College Algebra with Corequisite Support 2e.

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Preface

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Format

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About *College Algebra 2e with Corequisite Support*

College Algebra 2e with Corequisite Support integrates comprehensive algebraic principles with effective foundational review. Each *College Algebra 2e* textbook section is paired with a thoughtfully developed, topically aligned skills module that prepares students for the course material. The modules include conceptual overviews, worked examples, and guided practice; they incorporate relevant material from OpenStax's Developmental Math series. The modular approach and richness of content ensure that the book meets the needs of a variety of courses. *College Algebra 2e with Corequisite Support* offers a wealth of examples with detailed, conceptual explanations, building a strong foundation in the material before asking students to apply what they've learned.

Development Overview

College Algebra 2e with Corequisite Support is the product of a collaborative effort by a group of dedicated authors, and instructors experience and expertise led to a highly flexible and supportive resource for student at a range of levels. Special thanks is due the original *College Algebra* author, Jay Abramson of Arizona State University. Based on the widely used core text, Corequisite leader Sharon North (St. Louis Community College) developed a coordinated set of support resources, which provide review, instruction, and practice for algebra students.

The author team identified foundational skills and concepts, then mapped them to each module. These became the Corequisite Skills modules that precede each section of the text. In addition, Professor North authored a set of Labs designed for use in classes or workshops.

The collective experience of our author team allowed us to pinpoint the subtopics, exceptions, and individual connections that give students the most trouble. The textbook is therefore replete with well-designed features and highlights, which help students overcome these barriers. As the students read and practice, they are coached in methods of thinking through problems and internalizing mathematical processes.

Changes to the Second Edition

The *College Algebra 2e with Corequisite Support* revision focused on mathematical clarity and accuracy as well as inclusivity. Examples, Exercises, and Solutions were reviewed by multiple faculty experts. All improvement suggestions and errata updates, driven by faculty and students from several thousand colleges, were considered and unified across the different formats of the text.

OpenStax and our authors are aware of the difficulties posed by shifting problem and exercise numbers when textbooks are revised. In an effort to make the transition to the 2nd edition as seamless as possible, we have minimized any shifting of exercise numbers.

The revision also focused on supporting inclusive and welcoming learning experiences. The introductory narratives, example and problem contexts, and even many of the names used for fictional people in the text were all reviewed using a diversity, equity, and inclusion framework. Several hundred resulting revisions improve the balance and relevance to the students using the text, while maintaining a variety of applications to diverse careers and academic fields. In particular, explanations of scientific and historical aspects of mathematics have been expanded to include more contributors. For example, the authors added additional historical and multicultural context regarding what is widely known as Pascal’s Triangle, and similarly added details regarding the international process of decoding the Enigma machine (including the role of Polish college students). Several-chapter opening narratives and in-chapter references are completely new, and contexts across all chapters were specifically reviewed for equity in gender representation and connotation.

Finally, prior to the release of this edition, OpenStax published a version to support Corequisite instruction, which is described in more detail below.

Pedagogical Foundations and Features

Corequisite Skills Modules

Each *College Algebra 2e* section begins with a carefully developed skills component, designed to guide students though the prequisite material and provide the strongest possible foundation for their work. While the modules draw from some elements of OpenStax *Intermediate Algebra*, each one was adapted and enhance with new material to be most effective for students.

The skills modules consist of the following:

- Learning Objectives identifying the key concepts to be learned before students move on
- Conceptual explanations and connections to subsequent material
- Detailed worked examples
- Practice problems

The Corequisite skills modules are also available for download and printing on the *College Algebra 2e with Corequisite Support* book page.

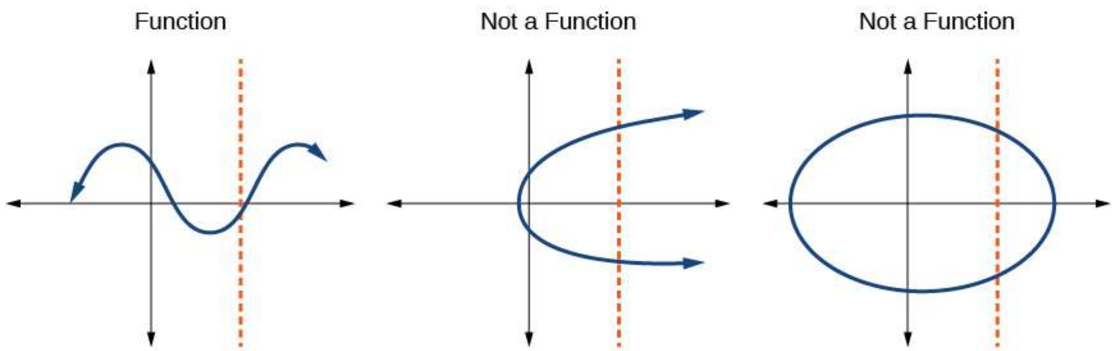
Examples

Each learning objective is supported by one or more worked examples that demonstrate the problem-solving approaches that students must master. The multiple Examples model different approaches to the same type of problem or introduce similar problems of increasing complexity.

All Examples follow a simple two- or three-part format. The question clearly lays out a mathematical problem to solve. The Solution walks through the steps, usually providing context for the approach — in other words, why the instructor is solving the problem in a specific manner. Finally, the Analysis (for select examples) reflects on the broader implications of the Solution just shown. Examples are followed by a “Try It” question, as explained below.

Figures

College Algebra 2e with Corequisite Support contains many figures and illustrations, the vast majority of which are graphs and diagrams. Art throughout the text adheres to a clear, understated style, drawing the eye to the most important information in each figure while minimizing visual distractions. Color contrast is employed with discretion to distinguish between the different functions or features of a graph.



Supporting Features

Several elements, each marked by a distinctive icon, contribute to and check understanding.

- A **How To** is a list of steps necessary to solve a certain type of problem. A How To typically precedes an Example that proceeds to demonstrate the steps in action.
- A **Try It** exercise immediately follows an Example or a set of related Examples, providing the student with an immediate opportunity to solve a similar problem. In the PDF and the Web View version of the core *College Algebra* portions of the text, answers to the Try It exercises are located in the Answer Key. Answers to the Try It Exercises for the Corequisite Skills subsections are provided to instructors only so that they can decide on the best way to use them.
- A **Q&A** may appear at any point in the narrative, but most often follows an Example. This feature pre-empts misconceptions by posing a commonly asked yes/no question, followed by a detailed answer and explanation.
- The **Media** icon appears at the conclusion of each section, just prior to the Section Exercises. This icon marks a list of links to online video tutorials that reinforce the concepts and skills introduced in the section.

While we have selected tutorials that closely align to our learning objectives, we did not produce these tutorials, nor were they specifically produced or tailored to accompany *College Algebra 2e with Corequisite Support*.

Section Exercises

Each section of every chapter concludes with a well-rounded set of exercises that can be assigned as homework or used selectively for guided practice. With over 4600 exercises across the 9 chapters, instructors should have plenty from which to choose.

Section Exercises are organized by question type, and generally appear in the following order:

- **Verbal** questions assess conceptual understanding of key terms and concepts.
- **Algebraic** problems require students to apply algebraic manipulations demonstrated in the section.
- **Graphical** problems assess students' ability to interpret or produce a graph.
- **Numeric** problems require the student to perform calculations or computations.
- **Technology** problems encourage exploration through use of a graphing utility, either to visualize or verify algebraic results or to solve problems via an alternative to the methods demonstrated in the section.
- **Extensions** pose problems more challenging than the Examples demonstrated in the section. They require students to synthesize multiple learning objectives or apply critical thinking to solve complex problems.
- **Real-World Applications** present realistic problem scenarios from fields such as physics, geology, biology, finance, and the social sciences.

Chapter Review Features

Each chapter concludes with a review of the most important takeaways, as well as additional practice problems that students can use to prepare for exams.

- **Key Terms** provides a formal definition for each bold-faced term in the chapter.
- **Key Equations** presents a compilation of formulas, theorems, and standard-form equations.
- **Key Concepts** summarizes the most important ideas introduced in each section, linking back to the relevant Example(s) in case students need to review.
- **Chapter Review Exercises** include 40-80 practice problems that recall the most important concepts from each section.
- **Practice Test** includes 25-50 problems assessing the most important learning objectives from the chapter. Note that the practice test is not organized by section, and may be more heavily weighted toward cumulative objectives as opposed to the foundational objectives covered in the opening sections.

- **Answer Key** includes the answers to all Try It exercises and every other exercise from the Section Exercises, Chapter Review Exercises, and Practice Test.

Accuracy of the Content

We understand that precision and accuracy are imperatives in mathematics, and undertook an accuracy program led by experienced faculty. Examples, art, problems, and solutions were reviewed by dedicated faculty, with a separate team evaluating the answer key and solutions.

The text also benefits from years of usage by thousands of faculty and students. A core aspect of the second edition revision process included consolidating and ensuring consistency with regard to any errata and corrections that have been implemented during the series' extensive usage and incorporation into homework systems.

Additional Resources

Student and Instructor Resources

We've compiled additional resources for both students and instructors, including Getting Started Guides, an instructor solution manual, and PowerPoint slides. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

The authors and OpenStax have provided substantial resources regarding teaching and learning in Corequisite environments. Any material that may be directly used by students -- including downloadable versions of the skills modules and the Labs -- will be included on the Student Resources page.

Community Hubs

OpenStax partners with the Institute for the Study of Knowledge Management in Education (ISKME) to offer Community Hubs on OER Commons—a platform for instructors to share community-created resources that support OpenStax books, free of charge. Through our Community Hubs, instructors can upload their own materials or download resources to use in their own courses, including additional ancillaries, teaching material, multimedia, and relevant course content. We encourage instructors to join the hubs for the subjects most relevant to your teaching and research as an opportunity both to enrich your courses and to engage with other faculty. To reach the Community Hubs, visit www.oercommons.org/hubs/openstax.

Technology Resources

As allies in making high-quality learning materials accessible, our technology partners offer optional low-cost tools that are integrated with OpenStax books. To access the technology options for your text, visit your book page on openstax.org.

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Sharon North, St. Louis Community College

Sharon North is a Professor of Mathematics at St. Louis Community College. She teaches gateway mathematics courses including Pre-calculus, Quantitative Reasoning, Introductory Statistics, College Algebra and Calculus. Sharon has also worked to develop and implement co-requisite offerings to these transfer level courses. Sharon designs math curricula and writes application-based content for students. She has served in leadership roles as a mathematics department chair and as district-wide developmental education coordinator. Sharon serves as the STLCC faculty representative to the Missouri Co-requisite at Scale Taskforce and serves on the state advisory board for mathematics pathways. She has worked in several states as part of the Charles A. Dana Center external team helping to mentor and provide support to faculty developing math pathways, corequisite courses and active learning activities for their classrooms.

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Jay Abramson has been teaching Precalculus for 38 years, the last 19 at Arizona State University, where he is a principal lecturer in the School of Mathematics and Statistics. His accomplishments at ASU include co-developing the university's first hybrid and online math courses as well as an extensive library of video lectures and tutorials. In addition, he has served as a contributing author for two of Pearson Education's math programs, NovaNet Precalculus and Trigonometry. Prior to coming to ASU, Jay taught at Texas State Technical College and Amarillo College. He received Teacher of the Year awards at both institutions.

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1 PREREQUISITES

Credit: Andreas Kambanls

Chapter Outline

Introduction to Prerequisites

It’s a cold day in Antarctica. In fact, it’s always a cold day in Antarctica. Earth’s southernmost continent, Antarctica experiences the coldest, driest, and windiest conditions known. The coldest temperature ever recorded, over one hundred degrees below zero on the Celsius scale, was recorded by remote satellite. It is no surprise then, that no native human population can survive the harsh conditions. Only explorers and scientists brave the environment for any length of time.

Measuring and recording the characteristics of weather conditions in Antarctica requires a use of different kinds of numbers. For tens of thousands of years, humans have undertaken methods to tally, track, and record numerical information. While we don't know much about their usage, the Lebombo Bone (dated to about 35,000 BCE) and the Ishango Bone (dated to about 20,000 BCE) are among the earliest mathematical artifacts. Found in Africa, their clearly deliberate groupings of notches may have been used to track time, moon cycles, or other information. Performing calculations with them and using the results to make predictions requires an understanding of relationships among numbers. In this chapter, we will review sets of numbers and properties of operations used to manipulate numbers. This understanding will serve as prerequisite knowledge throughout our study of algebra and trigonometry.

1.1 Real Numbers: Algebra Essentials

Learning Objectives

In this section, you will:

- Classify a real number as a natural, whole, integer, rational, or irrational number.
- Perform calculations using order of operations.
- Use the following properties of real numbers: commutative, associative, distributive, inverse, and identity.
- Evaluate algebraic expressions.
- Simplify algebraic expressions.

COREQUISITE SKILLS

Learning Objectives

- Identify the study skills leading to success in a college level mathematics course.
- Reflect on your past math experiences and create a plan for improvement.

Objective 1: Identify the study skills leading to success in a college level mathematics course.

Welcome to your algebra course! This course will be challenging so now is the time to set up a plan for success. In this first chapter we will focus on important strategies for success including: math study skills, time management, note taking skills, smart test taking strategies, and the idea of a growth mindset. Each of these ideas will help you to be successful in your college level math course whether you are enrolled in a face-to-face traditional section or an online section virtual section.

Complete the following survey by checking a column for each behavior based on the frequency that you engage in the behavior during your last academic term.

Behavior or belief:	Always	Sometimes	Never
1. Arrive or log in early to class each session.			
2. Stay engaged for the entire class session or online meeting.			
3. Contact a fellow student and my instructor if I must miss class for notes or important announcements.			
4. Read through my class notes before beginning my homework.			
5. Connect with a study partner either virtually or in class.			
6. Keep my phone put away during classes to avoid distractions.			
7. Spend time on homework each day.			
8. Begin to review for exams a week prior to exam.			
9. Create a practice test and take it before an exam.			
10. Find my instructor’s office hours and stop in either face-to-face or virtually for help.			
11. Locate the math tutoring resources (on campus or virtually) for students and make note of available hours.			
12. Visit math tutoring services for assistance on a regular basis (virtual or face-to-face).			
13. Spend at least two hours studying outside of class for each hour in class (virtual or face-to-face).			
14. Check my progress in my math course through my college’s learning management system.			
15. Scan through my entire test before beginning and start off working on a problem I am confident in solving.			

Behavior or belief:	Always	Sometimes	Never
16. Gain access to my math courseware by the end of first week of classes.			
17. Send an email to my instructor when I need assistance.			
18. Create a schedule for each week including time in class, at work and study time.			
19. Read through my textbook on the section we are covering before I come to class or begin virtual sessions.			
20. Feel confident when I start a math exam.			
21. Keep a separate notebook for each class I am taking. Divide math notebooks or binders into separate sections for homework, PowerPoint slides, and notes.			
22. Talk honestly about classes with a friend or family member on a regular basis.			
23. Add test dates to a calendar at the beginning of the semester.			
24. Take notes each math class session.			
25. Ask my instructor questions in class (face-to-face or virtual) if I don’t understand.			
26. Complete nightly homework assignments.			
27. Engage in class discussions.(virtual or face-to-face)			
28. Recopy my class notes more neatly after class.			
29. Have a quiet, organized place to study.			
30. Avoid calls or texts from friends when I’m studying.			
31. Set study goals for myself each week.			
32. Think about my academic major and future occupation.			
33. Take responsibility for my study plan.			
34. Try different approaches to solve when I get stuck on a problem.			
35. Believe that I can be successful in any college math course.			
36. Search for instructional videos online when I get really stuck on a section or an exercise.			
37. Create flashcards to help in memorizing important formulas and strategies.			

Behavior or belief:	Always	Sometimes	Never
Total number in each column:			
Scoring:	**Always: 4 points each**	**Sometimes: 2 points each**	**Never: 0 points each**
Total Points:			**0**

Practice Makes Perfect

Identify the study skills leading to success in a college level mathematics course.

1. Each of the behaviors or attitudes listed in the table above are associated with success in college mathematics. This means that students who use these strategies or are open to these beliefs are successful learners. Share your total score with your study group in class and be supportive of your fellow students!

2. Based on this survey create a list of the top 5 strategies that you currently utilize, and feel are most helpful to you.
   1.
   2.
   3.
   4.
   5.

3. Based on this survey create a list of the top 5 strategies that interest you, and that you feel could be most helpful to you this term. Plan on implementing these strategies.
   1.
   2.
   3.
   4.
   5.

Objective 2: Reflect on your past math experiences and create a plan for improvement.

1. It's important to take the opportunity to reflect on your past experiences in math classes as you begin a new term. We can learn a lot from these reflections and thus work toward developing a strategy for improvement.
   In the table below list 5 challenges you had in past math courses and list a possible solution that you could try this semester.

Challenge	Possible solution
1.	
2.	
3.	
4.	
5.	

2. Write your math autobiography. Tell your math story by describing your past experiences as a learner of mathematics. Share how your attitudes have changed about math over the years if they have. Perhaps include what you love, hate, dread, appreciate, fear, look forward to, or find beauty in. This will help your teacher to better

understand you and your current feelings about the discipline.

3. Share your autobiographies with your study group members. This helps to create a community in the classroom when common themes emerge.

It is often said that mathematics is the language of science. If this is true, then an essential part of the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattle herders, and traders used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a “base state” while counting and used various symbols to represent this null condition. However, it was not until about the fifth century CE in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century CE, negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the **natural numbers**: 1, 2, 3, 4, 5, and so on. We describe them in set notation as $\{1, 2, 3, ...\}$ where the ellipsis (…) indicates that the numbers continue to infinity. The natural numbers are, of course, also called the *counting numbers*. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, ...\}$.

The set of **integers** adds the opposites of the natural numbers to the set of whole numbers: $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$. It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

$$\begin{array}{ccc} \text{negative integers} & \text{zero} & \text{positive integers} \\ \ldots, -3, -2, -1, & 0, & 1, 2, 3, \cdots \end{array}$$

The set of **rational numbers** is written as $\left\{\frac{m}{n} \middle| m \text{ and } n \text{ are integers and } n \neq 0\right\}$. Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

ⓐ a terminating decimal: $\frac{15}{8} = 1.875$, or ⓑ a repeating decimal: $\frac{4}{11} = 0.36363636\ldots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

EXAMPLE 1

Writing Integers as Rational Numbers

Write each of the following as a rational number.

ⓐ 7 ⓑ 0 ⓒ −8

Solution

Write a fraction with the integer in the numerator and 1 in the denominator.

ⓐ $7 = \frac{7}{1}$ ⓑ $0 = \frac{0}{1}$ ⓒ $-8 = -\frac{8}{1}$

TRY IT #1 Write each of the following as a rational number.

ⓐ 11 ⓑ 3 ⓒ −4

EXAMPLE 2

Identifying Rational Numbers

Write each of the following rational numbers as either a terminating or repeating decimal.

ⓐ $-\frac{5}{7}$ ⓑ $\frac{15}{5}$ ⓒ $\frac{13}{25}$

Solution

Write each fraction as a decimal by dividing the numerator by the denominator.

ⓐ $-\frac{5}{7}=-0.\overline{714285}$, a repeating decimal ⓑ $\frac{15}{5}=3$ (or 3.0), a terminating decimal
ⓒ $\frac{13}{25}=0.52$, a terminating decimal

TRY IT #2 Write each of the following rational numbers as either a terminating or repeating decimal.

ⓐ $\frac{68}{17}$ ⓑ $\frac{8}{13}$ ⓒ $-\frac{17}{20}$

Irrational Numbers

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not 2 or even $\frac{3}{2}$, but was something else. Or a garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than 3, but still not a rational number. Such numbers are said to be *irrational* because they cannot be written as fractions. These numbers make up the set of **irrational numbers**. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. So we write this as shown.

$$\{h|h \text{ is not a rational number}\}$$

EXAMPLE 3

Differentiating Rational and Irrational Numbers

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

ⓐ $\sqrt{25}$ ⓑ $\frac{33}{9}$ ⓒ $\sqrt{11}$ ⓓ $\frac{17}{34}$ ⓔ $0.3033033303333\ldots$

Solution

ⓐ $\sqrt{25}$: This can be simplified as $\sqrt{25}=5$. Therefore, $\sqrt{25}$ is rational.

ⓑ $\frac{33}{9}$: Because it is a fraction of integers, $\frac{33}{9}$ is a rational number. Next, simplify and divide.

$$\frac{33}{9}=\frac{\overset{11}{\cancel{33}}}{\underset{3}{\cancel{9}}}=\frac{11}{3}=3.\overline{6}$$

So, $\frac{33}{9}$ is rational and a repeating decimal.

ⓒ $\sqrt{11}$: This cannot be simplified any further. Therefore, $\sqrt{11}$ is an irrational number.

ⓓ $\frac{17}{34}$: Because it is a fraction of integers, $\frac{17}{34}$ is a rational number. Simplify and divide.

$$\frac{17}{34}=\frac{\overset{1}{\cancel{17}}}{\underset{2}{\cancel{34}}}=\frac{1}{2}=0.5$$

So, $\frac{17}{34}$ is rational and a terminating decimal.

ⓔ $0.3033033303333\ldots$ is not a terminating decimal. Also note that there is no repeating pattern because the group of 3s increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

TRY IT #3 Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

ⓐ $\frac{7}{77}$ ⓑ $\sqrt{81}$ ⓒ $4.27027002700027\ldots$ ⓓ $\frac{91}{13}$ ⓔ $\sqrt{39}$

Real Numbers

Given any number *n*, we know that *n* is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of **real numbers**. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or –). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line.The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the **real number line** as shown in Figure 1.

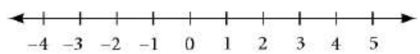


Figure 1 The real number line

EXAMPLE 4

Classifying Real Numbers

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

ⓐ $-\frac{10}{3}$ ⓑ $\sqrt{5}$ ⓒ $-\sqrt{289}$ ⓓ -6π ⓔ $0.615384615384\ldots$

Solution

ⓐ $-\frac{10}{3}$ is negative and rational. It lies to the left of 0 on the number line.
ⓑ $\sqrt{5}$ is positive and irrational. It lies to the right of 0.
ⓒ $-\sqrt{289} = -\sqrt{17^2} = -17$ is negative and rational. It lies to the left of 0.
ⓓ -6π is negative and irrational. It lies to the left of 0.
ⓔ $0.615384615384\ldots$ is a repeating decimal so it is rational and positive. It lies to the right of 0.

TRY IT #4 Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

ⓐ $\sqrt{73}$ ⓑ $-11.411411411\ldots$ ⓒ $\frac{47}{19}$ ⓓ $-\frac{\sqrt{5}}{2}$ ⓔ 6.210735

Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram, such as Figure 2.

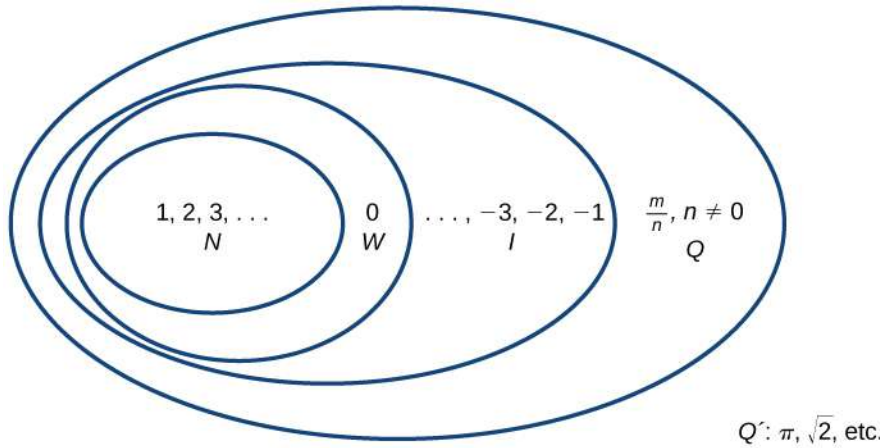


Figure 2 Sets of numbers
N: the set of natural numbers
W: the set of whole numbers
I: the set of integers
Q: the set of rational numbers
Q´: the set of irrational numbers

Sets of Numbers

The set of **natural numbers** includes the numbers used for counting: $\{1, 2, 3, ...\}$.

The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, ...\}$.

The set of **integers** adds the negative natural numbers to the set of whole numbers: $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

The set of **rational numbers** includes fractions written as $\left\{\frac{m}{n}\ \middle|\ m \text{ and } n \text{ are integers and } n \neq 0\right\}$.

The set of **irrational numbers** is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h | h \text{ is not a rational number}\}$.

EXAMPLE 5

Differentiating the Sets of Numbers

Classify each number as being a natural number (*N*), whole number (*W*), integer (*I*), rational number (*Q*), and/or irrational number (*Q'*).

ⓐ $\sqrt{36}$ ⓑ $\frac{8}{3}$ ⓒ $\sqrt{73}$ ⓓ -6 ⓔ $3.2121121112\ldots$

Solution

	N	W	I	Q	Q'
a. $\sqrt{36} = 6$	X	X	X	X	
b. $\frac{8}{3} = 2.\overline{6}$				X	
c. $\sqrt{73}$					X
d. –6			X	X	
e. 3.2121121112...					X

TRY IT #5 Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q').

ⓐ $-\frac{35}{7}$ ⓑ 0 ⓒ $\sqrt{169}$ ⓓ $\sqrt{24}$ ⓔ $4.763763763\ldots$

Performing Calculations Using the Order of Operations

When we multiply a number by itself, we square it or raise it to a power of 2. For example, $4^2 = 4 \cdot 4 = 16$. We can raise any number to any power. In general, the **exponential notation** a^n means that the number or variable a is used as a factor n times.

$$a^n = \overbrace{a \cdot a \cdot a \cdot \ldots \cdot a}^{n \text{ factors}}$$

In this notation, a^n is read as the nth power of a,or a to the n where a is called the **base** and n is called the **exponent.** A term in exponential notation may be part of a mathematical expression, which is a combination of numbers and operations. For example, $24 + 6 \cdot \frac{2}{3} - 4^2$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the **order of operations**. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses ( ), brackets [ ], and braces { } to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left to right.

Let's take a look at the expression provided.

$$24 + 6 \cdot \frac{2}{3} - 4^2$$

There are no grouping symbols, so we move on to exponents or radicals. The number 4 is raised to a power of 2, so simplify 4^2 as 16.

$$\begin{gathered} 24 + 6 \cdot \frac{2}{3} - 4^2 \\ 24 + 6 \cdot \frac{2}{3} - 16 \end{gathered}$$

Next, perform multiplication or division, left to right.

$$\begin{gathered} 24 + 6 \cdot \frac{2}{3} - 16 \\ 24 + 4 - 16 \end{gathered}$$

Lastly, perform addition or subtraction, left to right.

$$\begin{gathered} 24 + 4 - 16 \\ 28 - 16 \\ 12 \end{gathered}$$

Therefore, $24 + 6 \cdot \frac{2}{3} - 4^2 = 12$.

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

Order of Operations

Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the acronym **PEMDAS**:

P(arentheses)
E(xponents)
M(ultiplication) and **D**(ivision)
A(ddition) and **S**(ubtraction)

HOW TO

Given a mathematical expression, simplify it using the order of operations.

Step 1. Simplify any expressions within grouping symbols.
Step 2. Simplify any expressions containing exponents or radicals.
Step 3. Perform any multiplication and division in order, from left to right.
Step 4. Perform any addition and subtraction in order, from left to right.

EXAMPLE 6

Using the Order of Operations

Use the order of operations to evaluate each of the following expressions.

ⓐ $(3 \cdot 2)^2 - 4(6+2)$ ⓑ $\frac{5^2-4}{7} - \sqrt{11-2}$ ⓒ $6 - |5-8| + 3(4-1)$ ⓓ $\frac{14-3\cdot 2}{2\cdot 5-3^2}$
ⓔ $7(5 \cdot 3) - 2\left[(6-3) - 4^2\right] + 1$

Solution

ⓐ

$$\begin{aligned} (3 \cdot 2)^2 - 4(6+2) &= (6)^2 - 4(8) && \text{Simplify parentheses.} \\ &= 36 - 4(8) && \text{Simplify exponent.} \\ &= 36 - 32 && \text{Simplify multiplication.} \\ &= 4 && \text{Simplify subtraction.} \end{aligned}$$

ⓑ

$$\begin{aligned} \frac{5^2-4}{7} - \sqrt{11-2} &= \frac{5^2-4}{7} - \sqrt{9} && \text{Simplify grouping symbols (radical).} \\ &= \frac{5^2-4}{7} - 3 && \text{Simplify radical.} \\ &= \frac{25-4}{7} - 3 && \text{Simplify exponent.} \\ &= \frac{21}{7} - 3 && \text{Simplify subtraction in numerator.} \\ &= 3 - 3 && \text{Simplify division.} \\ &= 0 && \text{Simplify subtraction.} \end{aligned}$$

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.

ⓒ

$$\begin{aligned} 6 - |5-8| + 3(4-1) &= 6 - |-3| + 3(3) && \text{Simplify inside grouping symbols.} \\ &= 6 - 3 + 3(3) && \text{Simplify absolute value.} \\ &= 6 - 3 + 9 && \text{Simplify multiplication.} \\ &= 3 + 9 && \text{Simplify subtraction.} \\ &= 12 && \text{Simplify addition.} \end{aligned}$$

ⓓ

$$\begin{aligned}\frac{14-3\cdot 2}{2\cdot 5-3^2} &= \frac{14-3\cdot 2}{2\cdot 5-9} && \text{Simplify exponent.}\\ &= \frac{14-6}{10-9} && \text{Simplify products.}\\ &= \frac{8}{1} && \text{Simplify differences.}\\ &= 8 && \text{Simplify quotient.}\end{aligned}$$

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.

ⓔ

$$\begin{aligned}7(5\cdot 3)-2\left[(6-3)-4^2\right]+1 &= 7(15)-2\left[(3)-4^2\right]+1 && \text{Simplify inside parentheses.}\\ &= 7(15)-2(3-16)+1 && \text{Simplify exponent.}\\ &= 7(15)-2(-13)+1 && \text{Subtract.}\\ &= 105+26+1 && \text{Multiply.}\\ &= 132 && \text{Add.}\end{aligned}$$

TRY IT #6 Use the order of operations to evaluate each of the following expressions.

ⓐ $\sqrt{5^2-4^2}+7(5-4)^2$ ⓑ $1+\frac{7\cdot 5-8\cdot 4}{9-6}$ ⓒ $|1.8-4.3|+0.4\sqrt{15+10}$

ⓓ $\frac{1}{2}\left[5\cdot 3^2-7^2\right]+\frac{1}{3}\cdot 9^2$ ⓔ $\left[(3-8)^2-4\right]-(3-8)$

Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

Commutative Properties

The **commutative property of addition** states that numbers may be added in any order without affecting the sum.

$$a+b=b+a$$

We can better see this relationship when using real numbers.

$$(-2)+7=5 \quad \text{and} \quad 7+(-2)=5$$

Similarly, the **commutative property of multiplication** states that numbers may be multiplied in any order without affecting the product.

$$a\cdot b=b\cdot a$$

Again, consider an example with real numbers.

$$(-11)\cdot(-4)=44 \quad \text{and} \quad (-4)\cdot(-11)=44$$

It is important to note that neither subtraction nor division is commutative. For example, $17-5$ is not the same as $5-17$. Similarly, $20\div 5\neq 5\div 20$.

Associative Properties

The **associative property of multiplication** tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.

$$a(bc)=(ab)c$$

Consider this example.

$$(3\cdot 4)\cdot 5=60 \quad \text{and} \quad 3\cdot(4\cdot 5)=60$$

The **associative property of addition** tells us that numbers may be grouped differently without affecting the sum.

$$a+(b+c)=(a+b)+c$$

This property can be especially helpful when dealing with negative integers. Consider this example.

$$[15+(-9)]+23=29 \quad \text{and} \quad 15+[(-9)+23]=29$$

Are subtraction and division associative? Review these examples.

$$\begin{array}{rcl} 8-(3-15) & \stackrel{?}{=} & (8-3)-15 \\ 8-(-12) & = & 5-15 \\ 20 & \neq & -10 \end{array} \qquad \begin{array}{rcl} 64\div(8\div4) & \stackrel{?}{=} & (64\div8)\div4 \\ 64\div2 & \stackrel{?}{=} & 8\div4 \\ 32 & \neq & 2 \end{array}$$

As we can see, neither subtraction nor division is associative.

Distributive Property

The **distributive property** states that the product of a factor times a sum is the sum of the factor times each term in the sum.

$$a\cdot(b+c)=a\cdot b+a\cdot c$$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.

$$\begin{aligned} 4\cdot[12+(-7)] &= \overbrace{4\cdot 12}+\overbrace{4\cdot(-7)} \\ &= 48+(-28) \\ &= 20 \end{aligned}$$

Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12, multiplying it by –7, and adding the products.

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.

$$\begin{array}{rcl} 6+(3\cdot5) & \stackrel{?}{=} & (6+3)\cdot(6+5) \\ 6+(15) & \stackrel{?}{=} & (9)\cdot(11) \\ 21 & \neq & 99 \end{array}$$

A special case of the distributive property occurs when a sum of terms is subtracted.

$$a-b=a+(-b)$$

For example, consider the difference $12-(5+3)$. We can rewrite the difference of the two terms 12 and $(5+3)$ by turning the subtraction expression into addition of the opposite. So instead of subtracting $(5+3)$, we add the opposite.

$$12+(-1)\cdot(5+3)$$

Now, distribute -1 and simplify the result.

$$\begin{array}{rcl} 12-(5+3) & = & 12+(-1)\cdot(5+3) \\ & = & 12+[(-1)\cdot5+(-1)\cdot3] \\ & = & 12+(-8) \\ & = & 4 \end{array}$$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.

$$\begin{array}{rcl} 12-(5+3) & = & 12+(-5-3) \\ & = & 12+(-8) \\ & = & 4 \end{array}$$

Identity Properties

The **identity property of addition** states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.

$$a+0=a$$

The **identity property of multiplication** states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.

$$a\cdot1=a$$

For example, we have $(-6)+0=-6$ and $23\cdot1=23$. There are no exceptions for these properties; they work for every real number, including 0 and 1.

Inverse Properties

The **inverse property of addition** states that, for every real number *a*, there is a unique number, called the additive inverse (or opposite), denoted by (−*a*), that, when added to the original number, results in the additive identity, 0.

$$a+(-a)=0$$

For example, if $a=-8$, the additive inverse is 8, since $(-8)+8=0$.

The **inverse property of multiplication** holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number *a*, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, that, when multiplied by the original number, results in the multiplicative identity, 1.

$$a\cdot\frac{1}{a}=1$$

For example, if $a=-\frac{2}{3}$, the reciprocal, denoted $\frac{1}{a}$, is $-\frac{3}{2}$ because

$$a\cdot\frac{1}{a}=\left(-\frac{2}{3}\right)\cdot\left(-\frac{3}{2}\right)=1$$

Properties of Real Numbers

The following properties hold for real numbers *a*, *b*, and *c*.

	Addition	Multiplication
Commutative Property	$a+b=b+a$	$a\cdot b=b\cdot a$
Associative Property	$a+(b+c)=(a+b)+c$	$a\,(bc)=(ab)\,c$
Distributive Property	$a\cdot(b+c)=a\cdot b+a\cdot c$	
Identity Property	There exists a unique real number called the additive identity, 0, such that, for any real number *a* $a+0=a$	There exists a unique real number called the multiplicative identity, 1, such that, for any real number *a* $a\cdot1=a$
Inverse Property	Every real number a has an additive inverse, or opposite, denoted –*a*, such that $a+(-a)=0$	Every nonzero real number *a* has a multiplicative inverse, or reciprocal, denoted $\frac{1}{a}$, such that $a\cdot\left(\frac{1}{a}\right)=1$

EXAMPLE 7

Using Properties of Real Numbers

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

ⓐ $3\cdot6+3\cdot4$ ⓑ $(5+8)+(-8)$ ⓒ $6-(15+9)$ ⓓ $\frac{4}{7}\cdot\left(\frac{2}{3}\cdot\frac{7}{4}\right)$ ⓔ $100\cdot[0.75+(-2.38)]$

Solution

ⓐ

$$\begin{aligned} 3\cdot 6+3\cdot 4 &= 3\cdot(6+4) && \text{Distributive property.}\\ &= 3\cdot 10 && \text{Simplify.}\\ &= 30 && \text{Simplify.} \end{aligned}$$

ⓑ

$$\begin{aligned} (5+8)+(-8) &= 5+[8+(-8)] && \text{Associative property of addition.}\\ &= 5+0 && \text{Inverse property of addition.}\\ &= 5 && \text{Identity property of addition.} \end{aligned}$$

ⓒ

$$\begin{aligned} 6-(15+9) &= 6+[(-15)+(-9)] && \text{Distributive property.}\\ &= 6+(-24) && \text{Simplify.}\\ &= -18 && \text{Simplify.} \end{aligned}$$

ⓓ

$$\begin{aligned} \tfrac{4}{7}\cdot\left(\tfrac{2}{3}\cdot\tfrac{7}{4}\right) &= \tfrac{4}{7}\cdot\left(\tfrac{7}{4}\cdot\tfrac{2}{3}\right) && \text{Commutative property of multiplication.}\\ &= \left(\tfrac{4}{7}\cdot\tfrac{7}{4}\right)\cdot\tfrac{2}{3} && \text{Associative property of multiplication.}\\ &= 1\cdot\tfrac{2}{3} && \text{Inverse property of multiplication.}\\ &= \tfrac{2}{3} && \text{Identity property of multiplication.} \end{aligned}$$

ⓔ

$$\begin{aligned} 100\cdot[0.75+(-2.38)] &= 100\cdot 0.75+100\cdot(-2.38) && \text{Distributive property.}\\ &= 75+(-238) && \text{Simplify.}\\ &= -163 && \text{Simplify.} \end{aligned}$$

TRY IT #7 Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

ⓐ $\left(-\frac{23}{5}\right)\cdot\left[11\cdot\left(-\frac{5}{23}\right)\right]$ ⓑ $5\cdot(6.2+0.4)$ ⓒ $18-(7-15)$
ⓓ $\frac{17}{18}+\left[\frac{4}{9}+\left(-\frac{17}{18}\right)\right]$ ⓔ $6\cdot(-3)+6\cdot 3$

Evaluating Algebraic Expressions

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x+5$, $\frac{4}{3}\pi r^3$, or $\sqrt{2m^3n^2}$. In the expression $x+5$, 5 is called a **constant** because it does not vary and x is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.

$$\begin{aligned} (-3)^5 &= (-3)\cdot(-3)\cdot(-3)\cdot(-3)\cdot(-3) & x^5 &= x\cdot x\cdot x\cdot x\cdot x\\ (2\cdot 7)^3 &= (2\cdot 7)\cdot(2\cdot 7)\cdot(2\cdot 7) & (yz)^3 &= (yz)\cdot(yz)\cdot(yz) \end{aligned}$$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

EXAMPLE 8

Describing Algebraic Expressions

List the constants and variables for each algebraic expression.

ⓐ $x + 5$ ⓑ $\frac{4}{3}\pi r^3$ ⓒ $\sqrt{2m^3 n^2}$

Solution

	Constants	Variables
a. $x + 5$	5	x
b. $\frac{4}{3}\pi r^3$	$\frac{4}{3}, \pi$	r
c. $\sqrt{2m^3 n^2}$	2	m, n

TRY IT #8 List the constants and variables for each algebraic expression.

ⓐ $2\pi r\,(r + h)$ ⓑ $2(L + W)$ ⓒ $4y^3 + y$

EXAMPLE 9

Evaluating an Algebraic Expression at Different Values

Evaluate the expression $2x - 7$ for each value for x.

ⓐ $x = 0$ ⓑ $x = 1$ ⓒ $x = \frac{1}{2}$ ⓓ $x = -4$

Solution

ⓐ Substitute 0 for x.

$$\begin{aligned} 2x - 7 &= 2(0) - 7 \\ &= 0 - 7 \\ &= -7 \end{aligned}$$

ⓑ Substitute 1 for x.

$$\begin{aligned} 2x - 7 &= 2(1) - 7 \\ &= 2 - 7 \\ &= -5 \end{aligned}$$

ⓒ Substitute $\frac{1}{2}$ for x.

$$\begin{aligned} 2x - 7 &= 2\left(\frac{1}{2}\right) - 7 \\ &= 1 - 7 \\ &= -6 \end{aligned}$$

ⓓ Substitute -4 for x.

$$\begin{aligned} 2x - 7 &= 2(-4) - 7 \\ &= -8 - 7 \\ &= -15 \end{aligned}$$

TRY IT #9 Evaluate the expression $11 - 3y$ for each value for y.

ⓐ $y = 2$ ⓑ $y = 0$ ⓒ $y = \frac{2}{3}$ ⓓ $y = -5$

EXAMPLE 10

Evaluating Algebraic Expressions

Evaluate each expression for the given values.

ⓐ $x + 5$ for $x = -5$ ⓑ $\frac{t}{2t-1}$ for $t = 10$ ⓒ $\frac{4}{3}\pi r^3$ for $r = 5$ ⓓ $a + ab + b$ for $a = 11, b = -8$

ⓔ $\sqrt{2m^3 n^2}$ for $m = 2, n = 3$

Solution

ⓐ Substitute -5 for x.

$$\begin{aligned} x + 5 &= (-5) + 5 \\ &= 0 \end{aligned}$$

ⓑ Substitute 10 for t.

$$\begin{aligned} \frac{t}{2t-1} &= \frac{(10)}{2(10)-1} \\ &= \frac{10}{20-1} \\ &= \frac{10}{19} \end{aligned}$$

ⓒ Substitute 5 for r.

$$\begin{aligned} \frac{4}{3}\pi r^3 &= \frac{4}{3}\pi(5)^3 \\ &= \frac{4}{3}\pi(125) \\ &= \frac{500}{3}\pi \end{aligned}$$

ⓓ Substitute 11 for a and –8 for b.

$$\begin{aligned} a + ab + b &= (11) + (11)(-8) + (-8) \\ &= 11 - 88 - 8 \\ &= -85 \end{aligned}$$

ⓔ Substitute 2 for m and 3 for n.

$$\begin{aligned} \sqrt{2m^3n^2} &= \sqrt{2(2)^3(3)^2} \\ &= \sqrt{2(8)(9)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

TRY IT #10 Evaluate each expression for the given values.

ⓐ $\frac{y+3}{y-3}$ for $y = 5$ ⓑ $7 - 2t$ for $t = -2$ ⓒ $\frac{1}{3}\pi r^2$ for $r = 11$

ⓓ $\left(p^2 q\right)^3$ for $p = -2, q = 3$ ⓔ $4\,(m - n) - 5\,(n - m)$ for $m = \frac{2}{3}, n = \frac{1}{3}$

Formulas

An **equation** is a mathematical statement indicating that two expressions are equal. The expressions can be numerical or algebraic. The equation is not inherently true or false, but only a proposition. The values that make the equation true, the solutions, are found using the properties of real numbers and other results. For example, the equation $2x + 1 = 7$ has the solution of 3 because when we substitute 3 for x in the equation, we obtain the true statement $2\,(3) + 1 = 7$.

A **formula** is an equation expressing a relationship between constant and variable quantities. Very often, the equation is a means of finding the value of one quantity (often a single variable) in terms of another or other quantities. One of the most common examples is the formula for finding the area A of a circle in terms of the radius r of the circle: $A = \pi r^2$. For any value of r, the area A can be found by evaluating the expression πr^2.

EXAMPLE 11

Using a Formula

A right circular cylinder with radius r and height h has the surface area S (in square units) given by the formula $S = 2\pi r\,(r + h)$. See Figure 3. Find the surface area of a cylinder with radius 6 in. and height 9 in. Leave the answer in terms of π.

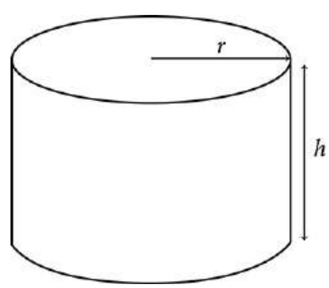


Figure 3 Right circular cylinder

Solution

Evaluate the expression $2\pi r\,(r + h)$ for $r = 6$ and $h = 9$.

$$\begin{aligned} S &= 2\pi r(r + h) \\ &= 2\pi(6)[(6) + (9)] \\ &= 2\pi(6)(15) \\ &= 180\pi \end{aligned}$$

The surface area is 180π square inches.

TRY IT #11 A photograph with length L and width W is placed in a mat of width 8 centimeters (cm). The area of the mat (in square centimeters, or cm^2) is found to be $A = (L + 16)\,(W + 16) - L \cdot W$. See Figure 4. Find the area of a mat for a photograph with length 32 cm and width 24 cm.

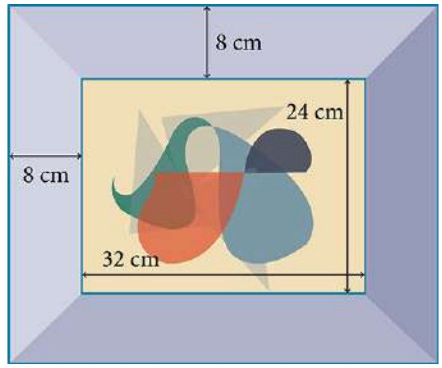


Figure 4

Simplifying Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

EXAMPLE 12

Simplifying Algebraic Expressions

Simplify each algebraic expression.

ⓐ $3x - 2y + x - 3y - 7$ ⓑ $2r - 5(3 - r) + 4$ ⓒ $\left(4t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right)$ ⓓ $2mn - 5m + 3mn + n$

Solution

ⓐ

$$\begin{aligned} 3x - 2y + x - 3y - 7 &= 3x + x - 2y - 3y - 7 && \text{Commutative property of addition.} \\ &= 4x - 5y - 7 && \text{Simplify.} \end{aligned}$$

ⓑ

$$\begin{aligned} 2r - 5(3 - r) + 4 &= 2r - 15 + 5r + 4 && \text{Distributive property.} \\ &= 2r + 5r - 15 + 4 && \text{Commutative property of addition.} \\ &= 7r - 11 && \text{Simplify.} \end{aligned}$$

ⓒ

$$\begin{aligned} \left(4t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right) &= 4t - \frac{5}{4}s - \frac{2}{3}t - 2s && \text{Distributive property.} \\ &= 4t - \frac{2}{3}t - \frac{5}{4}s - 2s && \text{Commutative property of addition.} \\ &= \frac{10}{3}t - \frac{13}{4}s && \text{Simplify.} \end{aligned}$$

ⓓ

$$\begin{aligned} 2mn - 5m + 3mn + n &= 2mn + 3mn - 5m + n && \text{Commutative property of addition.} \\ &= 5mn - 5m + n && \text{Simplify.} \end{aligned}$$

TRY IT #12 Simplify each algebraic expression.

ⓐ $\frac{2}{3}y - 2\left(\frac{4}{3}y + z\right)$ ⓑ $\frac{5}{t} - 2 - \frac{3}{t} + 1$ ⓒ $4p(q - 1) + q(1 - p)$
ⓓ $9r - (s + 2r) + (6 - s)$

EXAMPLE 13

Simplifying a Formula

A rectangle with length L and width W has a perimeter P given by $P = L + W + L + W$. Simplify this expression.

Solution

$$\begin{aligned} P &= L + W + L + W \\ P &= L + L + W + W && \text{Commutative property of addition} \\ P &= 2L + 2W && \text{Simplify} \\ P &= 2(L + W) && \text{Distributive property} \end{aligned}$$

TRY IT #13 If the amount P is deposited into an account paying simple interest r for time t, the total value of the deposit A is given by $A = P + Prt$. Simplify the expression. (This formula will be explored in more detail later in the course.)

MEDIA

Access these online resources for additional instruction and practice with real numbers.

[Simplify an Expression. (http://openstax.org/l/simexpress)](http://openstax.org/l/simexpress)
[Evaluate an Expression 1. (http://openstax.org/l/ordofoper1)](http://openstax.org/l/ordofoper1)
[Evaluate an Expression 2. (http://openstax.org/l/ordofoper2)](http://openstax.org/l/ordofoper2)

1.1 SECTION EXERCISES

Verbal

1. Is $\sqrt{2}$ an example of a rational terminating, rational repeating, or irrational number? Tell why it fits that category.

2. What is the order of operations? What acronym is used to describe the order of operations, and what does it stand for?

3. What do the Associative Properties allow us to do when following the order of operations? Explain your answer.

Numeric

For the following exercises, simplify the given expression.

4. $10 + 2 \times (5 - 3)$

5. $6 \div 2 - \left(81 \div 3^2\right)$

6. $18 + (6 - 8)^3$

7. $-2 \times \left[16 \div (8 - 4)^2\right]^2$

8. $4 - 6 + 2 \times 7$

9. $3(5 - 8)$

10. $4 + 6 - 10 \div 2$

11. $12 \div (36 \div 9) + 6$

12. $(4 + 5)^2 \div 3$

13. $3 - 12 \times 2 + 19$

14. $2 + 8 \times 7 \div 4$

15. $5 + (6 + 4) - 11$

16. $9 - 18 \div 3^2$

17. $14 \times 3 \div 7 - 6$

18. $9 - (3 + 11) \times 2$

19. $6 + 2 \times 2 - 1$

20. $64 \div (8 + 4 \times 2)$

21. $9 + 4\left(2^2\right)$

22. $(12 \div 3 \times 3)^2$

23. $25 \div 5^2 - 7$

24. $(15 - 7) \times (3 - 7)$

25. $2 \times 4 - 9(-1)$

26. $4^2 - 25 \times \frac{1}{5}$

27. $12(3 - 1) \div 6$

Algebraic

For the following exercises, evaluate the expressions using the given variable.

28. $8(x+3)–64$ for $x=2$

29. $4y+8–2y$ for $y=3$

30. $(11a+3)-18a+4$ for $a=–2$

31. $4z-2z(1+4)–36$ for $z=5$

32. $4y(7-2)^2+200$ for $y=–2$

33. $-(2x)^2+1+3$ for $x=2$

34. For the $8\,(2+4)-15b+b$ for $b=–3$

35. $2(11c-4)–36$ for $c=0$

36. $4(3-1)x–4$ for $x=10$

37. $\frac{1}{4}\left(8w-4^2\right)$ for $w=1$

For the following exercises, simplify the expression.

38. $4x+x\,(13-7)$

39. $2y-(4)^2y-11$

40. $\frac{a}{2^3}(64)-12a\div 6$

41. $8b-4b\,(3)+1$

42. $5l\div 3l \times (9-6)$

43. $7z-3+z \times 6^2$

44. $4 \times 3+18x\div 9-12$

45. $9\,(y+8)-27$

46. $\left(\frac{9}{6}t-4\right)2$

47. $6+12b-3 \times 6b$

48. $18y-2\,(1+7y)$

49. $\left(\frac{4}{9}\right)^2 \times 27x$

50. $8\,(3-m)+1\,(-8)$

51. $9x+4x\,(2+3)-4\,(2x+3x)$

52. $5^2-4\,(3x)$

Real-World Applications

For the following exercises, consider this scenario: Fred earns $40 at the community garden. He spends $10 on a streaming subscription, puts half of what is left in a savings account, and gets another $5 for walking his neighbor's dog.

53. Write the expression that represents the number of dollars Fred keeps (and does not put in his savings account). Remember the order of operations.

54. How much money does Fred keep?

For the following exercises, solve the given problem.

55. According to the U.S. Mint, the diameter of a quarter is 0.955 inches. The circumference of the quarter would be the diameter multiplied by π. Is the circumference of a quarter a whole number, a rational number, or an irrational number?

56. Jessica and her roommate, Adriana, have decided to share a change jar for joint expenses. Jessica put her loose change in the jar first, and then Adriana put her change in the jar. We know that it does not matter in which order the change was added to the jar. What property of addition describes this fact?

For the following exercises, consider this scenario: There is a mound of g pounds of gravel in a quarry. Throughout the day, 400 pounds of gravel is added to the mound. Two orders of 600 pounds are sold and the gravel is removed from the mound. At the end of the day, the mound has 1,200 pounds of gravel.

57. Write the equation that describes the situation.

58. Solve for *g*.

For the following exercise, solve the given problem.

59. Ramon runs the marketing department at their company. Their department gets a budget every year, and every year, they must spend the entire budget without going over. If they spend less than the budget, then the department gets a smaller budget the following year. At the beginning of this year, Ramon got $2.5 million for the annual marketing budget. They must spend the budget such that $2{,}500{,}000 - x = 0$. What property of addition tells us what the value of *x* must be?

Technology

For the following exercises, use a graphing calculator to solve for x. Round the answers to the nearest hundredth.

60. $0.5(12.3)^2 - 48x = \frac{3}{5}$

61. $(0.25 - 0.75)^2 x - 7.2 = 9.9$

Extensions

62. If a whole number is not a natural number, what must the number be?

63. Determine whether the statement is true or false: The multiplicative inverse of a rational number is also rational.

64. Determine whether the statement is true or false: The product of a rational and irrational number is always irrational.

65. Determine whether the simplified expression is rational or irrational: $\sqrt{-18 - 4(5)(-1)}$.

66. Determine whether the simplified expression is rational or irrational: $\sqrt{-16 + 4(5) + 5}$.

67. The division of two natural numbers will always result in what type of number?

68. What property of real numbers would simplify the following expression: $4 + 7(x - 1)$?

1.2 Exponents and Scientific Notation

Learning Objectives

In this section, you will:

- Use the product rule of exponents.
- Use the quotient rule of exponents.
- Use the power rule of exponents.
- Use the zero exponent rule of exponents.
- Use the negative rule of exponents.
- Find the power of a product and a quotient.
- Simplify exponential expressions.
- Use scientific notation.

COREQUISITE SKILLS

Learning Objective:

- Plan your weekly academic schedule for the term.

Objective 1: Plan your weekly academic schedule for the term.

1. Most college instructors advocate studying at least 2 hours for each hour in class. With this recommendation in mind, complete the following table showing credit hours enrolled in, the study time required, and total time to be devoted to college work. Assume 2 hours of study time for each hour in class to complete this table, and after your first exam you can fine tune this estimate based on your performance.

Credit hours (hours in class)	Study time outside of class	Total time spent in class and studying
9	(2,2)	(2,3)
12	(3,2)	(3,3)
15	(3,2)	(3,3)
18	(4,2)	(4,3)
21	(5,2)	(5,3)

Consider spending at least 2 hours of your study time each week at your campus (or virtual) math tutoring center or with a study group, the time will be well spent!

2. Another way to optimize your class and study time is to have a plan for efficiency, meaning make every minute count. Below is a list of good practices, check off those you feel you could utilize this term.

Best practices:	Will consider:	Not for me:
1. Attend each class session. It will take much more time to teach yourself the content.		
2. Ask your instructor. If you are unsure of a concept being taught in class, ask for clarification right away. Your instructor is an expert in their field and can provide the most efficient path to understanding.		

Best practices:	Will consider:	Not for me:
3. Be prepared for each class. Having completed prior assignments can go a long way in math understanding since mastery of most learning objectives depends on knowledge of prior concepts. Also, reading through a section prior to class will help to make concepts much clearer.		
4. Stay organized. Keeping your math materials in a 3-ring binder organized by lecture notes, class handouts, PowerPoint slides, and homework problems will save you time in finding materials when you need them. Having two spiral notebooks dedicated to math works well too, use one for class notes and one for homework assignments.		
5. Find a study partner. Making a connection either in class or virtually with a fellow student can save time in that now there are two sources for gathering important information. If you have to miss class or an online session for an important appointment, your study partner can provide you class notes, share in-class handouts, or relay announcements for your instructor. Study partner's name: Study partner's number: Study partner's email address:		
6. Begin exam review time by reworking each of the examples your instructor worked in class. Your instructor will emphasize the same topics in both lecture and on exams based on student learning objectives required by your college or university or even the state where the course is offered. Follow their lead in assigning importance to an objective and master these topics first.		

3. Creating your Semester Calendar- complete the following weekly schedule being sure to label
    - time in classes
    - study time for classes
    - time at work.

   Optional: also include if you want a more comprehensive view of your time commitments

    - time spent exercising
    - time with family and friends.

Term: ___________________________

Name: __________________________________

Date: __________________________________

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
6:30-7:00am							
7:00-7:30am							
7:30-8am							
8-8:30am							

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
8:30-9am							
9-9:30am							
9:30-10am							
10-10:30am							
10:30-11am							
11-11:30am							
11:30-12pm							
12-12:30pm							
12:30-1pm							
1-1:30pm							
1:30-2pm							
2-2:30pm							
2:30-3pm							
3-3:30pm							
3:30-4pm							
4-4:30pm							
4:30-5pm							
5-5:30pm							
5:30-6pm							
6-6:30pm							
6:30-7pm							
7-7:30pm							
7:30-8pm							
8-8:30pm							
8:30-9pm							

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
9-9:30pm							
9:30-10pm							
10-10:30pm							
10:30-11pm							
11-11:30pm							

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is 2,048 pixels by 1,536 pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour (3,600-second) digital film is then an extremely large number.

Using a calculator, we enter $2{,}048 \times 1{,}536 \times 48 \times 24 \times 3{,}600$ and press ENTER. The calculator displays 1.304596316E13. What does this mean? The "E13" portion of the result represents the exponent 13 of ten, so there are a maximum of approximately 1.3×10^{13} bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

Using the Product Rule of Exponents

Consider the product $x^3 \cdot x^4$. Both terms have the same base, *x*, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$\begin{aligned} x^3 \cdot x^4 &= \overbrace{x \cdot x \cdot x}^{3 \text{ factors}} \cdot \overbrace{x \cdot x \cdot x \cdot x}^{4 \text{ factors}} \\ &= \overbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}^{7 \text{ factors}} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \cdot x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the *product rule of exponents.*

$$a^m \cdot a^n = a^{m+n}$$

Now consider an example with real numbers.

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product $8 \cdot 16$ equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

The Product Rule of Exponents

For any real number a and natural numbers m and n, the product rule of exponents states that

$$a^m \cdot a^n = a^{m+n}$$

EXAMPLE 1

Using the Product Rule

Write each of the following products with a single base. Do not simplify further.

ⓐ $t^5 \cdot t^3$ ⓑ $(-3)^5 \cdot (-3)$ ⓒ $x^2 \cdot x^5 \cdot x^3$

Solution

Use the product rule to simplify each expression.

ⓐ $t^5 \cdot t^3 = t^{5+3} = t^8$ ⓑ $(-3)^5 \cdot (-3) = (-3)^5 \cdot (-3)^1 = (-3)^{5+1} = (-3)^6$ ⓒ $x^2 \cdot x^5 \cdot x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$x^2 \cdot x^5 \cdot x^3 = \left(x^2 \cdot x^5\right) \cdot x^3 = \left(x^{2+5}\right) \cdot x^3 = x^7 \cdot x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.

$$x^2 \cdot x^5 \cdot x^3 = x^{2+5+3} = x^{10}$$

TRY IT #1 Write each of the following products with a single base. Do not simplify further.

ⓐ $k^6 \cdot k^9$ ⓑ $\left(\frac{2}{y}\right)^4 \cdot \left(\frac{2}{y}\right)$ ⓒ $t^3 \cdot t^6 \cdot t^5$

Using the Quotient Rule of Exponents

The *quotient rule of exponents* allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$, where $m > n$. Consider the example $\frac{y^9}{y^5}$. Perform the division by canceling common factors.

$$\begin{aligned}\frac{y^9}{y^5} &= \frac{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}{y \cdot y \cdot y \cdot y \cdot y} \\ &= \frac{\cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot y \cdot y \cdot y \cdot y}{\cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y}} \\ &= \frac{y \cdot y \cdot y \cdot y}{1} \\ &= y^4\end{aligned}$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.

$$\frac{a^m}{a^n} = a^{m-n}$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

For the time being, we must be aware of the condition $m > n$. Otherwise, the difference $m - n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

The Quotient Rule of Exponents

For any real number a and natural numbers m and n, such that $m > n$, the quotient rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n}$$

EXAMPLE 2

Using the Quotient Rule

Write each of the following products with a single base. Do not simplify further.

ⓐ $\frac{(-2)^{14}}{(-2)^{9}}$ ⓑ $\frac{t^{23}}{t^{15}}$ ⓒ $\frac{\left(z\sqrt{2}\right)^{5}}{z\sqrt{2}}$

Solution

Use the quotient rule to simplify each expression.

ⓐ $\frac{(-2)^{14}}{(-2)^{9}} = (-2)^{14-9} = (-2)^{5}$ ⓑ $\frac{t^{23}}{t^{15}} = t^{23-15} = t^{8}$ ⓒ $\frac{\left(z\sqrt{2}\right)^{5}}{z\sqrt{2}} = \left(z\sqrt{2}\right)^{5-1} = \left(z\sqrt{2}\right)^{4}$

TRY IT #2 Write each of the following products with a single base. Do not simplify further.

ⓐ $\frac{s^{75}}{s^{68}}$ ⓑ $\frac{(-3)^{6}}{-3}$ ⓒ $\frac{\left(ef^{2}\right)^{5}}{\left(ef^{2}\right)^{3}}$

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the *power rule of exponents*. Consider the expression $\left(x^2\right)^3$. The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

$$\begin{aligned}
\left(x^2\right)^3 &= \overbrace{\left(x^2\right)\cdot\left(x^2\right)\cdot\left(x^2\right)}^{3\ \text{factors}} \\
&= \overbrace{\left(\overbrace{x\cdot x}^{2\ \text{factors}}\right)\cdot\left(\overbrace{x\cdot x}^{2\ \text{factors}}\right)\cdot\left(\overbrace{x\cdot x}^{2\ \text{factors}}\right)}^{3\ \text{factors}} \\
&= x\cdot x\cdot x\cdot x\cdot x\cdot x \\
&= x^6
\end{aligned}$$

The exponent of the answer is the product of the exponents: $\left(x^2\right)^3 = x^{2\cdot 3} = x^6$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.

$$\left(a^m\right)^n = a^{m\cdot n}$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

$$\begin{array}{rclclcrclcl}
 & & \text{Product Rule} & & & & & & \text{Power Rule} & & \\
5^3\cdot 5^4 & = & 5^{3+4} & = & 5^7 & \text{but} & \left(5^3\right)^4 & = & 5^{3\cdot 4} & = & 5^{12} \\
x^5\cdot x^2 & = & x^{5+2} & = & x^7 & \text{but} & \left(x^5\right)^2 & = & x^{5\cdot 2} & = & x^{10} \\
(3a)^7\cdot(3a)^{10} & = & (3a)^{7+10} & = & (3a)^{17} & \text{but} & \left((3a)^7\right)^{10} & = & (3a)^{7\cdot 10} & = & (3a)^{70}
\end{array}$$

The Power Rule of Exponents

For any real number a and positive integers m and n, the power rule of exponents states that

$$\left(a^m\right)^n = a^{m\cdot n}$$

EXAMPLE 3

Using the Power Rule

Write each of the following products with a single base. Do not simplify further.

ⓐ $\left(x^2\right)^7$ ⓑ $\left((2t)^5\right)^3$ ⓒ $\left((-3)^5\right)^{11}$

Solution

Use the power rule to simplify each expression.

ⓐ $\left(x^2\right)^7 = x^{2\cdot 7} = x^{14}$ ⓑ $\left((2t)^5\right)^3 = (2t)^{5\cdot 3} = (2t)^{15}$ ⓒ $\left((-3)^5\right)^{11} = (-3)^{5\cdot 11} = (-3)^{55}$

TRY IT #3 Write each of the following products with a single base. Do not simplify further.

ⓐ $\left((3y)^8\right)^3$ ⓑ $\left(t^5\right)^7$ ⓒ $\left((-g)^4\right)^4$

Using the Zero Exponent Rule of Exponents

Return to the quotient rule. We made the condition that $m > n$ so that the difference $m - n$ would never be zero or negative. What would happen if $m = n$? In this case, we would use the *zero exponent rule of exponents* to simplify the expression to 1. To see how this is done, let us begin with an example.

$$\frac{t^8}{t^8} = \frac{\cancel{t^8}}{\cancel{t^8}} = 1$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number, or any variable representing a real number.

$$a^0 = 1$$

The sole exception is the expression 0^0. This appears later in more advanced courses, but for now, we will consider the value to be undefined.

The Zero Exponent Rule of Exponents

For any nonzero real number a, the zero exponent rule of exponents states that

$$a^0 = 1$$

EXAMPLE 4

Using the Zero Exponent Rule

Simplify each expression using the zero exponent rule of exponents.

ⓐ $\frac{c^3}{c^3}$ ⓑ $\frac{-3x^5}{x^5}$ ⓒ $\frac{\left(j^2k\right)^4}{\left(j^2k\right)\cdot\left(j^2k\right)^3}$ ⓓ $\frac{5\left(rs^2\right)^2}{\left(rs^2\right)^2}$

Solution

Use the zero exponent and other rules to simplify each expression.

ⓐ

$$\begin{aligned}\frac{c^3}{c^3} &= c^{3-3}\\ &= c^0\\ &= 1\end{aligned}$$

ⓑ

$$\begin{aligned}\frac{-3x^5}{x^5} &= -3\cdot\frac{x^5}{x^5}\\ &= -3\cdot x^{5-5}\\ &= -3\cdot x^0\\ &= -3\cdot 1\\ &= -3\end{aligned}$$

ⓒ

$$\begin{aligned}\frac{\left(j^2k\right)^4}{\left(j^2k\right)\cdot\left(j^2k\right)^3} &= \frac{\left(j^2k\right)^4}{\left(j^2k\right)^{1+3}} && \text{Use the product rule in the denominator.}\\ &= \frac{\left(j^2k\right)^4}{\left(j^2k\right)^4} && \text{Simplify.}\\ &= \left(j^2k\right)^{4-4} && \text{Use the quotient rule.}\\ &= \left(j^2k\right)^0 && \text{Simplify.}\\ &= 1\end{aligned}$$

ⓓ

$$\begin{aligned}\frac{5\left(rs^2\right)^2}{\left(rs^2\right)^2} &= 5\left(rs^2\right)^{2-2} && \text{Use the quotient rule.}\\ &= 5\left(rs^2\right)^0 && \text{Simplify.}\\ &= 5\cdot 1 && \text{Use the zero exponent rule.}\\ &= 5 && \text{Simplify.}\end{aligned}$$

> TRY IT #4 Simplify each expression using the zero exponent rule of exponents.

ⓐ $\frac{t^7}{t^7}$ ⓑ $\frac{\left(de^2\right)^{11}}{2\left(de^2\right)^{11}}$ ⓒ $\frac{w^4\cdot w^2}{w^6}$ ⓓ $\frac{t^3\cdot t^4}{t^2\cdot t^5}$

Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m > n$ in the quotient rule even further. For example, can we simplify $\frac{h^3}{h^5}$? When $m < n$ —that is, where the difference $m - n$ is negative—we can use the *negative rule of exponents* to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{h^3}{h^5}$.

$$\begin{aligned}\frac{h^3}{h^5} &= \frac{h\cdot h\cdot h}{h\cdot h\cdot h\cdot h\cdot h}\\ &= \frac{\cancel{h}\cdot\cancel{h}\cdot\cancel{h}}{\cancel{h}\cdot\cancel{h}\cdot\cancel{h}\cdot h\cdot h}\\ &= \frac{1}{h\cdot h}\\ &= \frac{1}{h^2}\end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\begin{aligned}\frac{h^3}{h^5} &= h^{3-5}\\ &= h^{-2}\end{aligned}$$

Putting the answers together, we have $h^{-2} = \frac{1}{h^2}$. This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression a^n is defined when n is a natural number, 0, or the negative of a natural number. That means that a^n is defined for any integer n. Also, the product and quotient rules and all of the rules we will look at soon hold for any integer n.

The Negative Rule of Exponents

For any nonzero real number a and natural number n, the negative rule of exponents states that

$$a^{-n} = \frac{1}{a^n}$$

EXAMPLE 5

Using the Negative Exponent Rule

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

ⓐ $\frac{\theta^3}{\theta^{10}}$ ⓑ $\frac{z^2 \cdot z}{z^4}$ ⓒ $\frac{\left(-5t^3\right)^4}{\left(-5t^3\right)^8}$

Solution

ⓐ $\frac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \frac{1}{\theta^7}$ ⓑ $\frac{z^2 \cdot z}{z^4} = \frac{z^{2+1}}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$

ⓒ $\frac{\left(-5t^3\right)^4}{\left(-5t^3\right)^8} = \left(-5t^3\right)^{4-8} = \left(-5t^3\right)^{-4} = \frac{1}{\left(-5t^3\right)^4}$

TRY IT #5 Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

ⓐ $\frac{(-3t)^2}{(-3t)^8}$ ⓑ $\frac{f^{47}}{f^{49} \cdot f}$ ⓒ $\frac{2k^4}{5k^7}$

EXAMPLE 6

Using the Product and Quotient Rules

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

ⓐ $b^2 \cdot b^{-8}$ ⓑ $(-x)^5 \cdot (-x)^{-5}$ ⓒ $\frac{-7z}{(-7z)^5}$

Solution

ⓐ $b^2 \cdot b^{-8} = b^{2-8} = b^{-6} = \frac{1}{b^6}$ ⓑ $(-x)^5 \cdot (-x)^{-5} = (-x)^{5-5} = (-x)^0 = 1$

ⓒ $\frac{-7z}{(-7z)^5} = \frac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \frac{1}{(-7z)^4}$

TRY IT #6 Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

ⓐ $t^{-11} \cdot t^6$ ⓑ $\frac{25^{12}}{25^{13}}$

Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the *power of a product rule of exponents,* which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider $(pq)^3$. We begin by using the associative and commutative properties of multiplication to regroup the factors.

$$\begin{aligned}(pq)^3 &= \overbrace{(pq)\cdot(pq)\cdot(pq)}^{3 \text{ factors}} \\ &= p\cdot q\cdot p\cdot q\cdot p\cdot q \\ &= \overbrace{p\cdot p\cdot p}^{3 \text{ factors}}\cdot\overbrace{q\cdot q\cdot q}^{3 \text{ factors}} \\ &= p^3\cdot q^3\end{aligned}$$

In other words, $(pq)^3 = p^3 \cdot q^3$.

The Power of a Product Rule of Exponents

For any real numbers a and b and any integer n, the power of a product rule of exponents states that

$$(ab)^n = a^n b^n$$

EXAMPLE 7

Using the Power of a Product Rule

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

ⓐ $\left(ab^2\right)^3$ ⓑ $(2t)^{15}$ ⓒ $\left(-2w^3\right)^3$ ⓓ $\frac{1}{(-7z)^4}$ ⓔ $\left(e^{-2}f^2\right)^7$

Solution

Use the product and quotient rules and the new definitions to simplify each expression.

ⓐ $\left(ab^2\right)^3 = (a)^3\cdot\left(b^2\right)^3 = a^{1\cdot 3}\cdot b^{2\cdot 3} = a^3b^6$ ⓑ $(2t)^{15} = (2)^{15}\cdot(t)^{15} = 2^{15}t^{15} = 32,768t^{15}$

ⓒ $\left(-2w^3\right)^3 = (-2)^3\cdot\left(w^3\right)^3 = -8\cdot w^{3\cdot 3} = -8w^9$ ⓓ $\frac{1}{(-7z)^4} = \frac{1}{(-7)^4\cdot(z)^4} = \frac{1}{2{,}401z^4}$

ⓔ $\left(e^{-2}f^2\right)^7 = \left(e^{-2}\right)^7\cdot\left(f^2\right)^7 = e^{-2\cdot 7}\cdot f^{2\cdot 7} = e^{-14}f^{14} = \frac{f^{14}}{e^{14}}$

TRY IT #7 Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

ⓐ $\left(g^2h^3\right)^5$ ⓑ $(5t)^3$ ⓒ $\left(-3y^5\right)^3$ ⓓ $\frac{1}{\left(a^6b^7\right)^3}$ ⓔ $\left(r^3s^{-2}\right)^4$

Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the *power of a quotient rule,* which states that the power of a quotient of factors is the quotient of the powers of the factors. For example, let's look at the following example.

$$\left(e^{-2}f^2\right)^7 = \frac{f^{14}}{e^{14}}$$

Let's rewrite the original problem differently and look at the result.

$$\begin{aligned}\left(e^{-2}f^2\right)^7 &= \left(\frac{f^2}{e^2}\right)^7 \\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

$$\begin{aligned}(e^{-2}f^{2})^{7} &= \left(\frac{f^{2}}{e^{2}}\right)^{7}\\ &= \frac{(f^{2})^{7}}{(e^{2})^{7}}\\ &= \frac{f^{2\cdot 7}}{e^{2\cdot 7}}\\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

The Power of a Quotient Rule of Exponents

For any real numbers a and b and any integer n, the power of a quotient rule of exponents states that

$$\left(\frac{a}{b}\right)^{n} = \frac{a^{n}}{b^{n}}$$

EXAMPLE 8

Using the Power of a Quotient Rule

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

ⓐ $\left(\frac{4}{z^{11}}\right)^{3}$ ⓑ $\left(\frac{p}{q^{3}}\right)^{6}$ ⓒ $\left(\frac{-1}{t^{2}}\right)^{27}$ ⓓ $\left(j^{3}k^{-2}\right)^{4}$ ⓔ $\left(m^{-2}n^{-2}\right)^{3}$

Solution

ⓐ $\left(\frac{4}{z^{11}}\right)^{3} = \frac{(4)^{3}}{\left(z^{11}\right)^{3}} = \frac{64}{z^{11\cdot 3}} = \frac{64}{z^{33}}$ ⓑ $\left(\frac{p}{q^{3}}\right)^{6} = \frac{(p)^{6}}{\left(q^{3}\right)^{6}} = \frac{p^{1\cdot 6}}{q^{3\cdot 6}} = \frac{p^{6}}{q^{18}}$

ⓒ $\left(\frac{-1}{t^{2}}\right)^{27} = \frac{(-1)^{27}}{\left(t^{2}\right)^{27}} = \frac{-1}{t^{2\cdot 27}} = \frac{-1}{t^{54}} = -\frac{1}{t^{54}}$ ⓓ $\left(j^{3}k^{-2}\right)^{4} = \left(\frac{j^{3}}{k^{2}}\right)^{4} = \frac{\left(j^{3}\right)^{4}}{\left(k^{2}\right)^{4}} = \frac{j^{3\cdot 4}}{k^{2\cdot 4}} = \frac{j^{12}}{k^{8}}$

ⓔ $\left(m^{-2}n^{-2}\right)^{3} = \left(\frac{1}{m^{2}n^{2}}\right)^{3} = \frac{(1)^{3}}{\left(m^{2}n^{2}\right)^{3}} = \frac{1}{\left(m^{2}\right)^{3}\left(n^{2}\right)^{3}} = \frac{1}{m^{2\cdot 3}\cdot n^{2\cdot 3}} = \frac{1}{m^{6}n^{6}}$

TRY IT #8 Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

ⓐ $\left(\frac{b^{5}}{c}\right)^{3}$ ⓑ $\left(\frac{5}{u^{8}}\right)^{4}$ ⓒ $\left(\frac{-1}{w^{3}}\right)^{35}$ ⓓ $\left(p^{-4}q^{3}\right)^{8}$ ⓔ $\left(c^{-5}d^{-3}\right)^{4}$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combing terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

EXAMPLE 9

Simplifying Exponential Expressions

Simplify each expression and write the answer with positive exponents only.

