Therefore, if the probability of the horse winning the race is $\frac{1}{9}$, the probability of the horse losing the race is simply

$$1 - \frac{1}{9} = \frac{8}{9}$$

The Complement Rule

The probability that the **complement of an event** will occur is given by

$$P(E') = 1 - P(E)$$

EXAMPLE 5

Using the Complement Rule to Calculate Probabilities

Two six-sided number cubes are rolled.

ⓐ Find the probability that the sum of the numbers rolled is less than or equal to 3.

ⓑ Find the probability that the sum of the numbers rolled is greater than 3.

Solution

The first step is to identify the sample space, which consists of all the possible outcomes. There are two number cubes, and each number cube has six possible outcomes. Using the Multiplication Principle, we find that there are 6×6, or 36 total possible outcomes. So, for example, 1-1 represents a 1 rolled on each number cube.

1-1	1-2	1-3	1-4	1-5	1-6
2-1	2-2	2-3	2-4	2-5	2-6
3-1	3-2	3-3	3-4	3-5	3-6
4-1	4-2	4-3	4-4	4-5	4-6
5-1	5-2	5-3	5-4	5-5	5-6
6-1	6-2	6-3	6-4	6-5	6-6

Table 3

ⓐ We need to count the number of ways to roll a sum of 3 or less. These would include the following outcomes: 1-1, 1-2, and 2-1. So there are only three ways to roll a sum of 3 or less. The probability is

$$\frac{3}{36} = \frac{1}{12}$$

ⓑ Rather than listing all the possibilities, we can use the Complement Rule. Because we have already found the probability of the complement of this event, we can simply subtract that probability from 1 to find the probability that the sum of the numbers rolled is greater than 3.

$$\begin{aligned} P(E') &= 1 - P(E) \\ &= 1 - \tfrac{1}{12} \\ &= \tfrac{11}{12} \end{aligned}$$

TRY IT #5 Two number cubes are rolled. Use the Complement Rule to find the probability that the sum is less than 10.

Computing Probability Using Counting Theory

Many interesting probability problems involve counting principles, permutations, and combinations. In these problems, we will use permutations and combinations to find the number of elements in events and sample spaces. These problems can be complicated, but they can be made easier by breaking them down into smaller counting problems.

Assume, for example, that a store has 8 cellular phones and that 3 of those are defective. We might want to find the probability that a couple purchasing 2 phones receives 2 phones that are not defective. To solve this problem, we need to calculate all of the ways to select 2 phones that are not defective as well as all of the ways to select 2 phones. There are 5 phones that are not defective, so there are $C(5,2)$ ways to select 2 phones that are not defective. There are 8 phones, so there are $C(8,2)$ ways to select 2 phones. The probability of selecting 2 phones that are not defective is:

$$\begin{aligned}\frac{\text{ways to select 2 phones that are not defective}}{\text{ways to select 2 phones}} &= \frac{C(5,2)}{C(8,2)}\\ &= \frac{10}{28}\\ &= \frac{5}{14}\end{aligned}$$

EXAMPLE 6

Computing Probability Using Counting Theory

A child randomly selects 5 toys from a bin containing 3 bunnies, 5 dogs, and 6 bears.

ⓐ Find the probability that only bears are chosen. ⓑ Find the probability that 2 bears and 3 dogs are chosen.

ⓒ Find the probability that at least 2 dogs are chosen.

Solution

ⓐ We need to count the number of ways to choose only bears and the total number of possible ways to select 5 toys. There are 6 bears, so there are $C(6,5)$ ways to choose 5 bears. There are 14 toys, so there are $C(14,5)$ ways to choose any 5 toys.

$$\frac{C(6,5)}{C(14,5)} = \frac{6}{2{,}002} = \frac{3}{1{,}001}$$

ⓑ We need to count the number of ways to choose 2 bears and 3 dogs and the total number of possible ways to select 5 toys. There are 6 bears, so there are $C(6,2)$ ways to choose 2 bears. There are 5 dogs, so there are $C(5,3)$ ways to choose 3 dogs. Since we are choosing both bears and dogs at the same time, we will use the Multiplication Principle. There are $C(6,2) \cdot C(5,3)$ ways to choose 2 bears and 3 dogs. We can use this result to find the probability.

$$\frac{C(6,2)C(5,3)}{C(14,5)} = \frac{15 \cdot 10}{2{,}002} = \frac{75}{1{,}001}$$

ⓒ It is often easiest to solve "at least" problems using the Complement Rule. We will begin by finding the probability that fewer than 2 dogs are chosen. If less than 2 dogs are chosen, then either no dogs could be chosen, or 1 dog could be chosen.

When no dogs are chosen, all 5 toys come from the 9 toys that are not dogs. There are $C(9,5)$ ways to choose toys from the 9 toys that are not dogs. Since there are 14 toys, there are $C(14,5)$ ways to choose the 5 toys from all of the toys.

$$\frac{C(9,5)}{C(14,5)} = \frac{63}{1{,}001}$$

If there is 1 dog chosen, then 4 toys must come from the 9 toys that are not dogs, and 1 must come from the 5 dogs. Since we are choosing both dogs and other toys at the same time, we will use the Multiplication Principle. There are $C(5,1) \cdot C(9,4)$ ways to choose 1 dog and 1 other toy.

$$\frac{C(5,1)C(9,4)}{C(14,5)} = \frac{5 \cdot 126}{2{,}002} = \frac{315}{1{,}001}$$

Because these events would not occur together and are therefore mutually exclusive, we add the probabilities to find the probability that fewer than 2 dogs are chosen.

$$\frac{63}{1{,}001} + \frac{315}{1{,}001} = \frac{378}{1{,}001}$$

We then subtract that probability from 1 to find the probability that at least 2 dogs are chosen.

$$1 - \frac{378}{1{,}001} = \frac{623}{1{,}001}$$

TRY IT #6 A child randomly selects 3 gumballs from a container holding 4 purple gumballs, 8 yellow gumballs, and 2 green gumballs.

ⓐ Find the probability that all 3 gumballs selected are purple.
ⓑ Find the probability that no yellow gumballs are selected.
ⓒ Find the probability that at least 1 yellow gumball is selected.

MEDIA

Access these online resources for additional instruction and practice with probability.

[Introduction to Probability (http://openstax.org/l/introprob)](http://openstax.org/l/introprob)
[Determining Probability (http://openstax.org/l/determineprob)](http://openstax.org/l/determineprob)

9.7 SECTION EXERCISES

Verbal

1. What term is used to express the likelihood of an event occurring? Are there restrictions on its values? If so, what are they? If not, explain.

2. What is a sample space?

3. What is an experiment?

4. What is the difference between events and outcomes? Give an example of both using the sample space of tossing a coin 50 times.

5. The *union of two sets* is defined as a set of elements that are present in at least one of the sets. How is this similar to the definition used for the *union of two events* from a probability model? How is it different?

Numeric

For the following exercises, use the spinner shown in Figure 3 to find the probabilities indicated.

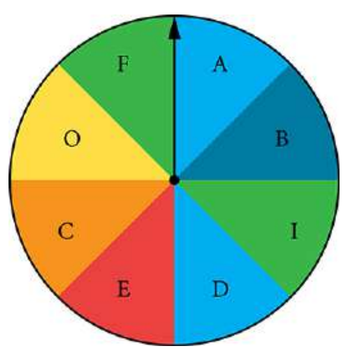


Figure 3

6. Landing on red

7. Landing on a vowel

8. Not landing on blue

9. Landing on purple or a vowel

10. Landing on blue or a vowel

11. Landing on green or blue

12. Landing on yellow or a consonant

13. Not landing on yellow or a consonant

For the following exercises, two coins are tossed.

14. What is the sample space?

15. Find the probability of tossing two heads.

16. Find the probability of tossing exactly one tail.

17. Find the probability of tossing at least one tail.

For the following exercises, four coins are tossed.

18. What is the sample space?

19. Find the probability of tossing exactly two heads.

20. Find the probability of tossing exactly three heads.

21. Find the probability of tossing four heads or four tails.

22. Find the probability of tossing all tails.

23. Find the probability of tossing not all tails.

24. Find the probability of tossing exactly two heads or at least two tails.

25. Find the probability of tossing either two heads or three heads.

For the following exercises, one card is drawn from a standard deck of 52 cards. Find the probability of drawing the following:

26. A club

27. A two

28. Six or seven

29. Red six

30. An ace or a diamond

31. A non-ace

32. A heart or a non-jack

For the following exercises, two dice are rolled, and the results are summed.

33. Construct a table showing the sample space of outcomes and sums.

34. Find the probability of rolling a sum of 3.

35. Find the probability of rolling at least one four or a sum of 8.

36. Find the probability of rolling an odd sum less than 9.

37. Find the probability of rolling a sum greater than or equal to 15.

38. Find the probability of rolling a sum less than 15.

39. Find the probability of rolling a sum less than 6 or greater than 9.

40. Find the probability of rolling a sum between 6 and 9, inclusive.

41. Find the probability of rolling a sum of 5 or 6.

42. Find the probability of rolling any sum other than 5 or 6.

For the following exercises, a coin is tossed, and a card is pulled from a standard deck. Find the probability of the following:

43. A head on the coin or a club

44. A tail on the coin or red ace

45. A head on the coin or a face card

46. No aces

For the following exercises, use this scenario: a bag of M&Ms contains 12 blue, 6 brown, 10 orange, 8 yellow, 8 red, and 4 green M&Ms. Reaching into the bag, a person grabs 5 M&Ms.

47. What is the probability of getting all blue M&Ms?

48. What is the probability of getting 4 blue M&Ms?

49. What is the probability of getting 3 blue M&Ms?

50. What is the probability of getting no brown M&Ms?

Extensions

Use the following scenario for the exercises that follow: In the game of Keno, a player starts by selecting 20 numbers from the numbers 1 to 80. After the player makes his selections, 20 winning numbers are randomly selected from numbers 1 to 80. A win occurs if the player has correctly selected $3, 4,$ or 5 of the 20 winning numbers. (Round all answers to the nearest hundredth of a percent.)

51. What is the percent chance that a player selects exactly 3 winning numbers?

52. What is the percent chance that a player selects exactly 4 winning numbers?

53. What is the percent chance that a player selects all 5 winning numbers?

54. What is the percent chance of winning?

55. How much less is a player's chance of selecting 3 winning numbers than the chance of selecting either 4 or 5 winning numbers?

Real-World Applications

Use this data for the exercises that follow: In 2013, there were roughly 317 million citizens in the United States, and about 40 million were elderly (aged 65 and over).[2]

56. If you meet a U.S. citizen, what is the percent chance that the person is elderly? (Round to the nearest tenth of a percent.)

57. If you meet five U.S. citizens, what is the percent chance that exactly one is elderly? (Round to the nearest tenth of a percent.)

58. If you meet five U.S. citizens, what is the percent chance that three are elderly? (Round to the nearest tenth of a percent.)

2 United States Census Bureau. http://www.census.gov

59. If you meet five U.S. citizens, what is the percent chance that four are elderly? (Round to the nearest thousandth of a percent.)

60. It is predicted that by 2030, one in five U.S. citizens will be elderly. How much greater will the chances of meeting an elderly person be at that time? What policy changes do you foresee if these statistics hold true?

Chapter Review

Key Terms

Addition Principle if one event can occur in m ways and a second event with no common outcomes can occur in n ways, then the first or second event can occur in $m+n$ ways
annuity an investment in which the purchaser makes a sequence of periodic, equal payments
arithmetic sequence a sequence in which the difference between any two consecutive terms is a constant
arithmetic series the sum of the terms in an arithmetic sequence
binomial coefficient the number of ways to choose r objects from n objects where order does not matter; equivalent to $C(n, r)$, denoted $\binom{n}{r}$
binomial expansion the result of expanding $(x+y)^n$ by multiplying
Binomial Theorem a formula that can be used to expand any binomial
combination a selection of objects in which order does not matter
common difference the difference between any two consecutive terms in an arithmetic sequence
common ratio the ratio between any two consecutive terms in a geometric sequence
complement of an event the set of outcomes in the sample space that are not in the event E
diverge a series is said to diverge if the sum is not a real number
event any subset of a sample space
experiment an activity with an observable result
explicit formula a formula that defines each term of a sequence in terms of its position in the sequence
finite sequence a function whose domain consists of a finite subset of the positive integers $\{1, 2, \ldots n\}$ for some positive integer n
Fundamental Counting Principle if one event can occur in m ways and a second event can occur in n ways after the first event has occurred, then the two events can occur in $m \times n$ ways; also known as the Multiplication Principle
geometric sequence a sequence in which the ratio of a term to a previous term is a constant
geometric series the sum of the terms in a geometric sequence
index of summation in summation notation, the variable used in the explicit formula for the terms of a series and written below the sigma with the lower limit of summation
infinite sequence a function whose domain is the set of positive integers
infinite series the sum of the terms in an infinite sequence
lower limit of summation the number used in the explicit formula to find the first term in a series
Multiplication Principle if one event can occur in m ways and a second event can occur in n ways after the first event has occurred, then the two events can occur in $m \times n$ ways; also known as the Fundamental Counting Principle
mutually exclusive events events that have no outcomes in common
n factorial the product of all the positive integers from 1 to n
nth partial sum the sum of the first n terms of a sequence
nth term of a sequence a formula for the general term of a sequence
outcomes the possible results of an experiment
permutation a selection of objects in which order matters
probability a number from 0 to 1 indicating the likelihood of an event
probability model a mathematical description of an experiment listing all possible outcomes and their associated probabilities
recursive formula a formula that defines each term of a sequence using previous term(s)
sample space the set of all possible outcomes of an experiment
sequence a function whose domain is a subset of the positive integers
series the sum of the terms in a sequence
summation notation a notation for series using the Greek letter sigma; it includes an explicit formula and specifies the first and last terms in the series
term a number in a sequence
union of two events the event that occurs if either or both events occur
upper limit of summation the number used in the explicit formula to find the last term in a series

Key Equations

Formula for a factorial	$0! = 1$ $1! = 1$ $n! = n(n-1)(n-2)\cdots(2)(1)$, for $n \geq 2$

recursive formula for nth term of an arithmetic sequence	$a_n = a_{n-1} + d, n \geq 2$
explicit formula for nth term of an arithmetic sequence	$a_n = a_1 + d(n-1)$

recursive formula for nth term of a geometric sequence	$a_n = ra_{n-1}, n \geq 2$
explicit formula for nth term of a geometric sequence	$a_n = a_1 r^{n-1}$

sum of the first n terms of an arithmetic series	$S_n = \frac{n(a_1+a_n)}{2}$
sum of the first n terms of a geometric series	$S_n = \frac{a_1(1-r^n)}{1-r}, r \neq 1$
sum of an infinite geometric series with $-1 < r < 1$	$S_n = \frac{a_1}{1-r}, r \neq 1$

number of permutations of n distinct objects taken r at a time	$P(n,r) = \frac{n!}{(n-r)!}$
number of combinations of n distinct objects taken r at a time	$C(n,r) = \frac{n!}{r!(n-r)!}$
number of permutations of n non-distinct objects	$\frac{n!}{r_1!r_2!\ldots r_k!}$

Binomial Theorem	$(x+y)^n = \sum\limits_{k-0}^{n} \binom{n}{k} x^{n-k} y^k$
$(r+1)th$ term of a binomial expansion	$\binom{n}{r} x^{n-r} y^r$

probability of an event with equally likely outcomes	$P(E) = \frac{n(E)}{n(S)}$
probability of the union of two events	$P(E \cup F) = P(E) + P(F) - P(E \cap F)$
probability of the union of mutually exclusive events	$P(E \cup F) = P(E) + P(F)$
probability of the complement of an event	$P(E') = 1 - P(E)$

Key Concepts

9.1 Sequences and Their Notations

- A sequence is a list of numbers, called terms, written in a specific order.
- Explicit formulas define each term of a sequence using the position of the term. See Example 1, Example 2, and Example 3.
- An explicit formula for the nth term of a sequence can be written by analyzing the pattern of several terms. See Example 4.
- Recursive formulas define each term of a sequence using previous terms.
- Recursive formulas must state the initial term, or terms, of a sequence.
- A set of terms can be written by using a recursive formula. See Example 5 and Example 6.
- A factorial is a mathematical operation that can be defined recursively.
- The factorial of n is the product of all integers from 1 to n See Example 7.

9.2 Arithmetic Sequences

- An arithmetic sequence is a sequence where the difference between any two consecutive terms is a constant.
- The constant between two consecutive terms is called the common difference.
- The common difference is the number added to any one term of an arithmetic sequence that generates the subsequent term. See Example 1.
- The terms of an arithmetic sequence can be found by beginning with the initial term and adding the common difference repeatedly. See Example 2 and Example 3.
- A recursive formula for an arithmetic sequence with common difference d is given by $a_n = a_{n-1} + d, n \geq 2$. See Example 4.
- As with any recursive formula, the initial term of the sequence must be given.
- An explicit formula for an arithmetic sequence with common difference d is given by $a_n = a_1 + d(n-1)$. See Example 5.
- An explicit formula can be used to find the number of terms in a sequence. See Example 6.
- In application problems, we sometimes alter the explicit formula slightly to $a_n = a_0 + dn$. See Example 7.

9.3 Geometric Sequences

- A geometric sequence is a sequence in which the ratio between any two consecutive terms is a constant.
- The constant ratio between two consecutive terms is called the common ratio.
- The common ratio can be found by dividing any term in the sequence by the previous term. See Example 1.
- The terms of a geometric sequence can be found by beginning with the first term and multiplying by the common ratio repeatedly. See Example 2 and Example 4.
- A recursive formula for a geometric sequence with common ratio r is given by $a_n = ra_{n-1}$ for $n \geq 2$.
- As with any recursive formula, the initial term of the sequence must be given. See Example 3.
- An explicit formula for a geometric sequence with common ratio r is given by $a_n = a_1 r^{n-1}$. See Example 5.
- In application problems, we sometimes alter the explicit formula slightly to $a_n = a_0 r^n$. See Example 6.

9.4 Series and Their Notations

- The sum of the terms in a sequence is called a series.
- A common notation for series is called summation notation, which uses the Greek letter sigma to represent the sum. See Example 1.
- The sum of the terms in an arithmetic sequence is called an arithmetic series.
- The sum of the first n terms of an arithmetic series can be found using a formula. See Example 2 and Example 3.
- The sum of the terms in a geometric sequence is called a geometric series.
- The sum of the first n terms of a geometric series can be found using a formula. See Example 4 and Example 5.
- The sum of an infinite series exists if the series is geometric with $-1 < r < 1$.
- If the sum of an infinite series exists, it can be found using a formula. See Example 6, Example 7, and Example 8.
- An annuity is an account into which the investor makes a series of regularly scheduled payments. The value of an annuity can be found using geometric series. See Example 9.

9.5 Counting Principles

- If one event can occur in m ways and a second event with no common outcomes can occur in n ways, then the first or second event can occur in $m + n$ ways. See Example 1.
- If one event can occur in m ways and a second event can occur in n ways after the first event has occurred, then the two events can occur in $m \times n$ ways. See Example 2.

- A permutation is an ordering of n objects.
- If we have a set of n objects and we want to choose r objects from the set in order, we write $P(n, r)$.
- Permutation problems can be solved using the Multiplication Principle or the formula for $P(n, r)$. See Example 3 and Example 4.
- A selection of objects where the order does not matter is a combination.
- Given n distinct objects, the number of ways to select r objects from the set is $\mathrm{C}(n, r)$ and can be found using a formula. See Example 5.
- A set containing n distinct objects has 2^n subsets. See Example 6.
- For counting problems involving non-distinct objects, we need to divide to avoid counting duplicate permutations. See Example 7.

9.6 Binomial Theorem

- $\binom{n}{r}$ is called a binomial coefficient and is equal to $C(n, r)$. See Example 1.
- The Binomial Theorem allows us to expand binomials without multiplying. See Example 2.
- We can find a given term of a binomial expansion without fully expanding the binomial. See Example 3.

9.7 Probability

- Probability is always a number between 0 and 1, where 0 means an event is impossible and 1 means an event is certain.
- The probabilities in a probability model must sum to 1. See Example 1.
- When the outcomes of an experiment are all equally likely, we can find the probability of an event by dividing the number of outcomes in the event by the total number of outcomes in the sample space for the experiment. See Example 2.
- To find the probability of the union of two events, we add the probabilities of the two events and subtract the probability that both events occur simultaneously. See Example 3.
- To find the probability of the union of two mutually exclusive events, we add the probabilities of each of the events. See Example 4.
- The probability of the complement of an event is the difference between 1 and the probability that the event occurs. See Example 5.
- In some probability problems, we need to use permutations and combinations to find the number of elements in events and sample spaces. See Example 6.

Exercises

Review Exercises

Sequences and Their Notation

1. Write the first four terms of the sequence defined by the recursive formula $a_1 = 2, \quad a_n = a_{n-1} + n$.

2. Evaluate $\frac{6!}{(5-3)!3!}$.

3. Write the first four terms of the sequence defined by the explicit formula $a_n = 10^n + 3$.

4. Write the first four terms of the sequence defined by the explicit formula $a_n = \frac{n!}{n(n+1)}$.

Arithmetic Sequences

5. Is the sequence $\frac{4}{7}, \frac{47}{21}, \frac{82}{21}, \frac{39}{7}, \ \ldots$ arithmetic? If so, find the common difference.

6. Is the sequence $2, 4, 8, 16, \ \ldots$ arithmetic? If so, find the common difference.

7. An arithmetic sequence has the first term $a_1 = 18$ and common difference $d = -8$. What are the first five terms?

8. An arithmetic sequence has terms $a_3 = 11.7$ and $a_8 = -14.6$. What is the first term?

9. Write a recursive formula for the arithmetic sequence $-20, -10, 0, 10, \ldots$

10. Write a recursive formula for the arithmetic sequence $0, \quad -\frac{1}{2}, \quad -1, \quad -\frac{3}{2}, \ldots,$ and then find the 31st term.

11. Write an explicit formula for the arithmetic sequence $\frac{7}{8}, \frac{29}{24}, \frac{37}{24}, \frac{15}{8}, \ldots$

12. How many terms are in the finite arithmetic sequence $12, 20, 28, \ldots, 172$?

Geometric Sequences

13. Find the common ratio for the geometric sequence $2.5, \quad 5, \quad 10, \quad 20, \ldots$

14. Is the sequence 4, 16, 28, 40 ... geometric? If so find the common ratio. If not, explain why.

15. A geometric sequence has terms $a_7 = 16{,}384$ and $a_9 = 262{,}144$. What are the first five terms?

16. A geometric sequence has the first term $a_1 = -3$ and common ratio $r = \frac{1}{2}$. What is the 8th term?

17. What are the first five terms of the geometric sequence $a_1 = 3, \quad a_n = 4 \cdot a_{n-1}$?

18. Write a recursive formula for the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

19. Write an explicit formula for the geometric sequence $-\frac{1}{5}, \quad -\frac{1}{15}, \quad -\frac{1}{45}, \quad -\frac{1}{135}, \ldots$

20. How many terms are in the finite geometric sequence $-5, \quad -\frac{5}{3}, \quad -\frac{5}{9}, \ldots, \quad -\frac{5}{59{,}049}$?

Series and Their Notation

21. Use summation notation to write the sum of terms $\frac{1}{2}m + 5$ from $m = 0$ to $m = 5$.

22. Use summation notation to write the sum that results from adding the number 13 twenty times.

23. Use the formula for the sum of the first n terms of an arithmetic series to find the sum of the first eleven terms of the arithmetic series 2.5, 4, 5.5, ...

24. A ladder has 15 tapered rungs, the lengths of which increase by a common difference. The first rung is 5 inches long, and the last rung is 20 inches long. What is the sum of the lengths of the rungs?

25. Use the formula for the sum of the first *n* terms of a geometric series to find S_9 for the series $12,\ 6,\ 3,\ \frac{3}{2}, \ldots$

26. The fees for the first three years of a hunting club membership are given in Table 1. If fees continue to rise at the same rate, how much will the total cost be for the first ten years of membership?

Year	Membership Fees
1	\$1500
2	\$1950
3	\$2535

Table 1

27. Find the sum of the infinite geometric series $\sum_{k=1}^{\infty} 45 \cdot \left(-\frac{1}{3}\right)^{k-1}$.

28. A ball has a bounce-back ratio of $\frac{3}{5}$ the height of the previous bounce. Write a series representing the total distance traveled by the ball, assuming it was initially dropped from a height of 5 feet. What is the total distance? (*Hint*: the total distance the ball travels on each bounce is the sum of the heights of the rise and the fall.)

29. Alejandro deposits \$80 of his monthly earnings into an annuity that earns 6.25% annual interest, compounded monthly. How much money will he have saved after 5 years?

30. The twins Hoa and Binh both opened retirement accounts on their 21st birthday. Hoa deposits \$4,800.00 each year, earning 5.5% annual interest, compounded monthly. Binh deposits \$3,600.00 each year, earning 8.5% annual interest, compounded monthly. Which twin will earn the most interest by the time they are 55 years old? How much more?

Counting Principles

31. How many ways are there to choose a number from the set $\{-10, -6, 4, 10, 12, 18, 24, 32\}$ that is divisible by either 4 or 6?

32. In a group of 20 musicians, 12 play piano, 7 play trumpet, and 2 play both piano and trumpet. How many musicians play either piano or trumpet?

33. How many ways are there to construct a 4-digit code if numbers can be repeated?

34. A palette of water color paints has 3 shades of green, 3 shades of blue, 2 shades of red, 2 shades of yellow, and 1 shade of black. How many ways are there to choose one shade of each color?

35. Calculate $P(18, 4)$.

36. In a group of 5 first-year, 10 second-year, 3 third-year, and 2 fourth-year students, how many ways can a president, vice president, and treasurer be elected?

37. Calculate $C(15, 6)$.

38. A coffee shop has 7 Guatemalan roasts, 4 Cuban roasts, and 10 Costa Rican roasts. How many ways can the shop choose 2 Guatemalan, 2 Cuban, and 3 Costa Rican roasts for a coffee tasting event?

39. How many subsets does the set $\{1, 3, 5, \ldots, 99\}$ have?

40. A day spa charges a basic day rate that includes use of a sauna, pool, and showers. For an extra charge, guests can choose from the following additional services: massage, body scrub, manicure, pedicure, facial, and straight-razor shave. How many ways are there to order additional services at the day spa?

41. How many distinct ways can the word DEADWOOD be arranged?

42. How many distinct rearrangements of the letters of the word DEADWOOD are there if the arrangement must begin and end with the letter D?

Binomial Theorem

43. Evaluate the binomial coefficient $\binom{23}{8}$.

44. Use the Binomial Theorem to expand $\left(3x + \frac{1}{2}y\right)^6$.

45. Use the Binomial Theorem to write the first three terms of $(2a + b)^{17}$.

46. Find the fourth term of $\left(3a^2 - 2b\right)^{11}$ without fully expanding the binomial.

Probability

For the following exercises, assume two die are rolled.

47. Construct a table showing the sample space.

48. What is the probability that a roll includes a 2?

49. What is the probability of rolling a pair?

50. What is the probability that a roll includes a 2 or results in a pair?

51. What is the probability that a roll doesn't include a 2 or result in a pair?

52. What is the probability of rolling a 5 or a 6?

53. What is the probability that a roll includes neither a 5 nor a 6?

For the following exercises, use the following data: An elementary school survey found that 350 of the 500 students preferred soda to milk. Suppose 8 children from the school are attending a birthday party. (Show calculations and round to the nearest tenth of a percent.)

54. What is the percent chance that all the children attending the party prefer soda?

55. What is the percent chance that at least one of the children attending the party prefers milk?

56. What is the percent chance that exactly 3 of the children attending the party prefer soda?

57. What is the percent chance that exactly 3 of the children attending the party prefer milk?

Practice Test

1. Write the first four terms of the sequence defined by the recursive formula $a =- 14, a_n = \frac{2+a_{n-1}}{2}$.

2. Write the first four terms of the sequence defined by the explicit formula $a_n = \frac{n^2-n-1}{n!}$.

3. Is the sequence $0.3,\ 1.2,\ 2.1,\ 3, \ldots$ arithmetic? If so find the common difference.

4. An arithmetic sequence has the first term $a_1 = -4$ and common difference $d =- \frac{4}{3}$. What is the 6th term?

5. Write a recursive formula for the arithmetic sequence $-2,\ -\frac{7}{2},\ -5,\ -\frac{13}{2}, \ldots$ and then find the 22nd term.

6. Write an explicit formula for the arithmetic sequence $15.6,\ 15,\ 14.4,\ 13.8, \ldots$ and then find the 32nd term.

7. Is the sequence $-2, -1, -\frac{1}{2}, -\frac{1}{4}, \ldots$ geometric? If so find the common ratio. If not, explain why.

8. What is the 11th term of the geometric sequence $-1.5, -3, -6, -12, \ldots$?

9. Write a recursive formula for the geometric sequence $1,\ -\frac{1}{2},\ \frac{1}{4},\ -\frac{1}{8}, \ldots$

10. Write an explicit formula for the geometric sequence $4, \ -\frac{4}{3}, \ \frac{4}{9}, \ -\frac{4}{27}, \ldots$

11. Use summation notation to write the sum of terms $3k^2 - \frac{5}{6}k$ from $k = -3$ to $k = 15$.

12. A community baseball stadium has 10 seats in the first row, 13 seats in the second row, 16 seats in the third row, and so on. There are 56 rows in all. What is the seating capacity of the stadium?

13. Use the formula for the sum of the first n terms of a geometric series to find $\sum_{k=1}^{7} -0.2 \cdot (-5)^{k-1}$.

14. Find the sum of the infinite geometric series $\sum_{k=1}^{\infty} \frac{1}{3} \cdot \left(-\frac{1}{5}\right)^{k-1}$.

15. Ramla deposits $3,600 into a retirement fund each year. The fund earns 7.5% annual interest, compounded monthly. If she opened her account when she was 20 years old, how much will she have by the time she’s 55? How much of that amount was interest earned?

16. In a competition of 50 professional ballroom dancers, 22 compete in the fox-trot competition, 18 compete in the tango competition, and 6 compete in both the fox-trot and tango competitions. How many dancers compete in the fox-trot or tango competitions?

17. A buyer of a new sedan can custom order the car by choosing from 5 different exterior colors, 3 different interior colors, 2 sound systems, 3 motor designs, and either manual or automatic transmission. How many choices does the buyer have?

18. To allocate annual bonuses, a manager must choose his top four employees and rank them first to fourth. In how many ways can he create the “Top-Four” list out of the 32 employees?

19. A music group needs to choose 3 songs to play at the annual Battle of the Bands. How many ways can they choose their set if have 15 songs to pick from?

20. A self-serve frozen yogurt shop has 8 candy toppings and 4 fruit toppings to choose from. How many ways are there to top a frozen yogurt?

21. How many distinct ways can the word EVANESCENCE be arranged if the anagram must end with the letter E?

22. Use the Binomial Theorem to expand $\left(\frac{3}{2}x - \frac{1}{2}y\right)^5$.

23. Find the seventh term of $\left(x^2 - \frac{1}{2}\right)^{13}$ without fully expanding the binomial.

For the following exercises, use the spinner in Figure 1.

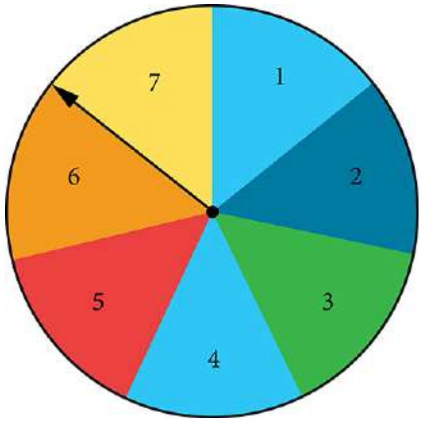


Figure 1

24. Construct a probability model showing each possible outcome and its associated probability. (Use the first letter for colors.)

25. What is the probability of landing on an odd number?

26. What is the probability of landing on blue?

27. What is the probability of landing on blue or an odd number?

28. What is the probability of landing on anything other than blue or an odd number?

29. A bowl of candy holds 16 peppermint, 14 butterscotch, and 10 strawberry flavored candies. Suppose a person grabs a handful of 7 candies. What is the percent chance that exactly 3 are butterscotch? (Show calculations and round to the nearest tenth of a percent.)

Answer Key

Chapter 1

Try It

1.1 Real Numbers: Algebra Essentials

1. ⓐ $\frac{11}{1}$ ⓑ $\frac{3}{1}$ ⓒ $-\frac{4}{1}$

2. ⓐ 4 (or 4.0), terminating; ⓑ $0.\overline{615384}$, repeating; ⓒ −0.85, terminating

3. ⓐ rational and repeating; ⓑ rational and terminating; ⓒ irrational; ⓓ rational and terminating; ⓔ irrational

4. ⓐ positive, irrational; right ⓑ negative, rational; left ⓒ positive, rational; right ⓓ negative, irrational; left ⓔ positive, rational; right

5.

	N	*W*	*I*	*Q*	*Q'*
a. $-\frac{35}{7}$			X	X	
b. 0		X	X	X	
c. $\sqrt{169}$	X	X	X	X	
d. $\sqrt{24}$					X
e. 4.763763763...				X	

6. ⓐ 10 ⓑ 2 ⓒ 4.5 ⓓ 25 ⓔ 26

7. ⓐ 11, commutative property of multiplication, associative property of multiplication, inverse property of multiplication, identity property of multiplication; ⓑ 33, distributive property; ⓒ 26, distributive property; ⓓ $\frac{4}{9}$, commutative property of addition, associative property of addition, inverse property of addition, identity property of addition; ⓔ 0, distributive property, inverse property of addition, identity property of addition

8.

	Constants	Variables
a. $2\pi r(r+h)$	$2, \pi$	r, h
b. 2(L + W)	2	L, W
c. $4y^3 + y$	4	y

9. ⓐ 5; ⓑ 11; ⓒ 9; ⓓ 26

10. ⓐ 4; ⓑ 11; ⓒ $\frac{121}{3}\pi$; ⓓ 1728; ⓔ 3

11. 1,152 cm^2

12. ⓐ $-2y-2z$ or $-2(y+z)$; ⓑ $\frac{2}{t}-1$; ⓒ $3pq-4p+q$; ⓓ $7r-2s+6$

13. $A = P(1+rt)$

1.2 Exponents and Scientific Notation

1. a k^{15} b $\left(\frac{2}{y}\right)^5$ c t^{14}

2. a s^7 b $(-3)^5$ c $\left(ef^2\right)^2$

3. a $(3y)^{24}$ b t^{35} c $(-g)^{16}$

4. a 1 b $\frac{1}{2}$ c 1 d 1

5. a $\frac{1}{(-3t)^6}$ b $\frac{1}{f^3}$ c $\frac{2}{5k^3}$

6. a $t^{-5} = \frac{1}{t^5}$ b $\frac{1}{25}$

7. a $g^{10}h^{15}$ b $125t^3$ c $-27y^{15}$ d $\frac{1}{a^{18}b^{21}}$ e $\frac{r^{12}}{s^8}$

8. a $\frac{b^{15}}{c^3}$ b $\frac{625}{u^{32}}$ c $\frac{-1}{w^{105}}$ d $\frac{q^{24}}{p^{32}}$ e $\frac{1}{c^{20}d^{12}}$

9. a $\frac{v^6}{8u^3}$ b $\frac{1}{x^3}$ c $\frac{e^4}{f^4}$ d $\frac{27r}{s}$ e 1 f $\frac{16h^{10}}{49}$

10. a $\$1.52 \times 10^5$ b 7.158×10^9 c $\$8.55 \times 10^{13}$ d 3.34×10^{-9} e 7.15×10^{-8}

11. a $703,000$ b $-816,000,000,000$ c $-0.000\ 000\ 000\ 000\ 39$ d 0.000008

12. a -8.475×10^6 b 8×10^{-8} c 2.976×10^{13} d -4.3×10^6 e $\approx 1.24 \times 10^{15}$

13. Number of cells: 3×10^{13}; length of a cell: 8×10^{-6} m; total length: 2.4×10^8 m or $240,000,000$ m.

1.3 Radicals and Rational Exponents

1. a 15 b 3 c 4 d 17

2. $5\,|x|\,|y|\,\sqrt{2yz}$. Notice the absolute value signs around *x* and *y*? That's because their value must be positive!

3. $10\,|x|$

4. $\frac{x\sqrt{2}}{3y^2}$. We do not need the absolute value signs for y^2 because that term will always be nonnegative.

5. $b^4\sqrt{3ab}$

6. $13\sqrt{5}$

7. 0

8. $6\sqrt{6}$

9. $14-7\sqrt{3}$

10. a -6 b 6 c $88\sqrt[3]{9}$

11. $\left(\sqrt{9}\right)^5 = 3^5 = 243$

12. $x(5y)^{\frac{9}{2}}$

13. $28x^{\frac{23}{15}}$

1.4 Polynomials

1. The degree is 6, the leading term is $-x^6$, and the leading coefficient is -1.

2. $2x^3 + 7x^2 - 4x - 3$

3. $-11x^3 - x^2 + 7x - 9$

4. $3x^4 - 10x^3 - 8x^2 + 21x + 14$

5. $3x^2 + 16x - 35$

6. $16x^2 - 8x + 1$

7. $4x^2 - 49$

8. $6x^2 + 21xy - 29x - 7y + 9$

1.5 Factoring Polynomials

1. $(b^2 - a)(x + 6)$

2. $(x - 6)(x - 1)$

3. ⓐ $(2x + 3)(x + 3)$
   ⓑ $(3x - 1)(2x + 1)$

4. $(7x - 1)^2$

5. $(9y + 10)(9y - 10)$

6. $(6a + b)(36a^2 - 6ab + b^2)$

7. $(10x - 1)\left(100x^2 + 10x + 1\right)$

8. $(5a - 1)^{-\frac{1}{4}}(17a - 2)$

1.6 Rational Expressions

1. $\frac{1}{x+6}$

2. $\frac{(x+5)(x+6)}{(x+2)(x+4)}$

3. 1

4. $\frac{2(x-7)}{(x+5)(x-3)}$

5. $\frac{x^2 - y^2}{xy^2}$

1.1 Section Exercises

1. irrational number. The square root of two does not terminate, and it does not repeat a pattern. It cannot be written as a quotient of two integers, so it is irrational.

3. The Associative Properties state that the sum or product of multiple numbers can be grouped differently without affecting the result. This is because the same operation is performed (either addition or subtraction), so the terms can be re-ordered.

5. -6

7. -2

9. -9

11. 9

13. -2

15. 4

17. 0

19. 9

21. 25

23. -6

25. 17

27. 4

29. 14

31. -66

33. -12

35. -44

37. -2

39. $-14y - 11$

41. $-4b + 1$

43. $43z - 3$

45. $9y + 45$

47. $-6b + 6$

49. $\frac{16x}{3}$

51. $9x$

53. $\frac{1}{2}(40-10)+5$

55. irrational number

57. $g+400-2(600)=1200$

59. inverse property of addition

61. 68.4

63. true

65. irrational

67. rational

1.2 Section Exercises

1. No, the two expressions are not the same. An exponent tells how many times you multiply the base. So 2^3 is the same as $2\times2\times2$, which is 8. 3^2 is the same as 3×3, which is 9.

3. It is a method of writing very small and very large numbers.

5. 81

7. 243

9. $\frac{1}{16}$

11. $\frac{1}{11}$

13. 1

15. 4^9

17. 12^{40}

19. $\frac{1}{7^9}$

21. 3.14×10^{-5}

23. 16,000,000,000

25. a^4

27. b^6c^8

29. ab^2d^3

31. m^4

33. $\frac{q^5}{p^6}$

35. $\frac{y^{21}}{x^{14}}$

37. 25

39. $72a^2$

41. $\frac{c^3}{b^9}$

43. $\frac{y}{81z^6}$

45. 0.00135 m

47. 1.0995×10^{12}

49. 0.00000000003397 in.

51. 12,230,590,464 m^{66}

53. $\frac{a^{14}}{1296}$

55. $\frac{n}{a^9c}$

57. $\frac{1}{a^6b^6c^6}$

59. 0.000000000000000000000000000000000662606957

1.3 Section Exercises

1. When there is no index, it is assumed to be 2 or the square root. The expression would only be equal to the radicand if the index were 1.

3. The principal square root is the nonnegative root of the number.

5. 16

7. 10

9. 14

11. $7\sqrt{2}$

13. $\frac{9\sqrt{5}}{5}$

15. 25

17. $\sqrt{2}$

19. $2\sqrt{6}$

21. $5\sqrt{6}$

23. $6\sqrt{35}$

25. $\frac{2}{15}$

27. $\frac{6\sqrt{10}}{19}$

29. $-\frac{1+\sqrt{17}}{2}$

31. $7\sqrt[3]{2}$

33. $15\sqrt{5}$

35. $20x^2$

37. $7\sqrt{p}$

39. $17m^2\sqrt{m}$

41. $2b\sqrt{a}$

43. $\frac{15x}{7}$

45. $5y^4\sqrt{2}$

47. $\frac{4\sqrt{7d}}{7d}$

49. $\frac{2\sqrt{2}+2\sqrt{6x}}{1-3x}$

51. $-w\sqrt{2w}$

53. $\frac{3\sqrt{x}-\sqrt{3x}}{2}$

55. $5n^5\sqrt{5}$

57. $\frac{9\sqrt{m}}{19m}$

59. $\frac{2}{3d}$

61. $\frac{3\sqrt[4]{2x^2}}{2}$

63. $6z\sqrt[3]{2}$

65. 500 feet

67. $\frac{-5\sqrt{2}-6}{7}$

69. $\frac{\sqrt{mnc}}{a^9cmn}$

71. $\frac{2\sqrt{2}x+\sqrt{2}}{4}$

73. $\frac{\sqrt{3}}{3}$

1.4 Section Exercises

1. The statement is true. In standard form, the polynomial with the highest value exponent is placed first and is the leading term. The degree of a polynomial is the value of the highest exponent, which in standard form is also the exponent of the leading term.

3. Use the distributive property, multiply, combine like terms, and simplify.

5. 2

7. 8

9. 2

11. $4x^2 + 3x + 19$

13. $3w^2 + 30w + 21$

15. $11b^4 - 9b^3 + 12b^2 - 7b + 8$

17. $24x^2 - 4x - 8$

19. $24b^4 - 48b^2 + 24$

21. $99v^2 - 202v + 99$

23. $8n^3 - 4n^2 + 72n - 36$

25. $9y^2 - 42y + 49$

27. $16p^2 + 72p + 81$

29. $9y^2 - 36y + 36$

31. $16c^2 - 1$

33. $225n^2 - 36$

35. $-16m^2 + 16$

37. $121q^2 - 100$

39. $16t^4 + 4t^3 - 32t^2 - t + 7$

41. $y^3 - 6y^2 - y + 18$

43. $3p^3 - p^2 - 12p + 10$

45. $a^2 - b^2$

47. $16t^2 - 40tu + 25u^2$

49. $4t^2 + x^2 + 4t - 5tx - x$

51. $24r^2 + 22rd - 7d^2$

53. $32x^2 - 4x - 3$ m^2

55. $32t^3 - 100t^2 + 40t + 38$

57. $a^4 + 4a^3c - 16ac^3 - 16c^4$

1.5 Section Exercises

1. The terms of a polynomial do not have to have a common factor for the entire polynomial to be factorable. For example, $4x^2$ and $-9y^2$ don't have a common factor, but the whole polynomial is still factorable: $4x^2 - 9y^2 = (2x + 3y)(2x - 3y)$.

3. Divide the x term into the sum of two terms, factor each portion of the expression separately, and then factor out the GCF of the entire expression.

5. $7m$

7. $10m^3$

9. y

11. $(2a - 3)(a + 6)$

13. $(3n - 11)(2n + 1)$

15. $(p + 1)(2p - 7)$

17. $(5h + 3)(2h - 3)$

19. $(9d - 1)(d - 8)$

21. $(12t + 13)(t - 1)$

23. $(4x + 10)(4x - 10)$

25. $(11p + 13)(11p - 13)$

27. $(19d + 9)(19d - 9)$

29. $(12b + 5c)(12b - 5c)$

31. $(7n + 12)^2$

33. $(15y + 4)^2$

35. $(5p - 12)^2$

37. $(x + 6)\left(x^2 - 6x + 36\right)$

39. $(5a + 7)\left(25a^2 - 35a + 49\right)$

41. $(4x - 5)\left(16x^2 + 20x + 25\right)$

43. $(5r + 12s)\left(25r^2 - 60rs + 144s^2\right)$

45. $(2c + 3)^{-\frac{1}{4}}(-7c - 15)$

47. $(x + 2)^{-\frac{2}{5}}(19x + 10)$

49. $(2z - 9)^{-\frac{3}{2}}(27z - 99)$

51. $(14x - 3)(7x + 9)$

53. $(3x + 5)(3x - 5)$

55. $(2x+5)^2(2x-5)^2$

57. $\left(4z^2+49a^2\right)(2z+7a)(2z-7a)$

59. $\frac{1}{(4x+9)(4x-9)(2x+3)}$

1.6 Section Exercises

1. You can factor the numerator and denominator to see if any of the terms can cancel one another out.

3. True. Multiplication and division do not require finding the LCD because the denominators can be combined through those operations, whereas addition and subtraction require like terms.

5. $\frac{y+5}{y+6}$

7. $3b+3$

9. $\frac{x+4}{2x+2}$

11. $\frac{a+3}{a-3}$

13. $\frac{3n-8}{7n-3}$

15. $\frac{c-6}{c+6}$

17. 1

19. $\frac{d^2-25}{25d^2-1}$

21. $\frac{t+5}{t+3}$

23. $\frac{6x-5}{6x+5}$

25. $\frac{p+6}{4p+3}$

27. $\frac{2d+9}{d+11}$

29. $\frac{12b+5}{3b-1}$

31. $\frac{4y-1}{y+4}$

33. $\frac{10x+4y}{xy}$

35. $\frac{9a-7}{a^2-2a-3}$

37. $\frac{2y^2-y+9}{y^2-y-2}$

39. $\frac{5z^2+z+5}{z^2-z-2}$

41. $\frac{x+2xy+y}{x+xy+y+1}$

43. $\frac{2b+7a}{ab^2}$

45. $\frac{18+ab}{4b}$

47. $a-b$

49. $\frac{3c^2+3c-2}{2c^2+5c+2}$

51. $\frac{15x+7}{x-1}$

53. $\frac{x+9}{x-9}$

55. $\frac{1}{y+2}$

57. 4

Review Exercises

1. -5

3. 53

5. $y=24$

7. $32m$

9. whole

11. irrational

13. 16

15. $3a^6$

17. $\frac{x^3}{32y^3}$

19. a

21. 1.634×10^7

23. 14

25. $5\sqrt{3}$

27. $\frac{4\sqrt{2}}{5}$

29. $\frac{7\sqrt{2}}{50}$

31. $10\sqrt{3}$

33. -3

35. $3x^3 + 4x^2 + 6$

37. $5x^2 - x + 3$

39. $k^2 - 3k - 18$

41. $x^3 + x^2 + x + 1$

43. $3a^2 + 5ab - 2b^2$

45. $9p$

47. $4a^2$

49. $(4a - 3)(2a + 9)$

51. $(x + 5)^2$

53. $(2h - 3k)^2$

55. $(p + 6)\left(p^2 - 6p + 36\right)$

57. $(4q - 3p)\left(16q^2 + 12pq + 9p^2\right)$

59. $(p + 3)^{\frac{1}{3}}(-5p - 24)$

61. $\frac{x+3}{x-4}$

63. $\frac{1}{2}$

65. $\frac{m+2}{m-3}$

67. $\frac{6x+10y}{xy}$

69. $\frac{1}{6}$

Practice Test

1. rational

3. $x = -2$

5. 3,141,500

7. 16

9. 9

11. $2x$

13. 21

15. $\frac{3\sqrt{x}}{4}$

17. $21\sqrt{6}$

19. $13q^3 - 4q^2 - 5q$

21. $n^3 - 6n^2 + 12n - 8$

23. $(4x + 9)(4x - 9)$

25. $(3c - 11)\left(9c^2 + 33c + 121\right)$

27. $\frac{4z-3}{2z-1}$

29. $\frac{3a+2b}{3b}$

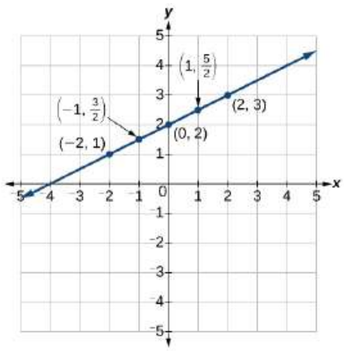
Chapter 2

Try It

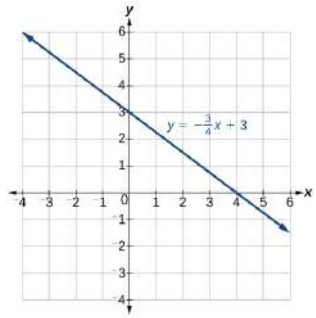
2.1 The Rectangular Coordinate Systems and Graphs

1.

x	$y = \frac{1}{2}x + 2$	(x, y)
-2	$y = \frac{1}{2}(-2) + 2 = 1$	$(-2, 1)$
-1	$y = \frac{1}{2}(-1) + 2 = \frac{3}{2}$	$\left(-1, \frac{3}{2}\right)$
0	$y = \frac{1}{2}(0) + 2 = 2$	$(0, 2)$
1	$y = \frac{1}{2}(1) + 2 = \frac{5}{2}$	$\left(1, \frac{5}{2}\right)$
2	$y = \frac{1}{2}(2) + 2 = 3$	$(2, 3)$



2. x-intercept is $(4, 0)$; y-intercept is $(0, 3)$.



3. $\sqrt{125} = 5\sqrt{5}$

4. $\left(-5, \frac{5}{2}\right)$

2.2 Linear Equations in One Variable

1. $x = -5$

2. $x = -3$

3. $x = \frac{10}{3}$

4. $x = 1$

5. $x = -\frac{7}{17}$. Excluded values are $x = -\frac{1}{2}$ and $x = -\frac{1}{3}$.

6. $x = \frac{1}{3}$

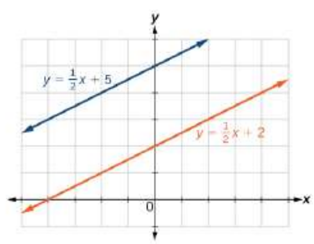
7. $m = -\frac{2}{3}$

8. $y = 4x-3$

9. $x + 3y = 2$

10. Horizontal line: $y = 2$

11. Parallel lines: equations are written in slope-intercept form.

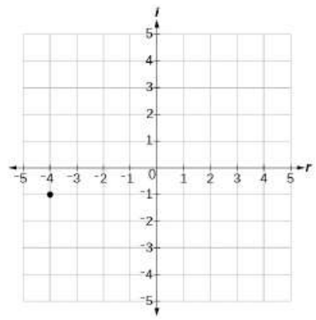


12. $y = 5x + 3$

2.3 Models and Applications

1. 11 and 25
2. $C = 2.5x + 3{,}650$
3. 45 mi/h
4. $L = 37$ cm, $W = 18$ cm
5. 250 ft^2

2.4 Complex Numbers

1. $\sqrt{-24} = 0 + 2i\sqrt{6}$
2. 

3. $(3-4i) - (2+5i) = 1-9i$
4. $\frac{5}{2} - i$
5. $18 + i$
6. $-3-4i$
7. -1

2.5 Quadratic Equations

1. $(x-6)(x+1) = 0; x = 6, x = -1$
2. $(x-7)(x+3) = 0, x = 7, x = -3.$
3. $(x+5)(x-5) = 0, x = -5, x = 5.$
4. $(3x+2)(4x+1) = 0, x = -\frac{2}{3}, x = -\frac{1}{4}$
5. $x = 0, x = -10, x = -1$
6. $x = 4 \pm \sqrt{5}$
7. $x = 3 \pm \sqrt{22}$
8. $x = -\frac{2}{3}, x = \frac{1}{3}$
9. 5 units

2.6 Other Types of Equations

1. $\frac{1}{4}$
2. 25
3. $\{-1\}$
4. $0, \frac{1}{2}, -\frac{1}{2}$
5. 1; extraneous solution $-\frac{2}{9}$
6. -2; extraneous solution -1
7. $-1, \frac{3}{2}$
8. $-3, 3, -i, i$
9. $2, 12$
10. $-1, 0$ is not a solution.

2.7 Linear Inequalities and Absolute Value Inequalities

1. $[-3, 5]$

2. $\left(-\infty, -2\right) \cup \left[3, \infty\right)$

3. $x < 1$

4. $x \geq -5$

5. $\left(2, \infty\right)$

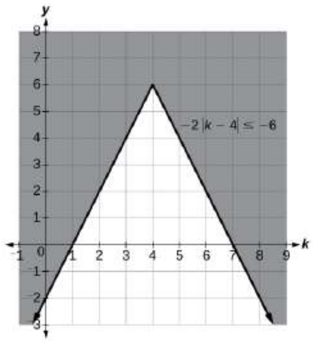
6. $\left[-\frac{3}{14}, \infty\right)$

7. $6 < x \leq 9$ or $(6, 9]$

8. $\left(-\frac{1}{8}, \frac{1}{2}\right)$

9. $|x-2| \leq 3$

10. $k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$.



2.1 Section Exercises

1. Answers may vary. Yes. It is possible for a point to be on the *x*-axis or on the *y*-axis and therefore is considered to NOT be in one of the quadrants.

3. The *y*-intercept is the point where the graph crosses the *y*-axis.

5. The *x*-intercept is $(2, 0)$ and the *y*-intercept is $(0, 6)$.

7. The *x*-intercept is $(2, 0)$ and the *y*-intercept is $(0, -3)$.

9. The *x*-intercept is $(3, 0)$ and the *y*-intercept is $\left(0, \frac{9}{8}\right)$.

11. $y = 4 - 2x$

13. $y = \frac{5-2x}{3}$

15. $y = 2x - \frac{4}{5}$

17. $d = \sqrt{74}$

19. $d = \sqrt{36} = 6$

21. $d \approx 62.97$

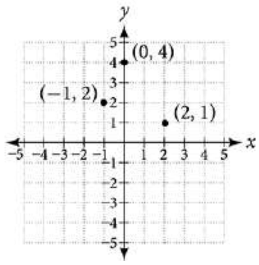
23. $\left(3, \frac{-3}{2}\right)$

25. $(2, -1)$

27. $(0, 0)$

29. $y = 0$

31.

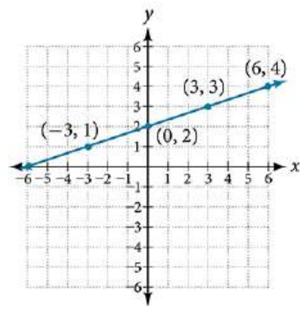


not collinear

33. A: $(-3,2)$, B: $(1,3)$, C: $(4,0)$

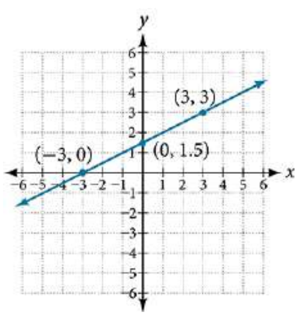
35.

x	y
-3	1
0	2
3	3
6	4

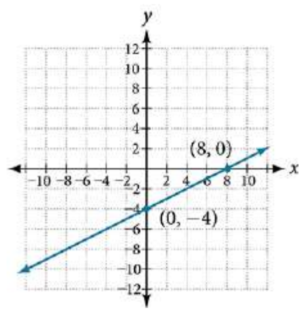


37.

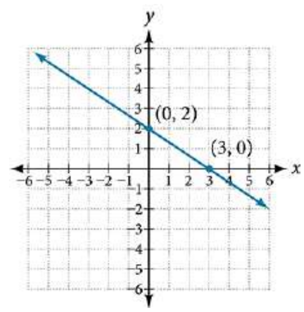
x	y
–3	0
0	1.5
3	3



39.



41.



43. $d = 8.246$

45. $d = 5$

47. $(-3,4)$

49. $x = 0$ $y = -2$

51. $x = 0.75 y = 0$

53. $x = -1.667 y = 0$

55. $15 - 11.2 = 3.8$mi shorter

57. 6.042

59. Midpoint of each diagonal is the same point $(2, –2)$. Note this is a characteristic of rectangles, but not other quadrilaterals.

61. 37mi

63. 54 ft

2.2 Section Exercises

1. It means they have the same slope.

3. The exponent of the x variable is 1. It is called a first-degree equation.

5. If we insert either value into the equation, they make an expression in the equation undefined (zero in the denominator).

7. $x = 2$

9. $x = \frac{2}{7}$

11. $x = 6$

13. $x = 3$

15. $x = -14$

17. $x \neq -4$; $x = -3$

19. $x \neq 1$; when we solve this we get $x = 1$, which is excluded, therefore NO solution

21. $x \neq 0$; $x = -\frac{5}{2}$

23. $y = -\frac{4}{5}x + \frac{14}{5}$

25. $y = -\frac{3}{4}x + 2$

27. $y = \frac{1}{2}x + \frac{5}{2}$

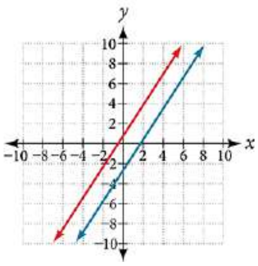
29. $y = -3x - 5$

31. $y = 7$

33. $y = -4$

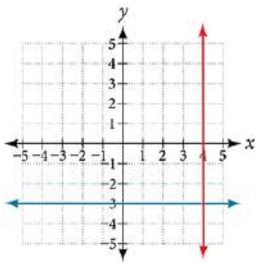
35. $8x + 5y = 7$

37.



Parallel

39.



Perpendicular

41. $m = -\frac{9}{7}$

43. $m = \frac{3}{2}$

45. $m_1 = -\frac{1}{3}, \quad m_2 = 3; \quad \text{Perpendicular.}$

47. $y = 0.245x - 45.662.$ Answers may vary. $y_{\min} = -50, \quad y_{\max} = -40$

49. $y = -2.333x + 6.667.$ Answers may vary. $y_{\min} = -10, \ y_{\max} = 10$

51. $y = -\frac{A}{B}x + \frac{C}{B}$

53.

$$\begin{aligned}
&\text{The slope for } (-1, 1) \text{ to } (0, 4) \text{ is } 3.\\
&\text{The slope for } (-1, 1) \text{ to } (2, 0) \text{ is } \tfrac{-1}{3}.\\
&\text{The slope for } (2, 0) \text{ to } (3, 3) \text{ is } 3.\\
&\text{The slope for } (0, 4) \text{ to } (3, 3) \text{ is } \tfrac{-1}{3}.
\end{aligned}$$

Yes they are perpendicular.

55. 30 ft

57. $57.50

59. 220 mi

2.3 Section Exercises

1. Answers may vary. Possible answers: We should define in words what our variable is representing. We should declare the variable. A heading.

3. $2,000 - x$

5. $v + 10$

7. Ann: 23; Beth: 46

9. $20 + 0.05m$

11. 300 min

13. $90 + 40P$

15. 6 devices

17. $50,000 - x$

19. 4 h

21. She traveled for 2 h at 20 mi/h, or 40 miles.

23. \$5,000 at 8% and \$15,000 at 12%

25. $B = 100 + .05x$

27. Plan A

29. $R = 9$

31. $r = \frac{4}{5}$ or 0.8

33. $W = \frac{P-2L}{2} = \frac{58-2(15)}{2} = 14$

35. $f = \frac{pq}{p+q} = \frac{8(13)}{8+13} = \frac{104}{21}$

37. $m = \frac{-5}{4}$

39. $h = \frac{2A}{b_1+b_2}$

41. length = 360 ft; width = 160 ft

43. 405 mi

45. $A = 88 \text{ in.}^2$

47. 28.7

49. $h = \frac{V}{\pi r^2}$

51. $r = \sqrt{\frac{V}{\pi h}}$

53. $C = 12\pi$

2.4 Section Exercises

1. Add the real parts together and the imaginary parts together.

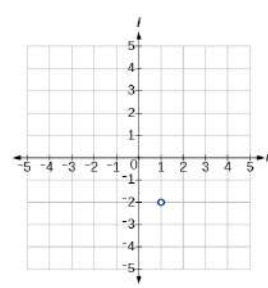
3. Possible answer: i times i equals -1, which is not imaginary.

5. $-8 + 2i$

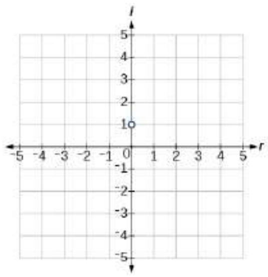
7. $14 + 7i$

9. $-\frac{23}{29} + \frac{15}{29}i$

11.



13.



15. $8 - i$

17. $-11 + 4i$

19. $2{-}5i$

21. $6 + 15i$

23. $-16 + 32i$

25. $-4{-}7i$

27. 25

29. $2 - \frac{2}{3}i$

31. $4 - 6i$

33. $\frac{2}{5} + \frac{11}{5}i$

35. $15i$

37. $1 + i\sqrt{3}$

39. 1

41. -1

43. 128i

45. $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^6 = -1$

47. $3i$

49. 0

51. $5{-}5i$

53. $-2i$

55. $\frac{9}{2} - \frac{9}{2}i$

2.5 Section Exercises

1. It is a second-degree equation (the highest variable exponent is 2).

3. We want to take advantage of the zero property of multiplication in the fact that if $a \cdot b = 0$ then it must follow that each factor separately offers a solution to the product being zero: $a = 0 \;\; or \;\; \mathrm{b} = 0.$

5. One, when no linear term is present (no *x* term), such as $x^2 = 16$. Two, when the equation is already in the form $(ax + b)^2 = d$.

7. $x = 6, x = 3$

9. $x = \frac{-5}{2}, x = \frac{-1}{3}$

11. $x = 5, x = -5$

13. $x = \frac{-3}{2}, x = \frac{3}{2}$

15. $x = -2, 3$

17. $x = 0, x = \frac{-3}{7}$

19. $x = -6, x = 6$

21. $x = 6, x = -4$

23. $x = 1, x = -2$

25. $x = -2, x = 11$

27. $x = 3 \pm \sqrt{22}$

29. $z = \frac{2}{3}, z = -\frac{1}{2}$

31. $x = \frac{3 \pm \sqrt{17}}{4}$

33. Not real

35. One rational

37. Two real; rational

39. $x=\frac{-1\pm\sqrt{17}}{2}$

41. $x=\frac{5\pm\sqrt{13}}{6}$

43. $x=\frac{-1\pm\sqrt{17}}{8}$

45. $x\approx0.131$ and $x\approx2.535$

47. $x\approx-6.7$ and $x\approx1.7$

49.
$$\begin{aligned} ax^2+bx+c &= 0\\ x^2+\frac{b}{a}x &= \frac{-c}{a}\\ x^2+\frac{b}{a}x+\frac{b^2}{4a^2} &= \frac{-c}{a}+\frac{b}{4a^2}\\ \left(x+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2}\\ x+\frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}}\\ x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\end{aligned}$$

51. $x(x+10)=119$; 7 ft. and 17 ft.

53. maximum at $x=70$

55. The quadratic equation would be $(100x-0.5x^2)-(60x+300)=300$. The two values of x are 20 and 60.

57. 3 feet

2.6 Section Exercises

1. This is not a solution to the radical equation, it is a value obtained from squaring both sides and thus changing the signs of an equation which has caused it not to be a solution in the original equation.

3. They are probably trying to enter negative 9, but taking the square root of -9 is not a real number. The negative sign is in front of this, so your friend should be taking the square root of 9, cubing it, and then putting the negative sign in front, resulting in -27.

5. A rational exponent is a fraction: the denominator of the fraction is the root or index number and the numerator is the power to which it is raised.

7. $x=81$

9. $x=17$

11. $x=8,\quad x=27$

13. $x=-2,1,-1$

15. $y=0,\ \frac{3}{2},\ \frac{-3}{2}$

17. $m=1,-1$

19. $x=\frac{2}{5},\pm3i$

21. $x=32$

23. $t=\frac{44}{3}$

25. $x=3$

27. $x=-2$

29. $x=4,\frac{-4}{3}$

31. $x=\frac{-5}{4},\frac{7}{4}$

33. $x=3,-2$

35. $x=-5$

37. $x=1,-1,3,-3$

39. $x=2,-2$

41. $x=1,5$

43. $x\geq0$

45. $x=4,6,-6,-8$

47. 10 in.

49. 90 kg

2.7 Section Exercises

1. When we divide both sides by a negative it changes the sign of both sides so the sense of the inequality sign changes.

3. $\left(-\infty, \infty\right)$

5. We start by finding the x-intercept, or where the function = 0. Once we have that point, which is $(3, 0)$, we graph to the right the straight line graph $y = x{-}3$, and then when we draw it to the left we plot positive y values, taking the absolute value of them.

7. $\left(-\infty, \frac{3}{4}\right]$

9. $\left[-\frac{13}{2}, \infty\right)$

11. $\left(-\infty, 3\right)$

13. $\left(-\infty, -\frac{37}{3}\right]$

15. All real numbers $\left(-\infty, \infty\right)$

17. $\left(-\infty, -\frac{10}{3}\right) \cup \left(4, \infty\right)$

19. $\left(-\infty, -4\right] \cup \left[8, +\infty\right)$

21. No solution

23. $(-5, 11)$

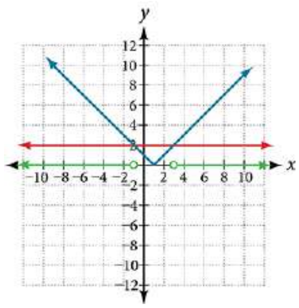
25. $[6, 12]$

27. $[-10, 12]$

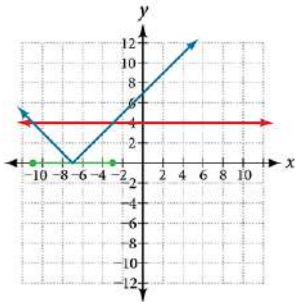
29.
$$\begin{array}{ll} x > -6 \ \text{ and } \ x > -2 & \text{Take the intersection of two sets.} \\ x > -2, \quad (-2, +\infty) & \end{array}$$

31.
$$\begin{array}{ll} x < -3 \quad \text{or} \quad x \geq 1 & \text{Take the union of the two sets.} \\ (-\infty, -3) \cup \quad [1, \infty) & \end{array}$$

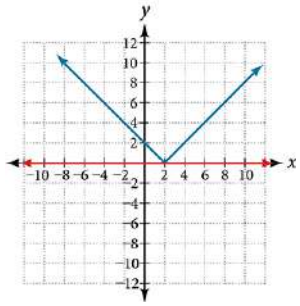
33. $\left(-\infty, -1\right) \cup \left(3, \infty\right)$



35. $[-11, -3]$

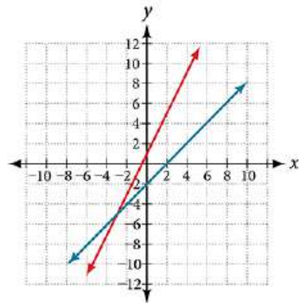


37. It is never less than zero. No solution.



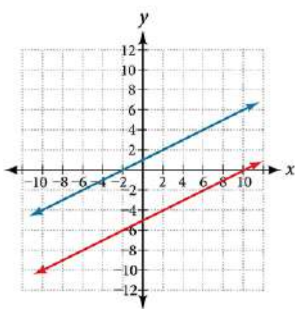
39. Where the blue line is above the orange line; point of intersection is $x = -3$.

$\left(-\infty, -3\right)$



41. Where the blue line is above the orange line; always. All real numbers.

$(-\infty, -\infty)$



43. $(-1, 3)$

45. $\left(-\infty, 4\right)$

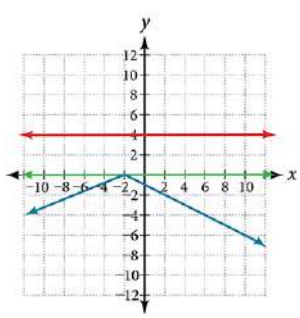
47. $\{x|x < 6\}$

49. $\{x|-3 \leq x < 5\}$

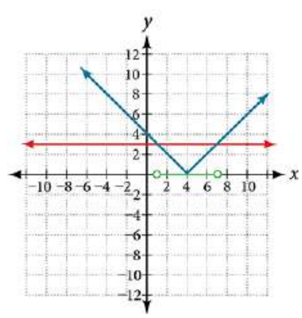
51. $(-2, 1]$

53. $\left(-\infty, 4\right]$

55. Where the blue is below the orange; always. All real numbers. $(-\infty, +\infty)$.



57. Where the blue is below the orange; $(1, 7)$.



59. $x = 2, \frac{-4}{5}$

61. $(-7, 5]$

63. $80 \le T \le 120$
$1,600 \le 20T \le 2,400$
$[1,600, 2,400]$

Review Exercises

1. *x*-intercept: $(3, 0)$; *y*-intercept: $(0, -4)$

3. $y = \frac{5}{3}x + 4$

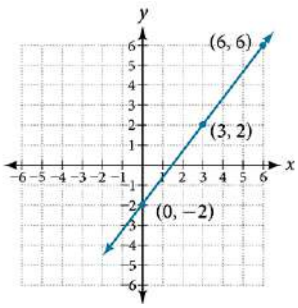
5. $\sqrt{72} = 6\sqrt{2}$

7. 620.097

9. midpoint is $\left(2, \frac{23}{2}\right)$

11.

x	*y*
0	−2
3	2
6	6



13. $x = 4$

15. $x = \frac{12}{7}$

17. No solution

19. $y = \frac{1}{6}x + \frac{4}{3}$

21. $y = \frac{2}{3}x + 6$

23. females 17, males 56

25. 84 mi

27. $x = -\frac{3}{4} \pm \frac{i\sqrt{47}}{4}$

29. horizontal component -2; vertical component -1

31. $7 + 11i$

33. $16i$

35. $-16 - 30i$

37. $-4 - i\sqrt{10}$

39. $x = 7 - 3i$

41. $x = -1, -5$

43. $x = 0, \frac{9}{7}$

45. $x = 10, -2$

47. $x = \frac{-1 \pm \sqrt{5}}{4}$

49. $x = \frac{2}{5}, \frac{-1}{3}$

51. $x = 5 \pm 2\sqrt{7}$

53. $x = 0, 256$

55. $x = 0, \pm\sqrt{2}$

57. $x = -2$

59. $x = \frac{11}{2}, \frac{-17}{2}$

61. $\left(-\infty, 4\right)$

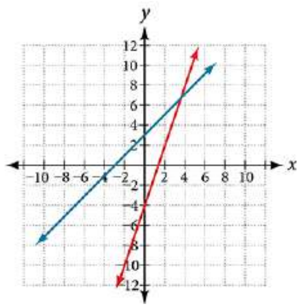
63. $\left[\frac{-10}{3}, 2\right]$

65. No solution

67. $\left(-\frac{4}{3}, \frac{1}{5}\right)$

69. Where the blue is below the orange line; point of intersection is $x = 3.5$.

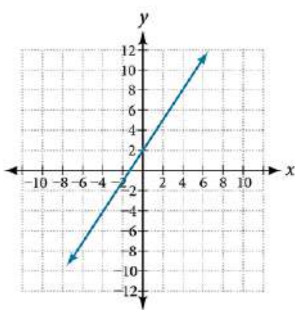
$\left(3.5, \infty\right)$



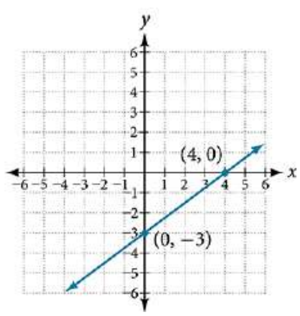
Practice Test

1. $y = \frac{3}{2}x + 2$

x	y
0	2
2	5
4	8



3. $(0, -3)\ (4, 0)$



5. $\left(-\infty, 9\right]$

7. $x = -15$

9. $x \neq -4, 2;\ x = \frac{-5}{2}, 1$

11. $x = \frac{3 \pm \sqrt{3}}{2}$

13. $(-4, 1)$

15. $y = \frac{-5}{9}x - \frac{2}{9}$

17. $y = \frac{5}{2}x - 4$

19. $14i$

21. $\frac{5}{13} - \frac{14}{13}i$

23. $x = 2, \frac{-4}{3}$

25. $x = \frac{1}{2} \pm \frac{\sqrt{2}}{2}$

27. 4

29. $x = \frac{1}{2}, 2, -2$

Chapter 3

Try It

3.1 Functions and Function Notation

1. (a) yes
(b) yes (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

2. $w = f(d)$

3. yes

4. $g(5) = 1$

5. $m = 8$

6. $y = f(x) = \frac{\sqrt[3]{x}}{2}$

7. $g(1) = 8$

8. $x = 0$ or $x = 2$

9. (a) yes, because each bank account has a single balance at any given time;
(b) no, because several bank account numbers may have the same balance;
(c) no, because the same output may correspond to more than one input.

10. (a) Yes, letter grade is a function of percent grade;
(b) No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

11. yes

12. No, because it does not pass the horizontal line test.

3.2 Domain and Range

1. $\{-5, \ 0, \ 5, \ 10, \ 15\}$

2. $\left(-\infty, \infty\right)$

3. $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

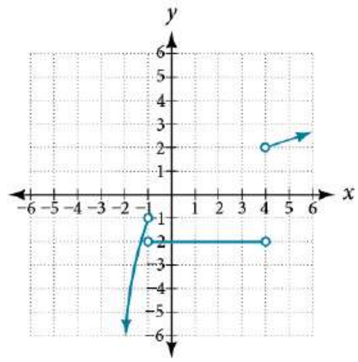
4. $\left[-\frac{5}{2}, \infty\right)$

5. (a) values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3
(b) $\{x | x \leq -2 \ \text{ or } \ -1 \leq x < 3\}$
(c) $(-\infty, -2] \cup [-1, 3)$

6. domain =[1950,2002] range = [47,000,000,89,000,000]

7. domain: $\left(-\infty, 2\right]$; range: $\left(-\infty, 0\right]$

8.



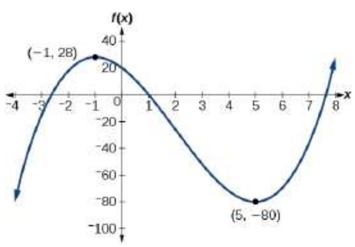
3.3 Rates of Change and Behavior of Graphs

1. $\frac{\$2.84-\$2.31}{5 \text{ years}} = \frac{\$0.53}{5 \text{ years}} = \$0.106$ per year.

2. $\frac{1}{2}$

3. $a+7$

4. The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.



3.4 Composition of Functions

1. $$(fg)(x) = f(x)g(x) = (x-1)\left(x^2-1\right) = x^3 - x^2 - x + 1$$
$$(f-g)(x) = f(x) - g(x) = (x-1) - \left(x^2-1\right) = x - x^2$$

No, the functions are not the same.

2. A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance *r* from the Sun (due to gravity), but $G(a(F))$ does not make sense.

3. $f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

4. $g(f(2)) = g(5) = 3$

5. ⓐ 8 ⓑ 20

6. $[-4, 0) \cup \left(0, \infty\right)$

7. Possible answer:

$g\left(x\right) = \sqrt{4 + x^2}$

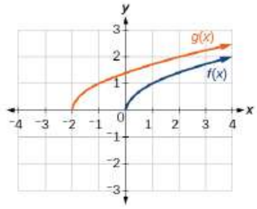
$h\left(x\right) = \frac{4}{3 - x}$

$f = h \circ g$

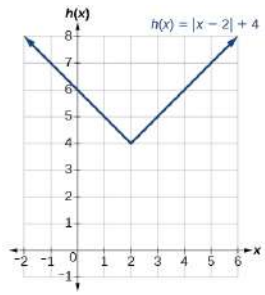
3.5 Transformation of Functions

1. $b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

2. The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

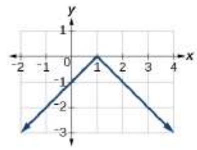


3.

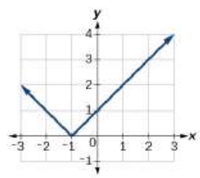


4. $g\left(x\right) = \frac{1}{x-1} + 1$

5. ⓐ



ⓑ



6. ⓐ

$g(x) = -f(x)$

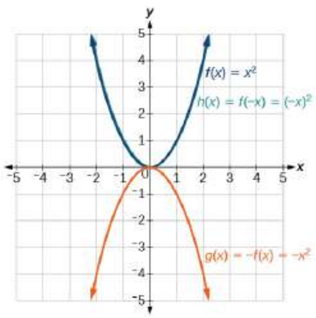
x	-2	0	2	4
$g(x)$	-5	-10	-15	-20

ⓑ

$h(x) = f(-x)$

x	-2	0	2	4
$h(x)$	15	10	5	unknown

7.



Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

8. even

9.

x	2	4	6	8
$g(x)$	9	12	15	0

10. $g(x) = 3x - 2$

11. $g(x) = f\left(\frac{1}{3}x\right)$ so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

3.6 Absolute Value Functions

1. using the variable p for passing, $|p - 80| \leq 20$

2. $f(x) = -|x + 2| + 3$

3. $x = -1$ or $x = 2$

3.7 Inverse Functions

1. $h(2) = 6$

2. Yes

3. Yes

4. The domain of function f^{-1} is $(-\infty, -2)$ and the range of function f^{-1} is $(1, \infty)$.

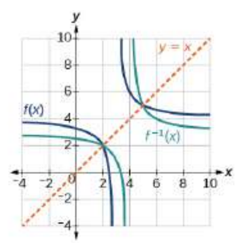
5. ⓐ $f(60) = 50$. In 60 minutes, 50 miles are traveled.
ⓑ $f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

6. ⓐ 3 ⓑ 5.6

7. $x = 3y + 5$

8. $f^{-1}(x) = (2 - x)^2$;
domain of f : $\left[0, \infty\right)$;
domain of f^{-1} : $\left(-\infty, 2\right]$

9.



3.1 Section Exercises

1. A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

3. When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

5. When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

7. function

9. function

11. function

13. function

15. function

17. function

19. function

21. function

23. function

25. not a function

27. $f(-3) = -11;$
$f(2) = -1;$
$f(-a) = -2a - 5;$
$-f(a) = -2a + 5;$
$f(a + h) = 2a + 2h - 5$

29. $f(-3) = \sqrt{5} + 5;$
$f(2) = 5;$
$f(-a) = \sqrt{2 + a} + 5;$
$-f(a) = -\sqrt{2 - a} - 5;$
$f(a + h) = \sqrt{2 - a - h} + 5$

31. $f(-3) = 2; f(2) = 1 - 3 = -2;$
$f(-a) = | - a - 1| - | - a + 1|;$
$-f(a) = -|a - 1| + |a + 1|;$
$f(a + h) = |a + h - 1| - |a + h + 1|$

33. $\frac{g(x)-g(a)}{x-a} = x + a + 2, \quad x \neq a$

35. a. $f(-2) = 14$; b. $x = 3$

37. a. $f(5) = 10$; b. $x = -1$ or $x = 4$

39. a. $f(t) = 6 - \frac{2}{3}t;$ b. $f(-3) = 8;$ c. $t = 6$

41. not a function

43. function

45. function

47. function

49. function

51. function

53. a. $f(0) = 1;$ b. $f(x) = -3, x = -2$ or $x = 2$

55. not a function so it is also not a one-to-one function

57. one-to- one function

59. function, but not one-to-one

61. function

63. function

65. not a function

67. $f(x) = 1, x = 2$

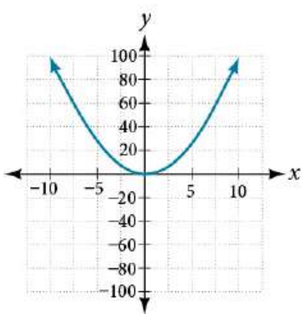
69. $f(-2) = 14; \quad f(-1) = 11; \quad f(0) = 8; \quad f(1) = 5; \quad f(2) = 2$

71. $f(-2) = 4; \quad f(-1) = 4.414; \quad f(0) = 4.732; \quad f(1) = 5; \quad f(2) = 5.236$

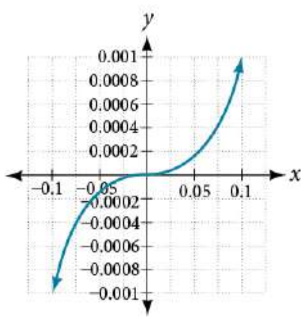
73. $f(-2) = \frac{1}{9}; \quad f(-1) = \frac{1}{3}; \quad f(0) = 1; \quad f(1) = 3; \quad f(2) = 9$

75. 20

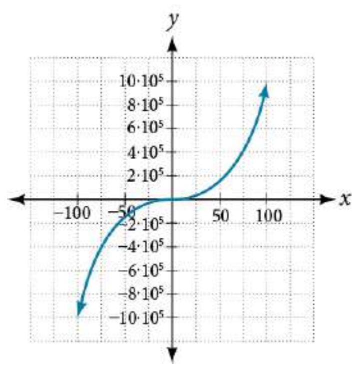
77. $[0, \quad 100]$



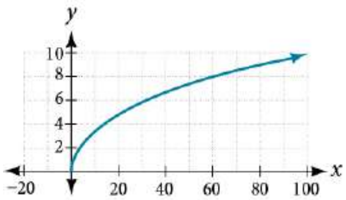
79. $[-0.001, \quad 0.001]$



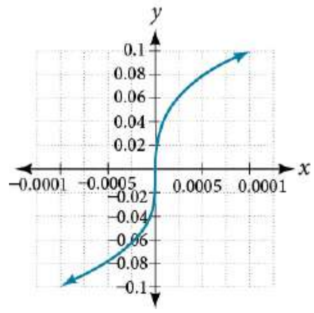
81. $[-1,000,000, \quad 1{,}000{,}000]$



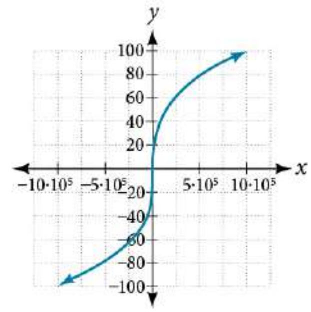
83. $[0, \quad 10]$



85. $[-0.1, 0.1]$



87. $[-100, \quad 100]$



89. ⓐ $g(5000) = 50$; ⓑ The number of cubic yards of dirt required for a garden of 100 square feet is 1.