ⓐ $\left(6m^{2}n^{-1}\right)^{3}$ ⓑ $17^{5}\cdot 17^{-4}\cdot 17^{-3}$ ⓒ $\left(\frac{u^{-1}v}{v^{-1}}\right)^{2}$ ⓓ $\left(-2a^{3}b^{-1}\right)\left(5a^{-2}b^{2}\right)$

ⓔ $\left(x^{2}\sqrt{2}\right)^{4}\left(x^{2}\sqrt{2}\right)^{-4}$ ⓕ $\frac{\left(3w^{2}\right)^{5}}{\left(6w^{-2}\right)^{2}}$

Solution

ⓐ

$$\begin{aligned}\left(6m^2n^{-1}\right)^3 &= (6)^3\left(m^2\right)^3\left(n^{-1}\right)^3 && \text{The power of a product rule}\\ &= 6^3m^{2\cdot 3}n^{-1\cdot 3} && \text{The power rule}\\ &= 216m^6n^{-3} && \text{Simplify.}\\ &= \frac{216m^6}{n^3} && \text{The negative exponent rule}\end{aligned}$$

ⓑ

$$\begin{aligned}17^5\cdot 17^{-4}\cdot 17^{-3} &= 17^{5-4-3} && \text{The product rule}\\ &= 17^{-2} && \text{Simplify.}\\ &= \frac{1}{17^2}\text{ or }\frac{1}{289} && \text{The negative exponent rule}\end{aligned}$$

ⓒ

$$\begin{aligned}\left(\frac{u^{-1}v}{v^{-1}}\right)^2 &= \frac{(u^{-1}v)^2}{(v^{-1})^2} && \text{The power of a quotient rule}\\ &= \frac{u^{-2}v^2}{v^{-2}} && \text{The power of a product rule}\\ &= u^{-2}v^{2-(-2)} && \text{The quotient rule}\\ &= u^{-2}v^4 && \text{Simplify.}\\ &= \frac{v^4}{u^2} && \text{The negative exponent rule}\end{aligned}$$

ⓓ

$$\begin{aligned}\left(-2a^3b^{-1}\right)\left(5a^{-2}b^2\right) &= -2\cdot 5\cdot a^3\cdot a^{-2}\cdot b^{-1}\cdot b^2 && \text{Commutative and associative laws of multiplication}\\ &= -10\cdot a^{3-2}\cdot b^{-1+2} && \text{The product rule}\\ &= -10ab && \text{Simplify.}\end{aligned}$$

ⓔ

$$\begin{aligned}\left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4} &= \left(x^2\sqrt{2}\right)^{4-4} && \text{The product rule}\\ &= \left(x^2\sqrt{2}\right)^0 && \text{Simplify.}\\ &= 1 && \text{The zero exponent rule}\end{aligned}$$

ⓕ

$$\begin{aligned}\frac{(3w^2)^5}{(6w^{-2})^2} &= \frac{(3)^5\cdot(w^2)^5}{(6)^2\cdot(w^{-2})^2} && \text{The power of a product rule}\\ &= \frac{3^5w^{2\cdot 5}}{6^2w^{-2\cdot 2}} && \text{The power rule}\\ &= \frac{243w^{10}}{36w^{-4}} && \text{Simplify.}\\ &= \frac{27w^{10-(-4)}}{4} && \text{The quotient rule and reduce fraction}\\ &= \frac{27w^{14}}{4} && \text{Simplify.}\end{aligned}$$

TRY IT #9 Simplify each expression and write the answer with positive exponents only.

ⓐ $\left(2uv^{-2}\right)^{-3}$ ⓑ $x^8\cdot x^{-12}\cdot x$ ⓒ $\left(\frac{e^2f^{-3}}{f^{-1}}\right)^2$ ⓓ $\left(9r^{-5}s^3\right)\left(3r^6s^{-4}\right)$

ⓔ $\left(\frac{4}{9}tw^{-2}\right)^{-3}\left(\frac{4}{9}tw^{-2}\right)^3$ ⓕ $\frac{\left(2h^2k\right)^4}{\left(7h^{-1}k^2\right)^2}$

Using Scientific Notation

Recall at the beginning of the section that we found the number 1.3×10^{13} when describing bits of information in digital images. Other extreme numbers include the width of a human hair, which is about 0.00005 m, and the radius of an electron, which is about 0.00000000000047 m. How can we effectively work read, compare, and calculate with numbers such as these?

A shorthand method of writing very small and very large numbers is called **scientific notation**, in which we express numbers in terms of exponents of 10. To write a number in scientific notation, move the decimal point to the right of the first digit in the number. Write the digits as a decimal number between 1 and 10. Count the number of places *n* that you moved the decimal point. Multiply the decimal number by 10 raised to a power of *n*. If you moved the decimal left as in a very large number, n is positive. If you moved the decimal right as in a small large number, n is negative.

For example, consider the number 2,780,418. Move the decimal left until it is to the right of the first nonzero digit, which is 2.

$$2{,}780{,}418 \longrightarrow 2.\overbrace{780418}^{\text{6 places left}}$$

We obtain 2.780418 by moving the decimal point 6 places to the left. Therefore, the exponent of 10 is 6, and it is positive because we moved the decimal point to the left. This is what we should expect for a large number.

$$2.780418 \times 10^6$$

Working with small numbers is similar. Take, for example, the radius of an electron, 0.00000000000047 m. Perform the same series of steps as above, except move the decimal point to the right.

$$0.00000000000047 \longrightarrow \overbrace{0000000000004}^{\text{13 places right}}.7$$

Be careful not to include the leading 0 in your count. We move the decimal point 13 places to the right, so the exponent of 10 is 13. The exponent is negative because we moved the decimal point to the right. This is what we should expect for a small number.

$$4.7 \times 10^{-13}$$

Scientific Notation

A number is written in **scientific notation** if it is written in the form $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer.

EXAMPLE 10

Converting Standard Notation to Scientific Notation

Write each number in scientific notation.

ⓐ Distance to Andromeda Galaxy from Earth: 24,000,000,000,000,000,000,000 m

ⓑ Diameter of Andromeda Galaxy: 1,300,000,000,000,000,000,000 m

ⓒ Number of stars in Andromeda Galaxy: 1,000,000,000,000

ⓓ Diameter of electron: 0.00000000000094 m

ⓔ Probability of being struck by lightning in any single year: 0.00000143

Solution

ⓐ

$$\begin{array}{l} 24{,}000{,}000{,}000{,}000{,}000{,}000{,}000 \ \text{ m} \\ 24{,}000{,}000{,}000{,}000{,}000{,}000{,}000 \ \text{ m} \\ \qquad\qquad \leftarrow 22 \ \text{ places} \\ 2.4 \times 10^{22} \ \text{ m} \end{array}$$

ⓑ

$$1{,}300{,}000{,}000{,}000{,}000{,}000{,}000 \text{ m}$$
$$1{,}300{,}000{,}000{,}000{,}000{,}000{,}000 \text{ m}$$
$$\leftarrow 21 \text{ places}$$
$$1.3 \times 10^{21} \text{ m}$$

ⓒ
$$1{,}000{,}000{,}000{,}000$$
$$1{,}000{,}000{,}000{,}000$$
$$\leftarrow 12 \text{ places}$$
$$1 \times 10^{12}$$

ⓓ
$$0.00000000000094 \text{ m}$$
$$0.00000000000094 \text{ m}$$
$$\rightarrow 13 \text{ places}$$
$$9.4 \times 10^{-13} \text{ m}$$

ⓔ
$$0.00000143$$
$$0.00000143$$
$$\rightarrow 6 \text{ places}$$
$$1.43 \times 10^{-6}$$

Analysis

Observe that, if the given number is greater than 1, as in examples a–c, the exponent of 10 is positive; and if the number is less than 1, as in examples d–e, the exponent is negative.

TRY IT #10 Write each number in scientific notation.

ⓐ U.S. national debt per taxpayer (April 2014): \$152,000
ⓑ World population (April 2014): 7,158,000,000
ⓒ World gross national income (April 2014): \$85,500,000,000,000
ⓓ Time for light to travel 1 m: 0.00000000334 s
ⓔ Probability of winning lottery (match 6 of 49 possible numbers): 0.0000000715

Converting from Scientific to Standard Notation

To convert a number in **scientific notation** to standard notation, simply reverse the process. Move the decimal n places to the right if n is positive or n places to the left if n is negative and add zeros as needed. Remember, if n is positive, the value of the number is greater than 1, and if n is negative, the value of the number is less than one.

EXAMPLE 11

Converting Scientific Notation to Standard Notation

Convert each number in scientific notation to standard notation.

ⓐ 3.547×10^{14} ⓑ -2×10^{6} ⓒ 7.91×10^{-7} ⓓ -8.05×10^{-12}

Solution

ⓐ	ⓑ	ⓒ	ⓓ
3.547×10^{14}	-2×10^{6}	7.91×10^{-7}	-8.05×10^{-12}
3.54700000000000	-2.000000	0000007.91	-000000000008.05
$\rightarrow 14$ places	$\rightarrow 6$ places	$\rightarrow 7$ places	$\rightarrow 12$ places
$354{,}700{,}000{,}000{,}000$	$-2{,}000{,}000$	0.000000791	-0.00000000000805

TRY IT #11 Convert each number in scientific notation to standard notation.

ⓐ 7.03×10^5 ⓑ -8.16×10^{11} ⓒ -3.9×10^{-13} ⓓ 8×10^{-6}

Using Scientific Notation in Applications

Scientific notation, used with the rules of exponents, makes calculating with large or small numbers much easier than doing so using standard notation. For example, suppose we are asked to calculate the number of atoms in 1 L of water. Each water molecule contains 3 atoms (2 hydrogen and 1 oxygen). The average drop of water contains around 1.32×10^{21} molecules of water and 1 L of water holds about 1.22×10^4 average drops. Therefore, there are approximately $3 \cdot \left(1.32 \times 10^{21}\right) \cdot \left(1.22 \times 10^4\right) \approx 4.83 \times 10^{25}$ atoms in 1 L of water. We simply multiply the decimal terms and add the exponents. Imagine having to perform the calculation without using scientific notation!

When performing calculations with scientific notation, be sure to write the answer in proper scientific notation. For example, consider the product $\left(7 \times 10^4\right) \cdot \left(5 \times 10^6\right) = 35 \times 10^{10}$. The answer is not in proper scientific notation because 35 is greater than 10. Consider 35 as 3.5×10. That adds a ten to the exponent of the answer.

$$(35) \times 10^{10} = (3.5 \times 10) \times 10^{10} = 3.5 \times \left(10 \times 10^{10}\right) = 3.5 \times 10^{11}$$

EXAMPLE 12

Using Scientific Notation

Perform the operations and write the answer in scientific notation.

ⓐ $\left(8.14 \times 10^{-7}\right)\left(6.5 \times 10^{10}\right)$ ⓑ $\left(4 \times 10^5\right) \div \left(-1.52 \times 10^9\right)$ ⓒ $\left(2.7 \times 10^5\right)\left(6.04 \times 10^{13}\right)$
ⓓ $\left(1.2 \times 10^8\right) \div \left(9.6 \times 10^5\right)$ ⓔ $\left(3.33 \times 10^4\right)\left(-1.05 \times 10^7\right)\left(5.62 \times 10^5\right)$

Solution

ⓐ

$$\begin{aligned} \left(8.14 \times 10^{-7}\right)\left(6.5 \times 10^{10}\right) &= (8.14 \times 6.5)\left(10^{-7} \times 10^{10}\right) && \text{Commutative and associative properties of multiplication} \\ &= (52.91)\left(10^3\right) && \text{Product rule of exponents} \\ &= 5.291 \times 10^4 && \text{Scientific notation} \end{aligned}$$

ⓑ

$$\begin{aligned} \left(4 \times 10^5\right) \div \left(-1.52 \times 10^9\right) &= \left(\frac{4}{-1.52}\right)\left(\frac{10^5}{10^9}\right) && \text{Commutative and associative properties of multiplication} \\ &\approx (-2.63)\left(10^{-4}\right) && \text{Quotient rule of exponents} \\ &= -2.63 \times 10^{-4} && \text{Scientific notation} \end{aligned}$$

ⓒ

$$\begin{aligned} \left(2.7 \times 10^5\right)\left(6.04 \times 10^{13}\right) &= (2.7 \times 6.04)\left(10^5 \times 10^{13}\right) && \text{Commutative and associative properties of multiplication} \\ &= (16.308)\left(10^{18}\right) && \text{Product rule of exponents} \\ &= 1.6308 \times 10^{19} && \text{Scientific notation} \end{aligned}$$

ⓓ

$$\begin{aligned} \left(1.2 \times 10^8\right) \div \left(9.6 \times 10^5\right) &= \left(\frac{1.2}{9.6}\right)\left(\frac{10^8}{10^5}\right) && \text{Commutative and associative properties of multiplication} \\ &= (0.125)\left(10^3\right) && \text{Quotient rule of exponents} \\ &= 1.25 \times 10^2 && \text{Scientific notation} \end{aligned}$$

ⓔ

$$\begin{aligned} \left(3.33 \times 10^4\right)\left(-1.05 \times 10^7\right)\left(5.62 \times 10^5\right) &= [3.33 \times (-1.05) \times 5.62]\left(10^4 \times 10^7 \times 10^5\right) \\ &\approx (-19.65)\left(10^{16}\right) \\ &= -1.965 \times 10^{17} \end{aligned}$$

TRY IT #12 Perform the operations and write the answer in scientific notation.

ⓐ $\left(-7.5 \times 10^{8}\right)\left(1.13 \times 10^{-2}\right)$ ⓑ $\left(1.24 \times 10^{11}\right) \div \left(1.55 \times 10^{18}\right)$
ⓒ $\left(3.72 \times 10^{9}\right)\left(8 \times 10^{3}\right)$ ⓓ $\left(9.933 \times 10^{23}\right) \div \left(-2.31 \times 10^{17}\right)$
ⓔ $\left(-6.04 \times 10^{9}\right)\left(7.3 \times 10^{2}\right)\left(-2.81 \times 10^{2}\right)$

EXAMPLE 13

Applying Scientific Notation to Solve Problems

In April 2014, the population of the United States was about 308,000,000 people. The national debt was about $17,547,000,000,000. Write each number in scientific notation, rounding figures to two decimal places, and find the amount of the debt per U.S. citizen. Write the answer in both scientific and standard notations.

Solution

The population was $308{,}000{,}000 = 3.08 \times 10^{8}$.

The national debt was $\$17{,}547{,}000{,}000{,}000 \approx \$1.75 \times 10^{13}$.

To find the amount of debt per citizen, divide the national debt by the number of citizens.

$$\begin{aligned}\left(1.75 \times 10^{13}\right) \div \left(3.08 \times 10^{8}\right) &= \left(\frac{1.75}{3.08}\right) \cdot \left(\frac{10^{13}}{10^{8}}\right) \\ &\approx 0.57 \times 10^{5} \\ &= 5.7 \times 10^{4}\end{aligned}$$

The debt per citizen at the time was about $\$5.7 \times 10^{4}$, or $57,000.

TRY IT #13 An average human body contains around 30,000,000,000,000 red blood cells. Each cell measures approximately 0.000008 m long. Write each number in scientific notation and find the total length if the cells were laid end-to-end. Write the answer in both scientific and standard notations.

MEDIA

Access these online resources for additional instruction and practice with exponents and scientific notation.

Exponential Notation (http://openstax.org/l/exponnot)
Properties of Exponents (http://openstax.org/l/exponprops)
Zero Exponent (http://openstax.org/l/zeroexponent)
Simplify Exponent Expressions (http://openstax.org/l/exponexpres)
Quotient Rule for Exponents (http://openstax.org/l/quotofexpon)
Scientific Notation (http://openstax.org/l/scientificnota)
Converting to Decimal Notation (http://openstax.org/l/decimalnota)

1.2 SECTION EXERCISES

Verbal

1. Is 2^{3} the same as 3^{2}? Explain.

2. When can you add two exponents?

3. What is the purpose of scientific notation?

4. Explain what a negative exponent does.

Numeric

For the following exercises, simplify the given expression. Write answers with positive exponents.

5. 9^2

6. 15^{-2}

7. $3^2 \times 3^3$

8. $4^4 \div 4$

9. $\left(2^2\right)^{-2}$

10. $(5-8)^0$

11. $11^3 \div 11^4$

12. $6^5 \times 6^{-7}$

13. $\left(8^0\right)^2$

14. $5^{-2} \div 5^2$

For the following exercises, write each expression with a single base. Do not simplify further. Write answers with positive exponents.

15. $4^2 \times 4^3 \div 4^{-4}$

16. $\frac{6^{12}}{6^9}$

17. $\left(12^3 \times 12\right)^{10}$

18. $10^6 \div \left(10^{10}\right)^{-2}$

19. $7^{-6} \times 7^{-3}$

20. $\left(3^3 \div 3^4\right)^5$

For the following exercises, express the decimal in scientific notation.

21. 0.0000314

22. 148,000,000

For the following exercises, convert each number in scientific notation to standard notation.

23. 1.6×10^{10}

24. 9.8×10^{-9}

Algebraic

For the following exercises, simplify the given expression. Write answers with positive exponents.

25. $\frac{a^3 a^2}{a}$

26. $\frac{mn^2}{m^{-2}}$

27. $\left(b^3 c^4\right)^2$

28. $\left(\frac{x^{-3}}{y^2}\right)^{-5}$

29. $ab^2 \div d^{-3}$

30. $\left(w^0 x^5\right)^{-1}$

31. $\frac{m^4}{n^0}$

32. $y^{-4}\left(y^2\right)^2$

33. $\frac{p^{-4} q^2}{p^2 q^{-3}}$

34. $(l \times w)^2$

35. $\left(y^7\right)^3 \div x^{14}$

36. $\left(\frac{a}{2^3}\right)^2$

37. $(25m) \div ({}^{5}_{0}m)$

38. $\frac{\left(16\sqrt{x}\right)^2}{y^{-1}}$

39. $\frac{2^3}{(3a)^{-2}}$

40. $\left(ma^6\right)^2 \frac{1}{m^3 a^2}$

41. $\left(b^{-3} c\right)^3$

42. $\left(x^2 y^{13} \div y^0\right)^2$

43. $\left(9z^3\right)^{-2}y$

Real-World Applications

44. To reach escape velocity, a rocket must travel at the rate of 2.2×10^6 ft/min. Rewrite the rate in standard notation.

45. A dime is the thinnest coin in U.S. currency. A dime's thickness measures 1.35×10^{-3} m. Rewrite the number in standard notation.

46. The average distance between Earth and the Sun is 92,960,000 mi. Rewrite the distance using scientific notation.

47. A terabyte is made of approximately 1,099,500,000,000 bytes. Rewrite in scientific notation.

48. The Gross Domestic Product (GDP) for the United States in the first quarter of 2014 was $\$1.71496 \times 10^{13}$. Rewrite the GDP in standard notation.

49. One picometer is approximately 3.397×10^{-11} in. Rewrite this length using standard notation.

50. The value of the services sector of the U.S. economy in the first quarter of 2012 was $10,633.6 billion. Rewrite this amount in scientific notation.

Technology

For the following exercises, use a graphing calculator to simplify. Round the answers to the nearest hundredth.

51. $\left(\frac{12^3 m^{33}}{4^{-3}}\right)^2$

52. $17^3 \div 15^2 x^3$

Extensions

For the following exercises, simplify the given expression. Write answers with positive exponents.

53. $\left(\frac{3^2}{a^3}\right)^{-2}\left(\frac{a^4}{2^2}\right)^2$

54. $\left(6^2 - 24\right)^2 \div \left(\frac{x}{y}\right)^{-5}$

55. $\frac{m^2 n^3}{a^2 c^{-3}} \cdot \frac{a^{-7} n^{-2}}{m^2 c^4}$

56. $\left(\frac{x^6 y^3}{x^3 y^{-3}} \cdot \frac{y^{-7}}{x^{-3}}\right)^{10}$

57. $\left(\frac{\left(ab^2 c\right)^{-3}}{b^{-3}}\right)^2$

58. Avogadro's constant is used to calculate the number of particles in a mole. A mole is a basic unit in chemistry to measure the amount of a substance. The constant is 6.0221413×10^{23}. Write Avogadro's constant in standard notation.

59. Planck's constant is an important unit of measure in quantum physics. It describes the relationship between energy and frequency. The constant is written as $6.62606957 \times 10^{-34}$. Write Planck's constant in standard notation.

1.3 Radicals and Rational Exponents

Learning Objectives

In this section, you will:

- Evaluate square roots.
- Use the product rule to simplify square roots.
- Use the quotient rule to simplify square roots.
- Add and subtract square roots.
- Rationalize denominators.
- Use rational roots.

COREQUISITE SKILLS

Learning Objective:

- Investigate the discipline called learning science and the idea of a knowledge space.

Objective 1: Investigate the discipline called learning science and the idea of a knowledge space.

The brain is a complex organ. It is the control center for our bodies, while the mind is where thinking and learning take place. In an attempt to understand the processes that occur in learning, researchers study a collection of disciplines called **learning sciences**. This interdisciplinary field includes study of psychological, sociological, anthropological, and computational approaches to learning.

In this skill sheet we will investigate the mathematics of mastery and knowledge spaces. A **knowledge space** includes the possible states of knowledge of a human learner. The theory of knowledge space was introduced in 1985 by mathematical psychologists Jean-Paul Doignon and Jean-Claude Falmagne and has since been studied by many researchers. [1]

Practice Makes Perfect

Investigation: There are 32 student-learning outcomes (SLO's) in a typical College Algebra course. These are topics a student needs to master to show proficiency in College Algebra. Let's begin by looking at just a few of these skills. Let's assign the variables, A, B, C, and D to the following topics. We will name the set containing each of these 4 topics, Q.

- A = Graph the basic functions listed in the library of functions.
- B = Find the domain of a function defined by an equation.
- C = Create a new function through composition of functions.
- D = Find linear functions that model data sets.

Using roster notation Q = {A, B, C, D}.

1. List each of the possible **subsets** of the 4 topics listed above using roster notation. Remember a subset is a collection of topics in which each topic listed is an element of the set Q we defined above. By including a topic, we are indicating that the student has mastered the topic.

2. Verify in your work above you have listed all 16 subsets to set Q. Remember that a subset may contain all of the topics listed in Q.

1 Doignon, J.-P.; Falmagne, J.-Cl. (1985), "Spaces for the assessment of knowledge", International Journal of Man-Machine Studies.

3. What formula could you use to help you determine the number of possible subsets? Remember that each topic could be mastered or not by a student. Show below that your formula would be equal to 16 for a list of 4 topics.

4. Now use the formula you found in #3 to find the number of subsets possible if we include all 32 student-learning outcomes.

Hint: In evaluating exponential terms, the function values increase very rapidly. To display very large (or very small) values, a calculator will use scientific notation. For example: 2.56 E6 is telling you to move the decimal point 6 places to the right and to insert zeros where you have missing values.

For example: 2.56 E6 = 2,560,000 or 2 million, five hundred, sixty thousand.

5. The subsets you created in #1 are referred to as knowledge spaces in the field of learning science. In this context mastery of one concept may depend on your mastery of another.

List one skill in mathematics that would help to master each of the following SLO's:

- A = Graph the basic functions listed in the library of functions.
- B = Find the domain of a function defined by an equation.
- C = Create a new function through composition of functions.
- D = Find linear functions that model data sets.

6. Mastery of what are called **linchpin topics** will make it easier to learn other topics. For example, the ability to solve linear equations with variables on both sides can "unlock" a whole set of new skills for a student to master.

List 3 other linchpin topics that would help you to master this math course. Discuss these with others in your class. Did they identify the same topics?

1.
2.
3.

7. A corequisite course in mathematics is designed to provide support to a student by reviewing linchpin topics right when and where students need the help. Review of these important foundational ideas allow the learner to move on and master the student learning objectives for the course.

Brainstorm ideas with your classmates about ways this corequisite support course could help you in your learning.

A hardware store sells 16-ft ladders and 24-ft ladders. A window is located 12 feet above the ground. A ladder needs to be purchased that will reach the window from a point on the ground 5 feet from the building. To find out the length of ladder needed, we can draw a right triangle as shown in Figure 1, and use the Pythagorean Theorem.

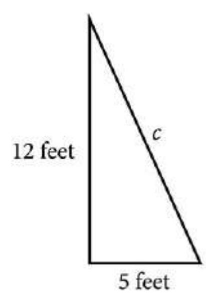


Figure 1

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + 12^2 &= c^2 \\ 169 &= c^2 \end{aligned}$$

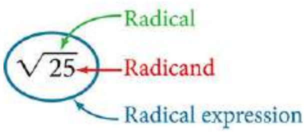
Now, we need to find out the length that, when squared, is 169, to determine which ladder to choose. In other words, we need to find a square root. In this section, we will investigate methods of finding solutions to problems such as this one.

Evaluating Square Roots

When the square root of a number is squared, the result is the original number. Since $4^2 = 16$, the square root of 16 is 4. The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if a is a positive real number, then the square root of a is a number that, when multiplied by itself, gives a. The square root could be positive or negative because multiplying two negative numbers gives a positive number. The **principal square root** is the nonnegative number that when multiplied by itself equals a. The square root obtained using a calculator is the principal square root.

The principal square root of a is written as $\sqrt{a}$. The symbol is called a **radical**, the term under the symbol is called the **radicand**, and the entire expression is called a **radical expression**.



Principal Square Root

The **principal square root** of a is the nonnegative number that, when multiplied by itself, equals a. It is written as a **radical expression,** with a symbol called a **radical** over the term called the **radicand**: $\sqrt{a}$.

Q&A **Does $\sqrt{25} = \pm 5$?**

No. Although both 5^2 and $(-5)^2$ are 25, the radical symbol implies only a nonnegative root, the principal square root. The principal square root of 25 is $\sqrt{25} = 5$.

EXAMPLE 1

Evaluating Square Roots

Evaluate each expression.

ⓐ $\sqrt{100}$ ⓑ $\sqrt{\sqrt{16}}$ ⓒ $\sqrt{25 + 144}$ ⓓ $\sqrt{49} - \sqrt{81}$

Solution

ⓐ $\sqrt{100} = 10$ because $10^2 = 100$ ⓑ $\sqrt{\sqrt{16}} = \sqrt{4} = 2$ because $4^2 = 16$ and $2^2 = 4$

ⓒ $\sqrt{25 + 144} = \sqrt{169} = 13$ because $13^2 = 169$ ⓓ $\sqrt{49} - \sqrt{81} = 7 - 9 = -2$ because $7^2 = 49$ and $9^2 = 81$

Q&A **For $\sqrt{25 + 144}$, can we find the square roots before adding?**

No. $\sqrt{25} + \sqrt{144} = 5 + 12 = 17$. This is not equivalent to $\sqrt{25 + 144} = 13$. The order of operations requires us to add the terms in the radicand before finding the square root.

TRY IT #1 Evaluate each expression.

ⓐ $\sqrt{225}$ ⓑ $\sqrt{\sqrt{81}}$ ⓒ $\sqrt{25 - 9}$ ⓓ $\sqrt{36} + \sqrt{121}$

Using the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the *product rule for simplifying square roots,* which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite $\sqrt{15}$ as $\sqrt{3} \cdot \sqrt{5}$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

The Product Rule for Simplifying Square Roots

If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b.

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

HOW TO

Given a square root radical expression, use the product rule to simplify it.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

EXAMPLE 2

Using the Product Rule to Simplify Square Roots
Simplify the radical expression.

ⓐ $\sqrt{300}$ ⓑ $\sqrt{162a^5b^4}$

Solution

ⓐ

$\sqrt{100 \cdot 3}$	Factor perfect square from radicand.
$\sqrt{100} \cdot \sqrt{3}$	Write radical expression as product of radical expressions.
$10\sqrt{3}$	Simplify.

ⓑ

$\sqrt{81a^4b^4 \cdot 2a}$	Factor perfect square from radicand.
$\sqrt{81a^4b^4} \cdot \sqrt{2a}$	Write radical expression as product of radical expressions.
$9a^2b^2\sqrt{2a}$	Simplify.

TRY IT #2 Simplify $\sqrt{50x^2y^3z}$.

HOW TO

Given the product of multiple radical expressions, use the product rule to combine them into one radical expression.

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

EXAMPLE 3

Using the Product Rule to Simplify the Product of Multiple Square Roots
Simplify the radical expression.
$\sqrt{12} \cdot \sqrt{3}$

Solution

$\sqrt{12 \cdot 3}$ Express the product as a single radical expression.

$\sqrt{36}$ Simplify.

6

TRY IT #3 Simplify $\sqrt{50x} \cdot \sqrt{2x}$ assuming $x > 0$.

Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square roots, using the *quotient rule for simplifying square roots.* It can be helpful to separate the numerator and denominator of a fraction under a radical so that we can take their square roots separately. We can rewrite $\sqrt{\frac{5}{2}}$ as $\frac{\sqrt{5}}{\sqrt{2}}$.

The Quotient Rule for Simplifying Square Roots

The square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b, where $b \neq 0$.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

HOW TO

Given a radical expression, use the quotient rule to simplify it.

1. Write the radical expression as the quotient of two radical expressions.
2. Simplify the numerator and denominator.

EXAMPLE 4

Using the Quotient Rule to Simplify Square Roots

Simplify the radical expression.

$$\sqrt{\frac{5}{36}}$$

Solution

$\frac{\sqrt{5}}{\sqrt{36}}$ Write as quotient of two radical expressions.

$\frac{\sqrt{5}}{6}$ Simplify denominator.

TRY IT #4 Simplify $\sqrt{\frac{2x^2}{9y^4}}$.

EXAMPLE 5

Using the Quotient Rule to Simplify an Expression with Two Square Roots

Simplify the radical expression.

$$\frac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}}$$

Solution

$\sqrt{\frac{234x^{11}y}{26x^7y}}$	Combine numerator and denominator into one radical expression.
$\sqrt{9x^4}$	Simplify fraction.
$3x^2$	Simplify square root.

TRY IT #5 Simplify $\frac{\sqrt{9a^5b^{14}}}{\sqrt{3a^4b^5}}$.

Adding and Subtracting Square Roots

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of $\sqrt{2}$ and $3\sqrt{2}$ is $4\sqrt{2}$. However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression $\sqrt{18}$ can be written with a 2 in the radicand, as $3\sqrt{2}$, so $\sqrt{2}+\sqrt{18}=\sqrt{2}+3\sqrt{2}=4\sqrt{2}$.

HOW TO

Given a radical expression requiring addition or subtraction of square roots, simplify.

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

EXAMPLE 6

Adding Square Roots

Add $5\sqrt{12}+2\sqrt{3}$.

Solution

We can rewrite $5\sqrt{12}$ as $5\sqrt{4\cdot 3}$. According the product rule, this becomes $5\sqrt{4}\sqrt{3}$. The square root of $\sqrt{4}$ is 2, so the expression becomes $5\,(2)\,\sqrt{3}$, which is $10\sqrt{3}$. Now the terms have the same radicand so we can add.

$$10\sqrt{3}+2\sqrt{3}=12\sqrt{3}$$

TRY IT #6 Add $\sqrt{5}+6\sqrt{20}$.

EXAMPLE 7

Subtracting Square Roots

Subtract $20\sqrt{72a^3b^4c} - 14\sqrt{8a^3b^4c}$.

Solution

Rewrite each term so they have equal radicands.

$$
\begin{aligned}
20\sqrt{72a^3b^4c} &= 20\sqrt{9}\sqrt{4}\sqrt{2}\sqrt{a}\sqrt{a^2}\sqrt{\left(b^2\right)^2}\sqrt{c} \\
&= 20(3)(2)|a|b^2\sqrt{2ac} \\
&= 120|a|b^2\sqrt{2ac}
\end{aligned}
$$

$$
\begin{aligned}
14\sqrt{8a^3b^4c} &= 14\sqrt{2}\sqrt{4}\sqrt{a}\sqrt{a^2}\sqrt{\left(b^2\right)^2}\sqrt{c} \\
&= 14(2)|a|b^2\sqrt{2ac} \\
&= 28|a|b^2\sqrt{2ac}
\end{aligned}
$$

Now the terms have the same radicand so we can subtract.

$$120|a|b^2\sqrt{2ac} - 28|a|b^2\sqrt{2ac} = 92|a|b^2\sqrt{2ac}$$

TRY IT #7 Subtract $3\sqrt{80x} - 4\sqrt{45x}$.

Rationalizing Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called *rationalizing the denominator*.

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b\sqrt{c}$, multiply by $\frac{\sqrt{c}}{\sqrt{c}}$.

For a denominator containing the sum or difference of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator. If the denominator is $a + b\sqrt{c}$, then the conjugate is $a - b\sqrt{c}$.

HOW TO

Given an expression with a single square root radical term in the denominator, rationalize the denominator.

a. Multiply the numerator and denominator by the radical in the denominator.
b. Simplify.

EXAMPLE 8

Rationalizing a Denominator Containing a Single Term

Write $\frac{2\sqrt{3}}{3\sqrt{10}}$ in simplest form.

Solution

The radical in the denominator is $\sqrt{10}$. So multiply the fraction by $\frac{\sqrt{10}}{\sqrt{10}}$. Then simplify.

$$\frac{2\sqrt{3}}{3\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$
$$\frac{2\sqrt{30}}{30}$$
$$\frac{\sqrt{30}}{15}$$

TRY IT #8 Write $\frac{12\sqrt{3}}{\sqrt{2}}$ in simplest form.

HOW TO

Given an expression with a radical term and a constant in the denominator, rationalize the denominator.

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

EXAMPLE 9

Rationalizing a Denominator Containing Two Terms

Write $\frac{4}{1+\sqrt{5}}$ in simplest form.

Solution

Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1 + \sqrt{5}$ is $1 - \sqrt{5}$. Then multiply the fraction by $\frac{1-\sqrt{5}}{1-\sqrt{5}}$.

$$\frac{4}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}}$$
$$\frac{4-4\sqrt{5}}{-4} \qquad \text{Use the distributive property.}$$
$$\sqrt{5} - 1 \qquad \text{Simplify.}$$

TRY IT #9 Write $\frac{7}{2+\sqrt{3}}$ in simplest form.

Using Rational Roots

Although square roots are the most common rational roots, we can also find cube roots, 4th roots, 5th roots, and more. Just as the square root function is the inverse of the squaring function, these roots are the inverse of their respective power functions. These functions can be useful when we need to determine the number that, when raised to a certain power, gives a certain number.

Understanding *n*th Roots

Suppose we know that $a^3 = 8$. We want to find what number raised to the 3rd power is equal to 8. Since $2^3 = 8$, we say that 2 is the cube root of 8.

The *n*th root of a is a number that, when raised to the *n*th power, gives a. For example, -3 is the 5th root of -243 because $(-3)^5 = -243$. If a is a real number with at least one *n*th root, then the **principal *n*th root** of a is the number with the same sign as a that, when raised to the *n*th power, equals a.

The principal *n*th root of a is written as $\sqrt[n]{a}$, where n is a positive integer greater than or equal to 2. In the radical expression, n is called the **index** of the radical.

Principal n th Root

If a is a real number with at least one nth root, then the **principal nth root** of a, written as $\sqrt[n]{a}$, is the number with the same sign as a that, when raised to the nth power, equals a. The **index** of the radical is n.

EXAMPLE 10

Simplifying nth Roots

Simplify each of the following:

ⓐ $\sqrt[5]{-32}$ ⓑ $\sqrt[4]{4} \cdot \sqrt[4]{1,024}$ ⓒ $-\sqrt[3]{\frac{8x^6}{125}}$ ⓓ $8\sqrt[4]{3} - \sqrt[4]{48}$

Solution

ⓐ $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

ⓑ First, express the product as a single radical expression. $\sqrt[4]{4,096} = 8$ because $8^4 = 4,096$

ⓒ
$\frac{-\sqrt[3]{8x^6}}{\sqrt[3]{125}}$ Write as quotient of two radical expressions.

$\frac{-2x^2}{5}$ Simplify.

ⓓ
$8\sqrt[4]{3} - 2\sqrt[4]{3}$ Simplify to get equal radicands.

$6\sqrt[4]{3}$ Add.

TRY IT #10 Simplify.

ⓐ $\sqrt[3]{-216}$ ⓑ $\frac{3\sqrt[4]{80}}{\sqrt[4]{5}}$ ⓒ $6\sqrt[3]{9,000} + 7\sqrt[3]{576}$

Using Rational Exponents

Radical expressions can also be written without using the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index n is even, then a cannot be negative.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

We can also have rational exponents with numerators other than 1. In these cases, the exponent must be a fraction in lowest terms. We raise the base to a power and take an nth root. The numerator tells us the power and the denominator tells us the root.

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

All of the properties of exponents that we learned for integer exponents also hold for rational exponents.

Rational Exponents

Rational exponents are another way to express principal nth roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

HOW TO

Given an expression with a rational exponent, write the expression as a radical.

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

EXAMPLE 11

Writing Rational Exponents as Radicals

Write $343^{\frac{2}{3}}$ as a radical. Simplify.

Solution

The 2 tells us the power and the 3 tells us the root.

$$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = \sqrt[3]{343^2}$$

We know that $\sqrt[3]{343} = 7$ because $7^3 = 343$. Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.

$$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = 7^2 = 49$$

TRY IT #11 Write $9^{\frac{5}{2}}$ as a radical. Simplify.

EXAMPLE 12

Writing Radicals as Rational Exponents

Write $\frac{4}{\sqrt[7]{a^2}}$ using a rational exponent.

Solution

The power is 2 and the root is 7, so the rational exponent will be $\frac{2}{7}$. We get $\frac{4}{a^{\frac{2}{7}}}$. Using properties of exponents, we get

$$\frac{4}{\sqrt[7]{a^2}} = 4a^{\frac{-2}{7}}.$$

TRY IT #12 Write $x\sqrt{(5y)^9}$ using a rational exponent.

EXAMPLE 13

Simplifying Rational Exponents

Simplify:

ⓐ $5\left(2x^{\frac{3}{4}}\right)\left(3x^{\frac{1}{5}}\right)$ ⓑ $\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

Solution

ⓐ

$30x^{\frac{3}{4}}x^{\frac{1}{5}}$	Multiply the coefficients.
$30x^{\frac{3}{4}+\frac{1}{5}}$	Use properties of exponents.
$30x^{\frac{19}{20}}$	Simplify.

ⓑ

$\left(\frac{9}{16}\right)^{\frac{1}{2}}$	Use definition of negative exponents.
$\sqrt{\frac{9}{16}}$	Rewrite as a radical.
$\frac{\sqrt{9}}{\sqrt{16}}$	Use the quotient rule.
$\frac{3}{4}$	Simplify.

TRY IT #13 Simplify $(8x)^{\frac{1}{3}}\left(14x^{\frac{6}{5}}\right)$.

MEDIA

Access these online resources for additional instruction and practice with radicals and rational exponents.

[Radicals (http://openstax.org/l/introradical)](http://openstax.org/l/introradical)
[Rational Exponents (http://openstax.org/l/rationexpon)](http://openstax.org/l/rationexpon)
[Simplify Radicals (http://openstax.org/l/simpradical)](http://openstax.org/l/simpradical)
[Rationalize Denominator (http://openstax.org/l/rationdenom)](http://openstax.org/l/rationdenom)

1.3 SECTION EXERCISES

Verbal

1. What does it mean when a radical does not have an index? Is the expression equal to the radicand? Explain.
2. Where would radicals come in the order of operations? Explain why.
3. Every number will have two square roots. What is the principal square root?
4. Can a radical with a negative radicand have a real square root? Why or why not?

Numeric

For the following exercises, simplify each expression.

5. $\sqrt{256}$
6. $\sqrt{\sqrt{256}}$
7. $\sqrt{4(9+16)}$
8. $\sqrt{289}-\sqrt{121}$
9. $\sqrt{196}$
10. $\sqrt{1}$
11. $\sqrt{98}$
12. $\sqrt{\frac{27}{64}}$
13. $\sqrt{\frac{81}{5}}$
14. $\sqrt{800}$
15. $\sqrt{169}+\sqrt{144}$
16. $\sqrt{\frac{8}{50}}$
17. $\frac{18}{\sqrt{162}}$
18. $\sqrt{192}$
19. $14\sqrt{6}-6\sqrt{24}$

20. $15\sqrt{5}+7\sqrt{45}$

21. $\sqrt{150}$

22. $\sqrt{\frac{96}{100}}$

23. $\left(\sqrt{42}\right)\left(\sqrt{30}\right)$

24. $12\sqrt{3}-4\sqrt{75}$

25. $\sqrt{\frac{4}{225}}$

26. $\sqrt{\frac{405}{324}}$

27. $\sqrt{\frac{360}{361}}$

28. $\frac{5}{1+\sqrt{3}}$

29. $\frac{8}{1-\sqrt{17}}$

30. $\sqrt[4]{16}$

31. $\sqrt[3]{128}+3\sqrt[3]{2}$

32. $\sqrt[5]{\frac{-32}{243}}$

33. $\frac{15\sqrt[4]{125}}{\sqrt[4]{5}}$

34. $3\sqrt[3]{-432}+\sqrt[3]{16}$

Algebraic

For the following exercises, simplify each expression.

35. $\sqrt{400x^4}$

36. $\sqrt{4y^2}$

37. $\sqrt{49p}$

38. $\left(144p^2q^6\right)^{\frac{1}{2}}$

39. $m^{\frac{5}{2}}\sqrt{289}$

40. $9\sqrt{3m^2}+\sqrt{27}$

41. $3\sqrt{ab^2}-b\sqrt{a}$

42. $\frac{4\sqrt{2n}}{\sqrt{16n^4}}$

43. $\sqrt{\frac{225x^3}{49x}}$

44. $3\sqrt{44z}+\sqrt{99z}$

45. $\sqrt{50y^8}$

46. $\sqrt{490bc^2}$

47. $\sqrt{\frac{32}{14d}}$

48. $q^{\frac{3}{2}}\sqrt{63p}$

49. $\frac{\sqrt{8}}{1-\sqrt{3x}}$

50. $\sqrt{\frac{20}{121d^4}}$

51. $w^{\frac{3}{2}}\sqrt{32}-w^{\frac{3}{2}}\sqrt{50}$

52. $\sqrt{108x^4}+\sqrt{27x^4}$

53. $\frac{\sqrt{12x}}{2+2\sqrt{3}}$

54. $\sqrt{147k^3}$

55. $\sqrt{125n^{10}}$

56. $\sqrt{\frac{42q}{36q^3}}$

57. $\sqrt{\frac{81m}{361m^2}}$

58. $\sqrt{72c}-2\sqrt{2c}$

59. $\sqrt{\frac{144}{324d^2}}$

60. $\sqrt[3]{24x^6}+\sqrt[3]{81x^6}$

61. $\sqrt[4]{\frac{162x^6}{16x^4}}$

62. $\sqrt[3]{64y}$

63. $\sqrt[3]{128z^3}-\sqrt[3]{-16z^3}$

64. $\sqrt[5]{1{,}024c^{10}}$

Real-World Applications

65. A guy wire for a suspension bridge runs from the ground diagonally to the top of the closest pylon to make a triangle. We can use the Pythagorean Theorem to find the length of guy wire needed. The square of the distance between the wire on the ground and the pylon on the ground is 90,000 feet. The square of the height of the pylon is 160,000 feet. So the length of the guy wire can be found by evaluating $\sqrt{90{,}000 + 160{,}000}$. What is the length of the guy wire?

66. A car accelerates at a rate of $6 - \frac{\sqrt{4}}{\sqrt{t}}\ \text{m/s}^2$ where t is the time in seconds after the car moves from rest. Simplify the expression.

Extensions

For the following exercises, simplify each expression.

67. $\frac{\sqrt{8}-\sqrt{16}}{4-\sqrt{2}} - 2^{\frac{1}{2}}$

68. $\frac{4^{\frac{3}{2}} - 16^{\frac{3}{2}}}{8^{\frac{1}{3}}}$

69. $\frac{\sqrt{mn^3}}{a^2\sqrt{c^{-3}}} \cdot \frac{a^{-7}n^{-2}}{\sqrt{m^2c^4}}$

70. $\frac{a}{a-\sqrt{c}}$

71. $\frac{x\sqrt{64y}+4\sqrt{y}}{\sqrt{128y}}$

72. $\left(\frac{\sqrt{250x^2}}{\sqrt{100b^3}}\right)\left(\frac{7\sqrt{b}}{\sqrt{125x}}\right)$

73. $\sqrt{\frac{\sqrt[3]{64}+\sqrt[4]{256}}{\sqrt{64}+\sqrt{256}}}$

1.4 Polynomials

Learning Objectives

In this section, you will:

- Identify the degree and leading coefficient of polynomials.
- Add and subtract polynomials.
- Multiply polynomials.
- Use FOIL to multiply binomials.
- Perform operations with polynomials of several variables.

COREQUISITE SKILLS

Learning Objectives

- Distinguish between a fixed and a growth mindset, and how these ideas may help in learning.

Objective 1: Distinguish between a fixed and a growth mindset, and how these ideas may help in learning.

Stanford University psychologist and researcher, Carol Dweck, PH.D., published a book in 2006 called *"Mindset, The New Psychology of Success"*, which changed how many people think about their talents and abilities. Based on decades of research Dr. Dweck outlined two mindsets and their influence on our learning.

Dr. Dweck's research found that people who believe that their abilities could change through learning and practice (**growth mindset**) more readily accepted learning challenges and persisted through these challenges. While individuals who believe that knowledge and abilities come from natural talent and cannot be changed (**fixed mindset**) more often become discouraged by failure and do not persist.

Her research shows that if we believe we can learn and master something new, this belief greatly improves our ability to learn.

1. Read through the following illustration based on Dr. Dweck's work.
2. It's important to note that we as individuals do not have a strict fixed or growth mindset at all times. We can lean one way or another in certain situations or when working in different disciplines or areas. For example, a person who often plays video games may feel they can learn any new game that is released and be confident in these abilities, but at the same time avoid sports and are fixed on the idea that they will never excel at physical activities. In terms of learning new skills in mathematics, which mindset, growth or fixed, best describes your beliefs as of today? Explain.

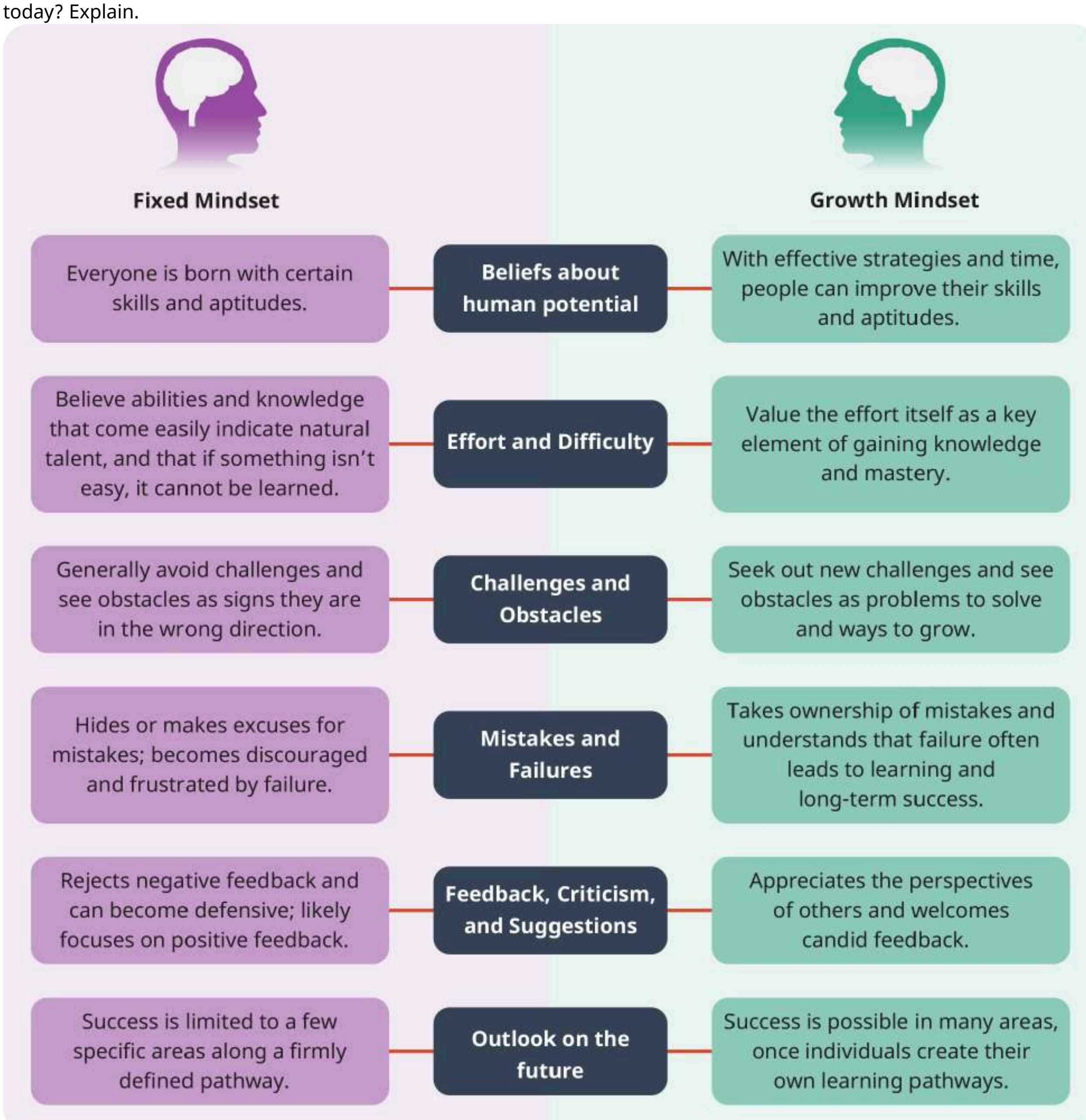


Figure 1 The differences between fixed and growth mindset are clear when aligned to key elements of learning and personality. (Credit: Adapted for OpenStax *College Success*, based on work by Dr. Carol Dweck)

3. Identify each of the following statements as coming from a student with a fixed mindset or with a growth mindset.

Statement	Fixed or Growth Mindset?
a. I've never been good at math, so I'll be happy just getting a D in this course.	
b. I hear that this instructor is really great, I'm excited to start this new term.	
c. I need to try harder in this class and put in more study time. I have the rest of the term to improve my performance.	
d. I hate math.	
e. This activity is dumb, I don't think it will help me.	
f. That exam was tough, but I'm going to rework it during my study time and get these concepts down before my final.	
g. I'm up for the challenge of this course.	
h. Some people are just better in math than me.	
i. Intelligence is something you have to work for.	
j. I find it best to erase every mistake I make in my homework and try to forget about it.	
k. I try to learn from my mistakes and make note of them.	
l. I'm not going to raise my hand to answer this question in class. I'll just be wrong.	

4. Mindsets can be changed. As Dr. Dweck would say "You have a choice. Mindsets are just beliefs. They are powerful beliefs, but they are something in your mind and you can change your mind."
   Think about what you would like to achieve in your classes this term and how a growth mindset can help you reach these goals. Write three goals for yourself below.
   1.
   2.
   3.

Maahi is building a little free library (a small house-shaped book repository), whose front is in the shape of a square topped with a triangle. There will be a rectangular door through which people can take and donate books. Maahi wants to find the area of the front of the library so that they can purchase the correct amount of paint. Using the measurements of the front of the house, shown in Figure 1, we can create an expression that combines several variable terms, allowing us to solve this problem and others like it.

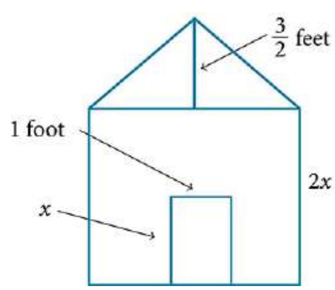


Figure 1

First find the area of the square in square feet.

$$\begin{aligned} A &= s^2 \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

Then find the area of the triangle in square feet.

$$\begin{aligned} A &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2}(2x)\left(\tfrac{3}{2}\right) \\ &= \tfrac{3}{2}x \end{aligned}$$

Next find the area of the rectangular door in square feet.

$$\begin{aligned} A &= lw \\ &= x \cdot 1 \\ &= x \end{aligned}$$

The area of the front of the library can be found by adding the areas of the square and the triangle, and then subtracting the area of the rectangle. When we do this, we get $4x^2 + \frac{3}{2}x - x\ \text{ft}^2$, or $4x^2 + \frac{1}{2}x\ \text{ft}^2$.

In this section, we will examine expressions such as this one, which combine several variable terms.

Identifying the Degree and Leading Coefficient of Polynomials

The formula just found is an example of a **polynomial**, which is a sum of or difference of terms, each consisting of a variable raised to a nonnegative integer power. A number multiplied by a variable raised to an exponent, such as 384π, is known as a **coefficient**. Coefficients can be positive, negative, or zero, and can be whole numbers, decimals, or fractions. Each product $a_i x^i$, such as $384\pi w$, is a **term of a polynomial**. If a term does not contain a variable, it is called a *constant*.

A polynomial containing only one term, such as $5x^4$, is called a **monomial**. A polynomial containing two terms, such as $2x - 9$, is called a **binomial**. A polynomial containing three terms, such as $-3x^2 + 8x - 7$, is called a **trinomial**.

We can find the **degree** of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the **leading term** because it is usually written first. The coefficient of the leading term is called the **leading coefficient**. When a polynomial is written so that the powers are descending, we say that it is in standard form.

Leading coefficient Degree

$$\underbrace{a_n x^n}_{\text{Leading term}} + \ldots + a_2 x^2 + a_1 x + a_0$$

Polynomials

A **polynomial** is an expression that can be written in the form

$$a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$$

Each real number a_i is called a **coefficient**. The number a_0 that is not multiplied by a variable is called a *constant*. Each product $a_i x^i$ is a **term of a polynomial**. The highest power of the variable that occurs in the polynomial is called the **degree** of a polynomial. The **leading term** is the term with the highest power, and its coefficient is called the **leading coefficient**.

HOW TO

Given a polynomial expression, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree.
2. Identify the term containing the highest power of x to find the leading term.

3. Identify the coefficient of the leading term.

EXAMPLE 1

Identifying the Degree and Leading Coefficient of a Polynomial

For the following polynomials, identify the degree, the leading term, and the leading coefficient.

ⓐ $3 + 2x^2 - 4x^3$ ⓑ $5t^5 - 2t^3 + 7t$ ⓒ $6p - p^3 - 2$

Solution

ⓐ The highest power of *x* is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

ⓑ The highest power of *t* is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

ⓒ The highest power of *p* is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$. The leading coefficient is the coefficient of that term, -1.

TRY IT #1 Identify the degree, leading term, and leading coefficient of the polynomial $4x^2 - x^6 + 2x - 6$.

Adding and Subtracting Polynomials

We can add and subtract polynomials by combining like terms, which are terms that contain the same variables raised to the same exponents. For example, $5x^2$ and $-2x^2$ are like terms, and can be added to get $3x^2$, but $3x$ and $3x^2$ are not like terms, and therefore cannot be added.

HOW TO

Given multiple polynomials, add or subtract them to simplify the expressions.

1. Combine like terms.
2. Simplify and write in standard form.

EXAMPLE 2

Adding Polynomials

Find the sum.

$$\left(12x^2 + 9x - 21\right) + \left(4x^3 + 8x^2 - 5x + 20\right)$$

Solution

$4x^3 + \left(12x^2 + 8x^2\right) + (9x - 5x) + (-21 + 20)$	Combine like terms.
$4x^3 + 20x^2 + 4x - 1$	Simplify.

Analysis

We can check our answers to these types of problems using a graphing calculator. To check, graph the problem as given along with the simplified answer. The two graphs should be equivalent. Be sure to use the same window to compare the graphs. Using different windows can make the expressions seem equivalent when they are not.

TRY IT #2 Find the sum.

$$\left(2x^3 + 5x^2 - x + 1\right) + \left(2x^2 - 3x - 4\right)$$

EXAMPLE 3

Subtracting Polynomials

Find the difference.

$$\left(7x^4 - x^2 + 6x + 1\right) - \left(5x^3 - 2x^2 + 3x + 2\right)$$

Solution

$7x^4 - x^2 + 6x + 1 - 5x^3 + 2x^3 - 2$	Distribute negative sign.
$7x^4 - 5x^3 + x^2 + 6x - 3x + 1 - 2$	Group like terms.
$7x^4 - 5x^3 + x^2 + 3x - 1$	Combine/simplify.

Analysis

Note that finding the difference between two polynomials is the same as adding the opposite of the second polynomial to the first.

TRY IT #3 Find the difference.

$$\left(-7x^3 - 7x^2 + 6x - 2\right) - \left(4x^3 - 6x^2 - x + 7\right)$$

Multiplying Polynomials

Multiplying polynomials is a bit more challenging than adding and subtracting polynomials. We must use the distributive property to multiply each term in the first polynomial by each term in the second polynomial. We then combine like terms. We can also use a shortcut called the **FOIL** method when multiplying binomials. Certain special products follow patterns that we can memorize and use instead of multiplying the polynomials by hand each time. We will look at a variety of ways to multiply polynomials.

Multiplying Polynomials Using the Distributive Property

To multiply a number by a polynomial, we use the distributive property. The number must be distributed to each term of the polynomial. We can distribute the 2 in $2(x + 7)$ to obtain the equivalent expression $2x + 14$. When multiplying polynomials, the distributive property allows us to multiply each term of the first polynomial by each term of the second. We then add the products together and combine like terms to simplify.

HOW TO

Given the multiplication of two polynomials, use the distributive property to simplify the expression.

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.
3. Simplify.

EXAMPLE 4

Multiplying Polynomials Using the Distributive Property

Find the product.

$$(2x + 1)\left(3x^2 - x + 4\right)$$

Solution

$$2x\left(3x^2 - x + 4\right) + 1\left(3x^2 - x + 4\right) \qquad \text{Use the distributive property.}$$

$$\left(6x^3 - 2x^2 + 8x\right) + \left(3x^2 - x + 4\right) \qquad \text{Multiply.}$$

$$6x^3 + \left(-2x^2 + 3x^2\right) + (8x - x) + 4 \qquad \text{Combine like terms.}$$

$$6x^3 + x^2 + 7x + 4 \qquad \text{Simplify.}$$

Analysis

We can use a table to keep track of our work, as shown in Table 1. Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify.

	$3x^2$	$-x$	$+4$
$2x$	$6x^3$	$-2x^2$	$8x$
$+1$	$3x^2$	$-x$	4

Table 1

TRY IT #4 Find the product.

$$(3x + 2)\left(x^3 - 4x^2 + 7\right)$$

Using FOIL to Multiply Binomials

A shortcut called FOIL is sometimes used to find the product of two binomials. It is called FOIL because we multiply the **f**irst terms, the **o**uter terms, the **i**nner terms, and then the **l**ast terms of each binomial.

First terms Last terms

$$(ax + b)(cx + d) = acx^2 + adx + bcx + bd$$

Inner terms

Outer terms

The FOIL method arises out of the distributive property. We are simply multiplying each term of the first binomial by each term of the second binomial, and then combining like terms.

HOW TO

Given two binomials, use FOIL to simplify the expression.

1. Multiply the first terms of each binomial.
2. Multiply the outer terms of the binomials.
3. Multiply the inner terms of the binomials.
4. Multiply the last terms of each binomial.
5. Add the products.
6. Combine like terms and simplify.

EXAMPLE 5

Using FOIL to Multiply Binomials

Use FOIL to find the product.

$(2x - 18)(3x + 3)$

Solution

Find the product of the first terms.

$$2x - 18 \quad 3x + 3 \qquad 2x \cdot 3x = 6x^2$$

Find the product of the outer terms.

$$2x - 18 \quad 3x + 3 \qquad 2x \cdot 3 = 6x$$

Find the product of the inner terms.

$$2x - 18 \quad 3x + 3 \qquad -18 \cdot 3x = -54x$$

Find the product of the last terms.

$$2x - 18 \quad 3x + 3 \qquad -18 \cdot 3 = -54$$

$6x^2 + 6x - 54x - 54$	Add the products.
$6x^2 + (6x - 54x) - 54$	Combine like terms.
$6x^2 - 48x - 54$	Simplify.

TRY IT #5 Use FOIL to find the product.

$(x + 7)(3x - 5)$

Perfect Square Trinomials

Certain binomial products have special forms. When a binomial is squared, the result is called a **perfect square trinomial**. We can find the square by multiplying the binomial by itself. However, there is a special form that each of these perfect square trinomials takes, and memorizing the form makes squaring binomials much easier and faster. Let’s look at a few perfect square trinomials to familiarize ourselves with the form.

$$\begin{aligned} (x+5)^2 &= x^2 + 10x + 25 \\ (x-3)^2 &= x^2 - 6x + 9 \\ (4x-1)^2 &= 16x^2 - 8x + 1 \end{aligned}$$

Notice that the first term of each trinomial is the square of the first term of the binomial and, similarly, the last term of each trinomial is the square of the last term of the binomial. The middle term is double the product of the two terms. Lastly, we see that the first sign of the trinomial is the same as the sign of the binomial.

Perfect Square Trinomials

When a binomial is squared, the result is the first term squared added to double the product of both terms and the last term squared.

$$(x + a)^2 = (x + a)(x + a) = x^2 + 2ax + a^2$$

HOW TO

Given a binomial, square it using the formula for perfect square trinomials.

1. Square the first term of the binomial.
2. Square the last term of the binomial.
3. For the middle term of the trinomial, double the product of the two terms.
4. Add and simplify.

EXAMPLE 6

Expanding Perfect Squares

Expand $(3x - 8)^2$.

Solution

Begin by squaring the first term and the last term. For the middle term of the trinomial, double the product of the two terms.

$$(3x)^2 - 2(3x)(8) + (-8)^2$$

Simplify.

$$9x^2 - 48x + 64.$$

TRY IT #6 Expand $(4x - 1)^2$.

Difference of Squares

Another special product is called the **difference of squares**, which occurs when we multiply a binomial by another binomial with the same terms but the opposite sign. Let's see what happens when we multiply $(x + 1)(x - 1)$ using the FOIL method.

$$\begin{aligned}(x+1)(x-1) &= x^2 - x + x - 1 \\ &= x^2 - 1\end{aligned}$$

The middle term drops out, resulting in a difference of squares. Just as we did with the perfect squares, let's look at a few examples.

$$\begin{aligned}(x+5)(x-5) &= x^2 - 25 \\ (x+11)(x-11) &= x^2 - 121 \\ (2x+3)(2x-3) &= 4x^2 - 9\end{aligned}$$

Because the sign changes in the second binomial, the outer and inner terms cancel each other out, and we are left only with the square of the first term minus the square of the last term.

Q&A **Is there a special form for the sum of squares?**

No. The difference of squares occurs because the opposite signs of the binomials cause the middle terms to disappear. There are no two binomials that multiply to equal a sum of squares.

Difference of Squares

When a binomial is multiplied by a binomial with the same terms separated by the opposite sign, the result is the square of the first term minus the square of the last term.

$$(a + b)(a - b) = a^2 - b^2$$

HOW TO

Given a binomial multiplied by a binomial with the same terms but the opposite sign, find the difference of squares.

1. Square the first term of the binomials.
2. Square the last term of the binomials.
3. Subtract the square of the last term from the square of the first term.

EXAMPLE 7

Multiplying Binomials Resulting in a Difference of Squares

Multiply $(9x+4)(9x-4)$.

Solution

Square the first term to get $(9x)^2 = 81x^2$. Square the last term to get $4^2 = 16$. Subtract the square of the last term from the square of the first term to find the product of $81x^2 - 16$.

TRY IT #7 Multiply $(2x+7)(2x-7)$.

Performing Operations with Polynomials of Several Variables

We have looked at polynomials containing only one variable. However, a polynomial can contain several variables. All of the same rules apply when working with polynomials containing several variables. Consider an example:

$$\begin{array}{ll}
(a+2b)(4a-b-c) & \\
a(4a-b-c)+2b(4a-b-c) & \text{Use the distributive property.} \\
4a^2-ab-ac+8ab-2b^2-2bc & \text{Multiply.} \\
4a^2+(-ab+8ab)-ac-2b^2-2bc & \text{Combine like terms.} \\
4a^2+7ab-ac-2bc-2b^2 & \text{Simplify.}
\end{array}$$

EXAMPLE 8

Multiplying Polynomials Containing Several Variables

Multiply $(x+4)(3x-2y+5)$.

Solution

Follow the same steps that we used to multiply polynomials containing only one variable.

$$\begin{array}{ll}
x(3x-2y+5)+4(3x-2y+5) & \text{Use the distributive property.} \\
3x^2-2xy+5x+12x-8y+20 & \text{Multiply.} \\
3x^2-2xy+(5x+12x)-8y+20 & \text{Combine like terms.} \\
3x^2-2xy+17x-8y+20 & \text{Simplify.}
\end{array}$$

TRY IT #8 Multiply $(3x-1)(2x+7y-9)$.

MEDIA

Access these online resources for additional instruction and practice with polynomials.

Adding and Subtracting Polynomials (http://openstax.org/l/addsubpoly)
Multiplying Polynomials (http://openstax.org/l/multiplpoly)
Special Products of Polynomials (http://openstax.org/l/specialpolyprod)

1.4 SECTION EXERCISES

Verbal

1. Evaluate the following statement: The degree of a polynomial in standard form is the exponent of the leading term. Explain why the statement is true or false.

2. Many times, multiplying two binomials with two variables results in a trinomial. This is not the case when there is a difference of two squares. Explain why the product in this case is also a binomial.

3. You can multiply polynomials with any number of terms and any number of variables using four basic steps over and over until you reach the expanded polynomial. What are the four steps?

4. State whether the following statement is true and explain why or why not: A trinomial is always a higher degree than a monomial.

Algebraic

For the following exercises, identify the degree of the polynomial.

5. $7x - 2x^2 + 13$

6. $14m^3 + m^2 - 16m + 8$

7. $-625a^8 + 16b^4$

8. $200p - 30p^2m + 40m^3$

9. $x^2 + 4x + 4$

10. $6y^4 - y^5 + 3y - 4$

For the following exercises, find the sum or difference.

11. $\left(12x^2 + 3x\right) - \left(8x^2 - 19\right)$

12. $\left(4z^3 + 8z^2 - z\right) + \left(-2z^2 + z + 6\right)$

13. $\left(6w^2 + 24w + 24\right) - \left(3w^2 - 6w + 3\right)$

14. $\left(7a^3 + 6a^2 - 4a - 13\right) + \left(-3a^3 - 4a^2 + 6a + 17\right)$

15. $\left(11b^4 - 6b^3 + 18b^2 - 4b + 8\right) - \left(3b^3 + 6b^2 + 3b\right)$

16. $\left(49p^2 - 25\right) + \left(16p^4 - 32p^2 + 16\right)$

For the following exercises, find the product.

17. $(4x + 2)(6x - 4)$

18. $\left(14c^2 + 4c\right)\left(2c^2 - 3c\right)$

19. $\left(6b^2 - 6\right)\left(4b^2 - 4\right)$

20. $(3d - 5)(2d + 9)$

21. $(9v - 11)(11v - 9)$

22. $\left(4t^2 + 7t\right)\left(-3t^2 + 4\right)$

23. $(8n - 4)\left(n^2 + 9\right)$

For the following exercises, expand the binomial.

24. $(4x + 5)^2$

25. $(3y - 7)^2$

26. $(12 - 4x)^2$

27. $(4p + 9)^2$

28. $(2m - 3)^2$

29. $(3y - 6)^2$

30. $(9b+1)^2$

For the following exercises, multiply the binomials.

31. $(4c+1)(4c-1)$

32. $(9a-4)(9a+4)$

33. $(15n-6)(15n+6)$

34. $(25b+2)(25b-2)$

35. $(4+4m)(4-4m)$

36. $(14p+7)(14p-7)$

37. $(11q-10)(11q+10)$

For the following exercises, multiply the polynomials.

38. $\left(2x^2+2x+1\right)(4x-1)$

39. $\left(4t^2+t-7\right)\left(4t^2-1\right)$

40. $(x-1)\left(x^2-2x+1\right)$

41. $(y-2)\left(y^2-4y-9\right)$

42. $(6k-5)\left(6k^2+5k-1\right)$

43. $\left(3p^2+2p-10\right)(p-1)$

44. $(4m-13)\left(2m^2-7m+9\right)$

45. $(a+b)(a-b)$

46. $(4x-6y)(6x-4y)$

47. $(4t-5u)^2$

48. $(9m+4n-1)(2m+8)$

49. $(4t-x)(t-x+1)$

50. $\left(b^2-1\right)\left(a^2+2ab+b^2\right)$

51. $(4r-d)(6r+7d)$

52. $(x+y)\left(x^2-xy+y^2\right)$

Real-World Applications

53. A developer wants to purchase a plot of land to build a house. The area of the plot can be described by the following expression: $(4x+1)(8x-3)$ where x is measured in meters. Multiply the binomials to find the area of the plot in standard form.

54. A prospective buyer wants to know how much grain a specific silo can hold. The area of the floor of the silo is $(2x+9)^2$. The height of the silo is $10x+10$, where x is measured in feet. Expand the square and multiply by the height to find the expression that shows how much grain the silo can hold.

Extensions

For the following exercises, perform the given operations.

55. $(4t-7)^2(2t+1)-\left(4t^2+2t+11\right)$

56. $(3b+6)(3b-6)\left(9b^2-36\right)$

57. $\left(a^2+4ac+4c^2\right)\left(a^2-4c^2\right)$

1.5 Factoring Polynomials

Learning Objectives

In this section, you will:

- Factor the greatest common factor of a polynomial.
- Factor a trinomial.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor the sum and difference of cubes.
- Factor expressions using fractional or negative exponents.

COREQUISITE SKILLS

Learning Objectives:

- Master a proven technique for note taking: The Cornell Method.
- Use the Cornell Process to study from your notes.

Objective 1: Master a proven technique for note taking: The Cornell Method.

The **Cornell Method** for taking notes was created by an education professor at Cornell University, Dr. Walter Pauk. The Cornell Method consists of two strategies, a format for notetaking and then the process of using your notes to study. The format for note taking involves separating out the pages of your notebook into three or four separate regions. This method can help summarize information from a lecture, a video, or a reading from your text. You don't need to purchase Cornell paper, just divide your sheets into regions like the illustration below. Write on just one side of the paper so that you can later fold back the left column and quiz yourself using practice test questions.

The top portion is called the **heading**. This is where you write your name, the class, the date, the section of the text and a main topic objective for the day.

The **right section** is where you will take **notes during class**. Try to summarize main ideas here without copying word for word everything your teacher is saying. Use bullet points or numbers to prioritize important ideas. Include here the definitions your teacher presents, and the examples worked in class. It is useful if you can group the content here by learning objectives.

The **left column** about 2 inches wide is used to write questions about the main concepts that were covered in class. For example, write sample **test questions** over the concepts discussed in class in this column. This is also where you should write **cues** about the importance of the information including vocabulary terms, diagrams, formulas or the methods being used to solve. If you like to sketch this is a place to add **illustrations** for key ideas or **icons** so you can quickly identify certain components.

The **summary** will be written in the lower section about 2 inches from the bottom. You will complete this after class **summarizing the important concepts** you were taught in a short compact way. Think of this summary as what you might describe to a friend who happened to miss class that day. **Reflect** on the important ideas here. Add illustrations or a **mind map** to your summary if this helps. You will use this section to find information later when studying.

Topic/Objective:	Name:
	Class/Period:
	Date:
Essential Question:	
Questions:	Notes:
Summary:	

Figure 1 The Cornell Method provides a straightforward, organized, and flexible approach.

Practice Makes Perfect

1. The following book section includes a variety of methods for factoring polynomials. Read the section carefully and complete Cornell notes for the process of factoring. This work will help you throughout the semester because the ability to factor polynomials is one of those linchpin topics that will continue to emerge throughout the term. Remember to write practice factoring examples in the left question or cues column.

   Need some inspiration? Search for videos on Cornell note taking on the web. There are some interesting videos of students showing off their beautiful notes that have been enhanced with highlighters and sketches. Many of the students talk about the difference this technique has made in their learning and attribute their success to this

Cornell format. Another topic to search is sketchnoting, which helps to visually enhance your Cornell notes and truly tell a story.

Objective 2: Use the Cornell Process to study from your notes.

The Cornell Way. Using your Cornell notes to study is referred to as the Cornell Way.
The process includes Cornell note taking (presented in Objective 1), note making, note interacting, and note reflecting. After taking notes in class, follow up with these study methods.

Note making: This is where you will fill in any gaps left in your Cornell notes meaning add in any missed details. Fill in your question column with sample test questions, any formulas used, and highlight, circle or star important any important ideas. Complete your summary row with a few sentences summarizing the important ideas. This should be completed right after class or within the next day. Be creative and add some color, make these notes into something you enjoy working with.

Note interacting: This is the ongoing process of studying from your notes. Fold your left question column back and ask yourself the practice test questions. Include note interactions in the review for your exams. These note interactions can be as short as 5 minutes in length but need to happen regularly and at least daily for the week before exams.

Note reflecting: This is where you assess how helpful your notes were. Do this right after you get back your graded exam. Did the regular note interactions help you to perform better on the exam? Were there problems similar to those you predicted on the exam? Enhance your notes with any important ideas you had initially left out. Then use what you learn from this assessment to improve your note taking in the future.

Practice Makes Perfect

2. Complete the Cornell Process by revisiting your factoring of polynomials summary notes. For now, follow the Note Making, and Note Interacting steps outlined above. Keep these Cornell notes in your divided binder for the course so you can easily refer back when you encounter an exercise requiring factoring throughout the term.

3. Share these completed Cornell notes with your classmates. Fill in any gaps in your work. Work on each other's practice problems.

Imagine that we are trying to find the area of a lawn so that we can determine how much grass seed to purchase. The lawn is the green portion in Figure 1.

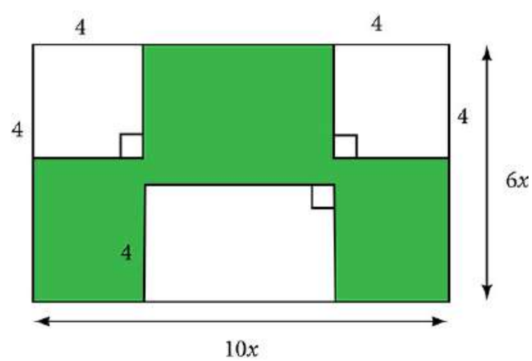


Figure 1

The area of the entire region can be found using the formula for the area of a rectangle.

$$\begin{aligned} A &= lw \\ &= 10x \cdot 6x \\ &= 60x^2 \ \text{units}^2 \end{aligned}$$

The areas of the portions that do not require grass seed need to be subtracted from the area of the entire region. The two square regions each have an area of $A = s^2 = 4^2 = 16$ units^2. The other rectangular region has one side of length $10x - 8$ and one side of length 4, giving an area of $A = lw = 4(10x - 8) = 40x - 32$ units^2. So the region that must be subtracted has an area of $2(16) + 40x - 32 = 40x$ units^2.

The area of the region that requires grass seed is found by subtracting $60x^2 - 40x$ units^2. This area can also be expressed in factored form as $20x(3x-2)$ units². We can confirm that this is an equivalent expression by multiplying.

Many polynomial expressions can be written in simpler forms by factoring. In this section, we will look at a variety of methods that can be used to factor polynomial expressions.