91. ⓐ The height of a rocket above ground after 1 second is 200 ft. ⓑ The height of a rocket above ground after 2 seconds is 350 ft.

3.2 Section Exercises

1. The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

3. There is no restriction on x for $f(x) = \sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x -values are restricted for $f(x) = \sqrt{x}$ to nonnegative numbers and the domain is $[0, \infty)$.

5. Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x -axis and y -axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints with an open circle. Use an arrow to indicate $-\infty$ or ∞. Combine the graphs to find the graph of the piecewise function.

7. $(-\infty, \infty)$

9. $(-\infty, 3]$

11. $(-\infty, \infty)$

13. $(-\infty, \infty)$

15. $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

17. $(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$

19. $(-\infty, -3) \cup (-3, 5) \cup (5, \infty)$

21. $(-\infty, 5)$

23. $[6, \infty)$

25. $\left(-\infty, -9\right) \cup (-9, 9) \cup \left(9, \infty\right)$

27. domain: $(2, 8]$, range $[6, 8)$

29. domain: $[-4, \quad 4]$, range: $[0, \quad 2]$

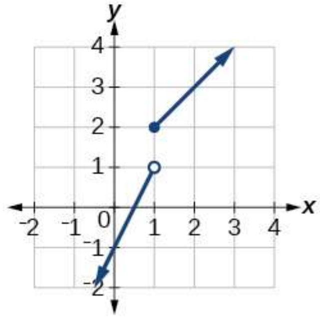
31. domain: $[-5, \quad 3)$, range: $[0, 2]$

33. domain: $(-\infty, 1]$, range: $[0, \infty)$

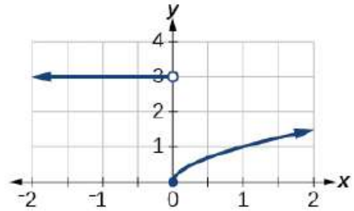
35. domain: $\left[-6, -\frac{1}{6}\right] \cup \left[\frac{1}{6}, 6\right]$; range: $\left[-6, -\frac{1}{6}\right] \cup \left[\frac{1}{6}, 6\right]$

37. domain: $[-3, \infty)$; range: $[0, \infty)$

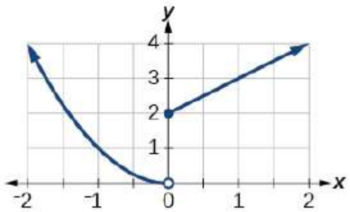
39. domain: $(-\infty, \infty)$



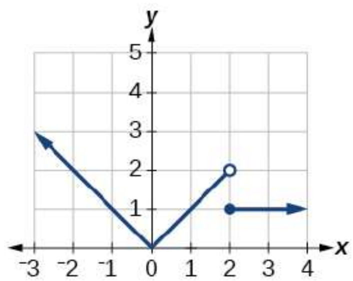
41. domain: $(-\infty, \infty)$



43. domain: $(-\infty, \infty)$



45. domain: $(-\infty, \infty)$



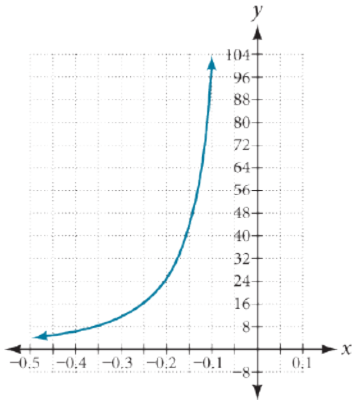
47. $f(-3) = 1;\quad f(-2) = 0;\quad f(-1) = 0;\quad f(0) = 0$

49. $f(-1) = -4;\quad f(0) = 6;\quad f(2) = 20;\quad f(4) = 34$

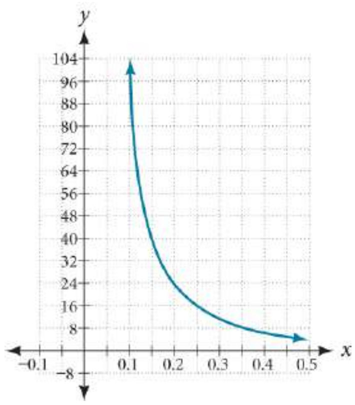
51. $f(-1) = -5;\quad f(0) = 3;\quad f(2) = 3;\quad f(4) = 16$

53. domain: $(-\infty, 1) \cup (1, \infty)$

55.



window: $[-0.5, -0.1]$; range: $[4, \ 100]$



window: $[0.1, \ 0.5]$; range: $[4, \ 100]$

57. $[0, \ 8]$

59. Many answers. One function is $f(x) = \frac{1}{\sqrt{x-2}}$.

61. (a) The fixed cost is \$500.
(b) The cost of making 25 items is \$750.
(c) The domain is [0, 100] and the range is [500, 1500].

3.3 Section Exercises

1. Yes, the average rate of change of all linear functions is constant.

3. The absolute maximum and minimum relate to the entire graph, whereas the local extrema relate only to a specific region around an open interval.

5. $4(b+1)$

7. 3

9. $4x + 2h$

11. $\frac{-1}{13(13+h)}$

13. $3h^2 + 9h + 9$

15. $4x + 2h - 3$

17. $\frac{4}{3}$

19. increasing on $\left(-\infty, -2.5\right) \cup \left(1, \infty\right)$, decreasing on $(-2.5,\ \ 1)$

21. increasing on $\left(-\infty, 1\right) \cup (3, 4)$, decreasing on $(1, 3) \cup \left(4, \infty\right)$

23. local maximum: $(-3,\ \ 60)$, local minimum: $(3,\ \ -60)$

25. absolute maximum at approximately $(7,\ \ 150)$, absolute minimum at approximately $(-7.5,\ \ -220)$

27. ⓐ –3000 ⓑ –1250

29. -4

31. 27

33. –0.167

35. Local minimum at $(3, -22)$, decreasing on $(-\infty,\ \ 3)$, increasing on $(3,\ \ \infty)$

37. Local minimum at $(-2, -2)$, decreasing on $(-3, -2)$, increasing on $(-2,\ \ \infty)$

39. Local maximum at $(-0.39,\ \ 6)$, local minima at $(-3.15, -47)$ and $(2.04, -32)$, decreasing on $(-\infty, -3.15)$ and $(-0.39,\ \ 2.04)$, increasing on $(-3.15,\ \ -0.39)$ and $(2.04,\ \ \infty)$

41. A

43. $b = 5$

45. 2.7 gallons per minute

47. approximately –0.6 milligrams per day

3.4 Section Exercises

1. Find the numbers that make the function in the denominator g equal to zero, and check for any other domain restrictions on f and g, such as an even-indexed root or zeros in the denominator.

3. Yes. Sample answer: Let $f(x) = x + 1$ and $g(x) = x - 1$. Then $f(g(x)) = f(x - 1) = (x - 1) + 1 = x$ and $g(f(x)) = g(x + 1) = (x + 1) - 1 = x$. So $f \circ g = g \circ f$.

5. $(f+g)(x)=2x+6$, domain: $(-\infty,\infty)$

$(f-g)(x)=2x^2+2x-6$, domain: $(-\infty,\infty)$

$(fg)(x)=-x^4-2x^3+6x^2+12x$, domain: $(-\infty,\infty)$

$\left(\frac{f}{g}\right)(x)=\frac{x^2+2x}{6-x^2}$, domain: $(-\infty,-\sqrt{6})\cup(-\sqrt{6},\sqrt{6})\cup(\sqrt{6},\infty)$

7. $(f+g)(x)=\frac{4x^3+8x^2+1}{2x}$, domain: $(-\infty,0)\cup(0,\infty)$

$(f-g)(x)=\frac{4x^3+8x^2-1}{2x}$, domain: $(-\infty,0)\cup(0,\infty)$

$(fg)(x)=x+2$, domain: $(-\infty,0)\cup(0,\infty)$

$\left(\frac{f}{g}\right)(x)=4x^3+8x^2$, domain: $(-\infty,0)\cup(0,\infty)$

9. $(f+g)(x)=3x^2+\sqrt{x-5}$, domain: $[5,\infty)$

$(f-g)(x)=3x^2-\sqrt{x-5}$, domain: $[5,\infty)$

$(fg)(x)=3x^2\sqrt{x-5}$, domain: $[5,\infty)$

$\left(\frac{f}{g}\right)(x)=\frac{3x^2}{\sqrt{x-5}}$, domain: $(5,\infty)$

11. ⓐ 3 ⓑ $f(g(x))=2(3x-5)^2+1$
ⓒ $f(g(x))=6x^2-2$
ⓓ
$(g\circ g)(x)=3(3x-5)-5=9x-20$
ⓔ $(f\circ f)(-2)=163$

13. $f(g(x))=\sqrt{x^2+3}+2,\quad g(f(x))=x+4\sqrt{x}+7$

15. $f(g(x))=\sqrt[3]{\frac{x+1}{x^3}}=\frac{\sqrt[3]{x+1}}{x},\quad g(f(x))=\frac{\sqrt[3]{x}+1}{x}$

17. $(f\circ g)(x)=\frac{1}{\frac{2}{x}+4-4}=\frac{x}{2},\quad (g\circ f)(x)=2x-4$

19. $f(g(h(x)))=\left(\frac{1}{x+3}\right)^2+1$

21. ⓐ $(g\circ f)(x)=-\frac{3}{\sqrt{2-4x}}$
ⓑ $\left(-\infty,\frac{1}{2}\right)$

23. ⓐ $(0,2)\cup(2,\infty)$;
ⓑ $(-\infty,-2)\cup(2,\infty)$;
ⓒ $(0,\infty)$

25. $(1,\infty)$

27. sample: $f(x)=x^3$
$g(x)=x-5$

29. sample: $f(x)=\frac{4}{x}$
$g(x)=(x+2)^2$

31. sample: $f(x)=\sqrt[3]{x}$
$g(x)=\frac{1}{2x-3}$

33. sample: $f(x)=\sqrt[4]{x}$
$g(x)=\frac{3x-2}{x+5}$

35. sample: $f(x)=\sqrt{x}$
$g(x)=2x+6$

37. sample: $f(x)=\sqrt[3]{x}$
$g(x)=(x-1)$

39. sample: $f(x)=x^3$
$g(x)=\frac{1}{x-2}$

41. sample: $f(x)=\sqrt{x}$
$g(x)=\frac{2x-1}{3x+4}$

43. 2

45. 5

47. 4

49. 0

51. 2

53. 1

55. 4

57. 4

59. 9

61. 4

63. 2

65. 3

67. 11

69. 0

71. 7

73. $f(g(0)) = 27, \ g(f(0)) = -94$

75. $f(g(0)) = \frac{1}{5}, \ g(f(0)) = 5$

77. $18x^2 + 60x + 51$

79. $g \circ g(x) = 9x + 20$

81. 2

83. $(-\infty, \infty)$

85. False

87. $(f \circ g)(6) = 6$; $(g \circ f)(6) = 6$

89. $(f \circ g)(11) = 11$, $(g \circ f)(11) = 11$

91. c

93. $A(t) = \pi\left(25\sqrt{t+2}\right)^2$ and $A(2) = \pi\left(25\sqrt{4}\right)^2 = 2500\pi$ square inches

95. $A(5) = \pi(2(5)+1)^2 = 121\pi$ square units

97. ⓐ $N(T(t)) = 23(5t + 1.5)^2 - 56(5t + 1.5) + 1$ ⓑ 3.38 hours

3.5 Section Exercises

1. A horizontal shift results when a constant is added to or subtracted from the input. A vertical shifts results when a constant is added to or subtracted from the output.

3. A horizontal compression results when a constant greater than 1 is multiplied by the input. A vertical compression results when a constant between 0 and 1 is multiplied by the output.

5. For a function f, substitute $(-x)$ for (x) in $f(x)$. Simplify. If the resulting function is the same as the original function, $f(-x) = f(x)$, then the function is even. If the resulting function is the opposite of the original function, $f(-x) = -f(x)$, then the original function is odd. If the function is not the same or the opposite, then the function is neither odd nor even.

7. $g(x) = |x - 1| - 3$

9. $g(x) = \frac{1}{(x+4)^2} + 2$

11. The graph of $f(x + 43)$ is a horizontal shift to the left 43 units of the graph of f.

13. The graph of $f(x - 4)$ is a horizontal shift to the right 4 units of the graph of f.

15. The graph of $f(x) + 8$ is a vertical shift up 8 units of the graph of f.

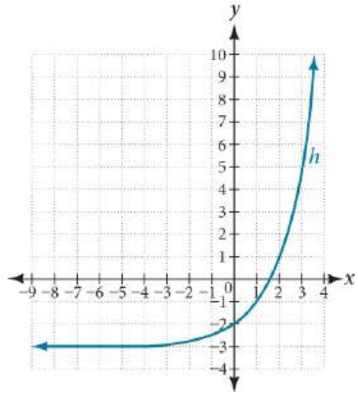
17. The graph of $f(x) - 7$ is a vertical shift down 7 units of the graph of f.

19. The graph of $f(x + 4) - 1$ is a horizontal shift to the left 4 units and a vertical shift down 1 unit of the graph of f.

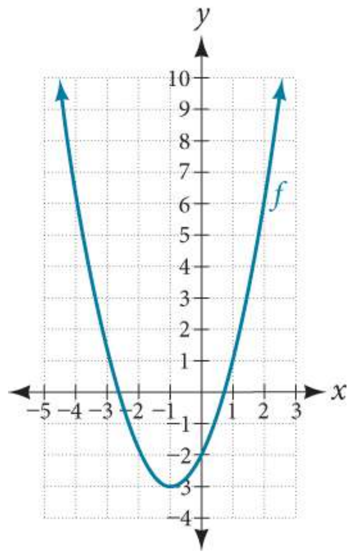
21. decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$

23. decreasing on $(0, \ \infty)$

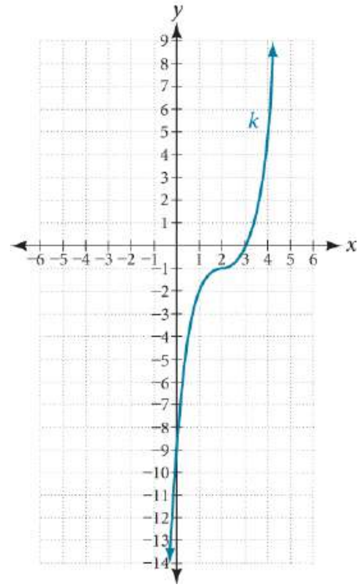
25.



27.



29.



31. $g(x) = f(x-1),\quad h(x) = f(x) + 1$

33. $f(x) = |x-3| - 2$

35. $f(x) = \sqrt{x+3} - 1$

37. $f(x) = (x-2)^2$

39. $f(x) = |x+3| - 2$

41. $f(x) = -\sqrt{x}$

43. $f(x) = -(x+1)^2 + 2$

45. $f(x) = \sqrt{-x} + 1$

47. even

49. odd

51. even

53. The graph of g is a vertical reflection (across the x -axis) of the graph of f.

55. The graph of g is a vertical stretch by a factor of 4 of the graph of f.

57. The graph of g is a horizontal compression by a factor of $\frac{1}{5}$ of the graph of f.

59. The graph of g is a horizontal stretch by a factor of 3 of the graph of f.

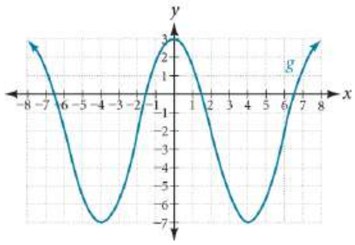
61. The graph of g is a horizontal reflection across the y -axis and a vertical stretch by a factor of 3 of the graph of f.

63. $g(x) = |-4x|$

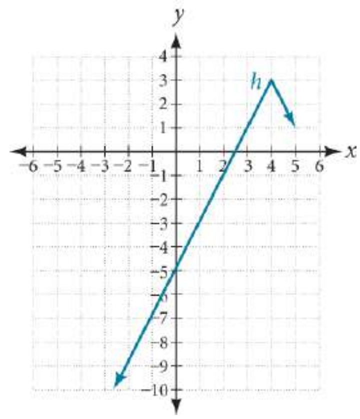
65. $g(x) = \frac{1}{3(x+2)^2} - 3$

67. $g(x) = \frac{1}{2}(x-5)^2 + 1$

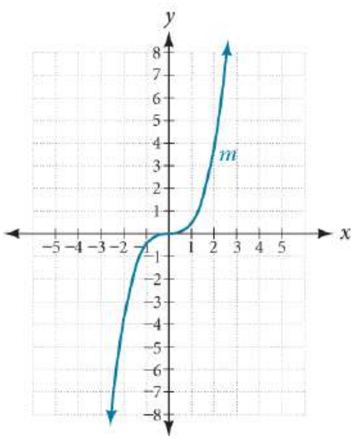
69. The graph of the function $f(x) = x^2$ is shifted to the left 1 unit, stretched vertically by a factor of 4, and shifted down 5 units.



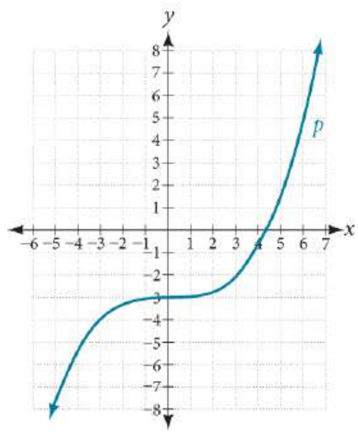
71. The graph of $f(x) = |x|$ is stretched vertically by a factor of 2, shifted horizontally 4 units to the right, reflected across the horizontal axis, and then shifted vertically 3 units up.



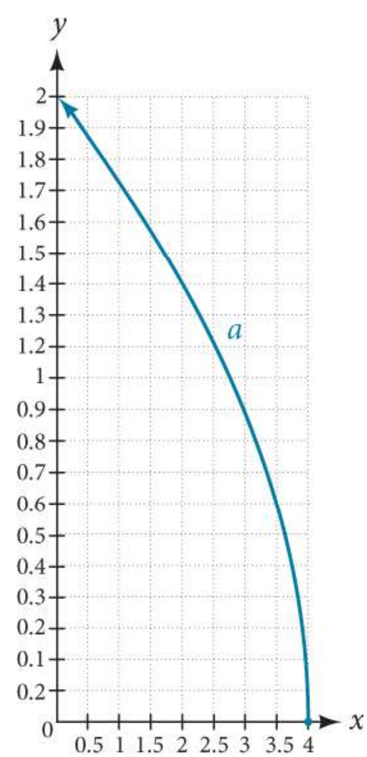
73. The graph of the function $f(x) = x^3$ is compressed vertically by a factor of $\frac{1}{2}$.



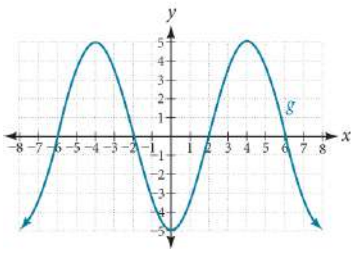
75. The graph of the function is stretched horizontally by a factor of 3 and then shifted vertically downward by 3 units.



77. The graph of $f(x) = \sqrt{x}$ is shifted right 4 units and then reflected across the vertical line $x = 4$.



79.



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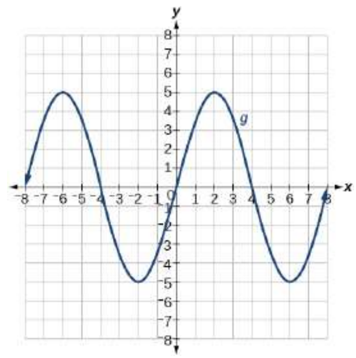


3.6 Section Exercises

1. Isolate the absolute value term so that the equation is of the form $|A| = B$. Form one equation by setting the expression inside the absolute value symbol, A, equal to the expression on the other side of the equation, B. Form a second equation by setting A equal to the opposite of the expression on the other side of the equation, $-B$. Solve each equation for the variable.

3. The graph of the absolute value function does not cross the x -axis, so the graph is either completely above or completely below the x -axis.

5. The distance from x to 8 can be represented using the absolute value statement: | x − 8 | = 4.

7. | x − 10 | ≥ 15

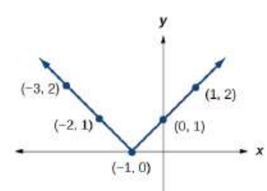
9. There are no x-intercepts.

11. (−4, 0) and (2, 0)

13. $(0, -4), (4, 0), (-2, 0)$

15. $(0, 7), (25, 0), (-7, 0)$

17.



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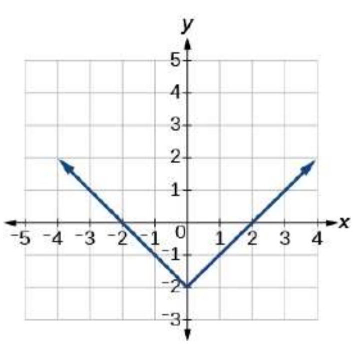

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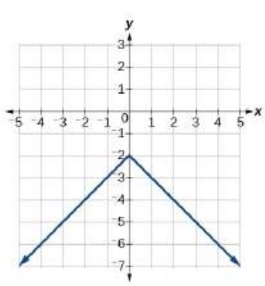

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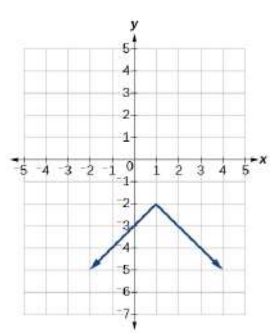

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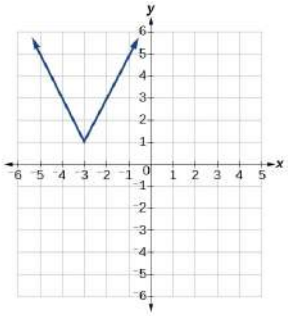

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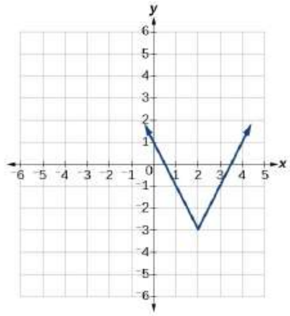

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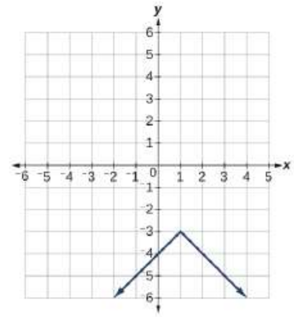

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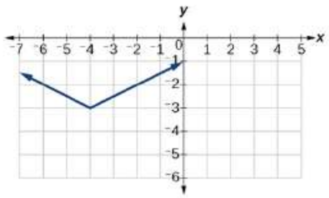
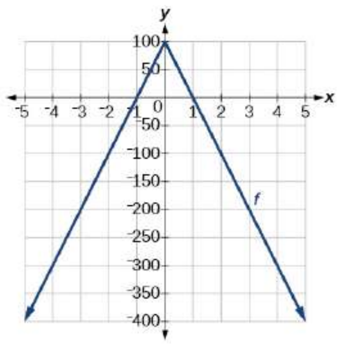
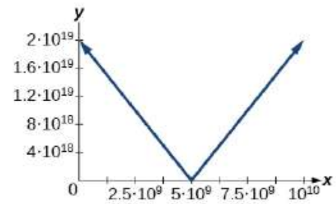


33. range: $[-400, 100]$



35.



37. There is no solution for a that will keep the function from having a y -intercept. The absolute value function always crosses the y -intercept when $x = 0$.

39. $|p - 0.08| \leq 0.015$

41. $|x - 5.0| \leq 0.01$

3.7 Section Exercises

1. Each output of a function must have exactly one output for the function to be one-to-one. If any horizontal line crosses the graph of a function more than once, that means that y -values repeat and the function is not one-to-one. If no horizontal line crosses the graph of the function more than once, then no y -values repeat and the function is one-to-one.

3. Yes. For example, $f(x) = \frac{1}{x}$ is its own inverse.

5. Given a function $y = f(x)$, solve for x in terms of y. Interchange the x and y. Solve the new equation for y. The expression for y is the inverse, $y = f^{-1}(x)$.

7. $f^{-1}(x) = x - 3$

9. $f^{-1}(x) = 2 - x$

11. $f^{-1}(x) = \frac{-2x}{x-1}$

13. domain of $f(x): \ [-7, \infty); \ f^{-1}(x) = \sqrt{x} - 7$

15. domain of $f(x): \ [0, \infty); \ f^{-1}(x) = \sqrt{x+5}$

16. ⓐ $f(g(x)) = x$ and $g(f(x)) = x$. ⓑ This tells us that f and g are inverse functions

17. $f(g(x)) = x, \ g(f(x)) = x$

19. one-to-one

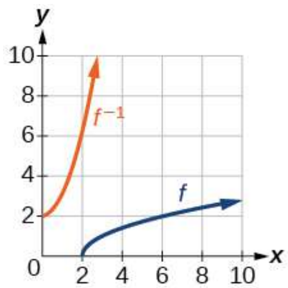
21. one-to-one

23. not one-to-one

25. 3

27. 2

29.



31. $[2, 10]$

33. 6

35. -4

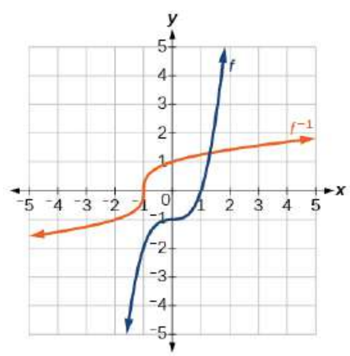
37. 0

39. 1

41.

x	1	4	7	12	16
$f^{-1}(x)$	3	6	9	13	14

43. $f^{-1}(x) = (1 + x)^{1/3}$



45. $f^{-1}(x) = \frac{5}{9}(x - 32)$. Given the Fahrenheit temperature, x, this formula allows you to calculate the Celsius temperature.

47. $t(d) = \frac{d}{50}, t(180) = \frac{180}{50}$. The time for the car to travel 180 miles is 3.6 hours.

Review Exercises

1. function

3. not a function

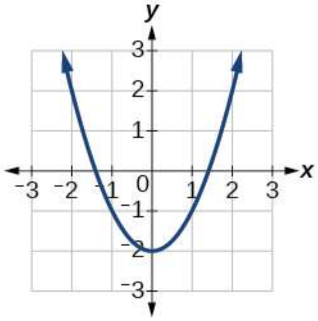
5. $f(-3) = -27; f(2) = -2;$
$f(-a) = -2a^2 - 3a;$
$-f(a) = 2a^2 - 3a;$
$f(a + h) = -2a^2 + 3a - 4ah + 3h - 2h^2$

7. one-to-one

9. function

11. function

13.



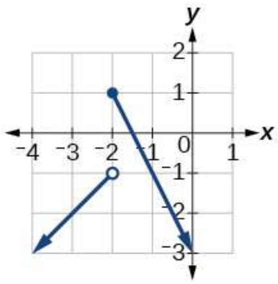
15. 2

17. $x = -1.8$ or or $x = 1.8$

19. $\frac{-64+80a-16a^2}{-1+a} = -16a + 64$

21. $\left(-\infty, -2\right) \cup (-2, 6) \cup \left(6, \infty\right)$

23.



25. 31

27. increasing $\left(2, \infty\right)$; decreasing $(-\infty, 2)$

29. increasing $(-3, 1)$; constant $(-\infty, -3) \cup \left(1, \infty\right)$

31. local minimum $(-2, -3)$; local maximum $(1, 3)$

33. $(-1.8, 10)$

35. $(f \circ g)(x) = 17 - 18x$; $(g \circ f)(x) = -7 - 18x$

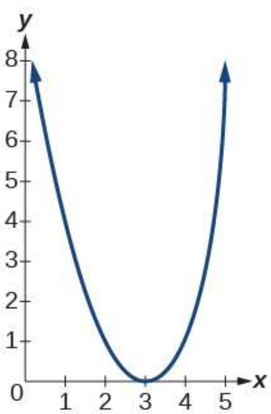
37. $(f \circ g)(x) = \sqrt{\frac{1}{x} + 2}$; $(g \circ f)(x) = \frac{1}{\sqrt{x+2}}$

39. $(f \circ g)(x) = \frac{1+x}{1+4x},\ x \neq 0,\ x \neq -\frac{1}{4}$

41. $(f \circ g)(x) = \frac{1}{\sqrt{x}},\quad x > 0$

43. sample: $g(x) = \frac{2x-1}{3x+4}$; $f(x) = \sqrt{x}$

45.



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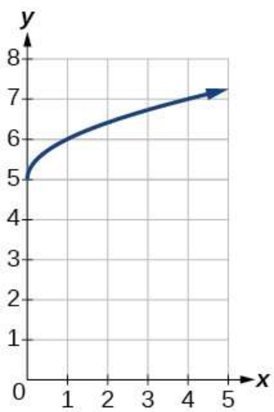

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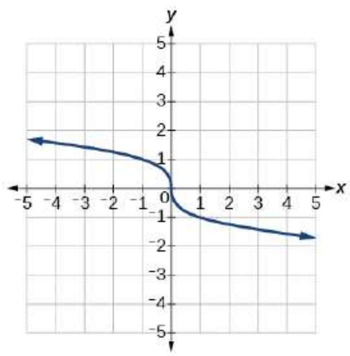

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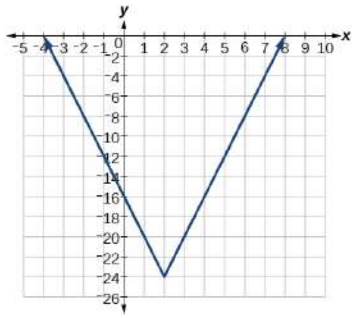

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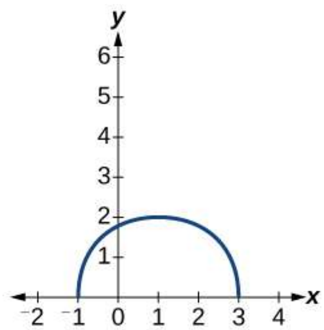


55. $f(x) = |x - 3|$

57. even

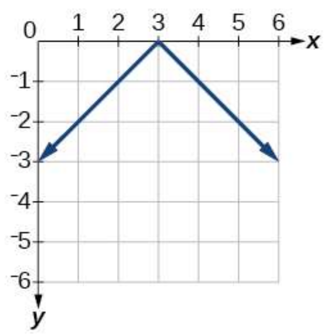
59. odd

61. even

63. $f(x) = \frac{1}{2}|x + 2| + 1$

65. $f(x) = -3\,|x - 3| + 3$

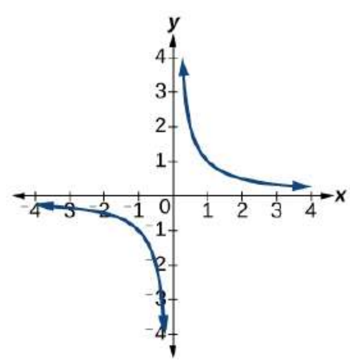
67.



69. $f^{-1}(x) = \frac{x-9}{10}$

71. $f^{-1}(x) = \sqrt{x - 1}$

73. The function is one-to-one.



75. 5

Practice Test

1. The relation is a function.

3. −16

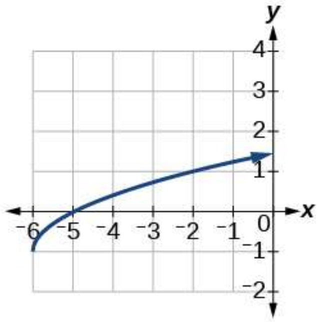
5. The graph is a parabola and the graph fails the horizontal line test.

7. $2a^2 - a$

9. $-2(a + b) + 1$

11. $\sqrt{2}$

13.



15. even

17. odd

19. $f^{-1}(x) = \frac{x+5}{3}$

21. $(-\infty, -1.1)$ and $(1.1, \infty)$

23. $(1.1, -0.9)$

25. $f(2) = 2$

27. $f(x) = \begin{cases} |x| & \text{if} \quad x \leq 2 \\ 3 & \text{if} \quad x > 2 \end{cases}$

29. $x = 2$

31. yes

33. $f^{-1}(x) = -\frac{x-11}{2}$

Chapter 4

Try It

4.1 Linear Functions

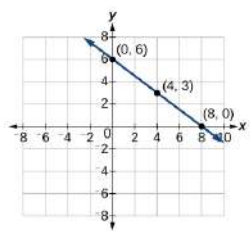
1. $m = \frac{4-3}{0-2} = \frac{1}{-2} = -\frac{1}{2}$; decreasing because $m < 0$.

2. $$m = \frac{1,868 - 1,442}{2,012 - 2,009} = \frac{426}{3} = 142 \text{ people per year}$$

3. $y = -7x + 3$

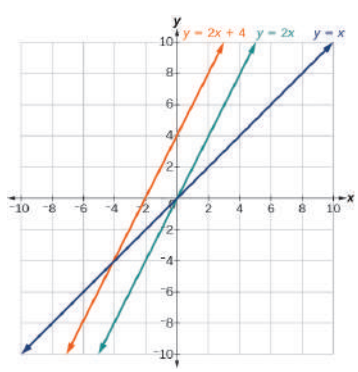
4. $H(x) = 0.5x + 12.5$

5.



6. Possible answers include $(-3, 7)$, $(-6, 9)$, or $(-9, 11)$.

7.



8. $(16, \ 0)$

9. ⓐ $f(x) = 2x$; ⓑ $g(x) = -\frac{1}{2}x$

10. $y = -\frac{1}{3}x + 6$

4.2 Modeling with Linear Functions

1. ⓐ $C(x) = 0.25x + 25{,}000$ ⓑ The y-intercept is $(0, 25{,}000)$. If the company does not produce a single doughnut, they still incur a cost of $25,000.

2. ⓐ 41,100 ⓑ 2020

3. 21.57 miles

4.3 Fitting Linear Models to Data

1. $54°\text{F}$

2. 150.871 billion gallons; extrapolation

4.1 Section Exercises

1. Terry starts at an elevation of 3000 feet and descends 70 feet per second.

3. $d(t) = 100 - 10t$

5. The point of intersection is $(a,\ a)$. This is because for the horizontal line, all of the y coordinates are a and for the vertical line, all of the x coordinates are a. The point of intersection is on both lines and therefore will have these two characteristics.

7. Yes

9. Yes

11. No

13. Yes

15. Increasing

17. Decreasing

19. Decreasing

21. Increasing

23. Decreasing

25. 2

27. –2

29. $y = \frac{3}{5}x - 1$

31. $y = 3x - 2$

33. $y = -\frac{1}{3}x + \frac{11}{3}$

35. $y = -1.5x - 3$

37. perpendicular

39. parallel

41. $$\begin{aligned} &f(0) = -(0) + 2 \\ &f(0) = 2 \\ &y - \text{int} : (0, 2) \\ &0 = -x + 2 \\ &x - \text{int} : (2, 0) \end{aligned}$$

43. $$\begin{aligned}h(0)&=3(0)-5\\h(0)&=-5\\y-\text{int}&:(0,-5)\\0&=3x-5\\x-\text{int}&:\left(\tfrac{5}{3},0\right)\end{aligned}$$

45. $$\begin{aligned}-2x+5y&=20\\-2(0)+5y&=20\\5y&=20\\y&=4\\y-\text{int}&:(0,4)\\-2x+5(0)&=20\\x&=-10\\x-\text{int}&:(-10,0)\end{aligned}$$

47. Line 1: m = –10 Line 2: m = –10 Parallel

49. Line 1: m = –2 Line 2: m = 1 Neither

51. $\text{Line 1}: m=-2 \quad \text{Line 2}: m=-2 \quad \text{Parallel}$

53. $y=3x-3$

55. $y=-\frac{1}{3}t+2$

57. 0

59. $y=-\frac{5}{4}x+5$

61. $y=3x-1$

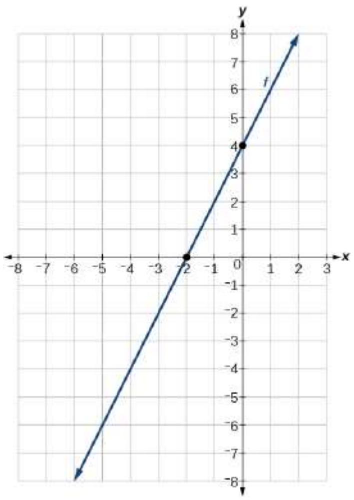
63. $y=-2.5$

65. F

67. C

69. A

71.



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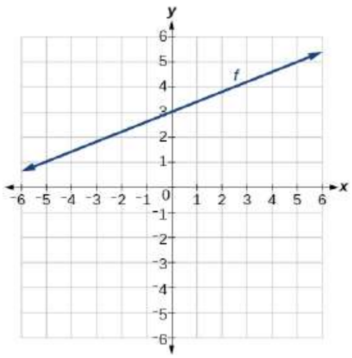

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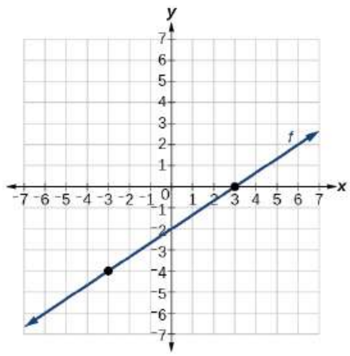

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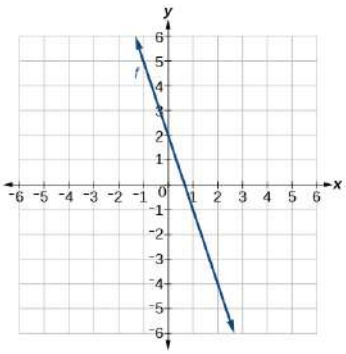

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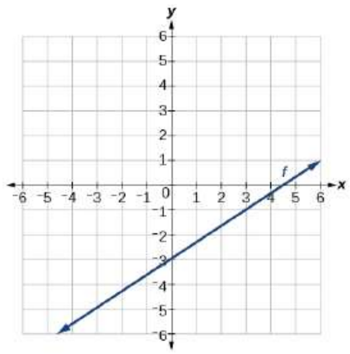

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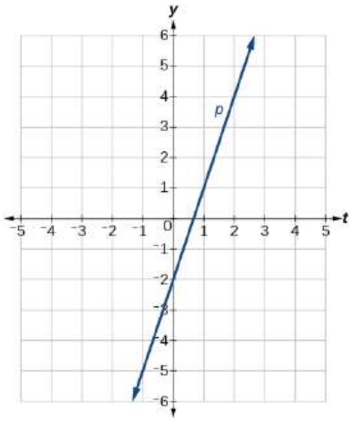

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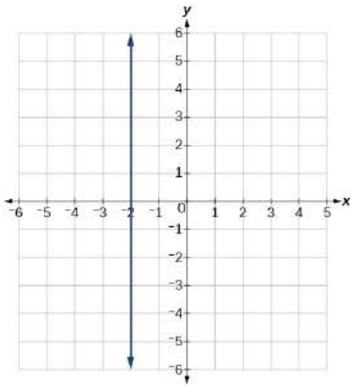


85. $y = 3$

87. $x = -3$

89. Linear, $g(x) = -3x + 5$

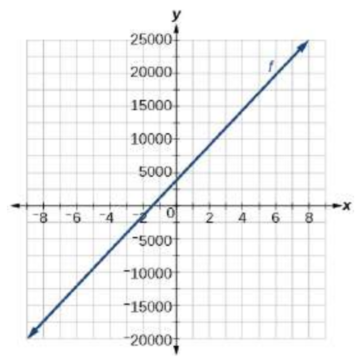
91. Linear, $f(x) = 5x - 5$

93. Linear, $g(x) = -\frac{25}{2}x + 6$

95. Linear, $f(x) = 10x - 24$

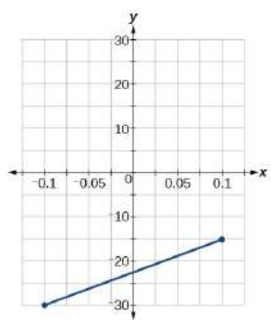
97. $f(x) = -58x + 17.3$

99.