Factoring the Greatest Common Factor of a Polynomial

When we study fractions, we learn that the **greatest common factor** (GCF) of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 The GCF of polynomials works the same way: $4x$ is the GCF of $16x$ and $20x^2$ because it is the largest polynomial that divides evenly into both $16x$ and $20x^2$.

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

Greatest Common Factor

The **greatest common factor** (GCF) of polynomials is the largest polynomial that divides evenly into the polynomials.

HOW TO

Given a polynomial expression, factor out the greatest common factor.

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

EXAMPLE 1

Factoring the Greatest Common Factor

Factor $6x^3y^3 + 45x^2y^2 + 21xy$.

Solution

First, find the GCF of the expression. The GCF of 6, 45, and 21 is 3. The GCF of x^3, x^2 , and x is x . (Note that the GCF of a set of expressions in the form x^n will always be the exponent of lowest degree.) And the GCF of y^3, y^2 , and y is y . Combine these to find the GCF of the polynomial, $3xy$.

Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that $3xy\left(2x^2y^2\right) = 6x^3y^3, 3xy(15xy) = 45x^2y^2$, and $3xy(7) = 21xy$.

Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.

$$(3xy)\left(2x^2y^2 + 15xy + 7\right)$$

Analysis

After factoring, we can check our work by multiplying. Use the distributive property to confirm that $(3xy)\left(2x^2y^2 + 15xy + 7\right) = 6x^3y^3 + 45x^2y^2 + 21xy$.

TRY IT #1 Factor $x\left(b^2 - a\right) + 6\left(b^2 - a\right)$ by pulling out the GCF.

Factoring a Trinomial with Leading Coefficient 1

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that polynomial expressions can be factored. The polynomial $x^2 + 5x + 6$ has a GCF of 1, but it can be written as the product of the factors $(x+2)$ and $(x+3)$.

Trinomials of the form $x^2 + bx + c$ can be factored by finding two numbers with a product of c and a sum of b. The trinomial $x^2 + 10x + 16$, for example, can be factored using the numbers 2 and 8 because the product of those numbers is 16 and their sum is 10. The trinomial can be rewritten as the product of $(x + 2)$ and $(x + 8)$.

Factoring a Trinomial with Leading Coefficient 1

A trinomial of the form $x^2 + bx + c$ can be written in factored form as $(x + p)(x + q)$ where $pq = c$ and $p + q = b$.

Q&A **Can every trinomial be factored as a product of binomials?**

No. Some polynomials cannot be factored. These polynomials are said to be prime.

HOW TO

Given a trinomial in the form $x^2 + bx + c$, factor it.

1. List factors of c.
2. Find p and q, a pair of factors of c with a sum of b.
3. Write the factored expression $(x + p)(x + q)$.

EXAMPLE 2

Factoring a Trinomial with Leading Coefficient 1

Factor $x^2 + 2x - 15$.

Solution

We have a trinomial with leading coefficient 1, $b = 2$, and $c = -15$. We need to find two numbers with a product of -15 and a sum of 2. In the table below, we list factors until we find a pair with the desired sum.

Factors of -15	Sum of Factors
$1, -15$	-14
$-1, 15$	14
$3, -5$	-2
$-3, 5$	2

Now that we have identified p and q as -3 and 5, write the factored form as $(x - 3)(x + 5)$.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(x - 3)(x + 5) = x^2 + 2x - 15$.

Q&A **Does the order of the factors matter?**

No. Multiplication is commutative, so the order of the factors does not matter.

TRY IT #2 Factor $x^2 - 7x + 6$.

Factoring by Grouping

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. For these trinomials, we can **factor by grouping** by dividing the *x* term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression. The trinomial $2x^2 + 5x + 3$ can be rewritten as $(2x + 3)(x + 1)$ using this process. We begin by rewriting the original expression as $2x^2 + 2x + 3x + 3$ and then factor each portion of the expression to obtain $2x(x + 1) + 3(x + 1)$. We then pull out the GCF of $(x + 1)$ to find the factored expression.

Factor by Grouping

To factor a trinomial in the form $ax^2 + bx + c$ by grouping, we find two numbers with a product of ac and a sum of b. We use these numbers to divide the x term into the sum of two terms and factor each portion of the expression separately, then factor out the GCF of the entire expression.

HOW TO

Given a trinomial in the form $ax^2 + bx + c$, factor by grouping.

1. List factors of ac.
2. Find p and q, a pair of factors of ac with a sum of b.
3. Rewrite the original expression as $ax^2 + px + qx + c$.
4. Pull out the GCF of $ax^2 + px$.
5. Pull out the GCF of $qx + c$.
6. Factor out the GCF of the expression.

EXAMPLE 3

Factoring a Trinomial by Grouping

Factor $5x^2 + 7x - 6$ by grouping.

Solution

We have a trinomial with $a = 5, b = 7$, and $c = -6$. First, determine $ac = -30$. We need to find two numbers with a product of -30 and a sum of 7. In the table below, we list factors until we find a pair with the desired sum.

Factors of -30	Sum of Factors
$1, -30$	-29
$-1, 30$	29
$2, -15$	-13
$-2, 15$	13
$3, -10$	-7
$-3, 10$	7

So $p = -3$ and $q = 10$.

$5x^2 - 3x + 10x - 6$	Rewrite the original expression as $ax^2 + px + qx + c$.
$x(5x - 3) + 2(5x - 3)$	Factor out the GCF of each part.
$(5x - 3)(x + 2)$	Factor out the GCF of the expression.

Analysis

We can check our work by multiplying. Use FOIL to confirm that $(5x - 3)(x + 2) = 5x^2 + 7x - 6$.

TRY IT #3 Factor

ⓐ $2x^2 + 9x + 9$ ⓑ $6x^2 + x - 1$

Factoring a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and the square of the last term.

$$\begin{aligned} a^2 + 2ab + b^2 &= (a + b)^2 \\ &\text{and} \\ a^2 - 2ab + b^2 &= (a - b)^2 \end{aligned}$$

We can use this equation to factor any perfect square trinomial.

Perfect Square Trinomials

A perfect square trinomial can be written as the square of a binomial:

$$a^2 + 2ab + b^2 = (a + b)^2$$

HOW TO

Given a perfect square trinomial, factor it into the square of a binomial.

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is twice the product of ab.
3. Write the factored form as $(a + b)^2$.

EXAMPLE 4

Factoring a Perfect Square Trinomial

Factor $25x^2 + 20x + 4$.

Solution

Notice that $25x^2$ and 4 are perfect squares because $25x^2 = (5x)^2$ and $4 = 2^2$. Then check to see if the middle term is twice the product of $5x$ and 2. The middle term is, indeed, twice the product: $2(5x)(2) = 20x$. Therefore, the trinomial is a perfect square trinomial and can be written as $(5x + 2)^2$.

TRY IT #4 Factor $49x^2 - 14x + 1$.

Factoring a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be rewritten as factors containing the same terms but opposite signs because the middle terms cancel each other out when the two factors are multiplied.

$$a^2 - b^2 = (a + b)(a - b)$$

We can use this equation to factor any differences of squares.

Differences of Squares

A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$a^2 - b^2 = (a + b)(a - b)$$

HOW TO

Given a difference of squares, factor it into binomials.

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as $(a + b)(a - b)$.

EXAMPLE 5

Factoring a Difference of Squares

Factor $9x^2 - 25$.

Solution

Notice that $9x^2$ and 25 are perfect squares because $9x^2 = (3x)^2$ and $25 = 5^2$. The polynomial represents a difference of squares and can be rewritten as $(3x + 5)(3x - 5)$.

TRY IT #5 Factor $81y^2 - 100$.

Q&A **Is there a formula to factor the sum of squares?**

No. A sum of squares cannot be factored.

Factoring the Sum and Difference of Cubes

Now, we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial.

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$$

Similarly, the difference of cubes can be factored into a binomial and a trinomial, but with different signs.

$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

We can use the acronym SOAP to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: **S**ame **O**pposite **A**lways **P**ositive. For example, consider the following example.

$$x^3 - 2^3 = (x - 2)\left(x^2 + 2x + 4\right)$$

The sign of the first 2 is the *same* as the sign between $x^3 - 2^3$. The sign of the $2x$ term is *opposite* the sign between $x^3 - 2^3$. And the sign of the last term, 4, is *always positive*.

Sum and Difference of Cubes

We can factor the sum of two cubes as

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$

We can factor the difference of two cubes as

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

HOW TO

Given a sum of cubes or difference of cubes, factor it.

1. Confirm that the first and last term are cubes, $a^3 + b^3$ or $a^3 - b^3$.
2. For a sum of cubes, write the factored form as $(a+b)\left(a^2 - ab + b^2\right)$. For a difference of cubes, write the factored form as $(a-b)\left(a^2 + ab + b^2\right)$.

EXAMPLE 6

Factoring a Sum of Cubes

Factor $x^3 + 512$.

Solution

Notice that x^3 and 512 are cubes because $8^3 = 512$. Rewrite the sum of cubes as $(x+8)\left(x^2 - 8x + 64\right)$.

Analysis

After writing the sum of cubes this way, we might think we should check to see if the trinomial portion can be factored further. However, the trinomial portion cannot be factored, so we do not need to check.

TRY IT #6 Factor the sum of cubes: $216a^3 + b^3$.

EXAMPLE 7

Factoring a Difference of Cubes

Factor $8x^3 - 125$.

Solution

Notice that $8x^3$ and 125 are cubes because $8x^3 = (2x)^3$ and $125 = 5^3$. Write the difference of cubes as $(2x-5)\left(4x^2 + 10x + 25\right)$.

Analysis

Just as with the sum of cubes, we will not be able to further factor the trinomial portion.

TRY IT #7 Factor the difference of cubes: $1{,}000x^3 - 1$.

Factoring Expressions with Fractional or Negative Exponents

Expressions with fractional or negative exponents can be factored by pulling out a GCF. Look for the variable or exponent that is common to each term of the expression and pull out that variable or exponent raised to the lowest power. These expressions follow the same factoring rules as those with integer exponents. For instance, $2x^{\frac{1}{4}} + 5x^{\frac{3}{4}}$ can be factored by pulling out $x^{\frac{1}{4}}$ and being rewritten as $x^{\frac{1}{4}}\left(2 + 5x^{\frac{1}{2}}\right)$.

EXAMPLE 8

Factoring an Expression with Fractional or Negative Exponents

Factor $3x(x+2)^{\frac{-1}{3}} + 4(x+2)^{\frac{2}{3}}$.

Solution

Factor out the term with the lowest value of the exponent. In this case, that would be $(x+2)^{-\frac{1}{3}}$.

$$\begin{array}{ll} (x+2)^{-\frac{1}{3}}(3x+4(x+2)) & \text{Factor out the GCF.} \\ (x+2)^{-\frac{1}{3}}(3x+4x+8) & \text{Simplify.} \\ (x+2)^{-\frac{1}{3}}(7x+8) & \end{array}$$

TRY IT #8 Factor $2(5a-1)^{\frac{3}{4}} + 7a(5a-1)^{-\frac{1}{4}}$.

MEDIA

Access these online resources for additional instruction and practice with factoring polynomials.

[Identify GCF (http://openstax.org/l/findgcftofact)](http://openstax.org/l/findgcftofact)
[Factor Trinomials when a Equals 1 (http://openstax.org/l/facttrinom1)](http://openstax.org/l/facttrinom1)
[Factor Trinomials when a is not equal to 1 (http://openstax.org/l/facttrinom2)](http://openstax.org/l/facttrinom2)
[Factor Sum or Difference of Cubes (http://openstax.org/l/sumdifcube)](http://openstax.org/l/sumdifcube)

1.5 SECTION EXERCISES

Verbal

1. If the terms of a polynomial do not have a GCF, does that mean it is not factorable? Explain.

2. A polynomial is factorable, but it is not a perfect square trinomial or a difference of two squares. Can you factor the polynomial without finding the GCF?

3. How do you factor by grouping?

Algebraic

For the following exercises, find the greatest common factor.

4. $14x + 4xy - 18xy^2$

5. $49mb^2 - 35m^2ba + 77ma^2$

6. $30x^3y - 45x^2y^2 + 135xy^3$

7. $200p^3m^3 - 30p^2m^3 + 40m^3$

8. $36j^4k^2 - 18j^3k^3 + 54j^2k^4$

9. $6y^4 - 2y^3 + 3y^2 - y$

For the following exercises, factor by grouping.

10. $6x^2 + 5x - 4$

11. $2a^2 + 9a - 18$

12. $6c^2 + 41c + 63$

13. $6n^2 - 19n - 11$

14. $20w^2 - 47w + 24$

15. $2p^2 - 5p - 7$

For the following exercises, factor the polynomial.

16. $7x^2 + 48x - 7$

17. $10h^2 - 9h - 9$

18. $2b^2 - 25b - 247$

19. $9d^2 - 73d + 8$

20. $90v^2 - 181v + 90$

21. $12t^2 + t - 13$

22. $2n^2 - n - 15$

23. $16x^2 - 100$

24. $25y^2 - 196$

25. $121p^2 - 169$

26. $4m^2 - 9$

27. $361d^2 - 81$

28. $324x^2 - 121$

29. $144b^2 - 25c^2$

30. $16a^2 - 8a + 1$

31. $49n^2 + 168n + 144$

32. $121x^2 - 88x + 16$

33. $225y^2 + 120y + 16$

34. $m^2 - 20m + 100$

35. $25p^2 - 120p + 144$

36. $36q^2 + 60q + 25$

For the following exercises, factor the polynomials.

37. $x^3 + 216$

38. $27y^3 - 8$

39. $125a^3 + 343$

40. $b^3 - 8d^3$

41. $64x^3 - 125$

42. $729q^3 + 1331$

43. $125r^3 + 1{,}728s^3$

44. $4x(x-1)^{-\frac{2}{3}} + 3(x-1)^{\frac{1}{3}}$

45. $3c(2c+3)^{-\frac{1}{4}} - 5(2c+3)^{\frac{3}{4}}$

46. $3t(10t+3)^{\frac{1}{3}} + 7(10t+3)^{\frac{4}{3}}$

47. $14x(x+2)^{-\frac{2}{5}} + 5(x+2)^{\frac{3}{5}}$

48. $9y(3y-13)^{\frac{1}{5}} - 2(3y-13)^{\frac{6}{5}}$

49. $5z(2z-9)^{-\frac{3}{2}} + 11(2z-9)^{-\frac{1}{2}}$

50. $6d(2d+3)^{-\frac{1}{6}} + 5(2d+3)^{\frac{5}{6}}$

Real-World Applications

For the following exercises, consider this scenario:

Charlotte has appointed a chairperson to lead a city beautification project. The first act is to install statues and fountains in one of the city's parks. The park is a rectangle with an area of $98x^2 + 105x - 27$ m², as shown in the figure below. The length and width of the park are perfect factors of the area.

$l \times w = 98x^2 + 105x - 27$

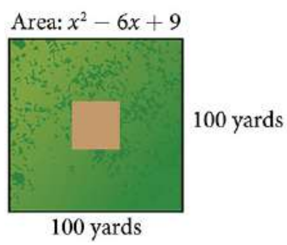
51. Factor by grouping to find the length and width of the park.

52. A statue is to be placed in the center of the park. The area of the base of the statue is $4x^2 + 12x + 9\text{m}^2$. Factor the area to find the lengths of the sides of the statue.

53. At the northwest corner of the park, the city is going to install a fountain. The area of the base of the fountain is $9x^2 - 25\text{m}^2$. Factor the area to find the lengths of the sides of the fountain.

For the following exercise, consider the following scenario:

A school is installing a flagpole in the central plaza. The plaza is a square with side length 100 yd. as shown in the figure below. The flagpole will take up a square plot with area $x^2 - 6x + 9$ yd².



54. Find the length of the base of the flagpole by factoring.

Extensions

For the following exercises, factor the polynomials completely.

55. $16x^4 - 200x^2 + 625$

56. $81y^4 - 256$

57. $16z^4 - 2{,}401a^4$

58. $5x(3x+2)^{-\frac{2}{4}} + (12x+8)^{\frac{3}{2}}$

59. $\left(32x^3 + 48x^2 - 162x - 243\right)^{-1}$

1.6 Rational Expressions

Learning Objectives

In this section, you will:

- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

COREQUISITE SKILLS

Learning Objectives

- Identify the skills leading to successful preparation for a college level mathematics exam.
- Create a plan for success when taking mathematics exams.

Objective 1: Identify the skills leading to successful preparation for a college level mathematics exam.

Complete the following surveys by placing a checkmark in the a column for each strategy based on the frequency that you engaged in the strategy during your last academic term.

Exam Preparation Strategies	Always	Sometimes	Never
1. Rework each of the examples my instructor did in class.			
2. Create note cards to help in memorizing important formulas and problem-solving strategies for the exam.			
3. Create a study schedule for each math exam and begin to study for the exam at least one week prior to the date. Spaced practice over 5-7 days is much more effective than cramming material in 1-2 sessions.			
4. Work the review exercises at the end of each chapter of the text.			
5. Visit my instructor’s office hours when I need assistance in preparing for an exam.			
6. Spend time on note interactions (see the section on Cornell notes) each day.			
7. Create a practice test using the questions I identified in my class notes (see the section on Cornell notes) and take it the week before the exam.			
8. Review each of the student learning objectives at the beginning of all sections covered on the exam and use this list as a checklist for exam preparation.			
9. Ask your instructor how many questions will be on the exam and if they award partial credit for work shown.			
10. Work through the practice test at the end of each chapter of the text.			
11. Get a good night’s sleep the night before my exam.			
12. Come to each exam prepared with a goal of earning an A.			

Exam Day Behaviors and Strategies	Always	Sometimes	Never
13. Make sure to grab a healthy breakfast the day of the exam.			
14. Arrive or log in early to class on exam days.			
15. Keep my phone put away in my bag during exams to avoid distractions.			
16. Try to relax and take a few deep breaths before beginning the exam.			
17. Use a pencil so that I can make corrections neatly.			
18. Read through all directions before beginning the exam.			
19. Write formulas that are memorized in the margins, top or back of the test to reference when needed.			

Exam Day Behaviors and Strategies	Always	Sometimes	Never
20. Scan through my entire test before beginning and start off working on a problem I am confident in solving.			
21. Work each of the questions that I find easier first.			
22. Keep track of time. Do a quick assessment of how much time should be spent on each question.			
23. Try different approaches to solve when I get stuck on a problem.			
24. Draw a diagram when solving an application problem.			
25. Do some work on each question.			
26. Work neatly and show all steps.			
27. Make sure to attach units to final answers when units are given in the problem. (for example: cm, $, or feet/second)			
28. Stay working for the entire class session or online exam session. If finished early I use the additional time to review my work and check answers.			
29. Circle final answers or write each on the answer blank.			

After the Exam Behaviors and Strategies	Always	Sometimes	Never
30. I work back through my exams after they are returned, writing corrections in another color or highlighting them for future reference.			
31. Keep my old exams in a binder or notebook and use this assessment to review for my final exam.			
32. Take responsibility for my exam performance and try to learn from the experience.			
33. Reflect on the test taking experience and make a list for yourself on what to do differently next time.			
34. Reflect on your feelings while taking the exam. Plan to replace any negative self-statements with positive ones on future exams.			
35. Celebrate my success after doing well on an exam! Talk to a friend or family member about my progress.			

Scoring			
Total Number in Each Column			
Scoring:	**Always: 4 points each**	**Sometimes: 2 points each**	**Never: 0 points each**
Total points:			**0**

Practice Makes Perfect

Practice: Identify the skills leading to successful preparation for a college level mathematics exam.

1. Each of the behaviors or attitudes listed in the table above are associated with successful college mathematics exam preparation. This means that students who use these strategies or are open to these beliefs pass their college math courses. Compute your total score and share your score with your study group in class. Be supportive of your fellow students and offer encouragement!

Total score =_________

2. Based on this survey, create a list of the top 5 test preparation and taking strategies that you currently utilize, and feel are most helpful to you.
1.
2.
3.
4.
5.

3. Based on this survey, create a list of the top 5 test preparation and taking strategies that interest you, and that you feel could be most helpful to you this term. Plan on implementing these strategies.
1.
2.
3.
4.
5.

Objective 2: Create a plan for success when taking mathematics exams.

1. It's important to take the opportunity to reflect on your past experiences in taking math exams as you begin a new term. We can learn a lot from these reflections and thus work toward developing a strategy for improvement. In the table below list 5 challenges you have had in past math courses when taking an exam and list a possible solution that you could try this semester.

Challenge:	Possible Solution:
1. _______	
2. _______	
3. _______	
4. _______	
5. _______	

2. Develop your plan for success. Keep in mind the idea of mindsets and try to approach your test taking strategies with a growth mindset. Now is the time for growth as you begin a new term. Share your plan with your study group

members.

A pastry shop has fixed costs of $\$280$ per week and variable costs of $\$9$ per box of pastries. The shop's costs per week in terms of x, the number of boxes made, is $280 + 9x$. We can divide the costs per week by the number of boxes made to determine the cost per box of pastries.

$$\frac{280 + 9x}{x}$$

Notice that the result is a polynomial expression divided by a second polynomial expression. In this section, we will explore quotients of polynomial expressions.

Simplifying Rational Expressions

The quotient of two polynomial expressions is called a **rational expression**. We can apply the properties of fractions to rational expressions, such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.

$$\frac{x^2 + 8x + 16}{x^2 + 11x + 28}$$

We can factor the numerator and denominator to rewrite the expression.

$$\frac{(x + 4)^2}{(x + 4)(x + 7)}$$

Then we can simplify that expression by canceling the common factor $(x + 4)$.

$$\frac{x + 4}{x + 7}$$

HOW TO

Given a rational expression, simplify it.

1. Factor the numerator and denominator.
2. Cancel any common factors.

EXAMPLE 1

Simplifying Rational Expressions

Simplify $\frac{x^2-9}{x^2+4x+3}$.

Solution

$\frac{(x+3)(x-3)}{(x+3)(x+1)}$	Factor the numerator and the denominator.
$\frac{x-3}{x+1}$	Cancel common factor $(x + 3)$.

Analysis

We can cancel the common factor because any expression divided by itself is equal to 1.

Q&A **Can the x^2 term be cancelled in Example 1?**

No. A factor is an expression that is multiplied by another expression. The x^2 term is not a factor of the numerator or the denominator.

TRY IT #1 Simplify $\frac{x-6}{x^2-36}$.

Multiplying Rational Expressions

Multiplication of rational expressions works the same way as multiplication of any other fractions. We multiply the numerators to find the numerator of the product, and then multiply the denominators to find the denominator of the product. Before multiplying, it is helpful to factor the numerators and denominators just as we did when simplifying rational expressions. We are often able to simplify the product of rational expressions.

HOW TO

Given two rational expressions, multiply them.

1. Factor the numerator and denominator.
2. Multiply the numerators.
3. Multiply the denominators.
4. Simplify.

EXAMPLE 2

Multiplying Rational Expressions

Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2+4x-5}{3x+18}\cdot\frac{2x-1}{x+5}$$

Solution

$\frac{(x+5)(x-1)}{3(x+6)}\cdot\frac{(2x-1)}{(x+5)}$	Factor the numerator and denominator.
$\frac{(x+5)(x-1)(2x-1)}{3(x+6)(x+5)}$	Multiply numerators and denominators.
$\frac{\cancel{(x+5)}(x-1)(2x-1)}{3(x+6)\cancel{(x+5)}}$	Cancel common factors to simplify.
$\frac{(x-1)(2x-1)}{3(x+6)}$	

TRY IT #2 Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2+11x+30}{x^2+5x+6}\cdot\frac{x^2+7x+12}{x^2+8x+16}$$

Dividing Rational Expressions

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite $\frac{1}{x}\div\frac{x^2}{3}$ as the product $\frac{1}{x}\cdot\frac{3}{x^2}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.

$$\frac{1}{x}\cdot\frac{3}{x^2}=\frac{3}{x^3}$$

HOW TO

Given two rational expressions, divide them.

1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators.

4. Multiply the denominators.
5. Simplify.

EXAMPLE 3

Dividing Rational Expressions

Divide the rational expressions and express the quotient in simplest form:

$$\frac{2x^2 + x - 6}{x^2 - 1} \div \frac{x^2 - 4}{x^2 + 2x + 1}$$

Solution

$\frac{2x^2+x-6}{x^2-1} \cdot \frac{x^2+2x+1}{x^2-4}$	Rewrite as multiplication.
$\frac{(2x-3)(x+2)}{(x+1)(x-1)} \cdot \frac{(x+1)^2}{(x+2)(x-2)}$	Factor.
$\frac{(2x-3)(x+2)(x+1)^2}{(x+1)(x-1)(x+2)(x-2)}$	Multiply.
$\frac{(2x-3)(x+1)}{(x-1)(x-2)}$	Cancel common factors to simplify.

TRY IT #3 Divide the rational expressions and express the quotient in simplest form:

$$\frac{9x^2 - 16}{3x^2 + 17x - 28} \div \frac{3x^2 - 2x - 8}{x^2 + 5x - 14}$$

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let's look at an example of fraction addition.

$$\begin{aligned} \frac{5}{24} + \frac{1}{40} &= \frac{25}{120} + \frac{3}{120} \\ &= \frac{28}{120} \\ &= \frac{7}{30} \end{aligned}$$

We have to rewrite the fractions so they share a common denominator before we are able to add. We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the **least common denominator**, or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, if the factored denominators were $(x+3)(x+4)$ and $(x+4)(x+5)$, then the LCD would be $(x+3)(x+4)(x+5)$.

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD. We would need to multiply the expression with a denominator of $(x+3)(x+4)$ by $\frac{x+5}{x+5}$ and the expression with a denominator of $(x+4)(x+5)$ by $\frac{x+3}{x+3}$.

HOW TO

Given two rational expressions, add or subtract them.

1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators.
5. Simplify.

EXAMPLE 4

Adding Rational Expressions

Add the rational expressions:

$$\frac{5}{x}+\frac{6}{y}$$

Solution

First, we have to find the LCD. In this case, the LCD will be xy. We then multiply each expression by the appropriate form of 1 to obtain xy as the denominator for each fraction.

$$\frac{5}{x}\cdot\frac{y}{y}+\frac{6}{y}\cdot\frac{x}{x}$$
$$\frac{5y}{xy}+\frac{6x}{xy}$$

Now that the expressions have the same denominator, we simply add the numerators to find the sum.

$$\frac{6x+5y}{xy}$$

Analysis

Multiplying by $\frac{y}{y}$ or $\frac{x}{x}$ does not change the value of the original expression because any number divided by itself is 1, and multiplying an expression by 1 gives the original expression.

EXAMPLE 5

Subtracting Rational Expressions

Subtract the rational expressions:

$$\frac{6}{x^2+4x+4}-\frac{2}{x^2-4}$$

Solution

$\frac{6}{(x+2)^2}-\frac{2}{(x+2)(x-2)}$	Factor.
$\frac{6}{(x+2)^2}\cdot\frac{x-2}{x-2}-\frac{2}{(x+2)(x-2)}\cdot\frac{x+2}{x+2}$	Multiply each fraction to get LCD as denominator.
$\frac{6(x-2)}{(x+2)^2(x-2)}-\frac{2(x+2)}{(x+2)^2(x-2)}$	Multiply.
$\frac{6x-12-(2x+4)}{(x+2)^2(x-2)}$	Apply distributive property.
$\frac{4x-16}{(x+2)^2(x-2)}$	Subtract.
$\frac{4(x-4)}{(x+2)^2(x-2)}$	Simplify.

Q&A **Do we have to use the LCD to add or subtract rational expressions?**

No. Any common denominator will work, but it is easiest to use the LCD.

TRY IT #4 Subtract the rational expressions: $\frac{3}{x+5}-\frac{1}{x-3}$.

Simplifying Complex Rational Expressions

A complex rational expression is a rational expression that contains additional rational expressions in the numerator, the denominator, or both. We can simplify complex rational expressions by rewriting the numerator and denominator as single rational expressions and dividing. The complex rational expression $\frac{a}{\frac{1}{b}+c}$ can be simplified by rewriting the numerator as the fraction $\frac{a}{1}$ and combining the expressions in the denominator as $\frac{1+bc}{b}$. We can then rewrite the expression as a multiplication problem using the reciprocal of the denominator. We get $\frac{a}{1} \cdot \frac{b}{1+bc}$, which is equal to $\frac{ab}{1+bc}$.

HOW TO

Given a complex rational expression, simplify it.

1. Combine the expressions in the numerator into a single rational expression by adding or subtracting.
2. Combine the expressions in the denominator into a single rational expression by adding or subtracting.
3. Rewrite as the numerator divided by the denominator.
4. Rewrite as multiplication.
5. Multiply.
6. Simplify.

EXAMPLE 6

Simplifying Complex Rational Expressions

Simplify: $\frac{y+\frac{1}{x}}{\frac{x}{y}}$.

Solution

Begin by combining the expressions in the numerator into one expression.

$y \cdot \frac{x}{x} + \frac{1}{x}$	Multiply by $\frac{x}{x}$ to get LCD as denominator.
$\frac{xy}{x} + \frac{1}{x}$	
$\frac{xy+1}{x}$	Add numerators.

Now the numerator is a single rational expression and the denominator is a single rational expression.

$$\frac{\frac{xy+1}{x}}{\frac{x}{y}}$$

We can rewrite this as division, and then multiplication.

$\frac{xy+1}{x} \div \frac{x}{y}$	
$\frac{xy+1}{x} \cdot \frac{y}{x}$	Rewrite as multiplication.
$\frac{y(xy+1)}{x^2}$	Multiply.

TRY IT #5 Simplify: $\frac{\frac{x}{y}-\frac{y}{x}}{y}$

Q&A **Can a complex rational expression always be simplified?**

Yes. We can always rewrite a complex rational expression as a simplified rational expression.

MEDIA

Access these online resources for additional instruction and practice with rational expressions.

[Simplify Rational Expressions (http://openstax.org/l/simpratexpress)](http://openstax.org/l/simpratexpress)
[Multiply and Divide Rational Expressions (http://openstax.org/l/multdivratex)](http://openstax.org/l/multdivratex)
[Add and Subtract Rational Expressions (http://openstax.org/l/addsubratex)](http://openstax.org/l/addsubratex)
[Simplify a Complex Fraction (http://openstax.org/l/complexfract)](http://openstax.org/l/complexfract)

1.6 SECTION EXERCISES

Verbal

1. How can you use factoring to simplify rational expressions?

2. How do you use the LCD to combine two rational expressions?

3. Tell whether the following statement is true or false and explain why: You only need to find the LCD when adding or subtracting rational expressions.

Algebraic

For the following exercises, simplify the rational expressions.

4. $\frac{x^2-16}{x^2-5x+4}$

5. $\frac{y^2+10y+25}{y^2+11y+30}$

6. $\frac{6a^2-24a+24}{6a^2-24}$

7. $\frac{9b^2+18b+9}{3b+3}$

8. $\frac{m-12}{m^2-144}$

9. $\frac{2x^2+7x-4}{4x^2+2x-2}$

10. $\frac{6x^2+5x-4}{3x^2+19x+20}$

11. $\frac{a^2+9a+18}{a^2+3a-18}$

12. $\frac{3c^2+25c-18}{3c^2-23c+14}$

13. $\frac{12n^2-29n-8}{28n^2-5n-3}$

For the following exercises, multiply the rational expressions and express the product in simplest form.

14. $\frac{x^2-x-6}{2x^2+x-6} \cdot \frac{2x^2+7x-15}{x^2-9}$

15. $\frac{c^2+2c-24}{c^2+12c+36} \cdot \frac{c^2-10c+24}{c^2-8c+16}$

16. $\frac{2d^2+9d-35}{d^2+10d+21} \cdot \frac{3d^2+2d-21}{3d^2+14d-49}$

17. $\frac{10h^2-9h-9}{2h^2-19h+24} \cdot \frac{h^2-16h+64}{5h^2-37h-24}$

18. $\frac{6b^2+13b+6}{4b^2-9} \cdot \frac{6b^2+31b-30}{18b^2-3b-10}$

19. $\frac{2d^2+15d+25}{4d^2-25} \cdot \frac{2d^2-15d+25}{25d^2-1}$

20. $\frac{6x^2-5x-50}{15x^2-44x-20} \cdot \frac{20x^2-7x-6}{2x^2+9x+10}$

21. $\frac{t^2-1}{t^2+4t+3} \cdot \frac{t^2+2t-15}{t^2-4t+3}$

22. $\frac{2n^2-n-15}{6n^2+13n-5} \cdot \frac{12n^2-13n+3}{4n^2-15n+9}$

23. $\frac{36x^2-25}{6x^2+65x+50} \cdot \frac{3x^2+32x+20}{18x^2+27x+10}$

For the following exercises, divide the rational expressions.

24. $\frac{3y^2-7y-6}{2y^2-3y-9} \div \frac{y^2+y-2}{2y^2+y-3}$

25. $\frac{6p^2+p-12}{8p^2+18p+9} \div \frac{6p^2-11p+4}{2p^2+11p-6}$

26. $\frac{q^2-9}{q^2+6q+9} \div \frac{q^2-2q-3}{q^2+2q-3}$

27. $\frac{18d^2+77d-18}{27d^2-15d+2} \div \frac{3d^2+29d-44}{9d^2-15d+4}$

28. $\frac{16x^2+18x-55}{32x^2-36x-11} \div \frac{2x^2+17x+30}{4x^2+25x+6}$

29. $\frac{144b^2-25}{72b^2-6b-10} \div \frac{18b^2-21b+5}{36b^2-18b-10}$

30. $\frac{16a^2-24a+9}{4a^2+17a-15} \div \frac{16a^2-9}{4a^2+11a+6}$

31. $\frac{22y^2+59y+10}{12y^2+28y-5} \div \frac{11y^2+46y+8}{24y^2-10y+1}$

32. $\frac{9x^2+3x-20}{3x^2-7x+4} \div \frac{6x^2+4x-10}{x^2-2x+1}$

For the following exercises, add and subtract the rational expressions, and then simplify.

33. $\frac{4}{x}+\frac{10}{y}$

34. $\frac{12}{2q}-\frac{6}{3p}$

35. $\frac{4}{a+1}+\frac{5}{a-3}$

36. $\frac{c+2}{3}-\frac{c-4}{4}$

37. $\frac{y+3}{y-2}+\frac{y-3}{y+1}$

38. $\frac{x-1}{x+1}-\frac{2x+3}{2x+1}$

39. $\frac{3z}{z+1}+\frac{2z+5}{z-2}$

40. $\frac{4p}{p+1}-\frac{p+1}{4p}$

41. $\frac{x}{x+1}+\frac{y}{y+1}$

For the following exercises, simplify the rational expression.

42. $\frac{\frac{6}{y}-\frac{4}{x}}{y}$

43. $\frac{\frac{2}{a}+\frac{7}{b}}{b}$

44. $\frac{\frac{x}{4}-\frac{p}{8}}{p}$

45. $\frac{\frac{3}{a}+\frac{b}{6}}{\frac{2b}{3a}}$

46. $\frac{\frac{3}{x+1}+\frac{2}{x-1}}{\frac{x-1}{x+1}}$

47. $\frac{\frac{a}{b}-\frac{b}{a}}{\frac{a+b}{ab}}$

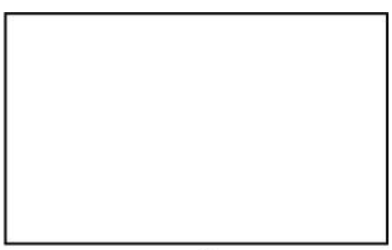
48. $\frac{\frac{2x}{3}+\frac{4x}{7}}{\frac{x}{2}}$

49. $\frac{\frac{2c}{c+2}+\frac{c-1}{c+1}}{\frac{2c+1}{c+1}}$

50. $\frac{\frac{x}{y}-\frac{y}{x}}{\frac{x}{y}+\frac{y}{x}}$

Real-World Applications

51. Brenda is placing tile on her bathroom floor. The area of the floor is $15x^2-8x-7$ ft^2. The area of one tile is x^2-2x+1ft^2. To find the number of tiles needed, simplify the rational expression: $\frac{15x^2-8x-7}{x^2-2x+1}$.



Area = $15x^2-8x-7$

52. The area of Lijuan's yard is $25x^2-625$ ft^2. A patch of sod has an area of $x^2-10x+25$ ft^2. Divide the two areas and simplify to find how many pieces of sod Lijuan needs to cover her yard.

53. Elroi wants to mulch his garden. His garden is $x^2+18x+81$ ft^2. One bag of mulch covers x^2-81 ft^2. Divide the expressions and simplify to find how many bags of mulch Elroi needs to mulch his garden.

Extensions

For the following exercises, perform the given operations and simplify.

54. $\frac{x^2+x-6}{x^2-2x-3}\cdot\frac{2x^2-3x-9}{x^2-x-2}\div\frac{10x^2+27x+18}{x^2+2x+1}$

55. $\frac{\frac{3y^2-10y+3}{3y^2+5y-2}\cdot\frac{2y^2-3y-20}{2y^2-y-15}}{y-4}$

56. $\frac{\frac{4a+1}{2a-3}+\frac{2a-3}{2a+3}}{\frac{4a^2+9}{a}}$

57. $\frac{x^2+7x+12}{x^2+x-6}\div\frac{3x^2+19x+28}{8x^2-4x-24}\div\frac{2x^2+x-3}{3x^2+4x-7}$

Chapter Review

Key Terms

algebraic expression constants and variables combined using addition, subtraction, multiplication, and division

associative property of addition the sum of three numbers may be grouped differently without affecting the result; in symbols, $a + (b + c) = (a + b) + c$

associative property of multiplication the product of three numbers may be grouped differently without affecting the result; in symbols, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

base in exponential notation, the expression that is being multiplied

binomial a polynomial containing two terms

coefficient any real number a_i in a polynomial in the form $a_n x^n + ... + a_2 x^2 + a_1 x + a_0$

commutative property of addition two numbers may be added in either order without affecting the result; in symbols, $a + b = b + a$

commutative property of multiplication two numbers may be multiplied in any order without affecting the result; in symbols, $a \cdot b = b \cdot a$

constant a quantity that does not change value

degree the highest power of the variable that occurs in a polynomial

difference of squares the binomial that results when a binomial is multiplied by a binomial with the same terms, but the opposite sign

distributive property the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols, $a \cdot (b + c) = a \cdot b + a \cdot c$

equation a mathematical statement indicating that two expressions are equal

exponent in exponential notation, the raised number or variable that indicates how many times the base is being multiplied

exponential notation a shorthand method of writing products of the same factor

factor by grouping a method for factoring a trinomial in the form $ax^2 + bx + c$ by dividing the *x* term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression

formula an equation expressing a relationship between constant and variable quantities

greatest common factor the largest polynomial that divides evenly into each polynomial

identity property of addition there is a unique number, called the additive identity, 0, which, when added to a number, results in the original number; in symbols, $a + 0 = a$

identity property of multiplication there is a unique number, called the multiplicative identity, 1, which, when multiplied by a number, results in the original number; in symbols, $a \cdot 1 = a$

index the number above the radical sign indicating the *n*th root

integers the set consisting of the natural numbers, their opposites, and 0: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

inverse property of addition for every real number a, there is a unique number, called the additive inverse (or opposite), denoted $-a$, which, when added to the original number, results in the additive identity, 0; in symbols, $a + (-a) = 0$

inverse property of multiplication for every non-zero real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, which, when multiplied by the original number, results in the multiplicative identity, 1; in symbols, $a \cdot \frac{1}{a} = 1$

irrational numbers the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers

leading coefficient the coefficient of the leading term

leading term the term containing the highest degree

least common denominator the smallest multiple that two denominators have in common

monomial a polynomial containing one term

natural numbers the set of counting numbers: $\{1, 2, 3, \dots\}$

order of operations a set of rules governing how mathematical expressions are to be evaluated, assigning priorities to operations

perfect square trinomial the trinomial that results when a binomial is squared

polynomial a sum of terms each consisting of a variable raised to a nonnegative integer power

principal *n*th root the number with the same sign as a that when raised to the *n*th power equals a

principal square root the nonnegative square root of a number a that, when multiplied by itself, equals a

radical the symbol used to indicate a root

radical expression an expression containing a radical symbol

radicand the number under the radical symbol

rational expression the quotient of two polynomial expressions
rational numbers the set of all numbers of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$. Any rational number may be written as a fraction or a terminating or repeating decimal.
real number line a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0; positive numbers lie to the right of 0 and negative numbers to the left.
real numbers the sets of rational numbers and irrational numbers taken together
scientific notation a shorthand notation for writing very large or very small numbers in the form $a \times 10^n$ where $1 \leq |a| < 10$ and n is an integer
term of a polynomial any $a_i x^i$ of a polynomial in the form $a_n x^n + ... + a_2 x^2 + a_1 x + a_0$
trinomial a polynomial containing three terms
variable a quantity that may change value
whole numbers the set consisting of 0 plus the natural numbers: $\{0, 1, 2, 3, \ldots\}$

Key Equations

Rules of Exponents For nonzero real numbers a and b and integers m and n	
Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}$
Power rule	$(a^m)^n = a^{m \cdot n}$
Zero exponent rule	$a^0 = 1$
Negative rule	$a^{-n} = \frac{1}{a^n}$
Power of a product rule	$(a \cdot b)^n = a^n \cdot b^n$
Power of a quotient rule	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

perfect square trinomial	$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2$
difference of squares	$(a+b)(a-b) = a^2 - b^2$

difference of squares	$a^2 - b^2 = (a+b)(a-b)$
perfect square trinomial	$a^2 + 2ab + b^2 = (a+b)^2$
sum of cubes	$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$
difference of cubes	$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

Key Concepts

1.1 Real Numbers: Algebra Essentials

- Rational numbers may be written as fractions or terminating or repeating decimals. See Example 1 and Example 2.
- Determine whether a number is rational or irrational by writing it as a decimal. See Example 3.
- The rational numbers and irrational numbers make up the set of real numbers. See Example 4. A number can be classified as natural, whole, integer, rational, or irrational. See Example 5.

- The order of operations is used to evaluate expressions. See Example 6.
- The real numbers under the operations of addition and multiplication obey basic rules, known as the properties of real numbers. These are the commutative properties, the associative properties, the distributive property, the identity properties, and the inverse properties. See Example 7.
- Algebraic expressions are composed of constants and variables that are combined using addition, subtraction, multiplication, and division. See Example 8. They take on a numerical value when evaluated by replacing variables with constants. See Example 9, Example 10, and Example 12
- Formulas are equations in which one quantity is represented in terms of other quantities. They may be simplified or evaluated as any mathematical expression. See Example 11 and Example 13.

1.2 Exponents and Scientific Notation

- Products of exponential expressions with the same base can be simplified by adding exponents. See Example 1.
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents. See Example 2.
- Powers of exponential expressions with the same base can be simplified by multiplying exponents. See Example 3.
- An expression with exponent zero is defined as 1. See Example 4.
- An expression with a negative exponent is defined as a reciprocal. See Example 5 and Example 6.
- The power of a product of factors is the same as the product of the powers of the same factors. See Example 7.
- The power of a quotient of factors is the same as the quotient of the powers of the same factors. See Example 8.
- The rules for exponential expressions can be combined to simplify more complicated expressions. See Example 9.
- Scientific notation uses powers of 10 to simplify very large or very small numbers. See Example 10 and Example 11.
- Scientific notation may be used to simplify calculations with very large or very small numbers. See Example 12 and Example 13.

1.3 Radicals and Rational Exponents

- The principal square root of a number a is the nonnegative number that when multiplied by itself equals a. See Example 1.
- If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b See Example 2 and Example 3.
- If a and b are nonnegative, the square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b See Example 4 and Example 5.
- We can add and subtract radical expressions if they have the same radicand and the same index. See Example 6 and Example 7.
- Radical expressions written in simplest form do not contain a radical in the denominator. To eliminate the square root radical from the denominator, multiply both the numerator and the denominator by the conjugate of the denominator. See Example 8 and Example 9.
- The principal nth root of a is the number with the same sign as a that when raised to the nth power equals a. These roots have the same properties as square roots. See Example 10.
- Radicals can be rewritten as rational exponents and rational exponents can be rewritten as radicals. See Example 11 and Example 12.
- The properties of exponents apply to rational exponents. See Example 13.

1.4 Polynomials

- A polynomial is a sum of terms each consisting of a variable raised to a non-negative integer power. The degree is the highest power of the variable that occurs in the polynomial. The leading term is the term containing the highest degree, and the leading coefficient is the coefficient of that term. See Example 1.
- We can add and subtract polynomials by combining like terms. See Example 2 and Example 3.
- To multiply polynomials, use the distributive property to multiply each term in the first polynomial by each term in the second. Then add the products. See Example 4.
- FOIL (First, Outer, Inner, Last) is a shortcut that can be used to multiply binomials. See Example 5.
- Perfect square trinomials and difference of squares are special products. See Example 6 and Example 7.
- Follow the same rules to work with polynomials containing several variables. See Example 8.

1.5 Factoring Polynomials

- The greatest common factor, or GCF, can be factored out of a polynomial. Checking for a GCF should be the first step in any factoring problem. See Example 1.
- Trinomials with leading coefficient 1 can be factored by finding numbers that have a product of the third term and a sum of the second term. See Example 2.

- Trinomials can be factored using a process called factoring by grouping. See Example 3.
- Perfect square trinomials and the difference of squares are special products and can be factored using equations. See Example 4 and Example 5.
- The sum of cubes and the difference of cubes can be factored using equations. See Example 6 and Example 7.
- Polynomials containing fractional and negative exponents can be factored by pulling out a GCF. See Example 8.

1.6 Rational Expressions

- Rational expressions can be simplified by cancelling common factors in the numerator and denominator. See Example 1.
- We can multiply rational expressions by multiplying the numerators and multiplying the denominators. See Example 2.
- To divide rational expressions, multiply by the reciprocal of the second expression. See Example 3.
- Adding or subtracting rational expressions requires finding a common denominator. See Example 4 and Example 5.
- Complex rational expressions have fractions in the numerator or the denominator. These expressions can be simplified. See Example 6.

Exercises

Review Exercises

Real Numbers: Algebra Essentials

For the following exercises, perform the given operations.

1. $(5-3\cdot 2)^2-6$

2. $64\div(2\cdot 8)+14\div 7$

3. $2\cdot 5^2+6\div 2$

For the following exercises, solve the equation.

4. $5x+9=-11$

5. $2y+4^2=64$

For the following exercises, simplify the expression.

6. $9\,(y+2)\div 3\cdot 2+1$

7. $3m\,(4+7)-m$

For the following exercises, identify the number as rational, irrational, whole, or natural. Choose the most descriptive answer.

8. 11

9. 0

10. $\frac{5}{6}$

11. $\sqrt{11}$

Exponents and Scientific Notation

For the following exercises, simplify the expression.

12. $2^2\cdot 2^4$

13. $\frac{4^5}{4^3}$

14. $\left(\frac{a^2}{b^3}\right)^4$

15. $\frac{6a^2\cdot a^0}{2a^{-4}}$

16. $\frac{(xy)^4}{y^3}\cdot\frac{2}{x^5}$

17. $\frac{4^{-2}x^3y^{-3}}{2x^0}$

18. $\left(\frac{2x^2}{y}\right)^{-2}$

19. $\left(\frac{16a^3}{b^2}\right)\left(4ab^{-1}\right)^{-2}$

20. Write the number in standard notation: $2.1314\ \times\ 10^{-6}$

21. Write the number in scientific notation: 16,340,000

Radicals and Rational Expressions

For the following exercises, find the principal square root.

22. $\sqrt{121}$

23. $\sqrt{196}$

24. $\sqrt{361}$

25. $\sqrt{75}$

26. $\sqrt{162}$

27. $\sqrt{\frac{32}{25}}$

28. $\sqrt{\frac{80}{81}}$

29. $\sqrt{\frac{49}{1250}}$

30. $\frac{2}{4+\sqrt{2}}$

31. $4\sqrt{3}+6\sqrt{3}$

32. $12\sqrt{5}-13\sqrt{5}$

33. $\sqrt[5]{-243}$

34. $\frac{\sqrt[3]{250}}{\sqrt[3]{-8}}$

Polynomials

For the following exercises, perform the given operations and simplify.

35. $\left(3x^3+2x-1\right)+\left(4x^2-2x+7\right)$

36. $(2y+1)-\left(2y^2-2y-5\right)$

37. $\left(2x^2+3x-6\right)+\left(3x^2-4x+9\right)$

38. $\left(6a^2+3a+10\right)-\left(6a^2-3a+5\right)$

39. $(k+3)(k-6)$

40. $(2h+1)(3h-2)$

41. $(x+1)\left(x^2+1\right)$

42. $(m-2)\left(m^2+2m-3\right)$

43. $(a+2b)(3a-b)$

44. $(x+y)(x-y)$

Factoring Polynomials

For the following exercises, find the greatest common factor.

45. $81p+9pq-27p^2q^2$

46. $12x^2y+4xy^2-18xy$

47. $88a^3b+4a^2b-144a^2$

For the following exercises, factor the polynomial.

48. $2x^2-9x-18$

49. $8a^2+30a-27$

50. $d^2-5d-66$

51. $x^2+10x+25$

52. y^2-6y+9

53. $4h^2-12hk+9k^2$

54. $361x^2-121$

55. p^3+216

56. $8x^3-125$

57. $64q^3-27p^3$

58. $4x(x-1)^{-\frac{1}{4}}+3(x-1)^{\frac{3}{4}}$

59. $3p(p+3)^{\frac{1}{3}}-8(p+3)^{\frac{4}{3}}$

60. $4r(2r-1)^{-\frac{2}{3}} - 5(2r-1)^{\frac{1}{3}}$

Rational Expressions

For the following exercises, simplify the expression.

61. $\frac{x^2-x-12}{x^2-8x+16}$

62. $\frac{4y^2-25}{4y^2-20y+25}$

63. $\frac{2a^2-a-3}{2a^2-6a-8} \cdot \frac{5a^2-19a-4}{10a^2-13a-3}$

64. $\frac{d-4}{d^2-9} \cdot \frac{d-3}{d^2-16}$

65. $\frac{m^2+5m+6}{2m^2-5m-3} \div \frac{2m^2+3m-9}{4m^2-4m-3}$

66. $\frac{4d^2-7d-2}{6d^2-17d+10} \div \frac{8d^2+6d+1}{6d^2+7d-10}$

67. $\frac{10}{x} + \frac{6}{y}$

68. $\frac{12}{a^2+2a+1} - \frac{3}{a^2-1}$

69. $\frac{\frac{1}{d}+\frac{2}{c}}{\frac{6c+12d}{dc}}$

70. $\frac{\frac{3}{x}-\frac{7}{y}}{\frac{2}{x}}$

Practice Test

For the following exercises, identify the number as rational, irrational, whole, or natural. Choose the most descriptive answer.

1. -13

2. $\sqrt{2}$

For the following exercises, evaluate the expression.

3. $2(x+3)-12; x=2$

4. $y(3+3)^2-26; y=1$

5. Write the number in standard notation: 3.1415×10^6

6. Write the number in scientific notation: 0.0000000212.

For the following exercises, simplify the expression.

7. $-2 \cdot (2+3 \cdot 2)^2 + 144$

8. $4\,(x+3)-(6x+2)$

9. $3^5 \cdot 3^{-3}$

10. $\left(\frac{2}{3}\right)^3$

11. $\frac{8x^3}{(2x)^2}$

12. $\left(16y^0\right)2y^{-2}$

13. $\sqrt{441}$

14. $\sqrt{490}$

15. $\sqrt{\frac{9x}{16}}$

16. $\frac{\sqrt{121b^2}}{1+\sqrt{b}}$

17. $6\sqrt{24}+7\sqrt{54}-12\sqrt{6}$

18. $\frac{\sqrt[3]{-8}}{\sqrt[4]{625}}$

19. $\left(13q^3+2q^2-3\right)-\left(6q^2+5q-3\right)$

20. $\left(6p^2+2p+1\right)+\left(9p^2-1\right)$

21. $(n-2)\left(n^2-4n+4\right)$

22. $(a-2b)(2a+b)$

For the following exercises, factor the polynomial.

23. $16x^2-81$

24. $y^2+12y+36$

25. $27c^3-1331$

26. $3x(x-6)^{-\frac{1}{4}}+2(x-6)^{\frac{3}{4}}$

For the following exercises, simplify the expression.

27. $\frac{2z^2+7z+3}{z^2-9}\cdot\frac{4z^2-15z+9}{4z^2-1}$

28. $\frac{x}{y}+\frac{2}{x}$

29. $\frac{\frac{a}{2b}-\frac{2b}{9a}}{\frac{3a-2b}{6a}}$

2 EQUATIONS AND INEQUALITIES

From the air, a landscape of circular crop fields may seem random, but they are laid out and irrigated very precisely. Farmers and irrigation providers combining agricultural science, engineering, and mathematics to achieve the most productive and efficient array. (Credit: Modification of "Aerial Phot of Center Pivot Irrigations Systems (1)" by Soil Science/flickr)

Chapter Outline

Introduction to Equations and Inequalities

Irrigation is a critical aspect of agriculture, which can expand the yield of farms and enable farming in areas not naturally viable for crops. But the materials, equipment, and the water itself are expensive and complex. To be efficient and productive, farm owners and irrigation specialists must carefully lay out the network of pipes, pumps, and related equipment. The available land can be divided into regular portions (similar to a grid), and the different sizes of irrigation systems and conduits can be installed within the plotted area.

2.1 The Rectangular Coordinate Systems and Graphs

Learning Objectives

In this section, you will:

- Plot ordered pairs in a Cartesian coordinate system.
- Graph equations by plotting points.
- Graph equations with a graphing utility.
- Find x-intercepts and y-intercepts.
- Use the distance formula.
- Use the midpoint formula.

COREQUISITE SKILLS

Learning Objectives

- Plot points on a real number line (IA 1.4.7)
- Plot points in a rectangular coordinate system (IA 3.1.1)

Objective 1: Plot points on a real number line (IA 1.4.7)

EXAMPLE 1

Locate Fractions and Decimals on the Number Line

We now want to include fractions and decimals on the number line.

In this example we will locate and plot the following points: $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5$ and $\frac{8}{3}$

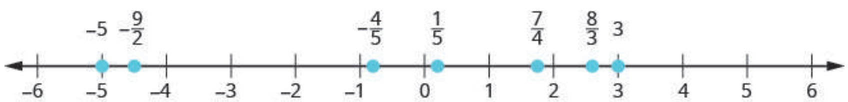
We'll start with the whole numbers 3 and -5 because they are the easiest to plot. Use zero as your starting point, move to the right for positive numbers and to the left for negative numbers.

The proper fractions listed are $\frac{1}{5}$ and $-\frac{4}{5}$. We know the proper fraction $\frac{1}{5}$ has value less than one and so would be located between 0 and 1. The denominator is 5, so imagine dividing the unit from 0 to 1 into 5 equal parts.

Similarly, $-\frac{4}{5}$ is between 0 and -1. After dividing the unit into 5 equal parts we plot $-\frac{4}{5}$.

Finally, look at the improper fractions $\frac{7}{4}, -\frac{9}{2}$, and $\frac{8}{3}$. Locating these points may be easier if you change each of them to a mixed number or use your calculator to get a decimal approximation.

The figure below shows the number line with all the points plotted.



Practice Makes Perfect

Plot each set of points on a real number line.

1. $-5, \sqrt{36}, \frac{5}{3}, 3\frac{1}{2}, -\frac{8}{10}$

2. $-\pi, \sqrt{25}, 1\frac{3}{4}, 0.8, 2.55, -\frac{15}{16}$

3. $e, \sqrt{5}, -2\frac{1}{4}, -0.75, -\frac{5}{2}$

Objective 2: Plot points in a rectangular coordinate system (IA 3.1.1)

EXAMPLE 2

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-5, 4)$ ⓑ $(-3, -4)$ ⓒ $(2, -3)$ ⓓ $(0, -1)$ ⓔ $\left(3, \frac{5}{2}\right)$.

Solution

The first number of the coordinate pair is the *x*-coordinate, and the second number is the *y*-coordinate. To plot each point, sketch a vertical line through the *x*-coordinate and a horizontal line through the *y*-coordinate. Their intersection is the point.

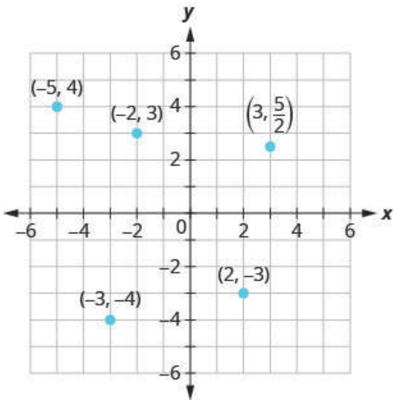
ⓐ Since $x = -5$, the point is to the left of the *y*-axis. Also, since $y = 4$, the point is above the *x*-axis. The point $(-5, 4)$ is in Quadrant II.

ⓑ Since $x = -3$, the point is to the left of the *y*-axis. Also, since $y = -4$, the point is below the *x*-axis. The point $(-3, -4)$ is in Quadrant III.

ⓒ Since $x = 2$, the point is to the right of the *y*-axis. Since $y = -3$, the point is below the *x*-axis. The point $(2, -3)$ is in Quadrant IV.

ⓓ Since $x = 0$, the point whose coordinates are $(0, -1)$ is on the *y*-axis.

ⓔ Since $x = 3$, the point is to the right of the *y*-axis. Since $y = \frac{5}{2}$, the point is above the *x*-axis. (It may be helpful to write $\frac{5}{2}$ as a mixed number or decimal.) The point $\left(3, \frac{5}{2}\right)$ is in Quadrant I.



Practice Makes Perfect

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

4. ⓐ $(-4, 2)$ ⓑ $(-1, -2)$ ⓒ $(3, -5)$ ⓓ $(-3, 0)$ ⓔ $\left(\frac{5}{3}, 2\right)$

5. ⓐ $(-2, -3)$ ⓑ $(3, -3)$ ⓒ $(-4, 1)$ ⓓ $(4, -1)$ ⓔ $\left(\frac{3}{2}, 1\right)$

6. ⓐ $(3, -1)$ ⓑ $(-3, 1)$ ⓒ $(-2, 0)$ ⓓ $(-4, -3)$ ⓔ $\left(1, \frac{14}{5}\right)$

7. ⓐ $(-1, 1)$ ⓑ $(-2, -1)$ ⓒ $(2, 0)$ ⓓ $(1, -4)$ ⓔ $\left(3, \frac{7}{2}\right)$

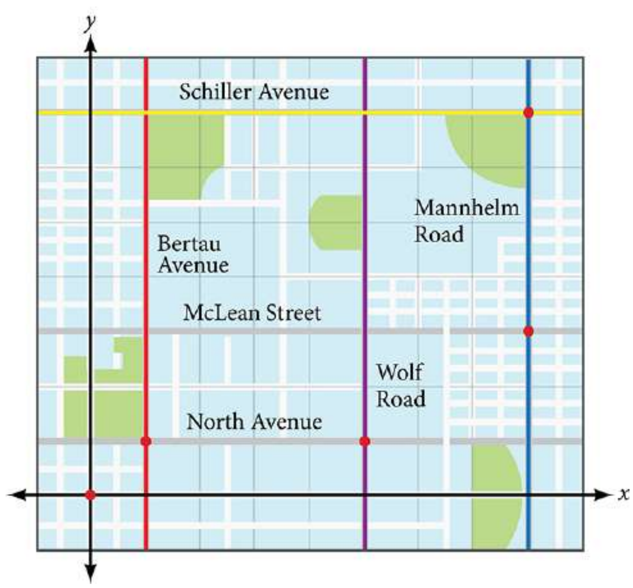


Figure 1

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in Figure 1. Laying a rectangular coordinate grid over the map, we can see that each stop aligns with an intersection of grid lines. In this section, we will learn how to use grid lines to describe locations and changes in locations.

Plotting Ordered Pairs in the Cartesian Coordinate System

An old story describes how seventeenth-century philosopher/mathematician René Descartes, while sick in bed, invented the system that has become the foundation of algebra. According to the story, Descartes was staring at a fly crawling on the ceiling when he realized that he could describe the fly's location in relation to the perpendicular lines formed by the adjacent walls of his room. He viewed the perpendicular lines as horizontal and vertical axes. Further, by dividing each axis into equal unit lengths, Descartes saw that it was possible to locate any object in a two-dimensional plane using just two numbers—the displacement from the horizontal axis and the displacement from the vertical axis.

While there is evidence that ideas similar to Descartes' grid system existed centuries earlier, it was Descartes who introduced the components that comprise the **Cartesian coordinate system**, a grid system having perpendicular axes. Descartes named the horizontal axis the ***x*-axis** and the vertical axis the ***y*-axis**.

The Cartesian coordinate system, also called the rectangular coordinate system, is based on a two-dimensional plane consisting of the *x*-axis and the *y*-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a **quadrant**; the quadrants are numbered counterclockwise as shown in Figure 2

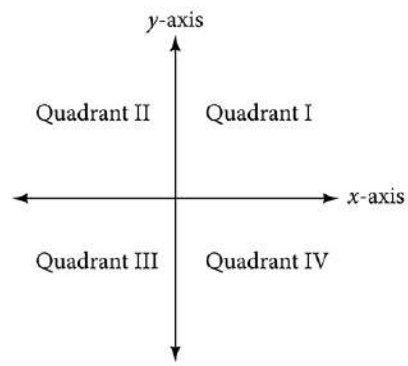


Figure 2

The center of the plane is the point at which the two axes cross. It is known as the **origin**, or point $(0, 0)$. From the origin, each axis is further divided into equal units: increasing, positive numbers to the right on the *x*-axis and up the *y*-axis; decreasing, negative numbers to the left on the *x*-axis and down the *y*-axis. The axes extend to positive and negative infinity as shown by the arrowheads in Figure 3.

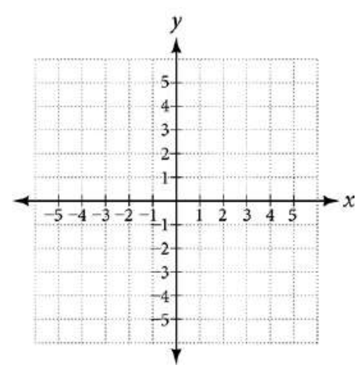


Figure 3

Each point in the plane is identified by its ***x*-coordinate**, or horizontal displacement from the origin, and its ***y*-coordinate**, or vertical displacement from the origin. Together, we write them as an **ordered pair** indicating the combined distance from the origin in the form (x, y) . An ordered pair is also known as a coordinate pair because it consists of *x*- and *y*-coordinates. For example, we can represent the point $(3, -1)$ in the plane by moving three units to the right of the origin in the horizontal direction, and one unit down in the vertical direction. See Figure 4.

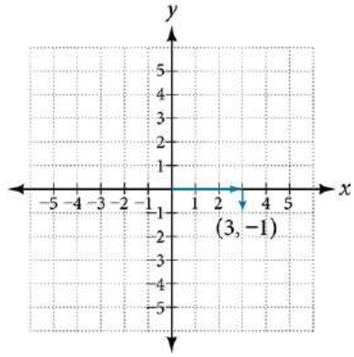


Figure 4

When dividing the axes into equally spaced increments, note that the *x*-axis may be considered separately from the *y*-axis. In other words, while the *x*-axis may be divided and labeled according to consecutive integers, the *y*-axis may be divided and labeled by increments of 2, or 10, or 100. In fact, the axes may represent other units, such as years against the balance in a savings account, or quantity against cost, and so on. Consider the rectangular coordinate system primarily as a method for showing the relationship between two quantities.

Cartesian Coordinate System

A two-dimensional plane where the

- *x*-axis is the horizontal axis
- *y*-axis is the vertical axis

A point in the plane is defined as an ordered pair, (x, y), such that *x* is determined by its horizontal distance from the origin and *y* is determined by its vertical distance from the origin.

EXAMPLE 1

Plotting Points in a Rectangular Coordinate System

Plot the points $(-2, 4)$, $(3, 3)$, and $(0, -3)$ in the plane.

Solution

To plot the point $(-2, 4)$, begin at the origin. The *x*-coordinate is –2, so move two units to the left. The *y*-coordinate is 4, so then move four units up in the positive *y* direction.

To plot the point $(3, 3)$, begin again at the origin. The *x*-coordinate is 3, so move three units to the right. The *y*-coordinate is also 3, so move three units up in the positive *y* direction.

To plot the point $(0, -3)$, begin again at the origin. The *x*-coordinate is 0. This tells us not to move in either direction along the *x*-axis. The *y*-coordinate is –3, so move three units down in the negative *y* direction. See the graph in Figure 5.

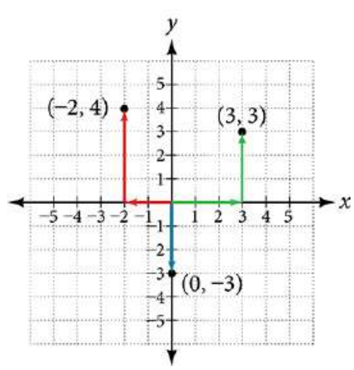


Figure 5

Analysis

Note that when either coordinate is zero, the point must be on an axis. If the *x*-coordinate is zero, the point is on the *y*-axis. If the *y*-coordinate is zero, the point is on the *x*-axis.

Graphing Equations by Plotting Points

We can plot a set of points to represent an equation. When such an equation contains both an *x* variable and a *y* variable, it is called an **equation in two variables**. Its graph is called a **graph in two variables**. Any graph on a two-dimensional plane is a graph in two variables.

Suppose we want to graph the equation $y = 2x - 1$. We can begin by substituting a value for *x* into the equation and determining the resulting value of *y*. Each pair of *x*- and *y*-values is an ordered pair that can be plotted. Table 1 lists values of *x* from –3 to 3 and the resulting values for *y*.

x	$y = 2x - 1$	(x, y)
-3	$y = 2(-3) - 1 = -7$	$(-3, -7)$
-2	$y = 2(-2) - 1 = -5$	$(-2, -5)$
-1	$y = 2(-1) - 1 = -3$	$(-1, -3)$
0	$y = 2(0) - 1 = -1$	$(0, -1)$
1	$y = 2(1) - 1 = 1$	$(1, 1)$
2	$y = 2(2) - 1 = 3$	$(2, 3)$
3	$y = 2(3) - 1 = 5$	$(3, 5)$

Table 1

We can plot the points in the table. The points for this particular equation form a line, so we can connect them. See Figure 6**.** This is not true for all equations.

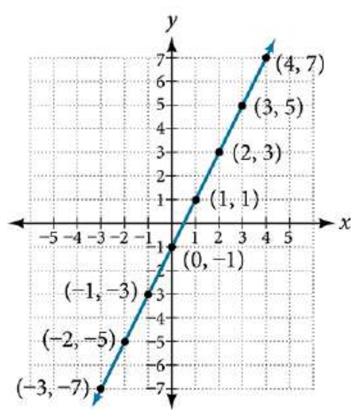


Figure 6

Note that the *x*-values chosen are arbitrary, regardless of the type of equation we are graphing. Of course, some situations may require particular values of *x* to be plotted in order to see a particular result. Otherwise, it is logical to choose values that can be calculated easily, and it is always a good idea to choose values that are both negative and positive. There is no rule dictating how many points to plot, although we need at least two to graph a line. Keep in mind, however, that the more points we plot, the more accurately we can sketch the graph.

HOW TO

Given an equation, graph by plotting points.

1. Make a table with one column labeled *x*, a second column labeled with the equation, and a third column listing the resulting ordered pairs.
2. Enter *x*-values down the first column using positive and negative values. Selecting the *x*-values in numerical order will make the graphing simpler.
3. Select *x*-values that will yield *y*-values with little effort, preferably ones that can be calculated mentally.
4. Plot the ordered pairs.
5. Connect the points if they form a line.

EXAMPLE 2

Graphing an Equation in Two Variables by Plotting Points

Graph the equation $y = -x + 2$ by plotting points.

Solution

First, we construct a table similar to Table 2. Choose *x* values and calculate *y*.

x	$y = -x + 2$	(x, y)
-5	$y = -(-5) + 2 = 7$	$(-5, 7)$
-3	$y = -(-3) + 2 = 5$	$(-3, 5)$
-1	$y = -(-1) + 2 = 3$	$(-1, 3)$
0	$y = -(0) + 2 = 2$	$(0, 2)$
1	$y = -(1) + 2 = 1$	$(1, 1)$
3	$y = -(3) + 2 = -1$	$(3, -1)$
5	$y = -(5) + 2 = -3$	$(5, -3)$

Table 2

Now, plot the points. Connect them if they form a line. See Figure 7

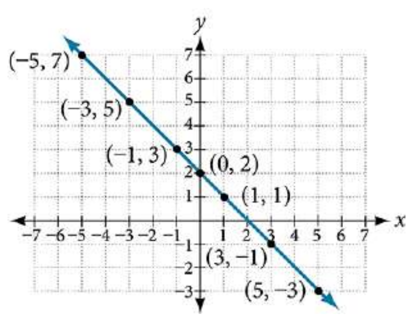


Figure 7

TRY IT #1 Construct a table and graph the equation by plotting points: $y = \frac{1}{2}x + 2$.

Graphing Equations with a Graphing Utility

Most graphing calculators require similar techniques to graph an equation. The equations sometimes have to be manipulated so they are written in the style $y =$______. The TI-84 Plus, and many other calculator makes and models, have a mode function, which allows the window (the screen for viewing the graph) to be altered so the pertinent parts of a graph can be seen.

For example, the equation $y = 2x - 20$ has been entered in the TI-84 Plus shown in Figure 8**a.** In Figure 8**b,** the resulting graph is shown. Notice that we cannot see on the screen where the graph crosses the axes. The standard window screen on the TI-84 Plus shows $-10 \leq x \leq 10$, and $-10 \leq y \leq 10$. See Figure 8**c**.

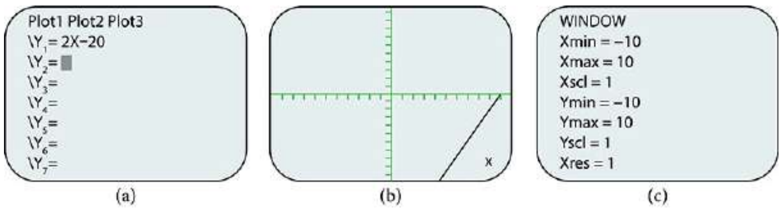


Figure 8 a. Enter the equation. b. This is the graph in the original window. c. These are the original settings.

By changing the window to show more of the positive *x*-axis and more of the negative *y*-axis, we have a much better view of the graph and the *x*- and *y*-intercepts. See Figure 9**a** and Figure 9**b.**

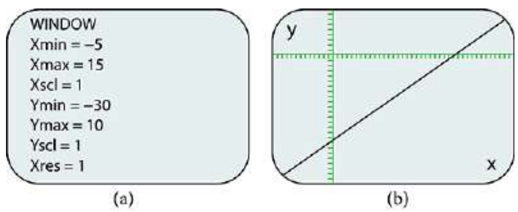


Figure 9 a. This screen shows the new window settings. b. We can clearly view the intercepts in the new window.

EXAMPLE 3

Using a Graphing Utility to Graph an Equation

Use a graphing utility to graph the equation: $y = -\frac{2}{3}x + \frac{4}{3}$.

Solution

Enter the equation in the *y*= function of the calculator. Set the window settings so that both the *x*- and *y*- intercepts are showing in the window. See Figure 10.

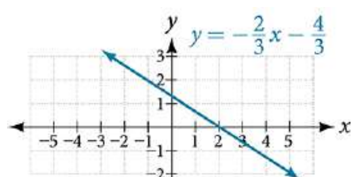


Figure 10

Finding x-intercepts and y-intercepts

The **intercepts** of a graph are points at which the graph crosses the axes. The ***x*-intercept** is the point at which the graph crosses the *x*-axis. At this point, the *y*-coordinate is zero. The ***y*-intercept** is the point at which the graph crosses the *y*-axis. At this point, the *x*-coordinate is zero.

To determine the *x*-intercept, we set *y* equal to zero and solve for *x*. Similarly, to determine the *y*-intercept, we set *x* equal to zero and solve for *y*. For example, lets find the intercepts of the equation $y = 3x - 1$.

To find the *x*-intercept, set $y = 0$.

$$\begin{aligned} y &= 3x - 1 \\ 0 &= 3x - 1 \\ 1 &= 3x \\ \tfrac{1}{3} &= x \\ \left(\tfrac{1}{3}, 0\right) & \qquad x\text{−intercept} \end{aligned}$$

To find the *y*-intercept, set $x = 0$.

$$\begin{aligned} y &= 3x - 1 \\ y &= 3(0) - 1 \\ y &= -1 \\ (0, -1) & \qquad y\text{−intercept} \end{aligned}$$

We can confirm that our results make sense by observing a graph of the equation as in Figure 11. Notice that the graph crosses the axes where we predicted it would.

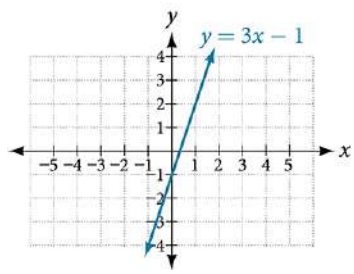


Figure 11

Given an equation, find the intercepts.

- Find the *x*-intercept by setting $y = 0$ and solving for x.
- Find the *y*-intercept by setting $x = 0$ and solving for y.

EXAMPLE 4

Finding the Intercepts of the Given Equation

Find the intercepts of the equation $y = -3x - 4$. Then sketch the graph using only the intercepts.

Solution

Set $y = 0$ to find the *x*-intercept.

$$\begin{aligned} y &= -3x - 4 \\ 0 &= -3x - 4 \\ 4 &= -3x \\ -\tfrac{4}{3} &= x \\ \left(-\tfrac{4}{3}, 0\right) & \qquad x\text{−intercept} \end{aligned}$$

Set $x = 0$ to find the *y*-intercept.

$$\begin{aligned} y &= -3x - 4 \\ y &= -3(0) - 4 \\ y &= -4 \\ (0, -4) & \qquad y\text{−intercept} \end{aligned}$$

Plot both points, and draw a line passing through them as in Figure 12.

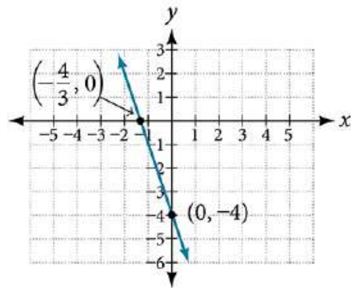


Figure 12

TRY IT #2 Find the intercepts of the equation and sketch the graph: $y = -\frac{3}{4}x + 3$.

Using the Distance Formula

Derived from the Pythagorean Theorem, the **distance formula** is used to find the distance between two points in the plane. The Pythagorean Theorem, $a^2 + b^2 = c^2$, is based on a right triangle where *a* and *b* are the lengths of the legs adjacent to the right angle, and *c* is the length of the hypotenuse. See Figure 13.