101. ⓐ $a = 11{,}900, b = 1000.1$
ⓑ $q(p) = 1000p - 100$

103.



105. $y = -\frac{16}{3}$

107. $x = a$

109. $y = \frac{d}{c-a}x - \frac{ad}{c-a}$

111. $y = 100x - 98$

113. $x < \frac{1999}{201}, x > \frac{1999}{201}$

115. $45 per training session.

117. The rate of change is 0.1. For every additional minute talked, the monthly charge increases by $0.1 or 10 cents. The initial value is 24. When there are no minutes talked, initially the charge is $24.

119. The slope is –400. this means for every year between 1960 and 1989, the population dropped by 400 per year in the city.

121. C

4.2 Section Exercises

1. Determine the independent variable. This is the variable upon which the output depends.

3. To determine the initial value, find the output when the input is equal to zero.

5. 6 square units

7. 20.01 square units

9. 2,300

11. 64,170

13. $P(t) = 75,000 + 2500t$

15. (–30, 0) Thirty years before the start of this model, the town had no citizens. (0, 75,000) Initially, the town had a population of 75,000.

17. Ten years after the model began

19. $W(t) = 0.5t + 7.5$

21. $(-15, 0)$: The *x*-intercept is not a plausible set of data for this model because it means the baby weighed 0 pounds 15 months prior to birth. $(0, 7.5)$: The baby weighed 7.5 pounds at birth.

23. At age 5.8 months

25. $C(t) = 12,025 - 205t$

27. $(58.7, 0)$: In roughly 59 years, the number of people inflicted with the common cold would be 0. $(0, 12,025)$ Initially there were 12,025 people afflicted by the common cold.

29. 2063

31. $y = -2t+180$

33. In 2070, the company's profit will be zero.

35. $y = 30t - 300$

37. (10, 0) In the year 1990, the company's profits were zero

39. Hawaii

41. During the year 1933

43. $105,620

45. a 696 people b 4 years
c 174 people per year
d 305 people
e P(t) = 305 + 174t
f 2,219 people

47. a C(x) = 0.15x + 10
b The flat monthly fee is $10 and there is a $0.15 fee for each additional minute used
c $113.05

49. P(t) = 190t + 4,360

51. ⓐ $R(t) = -2.1t + 16$
ⓑ 5.5 billion cubic feet
ⓒ During the year 2017

53. More than 133 minutes

55. More than $42,857.14 worth of jewelry

57. More than $66,666.67 in sales

4.3 Section Exercises

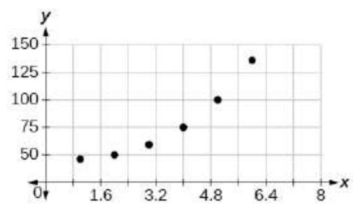
1. When our model no longer applies, after some value in the domain, the model itself doesn't hold.

3. We predict a value outside the domain and range of the data.

5. The closer the number is to 1, the less scattered the data, the closer the number is to 0, the more scattered the data.

7. 61.966 years

9.



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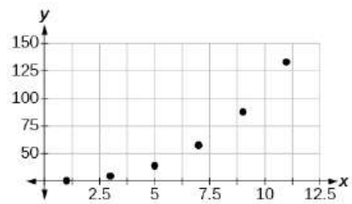
11.



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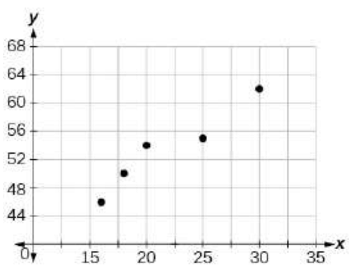
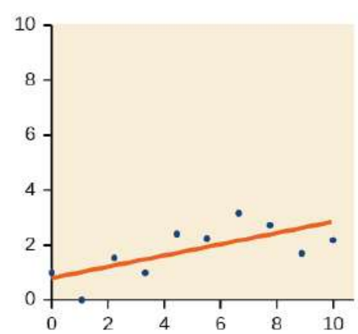


Interpolation. About $60°F$.

15. This value of r indicates a strong negative correlation or slope, so C

17. This value of r indicates a weak negative correlation, so B

19.



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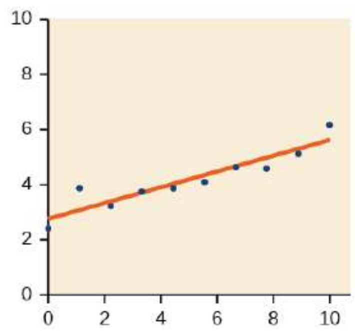


23. Yes, trend appears linear because $r = 0.985$ and will exceed 12,000 near midyear, 2016, 24.6 years since 1992.

25. $y = 1.640x + 13.800$, $r = 0.987$

27. $y = -0.962x + 26.86, \quad r = -0.965$

29. $y = -1.981x + 60.197$; $r = -0.998$

31. $y = 0.121x - 38.841, r = 0.998$

33. $(-2, -6), (1, -12), (5, -20), (6, -22), (9, -28)$; Yes, the function is a good fit.

35. $(189.8, 0)$ If 18,980 units are sold, the company will have a profit of zero dollars.

37. $y = 0.00587x + 1985.41$

39. $y = 20.25x - 671.5$

41. $y = -10.75x + 742.50$

Review Exercises

1. Yes

3. Increasing

5. $y = -3x + 26$

7. 3

9. $y = 2x - 2$

11. Not linear.

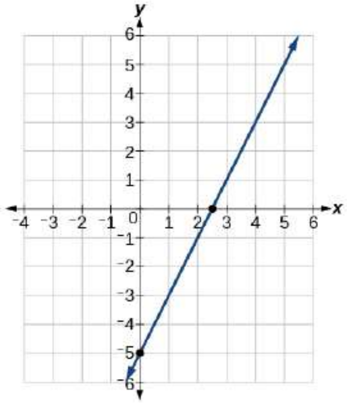
13. parallel

15. $(-9, 0); (0, -7)$

17. Line 1: $m = -2$; Line 2: $m = -2$; Parallel

19. $y = -0.2x + 21$

21.



23. More than 250

25. 118,000

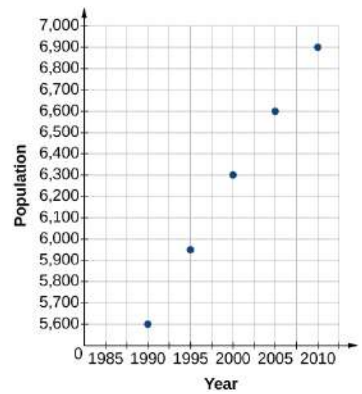
27. $y = -300x + 11,500$

29. (a) 800
(b) 100 students per year
(c) $P(t) = 100t + 1700$

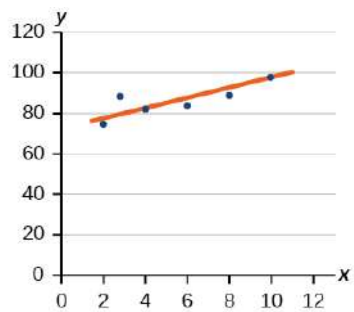
31. 18,500

33. $91,625

35. Extrapolation



37.



39. 2023

41. $y = -1.294x + 49.412;\ r = -0.974$

43. 2027

45. 7,660

Practice Test

1. Yes

3. Increasing

5. y = −1.5x − 6

7. y = −2x − 1

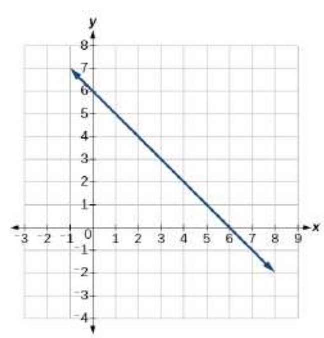
9. No

11. Perpendicular

13. (−7, 0); (0, −2)

15. y = −0.25x + 12

17.



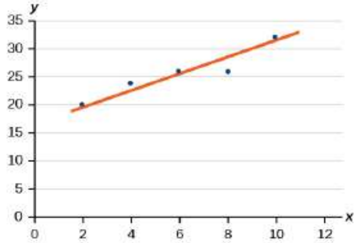
Slope = −1 and y-intercept = 6

19. 150

21. 165,000

23. y = 875x + 10,625

25. ⓐ 375
ⓑ dropped an average of 46.875, or about 47 people per year
ⓒ y = −46.875t + 1250

27.


29. In early 2018

31. y = 0.00455x + 1979.5

33. r = 0.999

Chapter 5

Try It

5.1 Quadratic Functions

1. The path passes through the origin and has vertex at $(-4,\ \ 7)$, so $h(x) =-\frac{7}{16}(x+4)^2+7$. To make the shot, $h\left(-7.5\right)$ would need to be about 4 but $h(-7.5)\approx 1.64$; he doesn't make it.

2. $g(x)=x^2-6x+13$ in general form; $g(x)=(x-3)^2+4$ in standard form

3. The domain is all real numbers. The range is $f(x)\geq\frac{8}{11}$, or $\left[\frac{8}{11},\infty\right)$.

4. y-intercept at (0, 13), No x-intercepts

5. ⓐ 3 seconds ⓑ 256 feet ⓒ 7 seconds

5.2 Power Functions and Polynomial Functions

1. $f(x)$ is a power function because it can be written as $f(x)=8x^5$. The other functions are not power functions.

2. As x approaches positive or negative infinity, $f\left(x\right)$ decreases without bound: as

$$x\rightarrow\pm\infty,\ f(x)\rightarrow-\infty$$

because of the negative coefficient.

3. The degree is 6. The leading term is $-x^6$. The leading coefficient is -1.

4. As

$$x\rightarrow\infty,\ f(x)\rightarrow-\infty;\ as\ x\rightarrow-\infty,\ f(x)\rightarrow-\infty.$$

It has the shape of an even degree power function with a negative coefficient.

5. The leading term is $0.2x^3$, so it is a degree 3 polynomial. As x approaches positive infinity, $f\left(x\right)$ increases without bound; as x approaches negative infinity, $f\left(x\right)$ decreases without bound.

6. y-intercept $(0, 0)$; x-intercepts $(0, 0), (-2, 0)$, and $(5, 0)$

7. There are at most 12 x-intercepts and at most 11 turning points.

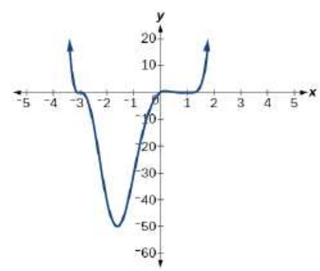
8. The end behavior indicates an odd-degree polynomial function; there are 3 x-intercepts and 2 turning points, so the degree is odd and at least 3. Because of the end behavior, we know that the lead coefficient must be negative.

9. The x- intercepts are $(2, 0), (-1, 0)$, and $(5, 0)$, the y-intercept is $(0, 2)$, and the graph has at most 2 turning points.

5.3 Graphs of Polynomial Functions

1. y-intercept $(0, 0)$; x-intercepts $(0, 0), (-5, 0), (2, 0)$, and $(3, 0)$

2. The graph has a zero of –5 with multiplicity 3, a zero of -1 with multiplicity 2, and a zero of 3 with multiplicity 4.

3. 

4. Because f is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x = 1$ and $x = 2$.

5. $f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

6. The minimum occurs at approximately the point $(0, -6.5)$, and the maximum occurs at approximately the point $(3.5, 7)$.

5.4 Dividing Polynomials

1. $4x^2 - 8x + 15 - \frac{78}{4x+5}$

2. $3x^3 - 3x^2 + 21x - 150 + \frac{1{,}090}{x+7}$

3. $3x^2 - 4x + 1$

5.5 Zeros of Polynomial Functions

1. $f(-3) = -412$

2. The zeros are 2, –2, and –4.

3. There are no rational zeros.

4. The zeros are –4, $\frac{1}{2}$, and 1.

5. $f(x) = -\frac{1}{2}x^3 + \frac{5}{2}x^2 - 2x + 10$

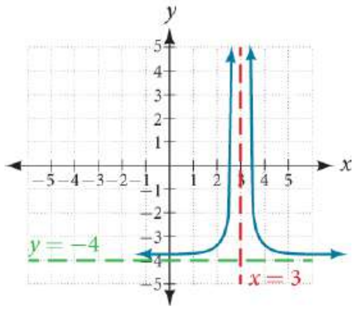
6. There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

7. 3 meters by 4 meters by 7 meters

5.6 Rational Functions

1. End behavior: as $x \to \pm\infty,\ f(x) \to 0$; Local behavior: as $x \to 0,\ f(x) \to \infty$ (there are no x- or y-intercepts)

2.



The function and the asymptotes are shifted 3 units right and 4 units down. As $x \to 3, f(x) \to \infty$, and as $x \to \pm\infty, f(x) \to -4$.

The function is $f(x) = \frac{1}{(x-3)^2} - 4$.

3. $\frac{12}{11}$

4. The domain is all real numbers except $x = 1$ and $x = 5$.

5. Removable discontinuity at $x = 5$. Vertical asymptotes: $x = 0,\ \ x = 1$.

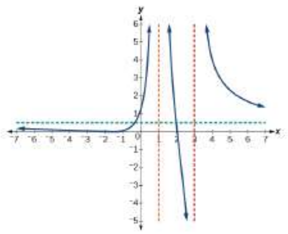
6. Vertical asymptotes at $x = 2$ and $x = -3$; horizontal asymptote at $y = 4$.

7. For the transformed reciprocal squared function, we find the rational form.
$f(x) = \frac{1}{(x-3)^2} - 4 = \frac{1-4(x-3)^2}{(x-3)^2} = \frac{1-4(x^2-6x+9)}{(x-3)(x-3)} = \frac{-4x^2+24x-35}{x^2-6x+9}$

Because the numerator is the same degree as the denominator we know that as $x \to \pm\infty,\ f(x) \to -4;$ so $y = -4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x = 3$, because as $x \to 3, f(x) \to \infty$. We then set the numerator equal to 0 and find the x-intercepts are at $(2.5, 0)$ and $(3.5, 0)$. Finally, we evaluate the function at 0 and find the y-intercept to be at $\left(0, \frac{-35}{9}\right)$.

8. Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $\left(0, \frac{4}{3}.\right)$

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.



5.7 Inverses and Radical Functions

1. $f^{-1}\left(f\left(x\right)\right)=f^{-1}\left(\frac{x+5}{3}\right)=3\left(\frac{x+5}{3}\right)-5=\left(x-5\right)+5=x$ and $f\left(f^{-1}\left(x\right)\right)=f\left(3x-5\right)=\frac{\left(3x-5\right)+5}{3}=\frac{3x}{3}=x$

2. $f^{-1}(x)=x^3-4$

3. $f^{-1}(x)=\sqrt{x-1}$

4. $f^{-1}(x)=\frac{x^2-3}{2},x\geq 0$

5. $f^{-1}(x)=\frac{2x+3}{x-1}$

5.8 Modeling Using Variation

1. $\frac{128}{3}$

2. $\frac{9}{2}$

3. $x=20$

5.1 Section Exercises

1. When written in that form, the vertex can be easily identified.

3. If $a=0$ then the function becomes a linear function.

5. If possible, we can use factoring. Otherwise, we can use the quadratic formula.

7. $g(x)=(x+1)^2-4,$ Vertex $(-1,-4)$

9. $f(x)=\left(x+\frac{5}{2}\right)^2-\frac{33}{4},$ Vertex $\left(-\frac{5}{2},-\frac{33}{4}\right)$

11. $f(x)=3(x-1)^2-12,$ Vertex $(1,-12)$

13. $f(x)=3\left(x-\frac{5}{6}\right)^2-\frac{37}{12},$ Vertex $\left(\frac{5}{6},-\frac{37}{12}\right)$

15. Minimum is $-\frac{17}{2}$ and occurs at $\frac{5}{2}$. Axis of symmetry is $x=\frac{5}{2}$.

17. Minimum is $-\frac{17}{16}$ and occurs at $-\frac{1}{8}$. Axis of symmetry is $x=-\frac{1}{8}$.

19. Minimum is $-\frac{7}{2}$ and occurs at -3. Axis of symmetry is $x=-3$.

21. Domain is $\left(-\infty,\infty\right)$. Range is $[2,\infty)$.

23. Domain is $\left(-\infty,\infty\right)$. Range is $[-5,\infty)$.

25. Domain is $\left(-\infty,\infty\right)$. Range is $[-12,\infty)$.

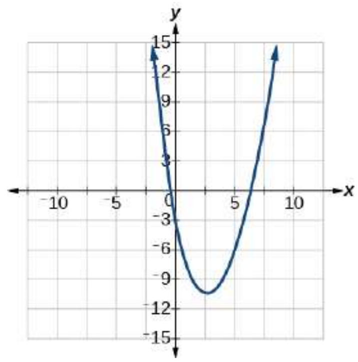
27. $f(x)=x^2+4x+3$

29. $f(x)=x^2-4x+7$

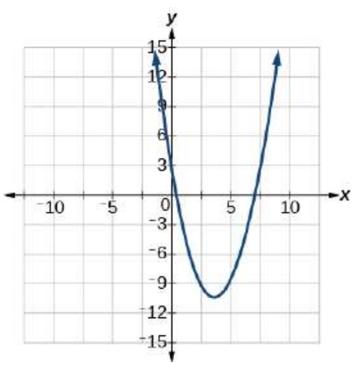
31. $f(x)=-\frac{1}{49}x^2+\frac{6}{49}x+\frac{89}{49}$

33. $f(x)=x^2-2x+1$

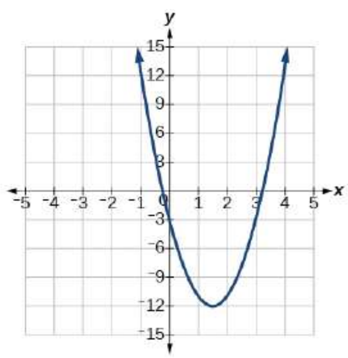
35. Vertex: (3, −10), axis of symmetry: x = 3, intercepts: $\left(3+\sqrt{10},0\right)$ and $\left(3-\sqrt{10},0\right)$



37. Vertex: $\left(\frac{3}{2},-12\right)$, axis of symmetry: $x=\frac{3}{2}$, intercept: $\left(\frac{3+2\sqrt{3}}{2},0\right)$ and $\left(\frac{3-2\sqrt{3}}{2},0\right)$



39.



41. $f(x)=x^2+2x+3$

43. $f(x)=-3x^2-6x-1$

45. $f(x)=-\frac{1}{4}x^2-x+2$

47. $f(x)=x^2+2x+1$

49. $f(x)=-\frac{x}{2}+2x$

50. $f(x)=2x^2$

53. The graph is shifted to the right or left (a horizontal shift).

55. The suspension bridge has 1,000 feet distance from the center.

57. Domain is $(-\infty,\infty)$. Range is $\left(-\infty,2\right]$.

59. Domain: $\left(-\infty,\infty\right)$; range: $\left[100,\infty\right)$

61. $f(x)=2x^2+2$

63. $f(x)=-x^2-2$

65. $f(x)=3x^2+6x-15$

67. 75 feet by 50 feet

69. 3 and 3; product is 9

71. The revenue reaches the maximum value when 1800 thousand phones are produced.

73. 2.449 seconds

75. 41 trees per acre

5.2 Section Exercises

1. The coefficient of the power function is the real number that is multiplied by the variable raised to a power. The degree is the highest power appearing in the function.

3. As x decreases without bound, so does $f(x)$. As x increases without bound, so does $f(x)$.

5. The polynomial function is of even degree and leading coefficient is negative.

7. Power function

9. Neither

11. Neither

13. Degree = 2, Coefficient = –2

15. Degree =4, Coefficient = –2

17. As $x \to \infty$,
$f(x) \to \infty, \text{ as } x \to -\infty, \ f(x) \to \infty$

19. As $x \to -\infty$,
$f(x) \to -\infty, \text{ as } x \to \infty, \ f(x) \to -\infty$

21. As $x \to -\infty$,
$f(x) \to -\infty, \text{ as } x \to \infty, \ f(x) \to -\infty$

23. As $x \to \infty$,
$f(x) \to \infty, \text{ as } x \to -\infty, \ f(x) \to -\infty$

25. y-intercept is $(0, 12)$, t-intercepts are $(1, 0); (-2, 0);$ and $(3, 0)$.

27. y-intercept is $(0, -16)$. x-intercepts are $(2, 0)$ and $(-2, 0)$.

29. y-intercept is $(0, 0)$. x-intercepts are $(0, 0), (4, 0)$, and $(-2, 0)$.

31. 3

33. 5

35. 3

37. 5

39. Yes. Number of turning points is 2. Least possible degree is 3.

41. Yes. Number of turning points is 1. Least possible degree is 2.

43. Yes. Number of turning points is 0. Least possible degree is 1.

45. Yes. Number of turning points is 0. Least possible degree is 1.

47.

x	$f(x)$
10	9,500
100	99,950,000
–10	9,500
–100	99,950,000

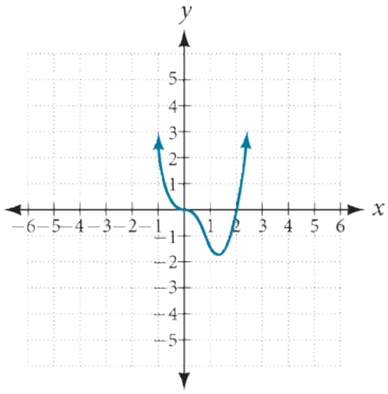
As $x \to -\infty$,
$f(x) \to \infty, \text{ as } x \to \infty, \ f(x) \to \infty$

49.

x	$f(x)$
10	–504
100	–941,094
–10	1,716
–100	1,061,106

As $x \to -\infty$,
$f(x) \to \infty, \text{ as } x \to \infty, \ f(x) \to -\infty$

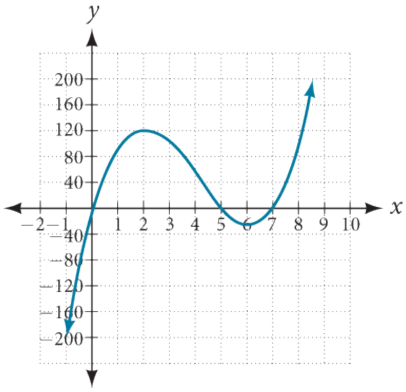
51.



The y- intercept is $(0,\ 0)$. The x- intercepts are $(0,\ 0)$, $(2,\ 0)$. As $x \to -\infty$,

$f(x) \to \infty,\ \text{ as }\ x \to \infty,\ \ f(x) \to \infty$

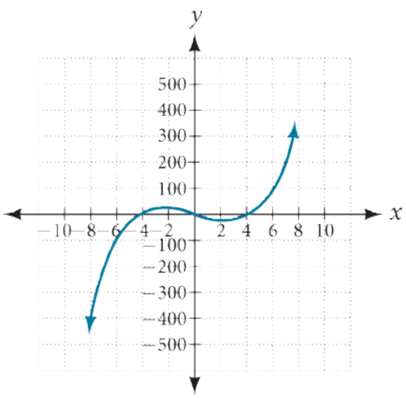
53.



The y- intercept is $(0, 0)$. The x- intercepts are $(0,\ 0)$, $(5,\ 0)$, $(7,\ 0)$. As $x \to -\infty$,

$f(x) \to -\infty,\ \text{ as }\ x \to \infty,\ \ f(x) \to \infty$

55.

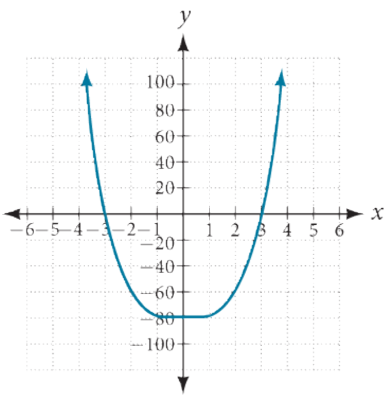


The y- intercept is $(0,\ 0)$. The x- intercept is $(-4,\ 0)$, $(0,\ 0)$, $(4,\ 0)$.

$As\ \ x \to -\infty,$

$f(x) \to -\infty,\ \text{ as }\ x \to \infty,\ \ f(x) \to \infty$

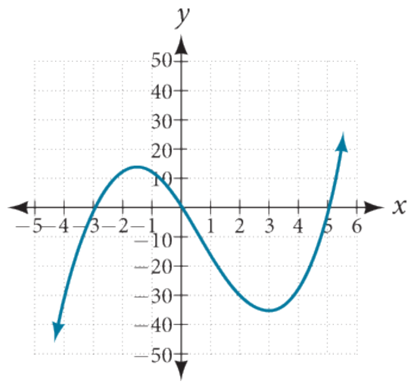
57.



The y- intercept is $(0,\ -81)$. The x- intercept are $(3,\ 0)$, $(-3,\ 0)$. As $x \to -\infty$,

$f(x) \to \infty,\ \text{ as }\ x \to \infty,\ \ f(x) \to \infty$

59.



The y- intercept is $(0,\ 0)$. The x- intercepts are $(-3,\ 0)$, $(0,\ 0)$, $(5,\ 0)$. As $x \to -\infty$,

$$f(x) \to -\infty,\ \text{ as }\ x \to \infty,\ \ f(x) \to \infty$$

61. $f(x) = x^2 - 4$

63. $f(x) = x^3 - 4x^2 + 4x$

65. $f(x) = x^4 + 1$

67. $V(m) = 8m^3 + 36m^2 + 54m + 27$

69. $V(x) = 4x^3 - 32x^2 + 64x$

5.3 Section Exercises

1. The x- intercept is where the graph of the function crosses the x- axis, and the zero of the function is the input value for which $f(x) = 0$.

3. If we evaluate the function at a and at b and the sign of the function value changes, then we know a zero exists between a and b.

5. There will be a factor raised to an even power.

7. $(-2, 0), (3, 0), (-5, 0)$

9. $(3, 0), (-1, 0), (0, 0)$

11. $(0, 0)$, $(-5, 0)$, $(2, 0)$

13. $(0, 0)$, $(-5, 0)$, $(4, 0)$

15. $(2, 0)$, $(-2, 0)$, $(-1, 0)$

17. $(-2, 0)$, $(2, 0)$, $\left(\frac{1}{2}, 0\right)$

19. $(1, 0)$, $(-1, 0)$

21. $(0, 0)$, $(\sqrt{3}, 0)$, $(-\sqrt{3}, 0)$

23. $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(2, 0)$, $(-2, 0)$

25. $f(2) = -10$ and $f(4) = 28$. Sign change confirms.

27. $f(1) = 3$ and $f(3) = -77$. Sign change confirms.

29. $f(0.01) = 1.000001$ and $f(0.1) = -7.999$. Sign change confirms.

31. 0 with multiplicity 2, $-\frac{3}{2}$ with multiplicity 5, 4 with multiplicity 2

33. 0 with multiplicity 2, –2 with multiplicity 2

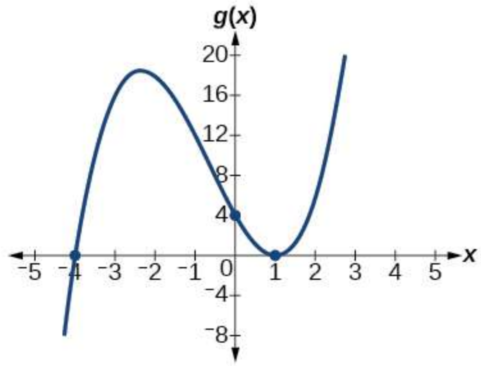
35. $-\frac{2}{3}$ with multiplicity 5, 5 with multiplicity 2

37. 0 with multiplicity 4, 2 with multiplicity 1, –1 with multiplicity 1

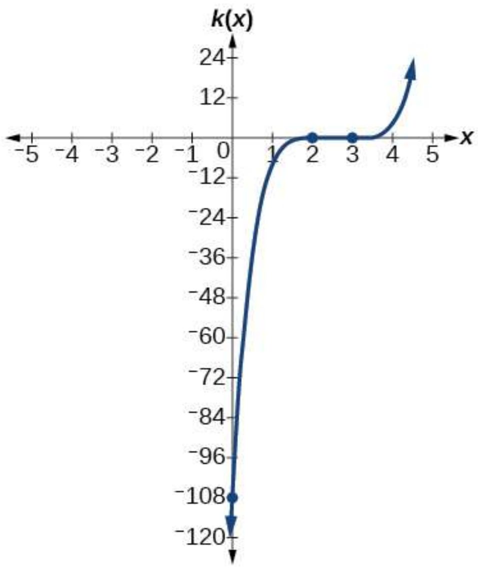
39. $\frac{3}{2}$ with multiplicity 2, 0 with multiplicity 3

41. 0 with multiplicity 6, $\frac{2}{3}$ with multiplicity 2

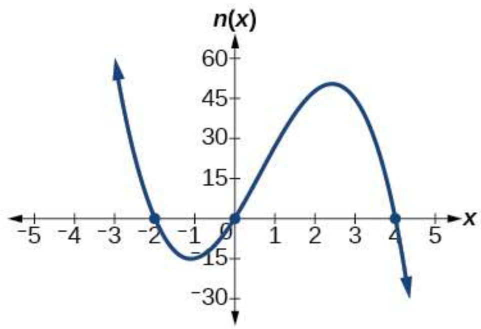
43. *x*-intercepts, $(1, 0)$ with multiplicity 2, $(-4, 0)$ with multiplicity 1, y- intercept $(0, 4)$. As $x \to -\infty, f(x) \to -\infty,$ as $x \to \infty, f(x) \to \infty$.



45. *x*-intercepts $(3, 0)$ with multiplicity 3, $(2, 0)$ with multiplicity 2, y- intercept $(0, -108)$. As $x \to -\infty, f(x) \to -\infty,$ as $x \to \infty, f(x) \to \infty$.



47. *x*-intercepts $(0, 0)$, $(-2, 0)$, $(4, 0)$ with multiplicity 1, y- intercept $(0, 0)$. As $x \to -\infty, f(x) \to \infty,$ as $x \to \infty, f(x) \to -\infty$.



49. $f(x) = -\frac{2}{9}(x-3)(x+1)(x+3)$

51. $f(x) = \frac{1}{4}(x+2)^2(x-3)$

53. –4, –2, 1, 3 with multiplicity 1

55. –2, 3 each with multiplicity 2

57. $f(x) = -\frac{2}{3}(x+2)(x-1)(x-3)$

59. $f(x) = \frac{1}{3}(x-3)^2(x-1)^2(x+3)$

61. $f(x) = -15(x-1)^2(x-3)^3$

63. $f(x) = -2(x+3)(x+2)(x-1)$

65. $f(x) = -\frac{3}{2}(2x-1)^2(x-6)(x+2)$

67. local max $(-.58, -.62)$, local min $(.58, -1.38)$

69. global min $(-.63, -.47)$

71. global min $(.75, .89)$

73. $f(x) = (x-500)^2(x+200)$

75. $f(x) = 4x^3 - 36x^2 + 80x$

77. $f(x) = 4x^3 - 36x^2 + 60x + 100$

79. $f(x) = 9\pi(x^3 + 5x^2 + 8x + 4)$

5.4 Section Exercises

1. The binomial is a factor of the polynomial.

3. $x + 6 + \frac{5}{x-1}$, quotient: $x + 6$, remainder: 5

5. $3x + 2$, quotient: $3x + 2$, remainder: 0

7. $x - 5$, quotient: $x - 5$, remainder: 0

9. $2x - 7 + \frac{16}{x+2}$, quotient: $2x - 7$, remainder: 16

11. $x - 2 + \frac{6}{3x+1}$, quotient: $x - 2$, remainder: 6

13. $2x^2 - 3x + 5$, quotient: $2x^2 - 3x + 5$, remainder: 0

15. $2x^2 + 2x + 1 + \frac{10}{x-4}$

17. $2x^2 - 7x + 1 - \frac{2}{2x+1}$

19. $3x^2 - 11x + 34 - \frac{106}{x+3}$

21. $x^2 + 5x + 1$

23. $4x^2 - 21x + 84 - \frac{323}{x+4}$

25. $x^2 - 14x + 49$

27. $3x^2 + x + \frac{2}{3x-1}$

29. $x^3 - 3x + 1$

31. $x^3 - x^2 + 2$

33. $x^3 - 6x^2 + 12x - 8$

35. $x^3 - 9x^2 + 27x - 27$

37. $2x^3 - 2x + 2$

39. Yes $(x - 2)(3x^3 - 5)$

41. Yes $(x - 2)(4x^3 + 8x^2 + x + 2)$

43. No

45. $(x - 1)(x^2 + 2x + 4)$

47. $(x - 5)(x^2 + x + 1)$

49. Quotient: $4x^2 + 8x + 16$, remainder: -1

51. Quotient: $3x^2 + 3x + 5$, remainder: 0

53. Quotient: $x^3 - 2x^2 + 4x - 8$, remainder: -6

55. $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$

57. $x^3 - x^2 + x - 1 + \frac{1}{x+1}$

59. $1 + \frac{1+i}{x-i}$

61. $1 + \frac{1-i}{x+i}$

63. $x^2 - ix - 1 + \frac{1-i}{x-i}$

65. $2x^2 + 3$

67. $2x + 3$

69. $x + 2$

71. $x - 3$

73. $3x^2 - 2$

5.5 Section Exercises

1. The theorem can be used to evaluate a polynomial.

3. Rational zeros can be expressed as fractions whereas real zeros include irrational numbers.

5. Polynomial functions can have repeated zeros, so the fact that number is a zero doesn’t preclude it being a zero again.

7. -106

9. 0

11. 255

13. -1

15. $-2,\ 1,\ \frac{1}{2}$

17. -2

19. -3

21. $-\frac{5}{2},\ \sqrt{6},\ -\sqrt{6}$

23. $2,\ -4,\ -\frac{3}{2}$

25. $4,\ -4,\ -5$

27. $5,\ -3,\ -\frac{1}{2}$

29. $\frac{1}{2},\ \frac{1+\sqrt{5}}{2},\ \frac{1-\sqrt{5}}{2}$

31. $\frac{3}{2}$

33. $2,\ 3,\ -1,\ -2$

35. $\frac{1}{2},\ -\frac{1}{2},\ 2,\ -3$

37. $-1,\ -1,\ \sqrt{5},\ -\sqrt{5}$

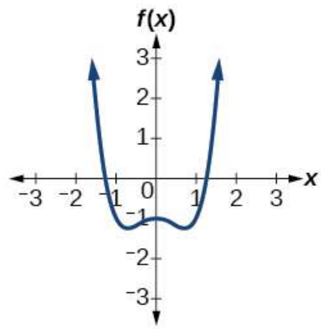
39. $-\frac{3}{4},\ -\frac{1}{2}$

41. $2,\ 3+2i,\ 3-2i$

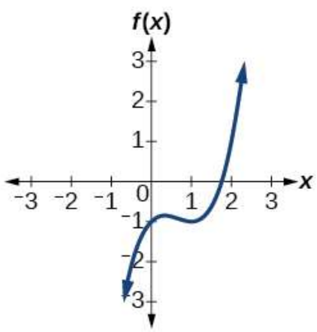
43. $-\frac{2}{3},\ 1+2i,\ 1-2i$

45. $-\frac{1}{2},\ 1+4i,\ 1-4i$

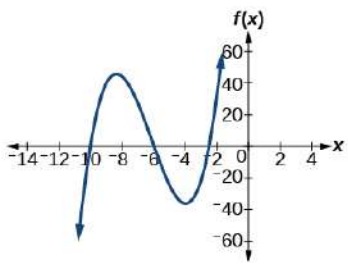
47. 1 positive, 1 negative



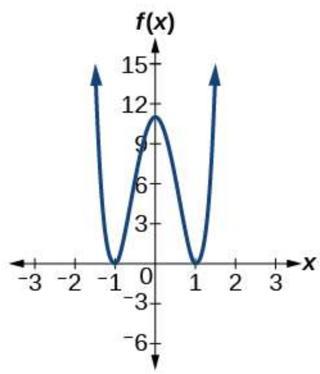
49. 3 or 1 positive, 0 negative



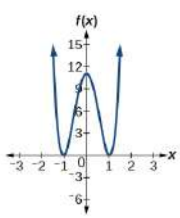
51. 0 positive, 3 or 1 negative



53. 2 or 0 positive, 2 or 0 negative



55. 2 or 0 positive, 2 or 0 negative



57. $\pm 5,\ \pm 1,\ \pm\frac{5}{2},\ \pm\frac{1}{2}$

59. $\pm 1,\ \pm\frac{1}{2},\ \pm\frac{1}{3},\ \pm\frac{1}{6}$

61. $1,\ \frac{1}{2},\ -\frac{1}{3}$

63. $2,\ \frac{1}{4},\ -\frac{3}{2}$

65. $\frac{5}{4}$

67. $f(x)=\frac{4}{9}\left(x^3+x^2-x-1\right)$

69. $f(x)=-\frac{1}{5}\left(4x^3-x\right)$

71. 8 by 4 by 6 inches

73. 5.5 by 4.5 by 3.5 inches

75. 8 by 5 by 3 inches

77. Radius = 6 meters, Height = 2 meters

79. Radius = 2.5 meters, Height = 4.5 meters

5.6 Section Exercises

1. The rational function will be represented by a quotient of polynomial functions.

3. The numerator and denominator must have a common factor.

5. Yes. The numerator of the formula of the functions would have only complex roots and/or factors common to both the numerator and denominator.