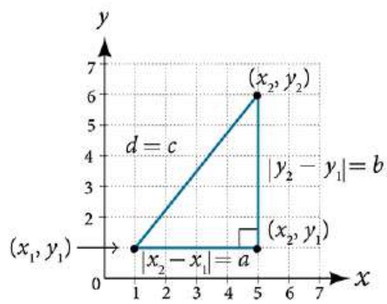


Figure 13

The relationship of sides $|x_2 - x_1|$ and $|y_2 - y_1|$ to side *d* is the same as that of sides *a* and *b* to side *c*. We use the absolute value symbol to indicate that the length is a positive number because the absolute value of any number is positive. (For example, $|-3| = 3$.) The symbols $|x_2 - x_1|$ and $|y_2 - y_1|$ indicate that the lengths of the sides of the triangle are positive. To find the length *c*, take the square root of both sides of the Pythagorean Theorem.

$$c^2 = a^2 + b^2 \rightarrow c = \sqrt{a^2 + b^2}$$

It follows that the distance formula is given as

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \rightarrow d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We do not have to use the absolute value symbols in this definition because any number squared is positive.

The Distance Formula

Given endpoints (x_1, y_1) and (x_2, y_2), the distance between two points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 5

Finding the Distance between Two Points

Find the distance between the points $(-3, -1)$ and $(2, 3)$.

Solution

Let us first look at the graph of the two points. Connect the points to form a right triangle as in Figure 14.

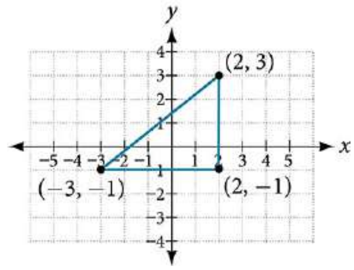


Figure 14

Then, calculate the length of *d* using the distance formula.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ d &= \sqrt{(2 - (-3))^2 + (3 - (-1))^2} \\ &= \sqrt{(5)^2 + (4)^2} \\ &= \sqrt{25 + 16} \\ &= \sqrt{41} \end{aligned}$$

TRY IT #3 Find the distance between two points: $(1, 4)$ and $(11, 9)$.

EXAMPLE 6

Finding the Distance between Two Locations

Let's return to the situation introduced at the beginning of this section.

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in Figure 1. Find the total distance that Tracie traveled. Compare this with the distance between her starting and final positions.

Solution

The first thing we should do is identify ordered pairs to describe each position. If we set the starting position at the origin, we can identify each of the other points by counting units east (right) and north (up) on the grid. For example, the first stop is 1 block east and 1 block north, so it is at $(1, 1)$. The next stop is 5 blocks to the east, so it is at $(5, 1)$. After that, she traveled 3 blocks east and 2 blocks north to $(8, 3)$. Lastly, she traveled 4 blocks north to $(8, 7)$. We can label these points on the grid as in Figure 15.

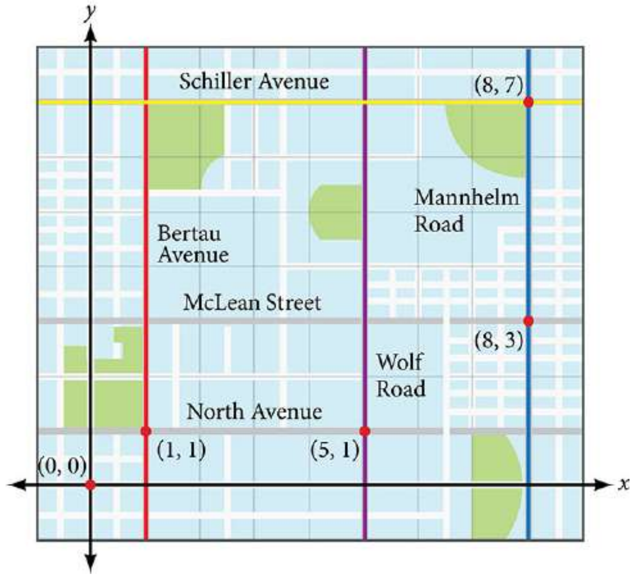


Figure 15

Next, we can calculate the distance. Note that each grid unit represents 1,000 feet.

- From her starting location to her first stop at $(1, 1)$, Tracie might have driven north 1,000 feet and then east 1,000 feet, or vice versa. Either way, she drove 2,000 feet to her first stop.
- Her second stop is at $(5, 1)$. So from $(1, 1)$ to $(5, 1)$, Tracie drove east 4,000 feet.
- Her third stop is at $(8, 3)$. There are a number of routes from $(5, 1)$ to $(8, 3)$. Whatever route Tracie decided to use, the distance is the same, as there are no angular streets between the two points. Let's say she drove east 3,000 feet and then north 2,000 feet for a total of 5,000 feet.
- Tracie's final stop is at $(8, 7)$. This is a straight drive north from $(8, 3)$ for a total of 4,000 feet.

Next, we will add the distances listed in Table 3.

From/To	Number of Feet Driven
$(0, 0)$ to $(1, 1)$	2,000
$(1, 1)$ to $(5, 1)$	4,000
$(5, 1)$ to $(8, 3)$	5,000
$(8, 3)$ to $(8, 7)$	4,000
Total	15,000

Table 3

The total distance Tracie drove is 15,000 feet, or 2.84 miles. This is not, however, the actual distance between her starting and ending positions. To find this distance, we can use the distance formula between the points $(0, 0)$ and $(8, 7)$.

$$\begin{aligned} d &= \sqrt{(8-0)^2 + (7-0)^2} \\ &= \sqrt{64 + 49} \\ &= \sqrt{113} \\ &\approx 10.63 \ \text{ units} \end{aligned}$$

At 1,000 feet per grid unit, the distance between Elmhurst, IL, to Franklin Park is 10,630.14 feet, or 2.01 miles. The distance formula results in a shorter calculation because it is based on the hypotenuse of a right triangle, a straight diagonal from the origin to the point $(8, 7)$. Perhaps you have heard the saying “as the crow flies,” which means the shortest distance between two points because a crow can fly in a straight line even though a person on the ground has to travel a longer distance on existing roadways.

Using the Midpoint Formula

When the endpoints of a line segment are known, we can find the point midway between them. This point is known as the midpoint and the formula is known as the **midpoint formula**. Given the endpoints of a line segment, (x_1, y_1) and (x_2, y_2) , the midpoint formula states how to find the coordinates of the midpoint M.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

A graphical view of a midpoint is shown in Figure 16. Notice that the line segments on either side of the midpoint are congruent.

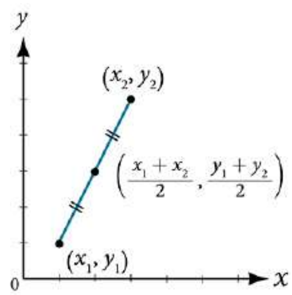


Figure 16

EXAMPLE 7

Finding the Midpoint of the Line Segment

Find the midpoint of the line segment with the endpoints $(7, -2)$ and $(9, 5)$.

Solution

Use the formula to find the midpoint of the line segment.

$$\begin{aligned}\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) &= \left(\frac{7+9}{2}, \frac{-2+5}{2}\right) \\ &= \left(8, \frac{3}{2}\right)\end{aligned}$$

TRY IT #4 Find the midpoint of the line segment with endpoints $(-2, -1)$ and $(-8, 6)$.

EXAMPLE 8

Finding the Center of a Circle

The diameter of a circle has endpoints $(-1, -4)$ and $(5, -4)$. Find the center of the circle.

Solution

The center of a circle is the center, or midpoint, of its diameter. Thus, the midpoint formula will yield the center point.

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\left(\frac{-1+5}{2}, \frac{-4-4}{2}\right) = \left(\frac{4}{2}, -\frac{8}{2}\right) = (2, -4)$$

MEDIA

Access these online resources for additional instruction and practice with the Cartesian coordinate system.

[Plotting points on the coordinate plane (http://openstax.org/l/coordplotpnts)](http://openstax.org/l/coordplotpnts)
[Find x and y intercepts based on the graph of a line (http://openstax.org/l/xyintsgraph)](http://openstax.org/l/xyintsgraph)

2.1 SECTION EXERCISES

Verbal

1. Is it possible for a point plotted in the Cartesian coordinate system to not lie in one of the four quadrants? Explain.

2. Describe the process for finding the *x*-intercept and the *y*-intercept of a graph algebraically.

3. Describe in your own words what the *y*-intercept of a graph is.

4. When using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, explain the correct order of operations that are to be performed to obtain the correct answer.

Algebraic

For each of the following exercises, find the x-intercept and the y-intercept without graphing. Write the coordinates of each intercept.

5. $y = -3x + 6$

6. $4y = 2x - 1$

7. $3x - 2y = 6$

8. $4x - 3 = 2y$

9. $3x + 8y = 9$

10. $2x - \frac{2}{3} = \frac{3}{4}y + 3$

For each of the following exercises, solve the equation for y in terms of x.

11. $4x + 2y = 8$

12. $3x - 2y = 6$

13. $2x = 5 - 3y$

14. $x - 2y = 7$

15. $5y + 4 = 10x$

16. $5x + 2y = 0$

For each of the following exercises, find the distance between the two points. Simplify your answers, and write the exact answer in simplest radical form for irrational answers.

17. $(-4, 1)$ and $(3, -4)$

18. $(2, -5)$ and $(7, 4)$

19. $(5, 0)$ and $(5, 6)$

20. $(-4, 3)$ and $(10, 3)$

21. Find the distance between the two points given using your calculator, and round your answer to the nearest hundredth.

$(19, 12)$ and $(41, 71)$

For each of the following exercises, find the coordinates of the midpoint of the line segment that joins the two given points.

22. $(-5, -6)$ and $(4, 2)$

23. $(-1, 1)$ and $(7, -4)$

24. $(-5, -3)$ and $(-2, -8)$

25. $(0, 7)$ and $(4, -9)$

26. $(-43, 17)$ and $(23, -34)$

Graphical

For each of the following exercises, identify the information requested.

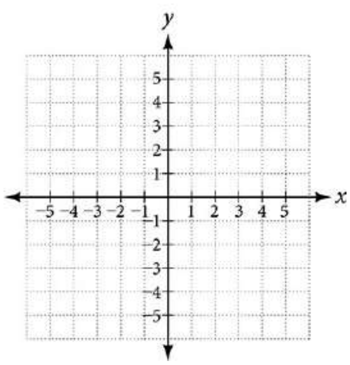
27. What are the coordinates of the origin?

28. If a point is located on the y-axis, what is the x-coordinate?

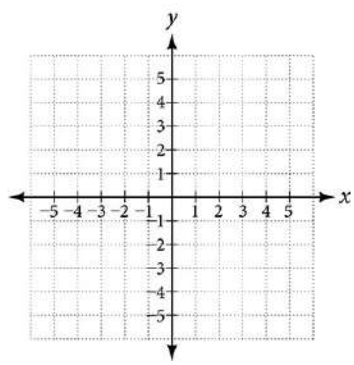
29. If a point is located on the x-axis, what is the y-coordinate?

For each of the following exercises, plot the three points on the given coordinate plane. State whether the three points you plotted appear to be collinear (on the same line).

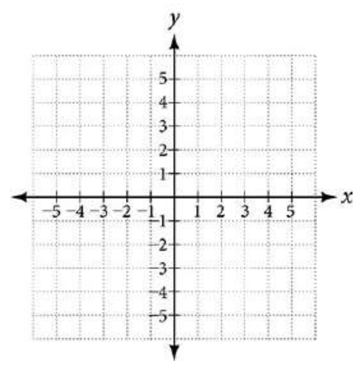
30. $(4, 1)\ (-2, -3)\ (5, 0)$



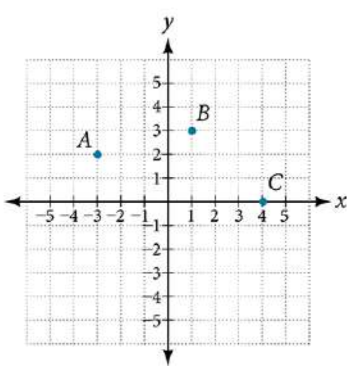
31. $(-1, 2)(0, 4)(2, 1)$



32. $(-3, 0)\ (-3, 4)\ (-3, -3)$



33. Name the coordinates of the points graphed.



34. Name the quadrant in which the following points would be located. If the point is on an axis, name the axis.

ⓐ $(-3, -4)$ ⓑ $(-5, 0)$
ⓒ $(1, -4)$ ⓓ $(-2, 7)$
ⓔ $(0, -3)$

For each of the following exercises, construct a table and graph the equation by plotting at least three points.

35. $y = \frac{1}{3}x + 2$

36. $y = -3x + 1$

37. $2y = x + 3$

Numeric

For each of the following exercises, find and plot the x- and y-intercepts, and graph the straight line based on those two points.

38. $4x - 3y = 12$

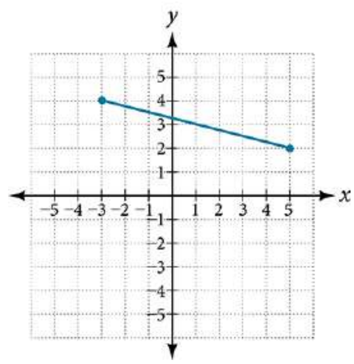
39. $x - 2y = 8$

40. $y - 5 = 5x$

41. $3y = -2x + 6$

42. $y = \frac{x-3}{2}$

For each of the following exercises, use the graph in the figure below.



43. Find the distance between the two endpoints using the distance formula. Round to three decimal places.

44. Find the coordinates of the midpoint of the line segment connecting the two points.

45. Find the distance that $(-3, 4)$ is from the origin.

46. Find the distance that $(5, 2)$ is from the origin. Round to three decimal places.

47. Which point is closer to the origin?

Technology

For the following exercises, use your graphing calculator to input the linear graphs in the Y= graph menu.

After graphing it, use the 2^nd CALC button and 1:value button, hit enter. At the lower part of the screen you will see "x=" and a blinking cursor. You may enter any number for x and it will display the y value for any x value you input. Use this and plug in x = 0, thus finding the y-intercept, for each of the following graphs.

48. $\mathrm{Y}_1 = -2x + 5$

49. $\mathrm{Y}_1 = \frac{3x-8}{4}$

50. $\mathrm{Y}_1 = \frac{x+5}{2}$

For the following exercises, use your graphing calculator to input the linear graphs in the Y= graph menu.

After graphing it, use the 2nd ***CALC*** *button and 2:zero button, hit* ***ENTER****. At the lower part of the screen you will see "left bound?" and a blinking cursor on the graph of the line. Move this cursor to the left of the x-intercept, hit* ***ENTER****. Now it says "right bound?" Move the cursor to the right of the x-intercept, hit* ***ENTER****. Now it says "guess?" Move your cursor to the left somewhere in between the left and right bound near the x-intercept. Hit* ***ENTER****. At the bottom of your screen it will display the coordinates of the x-intercept or the "zero" to the y-value. Use this to find the x-intercept.*

Note: With linear/straight line functions the zero is not really a "guess," but it is necessary to enter a "guess" so it will search and find the exact x-intercept between your right and left boundaries. With other types of functions (more than one x-intercept), they may be irrational numbers so "guess" is more appropriate to give it the correct limits to find a very close approximation between the left and right boundaries.

51. $\mathrm{Y}_1 = -8x + 6$

52. $\mathrm{Y}_1 = 4x - 7$

53. $\mathrm{Y}_1 = \frac{3x+5}{4}$ Round your answer to the nearest thousandth.

Extensions

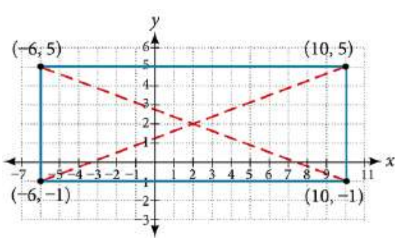
54. Someone drove 10 mi directly east from their home, made a left turn at an intersection, and then traveled 5 mi north to their place of work. If a road was made directly from the home to the place of work, what would its distance be to the nearest tenth of a mile?

55. If the road was made in the previous exercise, how much shorter would the person's one-way trip be every day?

56. Given these four points: $A(1,3)$, $B(-3,5)$, $C(4,7)$, and $D(5,-4)$ find the coordinates of the midpoint of line segments $\overline{\mathrm{AB}}$ and $\overline{\mathrm{CD}}$.

57. After finding the two midpoints in the previous exercise, find the distance between the two midpoints to the nearest thousandth.

58. Given the graph of the rectangle shown and the coordinates of its vertices, prove that the diagonals of the rectangle are of equal length.



59. In the previous exercise, find the coordinates of the midpoint for each diagonal.

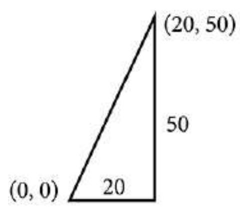
Real-World Applications

60. The coordinates on a map for San Francisco are $(53, 17)$ and those for Sacramento are $(128, 78)$. Note that coordinates represent miles. Find the distance between the cities to the nearest mile.

61. If San Jose's coordinates are $(76, -12)$, where the coordinates represent miles, find the distance between San Jose and San Francisco to the nearest mile.

62. A small craft in Lake Ontario sends out a distress signal. The coordinates of the boat in trouble were $(49, 64)$ One rescue boat is at the coordinates $(60, 82)$ and a second Coast Guard craft is at coordinates $(58, 47)$. Assuming both rescue craft travel at the same rate, which one would get to the distressed boat the fastest?

63. A person on the top of a building wants to have a guy wire extend to a point on the ground 20 ft from the building. To the nearest foot, how long will the wire have to be if the building is 50 ft tall?



64. If we rent a truck and pay a \$75/day fee plus \$.20 for every mile we travel, write a linear equation that would express the total cost per day y, using x to represent the number of miles we travel. Graph this function on your graphing calculator and find the total cost for one day if we travel 70 mi.

2.2 Linear Equations in One Variable

Learning Objectives

In this section, you will:

- Solve equations in one variable algebraically.
- Solve a rational equation.
- Find a linear equation.
- Given the equations of two lines, determine whether their graphs are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

COREQUISITE SKILLS

Learning Objectives

- Simplify expressions using order of operations (IA 1.1.3)
- Solve linear equations using a general strategy (IA 2.1.1)

Objective 1: Simplify expressions using order of operations (IA 1.1.3)

HOW TO

Use the order of operations

Step 1. Parentheses and Other Grouping Symbols
- Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.

Step 2. Exponents
- Simplify all expressions with exponents.

Step 3. Multiplication and Division
- Perform all multiplication and division in order from left to right. These operations have equal priority.

Step 4. Addition and Subtraction
- Perform all addition and subtraction in order from left to right. These operations have equal priority.

EXAMPLE 1

Simplify: $5 + 2^3 + 3[6 - 3(4 - 2)]$.

Solution

	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Are there any parentheses (or other grouping symbols)? Yes.	$5 + 2^3 + 3[6 - 3(4 - 2)]$
Focus on the parentheses that are inside the brackets. Subtract.	$5 + 2^3 + 3[6 - 3(2)]$
Continue inside the brackets and multiply.	$5 + 2^3 + 3[6 - 6]$
Continue inside the brackets and subtract.	$5 + 2^3 + 3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes. Simplify exponents.	$5 + 8 + 3[0]$
Is there any multiplication or division? Yes.	
Multiply.	$5 + 8 + 0$
Is there any addition of subtraction? Yes.	
Add.	$13 + 0$
Add.	13

Practice Makes Perfect

1. $3(1 + 9 \bullet 6) - 4^2$

2. $2^3 - 12 \div (9 - 5)$

3. $33 \div 3 + 4(7 - 2)$

4. $10 + 3[6 - 2(4 - 2)] - 2^4$

Evaluate the following expressions being sure to follow the order of operations:

5. When $x = 3$,

   ⓐ x^5 ⓑ 5^x ⓒ $3x^2 - 4x - 8$

6. When $x = 3, y = -2$
   $6x^2 + 3xy - 9y^2$

7. When $x = -8, y = 3$
   $(x + y)^2$

Simplify by combining like terms:

8. $10a + 7 + 5a - 2 + 7a - 4$

9. $5b + 9b + 10(2b + 3b) + 5$

Objective 2: Solve linear equations using a general strategy (IA 2.1.1)

HOW TO

Solve linear equations using a general strategy

Step 1. Simplify each side of the equation as much as possible. Use the Distributive Property to remove any parentheses. Combine like terms.
Step 2. Collect all the variable terms on one side of the equation. Use the Addition or Subtraction Property of Equality.
Step 3. Collect all the constant terms on the other side of the equation. Use the Addition or Subtraction Property of Equality.
Step 4. Make the coefficient of the variable term equal to 1. Use the Multiplication or Division Property of Equality. State the solution to the equation.
Step 5. Check the solution. Substitute the solution into the original equation to make sure the result is a true statement.

EXAMPLE 2

Solve linear equations using a general strategy.

Solve for w
$2(w + 5) + 1 = 10 + 4w + 2$

Solution

Use distributive property to remove parentheses:	$2w+10+1=10+4w+2$
Combine like terms on each side:	$2w+11=12+4w$
Subtract $2w$ from each side to bring variables to one side:	$2w-2w+11=12+4w-2w$
Combine like terms:	$11=12+2w$
Subtract 12 from each side to bring constants to one side:	$11-12=12-12+2w$
Combine like terms:	$-1=2w$
Divide each side by 2 to isolate the variable terms:	$\frac{-1}{2}=\frac{2w}{2}$
Simplify:	$\frac{-1}{2}=w$ or $w=-\frac{1}{2}$
To check your solution, replace w with $-\frac{1}{2}$ in the original equation and simplify:	$2(w+5)+1=10+4w+2$
	$2\left(-\frac{1}{2}+5\right)+1=10+4\left(-\frac{1}{2}\right)+2$
	$2\left(\frac{9}{2}\right)+1=10+\left(-2\right)+2$ $9+1=8+2$
The solution checks, we reached a true statement.	$10=10$

Practice Makes Perfect

Solve each linear equation using the general strategy.

10. $15(y-9)=-60$

11. $-2(11-7x)+54=4$

12. $3(4n-1)-2=8n+3$

13. $12+2(5-3y)=-9(y-1)-2$

14. $\frac{1}{4}\left(20x+12\right)=x+7$

15. $22(3m-4)=8(2m+9)$

16. $\frac{3x+4}{2}+1=\frac{5x+10}{8}$

17. $0.05n+0.10(n+8)=2.15$

Caroline is a full-time college student planning a spring break vacation. To earn enough money for the trip, she has taken a part-time job at the local bank that pays $15.00/hr, and she opened a savings account with an initial deposit of $400 on January 15. She arranged for direct deposit of her payroll checks. If spring break begins March 20 and the trip will cost approximately $2,500, how many hours will she have to work to earn enough to pay for her vacation? If she can only work 4 hours per day, how many days per week will she have to work? How many weeks will it take? In this section, we will investigate problems like this and others, which generate graphs like the line in Figure 1.

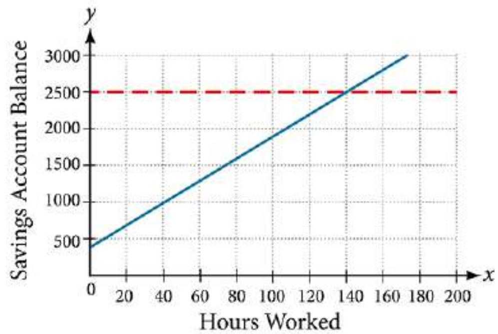


Figure 1

Solving Linear Equations in One Variable

A **linear equation** is an equation of a straight line, written in one variable. The only power of the variable is 1. Linear equations in one variable may take the form $ax + b = 0$ and are solved using basic algebraic operations.

We begin by classifying linear equations in one variable as one of three types: identity, conditional, or inconsistent. An **identity equation** is true for all values of the variable. Here is an example of an identity equation.

$$3x = 2x + x$$

The **solution set** consists of all values that make the equation true. For this equation, the solution set is all real numbers because any real number substituted for x will make the equation true.

A **conditional equation** is true for only some values of the variable. For example, if we are to solve the equation $5x + 2 = 3x - 6$, we have the following:

$$\begin{aligned} 5x + 2 &= 3x - 6 \\ 2x &= -8 \\ x &= -4 \end{aligned}$$

The solution set consists of one number: $\{-4\}$. It is the only solution and, therefore, we have solved a conditional equation.

An **inconsistent equation** results in a false statement. For example, if we are to solve $5x - 15 = 5(x - 4)$, we have the following:

$$\begin{array}{rcll} 5x - 15 &=& 5x - 20 & \\ 5x - 15 - 5x &=& 5x - 20 - 5x & \text{Subtract } 5x \text{ from both sides.} \\ -15 &\neq& -20 & \text{False statement} \end{array}$$

Indeed, $-15 \neq -20$. There is no solution because this is an inconsistent equation.

Solving linear equations in one variable involves the fundamental properties of equality and basic algebraic operations. A brief review of those operations follows.

Linear Equation in One Variable

A linear equation in one variable can be written in the form

$$ax + b = 0$$

where *a* and *b* are real numbers, $a \neq 0$.

HOW TO

Given a linear equation in one variable, use algebra to solve it.

The following steps are used to manipulate an equation and isolate the unknown variable, so that the last line reads $x =$___________, if x is the unknown. There is no set order, as the steps used depend on what is given:

1. We may add, subtract, multiply, or divide an equation by a number or an expression as long as we do the same thing to both sides of the equal sign. Note that we cannot divide by zero.
2. Apply the distributive property as needed: $a\left(b+c\right)=ab+ac$.
3. Isolate the variable on one side of the equation.
4. When the variable is multiplied by a coefficient in the final stage, multiply both sides of the equation by the reciprocal of the coefficient.

EXAMPLE 1

Solving an Equation in One Variable

Solve the following equation: $2x+7=19$.

Solution

This equation can be written in the form $ax+b=0$ by subtracting 19 from both sides. However, we may proceed to solve the equation in its original form by performing algebraic operations.

$$\begin{array}{rcll} 2x+7 & = & 19 & \\ 2x & = & 12 & \text{Subtract 7 from both sides.} \\ x & = & 6 & \text{Multiply both sides by } \frac{1}{2} \text{ or divide by 2.} \end{array}$$

The solution is 6.

TRY IT #1 Solve the linear equation in one variable: $2x+1=-9$.

EXAMPLE 2

Solving an Equation Algebraically When the Variable Appears on Both Sides

Solve the following equation: $4\left(x-3\right)+12=15-5\left(x+6\right)$.

Solution

Apply standard algebraic properties.

$$\begin{array}{rcll} 4(x-3)+12 & = & 15-5(x+6) & \\ 4x-12+12 & = & 15-5x-30 & \text{Apply the distributive property.} \\ 4x & = & -15-5x & \text{Combine like terms.} \\ 9x & = & -15 & \text{Place } x\text{-terms on one side and simplify.} \\ x & = & -\frac{15}{9} & \text{Multiply both sides by } \frac{1}{9}\text{, the reciprocal of 9.} \\ x & = & -\frac{5}{3} & \end{array}$$

Analysis

This problem requires the distributive property to be applied twice, and then the properties of algebra are used to reach the final line, $x=-\frac{5}{3}$.

TRY IT #2 Solve the equation in one variable: $-2\left(3x-1\right)+x=14-x$.

Solving a Rational Equation

In this section, we look at rational equations that, after some manipulation, result in a linear equation. If an equation contains at least one rational expression, it is a considered a **rational equation**.

Recall that a rational number is the ratio of two numbers, such as $\frac{2}{3}$ or $\frac{7}{2}$. A rational expression is the ratio, or quotient, of two polynomials. Here are three examples.

$$\frac{x+1}{x^2-4}, \quad \frac{1}{x-3}, \quad \text{or} \quad \frac{4}{x^2+x-2}$$

Rational equations have a variable in the denominator in at least one of the terms. Our goal is to perform algebraic operations so that the variables appear in the numerator. In fact, we will eliminate all denominators by multiplying both sides of the equation by the least common denominator (LCD).

Finding the LCD is identifying an expression that contains the highest power of all of the factors in all of the denominators. We do this because when the equation is multiplied by the LCD, the common factors in the LCD and in each denominator will equal one and will cancel out.

EXAMPLE 3

Solving a Rational Equation

Solve the rational equation: $\frac{7}{2x} - \frac{5}{3x} = \frac{22}{3}$.

Solution

We have three denominators; $2x$, $3x$, and 3. The LCD must contain $2x$, $3x$, and 3. An LCD of $6x$ contains all three denominators. In other words, each denominator can be divided evenly into the LCD. Next, multiply both sides of the equation by the LCD $6x$.

$$\begin{aligned}
(6x)\left(\frac{7}{2x} - \frac{5}{3x}\right) &= \left(\frac{22}{3}\right)(6x) && \\
(6x)\left(\frac{7}{2x}\right) - (6x)\left(\frac{5}{3x}\right) &= \left(\frac{22}{3}\right)(6x) && \text{Use the distributive property.} \\
(\cancel{6x})\left(\frac{7}{\cancel{2x}}\right) - (\cancel{6x})\left(\frac{5}{\cancel{3x}}\right) &= \left(\frac{22}{\cancel{3}}\right)(\cancel{6}x) && \text{Cancel out the common factors.} \\
3(7) - 2(5) &= 22(2x) && \text{Multiply remaining factors by each numerator.} \\
21 - 10 &= 44x && \\
11 &= 44x && \\
\frac{11}{44} &= x && \\
\frac{1}{4} &= x &&
\end{aligned}$$

A common mistake made when solving rational equations involves finding the LCD when one of the denominators is a binomial—two terms added or subtracted—such as $(x+1)$. Always consider a binomial as an individual factor—the terms cannot be separated. For example, suppose a problem has three terms and the denominators are x, $x-1$, and $3x-3$. First, factor all denominators. We then have x, $(x-1)$, and $3(x-1)$ as the denominators. (Note the parentheses placed around the second denominator.) Only the last two denominators have a common factor of $(x-1)$. The x in the first denominator is separate from the x in the $(x-1)$ denominators. An effective way to remember this is to write factored and binomial denominators in parentheses, and consider each parentheses as a separate unit or a separate factor. The LCD in this instance is found by multiplying together the x, one factor of $(x-1)$, and the 3. Thus, the LCD is the following:

$$x(x-1)3 = 3x(x-1)$$

So, both sides of the equation would be multiplied by $3x(x-1)$. Leave the LCD in factored form, as this makes it easier to see how each denominator in the problem cancels out.

Another example is a problem with two denominators, such as x and x^2+2x. Once the second denominator is factored as $x^2+2x = x(x+2)$, there is a common factor of x in both denominators and the LCD is $x(x+2)$.

Sometimes we have a rational equation in the form of a proportion; that is, when one fraction equals another fraction and there are no other terms in the equation.

$$\frac{a}{b} = \frac{c}{d}$$

We can use another method of solving the equation without finding the LCD: cross-multiplication. We multiply terms by crossing over the equal sign.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{b} \times \frac{c}{d}.$$

Multiply $a\,(d)$ and $b\,(c)$, which results in $ad = bc$.

Any solution that makes a denominator in the original expression equal zero must be excluded from the possibilities.

Rational Equations

A **rational equation** contains at least one rational expression where the variable appears in at least one of the denominators.

HOW TO

Given a rational equation, solve it.

1. Factor all denominators in the equation.
2. Find and exclude values that set each denominator equal to zero.
3. Find the LCD.
4. Multiply the whole equation by the LCD. If the LCD is correct, there will be no denominators left.
5. Solve the remaining equation.
6. Make sure to check solutions back in the original equations to avoid a solution producing zero in a denominator.

EXAMPLE 4

Solving a Rational Equation without Factoring

Solve the following rational equation:

$$\frac{2}{x} - \frac{3}{2} = \frac{7}{2x}$$

Solution

We have three denominators: x, 2, and $2x$. No factoring is required. The product of the first two denominators is equal to the third denominator, so, the LCD is $2x$. Only one value is excluded from a solution set, 0. Next, multiply the whole equation (both sides of the equal sign) by $2x$.

$$\begin{aligned}
2x\left(\tfrac{2}{x} - \tfrac{3}{2}\right) &= \left(\tfrac{7}{2x}\right)2x \\
2\cancel{x}\left(\frac{2}{\cancel{x}}\right) - \cancel{2}x\left(\frac{3}{\cancel{2}}\right) &= \left(\frac{7}{\cancel{2x}}\right)\cancel{2x} && \text{Distribute } 2x. \\
2(2) - 3x &= 7 && \text{Denominators cancel out.} \\
4 - 3x &= 7 \\
-3x &= 3 \\
x &= -1 \\
&\quad \text{or } \{-1\}
\end{aligned}$$

The proposed solution is −1, which is not an excluded value, so the solution set contains one number, -1, or $\{-1\}$ written in set notation.

TRY IT #3 Solve the rational equation: $\frac{2}{3x} = \frac{1}{4} - \frac{1}{6x}$.

EXAMPLE 5

Solving a Rational Equation by Factoring the Denominator

Solve the following rational equation: $\frac{1}{x}=\frac{1}{10}-\frac{3}{4x}.$

Solution

First find the common denominator. The three denominators in factored form are $x, 10=2\cdot 5,$ and $4x=2\cdot 2\cdot x.$ The smallest expression that is divisible by each one of the denominators is $20x.$ Only $x=0$ is an excluded value. Multiply the whole equation by $20x.$

$$\begin{aligned}20x\left(\frac{1}{x}\right) &= \left(\frac{1}{10}-\frac{3}{4x}\right)20x\\ 20 &= 2x-15\\ 35 &= 2x\\ \frac{35}{2} &= x\end{aligned}$$

The solution is $\frac{35}{2}.$

TRY IT #4 Solve the rational equation: $-\frac{5}{2x}+\frac{3}{4x}=-\frac{7}{4}.$

EXAMPLE 6

Solving Rational Equations with a Binomial in the Denominator

Solve the following rational equations and state the excluded values:

ⓐ $\frac{3}{x-6}=\frac{5}{x}$ ⓑ $\frac{x}{x-3}=\frac{5}{x-3}-\frac{1}{2}$ ⓒ $\frac{x}{x-2}=\frac{5}{x-2}-\frac{1}{2}$

Solution

ⓐ

The denominators x and $x-6$ have nothing in common. Therefore, the LCD is the product $x\left(x-6\right).$ However, for this problem, we can cross-multiply.

$$\begin{aligned}\frac{3}{x-6} &= \frac{5}{x}\\ 3x &= 5(x-6) && \text{Distribute.}\\ 3x &= 5x-30\\ -2x &= -30\\ x &= 15\end{aligned}$$

The solution is 15. The excluded values are 6 and $0.$

ⓑ

The LCD is $2\left(x-3\right).$ Multiply both sides of the equation by $2\left(x-3\right).$

$$\begin{aligned}2(x-3)\left(\frac{x}{x-3}\right) &= \left(\frac{5}{x-3}-\frac{1}{2}\right)2(x-3)\\ \frac{2\cancel{(x-3)}\,x}{\cancel{x-3}} &= \frac{2\cancel{(x-3)}\,5}{\cancel{x-3}}-\frac{\cancel{2}(x-3)}{\cancel{2}}\\ 2x &= 10-(x-3)\\ 2x &= 10-x+3\\ 2x &= 13-x\\ 3x &= 13\\ x &= \frac{13}{3}\end{aligned}$$

The solution is $\frac{13}{3}.$ The excluded value is $3.$

ⓒ
The least common denominator is $2(x-2)$. Multiply both sides of the equation by $x(x-2)$.

$$\begin{aligned} 2(x-2)\left(\frac{x}{x-2}\right) &= \left(\frac{5}{x-2} - \frac{1}{2}\right) 2(x-2) \\ 2x &= 10 - (x-2) \\ 2x &= 12 - x \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

The solution is 4. The excluded value is 2.

TRY IT #5 Solve $\frac{-3}{2x+1} = \frac{4}{3x+1}$. State the excluded values.

EXAMPLE 7

Solving a Rational Equation with Factored Denominators and Stating Excluded Values

Solve the rational equation after factoring the denominators: $\frac{2}{x+1} - \frac{1}{x-1} = \frac{2x}{x^2-1}$. State the excluded values.

Solution

We must factor the denominator x^2-1. We recognize this as the difference of squares, and factor it as $(x-1)(x+1)$. Thus, the LCD that contains each denominator is $(x-1)(x+1)$. Multiply the whole equation by the LCD, cancel out the denominators, and solve the remaining equation.

$$\begin{aligned} (x-1)(x+1)\left(\frac{2}{x+1} - \frac{1}{x-1}\right) &= \left(\frac{2x}{(x-1)(x+1)}\right)(x-1)(x+1) \\ 2(x-1) - 1(x+1) &= 2x \\ 2x - 2 - x - 1 &= 2x \quad \text{Distribute the negative sign.} \\ -3 - x &= 0 \\ -3 &= x \end{aligned}$$

The solution is -3. The excluded values are 1 and -1.

TRY IT #6 Solve the rational equation: $\frac{2}{x-2} + \frac{1}{x+1} = \frac{1}{x^2-x-2}$.

Finding a Linear Equation

Perhaps the most familiar form of a linear equation is the slope-intercept form, written as $y = mx + b$, where $m = \text{slope}$ and $b = y\text{-intercept}$. Let us begin with the slope.

The Slope of a Line

The **slope** of a line refers to the ratio of the vertical change in *y* over the horizontal change in *x* between any two points on a line. It indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

If the slope is positive, the line slants to the right. If the slope is negative, the line slants to the left. As the slope increases, the line becomes steeper. Some examples are shown in Figure 2. The lines indicate the following slopes: $m = -3$, $m = 2$, and $m = \frac{1}{3}$.

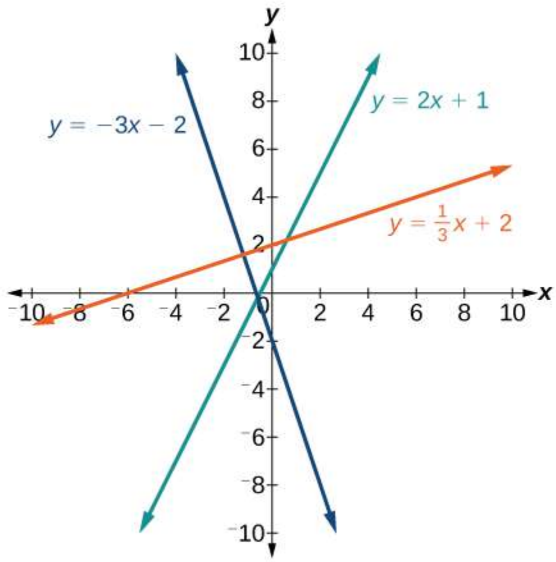


Figure 2

The Slope of a Line

The slope of a line, *m*, represents the change in *y* over the change in *x*. Given two points, (x_1, y_1) and (x_2, y_2), the following formula determines the slope of a line containing these points:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

EXAMPLE 8

Finding the Slope of a Line Given Two Points

Find the slope of a line that passes through the points $(2, -1)$ and $(-5, 3)$.

Solution

We substitute the *y*-values and the *x*-values into the formula.

$$\begin{aligned} m &= \frac{3-(-1)}{-5-2} \\ &= \frac{4}{-7} \\ &= -\frac{4}{7} \end{aligned}$$

The slope is $-\frac{4}{7}$.

Analysis

It does not matter which point is called (x_1, y_1) or (x_2, y_2). As long as we are consistent with the order of the *y* terms and the order of the *x* terms in the numerator and denominator, the calculation will yield the same result.

TRY IT #7 Find the slope of the line that passes through the points $(-2, 6)$ and $(1, 4)$.

EXAMPLE 9

Identifying the Slope and *y*-intercept of a Line Given an Equation

Identify the slope and *y*-intercept, given the equation $y = -\frac{3}{4}x - 4$.

Solution

As the line is in $y = mx + b$ form, the given line has a slope of $m = -\frac{3}{4}$. The *y*-intercept is $b = -4$.

Analysis

The *y*-intercept is the point at which the line crosses the *y*-axis. On the *y*-axis, $x = 0$. We can always identify the *y*-intercept when the line is in slope-intercept form, as it will always equal *b*. Or, just substitute $x = 0$ and solve for *y*.

The Point-Slope Formula

Given the slope and one point on a line, we can find the equation of the line using the point-slope formula.

$$y - y_1 = m(x - x_1)$$

This is an important formula, as it will be used in other areas of college algebra and often in calculus to find the equation of a tangent line. We need only one point and the slope of the line to use the formula. After substituting the slope and the coordinates of one point into the formula, we simplify it and write it in slope-intercept form.

The Point-Slope Formula

Given one point and the slope, the point-slope formula will lead to the equation of a line:

$$y - y_1 = m(x - x_1)$$

EXAMPLE 10

Finding the Equation of a Line Given the Slope and One Point

Write the equation of the line with slope $m = -3$ and passing through the point $(4, 8)$. Write the final equation in slope-intercept form.

Solution

Using the point-slope formula, substitute -3 for *m* and the point $(4, 8)$ for (x_1, y_1).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 8 &= -3(x - 4) \\ y - 8 &= -3x + 12 \\ y &= -3x + 20 \end{aligned}$$

Analysis

Note that any point on the line can be used to find the equation. If done correctly, the same final equation will be obtained.

TRY IT #8 Given $m = 4$, find the equation of the line in slope-intercept form passing through the point $(2, 5)$.

EXAMPLE 11

Finding the Equation of a Line Passing Through Two Given Points

Find the equation of the line passing through the points $(3, 4)$ and $(0, -3)$. Write the final equation in slope-intercept form.

Solution

First, we calculate the slope using the slope formula and two points.

$$\begin{aligned} m &= \frac{-3-4}{0-3} \\ &= \frac{-7}{-3} \\ &= \frac{7}{3} \end{aligned}$$

Next, we use the point-slope formula with the slope of $\frac{7}{3}$, and either point. Let's pick the point $(3, 4)$ for (x_1, y_1).

$$\begin{aligned} y - 4 &= \tfrac{7}{3}(x - 3) \\ y - 4 &= \tfrac{7}{3}x - 7 && \text{Distribute the } \tfrac{7}{3}. \\ y &= \tfrac{7}{3}x - 3 \end{aligned}$$

In slope-intercept form, the equation is written as $y = \frac{7}{3}x - 3$.

Analysis

To prove that either point can be used, let us use the second point $(0, -3)$ and see if we get the same equation.

$$\begin{aligned} y - (-3) &= \tfrac{7}{3}(x - 0) \\ y + 3 &= \tfrac{7}{3}x \\ y &= \tfrac{7}{3}x - 3 \end{aligned}$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

Standard Form of a Line

Another way that we can represent the equation of a line is in standard form. Standard form is given as

$$Ax + By = C$$

where A, B, and C are integers. The *x*- and *y*-terms are on one side of the equal sign and the constant term is on the other side.

EXAMPLE 12

Finding the Equation of a Line and Writing It in Standard Form

Find the equation of the line with $m = -6$ and passing through the point $\left(\frac{1}{4}, -2\right)$. Write the equation in standard form.

Solution

We begin using the point-slope formula.

$$\begin{aligned} y - (-2) &= -6\left(x - \tfrac{1}{4}\right) \\ y + 2 &= -6x + \tfrac{3}{2} \end{aligned}$$

From here, we multiply through by 2, as no fractions are permitted in standard form, and then move both variables to the left aside of the equal sign and move the constants to the right.

$$\begin{aligned} 2(y + 2) &= \left(-6x + \tfrac{3}{2}\right)2 \\ 2y + 4 &= -12x + 3 \\ 12x + 2y &= -1 \end{aligned}$$

This equation is now written in standard form.

TRY IT #9 Find the equation of the line in standard form with slope $m = -\frac{1}{3}$ and passing through the point $\left(1, \frac{1}{3}\right)$.

Vertical and Horizontal Lines

The equations of vertical and horizontal lines do not require any of the preceding formulas, although we can use the formulas to prove that the equations are correct. The equation of a vertical line is given as

$$x = c$$

where *c* is a constant. The slope of a vertical line is undefined, and regardless of the *y*-value of any point on the line, the *x*-coordinate of the point will be *c*.

Suppose that we want to find the equation of a line containing the following points: $(-3,-5)$, $(-3,1)$, $(-3,3)$, and $(-3,5)$. First, we will find the slope.

$$m=\frac{5-3}{-3-(-3)}=\frac{2}{0}$$

Zero in the denominator means that the slope is undefined and, therefore, we cannot use the point-slope formula. However, we can plot the points. Notice that all of the *x*-coordinates are the same and we find a vertical line through $x=-3$. See Figure 3.

The equation of a horizontal line is given as

$$y=c$$

where *c* is a constant. The slope of a horizontal line is zero, and for any *x*-value of a point on the line, the *y*-coordinate will be *c*.

Suppose we want to find the equation of a line that contains the following set of points: $(-2,-2)$, $(0,-2)$, $(3,-2)$, and $(5,-2)$. We can use the point-slope formula. First, we find the slope using any two points on the line.

$$\begin{aligned} m &= \tfrac{-2-(-2)}{0-(-2)} \\ &= \tfrac{0}{2} \\ &= 0 \end{aligned}$$

Use any point for (x_1,y_1) in the formula, or use the *y*-intercept.

$$\begin{aligned} y-(-2) &= 0(x-3) \\ y+2 &= 0 \\ y &= -2 \end{aligned}$$

The graph is a horizontal line through $y=-2$. Notice that all of the *y*-coordinates are the same. See Figure 3.

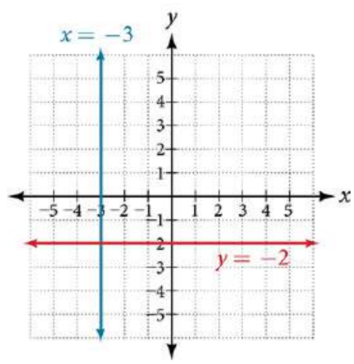


Figure 3 The line *x* = −3 is a vertical line. The line *y* = −2 is a horizontal line.

EXAMPLE 13

Finding the Equation of a Line Passing Through the Given Points

Find the equation of the line passing through the given points: $(1,-3)$ and $(1,4)$.

Solution

The *x*-coordinate of both points is 1. Therefore, we have a vertical line, $x=1$.

TRY IT #10 Find the equation of the line passing through $(-5,2)$ and $(2,2)$.

Determining Whether Graphs of Lines are Parallel or Perpendicular

Parallel lines have the same slope and different *y*-intercepts. Lines that are parallel to each other will never intersect. For example, Figure 4 shows the graphs of various lines with the same slope, $m=2$.

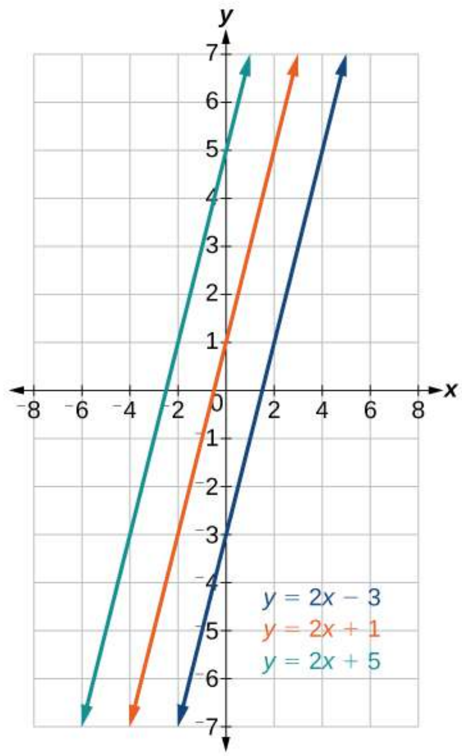


Figure 4 Parallel lines

All of the lines shown in the graph are parallel because they have the same slope and different *y*-intercepts.

Lines that are perpendicular intersect to form a $90°$ -angle. The slope of one line is the negative reciprocal of the other. We can show that two lines are perpendicular if the product of the two slopes is $-1 : m_1 \cdot m_2 = -1$. For example, Figure 5 shows the graph of two perpendicular lines. One line has a slope of 3; the other line has a slope of $-\frac{1}{3}$.

$$\begin{aligned} m_1 \cdot m_2 &= -1 \\ 3 \cdot \left(-\tfrac{1}{3}\right) &= -1 \end{aligned}$$

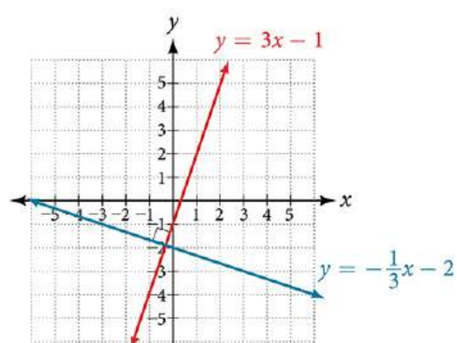


Figure 5 Perpendicular lines

EXAMPLE 14

Graphing Two Equations, and Determining Whether the Lines are Parallel, Perpendicular, or Neither

Graph the equations of the given lines, and state whether they are parallel, perpendicular, or neither: $3y = -4x + 3$ and $3x - 4y = 8$.

Solution

The first thing we want to do is rewrite the equations so that both equations are in slope-intercept form.

First equation:

$$\begin{aligned} 3y &= -4x+3 \\ y &= -\frac{4}{3}x+1 \end{aligned}$$

Second equation:

$$\begin{aligned} 3x-4y &= 8 \\ -4y &= -3x+8 \\ y &= \frac{3}{4}x-2 \end{aligned}$$

See the graph of both lines in Figure 6

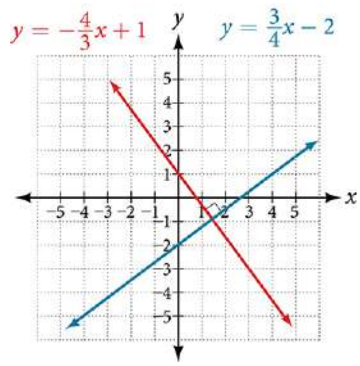


Figure 6

From the graph, we can see that the lines appear perpendicular, but we must compare the slopes.

$$\begin{aligned} m_1 &= -\frac{4}{3} \\ m_2 &= \frac{3}{4} \\ m_1 \cdot m_2 &= \left(-\frac{4}{3}\right)\left(\frac{3}{4}\right) = -1 \end{aligned}$$

The slopes are negative reciprocals of each other, confirming that the lines are perpendicular.

TRY IT #11 Graph the two lines and determine whether they are parallel, perpendicular, or neither: $2y - x = 10$ and $2y = x + 4$.

Writing the Equations of Lines Parallel or Perpendicular to a Given Line

As we have learned, determining whether two lines are parallel or perpendicular is a matter of finding the slopes. To write the equation of a line parallel or perpendicular to another line, we follow the same principles as we do for finding the equation of any line. After finding the slope, use the point-slope formula to write the equation of the new line.

HOW TO

Given an equation for a line, write the equation of a line parallel or perpendicular to it.

1. Find the slope of the given line. The easiest way to do this is to write the equation in slope-intercept form.
2. Use the slope and the given point with the point-slope formula.
3. Simplify the line to slope-intercept form and compare the equation to the given line.

EXAMPLE 15

Writing the Equation of a Line Parallel to a Given Line Passing Through a Given Point

Write the equation of line parallel to a $5x + 3y = 1$ and passing through the point $(3, 5)$.

Solution

First, we will write the equation in slope-intercept form to find the slope.

$$\begin{aligned} 5x + 3y &= 1 \\ 3y &= -5x + 1 \\ y &= -\tfrac{5}{3}x + \tfrac{1}{3} \end{aligned}$$

The slope is $m = -\frac{5}{3}$. The y-intercept is $\frac{1}{3}$, but that really does not enter into our problem, as the only thing we need for two lines to be parallel is the same slope. The one exception is that if the y-intercepts are the same, then the two lines are the same line. The next step is to use this slope and the given point with the point-slope formula.

$$\begin{aligned} y - 5 &= -\tfrac{5}{3}(x - 3) \\ y - 5 &= -\tfrac{5}{3}x + 5 \\ y &= -\tfrac{5}{3}x + 10 \end{aligned}$$

The equation of the line is $y = -\frac{5}{3}x + 10$. See Figure 7.

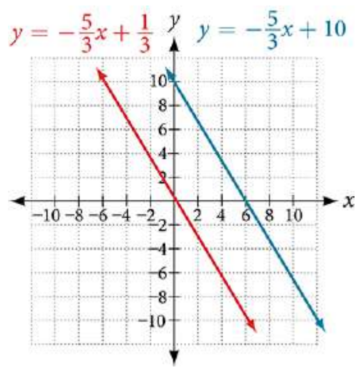


Figure 7

TRY IT #12 Find the equation of the line parallel to $5x = 7 + y$ and passing through the point $(-1, -2)$.

EXAMPLE 16

Finding the Equation of a Line Perpendicular to a Given Line Passing Through a Given Point

Find the equation of the line perpendicular to $5x - 3y + 4 = 0$ and passing through the point $(-4, 1)$.

Solution

The first step is to write the equation in slope-intercept form.

$$\begin{aligned} 5x - 3y + 4 &= 0 \\ -3y &= -5x - 4 \\ y &= \tfrac{5}{3}x + \tfrac{4}{3} \end{aligned}$$

We see that the slope is $m = \frac{5}{3}$. This means that the slope of the line perpendicular to the given line is the negative reciprocal, or $-\frac{3}{5}$. Next, we use the point-slope formula with this new slope and the given point.

$$\begin{aligned} y - 1 &= -\tfrac{3}{5}(x - (-4)) \\ y - 1 &= -\tfrac{3}{5}x - \tfrac{12}{5} \\ y &= -\tfrac{3}{5}x - \tfrac{12}{5} + \tfrac{5}{5} \\ y &= -\tfrac{3}{5}x - \tfrac{7}{5} \end{aligned}$$

MEDIA

Access these online resources for additional instruction and practice with linear equations.

Solving rational equations (http://openstax.org/l/rationaleqs)
Equation of a line given two points (http://openstax.org/l/twopointsline)
Finding the equation of a line perpendicular to another line through a given point (http://openstax.org/l/findperpline)
Finding the equation of a line parallel to another line through a given point (http://openstax.org/l/findparaline)

2.2 SECTION EXERCISES

Verbal

1. What does it mean when we say that two lines are parallel?

2. What is the relationship between the slopes of perpendicular lines (assuming neither is horizontal nor vertical)?

3. How do we recognize when an equation, for example $y = 4x + 3$, will be a straight line (linear) when graphed?

4. What does it mean when we say that a linear equation is inconsistent?

5. When solving the following equation:
$\frac{2}{x-5} = \frac{4}{x+1}$
explain why we must exclude $x = 5$ and $x = -1$ as possible solutions from the solution set.

Algebraic

For the following exercises, solve the equation for x.

6. $7x + 2 = 3x - 9$

7. $4x - 3 = 5$

8. $3(x + 2) - 12 = 5(x + 1)$

9. $12 - 5(x + 3) = 2x - 5$

10. $\frac{1}{2} - \frac{1}{3}x = \frac{4}{3}$

11. $\frac{x}{3} - \frac{3}{4} = \frac{2x+3}{12}$

12. $\frac{2}{3}x + \frac{1}{2} = \frac{31}{6}$

13. $3(2x - 1) + x = 5x + 3$

14. $\frac{2x}{3} - \frac{3}{4} = \frac{x}{6} + \frac{21}{4}$

15. $\frac{x+2}{4} - \frac{x-1}{3} = 2$

For the following exercises, solve each rational equation for x. State all x-values that are excluded from the solution set.

16. $\frac{3}{x} - \frac{1}{3} = \frac{1}{6}$

17. $2 - \frac{3}{x+4} = \frac{x+2}{x+4}$

18. $\frac{3}{x-2} = \frac{1}{x-1} + \frac{7}{(x-1)(x-2)}$

19. $\frac{3x}{x-1} + 2 = \frac{3}{x-1}$

20. $\frac{5}{x+1} + \frac{1}{x-3} = \frac{-6}{x^2-2x-3}$

21. $\frac{1}{x} = \frac{1}{5} + \frac{3}{2x}$

For the following exercises, find the equation of the line using the point-slope formula. Write all the final equations using the slope-intercept form.

22. $(0, 3)$ with a slope of $\frac{2}{3}$

23. $(1, 2)$ with a slope of $-\frac{4}{5}$

24. x-intercept is 1, and $(-2, 6)$

25. y-intercept is 2, and $(4, -1)$

26. $(-3, 10)$ and $(5, -6)$

27. $(1, 3) \quad \text{and} \quad (5, 5)$

28. parallel to $y = 2x + 5$ and passes through the point $(4, 3)$

29. perpendicular to $3y = x - 4$ and passes through the point $(-2, 1)$.

For the following exercises, find the equation of the line using the given information.

30. $(-2, 0)$ and $(-2, 5)$

31. $(1, 7)$ and $(3, 7)$

32. The slope is undefined and it passes through the point $(2, 3)$.

33. The slope equals zero and it passes through the point $(1, -4)$.

34. The slope is $\frac{3}{4}$ and it passes through the point $(1, 4)$.

35. $(–1, 3)$ and $(4, –5)$

Graphical

For the following exercises, graph the pair of equations on the same axes, and state whether they are parallel, perpendicular, or neither.

36. $$\begin{array}{l} y = 2x + 7 \\ y = -\frac{1}{2}x - 4 \end{array}$$

37. $$\begin{array}{l} 3x - 2y = 5 \\ 6y - 9x = 6 \end{array}$$

38. $$\begin{array}{l} y = \frac{3x+1}{4} \\ y = 3x + 2 \end{array}$$

39. $$\begin{array}{l} x = 4 \\ y = -3 \end{array}$$

Numeric

For the following exercises, find the slope of the line that passes through the given points.

40. $(5, 4)$ and $(7, 9)$

41. $(-3, 2)$ and $(4, -7)$

42. $(-5, 4)$ and $(2, 4)$

43. $(-1, -2)$ and $(3, 4)$

44. $(3, -2)$ and $(3, -2)$

For the following exercises, find the slope of the lines that pass through each pair of points and determine whether the lines are parallel or perpendicular.

45. $$\begin{array}{lll} (-1, 3) & \text{and} & (5, 1) \\ (-2, 3) & \text{and} & (0, 9) \end{array}$$

46. $$\begin{array}{lll} (2, 5) & \text{and} & (5, 9) \\ (-1, -1) & \text{and} & (2, 3) \end{array}$$

Technology

For the following exercises, express the equations in slope intercept form (rounding each number to the thousandths place). Enter this into a graphing calculator as Y1, then adjust the ymin and ymax values for your window to include where the y-intercept occurs. State your ymin and ymax values.

47. $0.537x - 2.19y = 100$

48. $4{,}500x - 200y = 9{,}528$

49. $\frac{200-30y}{x} = 70$

Extensions

50. Starting with the point-slope formula $y - y_1 = m(x - x_1)$, solve this expression for x in terms of x_1, y, y_1, and m.

51. Starting with the standard form of an equation $Ax + By = C$ solve this expression for y in terms of A, B, C and x. Then put the expression in slope-intercept form.

52. Use the above derived formula to put the following standard equation in slope intercept form: $7x - 5y = 25$.

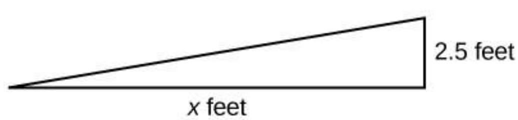
53. Given that the following coordinates are the vertices of a rectangle, prove that this truly is a rectangle by showing the slopes of the sides that meet are perpendicular.

$(-1, 1), (2, 0), (3, 3)$ and $(0, 4)$

54. Find the slopes of the diagonals in the previous exercise. Are they perpendicular?

Real-World Applications

55. The slope for a wheelchair ramp for a home has to be $\frac{1}{12}$. If the vertical distance from the ground to the door bottom is 2.5 ft, find the distance the ramp has to extend from the home in order to comply with the needed slope.



56. If the profit equation for a small business selling x number of item one and y number of item two is $p = 3x + 4y$, find the y value when $p = \$453$ and $x = 75$.

For the following exercises, use this scenario: The cost of renting a car is \$45/wk plus \$0.25/mi traveled during that week. An equation to represent the cost would be $y = 45 + .25x$, where x is the number of miles traveled.

57. What is your cost if you travel 50 mi?

58. If your cost were $\$63.75$, how many miles were you charged for traveling?

59. Suppose you have a maximum of \$100 to spend for the car rental. What would be the maximum number of miles you could travel?

2.3 Models and Applications

Learning Objectives

In this section, you will:

- Set up a linear equation to solve a real-world application.
- Use a formula to solve a real-world application.

COREQUISITE SKILLS

Learning Objectives

- Solve a formula for a specified variable (IA 2.3.1)
- Use a problem-solving strategy for word problems (IA 2.2.1)

Objective 1: Solve a formula for a specified variable (IA 2.3.1)

HOW TO

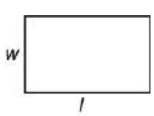
Solving a Formula for a Specified Variable.

Step 1. Refer to the appropriate formula and identify the variable you are solving for. Treat the other variable terms as if they were numbers.
Step 2. Bring all terms containing the specified variable to one side using the addition/subtraction property of equality.
Step 3. Isolate the variable you are solving for using the multiplication/division property of equality.

EXAMPLE 1

Solve a Formula for a Specific Variable

The formula for the perimeter of a rectangle is found using the formula: $P = 2l + 2w$. Solve this formula in terms of *l*.



Solution

$P = 2l + 2w$	Since we are solving for *l* we isolate the *l* term
$P - 2w = 2l + 2w - 2w$	Subtract 2*w* from both sides.
$P - 2w = 2l$	Combine like terms
$\frac{P-2w}{2} = \frac{2l}{2}$	Divide by 2 to isolate *l*.
$\frac{P-2w}{2} = l$	Simplify

Practice Makes Perfect

Solve each formula for the specific variable.

1. Solve for b
$P = a + b + c$

2. Solve for s

$P = 4s$

3. Solve for r
$C = 2\pi r$

4. Solve for b
$A = \frac{1}{2}bh$

5. Solve for W
$P = 2L + 2W$

6. Solve for m
$y = mx + b$

7. Solve for h
$A = 2\pi h + 2\pi r^2$

8. Solve for r
$A = \pi r^2$

9. Solve for s
$V = \frac{1}{3}s^2 h$

10. Solve for L
$A = 2LW + 2HW + 2LH$

Objective 2: Use a problem-solving strategy for word problems (IA 2.2.1)

HOW TO

Use a Problem-Solving Strategy for word problems.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
Step 2. **Identify** what you are looking for.
Step 3. **Name** what you are looking for. Choose a variable to represent that quantity.
Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
Step 5. **Solve** the equation using proper algebra techniques.
Step 6. **Check** the answer in the problem to make sure it makes sense.
Step 7. **Answer** the question with a complete sentence.

EXAMPLE 2

Use a Problem-Solving Strategy for word problems.
Hang borrowed \$7,500 from her parents to pay her tuition. In five years, she paid them \$1,500 interest in addition to the \$7,500 she borrowed. What was the rate of simple interest?

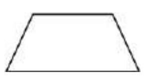
Solution

Write down the given information:	I = $1500
	P = $7500
	r = ?
	t = 5 years
Identify the unknown: let interest rate be represented by r	
Write a formula:	$I = Prt$
Substitute in the given information:	$1500 = (7500)r(5)$
Solve for r	$1500 = 37{,}500r$
	$\frac{1500}{37{,}500} = r$
	$0.04 = r$
	$4\% = r$

Practice Makes Perfect

Use a Problem-Solving Strategy for word problems.

11. The formula for area of a trapezoid is $A = \frac{1}{2}\left(B + b\right)h$ where B is the length of the base, b is the length of the other base and h is the height of the trapezoid.



If $B = 10\text{cm}$, $b = 8\text{cm}$ and $A = 45\text{cm}^2$, find the height of the trapezoid.

12. A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

13. The label on Audrey’s yogurt said that one serving provided 12 grams of protein, which is 24% of the recommended daily amount. What is the total recommended daily amount of protein?

14. Recently, the California governor proposed raising community college fees from $36 a unit to $46 a unit. Find the percent change. (Round to the nearest tenth of a percent.)

15. Sean’s new car loan statement said he would pay $4,866.25 in interest from a simple interest rate of 8.5% over five years. How much did he borrow to buy his new car?

16. At the campus coffee cart, a medium coffee costs $1.65. MaryAnne brings $2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

Figure 1 Credit: Kevin Dooley

Neka is hoping to get an A in his college algebra class. He has scores of 75, 82, 95, 91, and 94 on his first five tests. Only the final exam remains, and the maximum of points that can be earned is 100. Is it possible for Neka to end the course with an A? A simple linear equation will give Neka his answer.

Many real-world applications can be modeled by linear equations. For example, a cell phone package may include a monthly service fee plus a charge per minute of talk-time; it costs a widget manufacturer a certain amount to produce *x* widgets per month plus monthly operating charges; a car rental company charges a daily fee plus an amount per mile driven. These are examples of applications we come across every day that are modeled by linear equations. In this section, we will set up and use linear equations to solve such problems.

Setting up a Linear Equation to Solve a Real-World Application

To set up or model a linear equation to fit a real-world application, we must first determine the known quantities and define the unknown quantity as a variable. Then, we begin to interpret the words as mathematical expressions using mathematical symbols. Let us use the car rental example above. In this case, a known cost, such as $0.10/mi, is multiplied by an unknown quantity, the number of miles driven. Therefore, we can write $0.10x$. This expression represents a variable cost because it changes according to the number of miles driven.

If a quantity is independent of a variable, we usually just add or subtract it, according to the problem. As these amounts do not change, we call them fixed costs. Consider a car rental agency that charges $0.10/mi plus a daily fee of $50. We can use these quantities to model an equation that can be used to find the daily car rental cost C.

$$C = 0.10x + 50$$

When dealing with real-world applications, there are certain expressions that we can translate directly into math. Table 1 lists some common verbal expressions and their equivalent mathematical expressions.

Verbal	Translation to Math Operations
One number exceeds another by *a*	$x, \quad x + a$
Twice a number	$2x$
One number is *a* more than another number	$x, \quad x + a$
One number is *a* less than twice another number	$x, \quad 2x - a$
The product of a number and *a*, decreased by *b*	$ax - b$
The quotient of a number and the number plus *a* is three times the number	$\frac{x}{x+a} = 3x$
The product of three times a number and the number decreased by *b* is *c*	$3x\,(x - b) = c$

Table 1

HOW TO

Given a real-world problem, model a linear equation to fit it.

1. Identify known quantities.
2. Assign a variable to represent the unknown quantity.
3. If there is more than one unknown quantity, find a way to write the second unknown in terms of the first.
4. Write an equation interpreting the words as mathematical operations.
5. Solve the equation. Be sure the solution can be explained in words, including the units of measure.

EXAMPLE 1

Modeling a Linear Equation to Solve an Unknown Number Problem

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

Solution

Let x equal the first number. Then, as the second number exceeds the first by 17, we can write the second number as $x + 17$. The sum of the two numbers is 31. We usually interpret the word *is* as an equal sign.

$$\begin{aligned} x + (x + 17) &= 31 \\ 2x + 17 &= 31 \qquad \text{Simplify and solve.} \\ 2x &= 14 \\ x &= 7 \\ \\ x + 17 &= 7 + 17 \\ &= 24 \end{aligned}$$

The two numbers are 7 and 24.

TRY IT #1 Find a linear equation to solve for the following unknown quantities: One number is three more than twice another number. If the sum of the two numbers is 36, find the numbers.

EXAMPLE 2

Setting Up a Linear Equation to Solve a Real-World Application

There are two cell phone companies that offer different packages. Company A charges a monthly service fee of \$34 plus \$.05/min talk-time. Company B charges a monthly service fee of \$40 plus \$.04/min talk-time.

ⓐ Write a linear equation that models the packages offered by both companies.
ⓑ If the average number of minutes used each month is 1,160, which company offers the better plan?
ⓒ If the average number of minutes used each month is 420, which company offers the better plan?
ⓓ How many minutes of talk-time would yield equal monthly statements from both companies?

Solution

ⓐ The model for Company *A* can be written as $A = 0.05x + 34$. This includes the variable cost of $0.05x$ plus the monthly service charge of \$34. Company *B*'s package charges a higher monthly fee of \$40, but a lower variable cost of $0.04x$. Company *B*'s model can be written as $B = 0.04x + \$40$.

ⓑ
If the average number of minutes used each month is 1,160, we have the following:

$$\begin{aligned}\text{Company } A &= 0.05(1,160) + 34\\ &= 58 + 34\\ &= 92\end{aligned}$$

$$\begin{aligned}\text{Company } B &= 0.04(1,160) + 40\\ &= 46.4 + 40\\ &= 86.4\end{aligned}$$

So, Company *B* offers the lower monthly cost of $86.40 as compared with the $92 monthly cost offered by Company *A* when the average number of minutes used each month is 1,160.

ⓒ
If the average number of minutes used each month is 420, we have the following:

$$\begin{aligned}\text{Company } A &= 0.05(420) + 34\\ &= 21 + 34\\ &= 55\end{aligned}$$

$$\begin{aligned}\text{Company } B &= 0.04(420) + 40\\ &= 16.8 + 40\\ &= 56.8\end{aligned}$$

If the average number of minutes used each month is 420, then Company *A* offers a lower monthly cost of $55 compared to Company *B*’s monthly cost of $56.80.

ⓓ
To answer the question of how many talk-time minutes would yield the same bill from both companies, we should think about the problem in terms of (x, y) coordinates: At what point are both the *x*-value and the *y*-value equal? We can find this point by setting the equations equal to each other and solving for *x.*

$$\begin{aligned}0.05x + 34 &= 0.04x + 40\\ 0.01x &= 6\\ x &= 600\end{aligned}$$

Check the *x*-value in each equation.

$$\begin{aligned}0.05(600) + 34 &= 64\\ 0.04(600) + 40 &= 64\end{aligned}$$

Therefore, a monthly average of 600 talk-time minutes renders the plans equal. See Figure 2

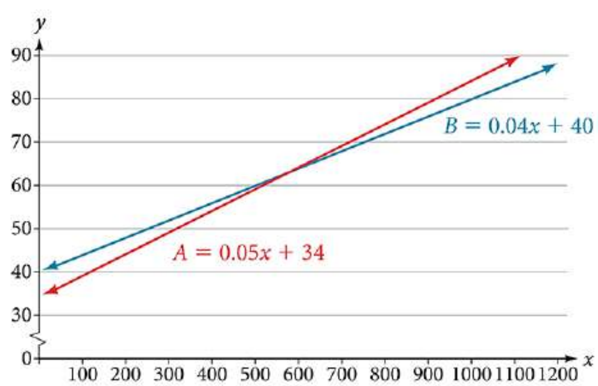


Figure 2

TRY IT #2 Find a linear equation to model this real-world application: It costs ABC electronics company \$2.50 per unit to produce a part used in a popular brand of desktop computers. The company has monthly operating expenses of \$350 for utilities and \$3,300 for salaries. What are the company's monthly expenses?

Using a Formula to Solve a Real-World Application

Many applications are solved using known formulas. The problem is stated, a formula is identified, the known quantities are substituted into the formula, the equation is solved for the unknown, and the problem's question is answered. Typically, these problems involve two equations representing two trips, two investments, two areas, and so on. Examples of formulas include the **area** of a rectangular region, $A = LW$; the **perimeter** of a rectangle, $P = 2L + 2W$; and the **volume** of a rectangular solid, $V = LWH$. When there are two unknowns, we find a way to write one in terms of the other because we can solve for only one variable at a time.

EXAMPLE 3

Solving an Application Using a Formula

It takes Andrew 30 min to drive to work in the morning. He drives home using the same route, but it takes 10 min longer, and he averages 10 mi/h less than in the morning. How far does Andrew drive to work?

Solution

This is a distance problem, so we can use the formula $d = rt$, where distance equals rate multiplied by time. Note that when rate is given in mi/h, time must be expressed in hours. Consistent units of measurement are key to obtaining a correct solution.

First, we identify the known and unknown quantities. Andrew's morning drive to work takes 30 min, or $\frac{1}{2}$ h at rate r. His drive home takes 40 min, or $\frac{2}{3}$ h, and his speed averages 10 mi/h less than the morning drive. Both trips cover distance d. A table, such as Table 2, is often helpful for keeping track of information in these types of problems.

	d	r	t
To Work	d	r	$\frac{1}{2}$
To Home	d	$r - 10$	$\frac{2}{3}$

Table 2

Write two equations, one for each trip.

$$\begin{aligned} d &= r\left(\tfrac{1}{2}\right) && \text{To work} \\ d &= (r-10)\left(\tfrac{2}{3}\right) && \text{To home} \end{aligned}$$

As both equations equal the same distance, we set them equal to each other and solve for r.

$$\begin{aligned} r\left(\tfrac{1}{2}\right) &= (r-10)\left(\tfrac{2}{3}\right) \\ \tfrac{1}{2}r &= \tfrac{2}{3}r - \tfrac{20}{3} \\ \tfrac{1}{2}r - \tfrac{2}{3}r &= -\tfrac{20}{3} \\ -\tfrac{1}{6}r &= -\tfrac{20}{3} \\ r &= -\tfrac{20}{3}(-6) \\ r &= 40 \end{aligned}$$

We have solved for the rate of speed to work, 40 mph. Substituting 40 into the rate on the return trip yields 30 mi/h. Now we can answer the question. Substitute the rate back into either equation and solve for d.

$$\begin{aligned} d &= 40\left(\tfrac{1}{2}\right) \\ &= 20 \end{aligned}$$

The distance between home and work is 20 mi.

Analysis

Note that we could have cleared the fractions in the equation by multiplying both sides of the equation by the LCD to solve for r.

$$\begin{aligned} r\left(\frac{1}{2}\right) &= (r-10)\left(\frac{2}{3}\right) \\ 6 \times r\left(\frac{1}{2}\right) &= 6 \times (r-10)\left(\frac{2}{3}\right) \\ 3r &= 4(r-10) \\ 3r &= 4r-40 \\ -r &= -40 \\ r &= 40 \end{aligned}$$

TRY IT #3 On Saturday morning, it took Jennifer 3.6 h to drive to her mother's house for the weekend. On Sunday evening, due to heavy traffic, it took Jennifer 4 h to return home. Her speed was 5 mi/h slower on Sunday than on Saturday. What was her speed on Sunday?

EXAMPLE 4

Solving a Perimeter Problem

The perimeter of a rectangular outdoor patio is 54 ft. The length is 3 ft greater than the width. What are the dimensions of the patio?

Solution

The perimeter formula is standard: $P = 2L + 2W$. We have two unknown quantities, length and width. However, we can write the length in terms of the width as $L = W + 3$. Substitute the perimeter value and the expression for length into the formula. It is often helpful to make a sketch and label the sides as in Figure 3.

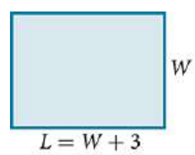


Figure 3

Now we can solve for the width and then calculate the length.

$$\begin{aligned} P &= 2L+2W \\ 54 &= 2(W+3)+2W \\ 54 &= 2W+6+2W \\ 54 &= 4W+6 \\ 48 &= 4W \\ 12 &= W \\ (12+3) &= L \\ 15 &= L \end{aligned}$$

The dimensions are $L = 15$ ft and $W = 12$ ft.

TRY IT #4 Find the dimensions of a rectangle given that the perimeter is 110 cm and the length is 1 cm more than twice the width.

EXAMPLE 5

Solving an Area Problem

The perimeter of a tablet of graph paper is 48 in. The length is 6 in. more than the width. Find the area of the graph paper.

Solution

The standard formula for area is $A = LW$; however, we will solve the problem using the perimeter formula. The reason we use the perimeter formula is because we know enough information about the perimeter that the formula will allow us to solve for one of the unknowns. As both perimeter and area use length and width as dimensions, they are often used together to solve a problem such as this one.

We know that the length is 6 in. more than the width, so we can write length as $L = W + 6$. Substitute the value of the perimeter and the expression for length into the perimeter formula and find the length.

$$\begin{aligned} P &= 2L + 2W \\ 48 &= 2(W+6) + 2W \\ 48 &= 2W + 12 + 2W \\ 48 &= 4W + 12 \\ 36 &= 4W \\ 9 &= W \\ (9+6) &= L \\ 15 &= L \end{aligned}$$

Now, we find the area given the dimensions of $L = 15$ in. and $W = 9$ in.

$$\begin{aligned} A &= LW \\ A &= 15(9) \\ &= 135 \ \text{in.}^2 \end{aligned}$$

The area is 135 in.^{2}.

TRY IT #5 A game room has a perimeter of 70 ft. The length is five more than twice the width. How many ft^2 of new carpeting should be ordered?

EXAMPLE 6

Solving a Volume Problem

Find the dimensions of a shipping box given that the length is twice the width, the height is 8 inches, and the volume is 1,600 in.^{3}.

Solution

The formula for the volume of a box is given as $V = LWH$, the product of length, width, and height. We are given that $L = 2W$, and $H = 8$. The volume is $1{,}600$ cubic inches.

$$\begin{aligned} V &= LWH \\ 1,600 &= (2W)W(8) \\ 1,600 &= 16W^2 \\ 100 &= W^2 \\ 10 &= W \end{aligned}$$

The dimensions are $L = 20$ in., $W = 10$ in., and $H = 8$ in.

Analysis

Note that the square root of W^2 would result in a positive and a negative value. However, because we are describing width, we can use only the positive result.

MEDIA

Access these online resources for additional instruction and practice with models and applications of linear equations.

Problem solving using linear equations (http://openstax.org/l/lineqprobsolve)
Problem solving using equations (http://openstax.org/l/equationprsolve)
Finding the dimensions of area given the perimeter (http://openstax.org/l/permareasolve)
Find the distance between the cities using the distance = rate * time formula (http://openstax.org/l/ratetimesolve)
Linear equation application (Write a cost equation) (http://openstax.org/l/lineqappl)

2.3 SECTION EXERCISES

Verbal

1. To set up a model linear equation to fit real-world applications, what should always be the first step?

2. Use your own words to describe this equation where n is a number: $5(n+3)=2n$

3. If the total amount of money you had to invest was $2,000 and you deposit x amount in one investment, how can you represent the remaining amount?

4. If a carpenter sawed a 10-ft board into two sections and one section was n ft long, how long would the other section be in terms of n ?

5. If Bill was traveling v mi/h, how would you represent Daemon's speed if he was traveling 10 mi/h faster?

Real-World Applications

For the following exercises, use the information to find a linear algebraic equation model to use to answer the question being asked.

6. Mark and Don are planning to sell each of their marble collections at a garage sale. If Don has 1 more than 3 times the number of marbles Mark has, how many does each boy have to sell if the total number of marbles is 113?

7. Beth and Ann are joking that their combined ages equal Sam's age. If Beth is twice Ann's age and Sam is 69 yr old, what are Beth and Ann's ages?

8. Ruden originally filled out 8 more applications than Hanh. Then each boy filled out 3 additional applications, bringing the total to 28. How many applications did each boy originally fill out?

For the following exercises, use this scenario: Two different telephone carriers offer the following plans that a person is considering. Company A has a monthly fee of $20 and charges of $.05/min for calls. Company B has a monthly fee of $5 and charges $.10/min for calls.

9. Find the model of the total cost of Company A's plan, using m for the minutes.

10. Find the model of the total cost of Company B's plan, using m for the minutes.

11. Find out how many minutes of calling would make the two plans equal.

12. If the person makes a monthly average of 200 min of calls, which plan should for the person choose?

For the following exercises, use this scenario: A wireless carrier offers the following plans that a person is considering. The Family Plan: $90 monthly fee, unlimited talk and text on up to 8 lines, and data charges of $40 for each device for up to 2 GB of data per device. The Mobile Share Plan: $120 monthly fee for up to 10 devices, unlimited talk and text for all the lines, and data charges of $35 for each device up to a shared total of 10 GB of data. Use P for the number of devices that need data plans as part of their cost.

13. Find the model of the total cost of the Family Plan.

14. Find the model of the total cost of the Mobile Share Plan.

15. Assuming they stay under their data limit, find the number of devices that would make the two plans equal in cost.

16. If a family has 3 smart phones, which plan should they choose?

For exercises 17 and 18, use this scenario: A retired woman has $50,000 to invest but needs to make $6,000 a year from the interest to meet certain living expenses. One bond investment pays 15% annual interest. The rest of it she wants to put in a CD that pays 7%.

17. If we let x be the amount the woman invests in the 15% bond, how much will she be able to invest in the CD?

18. Set up and solve the equation for how much the woman should invest in each option to sustain a $6,000 annual return.

19. Two planes fly in opposite directions. One travels 450 mi/h and the other 550 mi/h. How long will it take before they are 4,000 mi apart?

20. Ben starts walking along a path at 4 mi/h. One and a half hours after Ben leaves, his sister Amanda begins jogging along the same path at 6 mi/h. How long will it be before Amanda catches up to Ben?

21. Fiora starts riding her bike at 20 mi/h. After a while, she slows down to 12 mi/h, and maintains that speed for the rest of the trip. The whole trip of 70 mi takes her 4.5 h. For what distance did she travel at 20 mi/h?

22. A chemistry teacher needs to mix a 30% salt solution with a 70% salt solution to make 20 qt of a 40% salt solution. How many quarts of each solution should the teacher mix to get the desired result?

23. Raúl has $20,000 to invest. His intent is to earn 11% interest on his investment. He can invest part of his money at 8% interest and part at 12% interest. How much does Raúl need to invest in each option to make get a total 11% return on his $20,000?

For the following exercises, use this scenario: A truck rental agency offers two kinds of plans. Plan A charges $75/wk plus $.10/mi driven. Plan B charges $100/wk plus $.05/mi driven.

24. Write the model equation for the cost of renting a truck with plan A.

25. Write the model equation for the cost of renting a truck with plan B.

26. Find the number of miles that would generate the same cost for both plans.

27. If Tim knows he has to travel 300 mi, which plan should he choose?

For the following exercises, use the formula given to solve for the required value.

28. $A = P(1 + rt)$ is used to find the principal amount *P* deposited, earning *r*% interest, for *t* years. Use this to find what principal amount *P* David invested at a 3% rate for 20 yr if $A = \$8{,}000$.

29. The formula $F = \frac{mv^2}{R}$ relates force (F), velocity (v), mass , and resistance (m). Find R when $m = 45$, $v = 7$, and $F = 245$.

30. $F = ma$ indicates that force (*F*) equals mass (*m*) times acceleration (*a*). Find the acceleration of a mass of 50 kg if a force of 12 N is exerted on it.

31. $Sum = \frac{1}{1-r}$ is the formula for an infinite series sum. If the sum is 5, find r.

For the following exercises, solve for the given variable in the formula. After obtaining a new version of the formula, you will use it to solve a question.

32. Solve for *W*: $P = 2L + 2W$

33. Use the formula from the previous question to find the width, W, of a rectangle whose length is 15 and whose perimeter is 58.

34. Solve for f : $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$

35. Use the formula from the previous question to find f when $p = 8$and $q = 13$.

36. Solve for m in the slope-intercept formula: $y = mx + b$

37. Use the formula from the previous question to find m when the coordinates of the point are $(4, 7)$ and $b = 12$.

38. The area of a trapezoid is given by $A = \frac{1}{2}h\left(b_1 + b_2\right)$. Use the formula to find the area of a trapezoid with $h = 6, \quad b_1 = 14, \quad \text{and } b_2 = 8.$

39. Solve for *h:* $A = \frac{1}{2}h\left(b_1 + b_2\right)$

40. Use the formula from the previous question to find the height of a trapezoid with $A = 150, b_1 = 19,$ and $b_2 = 11.$

41. Find the dimensions of an American football field. The length is 200 ft more than the width, and the perimeter is 1,040 ft. Find the length and width. Use the perimeter formula $P = 2L + 2W$.

42. Distance equals rate times time, $d = rt$. Find the distance Tom travels if he is moving at a rate of 55 mi/h for 3.5 h.

43. Using the formula in the previous exercise, find the distance that Susan travels if she is moving at a rate of 60 mi/h for 6.75 h.

44. What is the total distance that two people travel in 3 h if one of them is riding a bike at 15 mi/h and the other is walking at 3 mi/h?

45. If the area model for a triangle is $A = \frac{1}{2}bh$, find the area of a triangle with a height of 16 in. and a base of 11 in.

46. Solve for *h*: $A = \frac{1}{2}bh$

47. Use the formula from the previous question to find the height to the nearest tenth of a triangle with a base of 15 and an area of 215.

48. The volume formula for a cylinder is $V = \pi r^2 h$. Using the symbol π in your answer, find the volume of a cylinder with a radius, r, of 4 cm and a height of 14 cm.

49. Solve for *h*: $V = \pi r^2 h$

50. Use the formula from the previous question to find the height of a cylinder with a radius of 8 and a volume of 16π

51. Solve for *r*: $V = \pi r^2 h$

52. Use the formula from the previous question to find the radius of a cylinder with a height of 36 and a volume of 324π.

53. The formula for the circumference of a circle is $C = 2\pi r$. Find the circumference of a circle with a diameter of 12 in. (diameter = 2*r*). Use the symbol π in your final answer.

54. Solve the formula from the previous question for π. Notice why π is sometimes defined as the ratio of the circumference to its diameter.

2.4 Complex Numbers

Learning Objectives

In this section, you will:

- Add and subtract complex numbers.
- Multiply and divide complex numbers.
- Simplify powers of i

COREQUISITE SKILLS

Learning Objectives

- Use the product property to simplify radical expressions (IA 8.2.1)
- Evaluate the square root of a negative number (IA 8.8.1)

Objective 1: Use the product property to simplify radical expressions (IA 8.2.1)

HOW TO

Simplify a radical expression using the Product Property.

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
Step 2. Use the product rule to rewrite the radical as the product of two radicals.
Step 3. Simplify the root of the perfect power.

EXAMPLE 1

Simplify a radical expression using the Product Property.

Simplify: ⓐ $\sqrt{500}$ ⓑ $\sqrt[3]{16}$ ⓒ $\sqrt[4]{243}$.

Solution

ⓐ

	$\sqrt{500}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{100 \cdot 5}$
Rewrite the radical as the product of two radicals.	$\sqrt{100} \cdot \sqrt{5}$
Simplify.	$10\sqrt{5}$

ⓑ

	$\sqrt[3]{16}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{8 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8} \cdot \sqrt[3]{2}$
Simplify.	$2\sqrt[3]{2}$

ⓒ

	$\sqrt[4]{243}$
Rewrite the radicand as a product using the largest perfect fourth power factor.	$\sqrt[4]{81 \cdot 3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{81} \cdot \sqrt[4]{3}$

Simplify.	$3\sqrt[4]{3}$

ⓓ

	$\sqrt[5]{x^8}$
Rewrite the radicand as a product using the largest perfect fifth power factor.	$\sqrt[5]{x^5 \cdot x^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[5]{x^5}\sqrt[5]{x^3}$
Simplify.	$x\sqrt[5]{x^3}$

Practice Makes Perfect

Simplify a radical expression using the Product Property.

1. $\sqrt{54}$

2. $\sqrt{125}$

3. $\sqrt{25x^3}$

4. $\sqrt[3]{625}$

5. $\sqrt[3]{128}$

6. $\sqrt[4]{x^{10}}$

7. $\sqrt[4]{16r^9}$

8. $\sqrt[4]{81s^{10}q^4}$

Objective 2: Evaluate the square root of a negative number (IA 8.8.1)

Imaginary numbers result when we try to take the square root of a negative number. They do not belong to the set of real numbers and so are called imaginary or complex. We will see complex solutions when we solve quadratic equations whose graph does not touch the x-axis. Imaginary numbers are used in many real-life applications including the study of electricity involving alternating current (AC) electronics. Wireless, cellular and radar technologies utilize imaginary numbers. Electrical engineers use imaginary numbers to measure the amplitude and phase of electrical oscillations.

Important Facts to Remember:

- $\sqrt{-1} = i$ (a complex, or imaginary number)
- $i^2 = -1$ (a real number)

- $a + bi =$ standard form for a complex number
- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ (product rule for radicals)
- $a - bi =$ the complex conjugate of $a + bi$

EXAMPLE 2

Evaluate the square root of a negative number

Simplify: ⓐ $\sqrt{-64}$ ⓑ $\sqrt{-98}$ ⓒ $\sqrt{-243}$

Solution

ⓐ $\sqrt{-64}$

Rewrite the radicand as a product. $\sqrt{-1 \cdot 64}$

Rewrite the radical as the product of two radicals. $\sqrt{-1} \cdot \sqrt{64}$

Use the definition of i and simplify radical terms. $i \cdot 8 = 8i$

ⓑ $\sqrt{-98}$

Rewrite the radicand as a product using the largest perfect square factor. $\sqrt{-1 \cdot 49 \cdot 2}$

Rewrite the radical as the product of three radicals. $\sqrt{-1} \cdot \sqrt{49} \cdot \sqrt{2}$

Use the definition of i and simplify radical terms. $7i\sqrt{2}$

ⓒ $\sqrt{-243}$

Rewrite the radicand as a product using the largest perfect square factor. $\sqrt{-1 \cdot 81 \cdot 3}$

Rewrite the radical as the product of three radicals. $\sqrt{-1} \cdot \sqrt{81} \cdot \sqrt{3}$

Use the definition of i and simplify radical terms. $9i\sqrt{3}$

Practice Makes Perfect

Evaluate the square root of a negative number. Perform any indicated operations and simplify.

9. $\sqrt{-100}$

10. $\sqrt{-12}$

11. $\sqrt{-75}$

12. Multiply: $\sqrt{-49} \cdot \sqrt{-9}$

13. Multiply: $\sqrt{36} \cdot \sqrt{-81}$

14. Multiply: $\left(2 - \sqrt{-16}\right)\left(3 + \sqrt{-4}\right)$

15. Multiply: $\left(2 + \sqrt{-25}\right)\left(2 - \sqrt{-25}\right)$

16. Add: $(1 + 3i) + (7 + 4i)$

17. Add: $\left(5 - \sqrt{-36}\right) + \left(2 - \sqrt{-49}\right)$

18. Subtract: $\left(5-\sqrt{-36}\right)-\left(2-\sqrt{-49}\right)$

19. Subtract: $\left(-7-\sqrt{-50}\right)-\left(-32-\sqrt{-18}\right)$

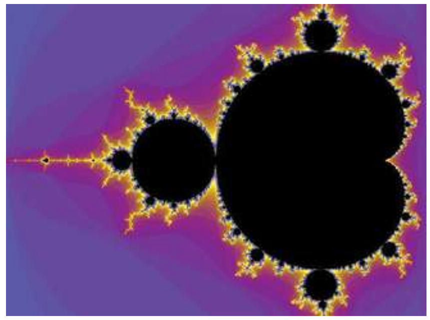


Figure 1

Discovered by Benoit Mandelbrot around 1980, the Mandelbrot Set is one of the most recognizable fractal images. The image is built on the theory of self-similarity and the operation of iteration. Zooming in on a fractal image brings many surprises, particularly in the high level of repetition of detail that appears as magnification increases. The equation that generates this image turns out to be rather simple.

In order to better understand it, we need to become familiar with a new set of numbers. Keep in mind that the study of mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. In this section, we will explore a set of numbers that fills voids in the set of real numbers and find out how to work within it.

Expressing Square Roots of Negative Numbers as Multiples of i

We know how to find the square root of any positive real number. In a similar way, we can find the square root of any negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number**.** The imaginary number i is defined as the square root of -1.

$$\sqrt{-1}=i$$

So, using properties of radicals,

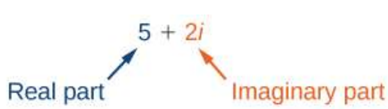
$$i^2=\left(\sqrt{-1}\right)^2=-1$$

We can write the square root of any negative number as a multiple of i. Consider the square root of -49.

$$\begin{aligned}\sqrt{-49}&=\sqrt{49\cdot(-1)}\\&=\sqrt{49}\sqrt{-1}\\&=7i\end{aligned}$$

We use $7i$ and not $-7i$ because the principal root of 49 is the positive root.

A complex number is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a+bi$ where a is the real part and b is the imaginary part. For example, $5+2i$ is a complex number. So, too, is $3+4i\sqrt{3}$.



Imaginary numbers differ from real numbers in that a squared imaginary number produces a negative real number. Recall that when a positive real number is squared, the result is a positive real number and when a negative real number is squared, the result is also a positive real number. Complex numbers consist of real and imaginary numbers.

Imaginary and Complex Numbers

A **complex number** is a number of the form $a + bi$ where

- a is the real part of the complex number.
- b is the imaginary part of the complex number.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and b is not equal to 0, the complex number is called a pure imaginary number. An **imaginary number** is an even root of a negative number.

HOW TO

Given an imaginary number, express it in the standard form of a complex number.

1. Write $\sqrt{-a}$ as $\sqrt{a}\sqrt{-1}$.
2. Express $\sqrt{-1}$ as i.
3. Write $\sqrt{a} \cdot i$ in simplest form.

EXAMPLE 1

Expressing an Imaginary Number in Standard Form

Express $\sqrt{-9}$ in standard form.

Solution

$$\begin{aligned} \sqrt{-9} &= \sqrt{9}\sqrt{-1} \\ &= 3i \end{aligned}$$

In standard form, this is $0 + 3i$.

TRY IT #1 Express $\sqrt{-24}$ in standard form.

Plotting a Complex Number on the Complex Plane

We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number, we need to address the two components of the number. We use the **complex plane**, which is a coordinate system in which the horizontal axis represents the real component and the vertical axis represents the imaginary component. Complex numbers are the points on the plane, expressed as ordered pairs (a, b), where a represents the coordinate for the horizontal axis and b represents the coordinate for the vertical axis.

Let's consider the number $-2 + 3i$. The real part of the complex number is -2 and the imaginary part is 3. We plot the ordered pair $(-2, 3)$ to represent the complex number $-2 + 3i$, as shown in Figure 2.

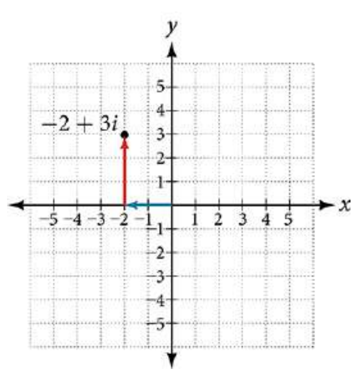


Figure 2

Complex Plane

In the complex plane, the horizontal axis is the real axis, and the vertical axis is the imaginary axis, as shown in Figure 3.

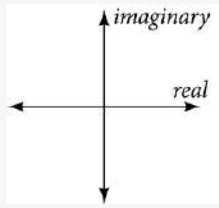


Figure 3

HOW TO

Given a complex number, represent its components on the complex plane.

1. Determine the real part and the imaginary part of the complex number.
2. Move along the horizontal axis to show the real part of the number.
3. Move parallel to the vertical axis to show the imaginary part of the number.
4. Plot the point.

EXAMPLE 2

Plotting a Complex Number on the Complex Plane

Plot the complex number $3 - 4i$ on the complex plane.

Solution

The real part of the complex number is 3, and the imaginary part is –4. We plot the ordered pair $(3, -4)$ as shown in Figure 4.

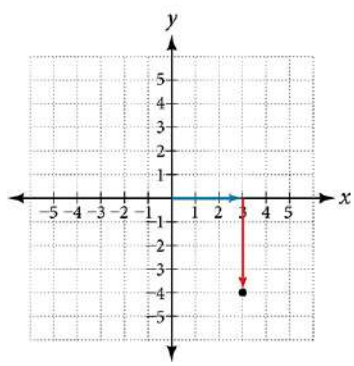


Figure 4

TRY IT #2 Plot the complex number $-4 - i$ on the complex plane.

Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and then combine the imaginary parts.

Complex Numbers: Addition and Subtraction

Adding complex numbers:

$$(a+bi)+(c+di)=(a+c)+(b+d)\,i$$

Subtracting complex numbers:

$$(a+bi)-(c+di)=(a-c)+(b-d)\,i$$

HOW TO

Given two complex numbers, find the sum or difference.

1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

EXAMPLE 3

Adding and Subtracting Complex Numbers

Add or subtract as indicated.

ⓐ $(3-4i)+(2+5i)$ ⓑ $(-5+7i)-(-11+2i)$

Solution

We add the real parts and add the imaginary parts.

ⓐ

$$\begin{aligned}(3-4i)+(2+5i) &= 3-4i+2+5i\\ &= 3+2+(-4i)+5i\\ &= (3+2)+(-4+5)i\\ &= 5+i\end{aligned}$$

ⓑ

$$\begin{aligned}(-5+7i)-(-11+2i) &= -5+7i+11-2i\\ &= -5+11+7i-2i\\ &= (-5+11)+(7-2)i\\ &= 6+5i\end{aligned}$$

TRY IT #3 Subtract $2+5i$ from $3–4i$.

Multiplying Complex Numbers

Multiplying complex numbers is much like multiplying binomials. The major difference is that we work with the real and imaginary parts separately.

Multiplying a Complex Number by a Real Number

Lets begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. Consider, for example, $3\,(6+2i)$:

$$\begin{aligned}3(6+2i) &= (3\cdot 6)+(3\cdot 2i) && \text{Distribute.}\\ &= 18+6i && \text{Simplify.}\end{aligned}$$

HOW TO

Given a complex number and a real number, multiply to find the product.

1. Use the distributive property.
2. Simplify.

EXAMPLE 4

Multiplying a Complex Number by a Real Number

Find the product $4(2 + 5i)$.

Solution

Distribute the 4.

$$\begin{aligned} 4(2+5i) &= (4 \cdot 2) + (4 \cdot 5i) \\ &= 8 + 20i \end{aligned}$$

TRY IT #4 Find the product: $\frac{1}{2}(5 - 2i)$.

Multiplying Complex Numbers Together

Now, let's multiply two complex numbers. We can use either the distributive property or more specifically the FOIL method because we are dealing with binomials. Recall that FOIL is an acronym for multiplying First, Inner, Outer, and Last terms together. The difference with complex numbers is that when we get a squared term, i^2, it equals -1.

$$\begin{aligned} (a+bi)(c+di) &= ac + adi + bci + bdi^2 & \\ &= ac + adi + bci - bd & i^2 = -1 \\ &= (ac - bd) + (ad + bc)i & \text{Group real terms and imaginary terms.} \end{aligned}$$

HOW TO

Given two complex numbers, multiply to find the product.

1. Use the distributive property or the FOIL method.
2. Remember that $i^2 = -1$.
3. Group together the real terms and the imaginary terms

EXAMPLE 5

Multiplying a Complex Number by a Complex Number

Multiply: $(4 + 3i)(2 - 5i)$.

Solution

$$\begin{aligned} (4+3i)(2-5i) &= 4(2) - 4(5i) + 3i(2) - (3i)(5i) \\ &= 8 - 20i + 6i - 15\left(i^2\right) \\ &= (8 + 15) + (-20 + 6)i \\ &= 23 - 14i \end{aligned}$$

TRY IT #5 Multiply: $(3 - 4i)(2 + 3i)$.

Dividing Complex Numbers

Dividing two complex numbers is more complicated than adding, subtracting, or multiplying because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator to write the answer in standard form $a + bi$. We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the complex conjugate of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of $a + bi$ is $a - bi$. For example, the product of $a + bi$ and $a - bi$ is

$$\begin{aligned}(a+bi)(a-bi) &= a^2-abi+abi-b^2i^2\\ &= a^2+b^2\end{aligned}$$

The result is a real number.

Note that complex conjugates have an opposite relationship: The complex conjugate of $a+bi$ is $a-bi$, and the complex conjugate of $a-bi$ is $a+bi$. Further, when a quadratic equation with real coefficients has complex solutions, the solutions are always complex conjugates of one another.

Suppose we want to divide $c+di$ by $a+bi$, where neither a nor b equals zero. We first write the division as a fraction, then find the complex conjugate of the denominator, and multiply.

$$\frac{c+di}{a+bi} \text{ where } a \neq 0 \text{ and } b \neq 0$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(c+di)}{(a+bi)} \cdot \frac{(a-bi)}{(a-bi)} = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)}$$

Apply the distributive property.

$$= \frac{ca-cbi+adi-bdi^2}{a^2-abi+abi-b^2i^2}$$

Simplify, remembering that $i^2=-1$.

$$\begin{aligned}&= \frac{ca-cbi+adi-bd(-1)}{a^2-abi+abi-b^2(-1)}\\ &= \frac{(ca+bd)+(ad-cb)i}{a^2+b^2}\end{aligned}$$

The Complex Conjugate

The **complex conjugate** of a complex number $a+bi$ is $a-bi$. It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

EXAMPLE 6

Finding Complex Conjugates

Find the complex conjugate of each number.

ⓐ $2+i\sqrt{5}$ ⓑ $-\frac{1}{2}i$

Solution

ⓐ The number is already in the form $a+bi$. The complex conjugate is $a-bi$, or $2-i\sqrt{5}$.

ⓑ We can rewrite this number in the form $a+bi$ as $0-\frac{1}{2}i$. The complex conjugate is $a-bi$, or $0+\frac{1}{2}i$. This can be written simply as $\frac{1}{2}i$.

Analysis

Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. To obtain a real number from an imaginary number, we can simply multiply by i.

TRY IT #6 Find the complex conjugate of $-3+4i$.

HOW TO

Given two complex numbers, divide one by the other.

1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

EXAMPLE 7

Dividing Complex Numbers

Divide: $(2+5i)$ by $(4-i)$.

Solution

We begin by writing the problem as a fraction.

$$\frac{(2+5i)}{(4-i)}$$

Then we multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)}$$

To multiply two complex numbers, we expand the product as we would with polynomials (using FOIL).

$$\begin{aligned} \frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)} &= \frac{8+2i+20i+5i^2}{16+4i-4i-i^2} \\ &= \frac{8+2i+20i+5(-1)}{16+4i-4i-(-1)} && \text{Because } i^2=-1. \\ &= \frac{3+22i}{17} \\ &= \frac{3}{17}+\frac{22}{17}i && \text{Separate real and imaginary parts.} \end{aligned}$$

Note that this expresses the quotient in standard form.

Simplifying Powers of i

The powers of i are cyclic. Let's look at what happens when we raise i to increasing powers.

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= i^2 \cdot i = -1 \cdot i = -i \\ i^4 &= i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1 \\ i^5 &= i^4 \cdot i = 1 \cdot i = i \end{aligned}$$

We can see that when we get to the fifth power of i, it is equal to the first power. As we continue to multiply i by increasing powers, we will see a cycle of four. Let's examine the next four powers of i.

$$\begin{aligned} i^6 &= i^5 \cdot i = i \cdot i = i^2 = -1 \\ i^7 &= i^6 \cdot i = i^2 \cdot i = i^3 = -i \\ i^8 &= i^7 \cdot i = i^3 \cdot i = i^4 = 1 \\ i^9 &= i^8 \cdot i = i^4 \cdot i = i^5 = i \end{aligned}$$

The cycle is repeated continuously: $i, -1, -i, 1,$ every four powers.

EXAMPLE 8

Simplifying Powers of i

Evaluate: i^{35}.

Solution

Since $i^4 = 1$, we can simplify the problem by factoring out as many factors of i^4 as possible. To do so, first determine how many times 4 goes into 35: $35 = 4 \cdot 8 + 3$.

$$i^{35} = i^{4 \cdot 8 + 3} = i^{4 \cdot 8} \cdot i^3 = \left(i^4\right)^8 \cdot i^3 = 1^8 \cdot i^3 = i^3 = -i$$

TRY IT #7 Evaluate: i^{18}

Q&A **Can we write i^{35} in other helpful ways?**

As we saw in Example 8, we reduced i^{35} to i^3 by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of i^{35} may be more useful. Table 1 shows some other possible factorizations.

Factorization of i^{35}	$i^{34} \cdot i$	$i^{33} \cdot i^2$	$i^{31} \cdot i^4$	$i^{19} \cdot i^{16}$
Reduced form	$\left(i^2\right)^{17} \cdot i$	$i^{33} \cdot (-1)$	$i^{31} \cdot 1$	$i^{19} \cdot \left(i^4\right)^4$
Simplified form	$(-1)^{17} \cdot i$	$-i^{33}$	i^{31}	i^{19}

Table 1

Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.

MEDIA

Access these online resources for additional instruction and practice with complex numbers.

Adding and Subtracting Complex Numbers (http://openstax.org/l/addsubcomplex)
Multiply Complex Numbers (http://openstax.org/l/multiplycomplex)
Multiplying Complex Conjugates (http://openstax.org/l/multcompconj)
Raising i to Powers (http://openstax.org/l/raisingi)

2.4 SECTION EXERCISES

Verbal

1. Explain how to add complex numbers.

2. What is the basic principle in multiplication of complex numbers?

3. Give an example to show that the product of two imaginary numbers is not always imaginary.

4. What is a characteristic of the plot of a real number in the complex plane?

Algebraic

For the following exercises, evaluate the algebraic expressions.

5. If $y = x^2 + x - 4$, evaluate y given $x = 2i$.

6. If $y = x^3 - 2$, evaluate y given $x = i$.

7. If $y = x^2 + 3x + 5$, evaluate y given $x = 2 + i$.

8. If $y = 2x^2 + x - 3$, evaluate y given $x = 2 - 3i$.

9. If $y = \frac{x+1}{2-x}$, evaluate y given $x = 5i$.

10. If $y = \frac{1+2x}{x+3}$, evaluate y given $x = 4i$.

Graphical

For the following exercises, plot the complex numbers on the complex plane.

11. $1 - 2i$

12. $-2 + 3i$

13. i

14. $-3 - 4i$

Numeric

For the following exercises, perform the indicated operation and express the result as a simplified complex number.

15. $(3 + 2i) + (5 - 3i)$

16. $(-2 - 4i) + (1 + 6i)$

17. $(-5 + 3i) - (6 - i)$

18. $(2 - 3i) - (3 + 2i)$

19. $(-4 + 4i) - (-6 + 9i)$

20. $(2 + 3i)(4i)$

21. $(5 - 2i)(3i)$

22. $(6 - 2i)(5)$

23. $(-2 + 4i)(8)$

24. $(2 + 3i)(4 - i)$

25. $(-1 + 2i)(-2 + 3i)$

26. $(4 - 2i)(4 + 2i)$

27. $(3 + 4i)(3 - 4i)$

28. $\frac{3+4i}{2}$

29. $\frac{6-2i}{3}$

30. $\frac{-5+3i}{2i}$

31. $\frac{6+4i}{i}$

32. $\frac{2-3i}{4+3i}$

33. $\frac{3+4i}{2-i}$

34. $\frac{2+3i}{2-3i}$

35. $\sqrt{-9} + 3\sqrt{-16}$

36. $-\sqrt{-4} - 4\sqrt{-25}$

37. $\frac{2+\sqrt{-12}}{2}$

38. $\frac{4+\sqrt{-20}}{2}$

39. i^8

40. i^{15}

41. i^{22}

Technology

For the following exercises, use a calculator to help answer the questions.

42. Evaluate $(1 + i)^k$ for $k = 4, 8,$ and $12.$ Predict the value if $k = 16.$

43. Evaluate $(1 - i)^k$ for $k = 2, 6,$ and $10.$ Predict the value if $k = 14.$

44. Evaluate $(1 + i)^k - (1 - i)^k$ for $k = 4, 8,$ and $12.$ Predict the value for $k = 16.$

45. Show that a solution of $x^6 + 1 = 0$ is $\frac{\sqrt{3}}{2} + \frac{1}{2}i$.

46. Show that a solution of $x^8 - 1 = 0$ is $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

Extensions

For the following exercises, evaluate the expressions, writing the result as a simplified complex number.

47. $\frac{1}{i} + \frac{4}{i^3}$

48. $\frac{1}{i^{11}} - \frac{1}{i^{21}}$

49. $i^7\left(1 + i^2\right)$

50. $i^{-3} + 5i^7$

51. $\frac{(2+i)(4-2i)}{(1+i)}$

52. $\frac{(1+3i)(2-4i)}{(1+2i)}$

53. $\frac{(3+i)^2}{(1+2i)^2}$

54. $\frac{3+2i}{2+i} + (4 + 3i)$

55. $\frac{4+i}{i} + \frac{3-4i}{1-i}$

56. $\frac{3+2i}{1+2i} - \frac{2-3i}{3+i}$

2.5 Quadratic Equations

Learning Objectives

In this section, you will:

- Solve quadratic equations by factoring.
- Solve quadratic equations by the square root property.
- Solve quadratic equations by completing the square.
- Solve quadratic equations by using the quadratic formula.

COREQUISITE SKILLS

Learning Objectives

- Recognize and use the appropriate method to factor a polynomial completely (IA 6.4.1)

Objective 1: Recognize and use the appropriate method to factor a polynomial completely (IA 6.4.1)

The following chart summarizes the factoring methods and outlines a strategy you should use when factoring polynomials.

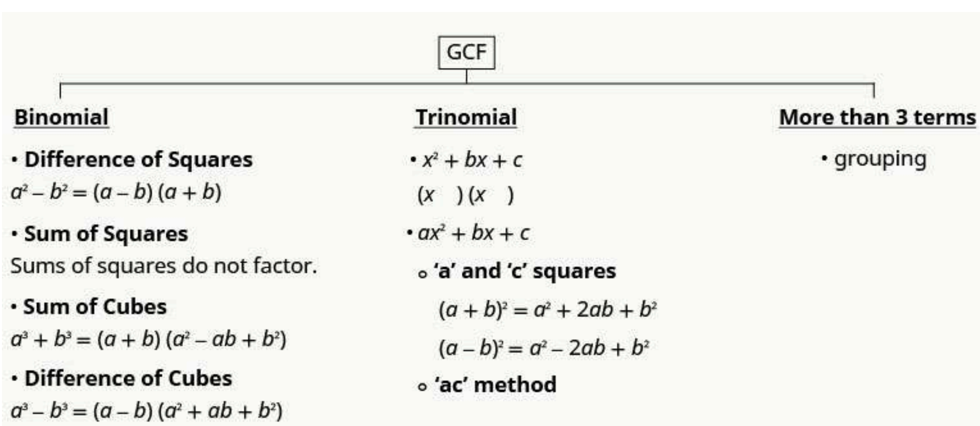


Figure 1 General Strategy for Factoring Polynomials

HOW TO

Step 1. Is there a factor common to all terms? Factor it out.

Step 2. Consider the number of terms in the original polynomial.

Two Terms? We refer to this as a binomial.

- If it is a sum of squared terms, this does not factor further. We refer to this as prime.
- If it is a difference of perfect square terms, factor using the difference of two squares form.
- If it is a sum of perfect cubed terms, factor using the sum of two cubes form.
- If it is a difference of perfect cubed terms, factor using the difference of two cubes form.

Three terms? We refer to this as a trinomial.

- If the coefficient of x^2 is 1 , $x^2 + bx + c$, try to find two factors of c that add to b . If you find these, the trinomial is factorable.
- If the coefficient of x^2 is not 1 , $ax^2 + bx + c$, try to find two factors of ac that add to b . If you find these, the trinomial is factorable. You can factor using trial and error or the ac method from here.
- If you notice that ax^2 and c are perfect squares, you are working with a perfect square trinomial which can be factored as the square of a binomial.

Four terms?

- Try to factor by grouping two terms, or three terms and finding factors common to those terms.

Step 3. Check your work. Is the polynomial factored completely? Do the factors multiply to give you the original polynomial?

EXAMPLE 1

Recognize and use the appropriate method to factor a polynomial completely.

ⓐ $5x^3 - 15x^2 - 50x$ ⓑ $4x^2 + 20xy + 25y^2$ ⓒ $9x^2 - 16$ ⓓ $8x^3 + 27y^3$
ⓔ $16x^2 + 24xy - 4x - 6y$

Solution

ⓐ	$5x^3 - 15x^2 - 50x$
First let's factor out the common factor of $5x$.	$5x(x^2 - 3x - 10)$
Consider the resulting trinomial with a leading coefficient of 1 . Are there two factors of -10 that add to -3 ? Yes, -5 and 2 , multiply to -10 and add to -3 . Use these factors.	$5x(x-5)(x+2)$

ⓑ	$4x^2 + 20xy + 25y^2$
There are no factors common to all three terms in the trinomial. Notice that the first term, $4x^2$ is a perfect square $(2x)^2$ and last term, $25y^2$, is a perfect square $(5y)^2$, so this may be a perfect square trinomial.	$(a+b)^2 = a^2 + 2ab + b^2$
Check the middle term to see if it is equivalent to $2ab$, or $2(2x)(5y) = 20xy$, and it is.	$ad(2x+5y)(2x+5y) = (2x+5y)^2$

ⓒ	$9x^2 - 16$

There are no factors common to both terms in the binomial. Notice that the first term, $9x^2$ is a perfect square $(3x)^2$ and last term, 16, is a perfect square 4^2, so this form is a difference of two squares.	$a^2 - b^2 = (a-b)(a+b)$
	$9x^2 - 16 = (3x-4)(3x+4)$

ⓓ	$8x^3 + 27y^3$
There are no factors common to both terms in the binomial. Notice that the first term, $8x^3$ is a perfect cube $(2x)^3$ and last term, $27y^3$, so this form is a sum of two cubes.	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
	$(2x)^3 + (3y)^3 = (2x+3y)(4x^2 - 6xy + 9y^2)$

ⓔ	$16x^2 + 24xy - 4x - 6y$
First, let's factor out the common factor of 2.	$2(8x^2 + 12xy - 2x - 3y)$
The first two terms in the parentheses have a 4x in common, let's factor that out. Since the second two terms have nothing in common, but are both negative, we'll factor out a negative 1.	$2[4x(2x+3y) - 1(2x+3y)]$
Inside the parentheses, the four terms have become two terms with a common factor of $(2x+3y)$ Factor this from the two terms	$2[(2x+3y)(4x-1)]$

Practice Makes Perfect

Recognize and use the appropriate method to factor a polynomial completely. Factor each of the following polynomials completely, if a polynomial does not factor label it as prime.

1. $x^2 + 10x + 24$

2. $y^2 - 20y + 36$

3. $2x^2 + 16x + 30$

4. $9x^2 + 42x + 49$

5. $25n^2 - 90n + 81$

6. $10x^3y + 65x^2y - 35xy$

7. $121q^2 - 100$

8. $50m^2 + 72$

9. $125z^3 + 27$

10. $12ab - 6a + 10b - 5$

11. $18x^2 - 12xy + 2y^2 - 98$

12. $(3x - 5)^2 - 7(3x - 5) + 12$

Figure 1

The computer monitor on the left in Figure 1 is a 23.6-inch model and the one on the right is a 27-inch model. Proportionally, the monitors appear very similar. If there is a limited amount of space and we desire the largest monitor possible, how do we decide which one to choose? In this section, we will learn how to solve problems such as this using four different methods.

Solving Quadratic Equations by Factoring

An equation containing a second-degree polynomial is called a quadratic equation. For example, equations such as $2x^2 + 3x - 1 = 0$ and $x^2 - 4 = 0$ are quadratic equations. They are used in countless ways in the fields of engineering, architecture, finance, biological science, and, of course, mathematics.

Often the easiest method of solving a quadratic equation is factoring. Factoring means finding expressions that can be multiplied together to give the expression on one side of the equation.

If a quadratic equation can be factored, it is written as a product of linear terms. Solving by factoring depends on the zero-product property, which states that if $a \cdot b = 0$, then $a = 0$ or $b = 0$, where *a* and *b* are real numbers or algebraic expressions. In other words, if the product of two numbers or two expressions equals zero, then one of the numbers or one of the expressions must equal zero because zero multiplied by anything equals zero.

Multiplying the factors expands the equation to a string of terms separated by plus or minus signs. So, in that sense, the operation of multiplication undoes the operation of factoring. For example, expand the factored expression $(x - 2)(x + 3)$ by multiplying the two factors together.

$$\begin{aligned}(x-2)(x+3) &= x^2 + 3x - 2x - 6 \\ &= x^2 + x - 6\end{aligned}$$

The product is a quadratic expression. Set equal to zero, $x^2 + x - 6 = 0$ is a quadratic equation. If we were to factor the equation, we would get back the factors we multiplied.

The process of factoring a quadratic equation depends on the leading coefficient, whether it is 1 or another integer. We will look at both situations; but first, we want to confirm that the equation is written in standard form, $ax^2 + bx + c = 0$, where *a*, *b*, and *c* are real numbers, and $a \neq 0$. The equation $x^2 + x - 6 = 0$ is in standard form.

We can use the zero-product property to solve quadratic equations in which we first have to factor out the greatest common factor (GCF), and for equations that have special factoring formulas as well, such as the difference of squares, both of which we will see later in this section.

The Zero-Product Property and Quadratic Equations

The **zero-product property** states

$$\text{If } a \cdot b = 0, \quad \text{then } a = 0 \ \text{ or } b = 0,$$

where *a* and *b* are real numbers or algebraic expressions.

A **quadratic equation** is an equation containing a second-degree polynomial; for example

$$ax^2 + bx + c = 0$$

where *a*, *b*, and *c* are real numbers, and if $a \neq 0$, it is in standard form.

Solving Quadratics with a Leading Coefficient of 1

In the quadratic equation $x^2 + x - 6 = 0$, the leading coefficient, or the coefficient of x^2, is 1. We have one method of factoring quadratic equations in this form.

HOW TO

Given a quadratic equation with the leading coefficient of 1, factor it.

1. Find two numbers whose product equals *c* and whose sum equals *b*.
2. Use those numbers to write two factors of the form $(x + k) \ \text{ or } \ (x - k)$, where *k* is one of the numbers found in step 1. Use the numbers exactly as they are. In other words, if the two numbers are 1 and -2, the factors are $(x + 1)(x - 2)$.
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

EXAMPLE 1

Factoring and Solving a Quadratic with Leading Coefficient of 1

Factor and solve the equation: $x^2 + x - 6 = 0$.

Solution

To factor $x^2 + x - 6 = 0$, we look for two numbers whose product equals -6 and whose sum equals 1. Begin by looking at the possible factors of -6.

$$\begin{gathered} 1 \cdot (-6) \\ (-6) \cdot 1 \\ 2 \cdot (-3) \\ 3 \cdot (-2) \end{gathered}$$

The last pair, $3 \cdot (-2)$ sums to 1, so these are the numbers. Note that only one pair of numbers will work. Then, write the factors.

$$(x - 2)(x + 3) = 0$$

To solve this equation, we use the zero-product property. Set each factor equal to zero and solve.

$$\begin{aligned} (x - 2)(x + 3) &= 0 \\ (x - 2) &= 0 \\ x &= 2 \\ (x + 3) &= 0 \\ x &= -3 \end{aligned}$$

The two solutions are 2 and -3. We can see how the solutions relate to the graph in Figure 2. The solutions are the *x*-intercepts of $y = x^2 + x - 6 = 0$.

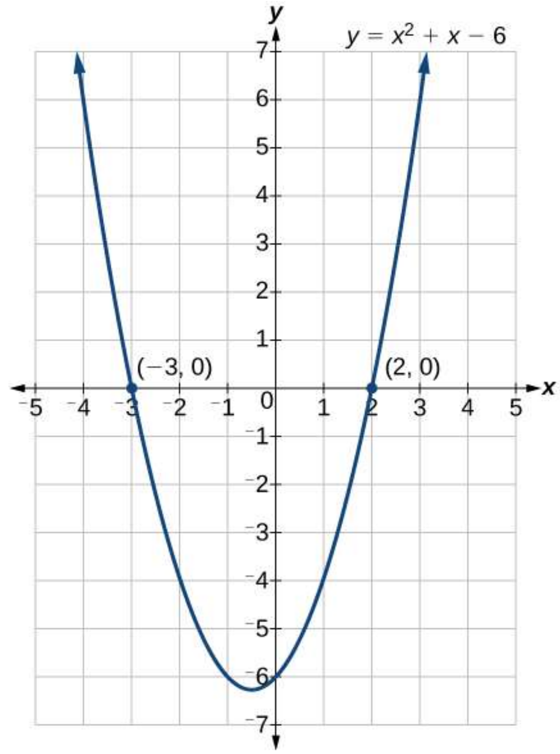


Figure 2

TRY IT #1 Factor and solve the quadratic equation: $x^2 - 5x - 6 = 0$.

EXAMPLE 2

Solve the Quadratic Equation by Factoring

Solve the quadratic equation by factoring: $x^2 + 8x + 15 = 0$.

Solution

Find two numbers whose product equals 15 and whose sum equals 8. List the factors of 15.

$$\begin{array}{l} 1 \cdot 15 \\ 3 \cdot 5 \\ (-1) \cdot (-15) \\ (-3) \cdot (-5) \end{array}$$

The numbers that add to 8 are 3 and 5. Then, write the factors, set each factor equal to zero, and solve.

$$\begin{aligned} (x+3)(x+5) &= 0 \\ (x+3) &= 0 \\ x &= -3 \\ (x+5) &= 0 \\ x &= -5 \end{aligned}$$

The solutions are -3 and -5.

TRY IT #2 Solve the quadratic equation by factoring: $x^2 - 4x - 21 = 0$.

EXAMPLE 3

Using the Zero-Product Property to Solve a Quadratic Equation Written as the Difference of Squares

Solve the difference of squares equation using the zero-product property: $x^2 - 9 = 0$.

Solution

Recognizing that the equation represents the difference of squares, we can write the two factors by taking the square root of each term, using a minus sign as the operator in one factor and a plus sign as the operator in the other. Solve using the zero-factor property.

$$\begin{aligned} x^2 - 9 &= 0 \\ (x-3)(x+3) &= 0 \\ \\ (x-3) &= 0 \\ x &= 3 \\ \\ (x+3) &= 0 \\ x &= -3 \end{aligned}$$

The solutions are 3 and -3.

TRY IT #3 Solve by factoring: $x^2 - 25 = 0$.

Solving a Quadratic Equation by Factoring when the Leading Coefficient is not 1

When the leading coefficient is not 1, we factor a quadratic equation using the method called grouping, which requires four terms. With the equation in standard form, let's review the grouping procedures:

1. With the quadratic in standard form, $ax^2 + bx + c = 0$, multiply $a \cdot c$.
2. Find two numbers whose product equals ac and whose sum equals b.
3. Rewrite the equation replacing the bx term with two terms using the numbers found in step 1 as coefficients of *x*.
4. Factor the first two terms and then factor the last two terms. The expressions in parentheses must be exactly the same to use grouping.
5. Factor out the expression in parentheses.
6. Set the expressions equal to zero and solve for the variable.

EXAMPLE 4

Solving a Quadratic Equation Using Grouping

Use grouping to factor and solve the quadratic equation: $4x^2 + 15x + 9 = 0$.

Solution

First, multiply $ac : 4(9) = 36$. Then list the factors of 36.

$$\begin{gathered} 1 \cdot 36 \\ 2 \cdot 18 \\ 3 \cdot 12 \\ 4 \cdot 9 \\ 6 \cdot 6 \end{gathered}$$

The only pair of factors that sums to 15 is $3 + 12$. Rewrite the equation replacing the *b* term, $15x$, with two terms using 3 and 12 as coefficients of *x*. Factor the first two terms, and then factor the last two terms.

$$\begin{aligned} 4x^2 + 3x + 12x + 9 &= 0 \\ x(4x+3) + 3(4x+3) &= 0 \\ (4x+3)(x+3) &= 0 \end{aligned}$$

Solve using the zero-product property.

$$\begin{aligned}(4x+3)(x+3) &= 0\\ \\ (4x+3) &= 0\\ x &= -\tfrac{3}{4}\\ \\ (x+3) &= 0\\ x &= -3\end{aligned}$$

The solutions are $-\frac{3}{4}$, and -3. See Figure 3.

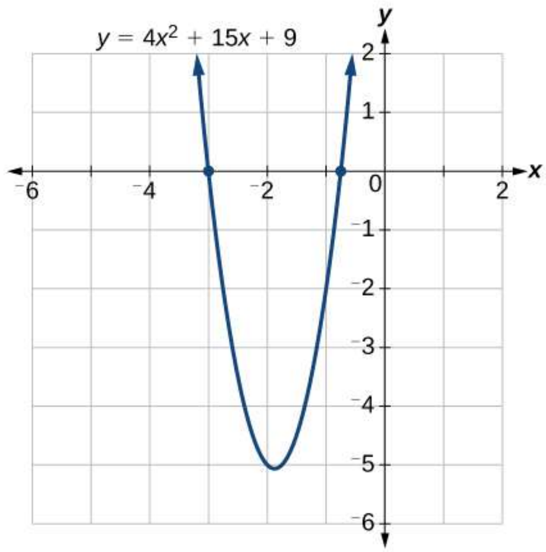


Figure 3

TRY IT #4 Solve using factoring by grouping: $12x^2 + 11x + 2 = 0$.

EXAMPLE 5

Solving a Polynomial of Higher Degree by Factoring

Solve the equation by factoring: $-3x^3 - 5x^2 - 2x = 0$.

Solution

This equation does not look like a quadratic, as the highest power is 3, not 2. Recall that the first thing we want to do when solving any equation is to factor out the GCF, if one exists. And it does here. We can factor out $-x$ from all of the terms and then proceed with grouping.

$$\begin{aligned}-3x^3 - 5x^2 - 2x &= 0\\ -x\left(3x^2 + 5x + 2\right) &= 0\end{aligned}$$

Use grouping on the expression in parentheses.

$$\begin{aligned}-x\left(3x^2 + 3x + 2x + 2\right) &= 0\\ -x[3x(x+1) + 2(x+1)] &= 0\\ -x(3x+2)(x+1) &= 0\end{aligned}$$

Now, we use the zero-product property. Notice that we have three factors.

$$\begin{aligned} -x &= 0 \\ x &= 0 \\ 3x+2 &= 0 \\ x &= -\tfrac{2}{3} \\ x+1 &= 0 \\ x &= -1 \end{aligned}$$

The solutions are $0, -\frac{2}{3}$, and -1.

TRY IT #5 Solve by factoring: $x^3 + 11x^2 + 10x = 0$.

Using the Square Root Property

When there is no linear term in the equation, another method of solving a quadratic equation is by using the **square root property**, in which we isolate the x^2 term and take the square root of the number on the other side of the equals sign. Keep in mind that sometimes we may have to manipulate the equation to isolate the x^2 term so that the square root property can be used.

The Square Root Property

With the x^2 term isolated, the square root property states that:

$$\text{if } \ x^2 = k, \text{then } \ x = \pm\sqrt{k}$$

where *k* is a nonzero real number.

HOW TO

Given a quadratic equation with an x^2 term but no x term, use the square root property to solve it.

1. Isolate the x^2 term on one side of the equal sign.
2. Take the square root of both sides of the equation, putting a $\pm$ sign before the expression on the side opposite the squared term.
3. Simplify the numbers on the side with the $\pm$ sign.

EXAMPLE 6

Solving a Simple Quadratic Equation Using the Square Root Property

Solve the quadratic using the square root property: $x^2 = 8$.

Solution

Take the square root of both sides, and then simplify the radical. Remember to use a $\pm$ sign before the radical symbol.

$$\begin{aligned} x^2 &= 8 \\ x &= \pm\sqrt{8} \\ &= \pm 2\sqrt{2} \end{aligned}$$

The solutions are $2\sqrt{2}, -2\sqrt{2}$.

EXAMPLE 7

Solving a Quadratic Equation Using the Square Root Property

Solve the quadratic equation: $4x^2 + 1 = 7$.

Solution

First, isolate the x^2 term. Then take the square root of both sides.

$$\begin{aligned} 4x^2 + 1 &= 7 \\ 4x^2 &= 6 \\ x^2 &= \frac{6}{4} \\ x &= \pm\frac{\sqrt{6}}{2} \end{aligned}$$

The solutions are $\frac{\sqrt{6}}{2}$, and $-\frac{\sqrt{6}}{2}$.

TRY IT #6 Solve the quadratic equation using the square root property: $3(x-4)^2 = 15$.

Completing the Square

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a quadratic equation known as **completing the square**. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, *a*, must equal 1. If it does not, then divide the entire equation by *a*. Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example $x^2 + 4x + 1 = 0$ to illustrate each step.

1. Given a quadratic equation that cannot be factored, and with $a = 1$, first add or subtract the constant term to the right side of the equal sign.

$$x^2 + 4x = -1$$

2. Multiply the *b* term by $\frac{1}{2}$ and square it.

$$\begin{aligned} \tfrac{1}{2}(4) &= 2 \\ 2^2 &= 4 \end{aligned}$$

3. Add $\left(\frac{1}{2}b\right)^2$ to both sides of the equal sign and simplify the right side. We have

$$\begin{aligned} x^2 + 4x + 4 &= -1 + 4 \\ x^2 + 4x + 4 &= 3 \end{aligned}$$

4. The left side of the equation can now be factored as a perfect square.

$$\begin{aligned} x^2 + 4x + 4 &= 3 \\ (x+2)^2 &= 3 \end{aligned}$$

5. Use the square root property and solve.

$$\begin{aligned} \sqrt{(x+2)^2} &= \pm\sqrt{3} \\ x + 2 &= \pm\sqrt{3} \\ x &= -2 \pm \sqrt{3} \end{aligned}$$

6. The solutions are $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$.

EXAMPLE 8

Solving a Quadratic by Completing the Square

Solve the quadratic equation by completing the square: $x^2 - 3x - 5 = 0$.

Solution

First, move the constant term to the right side of the equal sign.

$$x^2 - 3x = 5$$

Then, take $\frac{1}{2}$ of the b term and square it.

$$\begin{aligned} \tfrac{1}{2}(-3) &= -\tfrac{3}{2} \\ \left(-\tfrac{3}{2}\right)^2 &= \tfrac{9}{4} \end{aligned}$$

Add the result to both sides of the equal sign.

$$\begin{aligned} x^2 - 3x + \left(-\tfrac{3}{2}\right)^2 &= 5 + \left(-\tfrac{3}{2}\right)^2 \\ x^2 - 3x + \tfrac{9}{4} &= 5 + \tfrac{9}{4} \end{aligned}$$

Factor the left side as a perfect square and simplify the right side.

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$

Use the square root property and solve.

$$\begin{aligned} \sqrt{\left(x - \tfrac{3}{2}\right)^2} &= \pm\sqrt{\tfrac{29}{4}} \\ \left(x - \tfrac{3}{2}\right) &= \pm\tfrac{\sqrt{29}}{2} \\ x &= \tfrac{3}{2} \pm \tfrac{\sqrt{29}}{2} \end{aligned}$$

The solutions are $\frac{3+\sqrt{29}}{2}$ and $\frac{3-\sqrt{29}}{2}$.

TRY IT #7 Solve by completing the square: $x^2 - 6x = 13$.

Using the Quadratic Formula

The fourth method of solving a quadratic equation is by using the quadratic formula, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by completing the square. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by -1 and obtain a positive *a*. Given $ax^2 + bx + c = 0$, $a \neq 0$, we will complete the square as follows:

1. First, move the constant term to the right side of the equal sign:

$$ax^2 + bx = -c$$

2. As we want the leading coefficient to equal 1, divide through by *a*:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3. Then, find $\frac{1}{2}$ of the middle term, and add $\left(\frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$ to both sides of the equal sign:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

4. Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction:

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

5. Now, use the square root property, which gives

$$\begin{aligned} x+\tfrac{b}{2a} &= \pm\sqrt{\tfrac{b^2-4ac}{4a^2}} \\ x+\tfrac{b}{2a} &= \tfrac{\pm\sqrt{b^2-4ac}}{2a} \end{aligned}$$

6. Finally, add $-\frac{b}{2a}$ to both sides of the equation and combine the terms on the right side. Thus,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The Quadratic Formula

Written in standard form, $ax^2+bx+c=0$, any quadratic equation can be solved using the **quadratic formula**:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

where *a*, *b*, and *c* are real numbers and $a\neq 0$.

HOW TO

Given a quadratic equation, solve it using the quadratic formula

1. Make sure the equation is in standard form: $ax^2+bx+c=0$.
2. Make note of the values of the coefficients and constant term, a, b, and c.
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Calculate and solve.

EXAMPLE 9

Solve the Quadratic Equation Using the Quadratic Formula

Solve the quadratic equation: $x^2+5x+1=0$.

Solution

Identify the coefficients: $a=1, b=5, c=1$. Then use the quadratic formula.

$$\begin{aligned} x &= \tfrac{-(5)\pm\sqrt{(5)^2-4(1)(1)}}{2(1)} \\ &= \tfrac{-5\pm\sqrt{25-4}}{2} \\ &= \tfrac{-5\pm\sqrt{21}}{2} \end{aligned}$$

EXAMPLE 10

Solving a Quadratic Equation with the Quadratic Formula

Use the quadratic formula to solve $x^2+x+2=0$.

Solution

First, we identify the coefficients: $a = 1, b = 1$, and $c = 2$.

Substitute these values into the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(1) \pm \sqrt{(1)^2 - (4)\cdot(1)\cdot(2)}}{2\cdot 1} \\ &= \frac{-1 \pm \sqrt{1-8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $\frac{-1+i\sqrt{7}}{2}$ and $\frac{-1-i\sqrt{7}}{2}$

TRY IT #8 Solve the quadratic equation using the quadratic formula: $9x^2 + 3x - 2 = 0$.

The Discriminant

The quadratic formula not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the discriminant, or the expression under the radical, $b^2 - 4ac$. The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. Table 1 relates the value of the discriminant to the solutions of a quadratic equation.

Value of Discriminant	Results
$b^2 - 4ac = 0$	One rational solution (double solution)
$b^2 - 4ac > 0$, perfect square	Two rational solutions
$b^2 - 4ac > 0$, not a perfect square	Two irrational solutions
$b^2 - 4ac < 0$	Two complex solutions

Table 1

The Discriminant

For $ax^2 + bx + c = 0$, where a, b, and c are real numbers, the **discriminant** is the expression under the radical in the quadratic formula: $b^2 - 4ac$. It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

EXAMPLE 11

Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation

Use the discriminant to find the nature of the solutions to the following quadratic equations:

ⓐ $x^2 + 4x + 4 = 0$ ⓑ $8x^2 + 14x + 3 = 0$ ⓒ $3x^2 - 5x - 2 = 0$ ⓓ $3x^2 - 10x + 15 = 0$

Solution

Calculate the discriminant $b^2 - 4ac$ for each equation and state the expected type of solutions.

ⓐ
$x^2 + 4x + 4 = 0$

$b^2 - 4ac = (4)^2 - 4(1)(4) = 0$. There will be one rational double solution.

ⓑ
$8x^2 + 14x + 3 = 0$

$b^2 - 4ac = (14)^2 - 4(8)(3) = 100$. As 100 is a perfect square, there will be two rational solutions.

ⓒ
$3x^2 - 5x - 2 = 0$

$b^2 - 4ac = (-5)^2 - 4(3)(-2) = 49$. As 49 is a perfect square, there will be two rational solutions.

ⓓ
$3x^2 - 10x + 15 = 0$

$b^2 - 4ac = (-10)^2 - 4(3)(15) = -80$. There will be two complex solutions.

Using the Pythagorean Theorem

One of the most famous formulas in mathematics is the **Pythagorean Theorem**. It is based on a right triangle, and states the relationship among the lengths of the sides as $a^2 + b^2 = c^2$, where a and b refer to the legs of a right triangle adjacent to the $90°$ angle, and c refers to the hypotenuse. It has immeasurable uses in architecture, engineering, the sciences, geometry, trigonometry, and algebra, and in everyday applications.

We use the Pythagorean Theorem to solve for the length of one side of a triangle when we have the lengths of the other two. Because each of the terms is squared in the theorem, when we are solving for a side of a triangle, we have a quadratic equation. We can use the methods for solving quadratic equations that we learned in this section to solve for the missing side.

The Pythagorean Theorem is given as

$$a^2 + b^2 = c^2$$

where a and b refer to the legs of a right triangle adjacent to the $90°$ angle, and c refers to the hypotenuse, as shown in Figure 4.

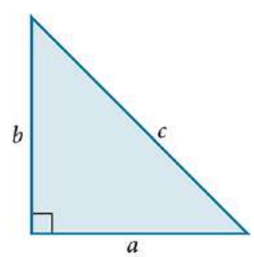


Figure 4

EXAMPLE 12

Finding the Length of the Missing Side of a Right Triangle

Find the length of the missing side of the right triangle in Figure 5.

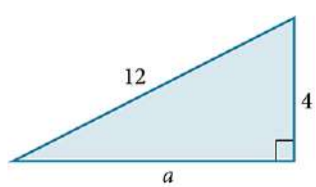


Figure 5

Solution

As we have measurements for side b and the hypotenuse, the missing side is a.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + (4)^2 &= (12)^2 \\ a^2 + 16 &= 144 \\ a^2 &= 128 \\ a &= \sqrt{128} \\ &= 8\sqrt{2} \end{aligned}$$

TRY IT #9 Use the Pythagorean Theorem to solve the right triangle problem: Leg a measures 4 units, leg b measures 3 units. Find the length of the hypotenuse.

MEDIA

Access these online resources for additional instruction and practice with quadratic equations.

Solving Quadratic Equations by Factoring (http://openstax.org/l/quadreqfactor)
The Zero-Product Property (http://openstax.org/l/zeroprodprop)
Completing the Square (http://openstax.org/l/complthesqr)
Quadratic Formula with Two Rational Solutions (http://openstax.org/l/quadrformrat)
Length of a leg of a right triangle (http://openstax.org/l/leglengthtri)

2.5 SECTION EXERCISES

Verbal

1. How do we recognize when an equation is quadratic?

2. When we solve a quadratic equation, how many solutions should we always start out seeking? Explain why when solving a quadratic equation in the form $ax^2 + bx + c = 0$ we may graph the equation $y = ax^2 + bx + c$ and have no zeroes (x-intercepts).

3. When we solve a quadratic equation by factoring, why do we move all terms to one side, having zero on the other side?

4. In the quadratic formula, what is the name of the expression under the radical sign $b^2 - 4ac$, and how does it determine the number of and nature of our solutions?

5. Describe two scenarios where using the square root property to solve a quadratic equation would be the most efficient method.

Algebraic

For the following exercises, solve the quadratic equation by factoring.

6. $x^2 + 4x - 21 = 0$

7. $x^2 - 9x + 18 = 0$

8. $2x^2 + 9x - 5 = 0$

9. $6x^2 + 17x + 5 = 0$

10. $4x^2 - 12x + 8 = 0$

11. $3x^2 - 75 = 0$

12. $8x^2 + 6x - 9 = 0$

13. $4x^2 = 9$

14. $2x^2 + 14x = 36$

15. $5x^2 = 5x + 30$

16. $4x^2 = 5x$

17. $7x^2 + 3x = 0$

18. $\frac{x}{3} - \frac{9}{x} = 2$

For the following exercises, solve the quadratic equation by using the square root property.

19. $x^2 = 36$

20. $x^2 = 49$

21. $(x-1)^2 = 25$

22. $(x-3)^2 = 7$

23. $(2x+1)^2 = 9$

24. $(x-5)^2 = 4$

For the following exercises, solve the quadratic equation by completing the square. Show each step.

25. $x^2 - 9x - 22 = 0$

26. $2x^2 - 8x - 5 = 0$

27. $x^2 - 6x = 13$

28. $x^2 + \frac{2}{3}x - \frac{1}{3} = 0$

29. $2 + z = 6z^2$

30. $6p^2 + 7p - 20 = 0$

31. $2x^2 - 3x - 1 = 0$

For the following exercises, determine the discriminant, and then state how many solutions there are and the nature of the solutions. Do not solve.

32. $2x^2 - 6x + 7 = 0$

33. $x^2 + 4x + 7 = 0$

34. $3x^2 + 5x - 8 = 0$

35. $9x^2 - 30x + 25 = 0$

36. $2x^2 - 3x - 7 = 0$

37. $6x^2 - x - 2 = 0$

For the following exercises, solve the quadratic equation by using the quadratic formula. If the solutions are not real, state No Real Solution.

38. $2x^2 + 5x + 3 = 0$

39. $x^2 + x = 4$

40. $2x^2 - 8x - 5 = 0$

41. $3x^2 - 5x + 1 = 0$

42. $x^2 + 4x + 2 = 0$

43. $4 + \frac{1}{x} - \frac{1}{x^2} = 0$

Technology

*For the following exercises, enter the expressions into your graphing utility and find the zeroes to the equation (the x-intercepts) by using **2nd CALC 2:zero**. Recall finding zeroes will ask left bound (move your cursor to the left of the zero,enter), then right bound (move your cursor to the right of the zero,enter), then guess (move your cursor between the bounds near the zero, enter). Round your answers to the nearest thousandth.*

44. $\mathrm{Y}_1 = 4x^2 + 3x - 2$

45. $\mathrm{Y}_1 = -3x^2 + 8x - 1$

46. $\mathrm{Y}_1 = 0.5x^2 + x - 7$

47. To solve the quadratic equation $x^2 + 5x - 7 = 4$, we can graph these two equations

$$\begin{aligned} Y_1 &= x^2 + 5x - 7 \\ Y_2 &= 4 \end{aligned}$$

and find the points of intersection. Recall 2nd CALC 5:intersection. Do this and find the solutions to the nearest tenth.

48. To solve the quadratic equation $0.3x^2 + 2x - 4 = 2$, we can graph these two equations

$$\begin{aligned} Y_1 &= 0.3x^2 + 2x - 4 \\ Y_2 &= 2 \end{aligned}$$

and find the points of intersection. Recall 2nd CALC 5:intersection. Do this and find the solutions to the nearest tenth.

Extensions

49. Beginning with the general form of a quadratic equation, $ax^2 + bx + c = 0$, solve for *x* by using the completing the square method, thus deriving the quadratic formula.

50. Show that the sum of the two solutions to the quadratic equation is $-\frac{b}{a}$.

51. A person has a garden that has a length 10 feet longer than the width. Set up a quadratic equation to find the dimensions of the garden if its area is 119 ft.^{2}. Solve the quadratic equation to find the length and width.

52. Abercrombie and Fitch stock had a price given as $P = 0.2t^2 - 5.6t + 50.2$, where t is the time in months from 1999 to 2001. ($t = 1$ is January 1999). Find the two months in which the price of the stock was $30.

53. Suppose that an equation is given $p = -2x^2 + 280x - 1000$, where x represents the number of items sold at an auction and p is the profit made by the business that ran the auction. How many items sold would make this profit a maximum? Solve this by graphing the expression in your graphing utility and finding the maximum using 2nd CALC maximum. To obtain a good window for the curve, set x [0,200] and y [0,10000].

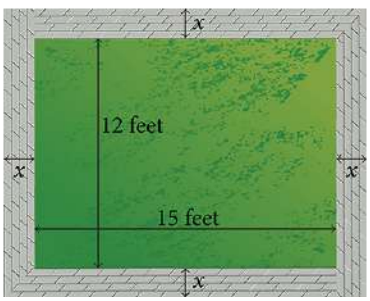
Real-World Applications

54. A formula for the normal systolic blood pressure for a man age A, measured in mmHg, is given as $P = 0.006A^2 - 0.02A + 120.$ Find the age to the nearest year of a man whose normal blood pressure measures 125 mmHg.

55. The cost function for a certain company is $C = 60x + 300$ and the revenue is given by $R = 100x - 0.5x^2$. Recall that profit is revenue minus cost. Set up a quadratic equation and find two values of x (production level) that will create a profit of \$300.

56. A falling object travels a distance given by the formula $d = 5t + 16t^2$ ft, where t is measured in seconds. How long will it take for the object to travel 74 ft?

57. A vacant lot is being converted into a community garden. The garden and the walkway around its perimeter have an area of 378 ft^2. Find the width of the walkway if the garden is 12 ft. wide by 15 ft. long.



58. An epidemiological study of the spread of a certain influenza strain that hit a small school population found that the total number of students, P, who contracted the flu t days after it broke out is given by the model $P = -t^2 + 13t + 130,$ where $1 \leq t \leq 6$. Find the day that 160 students had the flu. Recall that the restriction on t is at most 6.

2.6 Other Types of Equations

Learning Objectives

In this section, you will:

- Solve equations involving rational exponents.
- Solve equations using factoring.
- Solve radical equations.
- Solve absolute value equations.
- Solve other types of equations.

We have solved linear equations, rational equations, and quadratic equations using several methods. However, there are many other types of equations, and we will investigate a few more types in this section. We will look at equations involving rational exponents, polynomial equations, radical equations, absolute value equations, equations in quadratic form, and some rational equations that can be transformed into quadratics. Solving any equation, however, employs the same basic algebraic rules. We will learn some new techniques as they apply to certain equations, but the algebra never changes.

Solving Equations Involving Rational Exponents

Rational exponents are exponents that are fractions, where the numerator is a power and the denominator is a root. For example, $16^{\frac{1}{2}}$ is another way of writing $\sqrt{16}$; $8^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{8}$. The ability to work with rational exponents is a useful skill, as it is highly applicable in calculus.

We can solve equations in which a variable is raised to a rational exponent by raising both sides of the equation to the reciprocal of the exponent. The reason we raise the equation to the reciprocal of the exponent is because we want to eliminate the exponent on the variable term, and a number multiplied by its reciprocal equals 1. For example, $\frac{2}{3}\left(\frac{3}{2}\right) = 1,$

$3\left(\frac{1}{3}\right) = 1$, and so on.

Rational Exponents

A rational exponent indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

EXAMPLE 1

Evaluating a Number Raised to a Rational Exponent

Evaluate $8^{\frac{2}{3}}$.

Solution

Whether we take the root first or the power first depends on the number. It is easy to find the cube root of 8, so rewrite $8^{\frac{2}{3}}$ as $\left(8^{\frac{1}{3}}\right)^2$.

$$\begin{aligned}\left(8^{\frac{1}{3}}\right)^2 &= (2)^2 \\ &= 4\end{aligned}$$

TRY IT #1 Evaluate $64^{-\frac{1}{3}}$.

EXAMPLE 2

Solve the Equation Including a Variable Raised to a Rational Exponent

Solve the equation in which a variable is raised to a rational exponent: $x^{\frac{5}{4}} = 32$.

Solution

The way to remove the exponent on x is by raising both sides of the equation to a power that is the reciprocal of $\frac{5}{4}$, which is $\frac{4}{5}$.

$$\begin{aligned}x^{\frac{5}{4}} &= 32 \\ \left(x^{\frac{5}{4}}\right)^{\frac{4}{5}} &= (32)^{\frac{4}{5}} \\ x &= (2)^4 && \text{The fifth root of 32 is 2.} \\ &= 16\end{aligned}$$

TRY IT #2 Solve the equation $x^{\frac{3}{2}} = 125$.

EXAMPLE 3

Solving an Equation Involving Rational Exponents and Factoring

Solve $3x^{\frac{3}{4}} = x^{\frac{1}{2}}$.

Solution

This equation involves rational exponents as well as factoring rational exponents. Let us take this one step at a time. First, put the variable terms on one side of the equal sign and set the equation equal to zero.

$$\begin{aligned} 3x^{\frac{3}{4}} - \left(x^{\frac{1}{2}}\right) &= x^{\frac{1}{2}} - \left(x^{\frac{1}{2}}\right) \\ 3x^{\frac{3}{4}} - x^{\frac{1}{2}} &= 0 \end{aligned}$$

Now, it looks like we should factor the left side, but what do we factor out? We can always factor the term with the lowest exponent. Rewrite $x^{\frac{1}{2}}$ as $x^{\frac{2}{4}}$. Then, factor out $x^{\frac{2}{4}}$ from both terms on the left.

$$\begin{aligned} 3x^{\frac{3}{4}} - x^{\frac{2}{4}} &= 0 \\ x^{\frac{2}{4}}\left(3x^{\frac{1}{4}} - 1\right) &= 0 \end{aligned}$$

Where did $x^{\frac{1}{4}}$ come from? Remember, when we multiply two numbers with the same base, we add the exponents. Therefore, if we multiply $x^{\frac{2}{4}}$ back in using the distributive property, we get the expression we had before the factoring, which is what should happen. We need an exponent such that when added to $\frac{2}{4}$ equals $\frac{3}{4}$. Thus, the exponent on x in the parentheses is $\frac{1}{4}$.

Let us continue. Now we have two factors and can use the zero factor theorem.

$$\begin{aligned} x^{\frac{2}{4}}\left(3x^{\frac{1}{4}} - 1\right) &= 0 \\ x^{\frac{2}{4}} &= 0 \\ x &= 0 \\ 3x^{\frac{1}{4}} - 1 &= 0 \\ 3x^{\frac{1}{4}} &= 1 \\ x^{\frac{1}{4}} &= \frac{1}{3} && \text{Divide both sides by 3.} \\ \left(x^{\frac{1}{4}}\right)^4 &= \left(\frac{1}{3}\right)^4 && \text{Raise both sides to the reciprocal of } \frac{1}{4}. \\ x &= \frac{1}{81} \end{aligned}$$

The two solutions are 0 and $\frac{1}{81}$.

TRY IT #3 Solve: $(x+5)^{\frac{3}{2}} = 8.$

Solving Equations Using Factoring

We have used factoring to solve quadratic equations, but it is a technique that we can use with many types of polynomial equations, which are equations that contain a string of terms including numerical coefficients and variables. When we are faced with an equation containing polynomials of degree higher than 2, we can often solve them by factoring.

Polynomial Equations

A polynomial of degree *n* is an expression of the type

$$a_n x^n + a_{n-1} x^{n-1} + \quad \cdot \quad \cdot \quad \cdot \quad + a_2 x^2 + a_1 x + a_0$$

where *n* is a positive integer and $a_n, \ldots, a_0$ are real numbers and $a_n \neq 0$.

Setting the polynomial equal to zero gives a **polynomial equation**. The total number of solutions (real and complex) to a polynomial equation is equal to the highest exponent *n*.

EXAMPLE 4

Solving a Polynomial by Factoring

Solve the polynomial by factoring: $5x^4 = 80x^2$.

Solution

First, set the equation equal to zero. Then factor out what is common to both terms, the GCF.

$$\begin{aligned} 5x^4 - 80x^2 &= 0 \\ 5x^2\left(x^2 - 16\right) &= 0 \end{aligned}$$

Notice that we have the difference of squares in the factor $x^2 - 16$, which we will continue to factor and obtain two solutions. The first term, $5x^2$, generates, technically, two solutions as the exponent is 2, but they are the same solution.

$$\begin{aligned} 5x^2 &= 0 \\ x &= 0 \\ x^2 - 16 &= 0 \\ (x-4)(x+4) &= 0 \\ x &= 4 \\ x &= -4 \end{aligned}$$

The solutions are $0 \ \text{(double solution)}, 4,$ and -4.

Analysis

We can see the solutions on the graph in Figure 1. The *x*-coordinates of the points where the graph crosses the *x*-axis are the solutions—the *x*-intercepts. Notice on the graph that at the solution 0, the graph touches the *x*-axis and bounces back. It does not cross the *x*-axis. This is typical of double solutions.