7. All reals $x \neq -1,\ 1$

9. All reals $x \neq -1,\ -2,\ 1,\ 2$

11. V.A. at $x = -\frac{2}{5}$; H.A. at $y = 0$; Domain is all reals $x \neq -\frac{2}{5}$

13. V.A. at $x = 4,\ -9$; H.A. at $y = 0$; Domain is all reals $x \neq 4,\ -9$

15. V.A. at $x = 0,\ 4,\ \ -4$; H.A. at $y = 0$; Domain is all reals $x \neq 0, 4,\ -4$

17. V.A. at $x = 5$; H.A. at $y = 0$; Domain is all reals $x \neq 5, -5$

19. V.A. at $x = \frac{1}{3}$; H.A. at $y = -\frac{2}{3}$; Domain is all reals $x \neq \frac{1}{3}$.

21. none

23. x-intercepts none, y-intercept $\left(0, \frac{1}{4}\right)$

25. Local behavior:
$x \to -\frac{1}{2}^+, f(x) \to -\infty, x \to -\frac{1}{2}^-, f(x) \to \infty$

End behavior: $x \to \pm\infty, f(x) \to \frac{1}{2}$

27. Local behavior:
$x \to 6^+, f(x) \to -\infty, x \to 6^-, f(x) \to \infty,$

End behavior: $x \to \pm\infty, f(x) \to -2$

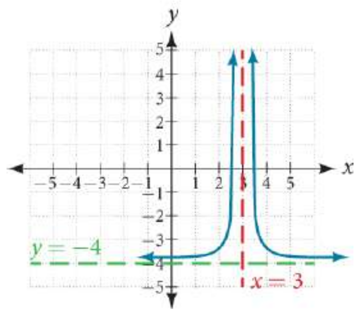
29. Local behavior: $x \to -\frac{1}{3}^+, f(x) \to \infty, x \to -\frac{1}{3}^-,$
$f(x) \to -\infty, x \to \frac{5}{2}^-, f(x) \to \infty, x \to \frac{5}{2}^+, f(x) \to -\infty$

End behavior: $x \to \pm\infty,\ f(x) \to \frac{1}{3}$

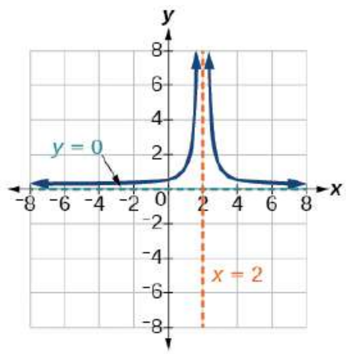
31. $y = 2x + 4$

33. $y = 2x$

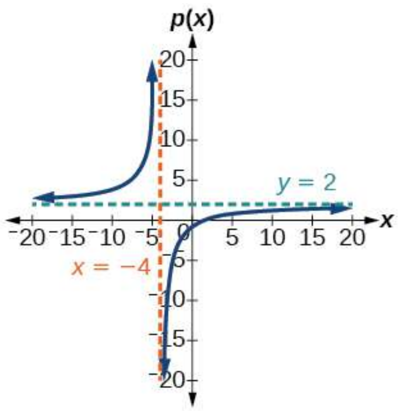
35. $V.A.\quad x = 0, H.A.\quad y = 2$



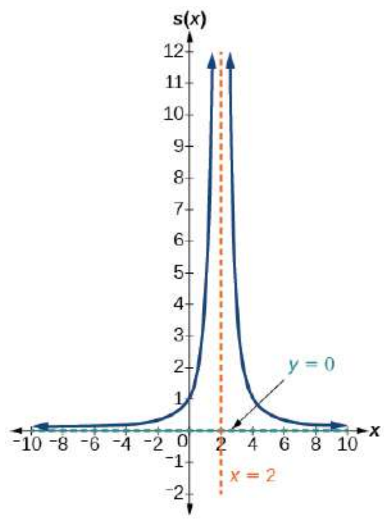
37. $V.A.\quad x = 2,\quad H.A.\quad y = 0$



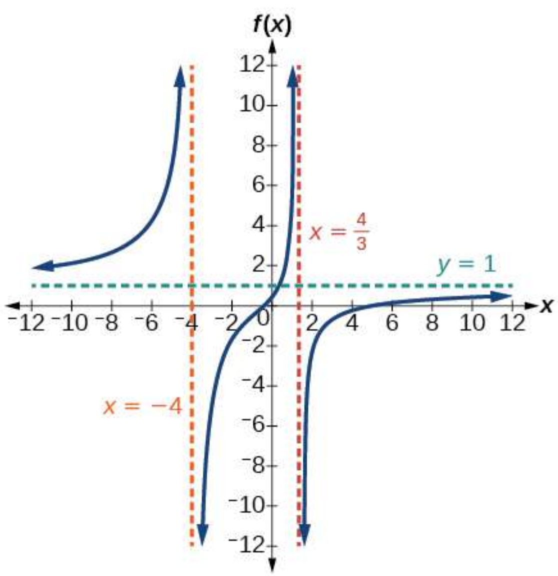
39. $V.A.\quad x = -4,\quad H.A.\quad y = 2; \left(\frac{3}{2}, 0\right); \left(0, -\frac{3}{4}\right)$



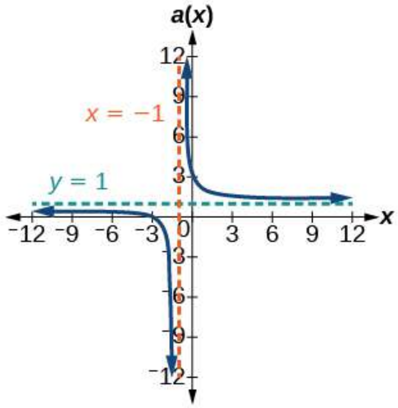
41. $V.A.\quad x = 2,\quad H.A.\quad y = 0,\quad (0, 1)$



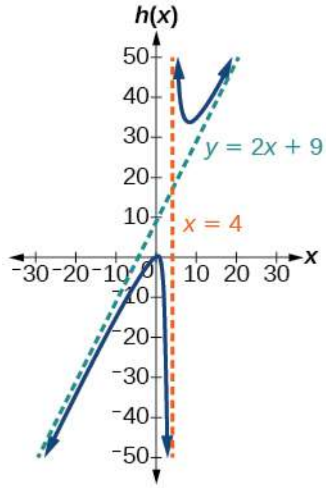
43. $V.A.\quad x = -4,\quad x = \frac{4}{3},\quad H.A.\quad y = 1; (5, 0); \left(-\frac{1}{3}, 0\right); \left(0, \frac{5}{16}\right)$



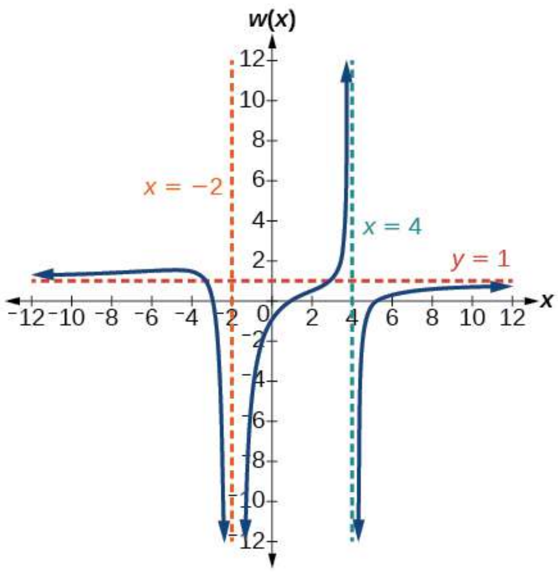
45. $V.A.\quad x=-1,\quad H.A.\quad y=1; (-3,0); (0,3)$



47. $V.A.\quad x=4,\quad S.A.\quad y=2x+9; (-1,0); \left(\frac{1}{2},0\right); \left(0,\frac{1}{4}\right)$



49. $V.A.\quad x=-2,\quad x=4,\quad H.A.\quad y=1, (1,0); (5,0); (-3,0); \left(0,-\frac{15}{16}\right)$



51. $y=50\frac{x^2-x-2}{x^2-25}$

53. $y=7\frac{x^2+2x-24}{x^2+9x+20}$

55. $y=\frac{1}{2}\frac{x^2-4x+4}{x+1}$

57. $y=4\frac{x-3}{x^2-x-12}$

59. $y=\frac{27(x-2)}{(x+3)(x-3)^2}$

61. $y=\frac{1}{3}\frac{x^2+x-6}{x-1}$

63. $y=-6\frac{(x-1)^2}{(x+3)(x-2)^2}$

65.

x	2.01	2.001	2.0001	1.99	1.999
y	100	1,000	10,000	–100	–1,000

x	10	100	1,000	10,000	100,000
y	.125	.0102	.001	.0001	.00001

Vertical asymptote $x = 2$, Horizontal asymptote $y = 0$

67.

x	–4.1	–4.01	–4.001	–3.99	–3.999
y	82	802	8,002	–798	–7998

x	10	100	1,000	10,000	100,000
y	1.4286	1.9331	1.992	1.9992	1.999992

Vertical asymptote $x = -4$, Horizontal asymptote $y = 2$

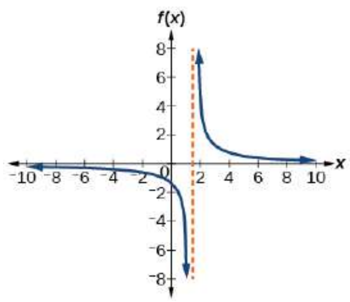
69.

x	–.9	–.99	–.999	–1.1	–1.01
y	81	9,801	998,001	121	10,201

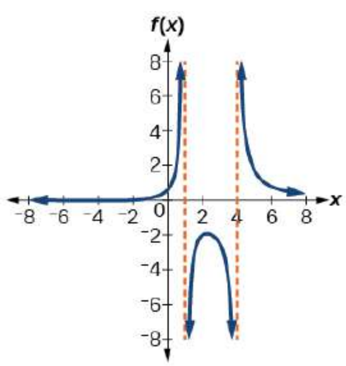
x	10	100	1,000	10,000	100,000
y	.82645	.9803	.998	.9998	

Vertical asymptote $x = -1$, Horizontal asymptote $y = 1$

71. $\left(\frac{3}{2}, \infty\right)$



73. $(-2, 1) \cup (4, \infty)$



75. $(2, 4)$

77. $(2, 5)$

79. $(-1, 1)$

81. $C(t) = \frac{8+2t}{300+20t}$

83. After about 6.12 hours.

85. $A(x) = 50x^2 + \frac{800}{x}$. 2 by 2 by 5 feet.

87. $A(x) = \pi x^2 + \frac{100}{x}$. Radius = 2.52 meters.

5.7 Section Exercises

1. It can be too difficult or impossible to solve for x in terms of y.

3. We will need a restriction on the domain of the answer.

5. $f^{-1}(x) = \sqrt{x} + 4$

7. $f^{-1}(x) = \sqrt{x+3} - 1$

9. $f^{-1}(x) = \sqrt{12 - x}$

11. $f^{-1}(x) = \pm\sqrt{\frac{x-4}{2}}$

13. $f^{-1}(x) = \sqrt{\frac{x-1}{3}3}$

15. $f^{-1}(x) = \sqrt{\frac{4-x}{2}3}$

17. $f^{-1}(x) = \frac{3-x^2}{4}, \quad \left[0, \infty\right)$

19. $f^{-1}(x) = \frac{(x-5)^2+8}{6}$

21. $f^{-1}(x) = (3-x)^2$

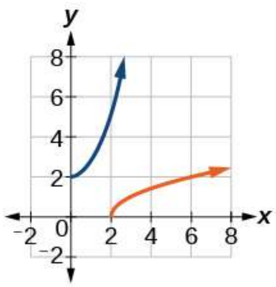
23. $f^{-1}(x) = \frac{4x+3}{x}$

25. $f^{-1}(x) = \frac{7x-3}{1-x}$

27. $f^{-1}(x) = \frac{2x-1}{5x+5}$

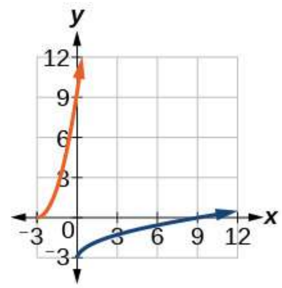
29. $f^{-1}(x) = \sqrt{x+3} - 2$

31.



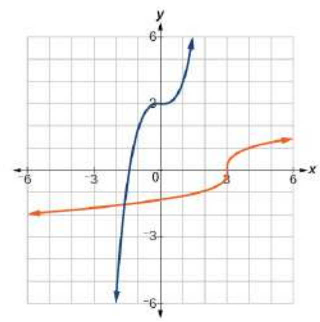
$f^{-1}(x) = \sqrt{x-2}$

33.



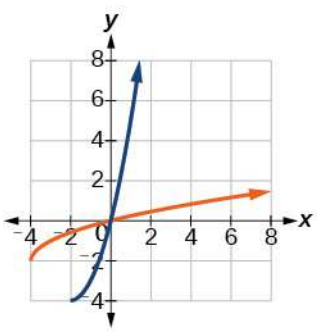
$f^{-1}(x) = \sqrt{x-3}$

35.



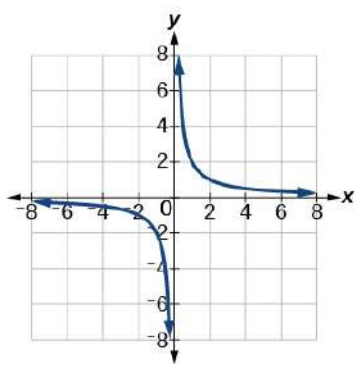
$f^{-1}(x) = \sqrt[3]{x-3}$

37.

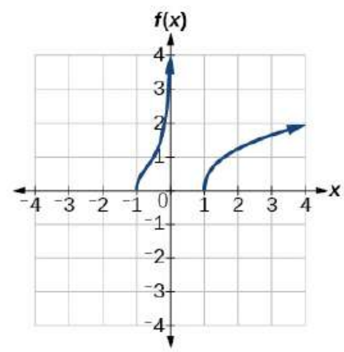


$f^{-1}(x) = \sqrt{x+4} - 2$

39.

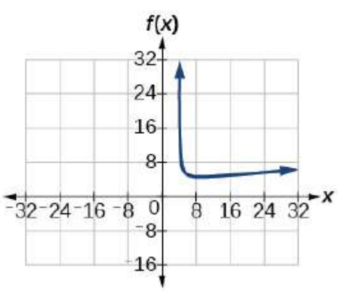


41.



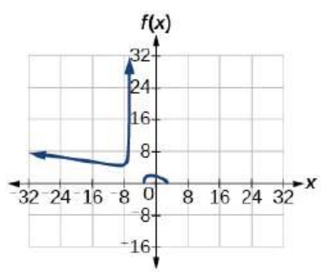
$[-1, 0) \cup [1, \infty)$

43.



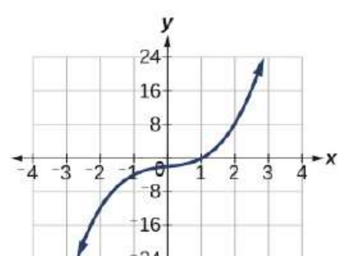
$[-3, 0] \cup (4, \infty)$

45.



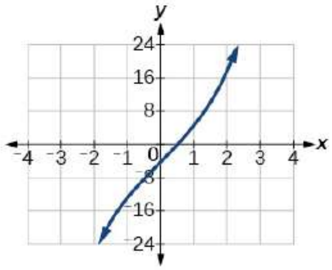
$[-\infty, -4] \cdot [-3, 3]$

47.



$(-2, 0), (0, 1), (8, 2)$

49.



$(-13,-1), (-4,0), (5,1)$

51. $f^{-1}(x) = \sqrt[3]{\frac{x-b}{a}}$

53. $f^{-1}(x) = \frac{\sqrt{x^2-b}}{a}$

55. $f^{-1}(x) = \frac{cx-b}{a-x}$

57. $t(h) = \sqrt{\frac{600-h}{16}}$, 3.54 seconds

59. $r(A) = \sqrt{\frac{A}{4\pi}}$, $\approx$ 8.92 in.

61. $l(T) = 32.2\left(\frac{T}{2\pi}\right)$, $\approx$ 3.26 ft

63. $r(A) = \sqrt{\frac{A+2\pi}{8\pi}}$, -2, 3.99 ft

65. $r(V) = \sqrt{\frac{V}{10\pi}}$, $\approx$ 5.64 ft

5.8 Section Exercises

1. The graph will have the appearance of a power function.

3. No. Multiple variables may jointly vary.

5. $y = 5x^2$

7. $y = \frac{1}{1944}x^3$

9. $y = 6x^4$

11. $y = \frac{18}{x^2}$

13. $y = \frac{81}{x^4}$

15. $y = \frac{20}{\sqrt[3]{x}}$

17. $y = 10xzw$

19. $y = 10x\sqrt{z}$

21. $y = 4\frac{xz}{w}$

23. $y = 40\frac{xz}{\sqrt{w}t^2}$

25. $y = 256$

27. $y = 6$

29. $y = 6$

31. $y = 27$

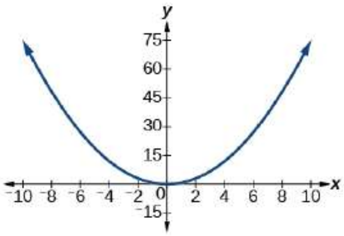
33. $y = 3$

35. $y = 18$

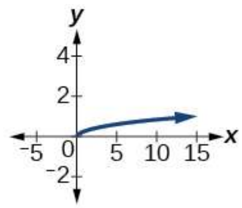
37. $y = 90$

39. $y = \frac{81}{2}$

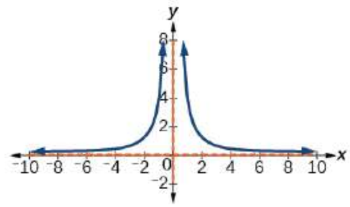
41. $y = \frac{3}{4}x^2$



43. $y = \frac{1}{3}\sqrt{x}$



45. $y = \frac{4}{x^2}$



47. 1.89 years

49. 0.61 years

51. 3 seconds

53. 48 inches

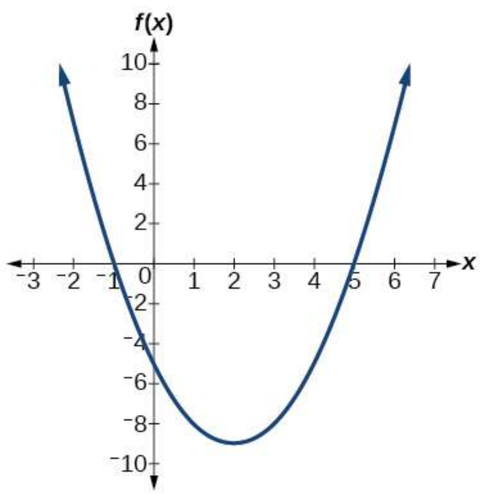
55. 49.75 pounds

57. 33.33 amperes

59. 2.88 inches

Review Exercises

1. $f(x) = (x-2)^2 - 9$ vertex $(2,-9)$, intercepts $(5,0)$; $(-1,0)$; $(0,-5)$



3. $f(x) = \frac{3}{25}(x+2)^2 + 3$

5. 300 meters by 150 meters, the longer side parallel to river.

7. Yes, degree = 5, leading coefficient = 4

9. Yes, degree = 4, leading coefficient = 1

11. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$

13. –3 with multiplicity 2, $-\frac{1}{2}$ with multiplicity 1, –1 with multiplicity 3

15. 4 with multiplicity 1

17. $\frac{1}{2}$ with multiplicity 1, 3 with multiplicity 3

19. $x^2 + 4$ with remainder 12

21. $x^2 - 5x + 20 - \frac{61}{x+3}$

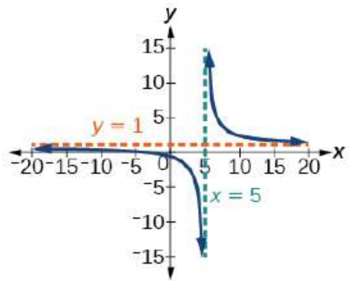
23. $2x^2 - 2x - 3$, so factored form is $(x+4)(2x^2 - 2x - 3)$

25. $\left\{-2,\ 4,\ -\frac{1}{2}\right\}$

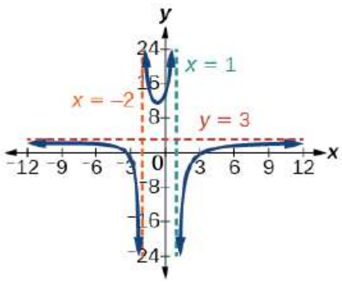
27. $\left\{1,\ 3,\ 4,\ \frac{1}{2}\right\}$

29. 0 or 2 positive, 1 negative

31. Intercepts $(-2, 0)$and $\left(0, -\frac{2}{5}\right)$, Asymptotes $x = 5$ and $y = 1$.



33. Intercepts (3, 0), (-3, 0), and $\left(0, \frac{27}{2}\right)$, Asymptotes $x = 1,\ x =-2,\ y = 3.$



35. $y = \quad x - 2$

37. $f^{-1}(x) = \sqrt{x} + 2$

39. $f^{-1}(x) = \sqrt{x + 11} - 3$

41. $f^{-1}(x) = \frac{(x+3)^2 - 5}{4},\quad x \geq -3$

43. $y = 64$

45. $y \quad = \quad 72$

47. 148.5 pounds

Practice Test

1. Degree: 5, leading coefficient: −2

3. $\text{As} x \to -\infty, f(x) \to \infty, \text{As} x \to \infty, f(x) \to \infty$

5. $f(x) = 3(x - 2)^2$

7. 3 with multiplicity 3, $\frac{1}{3}$ with multiplicity 1, 1 with multiplicity 2

9. $-\frac{1}{2}$ with multiplicity 3, 2 with multiplicity 2

11. $x^3 + 2x^2 + 7x + 14 + \frac{26}{x-2}$

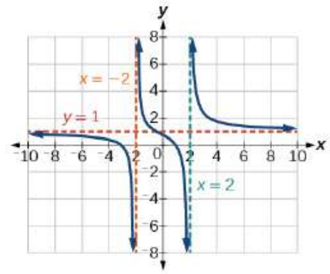
13. $\left\{-3, -1, \frac{3}{2}\right\}$

15. 1, −2, and – $\frac{3}{2}$ (multiplicity 2)

17. $f(x) = -\frac{2}{3}(x-3)^2\left(x-1\right)\left(x+2\right)$

19. 2 or 0 positive, 1 negative

21. $\left(-3,0\right)\left(1,0\right)\left(0,\frac{3}{4}\right)$



23. $f^{-1}(x) = (x-4)^2 + 2, x \geq 4$

25. $f^{-1}(x) = \frac{x+3}{3x-2}$

27. $y = 20$

Chapter 6

Try It

6.1 Exponential Functions

1. $g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions.

2. 5.5556

3. About 1.548 billion people; by the year 2031, India's population will exceed China's by about 0.001 billion, or 1 million people.

4. $(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

5. $f(x) = 2(1.5)^x$

6. $f(x) = \sqrt{2}\left(\sqrt{2}\right)^x$. Answers may vary due to round-off error. The answer should be very close to $1.4142(1.4142)^x$.

7. $y \approx 12 \cdot 1.85^x$

8. about $3,644,675.88

9. $13,693

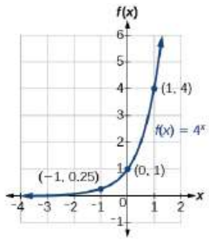
10. $e^{-0.5} \approx 0.60653$

11. $3,659,823.44

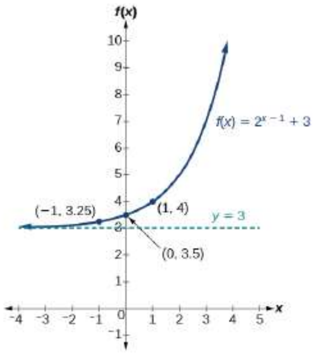
12. 3.77E-26 (This is calculator notation for the number written as 3.77×10^{-26} in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

6.2 Graphs of Exponential Functions

1. The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

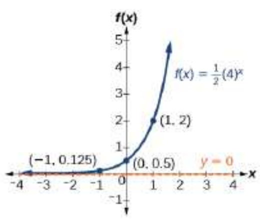


2. The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.

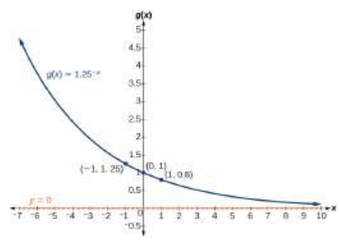


3. $x \approx -1.608$

4. The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



5. The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



6. $f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, -2)$; the horizontal asymptote is $y = -2$.

6.3 Logarithmic Functions

1. ⓐ $\log_{10}(1,000,000) = 6$ is equivalent to $10^6 = 1,000,000$
ⓑ $\log_5(25) = 2$ is equivalent to $5^2 = 25$

2. ⓐ $3^2 = 9$ is equivalent to $\log_3(9) = 2$
ⓑ $5^3 = 125$ is equivalent to $\log_5(125) = 3$
ⓒ $2^{-1} = \frac{1}{2}$ is equivalent to $\log_2(\frac{1}{2}) = -1$

3. $\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{\frac{1}{2}} = 11$)

4. $\log_2\left(\frac{1}{32}\right) = -5$

5. $\log(1,000,000) = 6$

6. $\log(123) \approx 2.0899$

7. The difference in magnitudes was about 3.929.

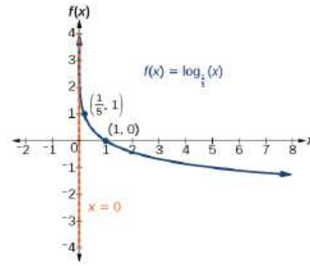
8. It is not possible to take the logarithm of a negative number in the set of real numbers.

6.4 Graphs of Logarithmic Functions

1. $(2, \infty)$

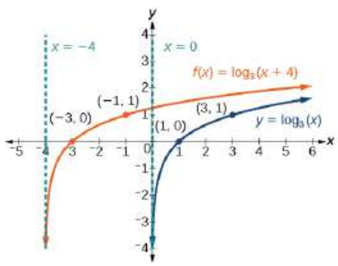
2. $(5, \infty)$

3.



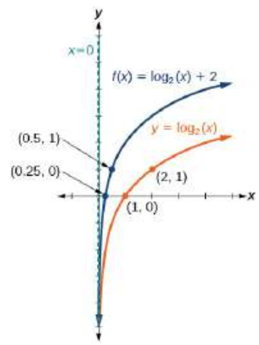
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

4.



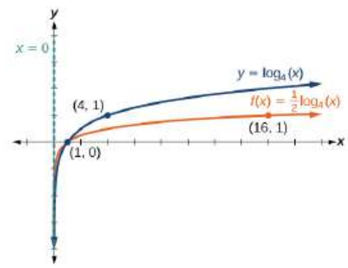
The domain is $(-4, \infty)$, the range $(-\infty, \infty)$, and the asymptote $x = -4$.

5.



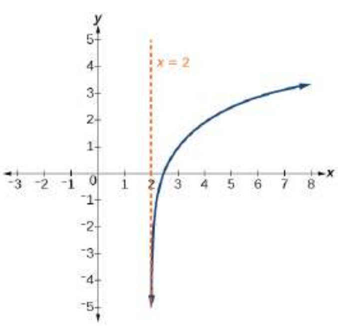
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

6.



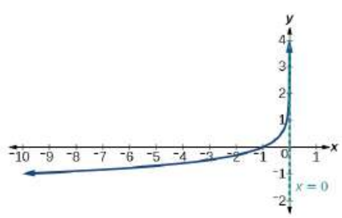
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

7.



The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

8.



The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

9. $x \approx 3.049$

10. $x = 1$

11. $f(x) = 2\ln(x + 3) - 1$

6.5 Logarithmic Properties

1. $\log_b 2 + \log_b 2 + \log_b 2 + \log_b k = 3\log_b 2 + \log_b k$

2. $\log_3 (x+3) - \log_3 (x-1) - \log_3 (x-2)$

3. $2 \ln x$

4. $-2 \ln(x)$

5. $\log_3 16$

6. $2 \log x + 3 \log y - 4 \log z$

7. $\frac{2}{3} \ln x$

8. $\frac{1}{2} \ln (x-1) + \ln (2x+1) - \ln (x+3) - \ln (x-3)$

9. $\log \left(\frac{3 \cdot 5}{4 \cdot 6} \right)$; can also be written $\log \left(\frac{5}{8} \right)$ by reducing the fraction to lowest terms.

10. $\log \left(\frac{5(x-1)^3 \sqrt{x}}{(7x-1)} \right)$

11. $\log \frac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log \left(\frac{x^3(x+5)}{(2x+3)} \right)^4$.

12. The pH increases by about 0.301.

13. $\frac{\ln 8}{\ln 0.5}$

14. $\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$

6.6 Exponential and Logarithmic Equations

1. $x = -2$

2. $x = -1$

3. $x = \frac{1}{2}$

4. The equation has no solution.

5. $x = \frac{\ln 3}{\ln\left(2/3 \right)}$

6. $t = 2 \ln \left(\frac{11}{3} \right)$ or $\ln \left(\frac{11}{3} \right)^2$

7. $t = \ln \left(\frac{1}{\sqrt{2}} \right) = -\frac{1}{2} \ln (2)$

8. $x = \ln 2$

9. $x = e^4$

10. $x = e^5 - 1$

11. $x \approx 9.97$

12. $x = 1$ or $x = -1$

13. $t = 703,800,000 \times \frac{\ln(0.8)}{\ln(0.5)}$ years $\approx$ $226,572,993$ years.

6.7 Exponential and Logarithmic Models

1. $f(t) = A_0 e^{-0.0000000087t}$

2. less than 230 years, 229.3157 to be exact

3. $f(t) = A_0 e^{\frac{\ln 2}{3} t}$

4. 6.026 hours

5. 895 cases on day 15

6. Exponential. $y = 2e^{0.5x}$.

7. $y = 3e^{(\ln 0.5)x}$

6.8 Fitting Exponential Models to Data

1. (a) The exponential regression model that fits these data is $y = 522.88585984(1.19645256)^x$. (b) If spending continues at this rate, the graduate's credit card debt will be $4,499.38 after one year.

2. (a) The logarithmic regression model that fits these data is $y = 141.91242949 + 10.45366573\ln(x)$ (b) If sales continue at this rate, about 171,000 games will be sold in the year 2015.

3. (a) The logistic regression model that fits these data is $y = \frac{25.65665979}{1+6.113686306e^{-0.3852149008x}}$. (b) If the population continues to grow at this rate, there will be about $25{,}634$ seals in 2020. (c) To the nearest whole number, the carrying capacity is 25,657.

6.1 Section Exercises

1. Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

3. When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of *nominal*.

5. exponential; the population decreases by a proportional rate. .

7. not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function. .

9. The forest represented by the function $B(t) = 82(1.029)^t$.

11. After $t = 20$ years, forest A will have 43 more trees than forest B.

13. Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

15. exponential growth; The growth factor, 1.06, is greater than 1.

17. exponential decay; The decay factor, 0.97, is between 0 and 1.

19. $f(x) = 2000(0.1)^x$

21. $f(x) = \left(\frac{1}{6}\right)^{-\frac{3}{5}}\left(\frac{1}{6}\right)^{\frac{x}{5}} \approx 2.93(0.699)^x$

23. Linear

25. Neither

27. Linear

29. $\$10,250$

31. $\$13,268.58$

33. $P = A(t) \cdot \left(1 + \frac{r}{n}\right)^{-nt}$

35. $\$4{,}572.56$

37. 4%

39. continuous growth; the growth rate is greater than 0.

41. continuous decay; the growth rate is less than 0.

43. $\$669.42$

45. $f(-1) = -4$

47. $f(-1) \approx -0.2707$

49. $f(3) \approx 483.8146$

51. $y = 3 \cdot 5^x$

53. $y \approx 18 \cdot 1.025^x$

55. $y \approx 0.2 \cdot 1.95^x$

57. $\text{APY} = \frac{A(t)-a}{a} = \frac{a\left(1+\frac{r}{365}\right)^{365(1)}-a}{a} = \frac{a\left[\left(1+\frac{r}{365}\right)^{365}-1\right]}{a} = \left(1 + \frac{r}{365}\right)^{365} - 1;$
$I(n) = \left(1 + \frac{r}{n}\right)^n - 1$

59. Let f be the exponential decay function $f(x) = a \cdot \left(\frac{1}{b}\right)^x$ such that $b > 1$. Then for some number $n > 0$,
$f(x) = a \cdot \left(\frac{1}{b}\right)^x = a\left(b^{-1}\right)^x = a\left((e^n)^{-1}\right)^x = a\left(e^{-n}\right)^x = a(e)^{-nx}.$

61. $47,622$ fox

63. 1.39%; $\$155,368.09$

65. $\$35,838.76$

67. $\$82,247.78$; $\$449.75$

6.2 Section Exercises

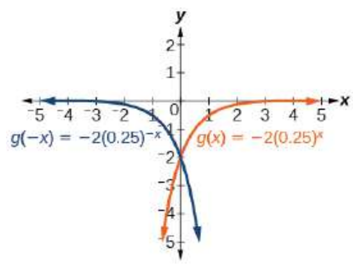
1. An asymptote is a line that the graph of a function approaches, as x either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

3. $g(x) = 4(3)^{-x}$; y-intercept: $(0, 4)$; Domain: all real numbers; Range: all real numbers greater than 0.

5. $g(x) = -10^{x} + 7$; y-intercept: $(0, 6)$; Domain: all real numbers; Range: all real numbers less than 7.

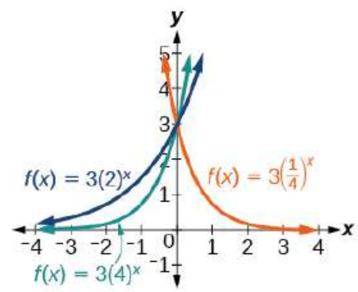
7. $g(x) = 2\left(\frac{1}{4}\right)^{x}$; y-intercept: $(0,\ \ 2)$; Domain: all real numbers; Range: all real numbers greater than 0.

9.



y-intercept: $(0, -2)$

11.



13. B

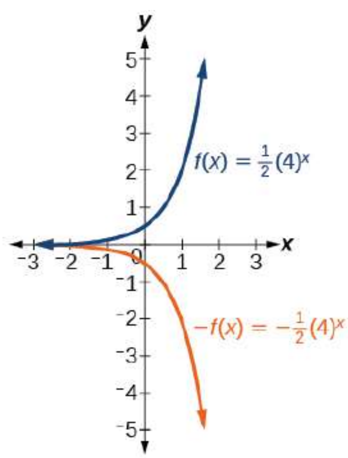
15. A

17. E

19. D

21. C

23.



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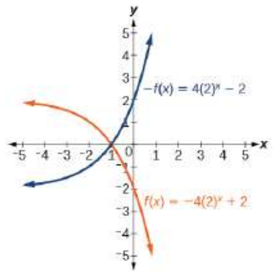
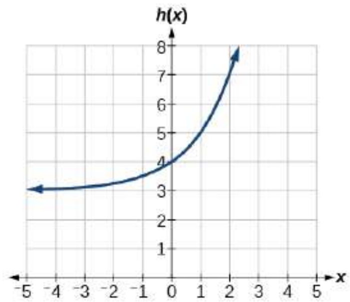


27.



Horizontal asymptote: $h(x) = 3$; Domain: all real numbers; Range: all real numbers strictly greater than 3.

29. As $x \to \infty$, $f(x) \to -\infty$;
As $x \to -\infty$, $f(x) \to -1$

31. As $x \to \infty$, $f(x) \to 2$;
As $x \to -\infty$, $f(x) \to \infty$

33. $f(x) = 4^x - 3$

35. $f(x) = 4^{x-5}$

37. $f(x) = 4^{-x}$

39. $y = -2^x + 3$

41. $y = -2(3)^x + 7$

43. $g(6) = 800 + \frac{1}{3} \approx 800.3333$

45. $h(-7) = -58$

47. $x \approx -2.953$

49. $x \approx -0.222$

51. The graph of $G(x) = \left(\frac{1}{b}\right)^x$ is the refelction about the *y*-axis of the graph of $F(x) = b^x$; For any real number $b > 0$ and function $f(x) = b^x$, the graph of $\left(\frac{1}{b}\right)^x$ is the the reflection about the *y*-axis, $F(-x)$.

53. The graphs of $g(x)$ and $h(x)$ are the same and are a horizontal shift to the right of the graph of $f(x)$; For any real number *n*, real number $b > 0$, and function $f(x) = b^x$, the graph of $\left(\frac{1}{b^n}\right) b^x$ is the horizontal shift $f(x - n)$.

6.3 Section Exercises

1. A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, y, in the expression b^y can also be written as the logarithm, $\log_b x$, and the value of x is the result of raising b to the power of y.

3. Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$, and then properties of exponents can be applied to solve for x.

5. The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_e(x)$, the notation used is $\ln(x)$.

7. $a^c = b$

9. $x^y = 64$

11. $15^b = a$

13. $13^a = 142$

15. $e^n = w$

17. $\log_c(k) = d$

19. $\log_{19} y = x$

21. $\log_n (103) = 4$

23. $\log_y \left(\frac{39}{100}\right) = x$

25. $\ln(h) = k$

27. $x = 2^{-3} = \frac{1}{8}$

29. $x = 3^3 = 27$

31. $x = 9^{\frac{1}{2}} = 3$

33. $x = 6^{-3} = \frac{1}{216}$

35. $x = e^2$

37. 32

39. 1.06

41. 14.125

43. $\frac{1}{2}$

45. 4

47. -3

49. -12

51. 0

53. 10

55. 2.708

57. 0.151

59. No, the function has no defined value for $x = 0$. To verify, suppose $x = 0$ is in the domain of the function $f(x) = \log(x)$. Then there is some number n such that $n = \log(0)$. Rewriting as an exponential equation gives: $10^n = 0$, which is impossible since no such real number n exists. Therefore, $x = 0$ is *not* the domain of the function $f(x) = \log(x)$.

61. Yes. Suppose there exists a real number x such that $\ln x = 2$. Rewriting as an exponential equation gives $x = e^2$, which is a real number. To verify, let $x = e^2$. Then, by definition, $\ln (x) = \ln \left(e^2\right) = 2$.

63. No; $\ln (1) = 0$, so $\frac{\ln\left(e^{1.725}\right)}{\ln(1)}$ is undefined.

65. 2

6.4 Section Exercises

1. Since the functions are inverses, their graphs are mirror images about the line $y = x$. So for every point (a, b) on the graph of a logarithmic function, there is a corresponding point (b, a) on the graph of its inverse exponential function.

3. Shifting the function right or left and reflecting the function about the y-axis will affect its domain.

5. No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

7. Domain: $\left(-\infty, \frac{1}{2}\right)$; Range: $\left(-\infty, \infty\right)$

9. Domain: $\left(-\frac{17}{4}, \infty\right)$; Range: $\left(-\infty, \infty\right)$

11. Domain: $\left(5, \infty\right)$; Vertical asymptote: $x = 5$

13. Domain: $\left(-\frac{1}{3}, \infty\right)$; Vertical asymptote: $x = -\frac{1}{3}$

15. Domain: $\left(-3, \infty\right)$; Vertical asymptote: $x = -3$

17. Domain: $\left(\frac{3}{7}, \infty\right)$; Vertical asymptote: $x = \frac{3}{7}$; End behavior: as $x \to \left(\frac{3}{7}\right)^+, f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$

19. Domain: $\left(-3, \infty\right)$; Vertical asymptote: $x = -3$; End behavior: as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

21. Domain: $\left(1, \infty\right)$; Range: $\left(-\infty, \infty\right)$; Vertical asymptote: $x = 1$; *x*-intercept: $\left(\frac{5}{4}, 0\right)$; *y*-intercept: DNE

23. Domain: $\left(-\infty, 0\right)$; Range: $\left(-\infty, \infty\right)$; Vertical asymptote: $x = 0$; *x*-intercept: $\left(-e^2, 0\right)$; *y*-intercept: DNE

25. Domain: $\left(0, \infty\right)$; Range: $\left(-\infty, \infty\right)$; Vertical asymptote: $x = 0$; *x*-intercept: $\left(e^3, 0\right)$; *y*-intercept: DNE

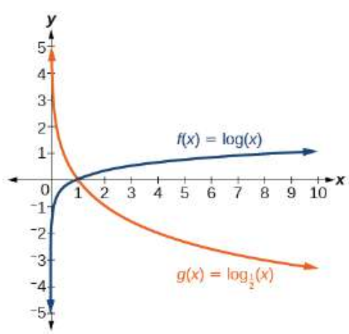
27. B

29. C

31. B

33. C

35.



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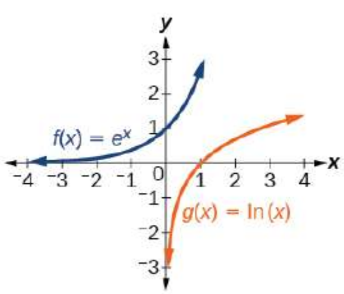
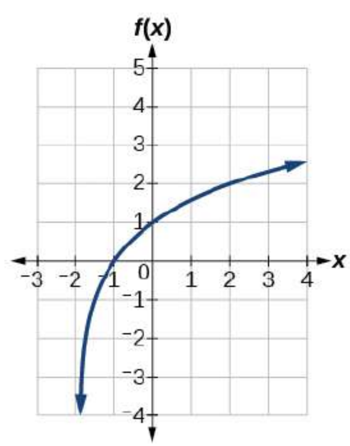


39. C

41.



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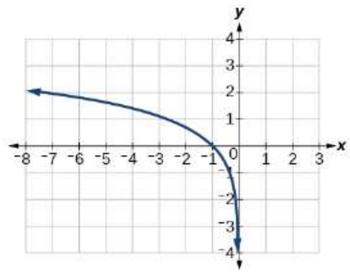

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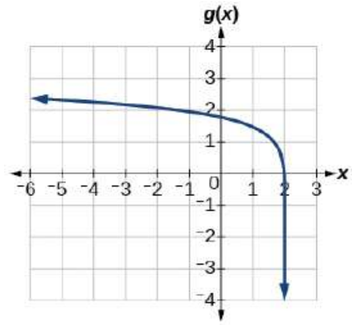


47. $f(x) = \log_2(-(x-1))$

49. $f(x) = 3\log_4(x+2)$

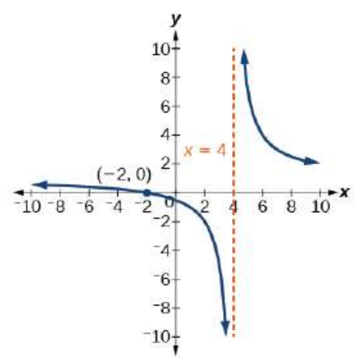
51. $x = 2$

53. $x \approx 2.303$

55. $x \approx -0.472$

57. The graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$ appear to be the same; Conjecture: for any positive base $b \neq 1$, $\log_b(x) = -\log_{\frac{1}{b}}(x)$.

59. Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4} > 0$. From the graph of the function $f(x) = \frac{x+2}{x-4}$, note that the graph lies above the *x*-axis on the interval $\left(-\infty, -2\right)$ and again to the right of the vertical asymptote, that is $\left(4, \infty\right)$. Therefore, the domain is $\left(-\infty, -2\right) \cup \left(4, \infty\right)$.



6.5 Section Exercises

1. Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_b(x)$.

3. $\log_b(2) + \log_b(7) + \log_b(x) + \log_b(y)$

5. $\log_b(13) - \log_b(17)$

7. $-k\ln(4)$

9. $\ln(7xy)$

11. $\log_b(4)$

13. $\log_b(7)$

15. $15\log(x) + 13\log(y) - 19\log(z)$

17. $\frac{3}{2}\log(x) - 2\log(y)$

19. $\frac{8}{3}\log(x) + \frac{14}{3}\log(y)$

21. $\ln(2x^7)$

23. $\log\left(\frac{xz^3}{\sqrt{y}}\right)$

25. $\log_7(15) = \frac{\ln(15)}{\ln(7)}$

27. $\log_{11}(5) = \frac{\log_5(5)}{\log_5(11)} = \frac{1}{b}$

29. $\log_{11}\left(\frac{6}{11}\right) = \frac{\log_5\left(\frac{6}{11}\right)}{\log_5(11)} = \frac{\log_5(6)-\log_5(11)}{\log_5(11)} = \frac{a-b}{b} = \frac{a}{b} - 1$

31. 3

33. 2.81359

35. 0.93913

37. -2.23266

39. $x = 4$; By the quotient rule:
$\log_6(x+2) - \log_6(x-3) = \log_6\left(\frac{x+2}{x-3}\right) = 1.$

Rewriting as an exponential equation and solving for x :

$$\begin{aligned}
6^1 &= \frac{x+2}{x-3} \\
0 &= \frac{x+2}{x-3} - 6 \\
0 &= \frac{x+2}{x-3} - \frac{6(x-3)}{(x-3)} \\
0 &= \frac{x+2-6x+18}{x-3} \\
0 &= \frac{x-4}{x-3} \\
x &= 4
\end{aligned}$$

Checking, we find that
$\log_6(4+2) - \log_6(4-3) = \log_6(6) - \log_6(1)$
is defined, so $x = 4$.

41. Let b and n be positive integers greater than 1. Then, by the change-of-base formula,
$\log_b(n) = \frac{\log_n(n)}{\log_n(b)} = \frac{1}{\log_n(b)}$.

6.6 Section Exercises

1. Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

3. The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

5. $x = -\frac{1}{3}$

7. $n = -1$

9. $b = \frac{6}{5}$

11. $x = 10$

13. No solution

15. $p=\log\left(\frac{17}{8}\right)-7$

17. $k=-\frac{\ln(38)}{3}$

19. $x=\frac{\ln\left(\frac{38}{3}\right)-8}{9}$

21. $x=\ln 12$

23. $x=\frac{\ln\left(\frac{3}{5}\right)-3}{8}$

25. no solution

27. $x=\ln(3)$

29. $10^{-2}=\frac{1}{100}$

31. $n=49$

33. $k=\frac{1}{36}$

35. $x=\frac{9-e}{8}$

37. $n=1$

39. No solution

41. No solution

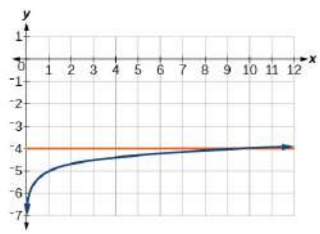
43. $x=\pm\frac{10}{3}$

45. $x=10$

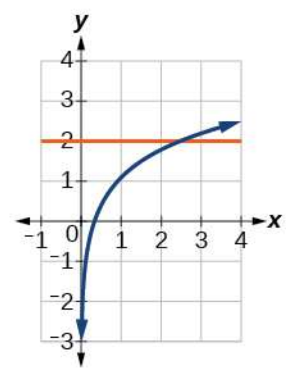
47. $x=0$

49. $x=\frac{3}{4}$

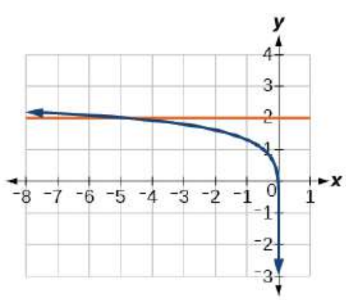
51. $x=9$



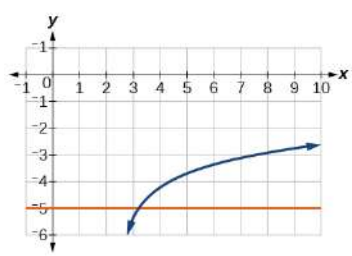
53. $x=\frac{e^2}{3}\approx 2.5$



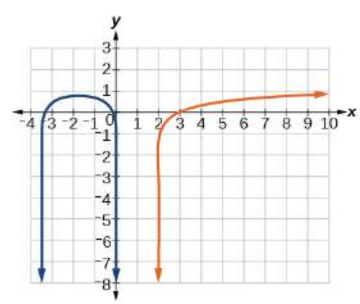
55. $x=-5$



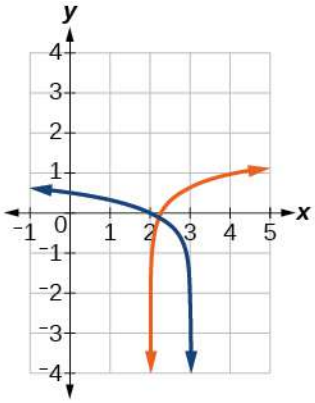
57. $x=\frac{e+10}{4}\approx 3.2$



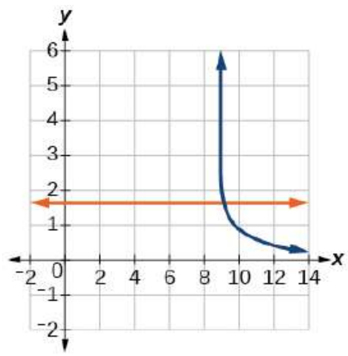
59. No solution



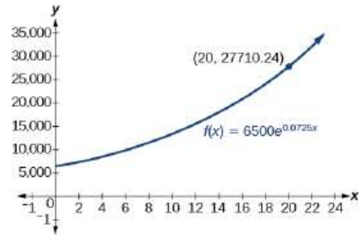
61. $x = \frac{11}{5} \approx 2.2$



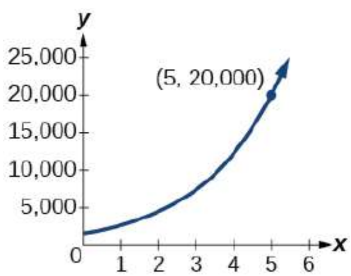
63. $x = \frac{101}{11} \approx 9.2$



65. about $\$27,710.24$



67. about 5 years



69. $\frac{\ln(17)}{5} \approx 0.567$

71. $x = \frac{\log(38)+5\log(3)}{4\log(3)} \approx 2.078$

73. $x \approx 2.2401$

75. $x \approx -44655.7143$

77. about 5.83

79. $t = \ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$

81. $t = \ln\left(\left(\frac{T-T_s}{T_0-T_s}\right)^{-\frac{1}{k}}\right)$

6.7 Section Exercises

1. Half-life is a measure of decay and is thus associated with exponential decay models. The half-life of a substance or quantity is the amount of time it takes for half of the initial amount of that substance or quantity to decay.

3. Doubling time is a measure of growth and is thus associated with exponential growth models. The doubling time of a substance or quantity is the amount of time it takes for the initial amount of that substance or quantity to double in size.

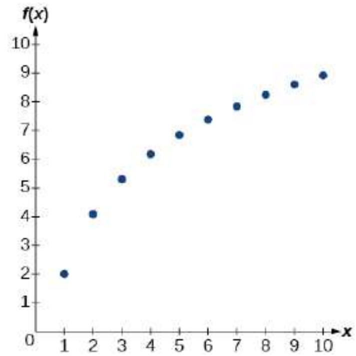
5. An order of magnitude is the nearest power of ten by which a quantity exponentially grows. It is also an approximate position on a logarithmic scale; Sample response: Orders of magnitude are useful when making comparisons between numbers that differ by a great amount. For example, the mass of Saturn is 95 times greater than the mass of Earth. This is the same as saying that the mass of Saturn is about 10^2 times, or *2 orders of magnitude* greater, than the mass of Earth.

7. $f(0) \approx 16.7$; The amount initially present is about 16.7 units.

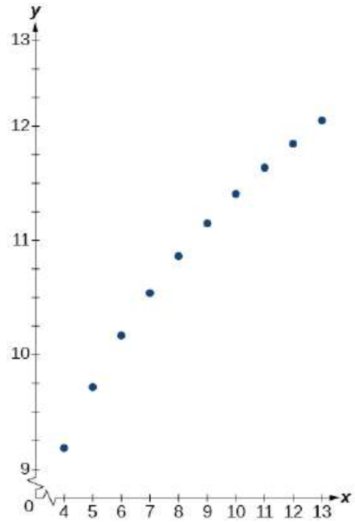
9. 150

11. exponential; $f(x) = 1.2^x$

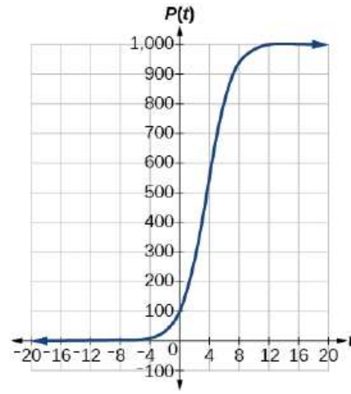
13. logarithmic



15. logarithmic



17.



19. about 1.4 years

21. about 7.3 years

23. 4 half-lives; 8.18 minutes

25.
$$M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$
$$\log\left(\frac{S}{S_0}\right) = \frac{3}{2}M$$
$$\frac{S}{S_0} = 10^{\frac{3M}{2}}$$
$$S = S_0 10^{\frac{3M}{2}}$$

27. Let $y = b^x$ for some non-negative real number b such that $b \neq 1$. Then,
$$\ln(y) = \ln(b^x)$$
$$\ln(y) = x\ln(b)$$
$$e^{\ln(y)} = e^{x\ln(b)}$$
$$y = e^{x\ln(b)}$$

29. $A = 125e^{(-0.3567t)}$; $A \approx 43$ mg

31. about 60 days

33. $A(t) = 250e^{(-0.00822t)}$; half-life: about 84 minutes

35. $r \approx -0.0667$, So the hourly decay rate is about 6.67%

37. $f(t) = 1350e^{(0.03466t)}$; after 3 hours: $P(180) \approx 691,200$

39. $f(t) = 256e^{(0.068110t)}$; doubling time: about 10 minutes

41. about 88 minutes

43. $T(t) = 90e^{(-0.008377t)} + 75$, where t is in minutes.

45. about 113 minutes

47. $\log(x) = 1.5; \quad x \approx 31.623$

49. MMS magnitude: 5.82

51. $N(3) \approx 71$

53. C

6.8 Section Exercises

1. Logistic models are best used for situations that have limited values. For example, populations cannot grow indefinitely since resources such as food, water, and space are limited, so a logistic model best describes populations.

3. Regression analysis is the process of finding an equation that best fits a given set of data points. To perform a regression analysis on a graphing utility, first list the given points using the STAT then EDIT menu. Next graph the scatter plot using the STAT PLOT feature. The shape of the data points on the scatter graph can help determine which regression feature to use. Once this is determined, select the appropriate regression analysis command from the STAT then CALC menu.

5. The *y*-intercept on the graph of a logistic equation corresponds to the initial population for the population model.

7. C

9. B

11. $P(0) = 22$; 175

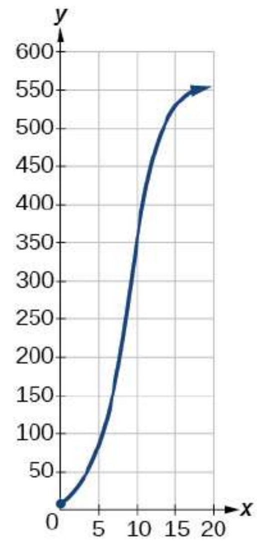
13. $p \approx 2.67$

15. *y*-intercept: $(0, 15)$

17. 4 koi

19. about 6.8 months.

21.

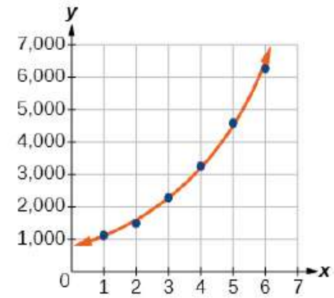


23. About 38 wolves

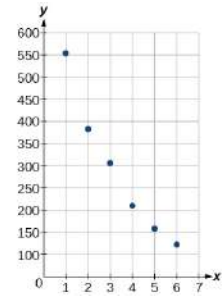
25. About 8.7 years

27. $f(x) = 776.682(1.426)^x$

29.



31.

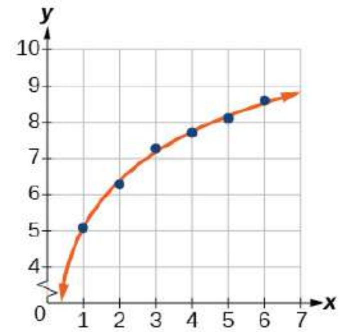


33. $f\left(x\right)=731.92e^{-0.3038x}$

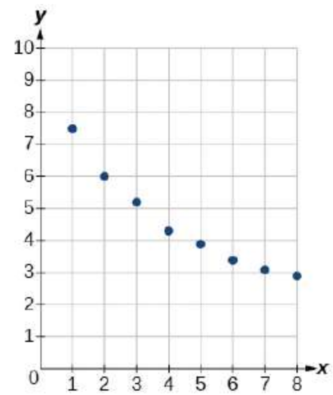
35. When $f(x)=250$, $x\approx 3.6$

37. $y=5.063+1.934\log(x)$

39.



41.



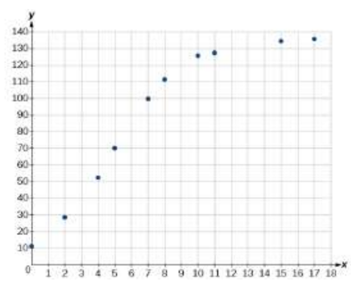
43. When $f(10)\approx 2.3$

45. When $f(x)=8$, $x\approx 0.82$

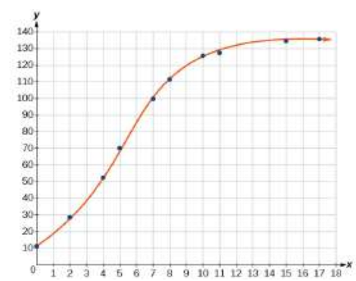
47. $f(x)=\frac{25.081}{1+3.182e^{-0.545x}}$

49. About 25

51.



53.



55. When $f(x)=68$, $x\approx 4.9$

57. $f(x)=1.034341(1.281204)^{x}$; $g(x)=4.035510$; the regression curves are symmetrical about $y=x$, so it appears that they are inverse functions.

59. $f^{-1}\left(x\right)=\frac{\ln(a)-\ln\left(\frac{c}{x}-1\right)}{b}$

Review Exercises

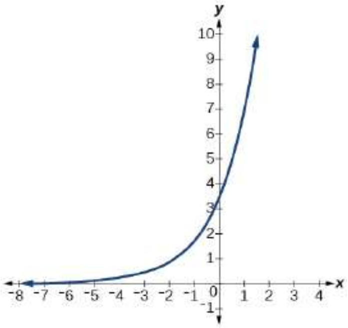
1. exponential decay; The growth factor, 0.825, is between 0 and 1.

3. $y=0.25(3)^{x}$

5. $\$42,888.18$

7. continuous decay; the growth rate is negative.

9. domain: all real numbers; range: all real numbers strictly greater than zero; y-intercept: (0, 3.5);



11. $g(x) = 7(6.5)^{-x}$; y-intercept: $(0, \ \ 7)$; Domain: all real numbers; Range: all real numbers greater than 0.

13. $17^x = 4913$

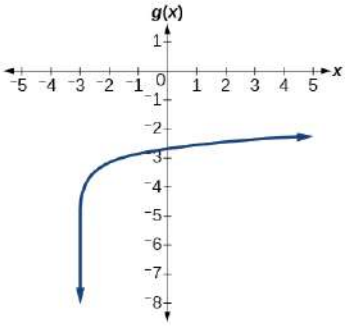
15. $\log_a b = -\frac{2}{5}$

17. $x = 64^{\frac{1}{3}} = 4$

19. $\log(0.000001) = -6$

21. $\ln\left(e^{-0.8648}\right) = -0.8648$

23.



25. Domain: $x > -5$; Vertical asymptote: $x = -5$; End behavior: as $x \to -5^+, f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$.

27. $\log_8(65xy)$

29. $\ln\left(\frac{z}{xy}\right)$

31. $\log_y(12)$

33. $\ln(2) + \ln(b) + \frac{\ln(b+1) - \ln(b-1)}{2}$

35. $\log_7\left(\frac{v^3 w^6}{\sqrt[3]{u}}\right)$

37. $x = \frac{\frac{\log(125)}{\log(5)} + 17}{12} = \frac{5}{3}$

39. $x = -3$

41. no solution

43. no solution

45. $x = \ln(11)$

47. $a = e^4 - 3$

49. $x = \pm\frac{9}{5}$

51. about 5.45 years

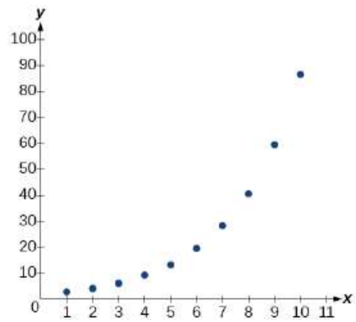
53. $f^{-1}(x) = \sqrt[3]{2^{4x} - 1}$

55. $f(t) = 300(0.83)^t$; $f(24) \approx 3.43 \quad g$

57. about 45 minutes

59. about 8.5 days

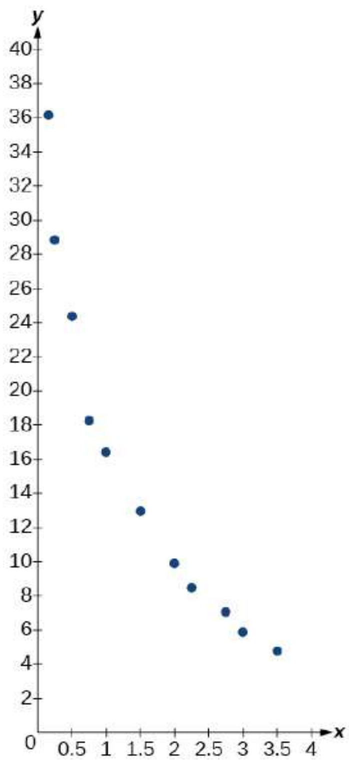
61. exponential



63. $y = 4(0.2)^x$;
$y = 4e^{-1.609438x}$

65. about 7.2 days

67. logarithmic;
$y = 16.68718 - 9.71860\ln(x)$

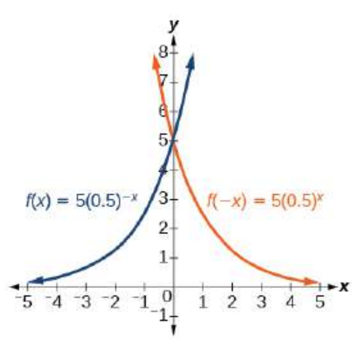


Practice Test

1. About 13 dolphins.

3. $\$1{,}947$

5. *y*-intercept: $(0,\ \ 5)$



7. $8.5^a = 614.125$

9. $x = \left(\frac{1}{7}\right)^2 = \frac{1}{49}$

11. $\ln(0.716) \approx -0.334$

13. Domain: $x < 3$; Vertical asymptote: $x = 3$; End behavior:
$x \to 3^-, f(x) \to -\infty$ and
$x \to -\infty, f(x) \to \infty$

15. $\log_t(12)$

17. $3\ \ln(y) + 2\ln(z) + \frac{\ln(x-4)}{3}$

19. $x = \frac{\frac{\ln(1000)}{\ln(16)}+5}{3} \approx 2.497$

21. $a = \frac{\ln(4)+8}{10}$

23. no solution

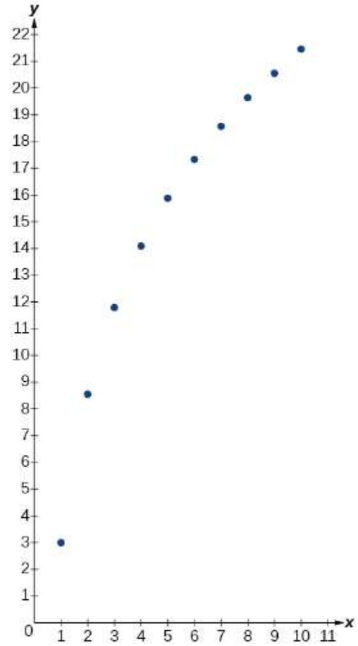
25. $x = \ln(9)$

27. $x = \pm\frac{3\sqrt{3}}{2}$

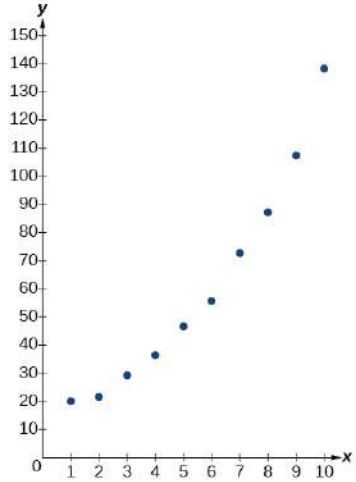
29. $f(t) = 112e^{-.019792t}$; half-life: about 35 days

31. $T(t) = 36e^{-0.025131t} + 35; T(60) \approx 43^{\mathrm{o}}\mathrm{F}$

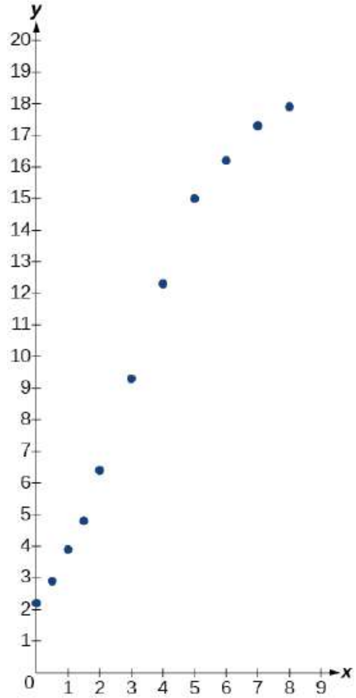
33. logarithmic



35. exponential;
$y = 15.10062(1.24621)^x$



37. logistic;
$y = \frac{18.41659}{1+7.54644e^{-0.68375x}}$



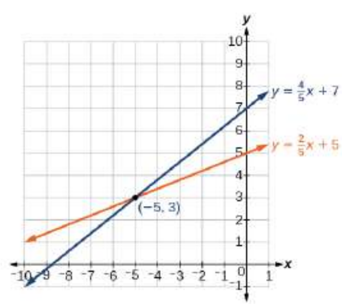
Chapter 7

Try It

7.1 Systems of Linear Equations: Two Variables

1. Not a solution.

2. The solution to the system is the ordered pair $(-5, 3)$.



3. $(-2, -5)$

4. $(-6, -2)$

5. $(10, -4)$

6. No solution. It is an inconsistent system.

7. The system is dependent so there are infinite solutions of the form $(x, 2x + 5)$.

8. 700 children, 950 adults

7.2 Systems of Linear Equations: Three Variables

1. $(1, -1, 1)$

2. No solution.

3. Infinite number of solutions of the form $(x, 4x-11, -5x+18)$.

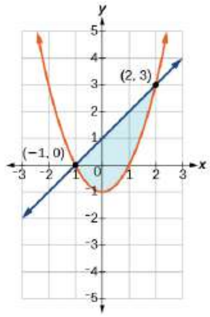
7.3 Systems of Nonlinear Equations and Inequalities: Two Variables

1. $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(2, 8)$

2. $(-1, 3)$

3. $\{(1, 3), (1, -3), (-1, 3), (-1, -3)\}$

4.



7.4 Partial Fractions

1. $\frac{3}{x-3} - \frac{2}{x-2}$

2. $\frac{6}{x-1} - \frac{5}{(x-1)^2}$

3. $\frac{3}{x-1} + \frac{2x-4}{x^2+1}$

4. $\frac{x-2}{x^2-2x+3} + \frac{2x+1}{\left(x^2-2x+3\right)^2}$

7.5 Matrices and Matrix Operations

1. $$A + B = \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2+3 & 6+(-2) \\ 1+1 & 0+5 \\ 1+(-4) & -3+3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 2 & 5 \\ -3 & 0 \end{bmatrix}$$

2. $-2B = \begin{bmatrix} -8 & -2 \\ -6 & -4 \end{bmatrix}$

7.6 Solving Systems with Gaussian Elimination

1. $\left[\begin{array}{cc|c} 4 & -3 & 11 \\ 3 & 2 & 4 \end{array}\right]$

2. $$\begin{aligned} x - y + z &= 5 \\ 2x - y + 3z &= 1 \\ y + z &= -9 \end{aligned}$$

3. $(2, \ 1)$

4. $\left[\begin{array}{ccc|c} 1 & -\frac{5}{2} & \frac{5}{2} & \frac{17}{2} \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right]$

5. $(1, \ 1, \ 1)$

6. \$150,000 at 7%, \$750,000 at 8%, \$600,000 at 10%

7.7 Solving Systems with Inverses

1. $$AB = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1(-3)+4(1) & 1(-4)+4(1) \\ -1(-3)+-3(1) & -1(-4)+-3(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$BA = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -3(1)+-4(-1) & -3(4)+-4(-3) \\ 1(1)+1(-1) & 1(4)+1(-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. $A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

3. $A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

4. $X = \begin{bmatrix} 4 \\ 38 \\ 58 \end{bmatrix}$

7.8 Solving Systems with Cramer's Rule

1. $(3, -7)$

2. -10

3. $\left(-2, \frac{3}{5}, \frac{12}{5}\right)$

7.1 Section Exercises

1. No, you can either have zero, one, or infinitely many. Examine graphs.

3. This means there is no realistic break-even point. By the time the company produces one unit they are already making profit.

5. You can solve by substitution (isolating x or y), graphically, or by addition.

7. Yes

9. Yes

11. $(-1, 2)$

13. $(-3, 1)$

15. $\left(-\frac{3}{5}, 0\right)$

17. No solutions exist.

19. $\left(\frac{72}{5}, \frac{132}{5}\right)$

21. $(6, -6)$

23. $\left(-\frac{1}{2}, \frac{1}{10}\right)$

25. No solutions exist.

27. $\left(-\frac{1}{5}, \frac{2}{3}\right)$

29. $\left(x, \frac{x+3}{2}\right)$

31. $(-4, 4)$

33. $\left(\frac{1}{2}, \frac{1}{8}\right)$

35. $\left(\frac{1}{6}, 0\right)$

37. $(x, 2(7x-6))$

39. $\left(-\frac{5}{6}, \frac{4}{3}\right)$

41. Consistent with one solution

43. Consistent with one solution

45. Dependent with infinitely many solutions

47. $(-3.08, 4.91)$

49. $(-1.52, 2.29)$

51. $\left(\frac{A+B}{2}, \frac{A-B}{2}\right)$

53. $\left(\frac{-1}{A-B}, \frac{A}{A-B}\right)$

55. $\left(\frac{CE-BF}{BD-AE}, \frac{AF-CD}{BD-AE}\right)$

57. They never turn a profit.

59. $(1,250, 100,000)$

61. The numbers are 7.5 and 20.5.

63. 24,000

65. 790 second-year students, 805 first-year students

67. 56 men, 74 women

69. 10 gallons of 10% solution, 15 gallons of 60% solution

71. Swan Peak: \$750,000, Riverside: \$350,000

73. \$12,500 in the first account, \$10,500 in the second account.

75. High-tops: 45, Low-tops: 15

77. Infinitely many solutions. We need more information.

7.2 Section Exercises

1. No, there can be only one, zero, or infinitely many solutions.

3. Not necessarily. There could be zero, one, or infinitely many solutions. For example, $(0, 0, 0)$ is not a solution to the system below, but that does not mean that it has no solution.

$$\begin{aligned} 2x + 3y - 6z &= 1 \\ -4x - 6y + 12z &= -2 \\ x + 2y + 5z &= 10 \end{aligned}$$

5. Every system of equations can be solved graphically, by substitution, and by addition. However, systems of three equations become very complex to solve graphically so other methods are usually preferable.

7. No

9. Yes

11. $(-1, 4, 2)$

13. $\left(-\frac{85}{107}, \frac{312}{107}, \frac{191}{107}\right)$

15. $\left(1, \frac{1}{2}, 0\right)$

17. $(4, -6, 1)$

19. $\left(x, \frac{1}{27}(65-16x), \frac{x+28}{27}\right)$

21. $\left(-\frac{45}{13}, \frac{17}{13}, -2\right)$

23. No solutions exist

25. $(0, 0, 0)$

27. $\left(\frac{4}{7}, -\frac{1}{7}, -\frac{3}{7}\right)$

29. $(7, 20, 16)$

31. $(-6, 2, 1)$

33. $(5, 12, 15)$

35. $(-5, -5, -5)$

37. $(10, 10, 10)$

39. $\left(\frac{1}{2}, \frac{1}{5}, \frac{4}{5}\right)$

41. $\left(\frac{1}{2}, \frac{2}{5}, \frac{4}{5}\right)$

43. $(2, 0, 0)$

45. $(1, 1, 1)$

47. $\left(\frac{128}{557}, \frac{23}{557}, \frac{28}{557}\right)$

49. $(6, -1, 0)$

51. 24, 36, 48

53. 70 grandparents, 140 parents, 190 children

55. Your share was \$19.95, Shani’s share was \$40, and your other roommate’s share was \$22.05.

57. There are infinitely many solutions; we need more information

59. 500 students, 225 children, and 450 adults

61. The BMW was \$49,636, the Jeep was \$42,636, and the Toyota was \$47,727.

63. \$400,000 in the account that pays 3% interest, \$500,000 in the account that pays 4% interest, and \$100,000 in the account that pays 2% interest.

65. The United States consumed 26.3%, Japan 7.1%, and China 6.4% of the world’s oil.

67. Saudi Arabia imported 16.8%, Canada imported 15.1%, and Mexico 15.0%

69. Birds were 19.3%, fish were 18.6%, and mammals were 17.1% of endangered species

7.3 Section Exercises

1. A nonlinear system could be representative of two circles that overlap and intersect in two locations, hence two solutions. A nonlinear system could be representative of a parabola and a circle, where the vertex of the parabola meets the circle and the branches also intersect the circle, hence three solutions.

3. No. There does not need to be a feasible region. Consider a system that is bounded by two parallel lines. One inequality represents the region above the upper line; the other represents the region below the lower line. In this case, no points in the plane are located in both regions; hence there is no feasible region.

5. Choose any number between each solution and plug into $C(x)$ and $R(x)$. If $C(x) < R(x)$, then there is profit.

7. $(0, -3), (3, 0)$

9. $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

11. $(-3, 0), (3, 0)$

13. $\left(\frac{1}{4}, -\frac{\sqrt{62}}{8}\right), \left(\frac{1}{4}, \frac{\sqrt{62}}{8}\right)$

15. $\left(-\frac{\sqrt{398}}{4}, \frac{199}{4}\right), \left(\frac{\sqrt{398}}{4}, \frac{199}{4}\right)$

17. $(0, 2), (1, 3)$

19. $\left(-\sqrt{\frac{1}{2}\left(\sqrt{5}-1\right)}, \frac{1}{2}\left(1-\sqrt{5}\right)\right), \left(\sqrt{\frac{1}{2}\left(\sqrt{5}-1\right)}, \frac{1}{2}\left(1-\sqrt{5}\right)\right)$

21. $(5, 0)$

23. $(0, 0)$

25. $(3, 0)$

27. No Solutions Exist

29. No Solutions Exist

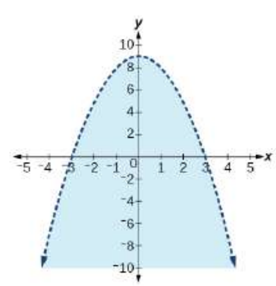
31. $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

33. $(2, 0)$

35. $\left(-\sqrt{7}, -3\right), \left(-\sqrt{7}, 3\right), \left(\sqrt{7}, -3\right), \left(\sqrt{7}, 3\right)$

37. $\left(-\sqrt{\frac{1}{2}\left(\sqrt{73}-5\right)}, \frac{1}{2}\left(7-\sqrt{73}\right)\right), \left(\sqrt{\frac{1}{2}\left(\sqrt{73}-5\right)}, \frac{1}{2}\left(7-\sqrt{73}\right)\right)$

39.



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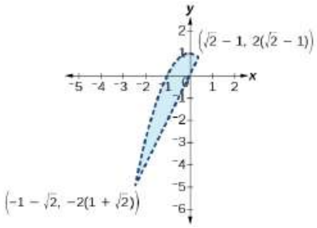

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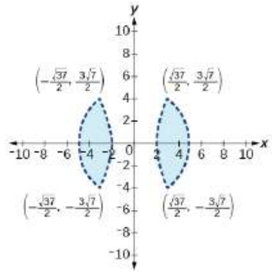

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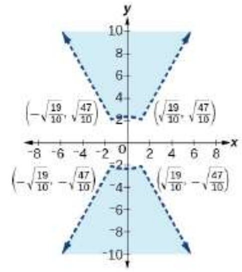

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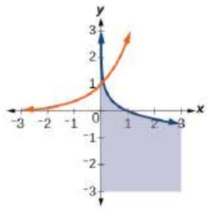


49. $\left(-2\sqrt{\frac{70}{383}}, -2\sqrt{\frac{35}{29}}\right), \left(-2\sqrt{\frac{70}{383}}, 2\sqrt{\frac{35}{29}}\right), \left(2\sqrt{\frac{70}{383}}, -2\sqrt{\frac{35}{29}}\right), \left(2\sqrt{\frac{70}{383}}, 2\sqrt{\frac{35}{29}}\right)$

51. No Solution Exists

53. $x = 0, y > 0$ and $0 < x < 1, \sqrt{x} < y < \frac{1}{x}$

55. 12, 288

57. 2–20 computers

7.4 Section Exercises

1. No, a quotient of polynomials can only be decomposed if the denominator can be factored. For example, $\frac{1}{x^2+1}$ cannot be decomposed because the denominator cannot be factored.

3. Graph both sides and ensure they are equal.

5. If we choose $x = -1$, then the *B*-term disappears, letting us immediately know that $A = 3$. We could alternatively plug in $x = -\frac{5}{3}$, giving us a *B*-value of -2.

7. $\frac{8}{x+3} - \frac{5}{x-8}$

9. $\frac{1}{x+5} + \frac{9}{x+2}$

11. $\frac{3}{5x-2} + \frac{4}{4x-1}$

13. $\frac{5}{2(x+3)} + \frac{5}{2(x-3)}$

15. $\frac{3}{x+2} + \frac{3}{x-2}$

17. $\frac{9}{5(x+2)} + \frac{11}{5(x-3)}$

19. $\frac{8}{x-3} - \frac{5}{x-2}$

21. $\frac{1}{x-2} + \frac{2}{(x-2)^2}$

23. $-\frac{6}{4x+5} + \frac{3}{(4x+5)^2}$

25. $-\frac{1}{x-7} - \frac{2}{(x-7)^2}$

27. $\frac{4}{x} - \frac{3}{2(x+1)} + \frac{7}{2(x+1)^2}$

29. $\frac{4}{x} + \frac{2}{x^2} - \frac{3}{3x+2} + \frac{7}{2(3x+2)^2}$

31. $\frac{x+1}{x^2+x+3} + \frac{3}{x+2}$

33. $\frac{4-3x}{x^2+3x+8} + \frac{1}{x-1}$

35. $\frac{2x-1}{x^2+6x+1} + \frac{2}{x+3}$

37. $\frac{1}{x^2+x+1} + \frac{4}{x-1}$

39. $\frac{2}{x^2-3x+9} + \frac{3}{x+3}$

41. $-\frac{1}{4x^2+6x+9} + \frac{1}{2x-3}$

43. $\frac{1}{x}+\frac{1}{x+6}-\frac{4x}{x^2-6x+36}$

45. $\frac{x+6}{x^2+1}+\frac{4x+3}{\left(x^2+1\right)^2}$

47. $\frac{x+1}{x+2}+\frac{2x+3}{(x+2)^2}$

49. $\frac{1}{x^2+3x+25}-\frac{3x}{\left(x^2+3x+25\right)^2}$

51. $\frac{1}{8x}-\frac{x}{8\left(x^2+4\right)}+\frac{10-x}{2\left(x^2+4\right)^2}$

53. $-\frac{16}{x}-\frac{9}{x^2}+\frac{16}{x-1}-\frac{7}{(x-1)^2}$

55. $\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{5}{(x+1)^3}$

57. $\frac{5}{x-2}-\frac{3}{10(x+2)}+\frac{7}{x+8}-\frac{7}{10(x-8)}$

59. $-\frac{5}{4x}-\frac{5}{2(x+2)}+\frac{11}{2(x+4)}+\frac{5}{4(x+4)}$

7.5 Section Exercises

1. No, they must have the same dimensions. An example would include two matrices of different dimensions. One cannot add the following two matrices because the first is a 2×2 matrix and the second is a 2×3 matrix. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ has no sum.

3. Yes, if the dimensions of A are $m \times n$ and the dimensions of B are $n \times m$, both products will be defined.

5. Not necessarily. To find AB, we multiply the first row of A by the first column of B to get the first entry of AB. To find BA, we multiply the first row of B by the first column of A to get the first entry of BA. Thus, if those are unequal, then the matrix multiplication does not commute.

7. $\begin{bmatrix} 11 & 19 \\ 15 & 94 \\ 17 & 67 \end{bmatrix}$

9. $\begin{bmatrix} -4 & 2 \\ 8 & 1 \end{bmatrix}$

11. Undidentified; dimensions do not match

13. $\begin{bmatrix} 9 & 27 \\ 63 & 36 \\ 0 & 192 \end{bmatrix}$

15. $\begin{bmatrix} -64 & -12 & -28 & -72 \\ -360 & -20 & -12 & -116 \end{bmatrix}$

17. $\begin{bmatrix} 1,800 & 1,200 & 1,300 \\ 800 & 1,400 & 600 \\ 700 & 400 & 2,100 \end{bmatrix}$

19. $\begin{bmatrix} 20 & 102 \\ 28 & 28 \end{bmatrix}$

21. $\begin{bmatrix} 60 & 41 & 2 \\ -16 & 120 & -216 \end{bmatrix}$

23. $\begin{bmatrix} -68 & 24 & 136 \\ -54 & -12 & 64 \\ -57 & 30 & 128 \end{bmatrix}$

25. Undefined; dimensions do not match.

27. $\begin{bmatrix} -8 & 41 & -3 \\ 40 & -15 & -14 \\ 4 & 27 & 42 \end{bmatrix}$

29. $\begin{bmatrix} -840 & 650 & -530 \\ 330 & 360 & 250 \\ -10 & 900 & 110 \end{bmatrix}$

31. $\begin{bmatrix} -350 & 1,050 \\ 350 & 350 \end{bmatrix}$

33. Undefined; inner dimensions do not match.

35. $\begin{bmatrix} 1,400 & 700 \\ -1,400 & 700 \end{bmatrix}$

37. $\begin{bmatrix} 332,500 & 927,500 \\ -227,500 & 87,500 \end{bmatrix}$

39. $\begin{bmatrix} 490,000 & 0 \\ 0 & 490,000 \end{bmatrix}$

41. $\begin{bmatrix} -2 & 3 & 4 \\ -7 & 9 & -7 \end{bmatrix}$

43. $\begin{bmatrix} -4 & 29 & 21 \\ -27 & -3 & 1 \end{bmatrix}$

45. $\begin{bmatrix} -3 & -2 & -2 \\ -28 & 59 & 46 \\ -4 & 16 & 7 \end{bmatrix}$

47. $\begin{bmatrix} 1 & -18 & -9 \\ -198 & 505 & 369 \\ -72 & 126 & 91 \end{bmatrix}$

49. $\begin{bmatrix} 0 & 1.6 \\ 9 & -1 \end{bmatrix}$

51. $\begin{bmatrix} 2 & 24 & -4.5 \\ 12 & 32 & -9 \\ -8 & 64 & 61 \end{bmatrix}$

53. $\begin{bmatrix} 0.5 & 3 & 0.5 \\ 2 & 1 & 2 \\ 10 & 7 & 10 \end{bmatrix}$

55. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

57. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

59. $B^n = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & n \text{ even}, \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, & n \text{ odd}. \end{cases}$

7.6 Section Exercises

1. Yes. For each row, the coefficients of the variables are written across the corresponding row, and a vertical bar is placed; then the constants are placed to the right of the vertical bar.