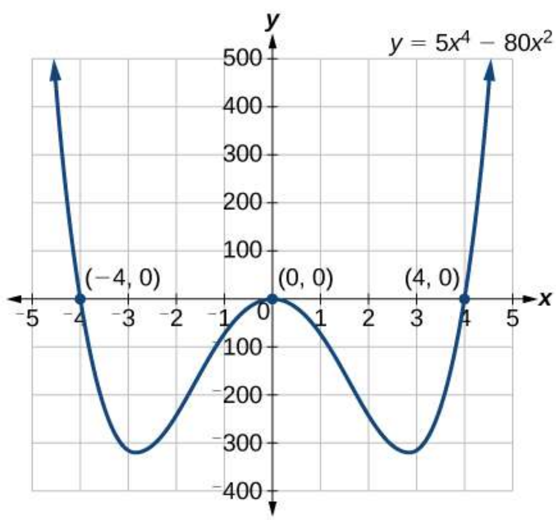


Figure 1

TRY IT #4 Solve by factoring: $12x^4 = 3x^2$.

EXAMPLE 5

Solve a Polynomial by Grouping

Solve a polynomial by grouping: $x^3 + x^2 - 9x - 9 = 0$.

Solution

This polynomial consists of 4 terms, which we can solve by grouping. Grouping procedures require factoring the first two terms and then factoring the last two terms. If the factors in the parentheses are identical, we can continue the process and solve, unless more factoring is suggested.

$$\begin{aligned} x^3 + x^2 - 9x - 9 &= 0 \\ x^2(x+1) - 9(x+1) &= 0 \\ \left(x^2 - 9\right)(x+1) &= 0 \end{aligned}$$

The grouping process ends here, as we can factor $x^2 - 9$ using the difference of squares formula.

$$\begin{aligned} \left(x^2 - 9\right)(x+1) &= 0 \\ (x-3)(x+3)(x+1) &= 0 \\ x &= 3 \\ x &= -3 \\ x &= -1 \end{aligned}$$

The solutions are 3, -3, and -1. Note that the highest exponent is 3 and we obtained 3 solutions. We can see the solutions, the *x*-intercepts, on the graph in Figure 2.

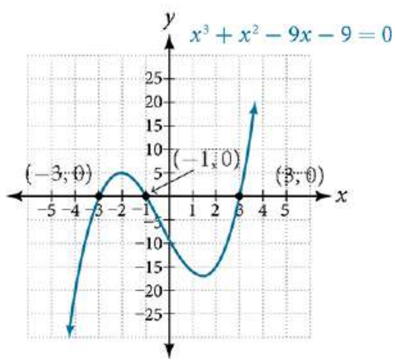


Figure 2

Analysis

We looked at solving quadratic equations by factoring when the leading coefficient is 1. When the leading coefficient is not 1, we solved by grouping. Grouping requires four terms, which we obtained by splitting the linear term of quadratic equations. We can also use grouping for some polynomials of degree higher than 2, as we saw here, since there were already four terms.

Solving Radical Equations

Radical equations are equations that contain variables in the radicand (the expression under a radical symbol), such as

$$\begin{aligned} \sqrt{3x+18} &= x \\ \sqrt{x+3} &= x-3 \\ \sqrt{x+5} - \sqrt{x-3} &= 2 \end{aligned}$$

Radical equations may have one or more radical terms, and are solved by eliminating each radical, one at a time. We have to be careful when solving radical equations, as it is not unusual to find **extraneous solutions**, roots that are not, in fact, solutions to the equation. These solutions are not due to a mistake in the solving method, but result from the process of raising both sides of an equation to a power. However, checking each answer in the original equation will confirm the true solutions.

Radical Equations

An equation containing terms with a variable in the radicand is called a **radical equation**.

HOW TO

Given a radical equation, solve it.

1. Isolate the radical expression on one side of the equal sign. Put all remaining terms on the other side.
2. If the radical is a square root, then square both sides of the equation. If it is a cube root, then raise both sides of the equation to the third power. In other words, for an *n*th root radical, raise both sides to the *n*th power. Doing so eliminates the radical symbol.
3. Solve the remaining equation.
4. If a radical term still remains, repeat steps 1–2.
5. Confirm solutions by substituting them into the original equation.

EXAMPLE 6

Solving an Equation with One Radical

Solve $\sqrt{15-2x}=x$.

Solution

The radical is already isolated on the left side of the equal side, so proceed to square both sides.

$$\begin{aligned}\sqrt{15-2x} &= x\\ \left(\sqrt{15-2x}\right)^2 &= (x)^2\\ 15-2x &= x^2\end{aligned}$$

We see that the remaining equation is a quadratic. Set it equal to zero and solve.

$$\begin{aligned}0 &= x^2+2x-15\\ &= (x+5)(x-3)\\ x &= -5\\ x &= 3\end{aligned}$$

The proposed solutions are -5 and 3. Let us check each solution back in the original equation. First, check $x=-5$.

$$\begin{aligned}\sqrt{15-2x} &= x\\ \sqrt{15-2(-5)} &= -5\\ \sqrt{25} &= -5\\ 5 &\neq -5\end{aligned}$$

This is an extraneous solution. While no mistake was made solving the equation, we found a solution that does not satisfy the original equation.

Check $x=3$.

$$\begin{aligned}\sqrt{15-2x} &= x\\ \sqrt{15-2(3)} &= 3\\ \sqrt{9} &= 3\\ 3 &= 3\end{aligned}$$

The solution is 3.

TRY IT #5 Solve the radical equation: $\sqrt{x+3} = 3x - 1$

EXAMPLE 7

Solving a Radical Equation Containing Two Radicals

Solve $\sqrt{2x+3} + \sqrt{x-2} = 4$.

Solution

As this equation contains two radicals, we isolate one radical, eliminate it, and then isolate the second radical.

$$\begin{aligned} \sqrt{2x+3} + \sqrt{x-2} &= 4 \\ \sqrt{2x+3} &= 4 - \sqrt{x-2} && \text{Subtract } \sqrt{x-2} \text{ from both sides.} \\ \left(\sqrt{2x+3}\right)^2 &= \left(4 - \sqrt{x-2}\right)^2 && \text{Square both sides.} \end{aligned}$$

Use the perfect square formula to expand the right side: $(a-b)^2 = a^2 - 2ab + b^2$.

$$\begin{aligned} 2x+3 &= (4)^2 - 2(4)\sqrt{x-2} + \left(\sqrt{x-2}\right)^2 \\ 2x+3 &= 16 - 8\sqrt{x-2} + (x-2) \\ 2x+3 &= 14 + x - 8\sqrt{x-2} && \text{Combine like terms.} \\ x - 11 &= -8\sqrt{x-2} && \text{Isolate the second radical.} \\ (x-11)^2 &= \left(-8\sqrt{x-2}\right)^2 && \text{Square both sides.} \\ x^2 - 22x + 121 &= 64(x-2) \end{aligned}$$

Now that both radicals have been eliminated, set the quadratic equal to zero and solve.

$$\begin{aligned} x^2 - 22x + 121 &= 64x - 128 \\ x^2 - 86x + 249 &= 0 \\ (x-3)(x-83) &= 0 && \text{Factor and solve.} \\ x &= 3 \\ x &= 83 \end{aligned}$$

The proposed solutions are 3 and 83. Check each solution in the original equation.

$$\begin{aligned} \sqrt{2x+3} + \sqrt{x-2} &= 4 \\ \sqrt{2x+3} &= 4 - \sqrt{x-2} \\ \sqrt{2(3)+3} &= 4 - \sqrt{(3)-2} \\ \sqrt{9} &= 4 - \sqrt{1} \\ 3 &= 3 \end{aligned}$$

One solution is 3.

Check $x = 83$.

$$\begin{aligned} \sqrt{2x+3} + \sqrt{x-2} &= 4 \\ \sqrt{2x+3} &= 4 - \sqrt{x-2} \\ \sqrt{2(83)+3} &= 4 - \sqrt{(83-2)} \\ \sqrt{169} &= 4 - \sqrt{81} \\ 13 &\neq -5 \end{aligned}$$

The only solution is 3. We see that 83 is an extraneous solution.

TRY IT #6 Solve the equation with two radicals: $\sqrt{3x+7} + \sqrt{x+2} = 1$.

Solving an Absolute Value Equation

Next, we will learn how to solve an absolute value equation. To solve an equation such as $|2x - 6| = 8$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value bars is 8 or -8. This leads to two different equations we can solve independently.

$$\begin{aligned} 2x - 6 &= 8 \\ 2x &= 14 \\ x &= 7 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2x - 6 &= -8 \\ 2x &= -2 \\ x &= -1 \end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

Absolute Value Equations

The absolute value of x is written as $|x|$. It has the following properties:

$$\begin{aligned} &\text{If } x \geq 0, \quad \text{then } |x| = x. \\ &\text{If } x < 0, \quad \text{then } |x| = -x. \end{aligned}$$

For real numbers A and B, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

An **absolute value equation** in the form $|ax + b| = c$ has the following properties:

$$\begin{aligned} &\text{If } \ c < 0, |ax + b| = c \ \text{ has no solution.} \\ &\text{If } \ c = 0, |ax + b| = c \ \text{ has one solution.} \\ &\text{If } \ c > 0, |ax + b| = c \ \text{ has two solutions.} \end{aligned}$$

HOW TO

Given an absolute value equation, solve it.

1. Isolate the absolute value expression on one side of the equal sign.
2. If $c > 0$, write and solve two equations: $ax + b = c$ and $ax + b = -c$.

EXAMPLE 8

Solving Absolute Value Equations

Solve the following absolute value equations:

ⓐ $|6x + 4| = 8$ ⓑ $|3x + 4| = -9$ ⓒ $|3x - 5| - 4 = 6$ ⓓ $|-5x + 10| = 0$

Solution

ⓐ

$|6x + 4| = 8$

Write two equations and solve each:

$$\begin{aligned} 6x + 4 &= 8 \\ 6x &= 4 \\ x &= \tfrac{2}{3} \end{aligned} \qquad\qquad \begin{aligned} 6x + 4 &= -8 \\ 6x &= -12 \\ x &= -2 \end{aligned}$$

The two solutions are $\frac{2}{3}$ and -2.

ⓑ

$|3x + 4| = -9$

There is no solution as an absolute value cannot be negative.

ⓒ
$|3x-5|-4=6$

Isolate the absolute value expression and then write two equations.

$$\begin{aligned} |3x-5|-4 &= 6 \\ |3x-5| &= 10 \end{aligned}$$

$$\begin{aligned} 3x-5 &= 10 \\ 3x &= 15 \\ x &= 5 \end{aligned} \qquad \begin{aligned} 3x-5 &= -10 \\ 3x &= -5 \\ x &= -\frac{5}{3} \end{aligned}$$

There are two solutions: 5, and $-\frac{5}{3}$.

ⓓ
(d) $|-5x+10|=0$

The equation is set equal to zero, so we have to write only one equation.

$$\begin{aligned} -5x+10 &= 0 \\ -5x &= -10 \\ x &= 2 \end{aligned}$$

There is one solution: 2.

TRY IT #7 Solve the absolute value equation: $|1-4x|+8=13$.

Solving Other Types of Equations

There are many other types of equations in addition to the ones we have discussed so far. We will see more of them throughout the text. Here, we will discuss equations that are in quadratic form, and rational equations that result in a quadratic.

Solving Equations in Quadratic Form

Equations in quadratic form are equations with three terms. The first term has a power other than 2. The middle term has an exponent that is one-half the exponent of the leading term. The third term is a constant. We can solve equations in this form as if they were quadratic. A few examples of these equations include $x^4-5x^2+4=0$, $x^6+7x^3-8=0$, and $x^{\frac{2}{3}}+4x^{\frac{1}{3}}+2=0$. In each one, doubling the exponent of the middle term equals the exponent on the leading term. We can solve these equations by substituting a variable for the middle term.

Quadratic Form

If the exponent on the middle term is one-half of the exponent on the leading term, we have an **equation in quadratic form**, which we can solve as if it were a quadratic. We substitute a variable for the middle term to solve equations in quadratic form.

HOW TO

Given an equation quadratic in form, solve it.

1. Identify the exponent on the leading term and determine whether it is double the exponent on the middle term.
2. If it is, substitute a variable, such as *u*, for the variable portion of the middle term.
3. Rewrite the equation so that it takes on the standard form of a quadratic.
4. Solve using one of the usual methods for solving a quadratic.
5. Replace the substitution variable with the original term.

6. Solve the remaining equation.

EXAMPLE 9

Solving a Fourth-degree Equation in Quadratic Form

Solve this fourth-degree equation: $3x^4 - 2x^2 - 1 = 0$.

Solution

This equation fits the main criteria, that the power on the leading term is double the power on the middle term. Next, we will make a substitution for the variable term in the middle. Let $u = x^2$. Rewrite the equation in *u*.

$$3u^2 - 2u - 1 = 0$$

Now solve the quadratic.

$$\begin{aligned} 3u^2 - 2u - 1 &= 0 \\ (3u + 1)(u - 1) &= 0 \end{aligned}$$

Solve each factor and replace the original term for *u.*

$$\begin{aligned} 3u + 1 &= 0 \\ 3u &= -1 \\ u &= -\tfrac{1}{3} \\ x^2 &= -\tfrac{1}{3} \\ x &= \pm i\sqrt{\tfrac{1}{3}} \end{aligned}$$

$$\begin{aligned} u - 1 &= 0 \\ u &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

The solutions are $\pm i\sqrt{\frac{1}{3}}$ and ± 1.

TRY IT #8 Solve using substitution: $x^4 - 8x^2 - 9 = 0$.

EXAMPLE 10

Solving an Equation in Quadratic Form Containing a Binomial

Solve the equation in quadratic form: $(x + 2)^2 + 11(x + 2) - 12 = 0$.

Solution

This equation contains a binomial in place of the single variable. The tendency is to expand what is presented. However, recognizing that it fits the criteria for being in quadratic form makes all the difference in the solving process. First, make a substitution, letting $u = x + 2$. Then rewrite the equation in *u.*

$$\begin{aligned} u^2 + 11u - 12 &= 0 \\ (u + 12)(u - 1) &= 0 \end{aligned}$$

Solve using the zero-factor property and then replace *u* with the original expression.

$$\begin{aligned} u + 12 &= 0 \\ u &= -12 \\ x + 2 &= -12 \\ x &= -14 \end{aligned}$$

The second factor results in

$$\begin{aligned} u-1 &= 0 \\ u &= 1 \\ x+2 &= 1 \\ x &= -1 \end{aligned}$$

We have two solutions: -14, and -1.

TRY IT #9 Solve: $(x-5)^2 - 4(x-5) - 21 = 0$.

Solving Rational Equations Resulting in a Quadratic

Earlier, we solved rational equations. Sometimes, solving a rational equation results in a quadratic. When this happens, we continue the solution by simplifying the quadratic equation by one of the methods we have seen. It may turn out that there is no solution.

EXAMPLE 11

Solving a Rational Equation Leading to a Quadratic

Solve the following rational equation: $\frac{-4x}{x-1} + \frac{4}{x+1} = \frac{-8}{x^2-1}$.

Solution

We want all denominators in factored form to find the LCD. Two of the denominators cannot be factored further. However, $x^2-1 = (x+1)(x-1)$. Then, the LCD is $(x+1)(x-1)$. Next, we multiply the whole equation by the LCD.

$$\begin{aligned} (x+1)(x-1)\left[\frac{-4x}{x-1} + \frac{4}{x+1}\right] &= \left[\frac{-8}{(x+1)(x-1)}\right](x+1)(x-1) \\ -4x(x+1) + 4(x-1) &= -8 \\ -4x^2 - 4x + 4x - 4 &= -8 \\ -4x^2 + 4 &= 0 \\ -4\left(x^2-1\right) &= 0 \\ -4(x+1)(x-1) &= 0 \\ x &= -1 \\ x &= 1 \end{aligned}$$

In this case, either solution produces a zero in the denominator in the original equation. Thus, there is no solution.

TRY IT #10 Solve $\frac{3x+2}{x-2} + \frac{1}{x} = \frac{-2}{x^2-2x}$.

MEDIA

Access these online resources for additional instruction and practice with different types of equations.

[Rational Equation with no Solution (http://openstax.org/l/rateqnosoln)](http://openstax.org/l/rateqnosoln)
[Solving equations with rational exponents using reciprocal powers (http://openstax.org/l/ratexprecpexp)](http://openstax.org/l/ratexprecpexp)
[Solving radical equations part 1 of 2 (http://openstax.org/l/radeqsolvepart1)](http://openstax.org/l/radeqsolvepart1)
[Solving radical equations part 2 of 2 (http://openstax.org/l/radeqsolvepart2)](http://openstax.org/l/radeqsolvepart2)

2.6 SECTION EXERCISES

Verbal

1. In a radical equation, what does it mean if a number is an extraneous solution?

2. Explain why possible solutions *must* be checked in radical equations.

3. Your friend tries to calculate the value $-9^{\frac{3}{2}}$ and keeps getting an ERROR message. What mistake are they probably making?

4. Explain why $|2x+5| = -7$ has no solutions.

5. Explain how to change a rational exponent into the correct radical expression.

Algebraic

For the following exercises, solve the rational exponent equation. Use factoring where necessary.

6. $x^{\frac{2}{3}} = 16$

7. $x^{\frac{3}{4}} = 27$

8. $2x^{\frac{1}{2}} - x^{\frac{1}{4}} = 0$

9. $(x-1)^{\frac{3}{4}} = 8$

10. $(x+1)^{\frac{2}{3}} = 4$

11. $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$

12. $x^{\frac{7}{3}} - 3x^{\frac{4}{3}} - 4x^{\frac{1}{3}} = 0$

For the following exercises, solve the following polynomial equations by grouping and factoring.

13. $x^3 + 2x^2 - x - 2 = 0$

14. $3x^3 - 6x^2 - 27x + 54 = 0$

15. $4y^3 - 9y = 0$

16. $x^3 + 3x^2 - 25x - 75 = 0$

17. $m^3 + m^2 - m - 1 = 0$

18. $2x^5 - 14x^3 = 0$

19. $5x^3 + 45x = 2x^2 + 18$

For the following exercises, solve the radical equation. Be sure to check all solutions to eliminate extraneous solutions.

20. $\sqrt{3x-1} - 2 = 0$

21. $\sqrt{x-7} = 5$

22. $\sqrt{x-1} = x - 7$

23. $\sqrt{3t+5} = 7$

24. $\sqrt{t+1} + 9 = 7$

25. $\sqrt{12-x} = x$

26. $\sqrt{2x+3} - \sqrt{x+2} = 2$

27. $\sqrt{3x+7} + \sqrt{x+2} = 1$

28. $\sqrt{2x+3} - \sqrt{x+1} = 1$

For the following exercises, solve the equation involving absolute value.

29. $|3x-4| = 8$

30. $|2x-3| = -2$

31. $|1-4x| - 1 = 5$

32. $|4x+1| - 3 = 6$

33. $|2x-1| - 7 = -2$

34. $|2x+1| - 2 = -3$

35. $|x+5| = 0$

36. $-|2x+1| = -3$

For the following exercises, solve the equation by identifying the quadratic form. Use a substitute variable and find all real solutions by factoring.

37. $x^4 - 10x^2 + 9 = 0$

38. $4(t-1)^2 - 9(t-1) = -2$

39. $\left(x^2-1\right)^2 + \left(x^2-1\right) - 12 = 0$

40. $(x+1)^2 - 8(x+1) - 9 = 0$

41. $(x-3)^2 - 4 = 0$

Extensions

For the following exercises, solve for the unknown variable.

42. $x^{-2} - x^{-1} - 12 = 0$

43. $\sqrt{|x|^2} = x$

44. $t^{10} - 2t^5 + 1 = 0$

45. $\left|x^2 + 2x - 36\right| = 12$

Real-World Applications

For the following exercises, use the model for the period of a pendulum, T, such that $T = 2\pi\sqrt{\frac{L}{g}}$, where the length of the pendulum is L and the acceleration due to gravity is g.

46. If the acceleration due to gravity is 9.8 m/s^2 and the period equals 1 s, find the length to the nearest cm (100 cm = 1 m).

47. If the gravity is 32 ft/s^2 and the period equals 1 s, find the length to the nearest in. (12 in. = 1 ft). Round your answer to the nearest in.

For the following exercises, use a model for body surface area, BSA, such that $BSA = \sqrt{\frac{wh}{3600}}$, where w = weight in kg and h = height in cm.

48. Find the height of a 72-kg female to the nearest cm whose $BSA = 1.8$.

49. Find the weight of a 177-cm male to the nearest kg whose $BSA = 2.1$.

2.7 Linear Inequalities and Absolute Value Inequalities

Learning Objectives

In this section, you will:

- Use interval notation
- Use properties of inequalities.
- Solve inequalities in one variable algebraically.
- Solve absolute value inequalities.

Figure 1

It is not easy to make the honor roll at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.

Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

We can use a number line as shown in Figure 2**.** The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.

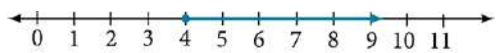


Figure 2

We can use set-builder notation: $\{x | x \geq 4\}$, which translates to "all real numbers *x* such that *x* is greater than or equal to 4." Notice that braces are used to indicate a set.

The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $\left[4, \infty\right)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-2, 6)$, or all numbers between -2 and 6, including -2, but not including 6; $(-1, 0)$, all real numbers between, but not including -1 and 0; and $\left(-\infty, 1\right]$, all real numbers less than and including 1. Table 1 outlines the possibilities.

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between *a* and *b*, but not including *a* or *b*	$\{x \mid a < x < b\}$	(a, b)
All real numbers greater than *a*, but not including *a*	$\{x \mid x > a\}$	$\left(a, \infty\right)$
All real numbers less than *b*, but not including *b*	$\{x \mid x < b\}$	$\left(-\infty, b\right)$
All real numbers greater than *a*, including *a*	$\{x \mid x \geq a\}$	$\left[a, \infty\right)$
All real numbers less than *b*, including *b*	$\{x \mid x \leq b\}$	$\left(-\infty, b\right]$
All real numbers between *a* and *b*, including *a*	$\{x \mid a \leq x < b\}$	$[a, b)$
All real numbers between *a* and *b*, including *b*	$\{x \mid a < x \leq b\}$	$(a, b]$

Table 1

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between *a* and *b*, including *a* and *b*	$\{x \mid a \leq x \leq b\}$	$[a, b]$
All real numbers less than *a* or greater than *b*	$\{x \mid x < a \text{ or } x > b\}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{x \mid x \text{ is all real numbers}\}$	$(-\infty, \infty)$

Table 1

EXAMPLE 1

Using Interval Notation to Express All Real Numbers Greater Than or Equal to *a*

Use interval notation to indicate all real numbers greater than or equal to -2.

Solution

Use a bracket on the left of -2 and parentheses after infinity: $[-2, \infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

TRY IT #1 Use interval notation to indicate all real numbers between and including -3 and 5.

EXAMPLE 2

Using Interval Notation to Express All Real Numbers Less Than or Equal to *a* or Greater Than or Equal to *b*

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1.

Solution

We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1. So, this interval begins at $-\infty$ and ends at -1, which is written as $(-\infty, -1]$.

The second interval must show all real numbers greater than or equal to 1, which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, ∪, between the two intervals.

$$(-\infty, -1] \cup [1, \infty)$$

TRY IT #2 Express all real numbers less than -2 or greater than or equal to 3 in interval notation.

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the addition property and the multiplication property to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

Properties of Inequalities

$$\textbf{Addition Property} \quad \text{If } a < b, \text{ then } a + c < b + c.$$

$$\begin{aligned} &\textbf{Multiplication Property} \quad &&\text{If } a < b \text{ and } c > 0, \text{ then } ac < bc. \\ & &&\text{If } a < b \text{ and } c < 0, \text{ then } ac > bc. \end{aligned}$$

These properties also apply to $a \leq b$, $a > b$, and $a \geq b$.

EXAMPLE 3

Demonstrating the Addition Property

Illustrate the addition property for inequalities by solving each of the following:

1. ⓐ $x - 15 < 4$
2. ⓑ $6 \geq x - 1$
3. ⓒ $x + 7 > 9$

Solution

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

ⓐ

$$\begin{array}{rl} x - 15 < 4 & \\ x - 15 + 15 < 4 + 15 & \text{Add 15 to both sides.} \\ x < 19 & \end{array}$$

ⓑ

$$\begin{array}{rl} 6 \geq x - 1 & \\ 6 + 1 \geq x - 1 + 1 & \text{Add 1 to both sides.} \\ 7 \geq x & \end{array}$$

ⓒ

$$\begin{array}{rl} x + 7 > 9 & \\ x + 7 - 7 > 9 - 7 & \text{Subtract 7 from both sides.} \\ x > 2 & \end{array}$$

TRY IT #3 Solve: $3x - 2 < 1$.

EXAMPLE 4

Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

1. ⓐ $3x < 6$
2. ⓑ $-2x - 1 \geq 5$
3. ⓒ $5 - x > 10$

Solution

ⓐ

$$\begin{array}{c} 3x < 6 \\ \frac{1}{3}(3x) < (6)\frac{1}{3} \\ x < 2 \end{array}$$

ⓑ

$$\begin{array}{rl} -2x - 1 \geq 5 & \\ -2x \geq 6 & \\ \left(-\frac{1}{2}\right)(-2x) \geq (6)\left(-\frac{1}{2}\right) & \text{Multiply by } -\frac{1}{2}. \\ x \leq -3 & \text{Reverse the inequality.} \end{array}$$

ⓒ

$$\begin{array}{rl} 5 - x > 10 & \\ -x > 5 & \\ (-1)(-x) > (5)(-1) & \text{Multiply by } -1. \\ x < -5 & \text{Reverse the inequality.} \end{array}$$

TRY IT #4 Solve: $4x + 7 \geq 2x - 3$.

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

EXAMPLE 5

Solving an Inequality Algebraically

Solve the inequality: $13 - 7x \geq 10x - 4$.

Solution

Solving this inequality is similar to solving an equation up until the last step.

$$\begin{array}{rl} 13 - 7x \geq 10x - 4 & \\ 13 - 17x \geq -4 & \text{Move variable terms to one side of the inequality.} \\ -17x \geq -17 & \text{Isolate the variable term.} \\ x \leq 1 & \text{Dividing both sides by} -17 \quad \text{reverses the inequality.} \end{array}$$

The solution set is given by the interval $\left(-\infty, 1\right]$, or all real numbers less than and including 1.

TRY IT #5 Solve the inequality and write the answer using interval notation: $-x + 4 < \frac{1}{2}x + 1$.

EXAMPLE 6

Solving an Inequality with Fractions

Solve the following inequality and write the answer in interval notation: $-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$.

Solution

We begin solving in the same way we do when solving an equation.

$$\begin{array}{rl} -\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x & \\ -\frac{3}{4}x - \frac{2}{3}x \geq -\frac{5}{8} & \text{Put variable terms on one side.} \\ -\frac{9}{12}x - \frac{8}{12}x \geq -\frac{5}{8} & \text{Write fractions with common denominator.} \\ -\frac{17}{12}x \geq -\frac{5}{8} & \\ x \leq -\frac{5}{8}\left(-\frac{12}{17}\right) & \text{Multiplying by a negative number reverses the inequality.} \\ x \leq \frac{15}{34} & \end{array}$$

The solution set is the interval $\left(-\infty, \frac{15}{34}\right]$.

TRY IT #6 Solve the inequality and write the answer in interval notation: $-\frac{5}{6}x \leq \frac{3}{4} + \frac{8}{3}x$.

Understanding Compound Inequalities

A **compound inequality** includes two inequalities in one statement. A statement such as $4 < x \leq 6$ means $4 < x$ and $x \leq 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

EXAMPLE 7

Solving a Compound Inequality

Solve the compound inequality: $3 \leq 2x + 2 < 6$.

Solution

The first method is to write two separate inequalities: $3 \leq 2x + 2$ and $2x + 2 < 6$. We solve them independently.

$$\begin{array}{lcl} 3 \leq 2x+2 & \text{and} & 2x+2<6 \\ 1 \leq 2x & & 2x<4 \\ \frac{1}{2} \leq x & & x<2 \end{array}$$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$$\frac{1}{2} \leq x < 2$$

In interval notation, the solution is written as $\left[\frac{1}{2}, 2\right)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

$$\begin{array}{ll} 3 \leq 2x+2<6 & \\ 1 \leq 2x<4 & \text{Isolate the variable term, and subtract 2 from all three parts.} \\ \frac{1}{2} \leq x<2 & \text{Divide through all three parts by 2.} \end{array}$$

We get the same solution: $\left[\frac{1}{2}, 2\right)$.

TRY IT #7 Solve the compound inequality: $4 < 2x-8 \leq 10$.

EXAMPLE 8

Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3+x>7x-2>5x-10$.

Solution

Let's try the first method. Write two inequalities:

$$\begin{array}{rcl} 3+x>7x-2 & \text{and} & 7x-2>5x-10 \\ 3>6x-2 & & 2x-2>-10 \\ 5>6x & & 2x>-8 \\ \frac{5}{6}>x & & x>-4 \\ x<\frac{5}{6} & & -4<x \end{array}$$

The solution set is $-4<x<\frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See Figure 3.

Figure 3

TRY IT #8 Solve the compound inequality: $3y<4-5y<5+3y$.

Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x, 0)$ has an absolute value of x, as it is x units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form

$$|A|<B, \quad |A| \leq B, \quad |A|>B, \quad \text{or} \quad |A| \geq B,$$

Where *A*, and sometimes *B*, represents an algebraic expression dependent on a variable *x*. Solving the inequality means finding the set of all x -values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

Suppose we want to know all possible returns on an investment if we could earn some amount of money within $200 of $600. We can solve algebraically for the set of *x*-values such that the distance between x and 600 is less than or equal to 200. We represent the distance between x and 600 as $|x-600|$, and therefore, $|x-600| \leq 200$ or

$$\begin{gathered} -200 \leq x-600 \leq 200 \\ -200+600 \leq x-600+600 \leq 200+600 \\ 400 \leq x \leq 800 \end{gathered}$$

This means our returns would be between $400 and $800.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

Absolute Value Inequalities

For an algebraic expression *X*, and $k>0$, an **absolute value inequality** is an inequality of the form

$$\begin{array}{ll} |X|<k & \text{is equivalent to} -k<X<k \\ |X|>k & \text{is equivalent to} X<-k \quad \text{or} X>k \end{array}$$

These statements also apply to $|X| \leq k$ and $|X| \geq k$.

EXAMPLE 9

Determining a Number within a Prescribed Distance

Describe all values x within a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as in Figure 4**,** to represent the condition to be satisfied.

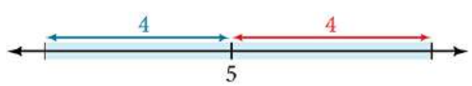


Figure 4

The distance from x to 5 can be represented using an absolute value symbol, $|x-5|$. Write the values of x that satisfy the condition as an absolute value inequality.

$$|x-5| \leq 4$$

We need to write two inequalities as there are always two solutions to an absolute value equation.

$$\begin{array}{ccc} x-5 \leq 4 & \text{and} & x-5 \geq -4 \\ x \leq 9 & & x \geq 1 \end{array}$$

If the solution set is $x \leq 9$ and $x \geq 1$, then the solution set is an interval including all real numbers between and including 1 and 9.

So $|x-5| \leq 4$ is equivalent to $[1,9]$ in interval notation.

TRY IT #9 Describe all *x*-values within a distance of 3 from the number 2.

EXAMPLE 10

Solving an Absolute Value Inequality

Solve $|x - 1| \leq 3$.

Solution

$$|x - 1| \leq 3$$

$$-3 \leq x - 1 \leq 3$$

$$-2 \leq x \leq 4$$

$$[-2, 4]$$

EXAMPLE 11

Using a Graphical Approach to Solve Absolute Value Inequalities

Given the equation $y = -\frac{1}{2}\left|4x - 5\right| + 3$, determine the *x*-values for which the *y*-values are negative.

Solution

We are trying to determine where $y < 0$, which is when $-\frac{1}{2}\left|4x - 5\right| + 3 < 0$. We begin by isolating the absolute value.

$$-\frac{1}{2}\left|4x - 5\right| < -3 \qquad \text{Multiply both sides by –2, and reverse the inequality.}$$
$$|4x - 5| > 6$$

Next, we solve for the equality $|4x - 5| = 6$.

$$\begin{aligned} 4x - 5 &= 6 \\ 4x &= 11 \\ x &= \tfrac{11}{4} \end{aligned} \qquad \text{or} \qquad \begin{aligned} 4x - 5 &= -6 \\ 4x &= -1 \\ x &= -\tfrac{1}{4} \end{aligned}$$

Now, we can examine the graph to observe where the *y*-values are negative. We observe where the branches are below the *x*-axis. Notice that it is not important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$, and that the graph opens downward. See Figure 5.

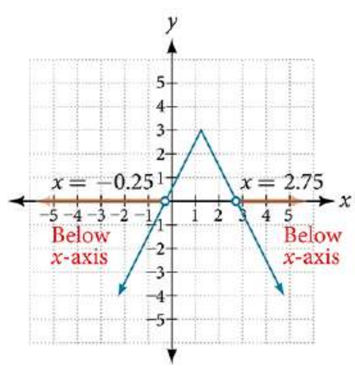


Figure 5

TRY IT #10 Solve $-2\,|k - 4| \leq -6$.

MEDIA

Access these online resources for additional instruction and practice with linear inequalities and absolute value inequalities.

[Interval notation (http://openstax.org/l/intervalnotn)](http://openstax.org/l/intervalnotn)
[How to solve linear inequalities (http://openstax.org/l/solvelinineq)](http://openstax.org/l/solvelinineq)
[How to solve an inequality (http://openstax.org/l/solveineq)](http://openstax.org/l/solveineq)
[Absolute value equations (http://openstax.org/l/absvaleq)](http://openstax.org/l/absvaleq)
[Compound inequalities (http://openstax.org/l/compndineqs)](http://openstax.org/l/compndineqs)
[Absolute value inequalities (http://openstax.org/l/absvalineqs)](http://openstax.org/l/absvalineqs)

2.7 SECTION EXERCISES

Verbal

1. When solving an inequality, explain what happened from Step 1 to Step 2:

Step 1 $\quad -2x > 6$
Step 2 $\quad x < -3$

2. When solving an inequality, we arrive at:

$$x + 2 < x + 3$$
$$2 < 3$$

Explain what our solution set is.

3. When writing our solution in interval notation, how do we represent all the real numbers?

4. When solving an inequality, we arrive at:

$$x + 2 > x + 3$$
$$2 > 3$$

Explain what our solution set is.

5. Describe how to graph $y = |x - 3|$

Algebraic

For the following exercises, solve the inequality. Write your final answer in interval notation.

6. $4x - 7 \leq 9$

7. $3x + 2 \geq 7x - 1$

8. $-2x + 3 > x - 5$

9. $4(x + 3) \geq 2x - 1$

10. $-\frac{1}{2}x \leq -\frac{5}{4} + \frac{2}{5}x$

11. $-5(x - 1) + 3 > 3x - 4 - 4x$

12. $-3(2x + 1) > -2(x + 4)$

13. $\frac{x+3}{8} - \frac{x+5}{5} \geq \frac{3}{10}$

14. $\frac{x-1}{3} + \frac{x+2}{5} \leq \frac{3}{5}$

For the following exercises, solve the inequality involving absolute value. Write your final answer in interval notation.

15. $|x + 9| \geq -6$

16. $|2x + 3| < 7$

17. $|3x - 1| > 11$

18. $|2x + 1| + 1 \leq 6$

19. $|x - 2| + 4 \geq 10$

20. $|-2x + 7| \leq 13$

21. $|x - 7| < -4$

22. $|x - 20| > -1$

23. $\left|\frac{x-3}{4}\right| < 2$

For the following exercises, describe all the x-values within or including a distance of the given values.

24. Distance of 5 units from the number 7

25. Distance of 3 units from the number 9

26. Distance of 10 units from the number 4

27. Distance of 11 units from the number 1

For the following exercises, solve the compound inequality. Express your answer using inequality signs, and then write your answer using interval notation.

28. $-4 < 3x + 2 \leq 18$

29. $3x + 1 > 2x - 5 > x - 7$

30. $3y < 5 - 2y < 7 + y$

31. $2x - 5 < -11 \quad \text{or} \quad 5x + 1 \geq 6$

32. $x + 7 < x + 2$

Graphical

For the following exercises, graph the function. Observe the points of intersection and shade the x-axis representing the solution set to the inequality. Show your graph and write your final answer in interval notation.

33. $|x - 1| > 2$

34. $|x + 3| \geq 5$

35. $|x + 7| \leq 4$

36. $|x - 2| < 7$

37. $|x - 2| < 0$

For the following exercises, graph both straight lines (left-hand side being y1 and right-hand side being y2) on the same axes. Find the point of intersection and solve the inequality by observing where it is true comparing the y-values of the lines.

38. $x + 3 < 3x - 4$

39. $x - 2 > 2x + 1$

40. $x + 1 > x + 4$

41. $\frac{1}{2}x + 1 > \frac{1}{2}x - 5$

42. $4x + 1 < \frac{1}{2}x + 3$

Numeric

For the following exercises, write the set in interval notation.

43. $\{x | -1 < x < 3\}$

44. $\{x | x \geq 7\}$

45. $\{x | x < 4\}$

46. $\{\ x |\ x \ \text{ is all real numbers}\}$

For the following exercises, write the interval in set-builder notation.

47. $\left(-\infty, 6\right)$

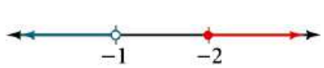
48. $\left(4, \infty\right)$

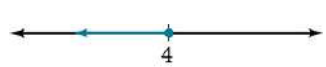
49. $[-3, 5)$

50. $[-4, 1] \cup [9, \infty)$

For the following exercises, write the set of numbers represented on the number line in interval notation.

51. 

52. 

53. 

Technology

For the following exercises, input the left-hand side of the inequality as a Y1 graph in your graphing utility. Enter y2 = the right-hand side. Entering the absolute value of an expression is found in the MATH menu, Num, 1:abs(. Find the points of intersection, recall (2nd CALC 5:intersection, 1st curve, enter, 2nd curve, enter, guess, enter). Copy a sketch of the graph and shade the x-axis for your solution set to the inequality. Write final answers in interval notation.

54. $|x+2|-5<2$

55. $\frac{-1}{2}|x+2|<4$

56. $|4x+1|-3>2$

57. $|x-4|<3$

58. $|x+2|\geq 5$

Extensions

59. Solve $|3x+1|=|2x+3|$

60. Solve $x^2-x>12$

61. $\frac{x-5}{x+7}\leq 0,\ x\neq -7$

62. $p=-x^2+130x-3000$ is a profit formula for a small business. Find the set of x-values that will keep this profit positive.

Real-World Applications

63. In chemistry the volume for a certain gas is given by $V=20T$, where V is measured in cc and T is temperature in ºC. If the temperature varies between 80ºC and 120ºC, find the set of volume values.

64. A basic cellular package costs $20/mo. for 60 min of calling, with an additional charge of $.30/min beyond that time.. The cost formula would be $C=20+.30(x-60)$. If you have to keep your bill no greater than $50, what is the maximum calling minutes you can use?

Chapter Review

Key Terms

absolute value equation an equation in which the variable appears in absolute value bars, typically with two solutions, one accounting for the positive expression and one for the negative expression

area in square units, the area formula used in this section is used to find the area of any two-dimensional rectangular region: $A = LW$

Cartesian coordinate system a grid system designed with perpendicular axes invented by René Descartes

completing the square a process for solving quadratic equations in which terms are added to or subtracted from both sides of the equation in order to make one side a perfect square

complex conjugate a complex number containing the same terms as another complex number, but with the opposite operator. Multiplying a complex number by its conjugate yields a real number.

complex number the sum of a real number and an imaginary number; the standard form is $a + bi$, where *a* is the real part and b is the complex part.

complex plane the coordinate plane in which the horizontal axis represents the real component of a complex number, and the vertical axis represents the imaginary component, labeled *i*.

compound inequality a problem or a statement that includes two inequalities

conditional equation an equation that is true for some values of the variable

discriminant the expression under the radical in the quadratic formula that indicates the nature of the solutions, real or complex, rational or irrational, single or double roots.

distance formula a formula that can be used to find the length of a line segment if the endpoints are known

equation in two variables a mathematical statement, typically written in *x* and *y*, in which two expressions are equal

equations in quadratic form equations with a power other than 2 but with a middle term with an exponent that is one-half the exponent of the leading term

extraneous solutions any solutions obtained that are not valid in the original equation

graph in two variables the graph of an equation in two variables, which is always shown in two variables in the two-dimensional plane

identity equation an equation that is true for all values of the variable

imaginary number the square root of $-1 : i = \sqrt{-1}$.

inconsistent equation an equation producing a false result

intercepts the points at which the graph of an equation crosses the *x*-axis and the *y*-axis

interval an interval describes a set of numbers within which a solution falls

interval notation a mathematical statement that describes a solution set and uses parentheses or brackets to indicate where an interval begins and ends

linear equation an algebraic equation in which each term is either a constant or the product of a constant and the first power of a variable

linear inequality similar to a linear equation except that the solutions will include sets of numbers

midpoint formula a formula to find the point that divides a line segment into two parts of equal length

ordered pair a pair of numbers indicating horizontal displacement and vertical displacement from the origin; also known as a coordinate pair, (x, y)

origin the point where the two axes cross in the center of the plane, described by the ordered pair $(0, 0)$

perimeter in linear units, the perimeter formula is used to find the linear measurement, or outside length and width, around a two-dimensional regular object; for a rectangle: $P = 2L + 2W$

polynomial equation an equation containing a string of terms including numerical coefficients and variables raised to whole-number exponents

Pythagorean Theorem a theorem that states the relationship among the lengths of the sides of a right triangle, used to solve right triangle problems

quadrant one quarter of the coordinate plane, created when the axes divide the plane into four sections

quadratic equation an equation containing a second-degree polynomial; can be solved using multiple methods

quadratic formula a formula that will solve all quadratic equations

radical equation an equation containing at least one radical term where the variable is part of the radicand

rational equation an equation consisting of a fraction of polynomials

slope the change in *y*-values over the change in *x*-values

solution set the set of all solutions to an equation

square root property one of the methods used to solve a quadratic equation, in which the x^2 term is isolated so that the square root of both sides of the equation can be taken to solve for *x*

volume in cubic units, the volume measurement includes length, width, and depth: $V = LWH$

***x*-axis** the common name of the horizontal axis on a coordinate plane; a number line increasing from left to right

***x*-coordinate** the first coordinate of an ordered pair, representing the horizontal displacement and direction from the origin
***x*-intercept** the point where a graph intersects the *x*-axis; an ordered pair with a *y*-coordinate of zero
***y*-axis** the common name of the vertical axis on a coordinate plane; a number line increasing from bottom to top
***y*-coordinate** the second coordinate of an ordered pair, representing the vertical displacement and direction from the origin
***y*-intercept** a point where a graph intercepts the *y*-axis; an ordered pair with an *x*-coordinate of zero
zero-product property the property that formally states that multiplication by zero is zero, so that each factor of a quadratic equation can be set equal to zero to solve equations

Key Equations

quadratic formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Key Concepts

2.1 The Rectangular Coordinate Systems and Graphs

- We can locate, or plot, points in the Cartesian coordinate system using ordered pairs, which are defined as displacement from the *x*-axis and displacement from the *y*-axis. See Example 1.
- An equation can be graphed in the plane by creating a table of values and plotting points. See Example 2.
- Using a graphing calculator or a computer program makes graphing equations faster and more accurate. Equations usually have to be entered in the form *y*=_____. See Example 3.
- Finding the *x*- and *y*-intercepts can define the graph of a line. These are the points where the graph crosses the axes. See Example 4.
- The distance formula is derived from the Pythagorean Theorem and is used to find the length of a line segment. See Example 5 and Example 6.
- The midpoint formula provides a method of finding the coordinates of the midpoint dividing the sum of the *x*-coordinates and the sum of the *y*-coordinates of the endpoints by 2. See Example 7 and Example 8.

2.2 Linear Equations in One Variable

- We can solve linear equations in one variable in the form $ax + b = 0$ using standard algebraic properties. See Example 1 and Example 2.
- A rational expression is a quotient of two polynomials. We use the LCD to clear the fractions from an equation. See Example 3 and Example 4.
- All solutions to a rational equation should be verified within the original equation to avoid an undefined term, or zero in the denominator. See Example 5 and Example 6 and Example 7.
- Given two points, we can find the slope of a line using the slope formula. See Example 8.
- We can identify the slope and *y*-intercept of an equation in slope-intercept form. See Example 9.
- We can find the equation of a line given the slope and a point. See Example 10.
- We can also find the equation of a line given two points. Find the slope and use the point-slope formula. See Example 11.
- The standard form of a line has no fractions. See Example 12.
- Horizontal lines have a slope of zero and are defined as $y = c$, where *c* is a constant.
- Vertical lines have an undefined slope (zero in the denominator), and are defined as $x = c$, where *c* is a constant. See Example 13.
- Parallel lines have the same slope and different *y*-intercepts. See Example 14 and Example 15.
- Perpendicular lines have slopes that are negative reciprocals of each other unless one is horizontal and the other is vertical. See Example 16.

2.3 Models and Applications

- A linear equation can be used to solve for an unknown in a number problem. See Example 1.
- Applications can be written as mathematical problems by identifying known quantities and assigning a variable to unknown quantities. See Example 2.
- There are many known formulas that can be used to solve applications. Distance problems, for example, are solved using the $d = rt$ formula. See Example 3.
- Many geometry problems are solved using the perimeter formula $P = 2L + 2W$, the area formula $A = LW$, or the

volume formula $V = LWH$. See Example 4**,** Example 5, and Example 6.

2.4 Complex Numbers

- The square root of any negative number can be written as a multiple of i. See Example 1.
- To plot a complex number, we use two number lines, crossed to form the complex plane. The horizontal axis is the real axis, and the vertical axis is the imaginary axis. See Example 2.
- Complex numbers can be added and subtracted by combining the real parts and combining the imaginary parts. See Example 3.
- Complex numbers can be multiplied and divided.
    - To multiply complex numbers, distribute just as with polynomials. See Example 4 and Example 5**.**
    - To divide complex numbers, multiply both numerator and denominator by the complex conjugate of the denominator to eliminate the complex number from the denominator. See Example 6 and Example 7**.**
- The powers of i are cyclic, repeating every fourth one. See Example 8.

2.5 Quadratic Equations

- Many quadratic equations can be solved by factoring when the equation has a leading coefficient of 1 or if the equation is a difference of squares. The zero-product property is then used to find solutions. See Example 1, Example 2, and Example 3.
- Many quadratic equations with a leading coefficient other than 1 can be solved by factoring using the grouping method. See Example 4 and Example 5.
- Another method for solving quadratics is the square root property. The variable is squared. We isolate the squared term and take the square root of both sides of the equation. The solution will yield a positive and negative solution. See Example 6 and Example 7.
- Completing the square is a method of solving quadratic equations when the equation cannot be factored. See Example 8.
- A highly dependable method for solving quadratic equations is the quadratic formula, based on the coefficients and the constant term in the equation. See Example 9 and Example 10.
- The discriminant is used to indicate the nature of the roots that the quadratic equation will yield: real or complex, rational or irrational, and how many of each. See Example 11.
- The Pythagorean Theorem, among the most famous theorems in history, is used to solve right-triangle problems and has applications in numerous fields. Solving for the length of one side of a right triangle requires solving a quadratic equation. See Example 12.

2.6 Other Types of Equations

- Rational exponents can be rewritten several ways depending on what is most convenient for the problem. To solve, both sides of the equation are raised to a power that will render the exponent on the variable equal to 1. See Example 1, Example 2, and Example 3.
- Factoring extends to higher-order polynomials when it involves factoring out the GCF or factoring by grouping. See Example 4 and Example 5.
- We can solve radical equations by isolating the radical and raising both sides of the equation to a power that matches the index. See Example 6 and Example 7.
- To solve absolute value equations, we need to write two equations, one for the positive value and one for the negative value. See Example 8.
- Equations in quadratic form are easy to spot, as the exponent on the first term is double the exponent on the second term and the third term is a constant. We may also see a binomial in place of the single variable. We use substitution to solve. See Example 9 and Example 10.
- Solving a rational equation may also lead to a quadratic equation or an equation in quadratic form. See Example 11.

2.7 Linear Inequalities and Absolute Value Inequalities

- Interval notation is a method to indicate the solution set to an inequality. Highly applicable in calculus, it is a system of parentheses and brackets that indicate what numbers are included in a set and whether the endpoints are included as well. See Table 1 and Example 2**.**
- Solving inequalities is similar to solving equations. The same algebraic rules apply, except for one: multiplying or dividing by a negative number reverses the inequality. See Example 3**,** Example 4, Example 5, and Example 6**.**
- Compound inequalities often have three parts and can be rewritten as two independent inequalities. Solutions are given by boundary values, which are indicated as a beginning boundary or an ending boundary in the solutions to the two inequalities. See Example 7 and Example 8.

- Absolute value inequalities will produce two solution sets due to the nature of absolute value. We solve by writing two equations: one equal to a positive value and one equal to a negative value. See Example 9 and Example 10.
- Absolute value inequalities can also be solved by graphing. At least we can check the algebraic solutions by graphing, as we cannot depend on a visual for a precise solution. See Example 11.

Exercises

Review Exercises

The Rectangular Coordinate Systems and Graphs

For the following exercises, find the x-intercept and the y-intercept without graphing.

1. $4x - 3y = 12$

2. $2y - 4 = 3x$

For the following exercises, solve for y in terms of x, putting the equation in slope–intercept form.

3. $5x = 3y - 12$

4. $2x - 5y = 7$

For the following exercises, find the distance between the two points.

5. $(-2, 5)\,(4, -1)$

6. $(-12, -3)\,(-1, 5)$

7. Find the distance between the two points $(-71,432)$ and $(511,218)$ using your calculator, and round your answer to the nearest thousandth.

For the following exercises, find the coordinates of the midpoint of the line segment that joins the two given points.

8. $(-1, 5)$ and $(4, 6)$

9. $(-13, 5)$ and $(17, 18)$

For the following exercises, construct a table and graph the equation by plotting at least three points.

10. $y = \frac{1}{2}x + 4$

11. $4x - 3y = 6$

Linear Equations in One Variable

For the following exercises, solve for x.

12. $5x + 2 = 7x - 8$

13. $3(x + 2) - 10 = x + 4$

14. $7x - 3 = 5$

15. $12 - 5(x + 1) = 2x - 5$

16. $\frac{2x}{3} - \frac{3}{4} = \frac{x}{6} + \frac{21}{4}$

For the following exercises, solve for x. State all x-values that are excluded from the solution set.

17. $\frac{x}{x^2-9} + \frac{4}{x+3} = \frac{3}{x^2-9}$
$x \neq 3, -3$

18. $\frac{1}{2} + \frac{2}{x} = \frac{3}{4}$

For the following exercises, find the equation of the line using the point-slope formula.

19. Passes through these two points: $(-2, 1), (4, 2)$.

20. Passes through the point $(-3, 4)$ and has a slope of $-\frac{1}{3}$.

21. Passes through the point $(-3, 4)$ and is parallel to the graph $y = \frac{2}{3}x + 5$.

22. Passes through these two points: $(5, 1), (5, 7)$.

Models and Applications

For the following exercises, write and solve an equation to answer each question.

23. The number of male fish in the tank is five more than three times the number of females. If the total number of fish is 73, how many of each sex are in the tank?

24. A landscaper has 72 ft. of fencing to put around a rectangular garden. If the length is 3 times the width, find the dimensions of the garden.

25. A truck rental is \$25 plus \$.30/mi. Find out how many miles Ken traveled if his bill was \$50.20.

Complex Numbers

For the following exercises, use the quadratic equation to solve.

26. $x^2 - 5x + 9 = 0$

27. $2x^2 + 3x + 7 = 0$

For the following exercises, name the horizontal component and the vertical component.

28. $4 - 3i$

29. $-2 - i$

For the following exercises, perform the operations indicated.

30. $(9 - i) - (4 - 7i)$

31. $(2 + 3i) - (-5 - 8i)$

32. $2\sqrt{-75} + 3\sqrt{25}$

33. $\sqrt{-16} + 4\sqrt{-9}$

34. $-6i(i - 5)$

35. $(3 - 5i)^2$

36. $\sqrt{-4} \cdot \sqrt{-12}$

37. $\sqrt{-2}\left(\sqrt{-8} - \sqrt{5}\right)$

38. $\frac{2}{5-3i}$

39. $\frac{3+7i}{i}$

Quadratic Equations

For the following exercises, solve the quadratic equation by factoring.

40. $2x^2 - 7x - 4 = 0$

41. $3x^2 + 18x + 15 = 0$

42. $25x^2 - 9 = 0$

43. $7x^2 - 9x = 0$

For the following exercises, solve the quadratic equation by using the square-root property.

44. $x^2 = 49$

45. $(x - 4)^2 = 36$

For the following exercises, solve the quadratic equation by completing the square.

46. $x^2 + 8x - 5 = 0$

47. $4x^2 + 2x - 1 = 0$

For the following exercises, solve the quadratic equation by using the quadratic formula. If the solutions are not real, state No real solution.

48. $2x^2 - 5x + 1 = 0$

49. $15x^2 - x - 2 = 0$

For the following exercises, solve the quadratic equation by the method of your choice.

50. $(x - 2)^2 = 16$

51. $x^2 = 10x + 3$

Other Types of Equations

For the following exercises, solve the equations.

52. $x^{\frac{3}{2}} = 27$

53. $x^{\frac{1}{2}} - 4x^{\frac{1}{4}} = 0$

54. $4x^3 + 8x^2 - 9x - 18 = 0$

55. $3x^5 - 6x^3 = 0$

56. $\sqrt{x + 9} = x - 3$

57. $\sqrt{3x + 7} + \sqrt{x + 2} = 1$

58. $|3x - 7| = 5$

59. $|2x + 3| - 5 = 9$

Linear Inequalities and Absolute Value Inequalities

For the following exercises, solve the inequality. Write your final answer in interval notation.

60. $5x - 8 \leq 12$

61. $-2x + 5 > x - 7$

62. $\frac{x-1}{3} + \frac{x+2}{5} \leq \frac{3}{5}$

63. $|3x + 2| + 1 \leq 9$

64. $|5x - 1| > 14$

65. $|x - 3| < -4$

For the following exercises, solve the compound inequality. Write your answer in interval notation.

66. $-4 < 3x + 2 \leq 18$

67. $3y < 1 - 2y < 5 + y$

For the following exercises, graph as described.

68. Graph the absolute value function and graph the constant function. Observe the points of intersection and shade the *x*-axis representing the solution set to the inequality. Show your graph and write your final answer in interval notation.

$|x + 3| \geq 5$

69. Graph both straight lines (left-hand side being y1 and right-hand side being y2) on the same axes. Find the point of intersection and solve the inequality by observing where it is true comparing the *y*-values of the lines. See the interval where the inequality is true.

$x + 3 < 3x - 4$

Practice Test

1. Graph the following: $2y = 3x + 4$.

2. Find the *x*- and *y*-intercepts for the following: $2x - 5y = 6$

3. Find the *x*- and *y*-intercepts of this equation, and sketch the graph of the line using just the intercepts plotted.

$3x - 4y = 12$

4. Find the exact distance between $(5, -3)$ and $(-2, 8)$. Find the coordinates of the midpoint of the line segment joining the two points.

5. Write the interval notation for the set of numbers represented by $\{x | x \leq 9\}$.

6. Solve for *x*: $5x + 8 = 3x - 10$.

7. Solve for x: $3\,(2x - 5) - 3\,(x - 7) = 2x - 9$.

8. Solve for *x*: $\frac{x}{2} + 1 = \frac{4}{x}$

9. Solve for *x*: $\frac{5}{x+4} = 4 + \frac{3}{x-2}$.

10. The perimeter of a triangle is 30 in. The longest side is 2 less than 3 times the shortest side and the other side is 2 more than twice the shortest side. Find the length of each side.

11. Solve for *x*. Write the answer in simplest radical form.

$\frac{x^2}{3} - x = -\frac{1}{2}$

12. Solve: $3x - 8 \leq 4$.

13. Solve: $|2x + 3| < 5$.

14. Solve: $|3x - 2| \geq 4$.

For the following exercises, find the equation of the line with the given information.

15. Passes through the points $(-4, 2)$ and $(5, -3)$.

16. Has an undefined slope and passes through the point $(4, 3)$.

17. Passes through the point $(2, 1)$ and is perpendicular to $y = -\frac{2}{5}x + 3$.

18. Add these complex numbers: $(3 - 2i) + (4 - i)$.

19. Simplify: $\sqrt{-4} + 3\sqrt{-16}$.

20. Multiply: $5i\,(5 - 3i)$.

21. Divide: $\frac{4-i}{2+3i}$.

22. Solve this quadratic equation and write the two complex roots in $a + bi$ form: $x^2 - 4x + 7 = 0$.

23. Solve: $(3x - 1)^2 - 1 = 24$.

24. Solve: $x^2 - 6x = 13$.

25. Solve: $4x^2 - 4x - 1 = 0$

26. Solve: $\sqrt{x - 7} = x - 7$

27. Solve: $2 + \sqrt{12 - 2x} = x$

28. Solve: $(x - 1)^{\frac{2}{3}} = 9$

For the following exercises, find the real solutions of each equation by factoring.

29. $2x^3 - x^2 - 8x + 4 = 0$

30. $(x + 5)^2 - 3\,(x + 5) - 4 = 0$

3 FUNCTIONS

Standard and Poor’s Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)

Chapter Outline

Introduction to Functions

Toward the end of the twentieth century, the values of stocks of Internet and technology companies rose dramatically. As a result, the Standard and Poor’s stock market average rose as well. The graph above tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1100 by the beginning of 2000. That five-year period became known as the “dot-com bubble” because so many Internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

3.1 Functions and Function Notation

Learning Objectives

In this section, you will:

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

COREQUISITE SKILLS

Learning Objectives

- Find the value of a function (IA 3.5.3)

Objective 1: Find the value of a function (IA 3.5.3)

A **relation** is any set of ordered pairs, (x,y). The collection of x-values in the ordered pairs together make up the **domain**. The collection of y-values in the ordered pairs together make up the **range**.

A special type of relation, called a function, is studied extensively in mathematics. A **function** is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each x-value is matched with only one y-value.

Function Notation

For the function $y = f(x)$

f is the name of the function
x is the domain value
$f(x)$ is the range value y corresponding to the value x

We read $f(x)$ as *f* of *x* or the value of *f* at *x*.

Representation of Functions, $y = f(x)$

There are many ways to represent functions including:

- Equations
- Tables of input and output values
- Collections of ordered pairs, or points (x,y) = (independent variable, dependent variable)
- Graphs
- Mappings
- Verbal descriptions

EXAMPLE 1

Find the value of a function

ⓐ For the function $f(x) = 2x - 5$
Find $f(4)$, $f(-6)$, $f(0)$, and $f(a)$.
Find the value of x that makes $f(x) = 11$

ⓑ Refer to the following table of values for the function $g(x)$.

x	$g(x)$
0	2
1	5
2	14
3	29

Find $g(1)$, $g(3)$, $g(0)$.

ⓒ
For a man of height $5'11$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5–24.9$ is considered healthy.

Find the BMI for a man of height $5'11$ who weighs 180 pounds. Find the weight of a man of height $5'11$ and who has a BMI of 22.3 .

Solution

ⓐ In part a we are working with an equation and will begin by substituting the input value for x. Then use the order of operations to evaluate.

$f(4) = 2(4) - 5 = 8 - 5 = 3$

$$f(-6) = 2(-6) - 5 = -12 - 5 = -17$$
$$f(0) = 2(0) - 5 = 0 - 5 = -5$$
$$f(a) = 2(a) - 5 = 2a - 5$$

Find the value of x that makes $f(x) = 11$ Now we are given a y value and need to solve the equation for the x that yielded an $f(x) = 11$.

$$\begin{aligned} f(x) &= 2x - 5 \\ 11 &= 2x - 5 \\ 11 + 5 &= 2x \\ 16 &= 2x \\ 8 &= x \end{aligned}$$

ⓑ Refer to the following table of values for the function $g(x)$.

x	$g(x)$
0	2
1	5
2	14
3	29

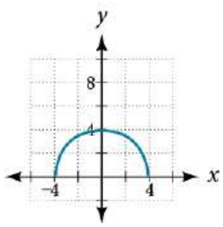
To find $g(1)$, find an x of 1 in your table and read $g(x)$ at this value, $g(1) = 5$
To find $g(3)$, find an x of 3 in your table and read $g(x)$ at this value, $g(3) = 29$
To find $g(0)$, find an x of 0 in your table and read $g(x)$ at this value, $g(0) = 2$
To find the value of x that makes $g(x) = 14$, look through the $g(x)$ column to find an output of 14 .
Notice $x = 2$ in this row, so $x = 2$.

ⓒ The representation shown in part c is called a mapping. Values in the domain (weight) map to values in the range (BMI).
The BMI for a man of height $5'11$ who weighs 180 pounds is 25.1 .
The weight for a man of height $5'11$ who has a BMI of 22.3 is 160 .

Practice Makes Perfect

Find the value of a function

1.



ⓐ Find: $f(0)$. ⓑ Find the values for x when $f(x) = 0$.

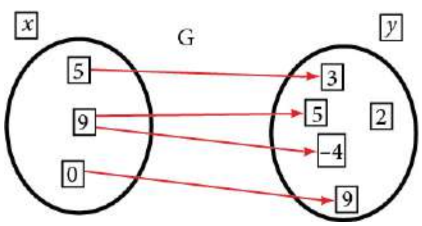
2. For the function $f(x) = 3x^2 - 2x + 1$ find

ⓐ $f(3)$ ⓑ $f(-2)$ ⓒ $f(t)$ ⓓ The value(s) of x that make $f(x) = 1$.

3. For the function $g(x) = -4 + \sqrt{3x + 19}$, find the following. Make sure to give exact values.

ⓐ $g(-5)$ ⓑ $g(2)$ ⓒ $g(0)$ ⓓ The value(s) of x that make $g(x) = 0$.

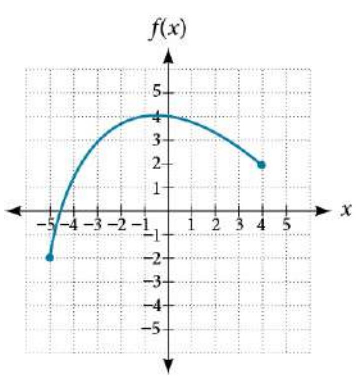
4.



Use the mapping $y = G(x)$ to find the following:

ⓐ $G(0) =$ ⓑ $G(9) =$ ⓒ If $G(x) = 9$, then $x =$ ______
ⓓ Is $G(x)$ a function? Explain why or why not.

5.



Use the mapping $y = F(x)$ to find the following:

ⓐ $F(-1) =$ ⓑ $F(4) =$ ⓒ $F(2) =$ ⓓ If $F(x) = 9$, then $x =$ ______
ⓔ Is $F(x)$ a function? Explain why or why not.

6. If $h(x) = 5x - 7$
Find:

ⓐ $h(-3) =$ ⓑ $h(0) =$ ⓒ $h(w + 4) =$ ⓓ $h(x) = 23$, then $x =$ ______
ⓔ $h(x) = -15$, then $x =$ ______

7. $g(t) = 2|t - 5| + 4$
Find:

ⓐ $g(-3) =$ ⓑ $g(0) =$ ⓒ $g(5) =$ ⓓ $g(w) =$ ⓔ $h(x) = 20$, then $x =$ ______

8.
1. ⓐ Use the following description to build a function called $f(x)$. "An input value is squared, multiplied by -2 and added to 3 ."
$f(x) =$
2. ⓑ Create a table of input/output values for $f(x)$ below. Show three numerical input values and one variable input value, and the corresponding output values in your table.

x	$f(x)$

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A **relation** is a set of ordered pairs. The set of the first components of each ordered pair is called the **domain** and the set of the second components of each ordered pair is called the **range**. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1, \ 2), \ (2, \ 4), \ (3, \ 6), \ (4, \ 8), \ (5, \ 10)\}$$

The domain is $\{1, \ 2, \ 3, \ 4, \ 5\}$. The range is $\{2, \ 4, \ 6, \ 8, \ 10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an **output** value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no *x*-values are repeated. For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, $\{1, \ 2, \ 3, \ 4, \ 5\}$, is paired with exactly one element in the range, $\{2, \ 4, \ 6, \ 8, \ 10\}$.

Now let’s consider the set of ordered pairs that relates the terms “even” and “odd” to the first five natural numbers. It would appear as

$$\{(\text{odd}, \ 1), \ (\text{even}, \ 2), \ (\text{odd}, \ 3), \ (\text{even}, \ 4), \ (\text{odd}, \ 5)\}$$

Notice that each element in the domain, $\{\text{even}, \ \text{odd}\}$ is *not* paired with exactly one element in the range, $\{1, \ 2, \ 3, \ 4, \ 5\}$. For example, the term “odd” corresponds to three values from the range, $\{1, \ 3, \ 5\}$ and the term “even” corresponds to two values from the range, $\{2, \ 4\}$. This violates the definition of a function, so this relation is not a function.

Figure 1 compares relations that are functions and not functions.

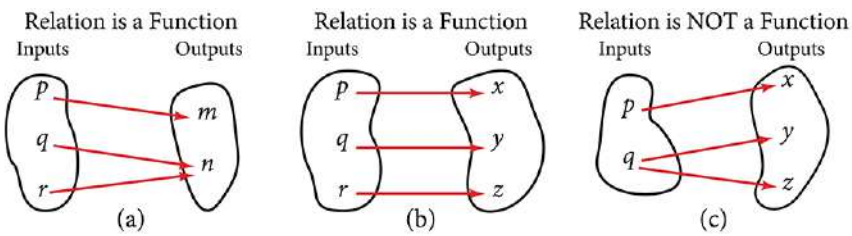


Figure 1 (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say “the output is a function of the input.”

The **input** values make up the **domain**, and the **output** values make up the **range**.

HOW TO

Given a relationship between two quantities, determine whether the relationship is a function.

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

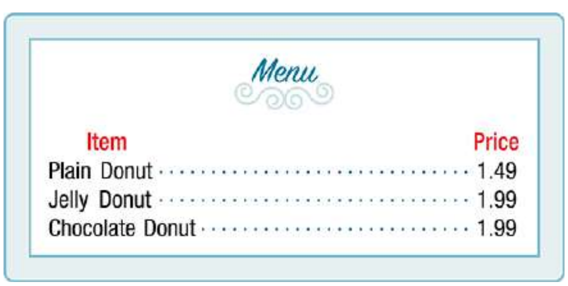
EXAMPLE 1

Determining If Menu Price Lists Are Functions

The coffee shop menu, shown below, consists of items and their prices.

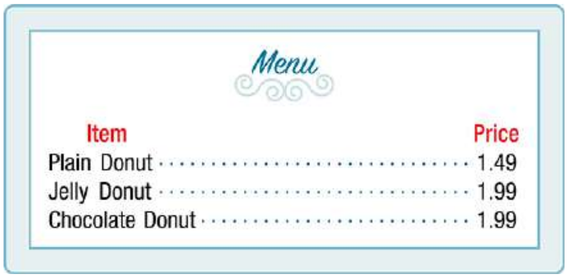
ⓐ Is price a function of the item?

ⓑ Is the item a function of the price?



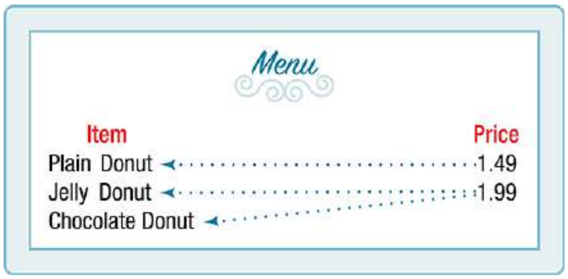
Solution

ⓐ Let’s begin by considering the input as the items on the menu. The output values are then the prices.



Each item on the menu has only one price, so the price is a function of the item.

ⓑ Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See the image below.



Therefore, the item is a not a function of price.

EXAMPLE 2

Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 1 shows a possible rule for assigning grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Table 1

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

TRY IT #1 Table 2[1] lists the five greatest baseball players of all time in order of rank.

Player	Rank
Babe Ruth	1

Table 2

Player	Rank
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

Table 2

ⓐ Is the rank a function of the player name? ⓑ Is the player name a function of the rank?

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters $f, \quad g$, and h are often used to represent functions just as we use $x, \quad y$, and z to represent numbers and $A, \quad B$, and C to represent sets.

$$\begin{array}{ll} h \text{ is } f \text{ of } a & \text{We name the function } f; \text{ height is a function of age.} \\ h = f(a) & \text{We use parentheses to indicate the function input.} \\ f(a) & \text{We name the function } f; \text{ the expression is read as "} f \text{ of } a\text{."} \end{array}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a + b)$ means "first add *a* and *b*, and the result is the input for the function *f*." The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

EXAMPLE 3

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month. Assume that the domain does not include leap years.

Solution

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

1 http://www.baseball-almanac.com/legendary/lisn100.shtml. Accessed 3/24/2014.

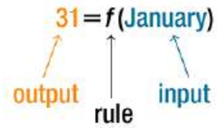


Figure 2

For example, $f\,(\text{March}) = 31$, because March has 31 days. The notation $d = f\,(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

EXAMPLE 4

Interpreting Function Notation

A function $N = f\,(y)$ gives the number of police officers, N, in a town in year y. What does $f\,(2005) = 300$ represent?

Solution

When we read $f\,(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N) , is 300. Remember, $N = f\,(y)$. The statement $f\,(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

TRY IT #2 Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Q&A **Instead of a notation such as $y = f(x\ )$, could we use the same symbol for the output as for the function, such as $y = y(x\ )$, meaning "*y* is a function of *x*?"**

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f\,(x)$, $P = W\,(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 3 lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f\,(m)$ identifies months by an integer rather than by name.

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table 3

Table 4 defines a function $Q = g\,(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

n	1	2	3	4	5
Q	8	6	7	6	8

Table 4

Table 5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

Table 5

HOW TO

Given a table of input and output values, determine whether the table represents a function.

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

EXAMPLE 5

Identifying Tables that Represent Functions

Which table, Table 6, Table 7, or Table 8, represents a function (if any)?

Input	Output
2	1
5	3
8	6

Table 6

Input	Output
–3	5
0	1
4	5

Table 7

Input	Output
1	0
5	2
5	4

Table 8

Solution

Table 6 and Table 7 define functions. In both, each input value corresponds to exactly one output value. Table 8 does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 6 can be represented by writing

$$f(2) = 1, f(5) = 3, \text{and } f(8) = 6$$

Similarly, the statements

$$g(-3) = 5, \quad g(0) = 1, \text{and } g(4) = 5$$

represent the function in Table 7.

Table 8 cannot be expressed in a similar way because it does not represent a function.

TRY IT #3 Does Table 9 represent a function?

Input	Output
1	10
2	100
3	1000

Table 9

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we *evaluate* the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and *solve* for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

HOW TO

Given the formula for a function, evaluate.

1. Substitute the input variable in the formula with the value provided.
2. Calculate the result.

EXAMPLE 6

Evaluating Functions at Specific Values

Evaluate $f(x) = x^2 + 3x - 4$ at:

ⓐ 2 ⓑ a ⓒ $a + h$ ⓓ Now evaluate $\frac{f(a+h)-f(a)}{h}$

Solution

Replace the x in the function with each specified value.

ⓐ Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

ⓑ In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

ⓒ In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h)-f(a)}{h} &= \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} \\ &= \frac{2ah+h^2+3h}{h} \\ &= \frac{h(2a+h+3)}{h} && \text{Factor out } h. \\ &= 2a + h + 3 && \text{Simplify.} \end{aligned}$$

EXAMPLE 7

Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p \\ h(4) &= (4)^2 + 2(4) \\ &= 16 + 8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

TRY IT #4 Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

EXAMPLE 8

Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

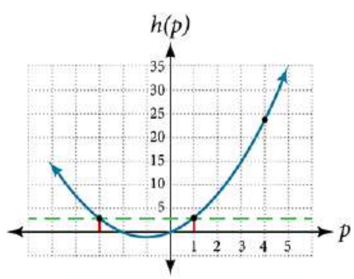
Solution

$$\begin{aligned} h(p) &= 3 && \\ p^2 + 2p &= 3 && \text{Substitute the original function } h(p) = p^2 + 2p. \\ p^2 + 2p - 3 &= 0 && \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 && \text{Factor.} \end{aligned}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$\begin{aligned} (p+3) &= 0, \quad p = -3 \\ (p-1) &= 0, \quad p = 1 \end{aligned}$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as in Figure 3. The graph verifies that $h(1) = h(-3) = 3$ and $h(4) = 24$.



p	−3	−2	0	1	4
$h(p)$	3	0	0	3	24

Figure 3

TRY IT #5 Given the function $g(m) = \sqrt{m-4}$, solve $g(m) = 2$.

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n + 6p = 12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

HOW TO

Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves *only* the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

EXAMPLE 9

Finding an Equation of a Function

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$, if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p = [\text{expression involving } n]$.

$$\begin{aligned} 2n + 6p &= 12 \\ 6p &= 12 - 2n && \text{Subtract } 2n \text{ from both sides.} \\ p &= \frac{12-2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6} - \frac{2n}{6} \\ p &= 2 - \frac{1}{3}n \end{aligned}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n$$

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

EXAMPLE 10

Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution

First we subtract x^2 from both sides.

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation.

$$\begin{aligned} y &= \pm\sqrt{1 - x^2} \\ &= +\sqrt{1 - x^2} \text{ and } -\sqrt{1 - x^2} \end{aligned}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$.

TRY IT #6 If $x - 8y^3 = 0$, express y as a function of x.

Q&A **Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?**

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a *unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.*

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table. See Table 10.[2]

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

Table 10

At times, evaluating a function in table form may be more useful than using equations.Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(\text{goldfish}) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

HOW TO

Given a function represented by a table, identify specific output and input values.

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

EXAMPLE 11

Evaluating and Solving a Tabular Function

Using Table 11,

ⓐ Evaluate $g(3)$.

ⓑ Solve $g(n) = 6$.

2 http://www.kgbanswers.com/how-long-is-a-dogs-memory-span/4221590. Accessed 3/24/2014.

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Table 11

Solution

ⓐ Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.

ⓑ Solving $g(n) = 6$ means identifying the input values, n, that produce an output value of 6. The table below shows two solutions: 2 and 4.

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

TRY IT #7 Using the table from **Evaluating and Solving a Tabular Function** above, evaluate $g(1)$.

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

EXAMPLE 12

Reading Function Values from a Graph

Given the graph in Figure 4,

ⓐ Evaluate $f(2)$.

ⓑ Solve $f(x) = 4$.

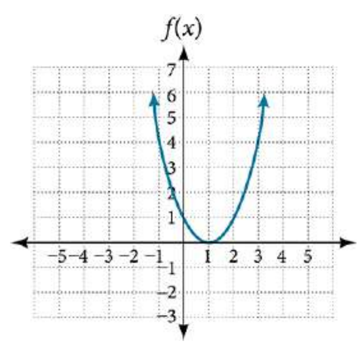


Figure 4

Solution

ⓐ To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 5.

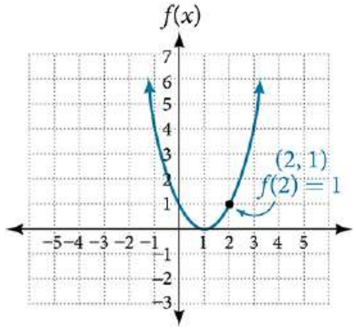


Figure 5

ⓑ To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: -1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4. See Figure 6.