3. No, there are numerous correct methods of using row operations on a matrix. Two possible ways are the following: (1) Interchange rows 1 and 2. Then $R_2 = R_2 - 9R_1$. (2) $R_2 = R_1 - 9R_2$. Then divide row 1 by 9.

5. No. A matrix with 0 entries for an entire row would have either zero or infinitely many solutions.

7. $\left[\begin{array}{cc|c} 0 & 16 & 4 \\ 9 & -1 & 2 \end{array}\right]$

9. $\left[\begin{array}{ccc|c} 1 & 5 & 8 & 16 \\ 12 & 3 & 0 & 4 \\ 3 & 4 & 9 & -7 \end{array}\right]$

11. $\begin{aligned} -2x + 5y &= 5 \\ 6x - 18y &= 26 \end{aligned}$

13. $\begin{aligned} 3x + 2y &= 3 \\ -x - 9y + 4z &= -1 \\ 8x + 5y + 7z &= 8 \end{aligned}$

15. $\begin{aligned} 4x + 5y - 2z &= 12 \\ y + 58z &= 2 \\ 8x + 7y - 3z &= -5 \end{aligned}$

17. No solutions

19. $(-1, -2)$

21. $(6, 7)$

23. $(3, 2)$

25. $\left(\frac{1}{5}, \frac{1}{2}\right)$

27. $\left(x, \frac{4}{15}(5x+1)\right)$

29. $(3, 4)$

31. $\left(\frac{196}{39}, -\frac{5}{13}\right)$

33. $(31, -42, 87)$

35. $\left(\frac{21}{40}, \frac{1}{20}, \frac{9}{8}\right)$

37. $\left(\frac{18}{13}, \frac{15}{13}, -\frac{15}{13}\right)$

39. $\left(x, y, \frac{1}{2}(1-2x-3y)\right)$

41. $\left(x, -\frac{x}{2}, -1\right)$

43. $(125, -25, 0)$

45. $(8, 1, -2)$

47. $(1, 2, 3)$

49. $\left(x, \frac{31}{28} - \frac{3x}{4}, \frac{1}{28}(-7x-3)\right)$

51. No solutions exist.

53. 860 red velvet, 1,340 chocolate

55. 4% for account 1, 6% for account 2

57. $126

59. Banana was 3%, pumpkin was 7%, and rocky road was 2%

61. 100 almonds, 200 cashews, 600 pistachios

7.7 Section Exercises

1. If A^{-1} is the inverse of A, then $AA^{-1} = I$, the identity matrix. Since A is also the inverse of A^{-1}, $A^{-1}A = I$. You can also check by proving this for a 2×2 matrix.

3. No, because ad and bc are both 0, so $ad - bc = 0$, which requires us to divide by 0 in the formula.

5. Yes. Consider the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The inverse is found with the following calculation: $A^{-1} = \frac{1}{0(0)-1(1)} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

7. $AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

9. $AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

11. $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

13. $\frac{1}{29} \begin{bmatrix} 9 & 2 \\ -1 & 3 \end{bmatrix}$

15. $\frac{1}{69} \begin{bmatrix} -2 & 7 \\ 9 & 3 \end{bmatrix}$

17. There is no inverse

19. $\frac{4}{7} \begin{bmatrix} 0.5 & 1.5 \\ 1 & -0.5 \end{bmatrix}$

21. $\frac{1}{17} \begin{bmatrix} -5 & 5 & -3 \\ 20 & -3 & 12 \\ 1 & -1 & 4 \end{bmatrix}$

23. $\frac{1}{209} \begin{bmatrix} 47 & -57 & 69 \\ 10 & 19 & -12 \\ -24 & 38 & -13 \end{bmatrix}$

25. $\begin{bmatrix} 18 & 60 & -168 \\ -56 & -140 & 448 \\ 40 & 80 & -280 \end{bmatrix}$

27. $(-5, 6)$

29. $(2, 0)$

31. $\left(\frac{1}{3}, -\frac{5}{2}\right)$

33. $\left(-\frac{2}{3}, -\frac{11}{6}\right)$

35. $\left(7, \frac{1}{2}, \frac{1}{5}\right)$

37. $(5, 0, -1)$

39. $\frac{1}{34}(-35, -97, -154)$

41. $\frac{1}{690}(65, -1136, -229)$

43. $\left(-\frac{37}{30}, \frac{8}{15}\right)$

45. $\left(\frac{10}{123}, -1, \frac{2}{5}\right)$

47. $\frac{1}{2} \begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$

49. $\frac{1}{39} \begin{bmatrix} 3 & 2 & 1 & -7 \\ 18 & -53 & 32 & 10 \\ 24 & -36 & 21 & 9 \\ -9 & 46 & -16 & -5 \end{bmatrix}$

51. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix}$

53. Infinite solutions.

55. 50% oranges, 25% bananas, 20% apples

57. 10 straw hats, 50 beanies, 40 cowboy hats

59. Micah ate 6, Joe ate 3, and Albert ate 3.

61. 124 oranges, 10 lemons, 8 pomegranates

7.8 Section Exercises

1. A determinant is the sum and products of the entries in the matrix, so you can always evaluate that product—even if it does end up being 0.

3. The inverse does not exist.

5. -2

7. 7

9. -4

11. 0

13. $-7,990.7$

15. 3

17. -1

19. 224

21. 15

23. -17.03

25. $(1,1)$

27. $\left(\frac{1}{2},\frac{1}{3}\right)$

29. $(2,5)$

31. $\left(-1,-\frac{1}{3}\right)$

33. $(15,12)$

35. $(1,3,2)$

37. $(-1,0,3)$

39. $\left(\frac{1}{2},1,2\right)$

41. $(2,1,4)$

43. Infinite solutions

45. 24

47. 1

49. Yes; 18, 38

51. Yes; 33, 36, 37

53. $7,000 in first account, $3,000 in second account.

55. 120 children, 1,080 adult

57. 4 gal yellow, 6 gal blue

59. 13 green tomatoes, 17 red tomatoes

61. Strawberries 18%, oranges 9%, kiwi 10%

63. 100 for movie 1, 230 for movie 2, 312 for movie 3

65. 300 almonds, 400 cranberries, 300 cashews

Review Exercises

1. No

3. $(-2,3)$

5. $(4,-1)$

7. No solutions exist.

9. $(300,60,000)$

11. Infinite solutions

13. No solutions exist.

15. $(-1,-2,3)$

17. $\left(x,\frac{8x}{5},\frac{14x}{5}\right)$

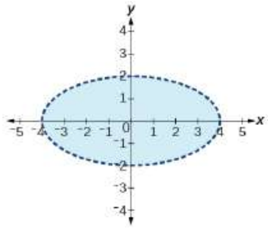
19. 11, 17, 33

21. $(2,-3),(3,2)$

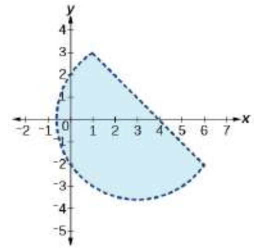
23. No solution

25. No solution

27.



29.



31. $\frac{2}{x+2}, \frac{-4}{x+1}$

33. $\frac{7}{x+5}, \frac{-15}{(x+5)^2}$

35. $\frac{3}{x-5}, \frac{-4x+1}{x^2+5x+25}$

37. $\frac{x-4}{(x^2-2)}, \frac{5x+3}{(x^2-2)^2}$

39. $\begin{bmatrix} -16 & 8 \\ -4 & -12 \end{bmatrix}$

41. undefined; dimensions do not match

43. undefined; inner dimensions do not match

45. $\begin{bmatrix} 113 & 28 & 10 \\ 44 & 81 & -41 \\ 84 & 98 & -42 \end{bmatrix}$

47. $\begin{bmatrix} -127 & -74 & 176 \\ -2 & 11 & 40 \\ 28 & 77 & 38 \end{bmatrix}$

49. undefined; inner dimensions do not match

51. $\begin{array}{l} x - 3z = 7 \\ y + 2z = -5 \end{array}$ with infinite solutions

53. $\left[\begin{array}{ccc|c} -2 & 2 & 1 & 7 \\ 2 & -8 & 5 & 0 \\ 19 & -10 & 22 & 3 \end{array}\right]$

55. $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 12 \\ -1 & 4 & 0 & 0 \\ 0 & 1 & 2 & -7 \end{array}\right]$

57. No solutions exist.

59. No solutions exist.

61. $\frac{1}{8}\begin{bmatrix} 2 & 7 \\ 6 & 1 \end{bmatrix}$

63. No inverse exists.

65. $(-20, 40)$

67. $(-1, 0.2, 0.3)$

69. 17% oranges, 34% bananas, 39% apples

71. 0

73. 6

75. $\left(6, \frac{1}{2}\right)$

77. $(x, 5x + 3)$

79. $\left(0, 0, -\frac{1}{2}\right)$

Practice Test

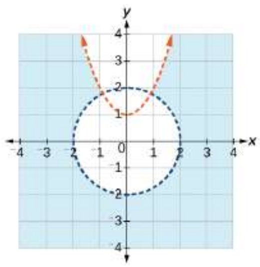
1. Yes

3. No solutions exist.

5. $\frac{1}{20}(10, 5, 4)$

7. $\left(x, \frac{16x}{5} - \frac{13x}{5}\right)$

9. $(-2\sqrt{2}, -\sqrt{17}), \left(-2\sqrt{2}, \sqrt{17}\right), \left(2\sqrt{2}, -\sqrt{17}\right), \left(2\sqrt{2}, \sqrt{17}\right)$

11. 

13. $\frac{5}{3x+1} - \frac{2x+3}{(3x+1)^2}$

15. $\begin{bmatrix} 17 & 51 \\ -8 & 11 \end{bmatrix}$

17. $\begin{bmatrix} 12 & -20 \\ -15 & 30 \end{bmatrix}$

19. $-\frac{1}{8}$

21. $\left[\begin{array}{ccc|c} 14 & -2 & 13 & 140 \\ -2 & 3 & -6 & -1 \\ 1 & -5 & 12 & 11 \end{array}\right]$

23. No solutions exist.

25. $(100, 90)$

27. $\left(\frac{1}{100}, 0\right)$

29. 32 or more cell phones per day

Chapter 8

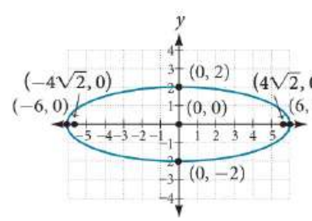
Try It

8.1 The Ellipse

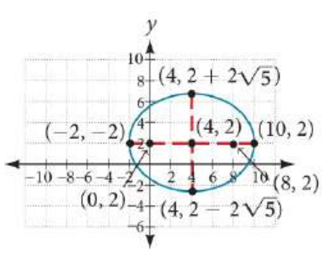
1. $x^2 + \frac{y^2}{16} = 1$

2. $\frac{(x-1)^2}{16} + \frac{(y-3)^2}{4} = 1$

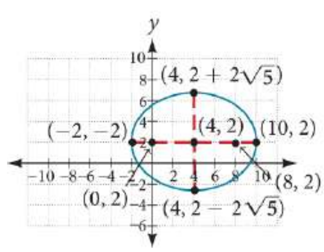
3. center: $(0, 0)$; vertices: $(\pm 6, 0)$; co-vertices: $(0, \pm 2)$; foci: $\left(\pm 4\sqrt{2}, 0\right)$



4. Standard form: $\frac{x^2}{16} + \frac{y^2}{49} = 1$; center: $(0, 0)$; vertices: $(0, \pm 7)$; co-vertices: $(\pm 4, 0)$; foci: $\left(0, \pm\sqrt{33}\right)$



5. Center: $(4, 2)$; vertices: $(-2, 2)$ and $(10, 2)$; co-vertices: $\left(4, 2 - 2\sqrt{5}\right)$ and $\left(4, 2 + 2\sqrt{5}\right)$; foci: $(0, 2)$ and $(8, 2)$



6. $\frac{(x-3)^2}{4} + \frac{(y+1)^2}{16} = 1$; center: $(3, -1)$; vertices: $(3, -5)$ and $(3, 3)$; co-vertices: $(1, -1)$ and $(5, -1)$; foci: $\left(3, -1 - 2\sqrt{3}\right)$ and $\left(3, -1+2\sqrt{3}\right)$

7. ⓐ $\frac{x^2}{57{,}600} + \frac{y^2}{25{,}600} = 1$
ⓑ The people are standing 358 feet apart.

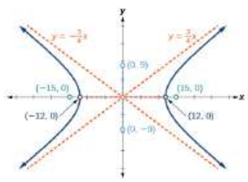
8.2 The Hyperbola

1. Vertices: $(\pm 3, 0)$; Foci: $\left(\pm\sqrt{34}, 0\right)$

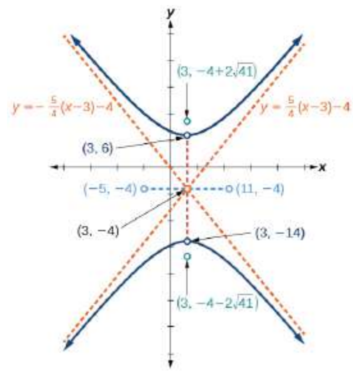
2. $\frac{y^2}{4} - \frac{x^2}{16} = 1$

3. $\frac{(y-3)^2}{25} + \frac{(x-1)^2}{144} = 1$

4. vertices: $(\pm 12, 0)$; co-vertices: $(0, \pm 9)$; foci: $(\pm 15, 0)$; asymptotes: $y = \pm\frac{3}{4}x$;



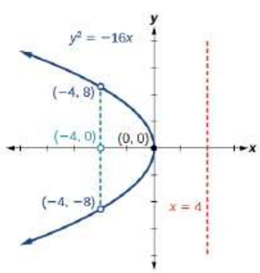
5. center: $(3, -4)$; vertices: $(3, -14)$ and $(3, 6)$; co-vertices: $(-5, -4)$; and $(11, -4)$; foci: $\left(3, -4 - 2\sqrt{41}\right)$ and $\left(3, -4 + 2\sqrt{41}\right)$; asymptotes: $y = \pm\frac{5}{4}(x-3) - 4$



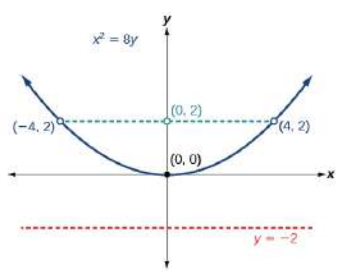
6. The sides of the tower can be modeled by the hyperbolic equation. $\frac{x^2}{400} - \frac{y^2}{3600} = 1 \text{or } \frac{x^2}{20^2} - \frac{y^2}{60^2} = 1.$

8.3 The Parabola

1. Focus: $(-4, 0)$; Directrix: $x = 4$; Endpoints of the latus rectum: $(-4, \pm 8)$

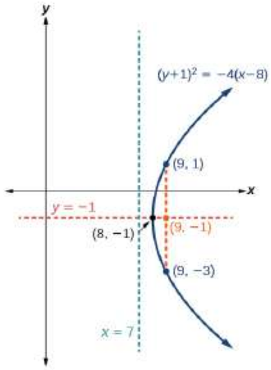


2. Focus: $(0, 2)$; Directrix: $y = -2$; Endpoints of the latus rectum: $(\pm 4, 2)$.

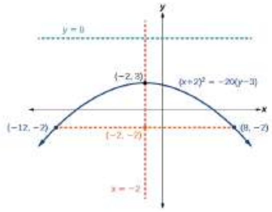


3. $x^2 = 14y.$

4. Vertex: $(8, -1)$; Axis of symmetry: $y = -1$; Focus: $(9, -1)$; Directrix: $x = 7$; Endpoints of the latus rectum: $(9, -3)$ and $(9, 1)$.



5. Vertex: $(-2, 3)$; Axis of symmetry: $x = -2$; Focus: $(-2, -2)$; Directrix: $y = 8$; Endpoints of the latus rectum: $(-12, -2)$ and $(8, -2)$.



6. ⓐ $y^2 = 1280x$
ⓑ The depth of the cooker is 500 mm

8.4 Rotation of Axes

1. ⓐ hyperbola ⓑ ellipse

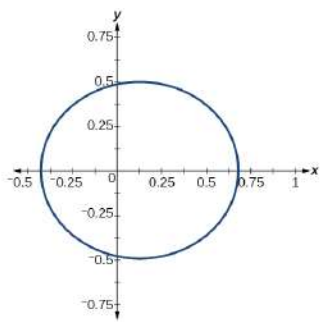
2. $\frac{x'^2}{4} + \frac{y'^2}{1} = 1$

3. ⓐ hyperbola ⓑ ellipse

8.5 Conic Sections in Polar Coordinates

1. ellipse; $e = \frac{1}{3}$; $x = -2$

2.



3. $r = \frac{1}{1 - \cos\theta}$

4. $4 - 8x + 3x^2 - y^2 = 0$

8.1 Section Exercises

1. An ellipse is the set of all points in the plane the sum of whose distances from two fixed points, called the foci, is a constant.

3. This special case would be a circle.

5. It is symmetric about the *x*-axis, *y*-axis, and the origin.

7. yes; $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$

9. yes; $\frac{x^2}{\left(\frac{1}{2}\right)^2} + \frac{y^2}{\left(\frac{1}{3}\right)^2} = 1$

11. $\frac{x^2}{2^2} + \frac{y^2}{7^2} = 1$; Endpoints of major axis $(0, 7)$ and $(0, -7)$. Endpoints of minor axis $(2, 0)$ and $(-2, 0)$. Foci at $\left(0, 3\sqrt{5}\right), \left(0, -3\sqrt{5}\right)$.

13. $\frac{x^2}{(1)^2} + \frac{y^2}{\left(\frac{1}{3}\right)^2} = 1;$ Endpoints of major axis $(1, 0)$ and $(-1, 0)$. Endpoints of minor axis $\left(0, \frac{1}{3}\right), \left(0, -\frac{1}{3}\right)$. Foci at $\left(\frac{2\sqrt{2}}{3}, 0\right), \left(-\frac{2\sqrt{2}}{3}, 0\right)$.

15. $\frac{(x-2)^2}{7^2} + \frac{(y-4)^2}{5^2} = 1;$ Endpoints of major axis $(9, 4), (-5, 4)$. Endpoints of minor axis $(2, 9), (2, -1)$. Foci at $\left(2 + 2\sqrt{6}, 4\right), \left(2 - 2\sqrt{6}, 4\right)$.

17. $\frac{(x+5)^2}{2^2} + \frac{(y-7)^2}{3^2} = 1;$ Endpoints of major axis $(-5, 10), (-5, 4)$. Endpoints of minor axis $(-3, 7), (-7, 7)$. Foci at $\left(-5, 7 + \sqrt{5}\right), \left(-5, 7 - \sqrt{5}\right)$.

19. $\frac{(x-1)^2}{3^2} + \frac{(y-4)^2}{2^2} = 1;$ Endpoints of major axis $(4, 4), (-2, 4)$. Endpoints of minor axis $(1, 6), (1, 2)$. Foci at $\left(1 + \sqrt{5}, 4\right), \left(1 - \sqrt{5}, 4\right)$.

21. $\frac{(x-3)^2}{\left(3\sqrt{2}\right)^2} + \frac{(y-5)^2}{\left(\sqrt{2}\right)^2} = 1;$ Endpoints of major axis $\left(3 + 3\sqrt{2}, 5\right), \left(3 - 3\sqrt{2}, 5\right)$. Endpoints of minor axis $\left(3, 5 + \sqrt{2}\right), \left(3, 5 - \sqrt{2}\right)$. Foci at $(7, 5), (-1, 5)$.

23. $\frac{(x+5)^2}{(5)^2} + \frac{(y-2)^2}{(2)^2} = 1$; Endpoints of major axis $(0, 2), (-10, 2)$. Endpoints of minor axis $(-5, 4), (-5, 0)$. Foci at $\left(-5 + \sqrt{21}, 2\right), \left(-5 - \sqrt{21}, 2\right)$.

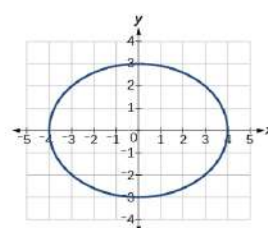
25. $\frac{(x+3)^2}{(5)^2} + \frac{(y+4)^2}{(2)^2} = 1$; Endpoints of major axis $(2, -4), (-8, -4)$. Endpoints of minor axis $(-3, -2), (-3, -6)$. Foci at $\left(-3 + \sqrt{21}, -4\right), \left(-3 - \sqrt{21}, -4\right)$.

27. Foci $\left(-3, -1 + \sqrt{11}\right), \left(-3, -1 - \sqrt{11}\right)$

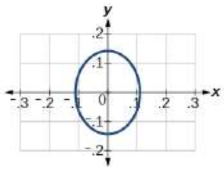
29. Focus $(0, 0)$

31. Foci $(-10, 30), (-10, -30)$

33. Center $(0, 0)$, Vertices $(4, 0), (-4, 0), (0, 3), (0, -3)$, Foci $\left(\sqrt{7}, 0\right), \left(-\sqrt{7}, 0\right)$

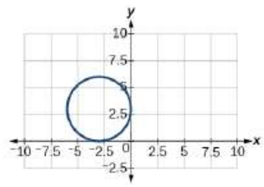


35. Center $(0, 0)$, Vertices $\left(\frac{1}{9}, 0\right)$, $\left(-\frac{1}{9}, 0\right)$, $\left(0, \frac{1}{7}\right)$, $\left(0, -\frac{1}{7}\right)$, Foci $\left(0, \frac{4\sqrt{2}}{63}\right)$, $\left(0, -\frac{4\sqrt{2}}{63}\right)$

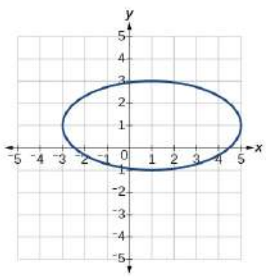


37. Center $(-3, 3)$, Vertices $(0, 3)$, $(-6, 3)$, $(-3, 0)$, $(-3, 6)$, Focus $(-3, 3)$

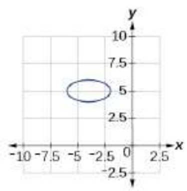
Note that this ellipse is a circle. The circle has only one focus, which coincides with the center.



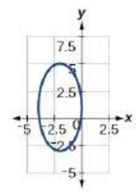
39. Center $(1, 1)$, Vertices $(5, 1)$, $(-3, 1)$, $(1, 3)$, $(1, -1)$, Foci $\left(1, 1 + 2\sqrt{3}\right)$, $\left(1, 1 - 2\sqrt{3}\right)$



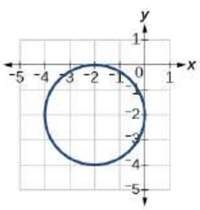
41. Center $(-4, 5)$, Vertices $(-2, 5)$, $(-6, 4)$, $(-4, 6)$, $(-4, 4)$, Foci $\left(-4 + \sqrt{3}, 5\right)$, $\left(-4 - \sqrt{3}, 5\right)$



43. Center $(-2, 1)$, Vertices $(0, 1)$, $(-4, 1)$, $(-2, 5)$, $(-2, -3)$, Foci $\left(-2, 1 + 2\sqrt{3}\right)$, $\left(-2, 1 - 2\sqrt{3}\right)$



45. Center $(-2, -2)$, Vertices $(0, -2)$, $(-4, -2)$, $(-2, 0)$, $(-2, -4)$, Focus $(-2, -2)$



47. $\frac{x^2}{25} + \frac{y^2}{29} = 1$

49. $\frac{(x-4)^2}{25} + \frac{(y-2)^2}{1} = 1$

51. $\frac{(x+3)^2}{16} + \frac{(y-4)^2}{4} = 1$

53. $\frac{x^2}{81} + \frac{y^2}{9} = 1$

55. $\frac{(x+2)^2}{4} + \frac{(y-2)^2}{9} = 1$

57. $\text{Area} = 12\pi$ square units

59. $\text{Area} = 2\sqrt{5}\pi$ square units.

61. $\text{Area} = 9\pi$ square units.

63. $\frac{x^2}{4h^2} + \frac{y^2}{\frac{1}{4}h^2} = 1$

65. $\frac{x^2}{400} + \frac{y^2}{144} = 1$. Distance = 17.32 feet

67. Approximately 51.96 feet

8.2 Section Exercises

1. A hyperbola is the set of points in a plane the difference of whose distances from two fixed points (foci) is a positive constant.

3. The foci must lie on the transverse axis and be in the interior of the hyperbola.

5. The center must be the midpoint of the line segment joining the foci.

7. yes $\frac{x^2}{6^2} - \frac{y^2}{3^2} = 1$

9. yes $\frac{x^2}{4^2} - \frac{y^2}{5^2} = 1$

11. $\frac{x^2}{5^2} - \frac{y^2}{6^2} = 1$; vertices: $(5, 0), (-5, 0)$; foci: $\left(\sqrt{61}, 0\right), \left(-\sqrt{61}, 0\right)$; asymptotes: $y = \frac{6}{5}x, y = -\frac{6}{5}x$

13. $\frac{y^2}{2^2} - \frac{x^2}{9^2} = 1$; vertices: $(0, 2), (0, -2)$; foci: $\left(0, \sqrt{85}\right), \left(0, -\sqrt{85}\right)$; asymptotes: $y = \frac{2}{9}x, y = -\frac{2}{9}x$

15. $\frac{(x-1)^2}{3^2} - \frac{(y-2)^2}{4^2} = 1$; vertices: $(4, 2), (-2, 2)$; foci: $(6, 2), (-4, 2)$; asymptotes: $y = \frac{4}{3}(x-1)+2, y = -\frac{4}{3}(x-1)+2$

17. $\frac{(x-2)^2}{7^2} - \frac{(y+7)^2}{7^2} = 1$; vertices: $(9, -7), (-5, -7)$; foci: $\left(2+7\sqrt{2}, -7\right), \left(2-7\sqrt{2}, -7\right)$; asymptotes: $y = x-9, y = -x-5$

19. $\frac{(x+3)^2}{3^2} - \frac{(y-3)^2}{3^2} = 1$; vertices: $(0, 3), (-6, 3)$; foci: $\left(-3+3\sqrt{2}, 1\right), \left(-3-3\sqrt{2}, 1\right)$; asymptotes: $y = x+6, y = -x$

21. $\frac{(y-4)^2}{2^2} - \frac{(x-3)^2}{4^2} = 1$; vertices: $(3, 6), (3, 2)$; foci: $\left(3, 4+2\sqrt{5}\right), \left(3, 4-2\sqrt{5}\right)$; asymptotes: $y = \frac{1}{2}(x-3)+4, y = -\frac{1}{2}(x-3)+4$

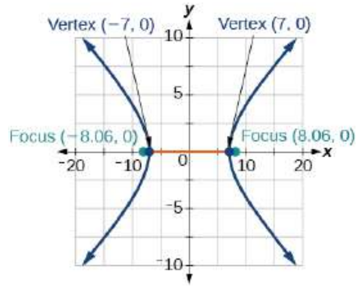
23. $\frac{(y+5)^2}{7^2} - \frac{(x+1)^2}{70^2} = 1$; vertices: $(-1, 2), (-1, -12)$; foci: $\left(-1, -5+7\sqrt{101}\right), \left(-1, -5-7\sqrt{101}\right)$; asymptotes: $y = \frac{1}{10}(x+1)-5, y = -\frac{1}{10}(x+1)-5$

25. $\frac{(x+3)^2}{5^2} - \frac{(y-4)^2}{2^2} = 1$; vertices: $(2, 4), (-8, 4)$; foci: $\left(-3+\sqrt{29}, 4\right), \left(-3-\sqrt{29}, 4\right)$; asymptotes: $y = \frac{2}{5}(x+3)+4, y = -\frac{2}{5}(x+3)+4$

27. $y = \frac{2}{5}(x-3)-4, y = -\frac{2}{5}(x-3)-4$

29. $y = \frac{3}{4}(x-1)+1,\, y = -\frac{3}{4}(x-1)+1$

31.



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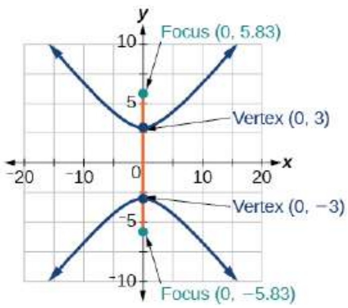

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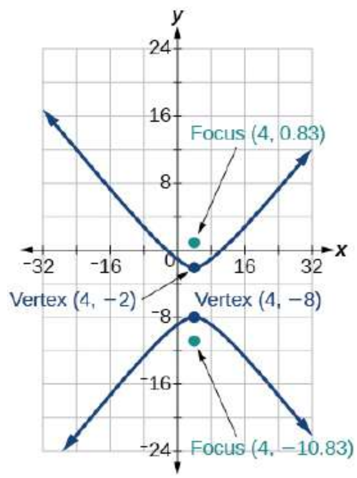

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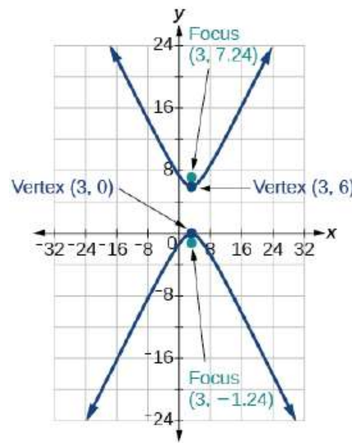

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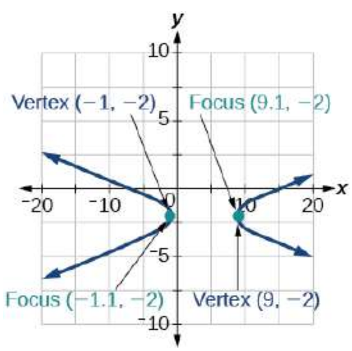

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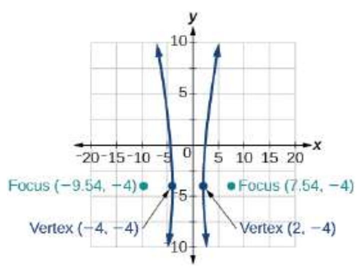

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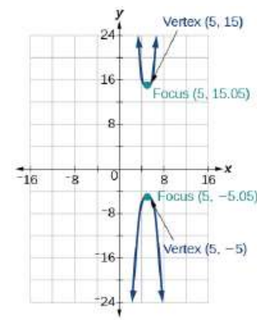


45. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

47. $\frac{(x-6)^2}{25} - \frac{(y-1)^2}{11} = 1$

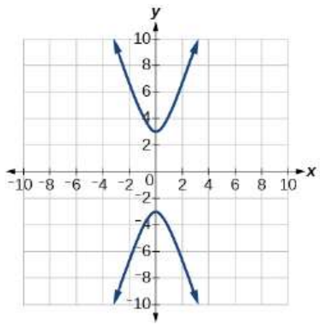
49. $\frac{(x-4)^2}{25} - \frac{(y-2)^2}{1} = 1$

51. $\frac{y^2}{16} - \frac{x^2}{25} = 1$

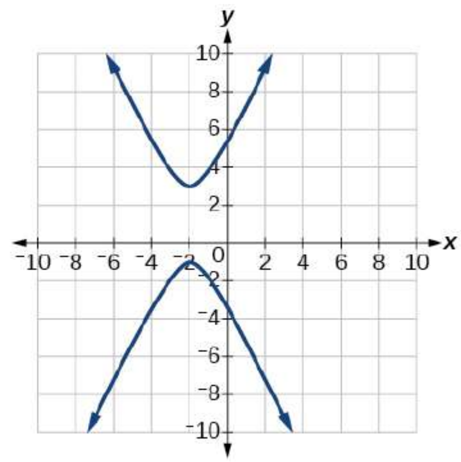
53. $\frac{y^2}{9} - \frac{(x+1)^2}{9} = 1$

55. $\frac{(x+3)^2}{25} - \frac{(y+3)^2}{25} = 1$

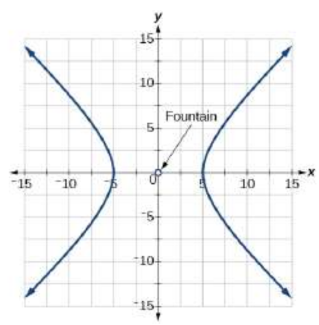
57. $y(x) = 3\sqrt{x^2+1}, y(x) = -3\sqrt{x^2+1}$



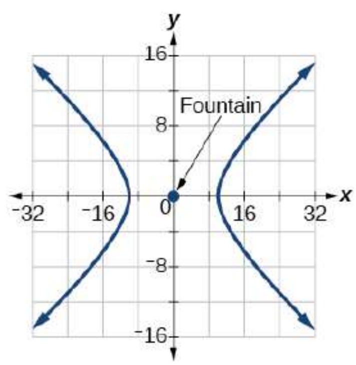
59. $y(x) = 1 + 2\sqrt{x^2+4x+5}, y(x) = 1 - 2\sqrt{x^2+4x+5}$



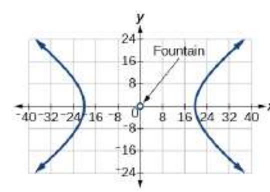
61. $\frac{x^2}{25} - \frac{y^2}{25} = 1$



63. $\frac{x^2}{100} - \frac{y^2}{25} = 1$



65. $\frac{x^2}{400} - \frac{y^2}{225} = 1$



67. $4\left(x-1\right)^2 - y2^2 = 16$

69. $\frac{(x-h)^2}{a^2} - \frac{\left(y-k\right)^2}{b^2} = \left(x-3\right)^2 - 9y^2 = 4$

8.3 Section Exercises

1. A parabola is the set of points in the plane that lie equidistant from a fixed point, the focus, and a fixed line, the directrix.

3. The graph will open down.

5. The distance between the focus and directrix will increase.

7. yes $x^2 = 4\left(\frac{1}{16}\right)y$

9. yes $(y-3)^2 = 4(2)(x-2)$

11. $y^2 = \frac{1}{8}x, V: (0,0); F: \left(\frac{1}{32}, 0\right); d: x = -\frac{1}{32}$

13. $x^2 = -\frac{1}{4}y, V: (0,0); F: \left(0, -\frac{1}{16}\right); d: y = \frac{1}{16}$

15. $y^2 = \frac{1}{36}x, V: (0,0); F: \left(\frac{1}{144}, 0\right); d: x = -\frac{1}{144}$

17. $(x-1)^2 = 4(y-1), V: (1,1); F: (1,2); d: y = 0$

19. $(y-4)^2 = 2(x+3), V: (-3,4); F: \left(-\frac{5}{2}, 4\right); d: x = -\frac{7}{2}$

21. $(x+4)^2 = 24(y+1)$, $V:(-4,-1)$; $F:(-4,5)$; $d: y=-7$

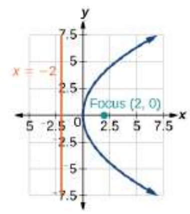
23. $(y-3)^2 = -12(x+1)$, $V:(-1,3)$; $F:(-4,3)$; $d: x=2$

25. $(x-5)^2 = \frac{4}{5}(y+3)$, $V:(5,-3)$; $F:\left(5,-\frac{14}{5}\right)$; $d: y=-\frac{16}{5}$

27. $(x-2)^2 = -2(y-5)$, $V:(2,5)$; $F:\left(2,\frac{9}{2}\right)$; $d: y=\frac{11}{2}$

29. $(y-1)^2 = \frac{4}{3}(x-5)$, $V:(5,1)$; $F:\left(\frac{16}{3},1\right)$; $d: x=\frac{14}{3}$

31.



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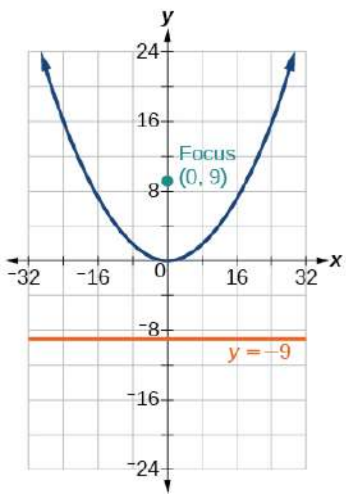

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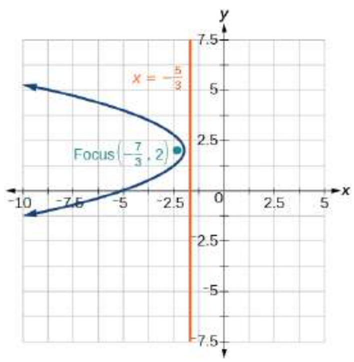

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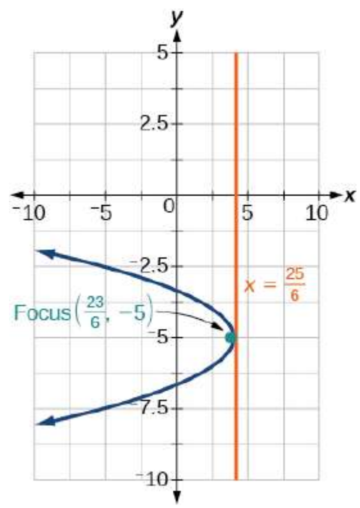

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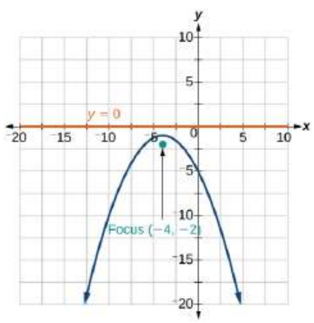
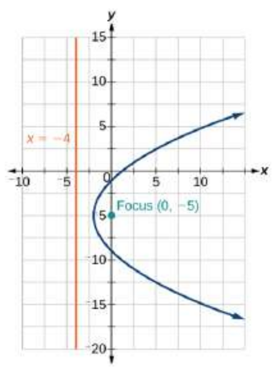


41.