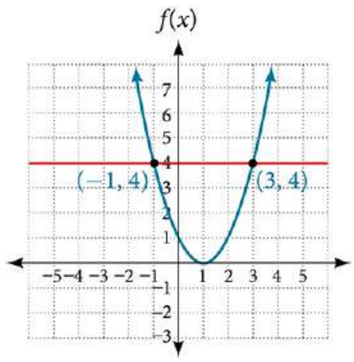


Figure 6

TRY IT #8 Using Figure 4, solve $f(x) = 1$.

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 12.

Letter grade	Grade point average
A	4.0
B	3.0
C	2.0
D	1.0

Table 12

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figure 1**(a)** and Figure 1**(b)**. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

One-to-One Function

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

EXAMPLE 13

Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

TRY IT #9

ⓐ Is a balance a function of the bank account number?
ⓑ Is a bank account number a function of the balance?
ⓒ Is a balance a one-to-one function of the bank account number?

TRY IT #10

Evaluate the following:
ⓐ If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
ⓑ If so, is the function one-to-one?

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 7 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points.

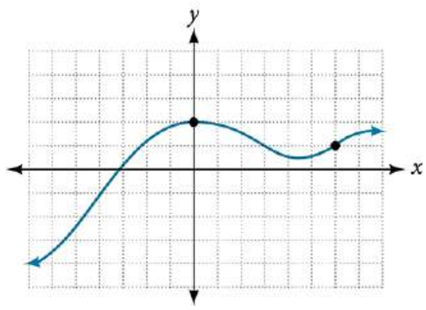


Figure 7

The **vertical line test** can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does *not* define a function because a function has only one output value for each input value. See Figure 8.

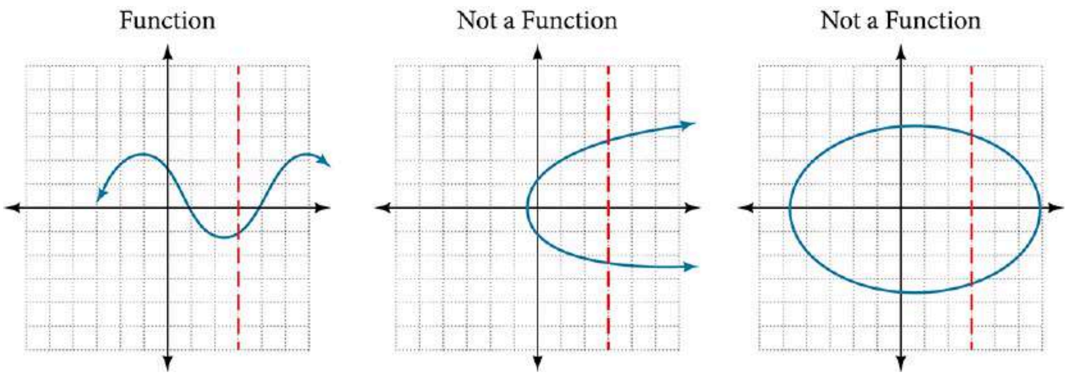


Figure 8

HOW TO

Given a graph, use the vertical line test to determine if the graph represents a function.

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

EXAMPLE 14

Applying the Vertical Line Test

Which of the graphs in Figure 9 represent(s) a function $y = f(x)$?

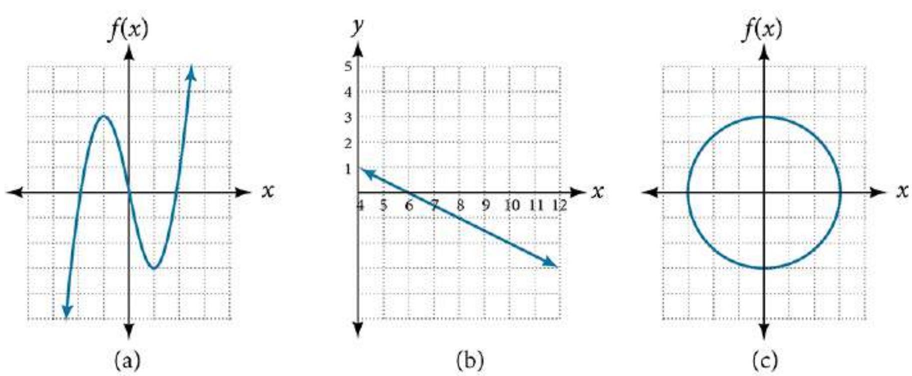


Figure 9

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 9. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most *x*-values, a vertical line would intersect the graph at more than one point, as shown in Figure 10.

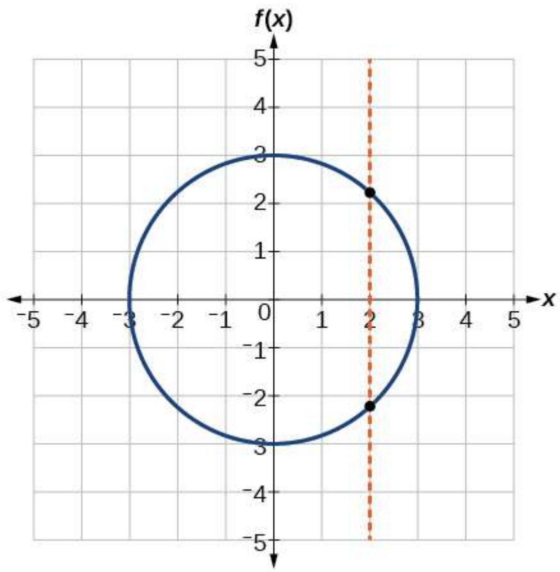


Figure 10

TRY IT #11 Does the graph in Figure 11 represent a function?

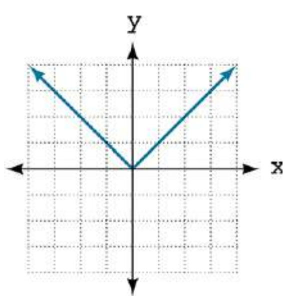


Figure 11

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

HOW TO

Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function.

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

EXAMPLE 15

Applying the Horizontal Line Test

Consider the functions shown in Figure 9**(a)** and Figure 9**(b)**. Are either of the functions one-to-one?

Solution

The function in Figure 9**(a)** is not one-to-one. The horizontal line shown in Figure 12 intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

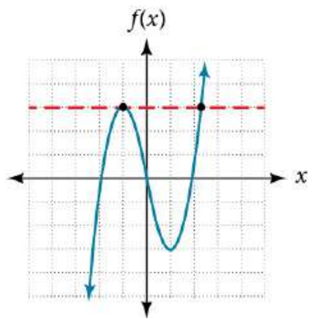


Figure 12

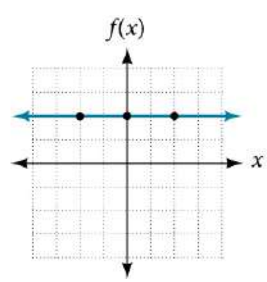
The function in Figure 9**(b)** is one-to-one. Any horizontal line will intersect a diagonal line at most once.

TRY IT #12 Is the graph shown in Figure 9 one-to-one?

Identifying Basic Toolkit Functions

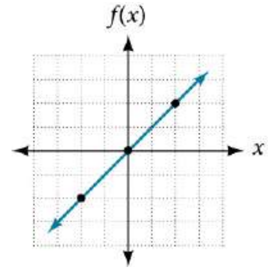
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our “toolkit functions,” which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

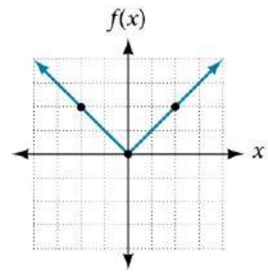
We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table 13.

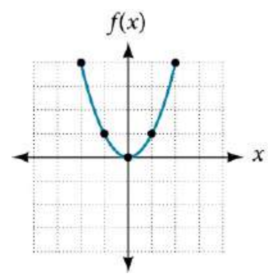
<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Constant</td><td>$f(x) = c$, where c is a constant</td><td>

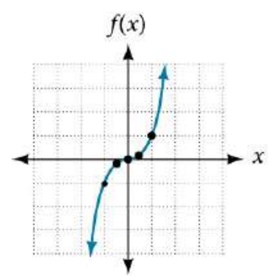
<table>
<tr><th>x</th><th>$f(x)$</th></tr>
<tr><td>−2</td><td>2</td></tr>
<tr><td>0</td><td>2</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
</table>

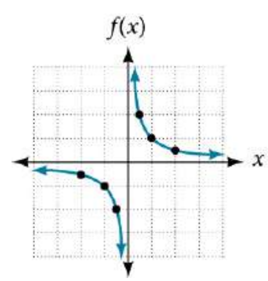
Table 13

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Identity</td><td>$f(x)=x$</td><td>

<table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–2</td><td>–2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Absolute value</td><td>$f(x)=|x|$</td><td>

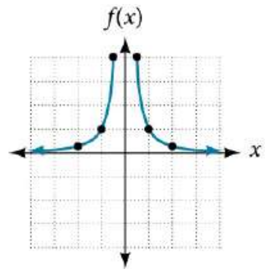
<table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Quadratic</td><td>$f(x)=x^2$</td><td>

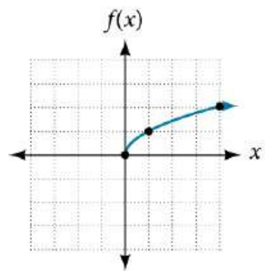
<table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–2</td><td>4</td></tr><tr><td>–1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table></td></tr>
<tr><td>Cubic</td><td>$f(x)=x^3$</td><td>

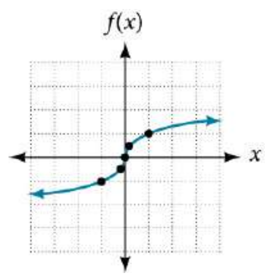
<table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–1</td><td>–1</td></tr><tr><td>–0.5</td><td>–0.125</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.5</td><td>0.125</td></tr><tr><td>1</td><td>1</td></tr></table></td></tr>
<tr><td>Reciprocal</td><td>$f(x)=\frac{1}{x}$</td><td>

<table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–2</td><td>–0.5</td></tr><tr><td>–1</td><td>–1</td></tr><tr><td>–0.5</td><td>–2</td></tr><tr><td>0.5</td><td>2</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.5</td></tr></table></td></tr>
</table>

Table 13

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Reciprocal squared</td><td>$f(x) = \frac{1}{x^2}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>–2</td><td>0.25</td></tr>
<tr><td>–1</td><td>1</td></tr>
<tr><td>–0.5</td><td>4</td></tr>
<tr><td>0.5</td><td>4</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>2</td><td>0.25</td></tr>
</table></td></tr>
<tr><td>Square root</td><td>$f(x) = \sqrt{x}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>4</td><td>2</td></tr>
</table></td></tr>
<tr><td>Cube root</td><td>$f(x) = \sqrt[3]{x}$</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>–1</td><td>–1</td></tr>
<tr><td>–0.125</td><td>–0.5</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>0.125</td><td>0.5</td></tr>
<tr><td>1</td><td>1</td></tr>
</table></td></tr>
</table>

Table 13

MEDIA

Access the following online resources for additional instruction and practice with functions.

Determine if a Relation is a Function (http://openstax.org/l/relationfunction)
Vertical Line Test (http://openstax.org/l/vertlinetest)
Introduction to Functions (http://openstax.org/l/introtofunction)
Vertical Line Test on Graph (http://openstax.org/l/vertlinegraph)
One-to-one Functions (http://openstax.org/l/onetoone)
Graphs as One-to-one Functions (http://openstax.org/l/graphonetoone)

3.1 SECTION EXERCISES

Verbal

1. What is the difference between a relation and a function?

2. What is the difference between the input and the output of a function?

3. Why does the vertical line test tell us whether the graph of a relation represents a function?

4. How can you determine if a relation is a one-to-one function?

5. Why does the horizontal line test tell us whether the graph of a function is one-to-one?

Algebraic

For the following exercises, determine whether the relation represents a function.

6. $\{(a,b),\ (c,d),\ (a,c)\}$

7. $\{(a,b),(b,c),(c,c)\}$

For the following exercises, determine whether the relation represents y as a function of x.

8. $5x+2y=10$

9. $y=x^2$

10. $x=y^2$

11. $3x^2+y=14$

12. $2x+y^2=6$

13. $y=-2x^2+40x$

14. $y=\frac{1}{x}$

15. $x=\frac{3y+5}{7y-1}$

16. $x=\sqrt{1-y^2}$

17. $y=\frac{3x+5}{7x-1}$

18. $x^2+y^2=9$

19. $2xy=1$

20. $x=y^3$

21. $y=x^3$

22. $y=\sqrt{1-x^2}$

23. $x=\pm\sqrt{1-y}$

24. $y=\pm\sqrt{1-x}$

25. $y^2=x^2$

26. $y^3=x^2$

For the following exercises, evaluate $f(-3), f(2), f(\text{-}a), \text{-}f(a), f(a+h)$.

27. $f(x)=2x-5$

28. $f(x)=-5x^2+2x-1$

29. $f(x)=\sqrt{2-x}+5$

30. $f(x)=\frac{6x-1}{5x+2}$

31. $f(x)=|x-1|-|x+1|$

32. Given the function $g(x)=5-x^2$, evaluate $\frac{g(x+h)-g(x)}{h}, h\neq 0.$

33. Given the function $g(x)=x^2+2x$, evaluate $\frac{g(x)-g(a)}{x-a}, x\neq a.$

34. Given the function $k(t)=2t-1$:
   ⓐ Evaluate $k(2)$.
   ⓑ Solve $k(t)=7$.

35. Given the function $f(x)=8-3x$:
   ⓐ Evaluate $f(-2)$.
   ⓑ Solve $f(x)=-1$.

36. Given the function $p(c)=c^2+c$:
   ⓐ Evaluate $p(-3)$.
   ⓑ Solve $p(c)=2$.

37. Given the function $f(x)=x^2-3x$:
   ⓐ Evaluate $f(5)$.
   ⓑ Solve $f(x)=4$.

38. Given the function $f(x)=\sqrt{x+2}$:
   ⓐ Evaluate $f(7)$.
   ⓑ Solve $f(x)=4$.

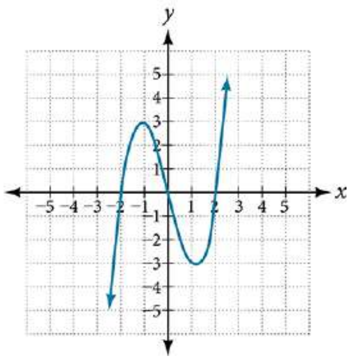
39. Consider the relationship $3r + 2t = 18$.

ⓐ Write the relationship as a function $r = f(t)$.
ⓑ Evaluate $f(-3)$.
ⓒ Solve $f(t) = 2$.

Graphical

For the following exercises, use the vertical line test to determine which graphs show relations that are functions.

40.



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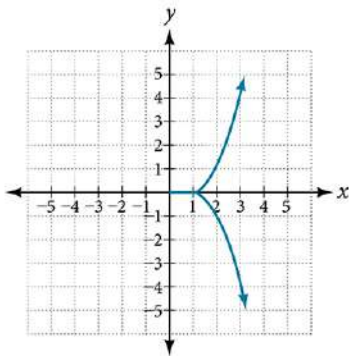

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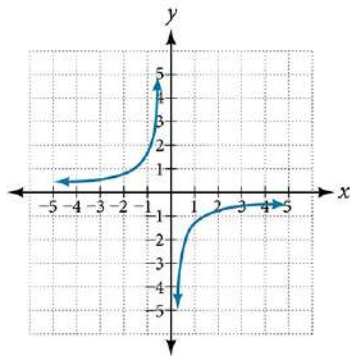

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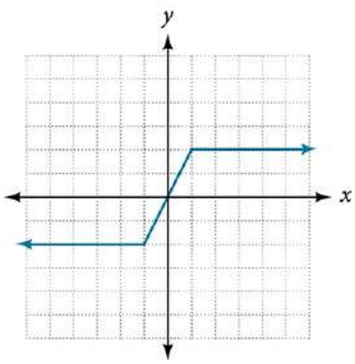

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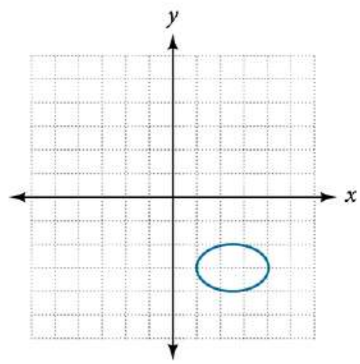

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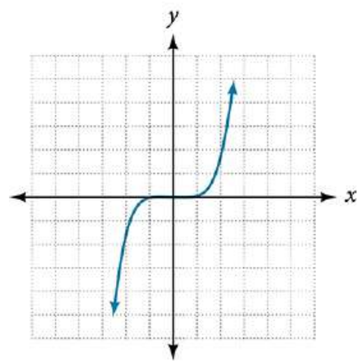

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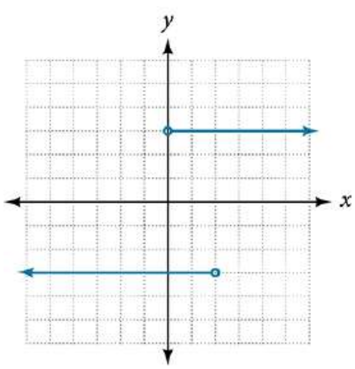

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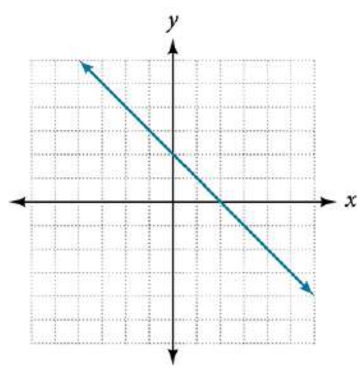

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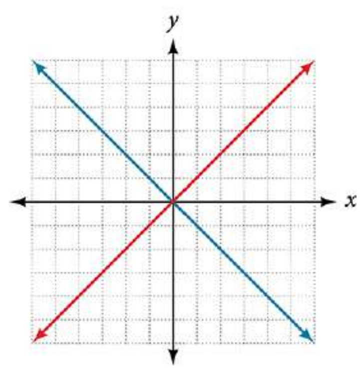

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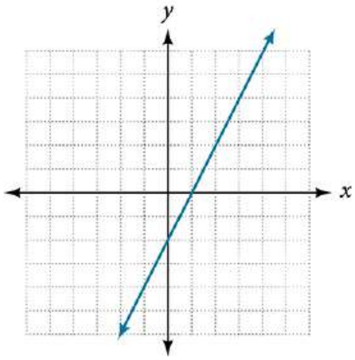

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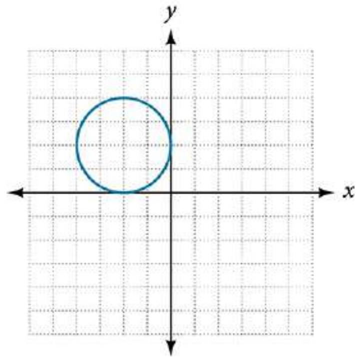

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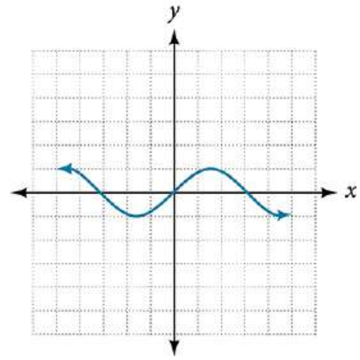
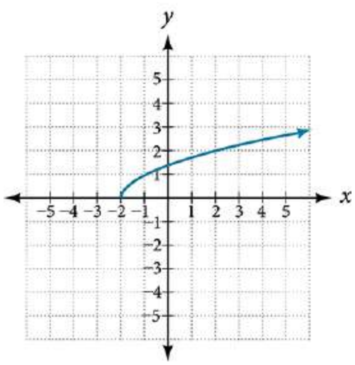


52. Given the following graph,

ⓐ Evaluate $f(-1)$.

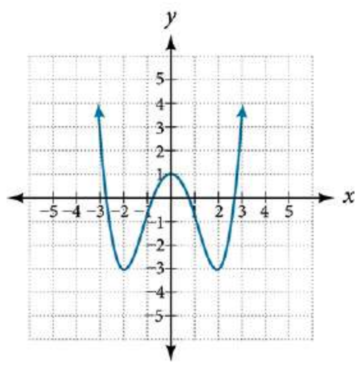
ⓑ Solve for $f(x) = 3$.



53. Given the following graph,

ⓐ Evaluate $f(0)$.

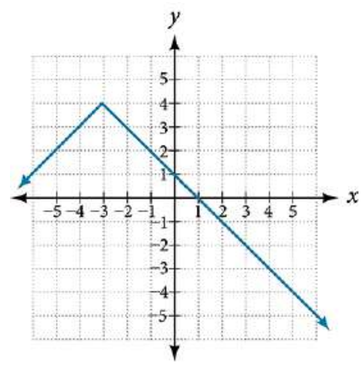
ⓑ Solve for $f(x) = -3$.



54. Given the following graph,

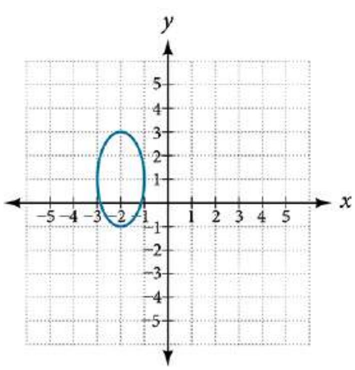
ⓐ Evaluate $f(4)$.

ⓑ Solve for $f(x) = 1$.



For the following exercises, determine if the given graph is a one-to-one function.

55.



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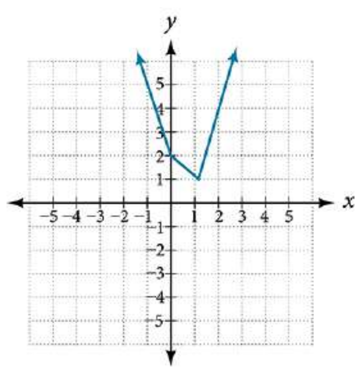

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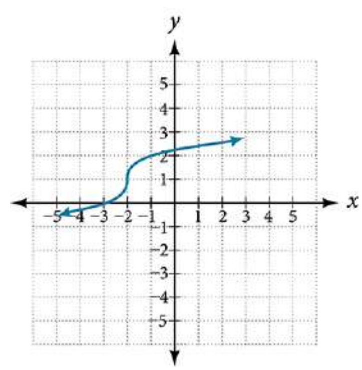

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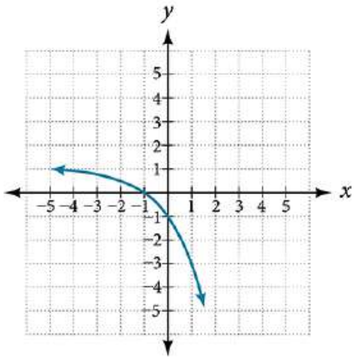

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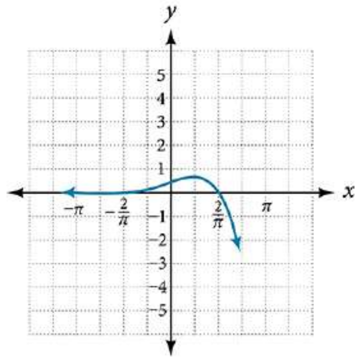


Numeric

For the following exercises, determine whether the relation represents a function.

60. $\{(-1,-1),(-2,-2),(-3,-3)\}$

61. $\{(3,4),(4,5),(5,6)\}$

62. $\{(2,5),(7,11),(15,8),(7,9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

63.

x	5	10	15
y	3	8	14

64.

x	5	10	15
y	3	8	8

65.

x	5	10	10
y	3	8	14

For the following exercises, use the function f represented in the table below.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

Table 14

66. Evaluate $f(3)$.

67. Solve $f(x)=1$.

For the following exercises, evaluate the function f at the values $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.

68. $f(x)=4-2x$

69. $f(x)=8-3x$

70. $f(x)=8x^2-7x+3$

71. $f(x)=3+\sqrt{x+3}$

72. $f(x)=\frac{x-2}{x+3}$

73. $f(x)=3^x$

For the following exercises, evaluate the expressions, given functions f, g, and h:

$f(x)=3x-2$
$g(x)=5-x^2$
$h(x)=-2x^2+3x-1$

74. $3f(1)-4g(-2)$

75. $f\left(\frac{7}{3}\right)-h(-2)$

Technology

For the following exercises, graph $y = x^2$ on the given domain. Determine the corresponding range. Show each graph.

76. $[-0.1,\ \ 0.1]$

77. $[-10,\ \ 10]$

78. $[-100, 100]$

For the following exercises, graph $y = x^3$ on the given domain. Determine the corresponding range. Show each graph.

79. $[-0.1,\ \ 0.1]$

80. $[-10,\ \ 10]$

81. $[-100,\ \ 100]$

For the following exercises, graph $y = \sqrt{x}$ on the given domain. Determine the corresponding range. Show each graph.

82. $[0,\ \ 0.01]$

83. $[0,\ \ 100]$

84. $[0,\ \ 10{,}000]$

For the following exercises, graph $y = \sqrt[3]{x}$ on the given domain. Determine the corresponding range. Show each graph.

85. $[-0.001, 0.001]$

86. $[-1000, 1000]$

87. $[-1{,}000{,}000, 1{,}000{,}000]$

Real-World Applications

88. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.

ⓐ The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
ⓑ Explain the meaning of the statement $f(5) = 2$.

89. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.

ⓐ A garden with area 5000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function g.
ⓑ Explain the meaning of the statement $g(100) = 1$.

90. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

ⓐ $f(5) = 30$
ⓑ $f(10) = 40$

91. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:

ⓐ $h(1) = 200$
ⓑ $h(2) = 350$

92. Show that the function $f(x) = 3(x-5)^2 + 7$ is <u>not</u> one-to-one.

3.2 Domain and Range

Learning Objectives

In this section, you will:

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

COREQUISITE SKILLS

Learning Objectives

- Find the domain and range of a function (IA 3.5.1)

A **relation** is any set of ordered pairs, (x,y). A special type of relation, called a function, is studied extensively in mathematics. A **function** is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each x-value is matched with only one y-value.

When studying functions, it's important to be able to identify potential input values, called the **domain**, and potential output values, called the **range**.

EXAMPLE 1

Find the domain of the following function: $\{(2, 10),(3, 10),(4, 20),(5, 30),(6, 40)\}$.

Solution

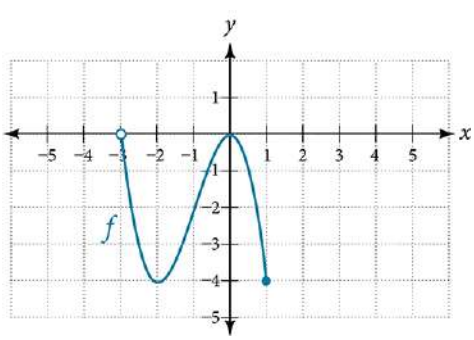
First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.
D: $\{2,3,4,5,6\}$

Notice here we are using set notation to represent this collection of input values.

A graph of a function can always help in identifying domain and range. When **graphing** basic functions, we can scan the x-axis just as we read in English from left to right to help determine the domain. We will scan the y-axis from bottom to top to help determine the range. So, in finding both domain and range, we scan axes from smallest to largest to see which values are defined. Typically, we will use **interval notation**, where you show the endpoints of defined sets using parentheses (endpoint not included) or brackets (endpoint is included) to express both the domain, D, and range, R, of a relation or function.

EXAMPLE 2

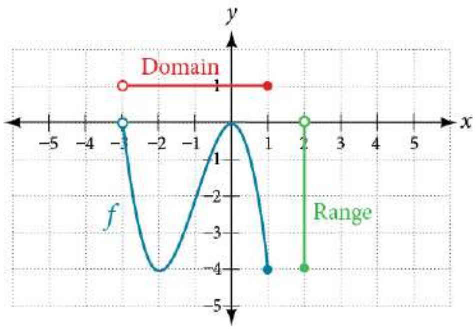
Find the domain and range from graphs
Find the domain and range of the function f whose graph is shown below.



Solution

Scanning the x-axis from left to right helps us to see the graph is defined for x-values between −3 to 1, so the domain of f is $(-3,1]$. (Note that open points translate to use of parentheses in interval notation, while included points translate to use of brackets in interval notation.)

Scanning the y-values from the bottom to top of the graph helps us to see the graph is defined for y-values between 0 to –4, so the range is [–4,0).



When working with functions expressed as an equation, the following steps can help to identify the domain.

HOW TO

Given a function written in equation form, find the domain.
Step 1. Identify the input values.
Step 2. Identify any restrictions on the input and exclude those values from the domain.
Step 3. Write the domain in interval notation form, if possible.

EXAMPLE 3

Find the domain and range from equations
Find the domain of the function $f\left(x\right) = x^2 - 3$

Solution
The input value, shown by the variable *x* in the equation, is squared and then the result is lowered by three. Any real number may be squared and then be lowered by three, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval notation form, the domain of *f* is (−∞,∞).

Activity: Restrictions on the domain of functions

Without a calculator, complete the following:

$\frac{6}{0} = \_\_\_\_$; $\frac{0}{0} = \_\_\_\_$; $\frac{0}{6} = \_\_\_\_$; $\sqrt{4} = \_\_\_\_$; $\sqrt{-4} = \_\_\_\_$; $\sqrt[3]{8} = \_\_\_\_$; $\sqrt[3]{-8} = \_\_\_\_$

Clearly describe the two “trouble spots” which prevent expressions from representing real numbers:

1. ____
2. ____
3. Keeping these trouble spots in mind, algebraically determine the domain of each function. Write each answer in interval notation below the function. Remember, looking at the graph of the function can always help in finding domain and range.

$f\left(x\right) = \frac{x-5}{x+3}$	$g\left(x\right) = 5\sqrt{x+7}$	$h\left(x\right) = -3\sqrt{5x-17}$	$f\left(x\right) = \frac{x^2-9}{x^2+7x+10}$
D:_______;	D:_______;	D:_______;	D:_______;

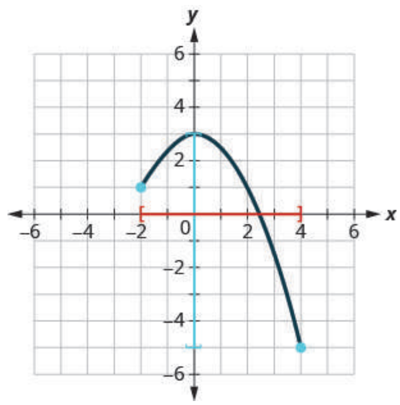
Practice Makes Perfect

Find the domain and range of a function.

1. For the relation {(1,3),(2,6),(3,9),(4,12),(5,15)}:

ⓐ Find the domain of the function, express using set notation.
ⓑ Find the range of the function, express using set notation.

2. Use the graph of the function to find its domain and range. Write the domain and range in interval notation.

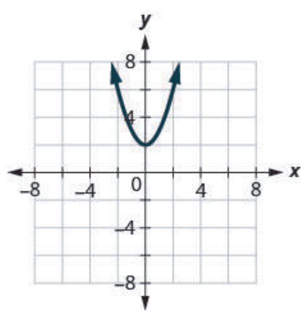


D: ________ R: ________

3. Graph the following function below. Use this graph to help determine the domain and range and express using interval notation. $f(x) = -4x - 3$

D: _______ R: ________

4. Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



D: _______ R: ________

5. Graph the following function. Use this graph to help determine the domain and range and express using interval notation. $f(x) = -2|x| + 3$

D: _______ R: ________

6. Graph the following function. Use this graph to help determine the domain and range and express using interval notation. $f\left(x\right) = \sqrt{x - 3}$

D: ________ R: _________

7. Graph the following function. Use this graph to help determine the domain and range and express using interval notation. $f\left(x\right) = \sqrt[3]{x+4}$

D: ________ R: _________

8. Graph the following function. Use this graph to help determine the domain and range and express using interval notation. $f\left(x\right) = \frac{(x-3)}{(2x+1)}$

D: ________ R: _________

Horror and thriller movies are both popular and, very often, extremely profitable. When big-budget actors, shooting locations, and special effects are included, however, studios count on even more viewership to be successful. Consider five major thriller/horror entries from the early 2000s—*I am Legend*, *Hannibal*, *The Ring*, *The Grudge*, and *The Conjuring*. Figure 1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

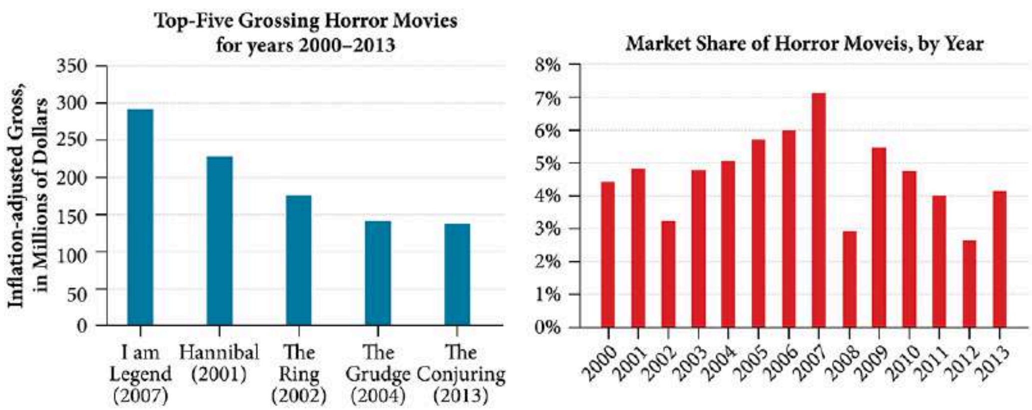


Figure 1 Based on data compiled by www.the-numbers.com.[3]

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of domain and range. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a “holding area” that contains “raw materials” for a “function machine” and the range as another “holding area” for the machine’s products. See Figure 2.

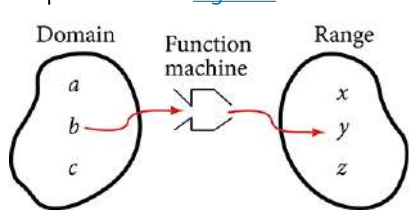


Figure 2

We can write the domain and range in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to

3 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014

indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has \$100 to spend, they would need to express the interval that is more than 0 and less than or equal to 100 and write $(0,\ \ 100]$. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an odd root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest number from the interval is written first.
- The largest number in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint value is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint value is included, called inclusive.

See Figure 3 for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure 3

EXAMPLE 1

Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: $\{(2, \ \ 10), (3, \ \ 10), (4, \ \ 20), (5, \ \ 30), (6, \ \ 40)\}$.

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

TRY IT #1 Find the domain of the function:

$\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$

HOW TO

Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

EXAMPLE 2

Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $\left(-\infty, \infty\right)$.

TRY IT #2 Find the domain of the function: $f(x) = 5 - x + x^3$.

HOW TO

Given a function written in an equation form that includes a fraction, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

EXAMPLE 3

Finding the Domain of a Function Involving a Denominator

Find the domain of the function $f(x) = \frac{x+1}{2-x}$.

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2 - x &= 0 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$ as shown in Figure 4. We can use a symbol known as the union, ∪, to combine the two sets. In interval notation, we write the solution: $(-\infty, 2) \cup \left(2, \infty\right)$.

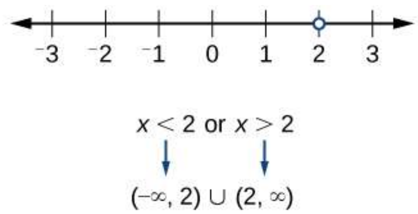


Figure 4

TRY IT #3 Find the domain of the function: $f(x) = \frac{1+4x}{2x-1}$.

HOW TO

Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

EXAMPLE 4

Finding the Domain of a Function with an Even Root

Find the domain of the function $f(x) = \sqrt{7 - x}$.

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7 - x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

TRY IT #4 Find the domain of the function $f(x) = \sqrt{5 + 2x}$.

Q&A **Can there be functions in which the domain and range do not intersect at all?**

Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in **set-builder notation**. For example, $\{x|10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces $\{\}$ are read as "the set of," and the vertical bar $|$ is read as "such that," so we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure 5 compares inequality notation, set-builder notation, and interval notation.

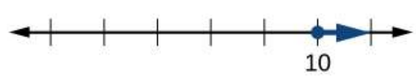
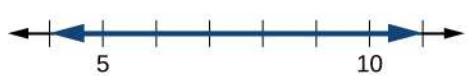
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 5

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$, to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one *or* the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x| \quad |x| \geq 3\} = \left(-\infty, -3\right] \cup \left[3, \infty\right)$$

Set-Builder Notation and Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x|\ \text{statement about } x\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x|4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

HOW TO

Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol ∪ to combine all intervals into one set.

EXAMPLE 5

Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure 6 using inequality notation, set-builder notation, and interval notation.

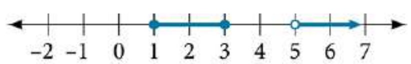


Figure 6

Solution

To describe the values, x, included in the intervals shown, we would say, " x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality	$1 \leq x \leq 3 \ \text{or}\ x > 5$
Set-builder notation	$\{x|1 \leq x \leq 3 \ \text{or}\ x > 5\}$
Interval notation	$[1, 3] \cup (5, \infty)$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

TRY IT #5 Given Figure 7, specify the graphed set in

ⓐ words ⓑ set-builder notation ⓒ interval notation

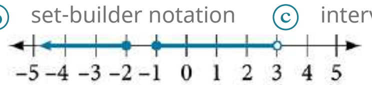


Figure 7

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the *x*-axis. The range is the set of possible output values, which are shown on the *y*-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure 8.

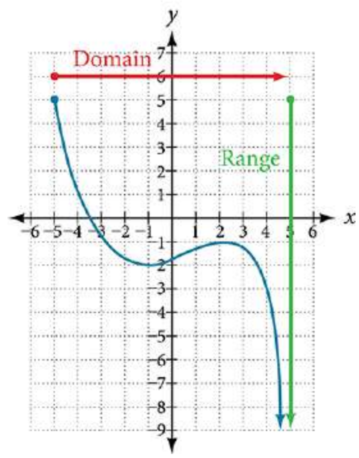


Figure 8

We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $\left[-5, \infty\right)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

EXAMPLE 6

Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 9.

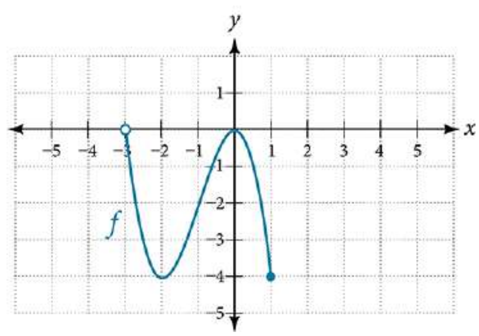


Figure 9

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4, so the range is $[-4, 0)$. See Figure 10.

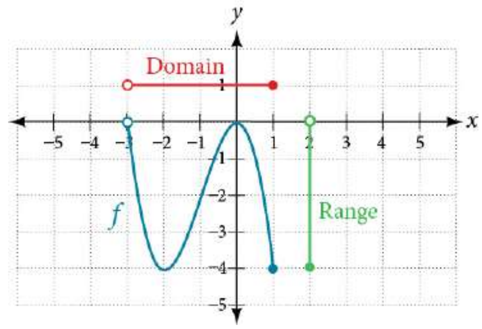


Figure 10

EXAMPLE 7

Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure 11.

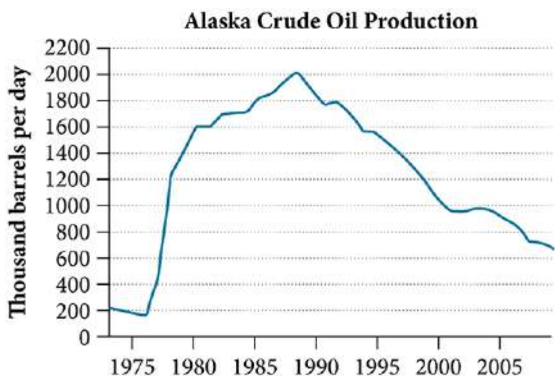


Figure 11 (credit: modification of work by the U.S. Energy Information Administration)[4]

Solution

The input quantity along the horizontal axis is "years," which we represent with the variable t for time. The output quantity is "thousands of barrels of oil per day," which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is [1973, 2008], and the range is about [180, 2010]. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

TRY IT #6 Given Figure 12, identify the domain and range using interval notation.

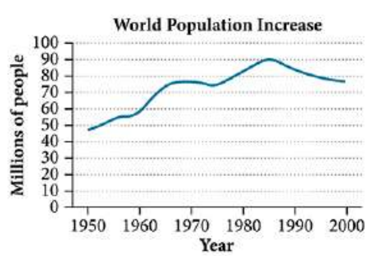


Figure 12

Q&A **Can a function's domain and range be the same?**

4 http://www.eia.gov/dnav/pet/hist/LeafHandler.ashx?n=PET&s=MCRFPAK2&f=A.

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.

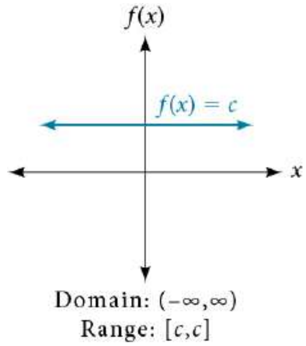


Figure 13 For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.

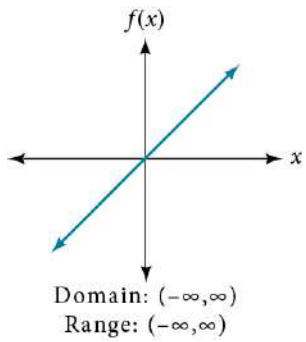


Figure 14 For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.

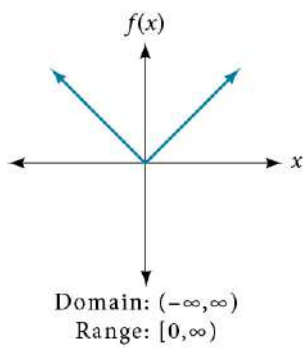


Figure 15 For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

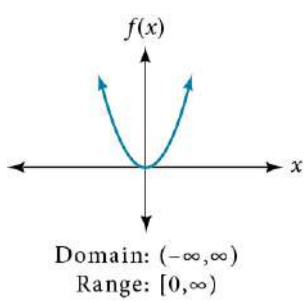


Figure 16 For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the

graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.

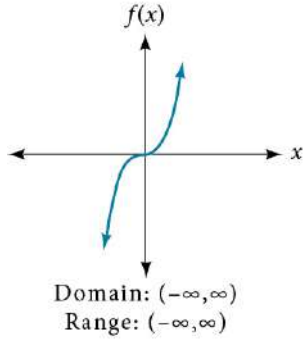


Figure 17 For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.

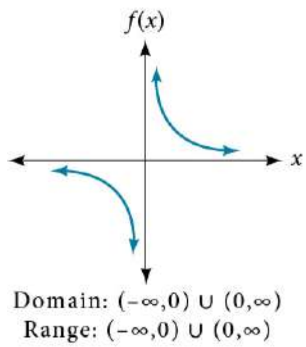


Figure 18 For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x|\ \ x \neq 0\}$, the set of all real numbers that are not zero.

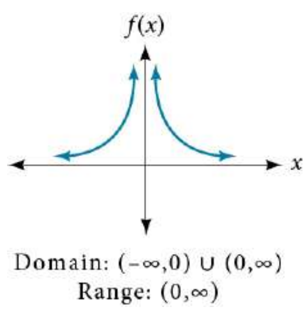


Figure 19 For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$, we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.

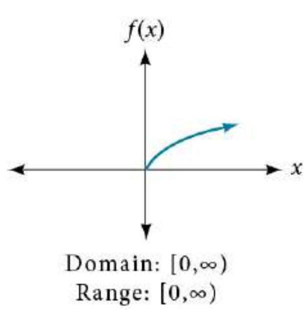


Figure 20 For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x

is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.

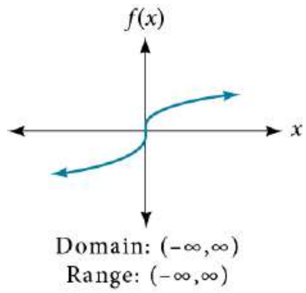


Figure 21 For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

HOW TO

Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

EXAMPLE 8

Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution

There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $\left(-\infty, \infty\right)$ and the range is also $\left(-\infty, \infty\right)$.

EXAMPLE 9

Finding the Domain and Range

Find the domain and range of $f(x) = \frac{2}{x+1}$.

Solution

We cannot evaluate the function at -1 because division by zero is undefined. The domain is $\left(-\infty, -1\right) \cup \left(-1, \infty\right)$. Because the function is never zero, we exclude 0 from the range. The range is $\left(-\infty, 0\right) \cup \left(0, \infty\right)$.

EXAMPLE 10

Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$$x + 4 \geq 0 \text{ when } x \geq -4$$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $\left[0, \infty\right)$.

Analysis
Figure 22 represents the function f.

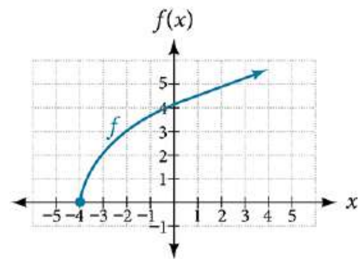


Figure 22

TRY IT #7 Find the domain and range of $f(x) = -\sqrt{2 - x}$.

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, absolute value can be defined as the magnitude, or modulus, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a piecewise function. A **piecewise function** is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10{,}000$ and $\$1000 + 0.2(S - \$10{,}000)$ if $S > \$10{,}000$.

Piecewise Function

A **piecewise function** is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1} & \text{if } x \text{ is in domain 1} \\ \text{formula 2} & \text{if } x \text{ is in domain 2} \\ \text{formula 3} & \text{if } x \text{ is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

HOW TO

Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

EXAMPLE 11

Writing a Piecewise Function

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people or a fixed \$50 fee for a group of 10 or more people. Write a function relating the number of people, n, to the cost, C.

Solution

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } 0 < n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Analysis

The function is represented in Figure 23. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

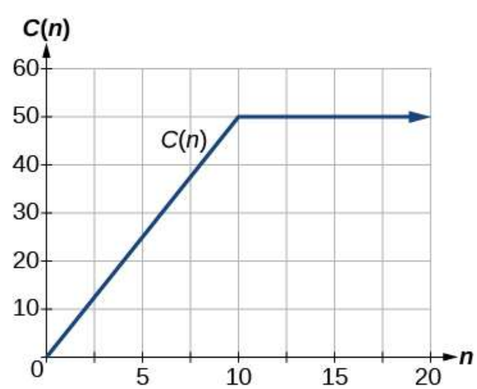


Figure 23

EXAMPLE 12

Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g - 2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Solution

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in.

Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4 - 2) = \$45$$

Analysis

The function is represented in Figure 24. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

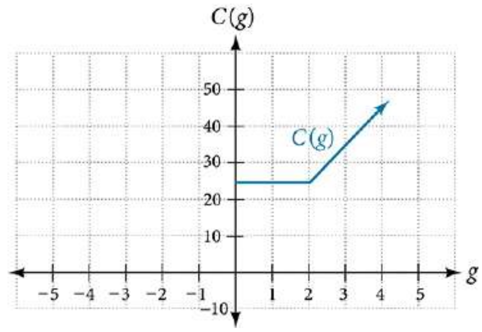


Figure 24

HOW TO

Given a piecewise function, sketch a graph.

1. Indicate on the *x*-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

EXAMPLE 13

Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \leq 1 \\ 3 & \text{if} \quad 1 < x \leq 2 \\ x & \text{if} \quad x > 2 \end{cases}$$

Solution

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 25 shows the three components of the piecewise function graphed on separate coordinate systems.

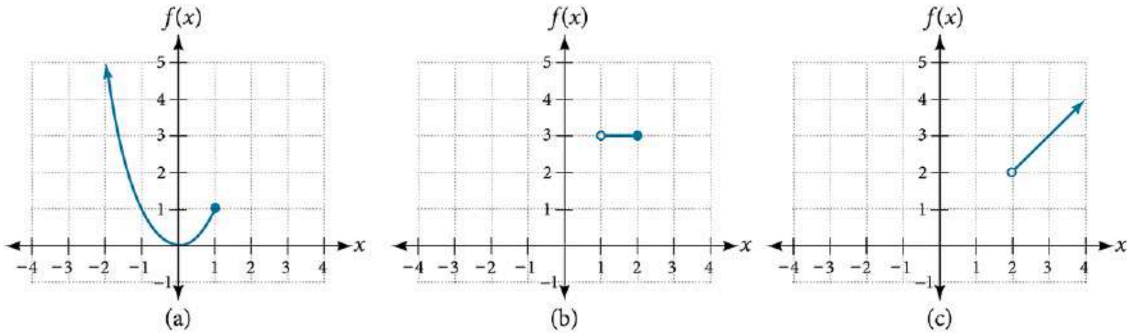


Figure 25 (a) $f(x) = x^2$ if $x \le 1$; (b) $f(x) = 3$ if $1 < x \le 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure 26.

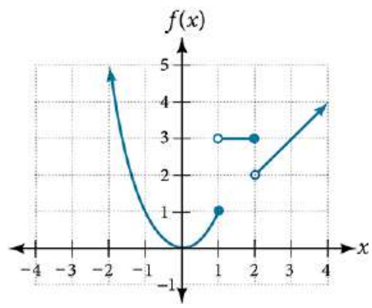


Figure 26

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, \ 3)$ are.

TRY IT #8 Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if} \quad x < -1 \\ -2 & \text{if} \quad -1 < x < 4 \\ \sqrt{x} & \text{if} \quad x > 4 \end{cases}$$

Q&A **Can more than one formula from a piecewise function be applied to a value in the domain?**

No. Each value corresponds to one equation in a piecewise formula.

MEDIA

Access these online resources for additional instruction and practice with domain and range.

Domain and Range of Square Root Functions (http://openstax.org/l/domainsqroot)
Determining Domain and Range (http://openstax.org/l/determinedomain)
Find Domain and Range Given the Graph (http://openstax.org/l/drgraph)
Find Domain and Range Given a Table (http://openstax.org/l/drtable)
Find Domain and Range Given Points on a Coordinate Plane (http://openstax.org/l/drcoordinate)

3.2 SECTION EXERCISES

Verbal

1. Why does the domain differ for different functions?

2. How do we determine the domain of a function defined by an equation?

3. Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.

4. When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?

5. How do you graph a piecewise function?

Algebraic

For the following exercises, find the domain of each function using interval notation.

6. $f(x) = -2x(x-1)(x-2)$

7. $f(x) = 5 - 2x^2$

8. $f(x) = 3\sqrt{x-2}$

9. $f(x) = 3 - \sqrt{6-2x}$

10. $f(x) = \sqrt{4-3x}$

11. $f(x) = \sqrt{x^2+4}$

12. $f(x) = \sqrt[3]{1-2x}$

13. $f(x) = \sqrt[3]{x-1}$

14. $f(x) = \frac{9}{x-6}$

15. $f(x) = \frac{3x+1}{4x+2}$

16. $f(x) = \frac{\sqrt{x+4}}{x-4}$

17. $f(x) = \frac{x-3}{x^2+9x-22}$

18. $f(x) = \frac{1}{x^2-x-6}$

19. $f(x) = \frac{2x^3-250}{x^2-2x-15}$

20. $\frac{5}{\sqrt{x-3}}$

21. $\frac{2x+1}{\sqrt{5-x}}$

22. $f(x) = \frac{\sqrt{x-4}}{\sqrt{x-6}}$

23. $f(x) = \frac{\sqrt{x-6}}{\sqrt{x-4}}$

24. $f(x) = \frac{x}{x}$

25. $f(x) = \frac{x^2-9x}{x^2-81}$

26. Find the domain of the function $f(x) = \sqrt{2x^3 - 50x}$ by:

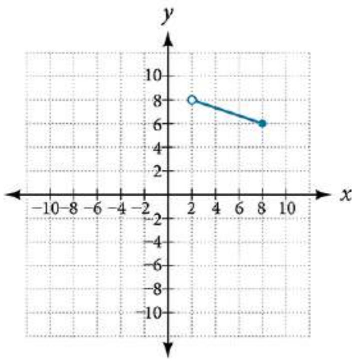
    a. using algebra.

    b. graphing the function in the radicand and determining intervals on the *x*-axis for which the radicand is nonnegative.

Graphical

For the following exercises, write the domain and range of each function using interval notation.

27.



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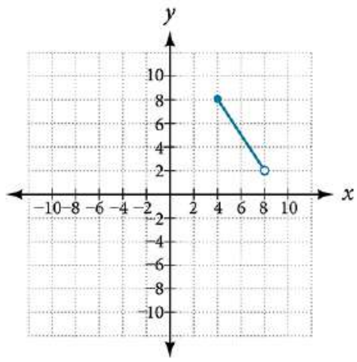

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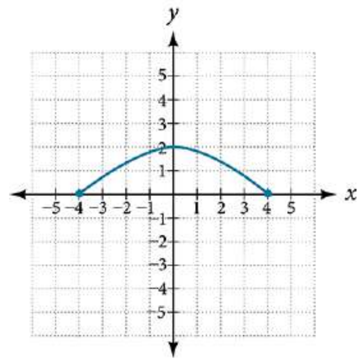

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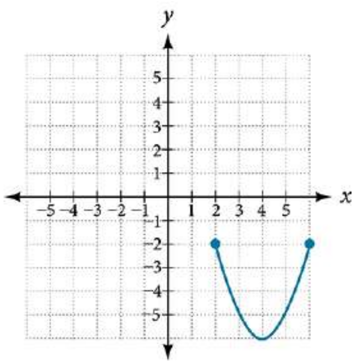

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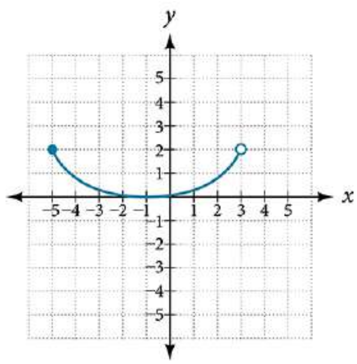

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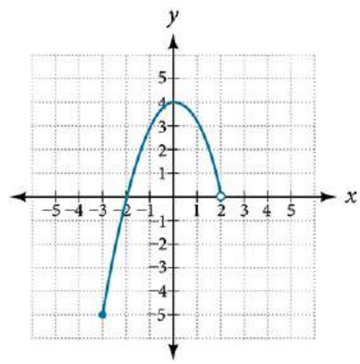

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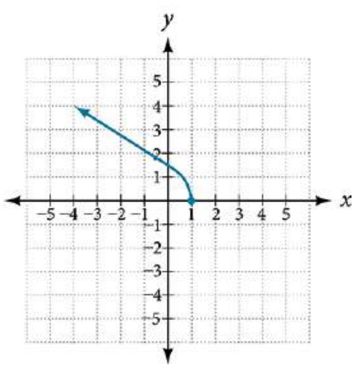

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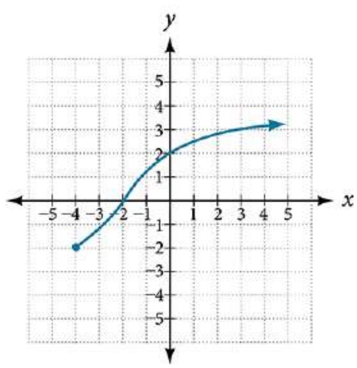

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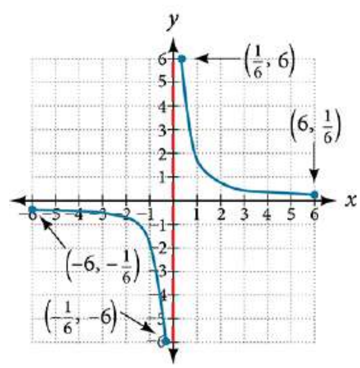

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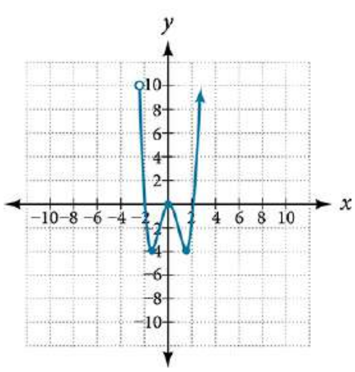
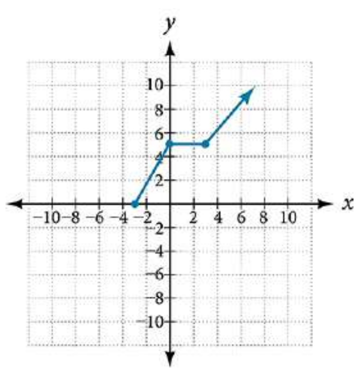


37.



For the following exercises, sketch a graph of the piecewise function. Write the domain in interval notation.

38. $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

39. $f(x) = \begin{cases} 2x-1 & \text{if } x < 1 \\ 1+x & \text{if } x \geq 1 \end{cases}$

40. $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x-1 & \text{if } x > 0 \end{cases}$

41. $f(x) = \begin{cases} 3 & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$

42. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1-x & \text{if } x > 0 \end{cases}$

43. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x+2 & \text{if } x \geq 0 \end{cases}$

44. $f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ x^3 & \text{if } x \geq 1 \end{cases}$

45. $f(x) = \begin{cases} |x| & \text{if } x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$

Numeric

For the following exercises, given each function f, evaluate $f(-3)$, $f(-2)$, $f(-1)$, and $f(0)$.

46. $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

47. $f(x) = \begin{cases} 1 & \text{if } x \leq -3 \\ 0 & \text{if } x > -3 \end{cases}$

48. $f(x) = \begin{cases} -2x^2+3 & \text{if } x \leq -1 \\ 5x-7 & \text{if } x > -1 \end{cases}$

For the following exercises, given each function f, evaluate $f(-1)$, $f(0)$, $f(2)$, and $f(4)$.

49. $f(x) = \begin{cases} 7x+3 & \text{if } x < 0 \\ 7x+6 & \text{if } x \geq 0 \end{cases}$

50. $f(x) = \begin{cases} x^2-2 & \text{if } x < 2 \\ 4+|x-5| & \text{if } x \geq 2 \end{cases}$

51. $f(x) = \begin{cases} 5x & \text{if } x < 0 \\ 3 & \text{if } 0 \leq x \leq 3 \\ x^2 & \text{if } x > 3 \end{cases}$

For the following exercises, write the domain for the piecewise function in interval notation.

52. $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

53. $f(x) = \begin{cases} x^2-2 & \text{if } x < 1 \\ -x^2+2 & \text{if } x > 1 \end{cases}$

54. $f(x) = \begin{cases} 2x-3 & \text{if } x < 0 \\ -3x^2 & \text{if } x \geq 2 \end{cases}$

Technology

55. Graph $y = \frac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

56. Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, \ 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Extension

57. Suppose the range of a function f is $[-5, \ 8]$. What is the range of $|f(x)|$?

58. Create a function in which the range is all nonnegative real numbers.

59. Create a function in which the domain is $x > 2$.

Real-World Applications

60. The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

61. The cost in dollars of making x items is given by the function $C(x) = 10x + 500$.

ⓐ The fixed cost is determined when zero items are produced. Find the fixed cost for this item.
ⓑ What is the cost of making 25 items?
ⓒ Suppose the maximum cost allowed is $1500. What are the domain and range of the cost function, $C(x)$?

3.3 Rates of Change and Behavior of Graphs

Learning Objectives

In this section, you will:

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

COREQUISITE SKILLS

Learning Objectives

- Find the slope of a line (IA 3.2.1)

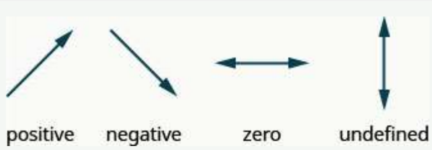
Objective 1: Find the slope of a line (IA 3.2.1)

In our work with functions we will make observations about when the function increases or decreases and how quickly this change takes place. The **average rate of change** is a measure of change of a function and tells us how an output quantity, or y value, changes relative to an input quantity, or x value. Finding the average rate of change between two points is equivalent to finding the **slope** of the line segment connecting these two data points.

$$\text{average rate of change} = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

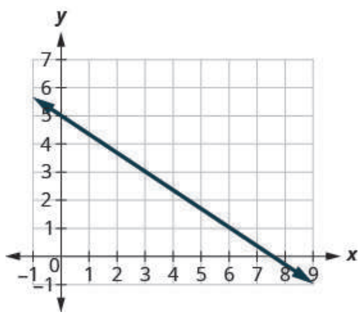
When interpreting an average rate of change it will be important to consider the units of measurement. Make sure to always attach these units to both the numerator and denominator when they are provided to you.

Quick Guide to the Slopes of Lines



EXAMPLE 1

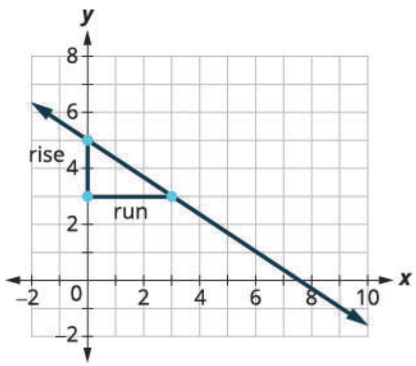
Find the slope of the line shown.



Solution

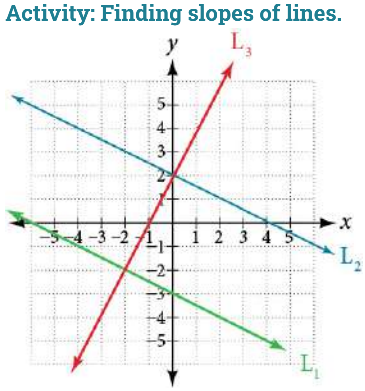
Locate two points on the graph whose coordinates are integers.

$(0, 5) = (x_1, y_1)$
$(3, 3) = (x_2, y_2)$



It may help to visualize this change as $m = \frac{\text{rise}}{\text{run}}$. Count the rise between the points: down 2 units. Then count the run, or horizontal change: to the right 3 units. Note, since the line goes down, the slope is negative. Use the slope formula:
$m = \frac{\text{rise}}{\text{run}} = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{3-5}{3-0} \frac{-2}{3}$

Activity: Finding slopes of lines.



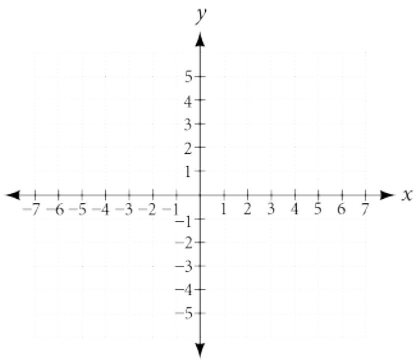
ⓐ Which pair of lines appear parallel? _______ and _______. Find their slopes: _______; ________
ⓑ Which pair of lines appear perpendicular? _______ and _______. Find their slopes: _______; ________
ⓒ Complete the following: Two lines are parallel if their slopes are ________. Two lines are perpendicular if their slopes are ________.

Practice Makes Perfect

Find the slope of a line and the average rate of change.

1. Complete this table's y-values, and then graph the line. $y = \frac{2}{3}x + 1$

x	y
-6	
-3	
0	
3	
6	



What is the slope of this line? _______ What is the y-intercept of this line? (_______ , _______)

2. The following table shows the number of Associates degrees awarded (in thousands) in the US for several years.

Source: U.S. National Center for Education Statistics.

Year	Number of Associate’s Degrees Earned (in thousands)
2000	569
2001	579
2005	668
2010	719
2014	1,003

Find the following average rates of change, being careful to attach units to your answers.

ⓐ Between 2005 and 2000. Average rate of change= $\frac{f(x_2)-f(x_1)}{x_2-x_1}$

ⓑ Between 2001 and 2010. Average rate of change= $\frac{f(x_2)-f(x_1)}{x_2-x_1}$

ⓒ Between 2014 and 2010. Average rate of change= $\frac{f(x_2)-f(x_1)}{x_2-x_1}$

3. ⓐ Complete the following table of values for $f(x) = x^2 + 2x - 8$

x	f(x)
−2	
2	
4	

ⓑ Use the table above to find the average rate of change between x=–2 and x=2.

ⓒ Use the table above to find the average rate of change between x=2 and x=4

ⓓ Sketch the graph of $f(x)$ below and graph the lines with slopes equal to the average rates of change found in parts ⓑ and ⓒ .

4. ⓐ Complete the following table of values for $g\left(x\right) = \frac{1}{x+3}$

x	g(x)
−1	
0	
2	

ⓑ Use the table above to find the average rate of change between x=–1 and x=0.
ⓒ Use the table above to find the average rate of change between x=–1 and x=2.
ⓓ Sketch the graph of $g(x)$ below and graph the lines with slopes equal to the average rates of change found in parts ⓑ and ⓒ.

Gasoline costs have experienced some wild fluctuations over the last several decades. Table 1[5] lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

y	2005	2006	2007	2008	2009	2010	2011	2012
$C\left(y\right)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

Table 1

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from \$2.31 to \$3.68, an increase of \$1.37. While this is interesting, it might be more useful to look at how much the price changed *per year*. In this section, we will investigate changes such as these.

Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned} \text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \end{aligned}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as "delta-y over delta-x" or "the change in y divided by the change in x." Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

5 http://www.eia.gov/totalenergy/data/annual/showtext.cfm?t=ptb0524. Accessed 3/5/2014.

In our example, the gasoline price increased by \$1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7 \text{ years}} \approx 0.196 \text{ dollars per year}$$

On average, the price of gas increased by about 19.6¢ each year.

Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by \$4,000 per quarter

Rate of Change

A rate of change describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

HOW TO

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values x_1 and x_2.

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

EXAMPLE 1

Computing an Average Rate of Change

Using the data in Table 1, find the average rate of change of the price of gasoline between 2007 and 2009.

Solution

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} \\ &= \frac{-\$0.43}{2 \text{ years}} \\ &= -\$0.22 \text{ per year}\end{aligned}$$

Analysis

Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

TRY IT #1 Using the data in Table 1, find the average rate of change between 2005 and 2010.

EXAMPLE 2

Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in Figure 1, find the average rate of change on the interval $[-1, 2]$.

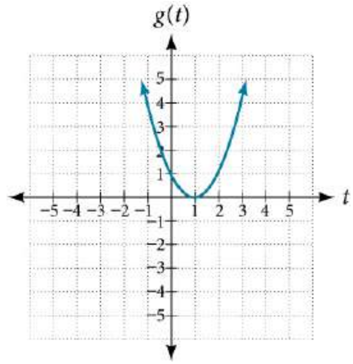


Figure 1

Solution

At $t = -1$, Figure 2 shows $g(-1) = 4$. At $t = 2$, the graph shows $g(2) = 1$.

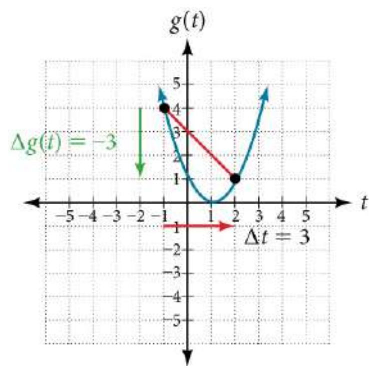


Figure 2

The horizontal change $\Delta t = 3$ is shown by the red arrow, and the vertical change $\Delta g(t) = -3$ is shown by the turquoise arrow. The average rate of change is shown by the slope of the orange line segment. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)} = \frac{-3}{3} = -1$$

Analysis

Note that the order we choose is very important. If, for example, we use $\frac{y_2-y_1}{x_1-x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1, y_1) and (x_2, y_2).

EXAMPLE 3

Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away and leaving on a trip, Anna records her distance from home over time. The values are shown in Table 2. Find her average speed over the first 6 hours.

***t* (hours)**	0	1	2	3	4	5	6	7

Table 2

***D*(*t*) (miles)**	10	55	90	153	214	240	292	300

Table 2

Solution

Here, the average speed is the average rate of change. She traveled 282 miles in 6 hours.

$$\begin{aligned} \frac{292-10}{6-0} &= \frac{282}{6} \\ &= 47 \end{aligned}$$

The average speed is 47 miles per hour.

Analysis

Because the speed is not constant, the average speed depends on the interval chosen. For the interval [2,3], the average speed is 63 miles per hour.

EXAMPLE 4

Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval $[2,\ 4]$.

Solution

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2 - \tfrac{1}{2} & f(4) &= 4^2 - \tfrac{1}{4} \\ &= 4 - \tfrac{1}{2} & &= 16 - \tfrac{1}{4} \\ &= \tfrac{7}{2} & &= \tfrac{63}{4} \end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned} \text{Average rate of change} &= \frac{f(4)-f(2)}{4-2} \\ &= \frac{\frac{63}{4}-\frac{7}{2}}{4-2} \\ &= \frac{\frac{49}{4}}{2} \\ &= \frac{49}{8} \end{aligned}$$

TRY IT #2 Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval $[1,\ 9]$.

EXAMPLE 5

Finding the Average Rate of Change of a Force

The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Solution

We are computing the average rate of change of $F(d) = \frac{2}{d^2}$ on the interval $[2, 6]$.

$$\begin{aligned}
\text{Average rate of change} &= \frac{F(6)-F(2)}{6-2} & \\
&= \frac{\frac{2}{6^2}-\frac{2}{2^2}}{6-2} & \text{Simplify.} \\
&= \frac{\frac{2}{36}-\frac{2}{4}}{4} & \\
&= \frac{-\frac{16}{36}}{4} & \text{Combine numerator terms.} \\
&= -\frac{1}{9} & \text{Simplify}
\end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

EXAMPLE 6

Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, \ a]$. The answer will be an expression involving a in simplest form.

Solution

We use the average rate of change formula.

$$\begin{aligned}
\text{Average rate of change} &= \frac{g(a)-g(0)}{a-0} & \text{Evaluate.} \\
&= \frac{\left(a^2+3a+1\right)-\left(0^2+3\left(0\right)+1\right)}{a-0} & \text{Simplify.} \\
&= \frac{a^2+3a+1-1}{a} & \text{Simplify and factor.} \\
&= \frac{a(a+3)}{a} & \text{Divide by the common factor } a. \\
&= a+3 &
\end{aligned}$$

This result tells us the average rate of change in terms of a between $t = 0$ and any other point $t = a$. For example, on the interval $[0, 5]$, the average rate of change would be $5 + 3 = 8$.

TRY IT #3 Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$ in simplest forms in terms of a.

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. Figure 3 shows examples of increasing and decreasing intervals on a function.

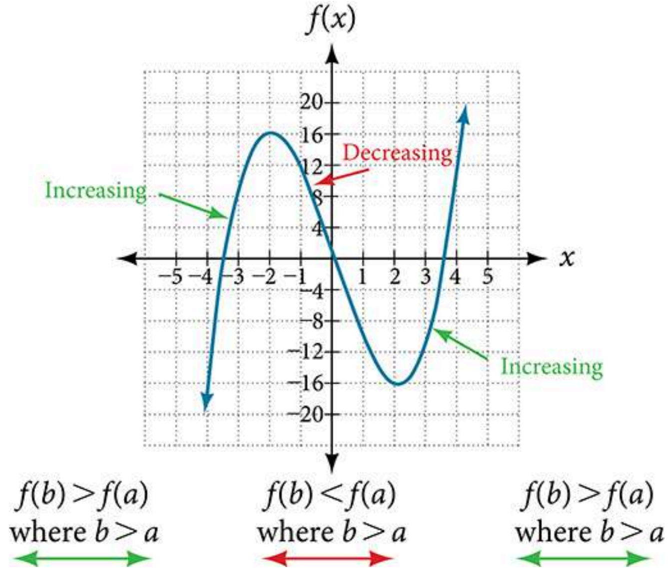


Figure 3 The function $f(x) = x^3 - 12x$ is increasing on $\left(-\infty,\ -2\right) \cup \left(2,\ \infty\right)$ and is decreasing on $(-2,\ 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is the location of a **local maximum**. The function value at that point is the local maximum. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is the location of a **local minimum**. The function value at that point is the local minimum. The plural form is "local minima." Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is "extremum.") Often, the term *local* is replaced by the term *relative*. In this text, we will use the term *local*.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of *local* extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in Figure 4, the local maximum is 16, and it occurs at $x = -2$. The local minimum is -16 and it occurs at $x = 2$.

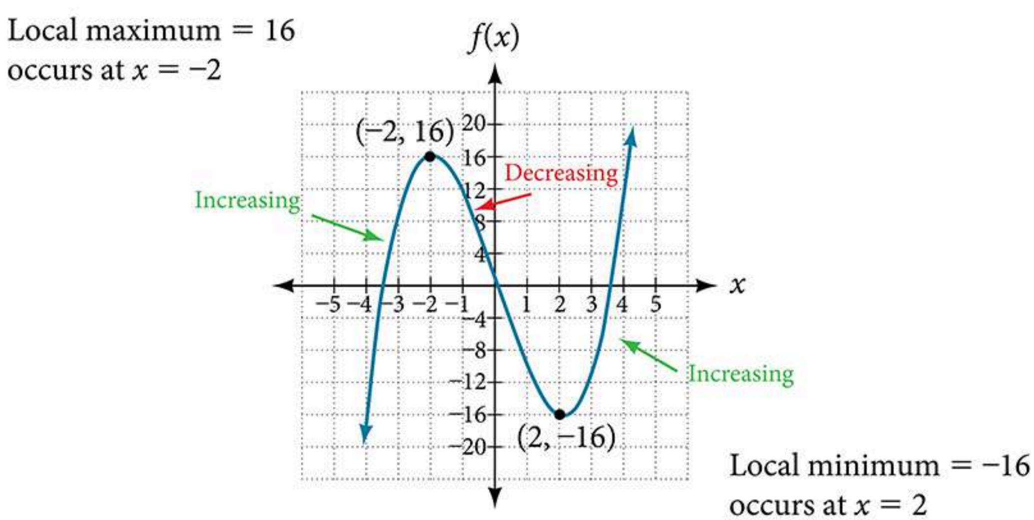


Figure 4

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure 5 illustrates these ideas for a local maximum.

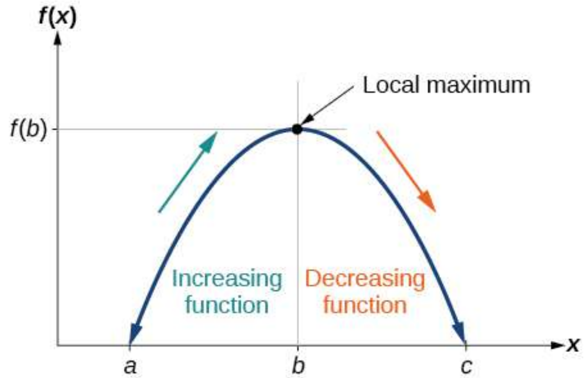


Figure 5 Definition of a local maximum

These observations lead us to a formal definition of local extrema.