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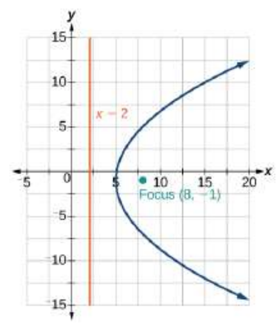


45. $x^2 = -16y$

47. $(y-2)^2 = 4\sqrt{2}(x-2)$

49. $\left(y+\sqrt{3}\right)^2 = -4\sqrt{2}\left(x-\sqrt{2}\right)$

51. $x^2 = y$

53. $(y-2)^2 = \frac{1}{4}(x+2)$

55. $\left(y-\sqrt{3}\right)^2 = 4\sqrt{5}\left(x+\sqrt{2}\right)$

57. $y^2 = -8x$

59. $(y+1)^2 = 12(x+3)$

61. $(0, 1)$

63. At the point 2.25 feet above the vertex.

65. 0.5625 feet

67. $x^2 = -125(y-20)$, height is 7.2 feet

69. 2304 feet

8.4 Section Exercises

1. The xy term causes a rotation of the graph to occur.

3. The conic section is a hyperbola.

5. It gives the angle of rotation of the axes in order to eliminate the xy term.

7. $AB = 0$, parabola

9. $AB = -4 < 0$, hyperbola

11. $AB = 6 > 0$, ellipse

13. $B^2 - 4AC = 0$, parabola

15. $B^2 - 4AC = 0$, parabola

17. $B^2 - 4AC = -96 < 0$, ellipse

19. $7x'^2 + 9y'^2 - 4 = 0$

21. $3x'^2 + 2x'y' - 5y'^2 + 1 = 0$

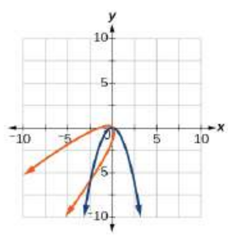
23. $\theta = 60^\circ, 11x'^2 - y'^2 + \sqrt{3}x' + y' - 4 = 0$

25. $\theta = 150^\circ, 21x'^2 + 9y'^2 + 4x' - 4\sqrt{3}y' - 6 = 0$

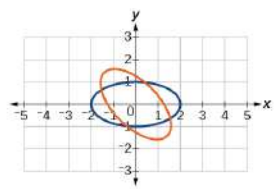
27. $\theta \approx 36.9^\circ, 125x'^2 + 6x' - 42y' + 10 = 0$

29. $\theta = 45^\circ, 3x'^2 - y'^2 - \sqrt{2}x' + \sqrt{2}y' + 1 = 0$

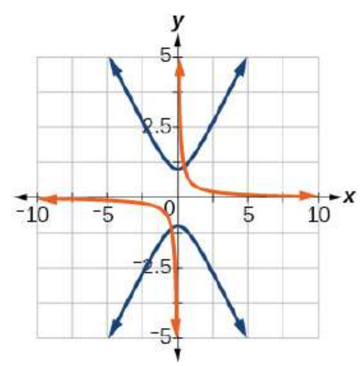
31. $\frac{\sqrt{2}}{2}(x' + y') = \frac{1}{2}(x' - y')^2$



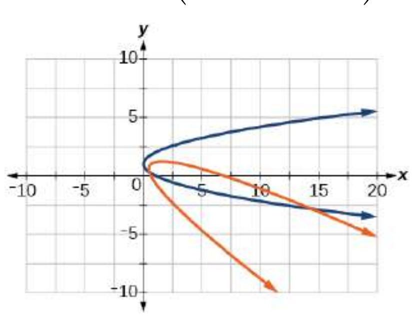
33. $\frac{(x'-y')^2}{8} + \frac{(x'+y')^2}{2} = 1$



35. $\frac{(x'+y')^2}{2} - \frac{(x'-y')^2}{2} = 1$



37. $\frac{\sqrt{3}}{2}x' - \frac{1}{2}y' = \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y' - 1\right)^2$



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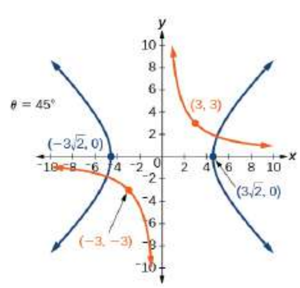

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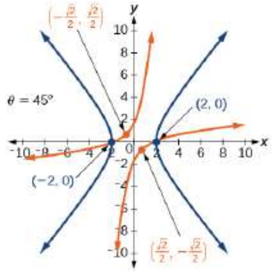

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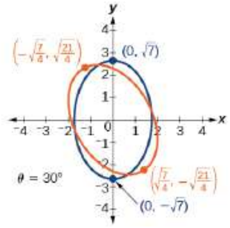

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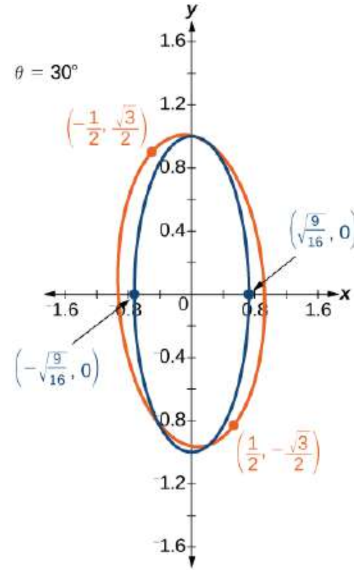

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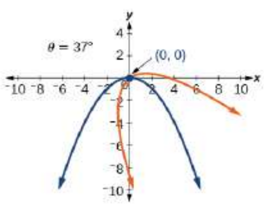
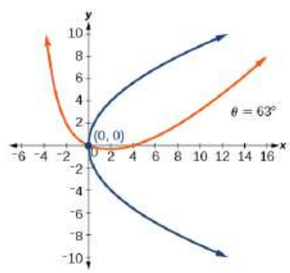
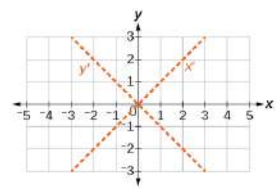


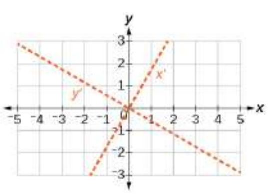
49.



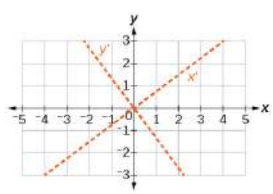
51. $\theta = 45°$



53. $\theta = 60°$



55. $\theta \approx 36.9°$



57. $-4\sqrt{6} < k < 4\sqrt{6}$

59. $k = 2$

8.5 Section Exercises

1. If eccentricity is less than 1, it is an ellipse. If eccentricity is equal to 1, it is a parabola. If eccentricity is greater than 1, it is a hyperbola.

3. The directrix will be parallel to the polar axis.

5. One of the foci will be located at the origin.

7. Parabola with $e = 1$ and directrix $\frac{3}{4}$ units below the pole.

9. Hyperbola with $e = 2$ and directrix $\frac{5}{2}$ units above the pole.

11. Parabola with $e = 1$ and directrix $\frac{3}{10}$ units to the right of the pole.

13. Ellipse with $e=\frac{2}{7}$ and directrix 2 units to the right of the pole.

15. Hyperbola with $e=\frac{5}{3}$ and directrix $\frac{11}{5}$ units above the pole.

17. Hyperbola with $e=\frac{8}{7}$ and directrix $\frac{7}{8}$ units to the right of the pole.

19. $25x^2+16y^2-12y-4=0$

21. $21x^2-4y^2-30x+9=0$

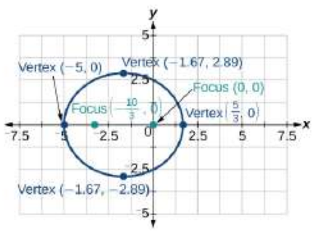
23. $64y^2=48x+9$

25. $96y^2-25x^2+110y+25=0$

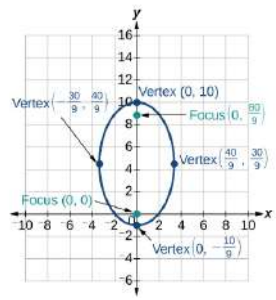
27. $3x^2+4y^2-2x-1=0$

29. $5x^2+9y^2-24x-36=0$

31.



33.



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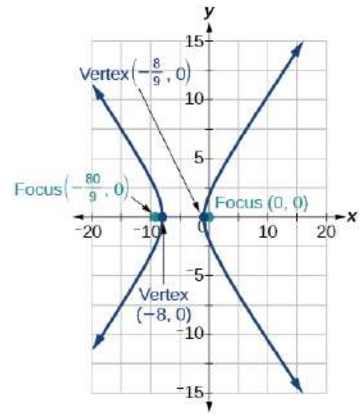

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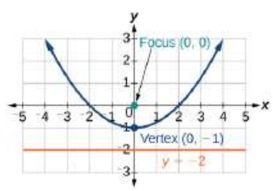

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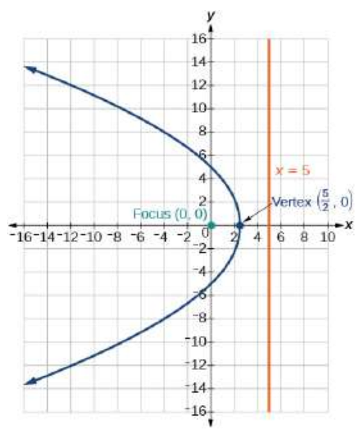

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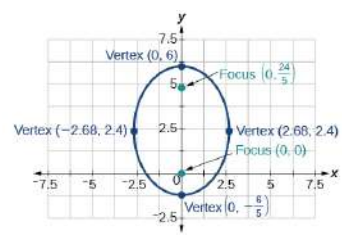


43. $r=\frac{4}{5+\cos\theta}$

45. $r=\frac{4}{1+2\sin\theta}$

47. $r=\frac{1}{1+\cos\theta}$

49. $r=\frac{7}{8-28\cos\theta}$

51. $r=\frac{12}{2+3\sin\theta}$

53. $r=\frac{15}{4-3\cos\theta}$

55. $r=\frac{3}{3-3\cos\theta}$

57. $r=\pm\frac{2}{\sqrt{1+\sin\theta\cos\theta}}$

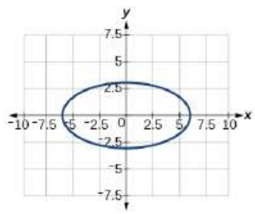
59. $r=\pm\frac{2}{4\cos\theta+3\sin\theta}$

Review Exercises

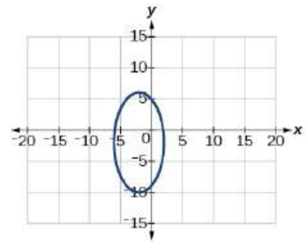
1. $\frac{x^2}{5^2}+\frac{y^2}{8^2}=1$; center: $(0,0)$; vertices: $(5,0),(-5,0),(0,8),(0,-8)$; foci: $\left(0,\sqrt{39}\right),\left(0,-\sqrt{39}\right)$

3. $\frac{(x+3)^2}{1^2} + \frac{(y-2)^2}{3^2} = 1 \quad (-3, 2); \quad (-2, 2), (-4, 2), (-3, 5), (-3, -1); \quad \left(-3, 2 + 2\sqrt{2}\right), \left(-3, 2 - 2\sqrt{2}\right)$

5. center: $(0, 0)$; vertices: $(6, 0)$, $(-6, 0)$, $(0, 3)$, $(0, -3)$; foci: $\left(3\sqrt{3}, 0\right)$, $\left(-3\sqrt{3}, 0\right)$



7. center: $(-2, -2)$; vertices: $(2, -2)$, $(-6, -2)$, $(-2, 6)$, $(-2, -10)$; foci: $\left(-2, -2 + 4\sqrt{3}, \right)$, $\left(-2, -2 - 4\sqrt{3}\right)$



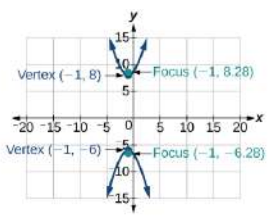
9. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

11. Approximately 35.71 feet

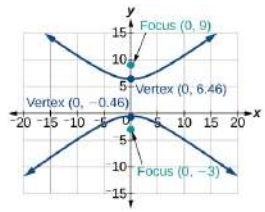
13. $\frac{(y+1)^2}{4^2} - \frac{(x-4)^2}{6^2} = 1$; center: $(4, -1)$; vertices: $(4, 3)$, $(4, -5)$; foci: $\left(4, -1 + 2\sqrt{13}\right)$, $\left(4, -1 - 2\sqrt{13}\right)$

15. $\frac{(x-2)^2}{2^2} - \frac{(y+3)^2}{\left(2\sqrt{3}\right)^2} = 1$; center: $(2, -3)$; vertices: $(4, -3)$, $(0, -3)$; foci: $(6, -3)$, $(-2, -3)$

17.



19.

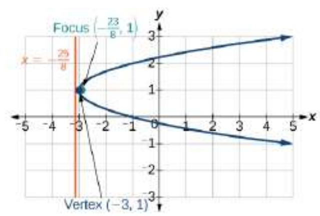


21. $\frac{(x-5)^2}{1} - \frac{(y-7)^2}{3} = 1$

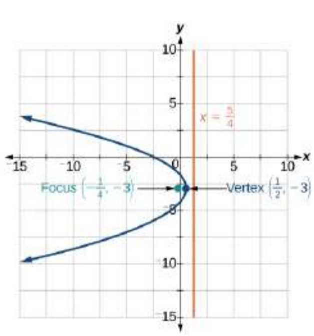
23. $(x + 2)^2 = \frac{1}{2}(y - 1)$; vertex: $(-2, 1)$; focus: $\left(-2, \frac{9}{8}\right)$; directrix: $y = \frac{7}{8}$

25. $(x + 5)^2 = (y + 2)$; vertex: $(-5, -2)$; focus: $\left(-5, -\frac{7}{4}\right)$; directrix: $y = -\frac{9}{4}$

27.



29.



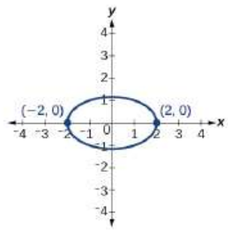
31. $(x - 2)^2 = \left(\frac{1}{2}\right)(y - 1)$

33. $B^2 - 4AC = 0$, parabola

35. $B^2 - 4AC = -31 < 0,$ ellipse

37. $\theta = 45°, x'^2 + 3y'^2 - 12 = 0$

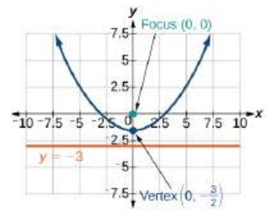
39. $\theta = 45°$



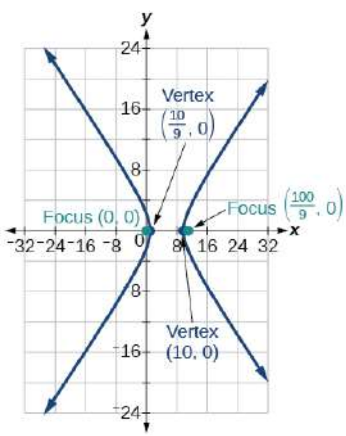
41. Hyperbola with $e = 5$ and directrix 2 units to the left of the pole.

43. Ellipse with $e = \frac{3}{4}$ and directrix $\frac{1}{3}$ unit above the pole.

45.



47.

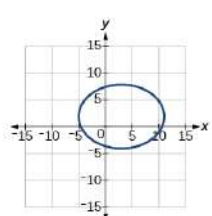


49. $r = \frac{3}{1+\cos\ \theta}$

Practice Test

1. $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$; center: $(0, 0)$; vertices: $(3, 0), (–3, 0), (0, 2), (0, –2)$; foci: $\left(\sqrt{5}, 0\right), \left(-\sqrt{5}, 0\right)$

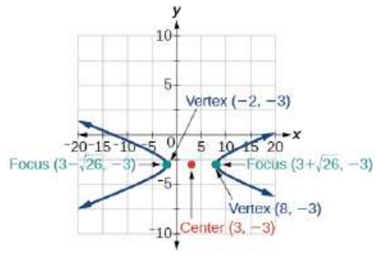
3. center: $(3, 2)$; vertices: $(11, 2), (–5, 2), (3, 8), (3, –4)$; foci: $\left(3 + 2\sqrt{7}, 2\right), \left(3 - 2\sqrt{7}, 2\right)$



5. $\frac{(x-1)^2}{36} + \frac{(y-2)^2}{27} = 1$

7. $\frac{x^2}{7^2} - \frac{y^2}{9^2} = 1$; center: $(0, 0)$; vertices $(7, 0)$, $(-7, 0)$; foci: $\left(\sqrt{130}, 0\right)$, $\left(-\sqrt{130}, 0\right)$; asymptotes: $y = \pm\frac{9}{7}x$

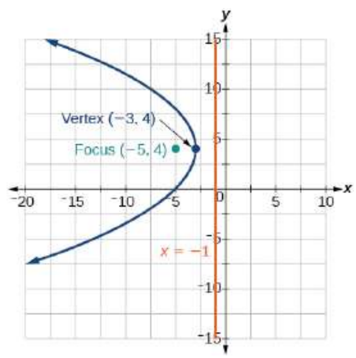
9. center: $(3, -3)$; vertices: $(8, -3)$, $(-2, -3)$; foci: $\left(3 + \sqrt{26}, -3\right)$, $\left(3 - \sqrt{26}, -3\right)$; asymptotes: $y = \pm\frac{1}{5}(x - 3) - 3$



11. $\frac{(y-3)^2}{1} - \frac{(x-1)^2}{8} = 1$

13. $(x - 2)^2 = \frac{1}{3}(y + 1)$; vertex: $(2, -1)$; focus: $\left(2, -\frac{11}{12}\right)$; directrix: $y = -\frac{13}{12}$

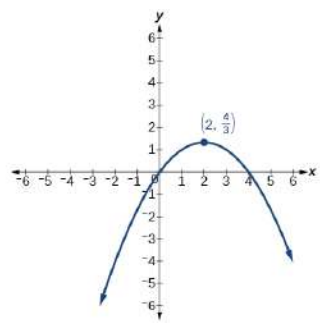
15.



17. Approximately 8.49 feet

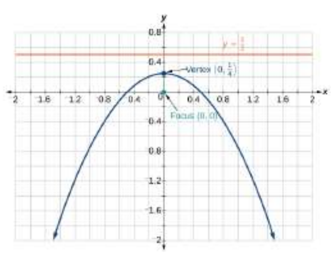
19. parabola; $\theta \approx 63.4°$

21. $x'^2 - 4x' + 3y' = 0$



23. Hyperbola with $e = \frac{3}{2}$, and directrix $\frac{5}{6}$ units to the right of the pole.

25.



Chapter 9

Try It

9.1 Sequences and Their Notations

1. The first five terms are $\{1, 6,\ 11,\ 16,\ 21\}$.
2. The first five terms are $\left\{-2,\ 2,\ -\frac{3}{2},\ 1,\ \ -\frac{5}{8}\right\}$.
3. The first six terms are $\{2,\ \ 5,\ \ 54,\ \ 10,\ \ 250,\ \ 15\}$.
4. $a_n = (-1)^{n+1}9^n$
5. $a_n = -\frac{3^n}{4n}$
6. $a_n = e^{n-3}$
7. $\{2,\ 5,\ 11,\ 23,\ 47\}$
8. $\left\{0,\ 1,\ 1,\ 1,\ 2,\ 3,\ \frac{5}{2},\ \ \frac{17}{6}\right\}$.
9. The first five terms are $\left\{1,\ \frac{3}{2},\ 4,\ \ 15,\ \ 72\right\}$.

9.2 Arithmetic Sequences

1. The sequence is arithmetic. The common difference is -2.
2. The sequence is not arithmetic because $3-1 \neq 6-3$.
3. $\{1,\ 6,\ 11,\ 16,\ 21\}$
4. $a_2 = 2$
5. $$\begin{aligned} a_1 &= 25 \\ a_n &= a_{n-1} + 12, \quad \text{for } n \geq 2 \end{aligned}$$
6. $a_n = 53 - 3n$
7. There are 11 terms in the sequence.
8. The formula is $T_n = 10 + 4n$, and it will take her 42 minutes.

9.3 Geometric Sequences

1. The sequence is not geometric because $\frac{10}{5} \neq \frac{15}{10}$.
2. The sequence is geometric. The common ratio is $\frac{1}{5}$.
3. $\left\{18, 6, 2, \frac{2}{3}, \frac{2}{9}\right\}$
4. $$\begin{aligned} a_1 &= 2 \\ a_n &= \frac{2}{3}a_{n-1} \quad \text{for } n \geq 2 \end{aligned}$$
5. $a_6 = 16{,}384$
6. $a_n = -(-3)^{n-1}$
7. ⓐ $P_n \ = \ 293 \cdot 1.026a^n$
   ⓑ The number of hits will be about 333.

9.4 Series and Their Notations

1. 38
2. 26.4
3. 328
4. -280
5. \$2,025
6. $\approx 2{,}000.00$
7. 9,840
8. \$275,513.31
9. The sum is not defined.

10. The sum of the infinite series is defined.

11. The sum of the infinite series is defined.

12. 3

13. The series is not geometric.

14. $-\frac{3}{11}$

15. $32,775.87

9.5 Counting Principles

1. 7

2. There are 60 possible breakfast specials.

3. 120

4. 60

5. 12

6. $P(7,7) = 5,040$

7. $P(7,5) = 2,520$

8. $C(10,3) = 120$

9. 64 sundaes

10. 840

9.6 Binomial Theorem

1. a 35 b 330

2. a $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$ b $8x^3 + 60x^2y + 150xy^2 + 125y^3$

3. $-10,206x^4y^5$

9.7 Probability

1.

Outcome	Probability
Heads	$\frac{1}{2}$
Tails	$\frac{1}{2}$

2. $\frac{2}{3}$

3. $\frac{7}{13}$

4. $\frac{2}{13}$

5. $\frac{5}{6}$

6. a. $\frac{1}{91}$; b. $\frac{5}{91}$; c. $\frac{86}{91}$

9.1 Section Exercises

1. A sequence is an ordered list of numbers that can be either finite or infinite in number. When a finite sequence is defined by a formula, its domain is a subset of the non-negative integers. When an infinite sequence is defined by a formula, its domain is all positive or all non-negative integers.

3. Yes, both sets go on indefinitely, so they are both infinite sequences.

5. A factorial is the product of a positive integer and all the positive integers below it. An exclamation point is used to indicate the operation. Answers may vary. An example of the benefit of using factorial notation is when indicating the product It is much easier to write than it is to write out $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1.$

7. First four terms: $-8, \ -\frac{16}{3}, \ -4, \ -\frac{16}{5}$

9. First four terms: $2, \ \frac{1}{2}, \ \frac{8}{27}, \ \frac{1}{4}.$

11. First four terms: $1.25, \ -5, \ 20, \ -80.$

13. First four terms: $\frac{1}{3}, \ \frac{4}{5}, \ \frac{9}{7}, \ \frac{16}{9}.$

15. First four terms: $-\frac{4}{5}, \ 4, \ -20, \ 100$

17. $\frac{1}{3}, \ \frac{4}{5}, \ \frac{9}{7}, \ \frac{16}{9}, \ \frac{25}{11}, \ 31, \ 44, \ 59$

19. $-0.6, -3, -15, -20, -375, -80, -9375, -320$

21. $a_n = n^2 + 3$

23. $a_n = \frac{2^n}{2n}$ or $\frac{2^{n-1}}{n}$

25. $a_n = \left(-\frac{1}{2}\right)^{n-1}$

27. First five terms: $3, \ -9, \ 27, \ -81, \ 243$

29. First five terms: $-1, \ 1, \ -9, \ \frac{27}{11}, \ \frac{891}{5}$

31. $\frac{1}{24}, \ 1, \frac{1}{4}, \ \frac{3}{2}, \ \frac{9}{4}, \ \frac{81}{4}, \ \frac{2187}{8}, \ \frac{531{,}441}{16}$

33. $2, \ 10, \ 12, \ \frac{14}{5}, \ \frac{4}{5}, \ 2, \ 10, \ 12$

35. $a_1 = -8, a_n = a_{n-1} + n$

37. $a_1 = 35, a_n = a_{n-1} + 3$

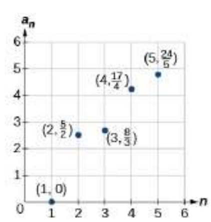
39. 720

41. $665, 280$

43. First four terms: $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{2}$

45. First four terms: $-1, 2, \frac{6}{5}, \frac{24}{11}$

47.



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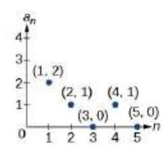

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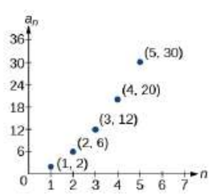


53. $a_n = 2^{n-2}$

55. $a_1 = 6, \ \ a_n = 2a_{n-1} - 5$

57. First five terms: $\frac{29}{37}, \frac{152}{111}, \frac{716}{333}, \frac{3188}{999}, \frac{13724}{2997}$

59. First five terms: 2, 3, 5, 17, 65537

61. $a_{10} = 7, 257, 600$

63. First six terms: 0.042, 0.146, 0.875, 2.385, 4.708

65. First four terms: 5.975, 2.765, 185.743, 1057.25, 6023.521

67. If $a_n = -421$ is a term in the sequence, then solving the equation $-421 = -6 - 8n$ for n will yield a non-negative integer. However, if $-421 = -6 - 8n$, then $n = 51.875$ so $a_n = -421$ is not a term in the sequence.

69. $a_1 = 1, a_2 = 0, a_n = a_{n-1} - a_{n-2}$

71. $\frac{(n+2)!}{(n-1)!} = \frac{(n+2)\cdot(n+1)\cdot(n)\cdot(n-1)\cdot\ldots\cdot3\cdot2\cdot1}{(n-1)\cdot\ldots\cdot3\cdot2\cdot1} = n(n+1)(n+2) = n^3 + 3n^2 + 2n$

9.2 Section Exercises

1. A sequence where each successive term of the sequence increases (or decreases) by a constant value.

3. We find whether the difference between all consecutive terms is the same. This is the same as saying that the sequence has a common difference.

5. Both arithmetic sequences and linear functions have a constant rate of change. They are different because their domains are not the same; linear functions are defined for all real numbers, and arithmetic sequences are defined for natural numbers or a subset of the natural numbers.

7. The common difference is $\frac{1}{2}$

9. The sequence is not arithmetic because $16 - 4 \neq 64 - 16$.

11. $0, \ \frac{2}{3}, \ \frac{4}{3}, \ 2, \ \frac{8}{3}$

13. $0, -5, -10, -15, -20$

15. $a_4 = 19$

17. $a_6 = 41$

19. $a_1 = 2$

21. $a_1 = 5$

23. $a_1 = 6$

25. $a_{21} = -13.5$

27. $-19, -20.4, -21.8, -23.2, -24.6$

29. $a_1 = 17; \ a_n = a_{n-1} + 9 \quad n \geq 2$

31. $a_1 = 12; \ a_n = a_{n-1} + 5 \quad n \geq 2$

33. $a_1 = 8.9; \ a_n = a_{n-1} + 1.4 \quad n \geq 2$

35. $a_1 = \frac{1}{5}; \ a_n = a_{n-1} + \frac{1}{4} \quad n \geq 2$

37. $_1 = \frac{1}{6}; \ a_n = a_{n-1} - \frac{13}{12} \quad n \geq 2$

39. $a_1 = 4; \ \ a_n = a_{n-1} + 7; \ \ a_{14} = 95$

41. First five terms: $20, 16, 12, 8, 4$.

43. $a_n = 1 + 2n$

45. $a_n = -105 + 100n$

47. $a_n = 1.8n$

49. $a_n = 13.1 + 2.7n$

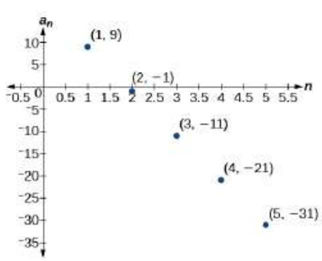
51. $a_n = \frac{1}{3}n - \frac{1}{3}$

53. There are 10 terms in the sequence.

55. There are 6 terms in the sequence.

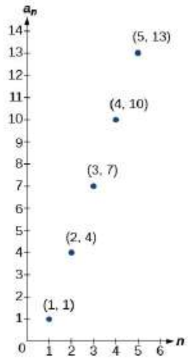
57. The graph does not represent an arithmetic sequence.

59.



61. $1, 4, 7, 10, 13, 16, 19$

63.



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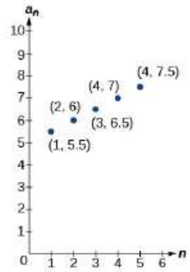


67. Answers will vary. Examples: $a_n = 20.6n$ and $a_n = 2 + 20.4\text{n}.$

69. $a_{11} = -17a + 38b$

71. The sequence begins to have negative values at the 13th term, $a_{13} = -\frac{1}{3}$

73. Answers will vary. Check to see that the sequence is arithmetic. Example: Recursive formula: $a_1 = 3, a_n = a_{n-1} - 3.$ First 4 terms: $3, 0, -3, -6 \quad a_{31} = -87$

9.3 Section Exercises

1. A sequence in which the ratio between any two consecutive terms is constant.

3. Divide each term in a sequence by the preceding term. If the resulting quotients are equal, then the sequence is geometric.

5. Both geometric sequences and exponential functions have a constant ratio. However, their domains are not the same. Exponential functions are defined for all real numbers, and geometric sequences are defined only for positive integers. Another difference is that the base of a geometric sequence (the common ratio) can be negative, but the base of an exponential function must be positive.

7. The common ratio is -2

9. The sequence is geometric. The common ratio is 2.

11. The sequence is geometric. The common ratio is $-\frac{1}{2}$.

13. The sequence is geometric. The common ratio is 5.

15. $5, 1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}$

17. $800, 400, 200, 100, 50$

19. $a_4 = -\frac{16}{27}$

21. $a_7 = -\frac{2}{729}$

23. $7, 1.4, 0.28, 0.056, 0.0112$

25. $a_1 = -32, \quad a_n = \frac{1}{2}a_{n-1}$

27. $a_1 = 10, \quad a_n = -0.3a_{n-1}$

29. $a_1 = \frac{3}{5}, \quad a_n = \frac{1}{6}a_{n-1}$

31. $a_1 = \frac{1}{512}, a_n = -4a_{n-1}$

33. $12, -6, 3, -\frac{3}{2}, \frac{3}{4}$

35. $a_n = 3^{n-1}$

37. $a_n = 0.8 \cdot (-5)^{n-1}$

39. $a_n = -\left(\frac{4}{5}\right)^{n-1}$

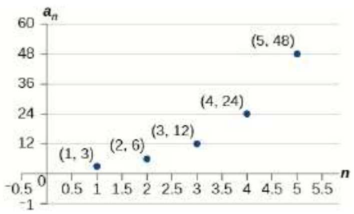
41. $a_n = 3 \cdot \left(-\frac{1}{3}\right)^{n-1}$

43. $a_{12} = \frac{1}{177{,}147}$

45. There are 12 terms in the sequence.

47. The graph does not represent a geometric sequence.

49.



51. Answers will vary. Examples:
$a_1 = 800, \quad a_n = 0.5a_{n-1}$
and
$a_1 = 12.5, \quad a_n = 4a_{n-1}$

53. $a_5 = 256b$

55. The sequence exceeds 100 at the 14th term, $a_{14} \approx 107$.

57. $a_4 = -\frac{32}{3}$ is the first non-integer value

59. Answers will vary. Example: Explicit formula with a decimal common ratio:
$a_n = 400 \cdot 0.5^{n-1}$; First 4 terms:
$400, 200, 100, 50; \quad a_8 = 3.125$

9.4 Section Exercises

1. An nth partial sum is the sum of the first n terms of a sequence.

3. A geometric series is the sum of the terms in a geometric sequence.

5. An annuity is a series of regular equal payments that earn a constant compounded interest.

7. $\sum_{n=0}^{4} 5n$

9. $\sum_{k=1}^{5} 4$

11. $\sum_{k=1}^{20} 8k + 2$

13. $S_5 = \frac{5\left(\frac{3}{2}+\frac{7}{2}\right)}{2}$

15. $S_{13} = \frac{13(3.2+5.6)}{2}$

17. $\sum_{k=1}^{7} 8 \cdot 0.5^{k-1}$

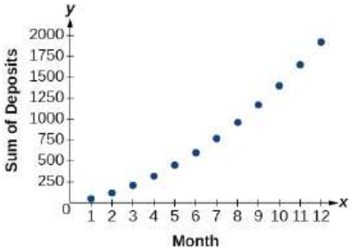
19. $S_5 = \frac{9\left(1-\left(\frac{1}{3}\right)^5\right)}{1-\frac{1}{3}} = \frac{121}{9} \approx 13.44$

21. $S_{11} = \frac{64\left(1-0.2^{11}\right)}{1-0.2} = \frac{781{,}249{,}984}{9{,}765{,}625} \approx 80$

23. The series is defined. $S = \frac{2}{1-0.8}$

25. The series is defined. $S = \frac{-1}{1-\left(-\frac{1}{2}\right)}$

27.



29. Sample answer: The graph of S_n seems to be approaching 1. This makes sense because $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ is a defined infinite geometric series with $S = \frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)} = 1.$

31. 49

33. 254

35. $S_7 = \frac{147}{2}$

37. $S_{11} = \frac{55}{2}$

39. $S_7 = 5208.4$

41. $S_{10} = -\frac{1023}{256}$

43. $S = -\frac{4}{3}$

45. $S = 9.2$

47. $3,705.42

49. $695,823.97

51. $a_k = 30 - k$

53. 9 terms

55. $r = \frac{4}{5}$

57. $400 per month

59. 420 feet

61. 12 feet

9.5 Section Exercises

1. There are $m + n$ ways for either event A or event B to occur.

3. The addition principle is applied when determining the total possible of outcomes of either event occurring. The multiplication principle is applied when determining the total possible outcomes of both events occurring. The word “or” usually implies an addition problem. The word “and” usually implies a multiplication problem.

5. A combination; $C(n, r) = \frac{n!}{(n-r)!r!}$

7. $4 + 2 = 6$

9. $5 + 4 + 7 = 16$

11. $2 \times 6 = 12$

13. $10^3 = 1000$

15. $P(5, 2) = 20$

17. $P(3, 3) = 6$

19. $P(11, 5) = 55,440$

21. $C(12, 4) = 495$

23. $C(7, 6) = 7$

25. $2^{10} = 1024$

27. $2^{12} = 4096$

29. $2^9 = 512$

31. $\frac{8!}{3!} = 6720$

33. $\frac{12!}{3!2!3!4!}$

35. 9

37. Yes, for the trivial cases $r = 0$ and $r = 1$. If $r = 0$, then $C(n, r) = P(n, r) = 1$. If $r = 1$, then $r = 1$, $C(n, r) = P(n, r) = n$.

39. $\frac{6!}{2!} \times 4! = 8640$

41. $6 - 3 + 8 - 3 = 8$

43. $4 \times 2 \times 5 = 40$

45. $4 \times 12 \times 3 = 144$

47. $P(15, 9) = 1,816,214,400$

49. $C(10, 3) \times C(6, 5) \times C(5, 2) = 7,200$

51. $2^{11} = 2048$

53. $\frac{20!}{6!6!8!} = 116,396,280$

9.6 Section Exercises

1. A binomial coefficient is an alternative way of denoting the combination $C(n, r)$. It is defined as $\binom{n}{r} = C(n, r) = \frac{n!}{r!(n-r)!}$.

3. The Binomial Theorem is defined as $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ and can be used to expand any binomial.

5. 15

7. 35

9. 10

11. 12,376

13. $64a^3 - 48a^2b + 12ab^2 - b^3$

15. $27a^3 + 54a^2b + 36ab^2 + 8b^3$

17. $1024x^5 + 2560x^4y + 2560x^3y^2 + 1280x^2y^3 + 320xy^4 + 32y^5$

19. $1024x^5 - 3840x^4y + 5760x^3y^2 - 4320x^2y^3 + 1620xy^4 - 243y^5$

21. $\frac{1}{x^4} + \frac{8}{x^3y} + \frac{24}{x^2y^2} + \frac{32}{xy^3} + \frac{16}{y^4}$

23. $a^{17} + 17a^{16}b + 136a^{15}b^2$

25. $a^{15} - 30a^{14}b + 420a^{13}b^2$

27. $3,486,784,401a^{20} + 23,245,229,340a^{19}b + 73,609,892,910a^{18}b^2$

29. $x^{24} - 8x^{21}\sqrt{y} + 28x^{18}y$

31. $-720x^2y^3$

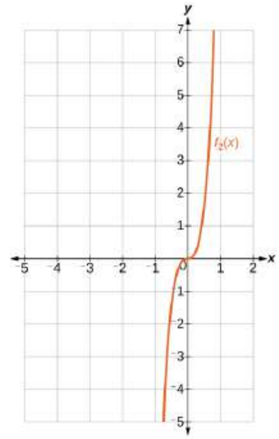
33. $220,812,466,875,000y^7$

35. $35x^3y^4$

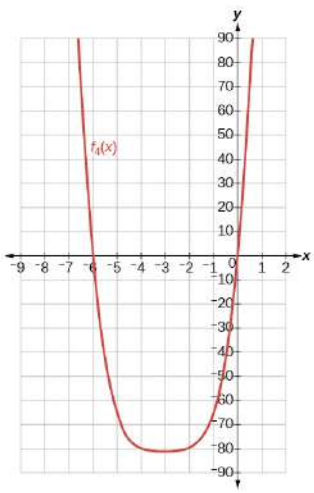
37. $1,082,565a^3b^{16}$

39. $\frac{1152y^2}{x^7}$

41. $f_2(x) = x^4 + 12x^3$



43. $f_4(x) = x^4 + 12x^3 + 54x^2 + 108x$



45. $590,625x^5y^2$

47. $k - 1$

49. The expression $(x^3 + 2y^2 - z)^5$ cannot be expanded using the Binomial Theorem because it cannot be rewritten as a binomial.

9.7 Section Exercises

1. probability; The probability of an event is restricted to values between 0 and 1, inclusive of 0 and 1.

3. An experiment is an activity with an observable result.

5. The probability of the *union of two events* occurring is a number that describes the likelihood that at least one of the events from a probability model occurs. In both a union of sets A and B and a union of events A and B, the union includes either A or B or both. The difference is that a union of sets results in another set, while the union of events is a probability, so it is always a numerical value between 0 and 1.

7. $\frac{1}{2}$.

9. $\frac{5}{8}$.

11. $\frac{1}{2}$.

13. $\frac{3}{8}$.

15. $\frac{1}{4}$.

17. $\frac{3}{4}$.

19. $\frac{3}{8}$.

21. $\frac{1}{8}$.

23. $\frac{15}{16}$.

25. $\frac{5}{8}$.

27. $\frac{1}{13}$.

29. $\frac{1}{26}$.

31. $\frac{12}{13}$.

33.

	1	2	3	4	5	6
1	(1,1) 2	(1,2) 3	(1,3) 4	(1,4) 5	(1,5) 6	(1,6) 7
2	(2,1) 3	(2,2) 4	(2,3) 5	(2,4) 6	(2,5) 7	(2,6) 8
3	(3,1) 4	(3,2) 5	(3,3) 6	(3,4) 7	(3,5) 8	(3,6) 9
4	(4,1) 5	(4,2) 6	(4,3) 7	(4,4) 8	(4,5) 9	(4,6) 10
5	(5,1) 6	(5,2) 7	(5,3) 8	(5,4) 9	(5,5) 10	(5,6) 11
6	(6,1) 7	(6,2) 8	(6,3) 9	(6,4) 10	(6,5) 11	(6,6) 12

35. $\frac{5}{12}$.

37. 0.

39. $\frac{4}{9}$.

41. $\frac{1}{4}$.

43. $\frac{5}{8}$

45. $\frac{8}{13}$

47. $\frac{C(12,5)}{C(48,5)} = \frac{1}{2162}$

49. $\frac{C(12,3)C(36,2)}{C(48,5)} = \frac{175}{2162}$

51. $\frac{C(20,3)C(60,17)}{C(80,20)} \approx 12.49\%$

53. $\frac{C(20,5)C(60,15)}{C(80,20)} \approx 23.33\%$

55. $20.50 + 23.33 - 12.49 = 31.34\%$

57. $\frac{C(40000000,1)C(277000000,4)}{C(317000000,5)} = 36.78\%$

59. $\frac{C(40000000,4)C(277000000,1)}{C(317000000,5)} = 0.11\%$

Review Exercises

1. $2, 4, 7, 11$

3. $13, 103, 1003, 10003$

5. The sequence is arithmetic. The common difference is $d = \frac{5}{3}$.

7. $18, 10, 2, -6, -14$

9. $a_1 = -20, \quad a_n = a_{n-1} + 10$

11. $a_n = \frac{1}{3}n + \frac{13}{24}$

13. $r = 2$

15. 4, 16, 64, 256, 1024

17. $3, \quad 12, \quad 48, \quad 192, \quad 768$

19. $a_n = -\frac{1}{5} \cdot \left(\frac{1}{3}\right)^{n-1}$

21. $\sum_{m=0}^{5} \left(\frac{1}{2}m + 5\right).$

23. $S_{11} = 110$

25. $S_9 \approx 23.95$

27. $S = \frac{135}{4}$

29. $5,617.61

31. 6

33. $10^4 = 10{,}000$

35. $P(18, 4) = 73{,}440$

37. $C(15, 6) = 5005$

39. $2^{50} = 1.13 \times 10^{15}$

41. $\frac{8!}{3!2!} = 3360$

43. $490{,}314$

45. $131{,}072a^{17} + 1{,}114{,}112a^{16}b + 4{,}456{,}448a^{15}b^2$

47.

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

49. $\frac{1}{6}$

51. $\frac{5}{9}$

53. $\frac{4}{9}$

55. $1 - \frac{C(350,8)}{C(500,8)} \approx 94.4\%$

57. $\frac{C(150,3)C(350,5)}{C(500,8)} \approx 25.6\%$

Practice Test

1. $-14, -6, -2, 0$

3. The sequence is arithmetic. The common difference is $d = 0.9$.

5. $a_1 = -2, \; a_n = a_{n-1} - \frac{3}{2}; \; a_{22} = -\frac{67}{2}$

7. The sequence is geometric. The common ratio is $r = \frac{1}{2}$.

9. $a_1 = 1, \; a_n = -\frac{1}{2} \cdot a_{n-1}$

11. $\sum_{k=-3}^{15} \left(3k^2 - \frac{5}{6}k\right)$

13. $S_7 = -2604.2$

15. Total in account: $\$140, 355.75$; Interest earned: $\$14, 355.75$

17. $5 \times 3 \times 2 \times 3 \times 2 = 180$

19. $C(15, 3) = 455$

21. $\frac{10!}{2!3!2!} = 151{,}200$

23. $\frac{429x^{14}}{16}$

25. $\frac{4}{7}$

27. $\frac{5}{7}$

29. $\frac{C(14,3)C(26,4)}{C(40,7)} \approx 29.2\%$

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