Local Minima and Local Maxima

A function f is an **increasing function** on an open interval if $f(b) > f(a)$ for any two input values a and b in the given interval where $b > a$.

A function f is a **decreasing function** on an open interval if $f(b) < f(a)$ for any two input values a and b in the given interval where $b > a$.

A function f has a local maximum at $x = b$ if there exists an interval (a, c) with $a < b < c$ such that, for any x in the interval (a, c), $f(x) \leq f(b)$. Likewise, f has a local minimum at $x = b$ if there exists an interval (a, c) with $a < b < c$ such that, for any x in the interval (a, c), $f(x) \geq f(b)$.

EXAMPLE 7

Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in Figure 6, identify the intervals on which the function appears to be increasing.

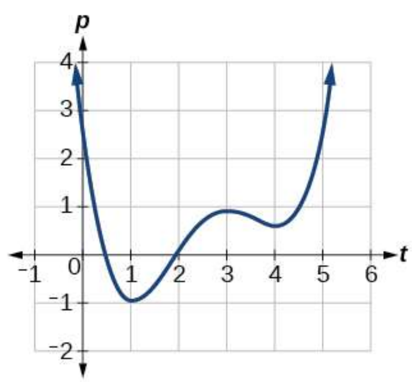


Figure 6

Solution

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval (1,3) and the interval $(4, \infty)$.

Analysis

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

EXAMPLE 8

Finding Local Extrema from a Graph

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution

Using technology, we find that the graph of the function looks like that in Figure 7. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$.

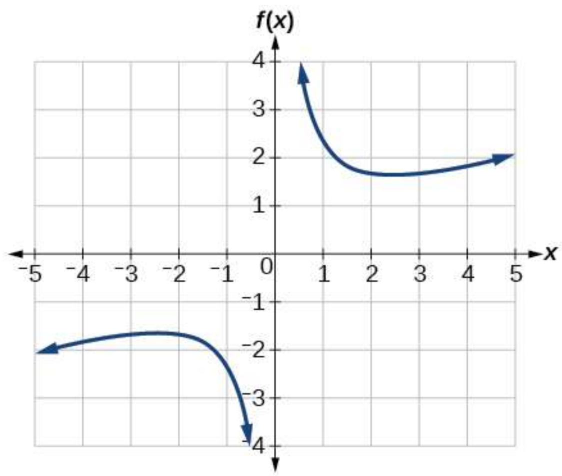


Figure 7

Analysis

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Figure 8 provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

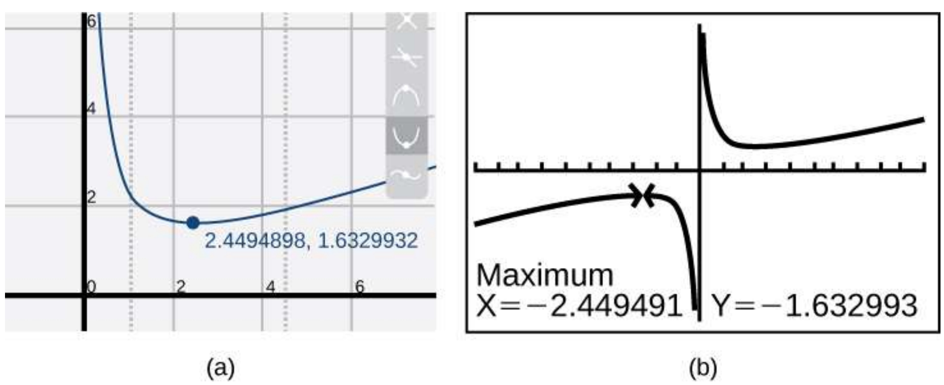


Figure 8

Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

TRY IT #4 Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

EXAMPLE 9

Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in Figure 9, find all local maxima and minima.

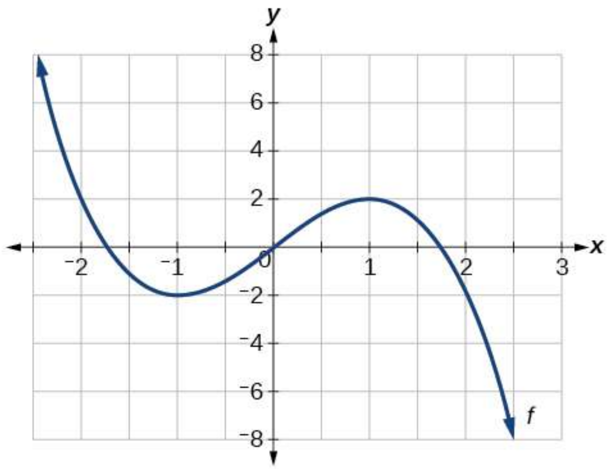


Figure 9

Solution

Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$. The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is -2.

Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in Figure 10, Figure 11, and Figure 12.

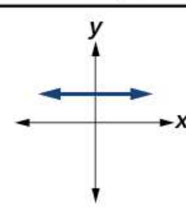
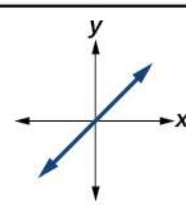
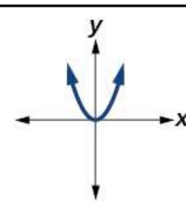
Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Figure 10

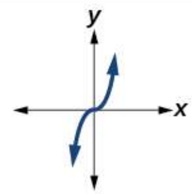
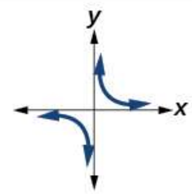
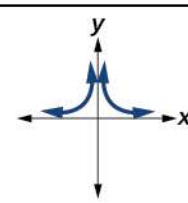
Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0)\cup(0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

Figure 11

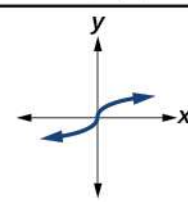
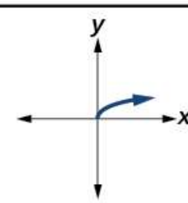
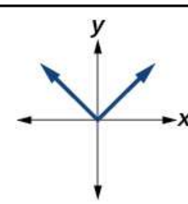
Function	Increasing/Decreasing	Example
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \lvert x \rvert$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$	

Figure 12

Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y- coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively.

To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function. See Figure 13.

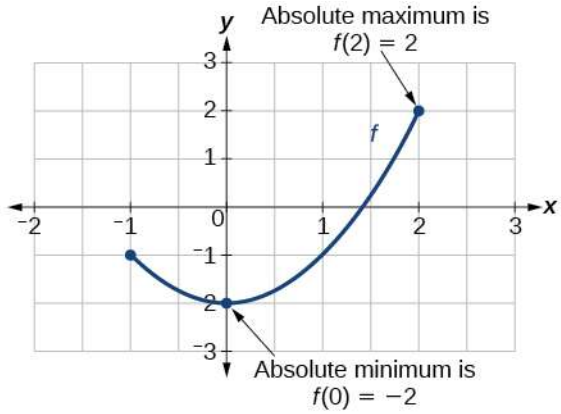


Figure 13

Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

Absolute Maxima and Minima

The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.

The **absolute minimum** of f at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all x in the domain of f.

EXAMPLE 10

Finding Absolute Maxima and Minima from a Graph

For the function f shown in Figure 14, find all absolute maxima and minima.

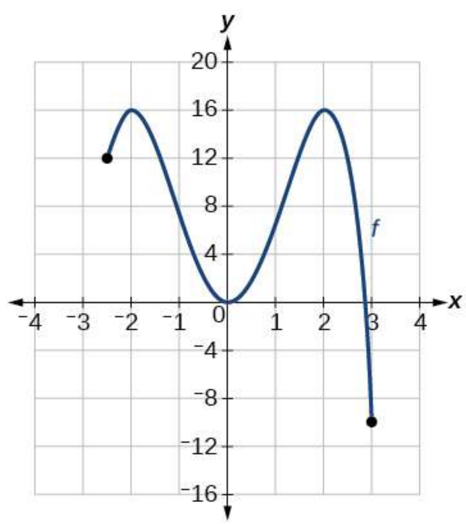


Figure 14

Solution

Observe the graph of f. The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the *y*-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at $x = 3$, because it is the lowest point on the domain of the function's graph. The absolute minimum is the *y*-coordinate at $x = 3$, which is -10.

MEDIA

Access this online resource for additional instruction and practice with rates of change.

[Average Rate of Change (http://openstax.org/l/aroc)](http://openstax.org/l/aroc)

3.3 SECTION EXERCISES

Verbal

1. Can the average rate of change of a function be constant?

2. If a function f is increasing on (a, b) and decreasing on (b, c), then what can be said about the local extremum of f on (a, c)?

3. How are the absolute maximum and minimum similar to and different from the local extrema?

4. How does the graph of the absolute value function compare to the graph of the quadratic function, $y = x^2$, in terms of increasing and decreasing intervals?

Algebraic

For the following exercises, find the average rate of change of each function on the interval specified for real numbers b or h in simplest form.

5. $f(x) = 4x^2 - 7$ on $[1, \ b]$

6. $g(x) = 2x^2 - 9$ on $[4, \ b]$

7. $p(x) = 3x + 4$ on $[2, \ 2 + h]$

8. $k(x) = 4x - 2$ on $[3, \ 3 + h]$

9. $f(x) = 2x^2 + 1$ on $[x, x + h]$

10. $g(x) = 3x^2 - 2$ on $[x, x + h]$

11. $a(t) = \frac{1}{t+4}$ on $[9, 9 + h]$

12. $b(x) = \frac{1}{x+3}$ on $[1, 1 + h]$

13. $j(x) = 3x^3$ on $[1, 1 + h]$

14. $r(t) = 4t^3$ on $[2, 2 + h]$

15. $\frac{f(x+h)-f(x)}{h}$ given $f(x) = 2x^2 - 3x$ on $[x, x + h]$

Graphical

For the following exercises, consider the graph of f shown in Figure 15.

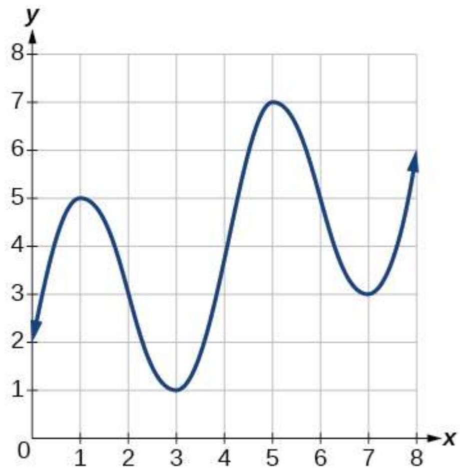


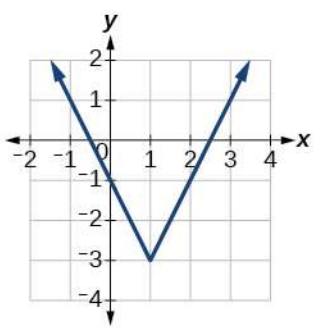
Figure 15

16. Estimate the average rate of change from $x = 1$ to $x = 4$.

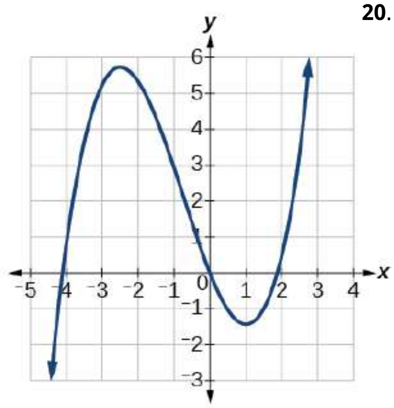
17. Estimate the average rate of change from $x = 2$ to $x = 5$.

For the following exercises, use the graph of each function to estimate the intervals on which the function is increasing or decreasing.

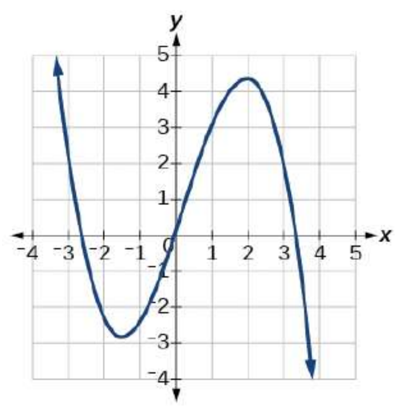
18.



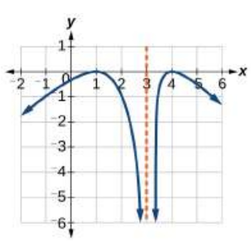
19.



20.



21.



For the following exercises, consider the graph shown in Figure 16.

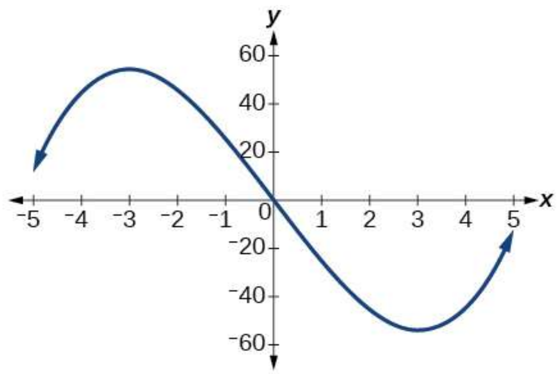


Figure 16

22. Estimate the intervals where the function is increasing or decreasing.

23. Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

For the following exercises, consider the graph in Figure 17.

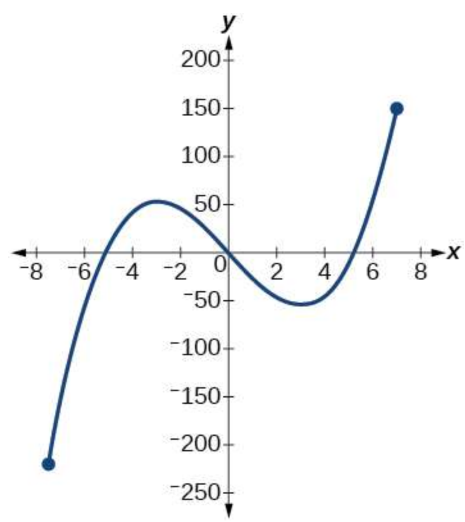


Figure 17

24. If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.

25. If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

Numeric

26. Table 3 gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (a) between 2001 and 2002, and (b) between 2001 and 2004?

Year	Sales (millions of dollars)
1998	201
1999	219
2000	233
2001	243
2002	249
2003	251
2004	249
2005	243
2006	233

Table 3

27. Table 4 gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of population (a) between 2002 and 2004, and (b) between 2002 and 2006?

Year	Population (thousands)
2000	87
2001	84
2002	83
2003	80
2004	77
2005	76
2006	78
2007	81
2008	85

Table 4

For the following exercises, find the average rate of change of each function on the interval specified.

28. $f(x) = x^2$ on $[1, \ 5]$

29. $h(x) = 5 - 2x^2$ on $[-2, 4]$

30. $q(x) = x^3$ on $[-4, 2]$

31. $g(x) = 3x^3 - 1$ on $[-3, 3]$

32. $y = \frac{1}{x}$ on $[1, \ 3]$

33. $p(t) = \frac{\left(t^2-4\right)(t+1)}{t^2+3}$ on $[-3, 1]$

34. $k(t) = 6t^2 + \frac{4}{t^3}$ on $[-1, 3]$

Technology

For the following exercises, use a graphing utility to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.

35. $f(x) = x^4 - 4x^3 + 5$

36. $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

37. $g(t) = t\sqrt{t+3}$

38. $k(t) = 3t^{\frac{2}{3}} - t$

39. $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

40. $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

Extension

41. The graph of the function f is shown in Figure 18.

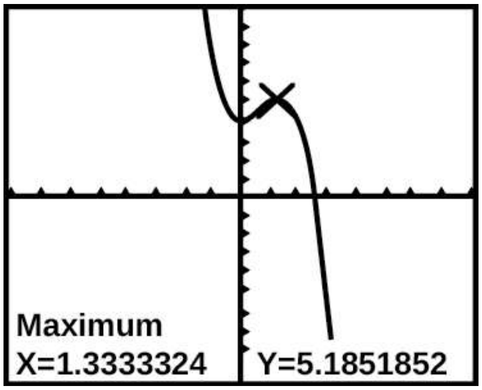


Figure 18

Based on the calculator screen shot, the point $(1.333,\ \ 5.185)$ is which of the following?

ⓐ a relative (local) maximum of the function
ⓑ the vertex of the function
ⓒ the absolute maximum of the function
ⓓ a zero of the function

42. Let $f(x) = \frac{1}{x}$. Find a number c such that the average rate of change of the function f on the interval $(1, c)$ is $-\frac{1}{4}$.

43. Let $f(x) = \frac{1}{x}$. Find the number b such that the average rate of change of f on the interval $(2, b)$ is $-\frac{1}{10}$.

Real-World Applications

44. At the start of a trip, the odometer on a car read 21,395. At the end of the trip, 13.5 hours later, the odometer read 22,125. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?

45. A driver of a car stopped at a gas station to fill up their gas tank. They looked at their watch, and the time read exactly 3:40 p.m. At this time, they started pumping gas into the tank. At exactly 3:44, the tank was full and the driver noticed that they had pumped 10.7 gallons. What is the average rate of flow of the gasoline into the gas tank?

46. Near the surface of the moon, the distance that an object falls is a function of time. It is given by $d(t) = 2.6667t^2$, where t is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.

47. The graph in Figure 19 illustrates the decay of a radioactive substance over t days.

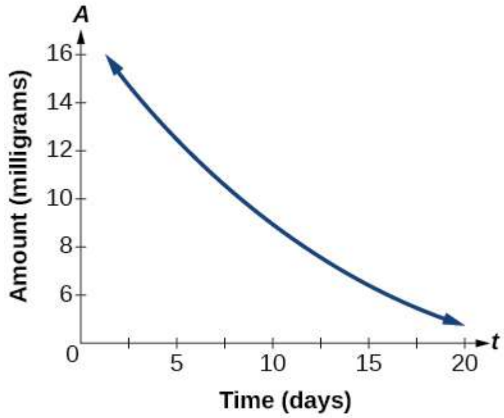


Figure 19

Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

3.4 Composition of Functions

Learning Objectives

In this section, you will:

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

COREQUISITE SKILLS

Learning Objectives

- Find the value of a function (IA 3.5.3), (CA 3.1.2)

Objective 1: Find the value of a function (IA 3.5.3), (CA 3.1.2)

A **function** is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each x -value is matched with only one y -value. The notation $y = f(x)$ defines a function named f . This is read as " y is a function of x ." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Function Notation

For the function $y = f(x)$

f is the name of the function.
x is the input value, the collection of possible input values make up the domain.
$f(x)$ is the output value, the collection of possible output values make up the range.

We read $f(x)$ as *f* of *x* or the value of *f* at *x* .

EXAMPLE 1

Evaluating and Solving a Function Represented in Table Form

ⓐ Evaluate $g(3)$ ⓑ Solve $g(n) = 6$

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Solution

ⓐ Evaluating *g*(3) means determining the output value of the function *g* for the input value of *n*=3. The table output value corresponding to *n*=3 is 7, so *g*(3)=7.

ⓑ Solving *g*(*n*)=6 means identifying the input values, *n*, that produce an output value of 6. The table shows two values where *g*(*n*) = 6 at *x*=2 and 4.

Practice Makes Perfect

Find the value of a function.

1. Given the function $k(t) = 2t{-}1$:

ⓐ Evaluate $k(2)$ ⓑ Solve $k(t) = 7$

2. Given the function $f(x) = \sqrt{x+2}$:

ⓐ Evaluate $f(7)$ ⓑ Solve $f(x) = 4$

3. For the function $f(x) = 2x^2 + 3x + 1$, find

ⓐ $f(3)$ ⓑ $f(-2)$ ⓒ $f(t)$ ⓓ The value(s) of *x* that make $f(x) = 1$

4. Use the table below to help answer the following:

x	1	2	3	4	5
$g(x)$	5	12	21	32	45

ⓐ Evaluate $g(4)$ ⓑ Solve $g(x) = 32$

A **composite** function is a two-step function and can have numerical or variable inputs.

$x \to \boxed{g} \to g\left(x\right) \to \boxed{f} \to f\left(g\left(x\right)\right)$

$(f \circ g)(x) = f(g(x))$ is read as "*f* of *g* of *x*"

To evaluate a composite function, we always start evaluating the inner function and then evaluate the outer function in terms of the inner function.

Let's use a table to help us organize our work in evaluating a two-step (composition) function in terms of some numerical inputs.

First evaluate *g* in terms of *x*, the *f* in terms of *g*(*x*).

Given that: $g(x) = 3x - 1$, and $f\left(x\right) = x^2 + 1$, complete the table below. Remember the output of *g*(*x*) becomes the input of *f*(*x*)!

x	$g(x) = 3x - 1$	$f\left(g\left(x\right)\right) = (g(x))^2 + 1$
–1		
–3		
0		
4		
10		

Practice Makes Perfect

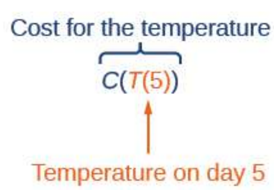
5. Use the table below showing values of $f(x)$ and $g(x)$ to find each of the following. Remember when working with composition functions we always evaluate the inner function first.

x	–3	–2	–1	0	1	2	3
$f(x)$	11	9	7	5	3	1	–1
$g(x)$	–8	–3	0	1	0	–3	–8

ⓐ $f(1) =$ ⓑ $g(f(1)) =$ ⓒ $g(0) =$ ⓓ $f(g(0)) =$ ⓔ $f(g(2)) =$ ⓕ $f(f(3)) =$

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C\,(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T\,(d)$ gives the average daily temperature on day d of the year. For any given day, $\text{Cost} = C\,(T\,(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T\,(d)$. For example, we could evaluate $T\,(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write $C\,(T\,(5))$.



By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T\,(y) = h\,(y) + w\,(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f + g,\ \ f - g,\ \ fg,$ and $\frac{f}{g}$ by the relations

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f-g)(x) &= f(x)-g(x)\\ (fg)(x) &= f(x)g(x)\\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \qquad \text{where } g(x) \neq 0\end{aligned}$$

EXAMPLE 1

Performing Algebraic Operations on Functions

Find and simplify the functions $(g - f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x) = x - 1$ and $g(x) = x^2 - 1$. Are they the same function?

Solution

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ (g-f)(x) &= x^2-x\\ (g-f)(x) &= x(x-1)\end{aligned}$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)}\\ \left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ \left(\frac{g}{f}\right)(x) &= \frac{(x+1)(x-1)}{x-1} \qquad \text{where} x \neq 1\\ \left(\frac{g}{f}\right)(x) &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x = 1$, the denominator is equal to 0, which makes the function undefined.

TRY IT #1 Find and simplify the functions $(fg)(x)$ and $(f - g)(x)$.

$$f(x) = x - 1 \quad \text{and} \quad g(x) = x^2 - 1$$

Are they the same function?

Create a Function by Composition of Functions

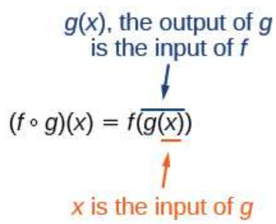
Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a composition of functions**.** The resulting function is known as a **composite**

function. We represent this combination by the following notation:

$$(f \circ g)(x) = f(g(x))$$

We read the left-hand side as "f composed with g at x," and the right-hand side as "f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.



In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g\left(x^2\right) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x))$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

EXAMPLE 2

Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is commutative.

$$f(x) = 2x + 1 \qquad\qquad g(x) = 3 - x$$

Solution

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3 - x) + 1 \\ &= 6 - 2x + 1 \\ &= 7 - 2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3 - (2x + 1) \\ &= 3 - 2x - 1 \\ &= -2x + 2 \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

EXAMPLE 3

Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t = 3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

EXAMPLE 4

Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y = f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f \ (\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g \ (\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Q&A Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the *same units also may give compositions that are meaningful in either order.*

TRY IT #2 The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

EXAMPLE 5

Using a Table to Evaluate a Composite Function

Using Table 1, evaluate $f(g(3))$ and $g(f(3))$.

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Table 1

Solution

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function $g: g(3) = 2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

$$\begin{aligned} g(3) &= 2 \\ f(g(3)) &= f(2) = 8 \end{aligned}$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

$$g(f(3)) = g(3) = 2$$

Table 2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

Table 2

TRY IT #3 Using Table 1, evaluate $f(g(1))$ and $g(f(4))$.

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y- axes of the graphs.

HOW TO

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x- axis of its graph.
2. Read off the output of the inner function from the y- axis of its graph.
3. Locate the inner function output on the x- axis of the graph of the outer function.
4. Read the output of the outer function from the y- axis of its graph. This is the output of the composite function.

EXAMPLE 6

Using a Graph to Evaluate a Composite Function

Using Figure 1, evaluate $f(g(1))$.

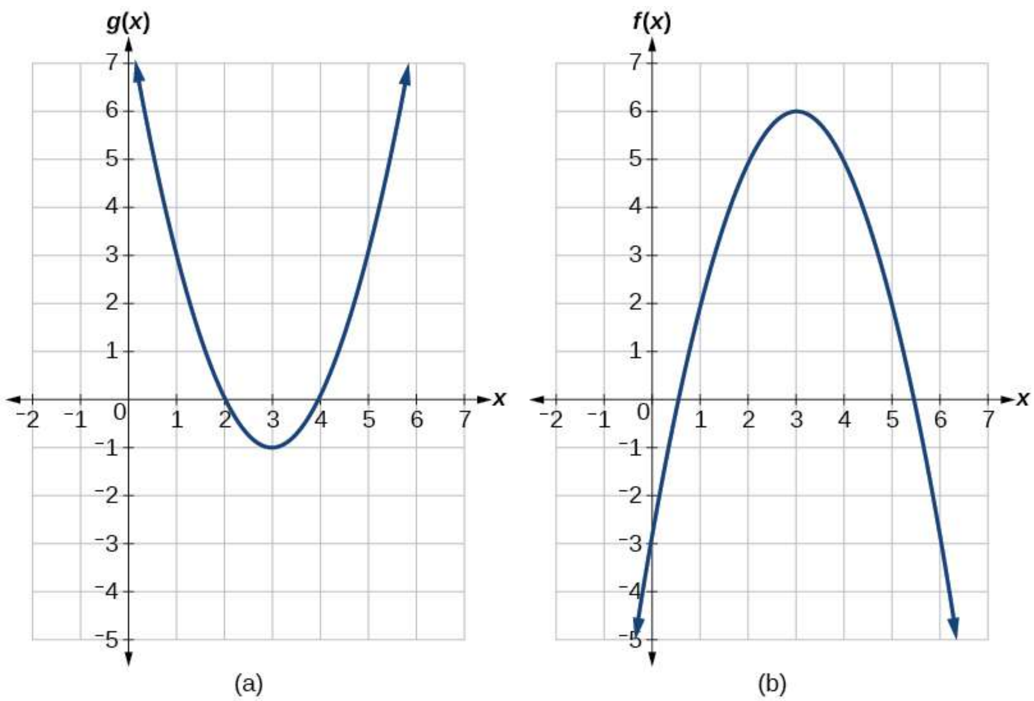


Figure 1

Solution

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 2.

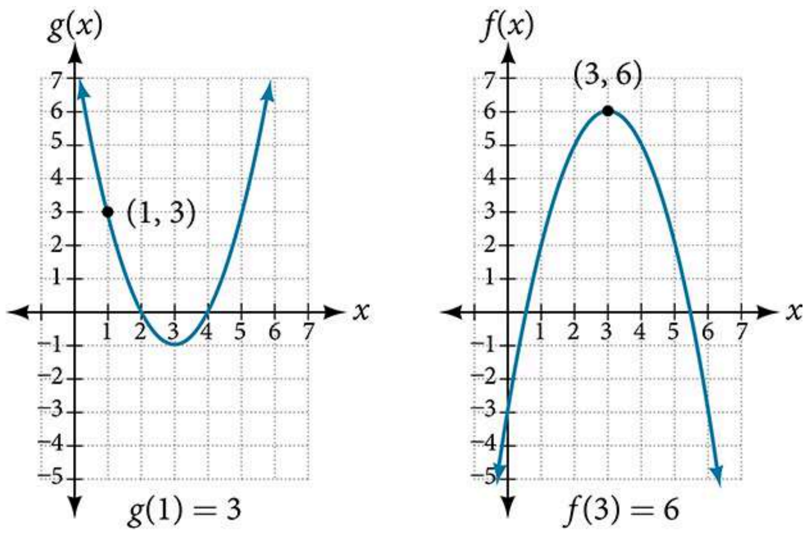


Figure 2

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x- axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x- axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis

Figure 3 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

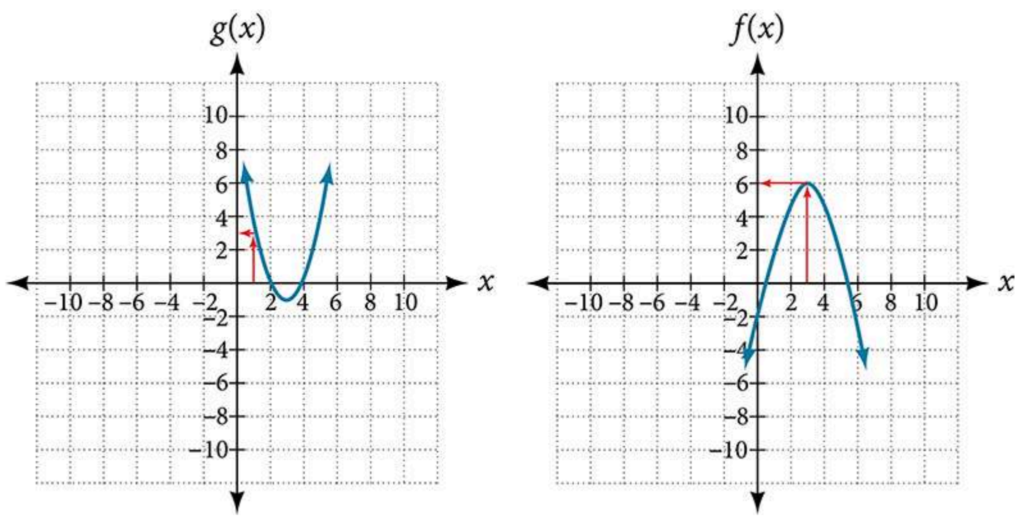


Figure 3

TRY IT #4 Using Figure 1, evaluate $g(f(2))$.

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

HOW TO

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

EXAMPLE 7

Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$\begin{aligned} h(1) &= 3(1) + 2 \\ h(1) &= 5 \end{aligned}$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$\begin{aligned} f(h(1)) &= f(5) \\ f(h(1)) &= 5^2 - 5 \\ f(h(1)) &= 20 \end{aligned}$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

TRY IT #5 Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

ⓐ $h(f(2))$ ⓑ $h(f(-2))$

Finding the Domain of a Composite Function

As we discussed previously, the domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Domain of a Composite Function

The domain of a composite function $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

HOW TO

Given a function composition $f(g(x))$, determine its domain.

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

EXAMPLE 8

Finding the Domain of a Composite Function

Find the domain of

$$(f \circ g)(x) \text{ where } \quad f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

Solution

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.

$$\begin{aligned} \tfrac{4}{3x-2} &= 1 \\ 4 &= 3x-2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup \left(2, \infty\right)$$

EXAMPLE 9

Finding the Domain of a Composite Function Involving Radicals

Find the domain of

$$(f \circ g)(x) \quad \text{where} \qquad f(x) = \sqrt{x+2} \text{ and } g(x) = \sqrt{3-x}$$

Solution

Because we cannot take the square root of a negative number, the domain of g is $\left(-\infty, 3\right]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$, $\sqrt{3-x}+2 \geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $\sqrt{3-x} \geq 0$, or, $3-x \geq 0$, which gives a domain of $(-\infty, 3]$.

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

TRY IT #6 Find the domain of

$$(f \circ g)(x) \quad \text{where} \qquad f(x) = \frac{1}{x-2} \quad \text{and} \quad g(x) = \sqrt{x+4}$$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

EXAMPLE 10

Decomposing a Function

Write $f(x) = \sqrt{5-x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5 - x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5 - x^2 \ \text{ and } g(x) = \sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g\left(5 - x^2\right) = \sqrt{5 - x^2}$$

TRY IT #7 Write $f(x) = \frac{4}{3-\sqrt{4+x^2}}$ as the composition of two functions.

MEDIA

Access these online resources for additional instruction and practice with composite functions.

[Composite Functions (http://openstax.org/l/compfunction)](http://openstax.org/l/compfunction)
[Composite Function Notation Application (http://openstax.org/l/compfuncnot)](http://openstax.org/l/compfuncnot)
[Composite Functions Using Graphs (http://openstax.org/l/compfuncgraph)](http://openstax.org/l/compfuncgraph)
[Decompose Functions (http://openstax.org/l/decompfunction)](http://openstax.org/l/decompfunction)
[Composite Function Values (http://openstax.org/l/compfuncvalue)](http://openstax.org/l/compfuncvalue)

3.4 SECTION EXERCISES

Verbal

1. How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

2. What is the composition of two functions, $f \circ g$?

3. If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

4. How do you find the domain for the composition of two functions, $f \circ g$?

Algebraic

For the following exercises, determine the domain for each function in interval notation.

5. Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$, find $f + g,\ \ f - g,\ \ fg$, and $\frac{f}{g}$.

6. Given $f(x) = -3x^2 + x$ and $g(x) = 5$, find $f + g,\ \ f - g,\ \ fg$, and $\frac{f}{g}$.

7. Given $f(x) = 2x^2 + 4x$ and $g(x) = \frac{1}{2x}$, find $f + g,\ \ f - g,\ \ fg$, and $\frac{f}{g}$.

8. Given $f(x) = \frac{1}{x-4}$ and $g(x) = \frac{1}{6-x}$, find $f + g,\ \ f - g,\ \ fg$, and $\frac{f}{g}$.

9. Given $f(x) = 3x^2$ and $g(x) = \sqrt{x-5}$, find $f + g,\ \ f - g,\ \ fg$, and $\frac{f}{g}$.

10. Given $f(x) = \sqrt{x}$ and $g(x) = |x - 3|$, find $\frac{g}{f}$.

11. For the following exercise, find the indicated function given $f(x) = 2x^2 + 1$ and $g(x) = 3x - 5$.

ⓐ $f(g(2))$ ⓑ $f(g(x))$ ⓒ $g(f(x))$
ⓓ $(g \circ g)(x)$ ⓔ $(f \circ f)(-2)$

For the following exercises, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.

12. $f(x) = x^2 + 1,\ \ g(x) = \sqrt{x+2}$

13. $f(x) = \sqrt{x} + 2,\ \ g(x) = x^2 + 3$

14. $f(x) = |x|,\ \ g(x) = 5x + 1$

15. $f(x) = \sqrt[3]{x},\ \ g(x) = \frac{x+1}{x^3}$

16. $f(x) = \frac{1}{x-6},\ \ g(x) = \frac{7}{x} + 6$

17. $f(x) = \frac{1}{x-4},\ \ g(x) = \frac{2}{x} + 4$

For the following exercises, use each set of functions to find $f(g(h(x)))$. Simplify your answers.

18. $f(x) = x^4 + 6$, $g(x) = x - 6$, and $h(x) = \sqrt{x}$

19. $f(x) = x^2 + 1$, $g(x) = \frac{1}{x}$, and $h(x) = x + 3$

20. Given $f(x) = \frac{1}{x}$ and $g(x) = x - 3$, find the following:

ⓐ $(f \circ g)(x)$
ⓑ the domain of $(f \circ g)(x)$ in interval notation
ⓒ $(g \circ f)(x)$
ⓓ the domain of $(g \circ f)(x)$
ⓔ $\left(\frac{f}{g}\right)(x)$

21. Given $f(x) = \sqrt{2 - 4x}$ and $g(x) = -\frac{3}{x}$, find the following:

ⓐ $(g \circ f)(x)$
ⓑ the domain of $(g \circ f)(x)$ in interval notation

22. Given the functions $f(x) = \frac{1-x}{x}$ and $g(x) = \frac{1}{1+x^2}$, find the following:

ⓐ $(g \circ f)(x)$ ⓑ $(g \circ f)(2)$

23. Given functions $p(x) = \frac{1}{\sqrt{x}}$ and $m(x) = x^2 - 4$, state the domain of each of the following functions using interval notation:

ⓐ $\frac{p(x)}{m(x)}$ ⓑ $p(m(x))$
ⓒ $m(p(x))$

24. Given functions $q(x) = \frac{1}{\sqrt{x}}$ and $h(x) = x^2 - 9$, state the domain of each of the following functions using interval notation.

ⓐ $\frac{q(x)}{h(x)}$ ⓑ $q(h(x))$ ⓒ $h(q(x))$

25. For $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x-1}$, write the domain of $(f \circ g)(x)$ in interval notation.

For the following exercises, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x) = f(g(x))$.

26. $h(x) = (x+2)^2$

27. $h(x) = (x-5)^3$

28. $h(x) = \frac{3}{x-5}$

29. $h(x) = \frac{4}{(x+2)^2}$

30. $h(x) = 4 + \sqrt[3]{x}$

31. $h(x) = \sqrt[3]{\frac{1}{2x-3}}$

32. $h(x) = \frac{1}{(3x^2-4)^{-3}}$

33. $h(x) = \sqrt[4]{\frac{3x-2}{x+5}}$

34. $h(x) = \left(\frac{8+x^3}{8-x^3}\right)^4$

35. $h(x) = \sqrt{2x+6}$

36. $h(x) = (5x-1)^3$

37. $h(x) = \sqrt[3]{x-1}$

38. $h(x) = \left|x^2+7\right|$

39. $h(x) = \frac{1}{(x-2)^3}$

40. $h(x) = \left(\frac{1}{2x-3}\right)^2$

41. $h(x) = \sqrt{\frac{2x-1}{3x+4}}$

Graphical

For the following exercises, use the graphs of f, shown in Figure 4, and g, shown in Figure 5, to evaluate the expressions.

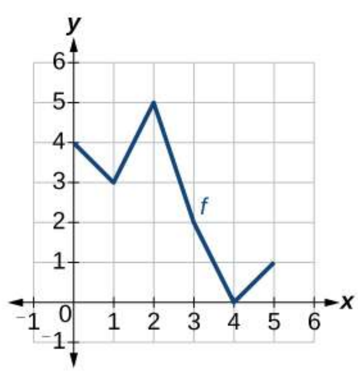


Figure 4

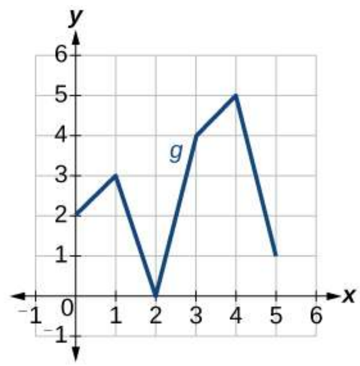


Figure 5

42. $f(g(3))$

43. $f(g(1))$

44. $g(f(1))$

45. $g(f(0))$

46. $f(f(5))$

47. $f(f(4))$

48. $g(g(2))$

49. $g(g(0))$

For the following exercises, use graphs of $f(x)$, shown in Figure 6, $g(x)$, shown in Figure 7, and $h(x)$, shown in Figure 8, to evaluate the expressions.

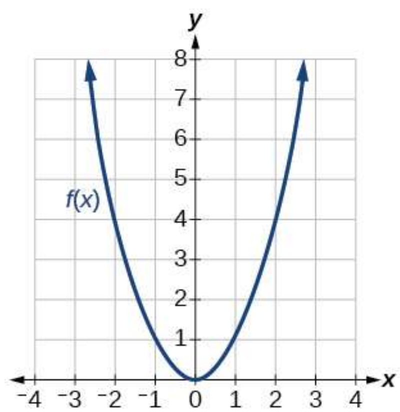


Figure 6

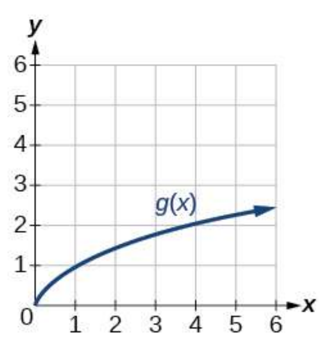


Figure 7

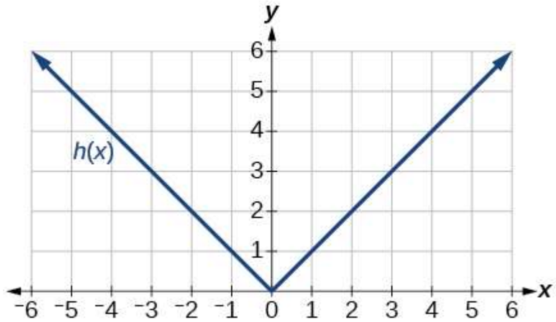


Figure 8

50. $g(f(1))$

51. $g(f(2))$

52. $f(g(4))$

53. $f(g(1))$

54. $f(h(2))$

55. $h(f(2))$

56. $f(g(h(4)))$

57. $f(g(f(-2)))$

Numeric

For the following exercises, use the function values for f and g shown in Table 3 to evaluate each expression.

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

Table 3

58. $f(g(8))$

59. $f(g(5))$

60. $g(f(5))$

61. $g(f(3))$

62. $f(f(4))$

63. $f(f(1))$

64. $g(g(2))$

65. $g(g(6))$

For the following exercises, use the function values for f and g shown in Table 4 to evaluate the expressions.

x	$f(x)$	$g(x)$
-3	11	-8
-2	9	-3
-1	7	0
0	5	1
1	3	0
2	1	-3
3	-1	-8

Table 4

66. $(f \circ g)(1)$

67. $(f \circ g)(2)$

68. $(g \circ f)(2)$

69. $(g \circ f)(3)$

70. $(g \circ g)(1)$

71. $(f \circ f)(3)$

For the following exercises, use each pair of functions to find $f(g(0))$ and $g(f(0))$.

72. $f(x) = 4x + 8, \quad g(x) = 7 - x^2$

73. $f(x) = 5x + 7, \quad g(x) = 4 - 2x^2$

74. $f(x) = \sqrt{x + 4}, \quad g(x) = 12 - x^3$

75. $f(x) = \frac{1}{x+2}, \quad g(x) = 4x + 3$

For the following exercises, use the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$ to evaluate or find the composite function as indicated.

76. $f(g(2))$

77. $f(g(x))$

78. $g(f(-3))$

79. $(g \circ g)(x)$

Extensions

For the following exercises, use $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x - 1}$.

80. Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Compare the two answers.

81. Find $(f \circ g)(2)$ and $(g \circ f)(2)$.

82. What is the domain of $(g \circ f)(x)$?

83. What is the domain of $(f \circ g)(x)$?

84. Let $f(x) = \frac{1}{x}$.

ⓐ Find $(f \circ f)(x)$.
ⓑ Is $(f \circ f)(x)$ for any function f the same result as the answer to part (a) for any function? Explain.

For the following exercises, let $F(x) = (x+1)^5$, $f(x) = x^5$, and $g(x) = x + 1$.

85. True or False: $(g \circ f)(x) = F(x)$.

86. True or False: $(f \circ g)(x) = F(x)$.

For the following exercises, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x-2}$.

87. $(f \circ g)(6)$; $(g \circ f)(6)$

88. $(g \circ f)(a)$; $(f \circ g)(a)$

89. $(f \circ g)(11)$; $(g \circ f)(11)$

Real-World Applications

90. The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost $C(x)$ is the cost of producing x items. To determine the cost of production when the price is \$6, you would do which of the following?

ⓐ Evaluate $D(C(6))$.
ⓑ Evaluate $C(D(6))$.
ⓒ Solve $D(C(x)) = 6$.
ⓓ Solve $C(D(p)) = 6$.

91. The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?

ⓐ Evaluate $A(m(4))$.
ⓑ Evaluate $m(A(4))$.
ⓒ Solve $A(m(t)) = 4$.
ⓓ Solve $m(A(d)) = 4$.

92. A store offers customers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price x. (Hint: Use function composition to find your answer.)

93. A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.

94. A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t) = 2t + 1$, express the area burned as a function of time, t (minutes).

95. Use the function you found in the previous exercise to find the total area burned after 5 minutes.

96. The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t$.

ⓐ Find the composite function $r\left(V(t)\right)$.
ⓑ Find the *exact* time when the radius reaches 10 inches.

97. The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where t is the time in hours.

ⓐ Find the composite function $N\left(T(t)\right)$.
ⓑ Find the time (round to two decimal places) when the bacteria count reaches 6752.

3.5 Transformation of Functions

Learning Objectives

In this section, you will:

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

COREQUISITE SKILLS

Learning Objectives

- Identify graphs of basic functions, (IA 3.6.2)
- Graph quadratic functions using transformations, (IA 9.7.4)

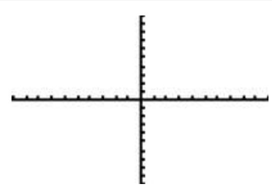
Objective 1: Identify graphs of basic functions, (IA 3.6.2)

Basic functions have unique shapes, characteristics, and algebraic equations. It will be helpful to recognize and identify these basic or “toolkit functions” in our work in algebra, precalculus and calculus. Remember functions can be represented in many ways including by name, equation, graph, and basic tables of values.

Practice Makes Perfect

Use a graphing program to help complete the following. Then, choose three values of x to evaluate for each. Add the x and y to the table for each exercise.

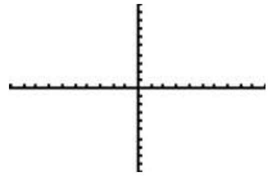
1.

Name	Equation	Graph
Constant	*y*=*c*, where *c* is a constant	

Choose 3 values of x to evaluate for each.

x	*y*

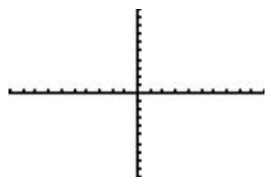
2.

Name	Equation	Graph
Identity	$y=x$	

Choose 3 values of x to evaluate for each.

x	y

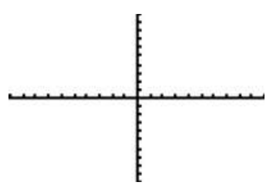
3.

Name	Equation	Graph
Absolute Value	$y=\|x\|$	

Choose 3 values of x to evaluate for each.

x	y

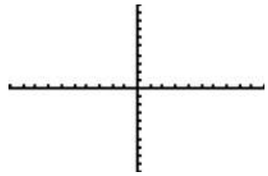
4.

Name	Equation	Graph
Quadratic	$y=x^2$	

Choose 3 values of x to evaluate for each.

x	y

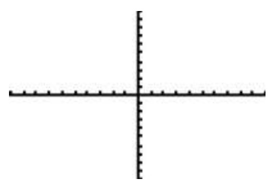
5.

Name	Equation	Graph
Cubic	$y{=}x^3$	

Choose 3 values of x to evaluate for each.

x	y

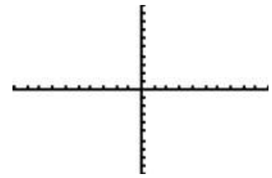
6.

Name	Equation	Graph
Reciprocal	$y = \frac{1}{x}$	

Choose 3 values of x to evaluate for each.

x	y

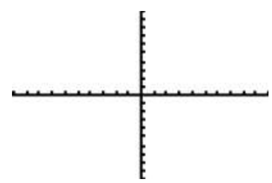
7.

Name	Equation	Graph
Square Root	$y = \sqrt{x}$	

Choose 3 values of x to evaluate for each.

x	y

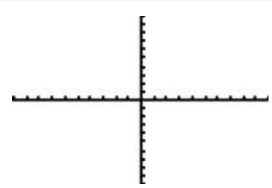
8.

Name	Equation	Graph
Cube Root	$y = \sqrt[3]{x}$	

Choose 3 values of x to evaluate for each.

x	y

9.

Name	Equation	Graph
Exponential	$y = e^x$	

Choose 3 values of x to evaluate for each.

x	y

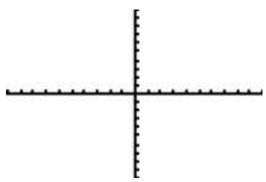
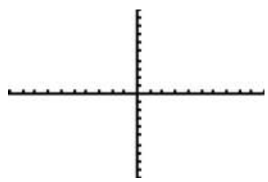
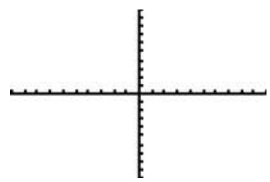
Objective 2: Graph quadratic functions using transformations (IA 9.7.4)

When we modify basic functions by adding, subtracting, or multiplying constants to the equation, very systematic changes take place. We call these **transformations** of basic functions. Here we will investigate the effects of vertical shifts, horizontal shifts, vertical stretches or compressions, and reflections on quadratic functions. We could use any basic function to illustrate transformations, but quadratics work nicely because we can easily keep track of a point called the vertex.

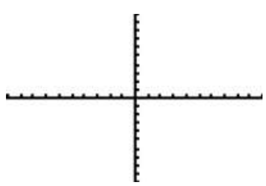
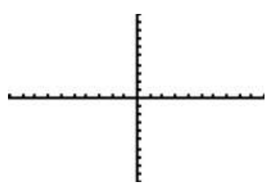
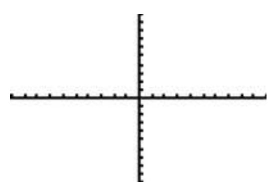
Practice Makes Perfect

The graphs of quadratic functions are called parabolas. Use a graphing program to graph each of the following quadratic functions. For each graph find the vertex (the minimum or maximum value) of the parabola and list its coordinates. Most importantly use the patterns observed to answer each of the given questions.

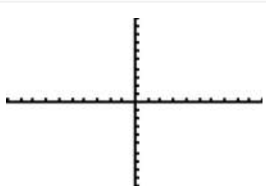
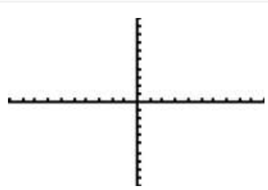
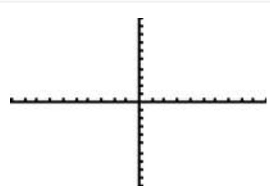
10. In general, what effect does adding or subtracting a constant have on the graph of $f\left(x\right) = x^2$?

		
$f\left(x\right) = x^2$ vertex: _______	$f\left(x\right) = x^2 + 2$ vertex: _______	$f\left(x\right) = x^2 - 4$ vertex: _______

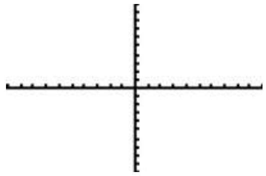
11. In general, what effect does adding or subtracting a value to x before it is squared have on the graph of $f\left(x\right) = x^2$?

		
$f\left(x\right) = (x - 2)^2$ vertex: _______	$f\left(x\right) = (x - 4)^2$ vertex: _______	$f\left(x\right) = (x + 3)^2$ vertex: _______

12. In general, what effect does multiplying by a constant have on the graph of $f\left(x\right) = x^2$?

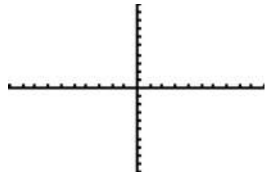
		
$f\left(x\right) = 2x^2$ vertex: _______	$f\left(x\right) = 5x^2$ vertex: _______	$f\left(x\right) = \frac{1}{10}x^2$ vertex: _______

13. In general, what effect does multiplying by a negative constant have on the graph of $f\left(x\right) = x^2$?



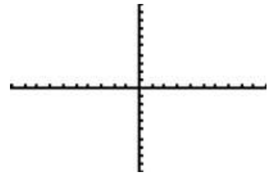
$f\left(x\right) = -x^2$

vertex: _______



$f\left(x\right) = -3x^2$

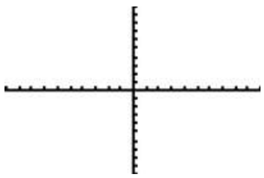
vertex: _______



$y = (x-2)^2$

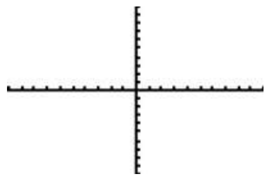
vertex: _______

14.



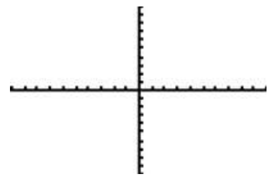
$f\left(x\right) = (x+3)^2 + 2$

vertex: _______



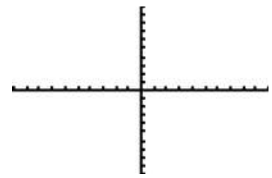
$f\left(x\right) = (x+3)^2 - 2$

vertex: _______



$f\left(x\right) = (x-3)^2 + 2$

vertex: _______



$f\left(x\right) = (x-3)^2 - 2$

vertex: _______

15. Answer each of the following based on the changes you saw in the graphs above.

ⓐ Based on your observations from the previous graphs, what are the coordinates of the vertex of the parabola $f\left(x\right) = (x + 200)^2 - 67$? Do not attempt to graph!

ⓑ Based on your observations from the previous graphs, what are the coordinates of the vertex of the parabola $f\left(x\right) = 12(x + 6)^2 + 111$? Do not attempt to graph!

16. Fill in the blanks:

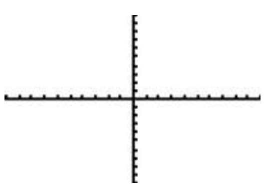
ⓐ If $c > 0$, the graph of $y = f(x) + c$ is obtained by shifting the graph of $y = f(x)$ to the _______ a distance of c units. The graph of $y = f(x) - c$ is obtained by shifting the graph of $y = f(x)$ to the _______ a distance of c units.

ⓑ If $c > 0$, the graph of $y = f(x) - c$ is obtained by shifting the graph of $y = f(x)$ to the _______ a distance of c units. The graph of $y = f(x) + c$ is obtained by shifting the graph of $y = f(x)$ to the _______ a distance of c units.

17. Apply what you have learned in this skill sheet regarding transformations. Write the equation of a quadratic function that has been transformed in the each of the ways described in parts ⓐ and ⓑ below. Write equations in $f(x)$ = form. After writing an equation check your answer using a graphing program and graph below. Be sure to label the vertex as an ordered pair. Does your graph match the description?

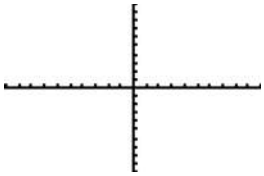
ⓐ Flipped upside down and shifted to the right 3 units and down 2 units.

$f(x) =$



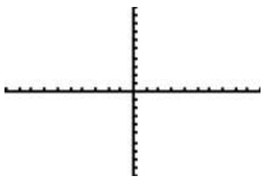
ⓑ Stretched vertically by a factor of 4 and shifted left 6 units and up 5 units.

$f(x) =$



18. Remember the basic transformations investigated in this activity apply to all basic functions. Apply what you have learned in this lab about transformations. Write the equation of a function that has been transformed in the following ways. Write equations in $f(x)$ = form. After writing an equation check your answer using a graphing program and graph below. Be sure to label a point on the graph. Does your graph match the description?

ⓐ Begin with a basic square root function. Reflect the graph over the x-axis and shift it to the right 2 units and down 1 unit.
$f(x) =$



ⓑ Begin with an absolute value function. Stretch the graph vertically by a factor of 3 and shift it left 4 units and up 5 units.
$f(x) =$

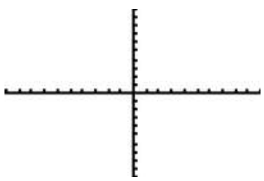


Figure 1 (credit: "Misko"/Flickr)

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

Graphing Functions Using Vertical and Horizontal Shifts

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure 2 for an example.

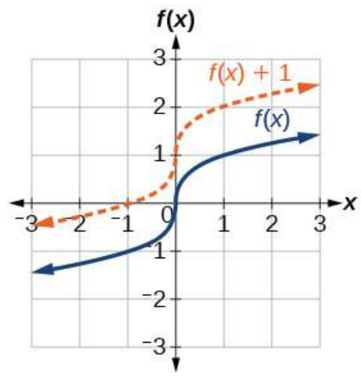


Figure 2 Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

EXAMPLE 1

Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

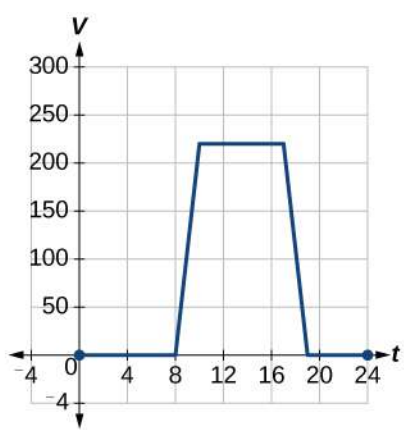


Figure 3

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 4.

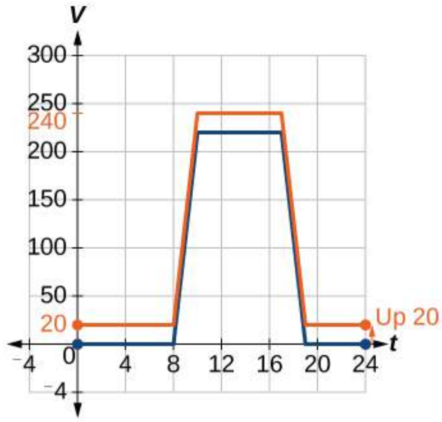


Figure 4

Notice that in Figure 4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$, we could write

$$S(t) = V(t) + 20$$

This notation tells us that, for any value of t, $S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 1.

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

Table 1

HOW TO

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

EXAMPLE 2

Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table 2. Create a table for the function $g(x) = f(x) - 3$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 2

Solution

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f.

For example:

$$\begin{aligned} f(2) &= 1 && \text{Given} \\ g(x) &= f(x) - 3 && \text{Given transformation} \\ g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table 3.

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	−2	0	4	8

Table 3

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

TRY IT #1 The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 5.

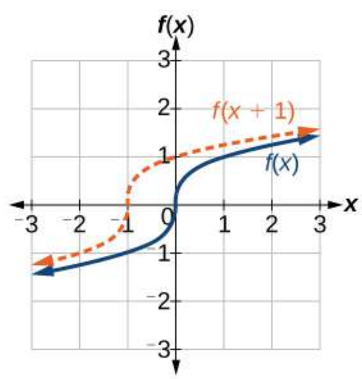


Figure 5 Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $(x+1)$ means $h = -1$, which shifts the graph to the left, that is, towards *negative* values of x.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Horizontal Shift

Given a function f, a new function $g(x) = f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

EXAMPLE 3

Adding a Constant to an Input

Returning to our building airflow example from Figure 3, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$\begin{aligned} V(t) &= \text{the original venting plan} \\ F(t) &= \text{starting 2 hrs sooner} \end{aligned}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 6. Notice also that the vents first opened to $220\ \text{ft}^2$ at 10 a.m. under the original plan, while under the new plan the vents reach $220\ \text{ft}^2$ at
8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t - (-2)) = V(t + 2)$.

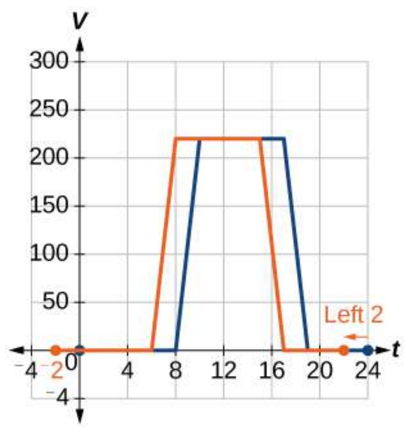


Figure 6

Analysis

Note that $V(t + 2)$ has the effect of shifting the graph to the *left*.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F : F(t) = V(t + 2)$.

HOW TO

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

EXAMPLE 4

Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 4. Create a table for the function $g(x) = f(x-3)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 4

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 *larger*. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 5.

x	5	7	9	11
$x-3$	2	4	6	8
$f(x-3)$	1	3	7	11
$g(x)$	1	3	7	11

Table 5

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis

Figure 7 represents both of the functions. We can see the horizontal shift in each point.

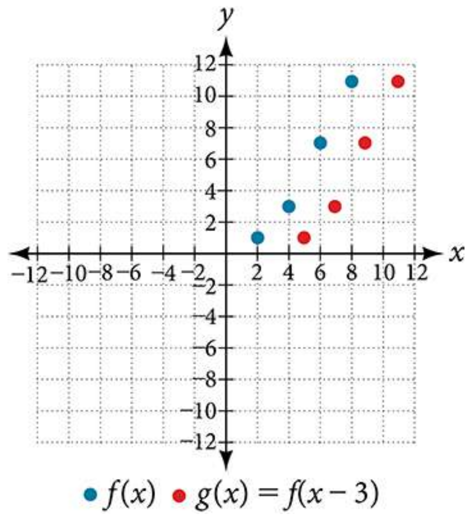


Figure 7

EXAMPLE 5

Identifying a Horizontal Shift of a Toolkit Function

Figure 8 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

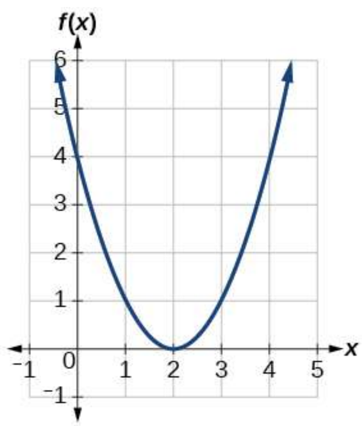


Figure 8

Solution

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at (0,0), but now the vertex is at (2,0). The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x - 2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x - 2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x - 2) \\ g(x) &= f(x - 2) = (x - 2)^2 \end{aligned}$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x - 2) = f(0) = 0$. For this to work, we will need to *subtract* 2 units from our input values.

EXAMPLE 6

Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m) + 10$ and $G(m + 10)$.

Solution

$G(m) + 10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m + 10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

TRY IT #2 Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them. Vertical shifts are outside changes that affect the output (*y*-) values and shift the function up or down. Horizontal shifts are inside changes that affect the input (*x*-) values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down *and* left or right.

HOW TO

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

EXAMPLE 7

Graphing Combined Vertical and Horizontal Shifts

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x + 1) - 3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x + 1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x + 1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 9.

Let us follow one point of the graph of $f(x) = |x|$.

- The point $(0, 0)$ is transformed first by shifting left 1 unit: $(0, 0) \to (-1, 0)$
- The point $(-1, 0)$ is transformed next by shifting down 3 units: $(-1, 0) \to (-1, -3)$

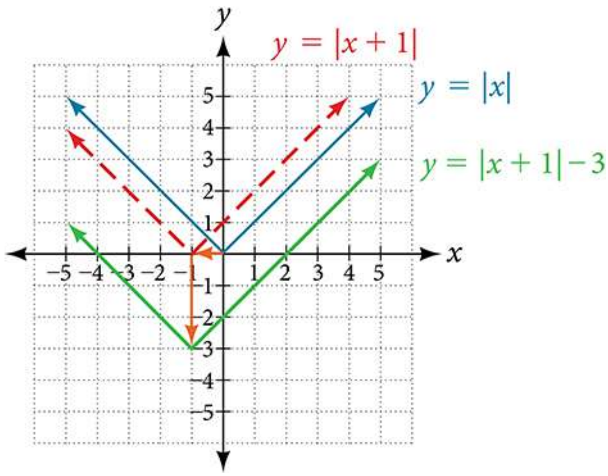


Figure 9

Figure 10 shows the graph of h.

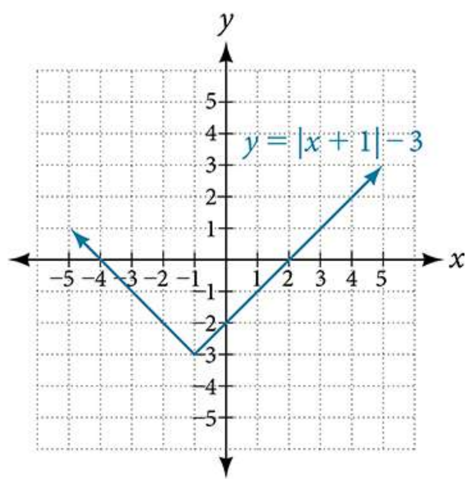


Figure 10

TRY IT #3 Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

EXAMPLE 8

Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure 11, which is a transformation of the toolkit square root function.

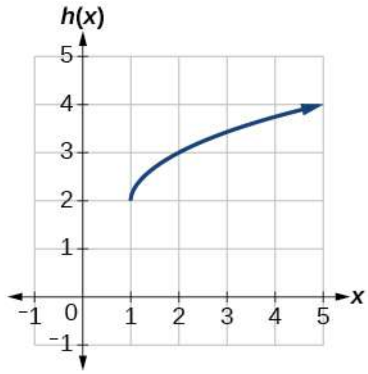


Figure 11

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x - 1) + 2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x - 1} + 2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

TRY IT #4 Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the *x*- or *y*-axis. A **vertical reflection** reflects a graph vertically across the *x*-axis, while a **horizontal reflection** reflects a graph horizontally across the *y*-axis. The reflections are shown in Figure 12.

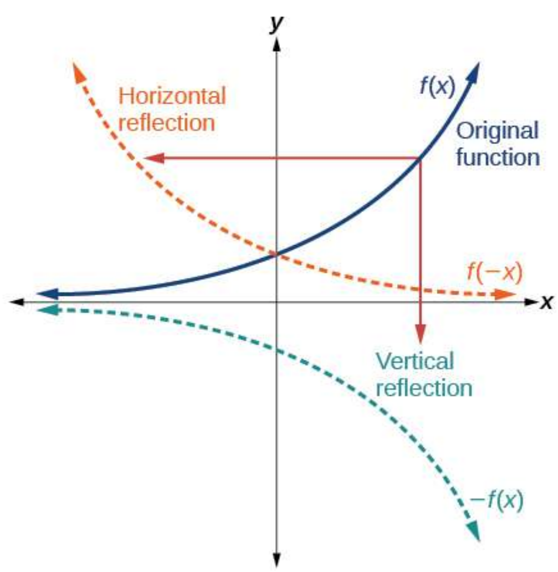


Figure 12 Vertical and horizontal reflections of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the *x*-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the *y*-axis.

Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the *x*-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the *y*-axis.

HOW TO

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the *x*-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the *y*-axis.

EXAMPLE 9

Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution

ⓐ

Reflecting the graph vertically means that each output value will be reflected over the horizontal *t*-axis as shown in Figure 13.

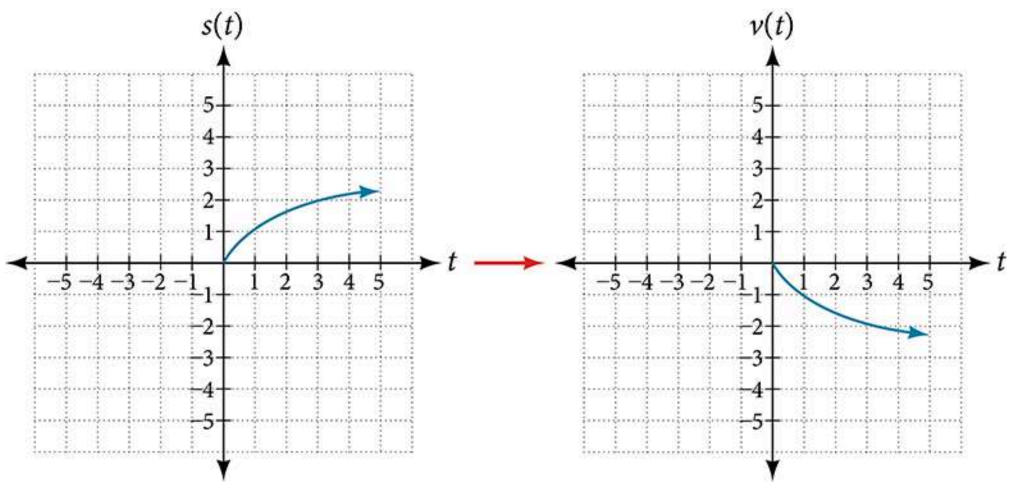


Figure 13 Vertical reflection of the square root function

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

(b)
Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 14.

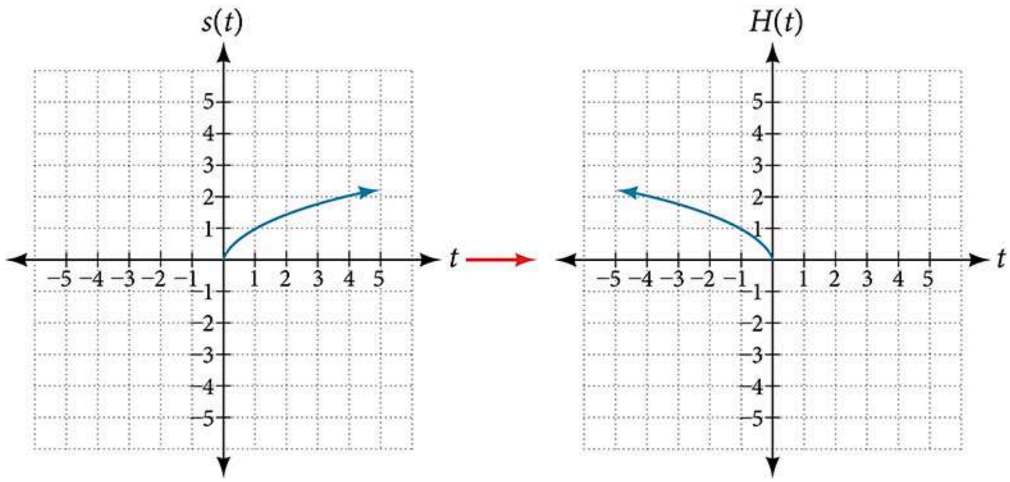


Figure 14 Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $\left(-\infty, \ 0\right]$ and the horizontal reflection gives the $H(t)$ function the domain $\left(-\infty, \ 0\right]$.

TRY IT #5 Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

EXAMPLE 10

Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table 6. Create a table for the functions below.

(a) $g(x) = -f(x)$ (b) $h(x) = f(-x)$

x	2	4	6	8
$f(x)$	1	3	7	11

Table 6

Solution

ⓐ

For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 7.

x	2	4	6	8
$g(x)$	−1	−3	−7	−11

Table 7

ⓑ

For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 8.

x	−2	−4	−6	−8
$h(x)$	1	3	7	11

Table 8

TRY IT #6 A function $f(x)$ is given as Table 9. Create a table for the functions below.

ⓐ $g(x) = -f(x)$ ⓑ $h(x) = f(-x)$

x	−2	0	2	4
$f(x)$	5	10	15	20

Table 9

EXAMPLE 11

Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 15. Sketch a graph of $k(t)$.

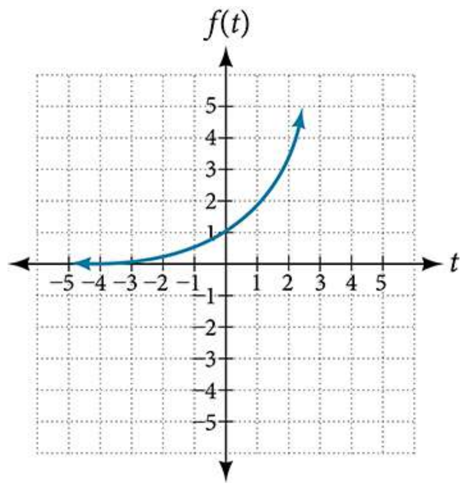


Figure 15

Solution

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points (0, 1) and (1, 2).

1. First, we apply a horizontal reflection: (0, 1) (–1, 2).
2. Then, we apply a vertical reflection: (0, -1) (-1, –2)
3. Finally, we apply a vertical shift: (0, 0) (-1, -1)).

This means that the original points, (0,1) and (1,2) become (0,0) and (-1,-1) after we apply the transformations.

In Figure 16, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

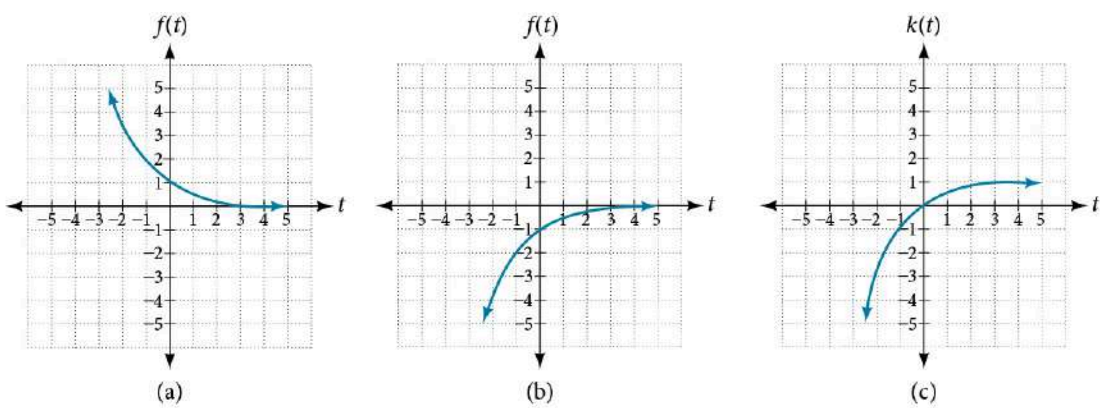


Figure 16

Analysis

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

TRY IT #7 Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the *y*-axis. A function whose graph is symmetric about the *y*-axis is called an **even function.**

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over *both* axes, the result would be the original graph, as shown in Figure 17.

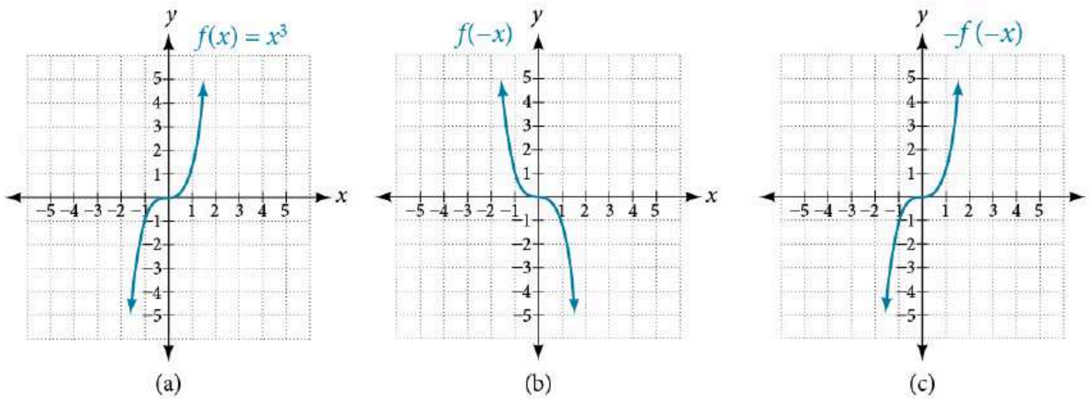


Figure 17 (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Even and Odd Functions

A function is called an **even function** if for every input x

$$f(x) = f(-x)$$

The graph of an even function is symmetric about the y- axis.

A function is called an **odd function** if for every input x

$$f(x) = -f(-x)$$

The graph of an odd function is symmetric about the origin.

HOW TO

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

EXAMPLE 12

Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -\left(-x^3 - 2x\right) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis

Consider the graph of f in Figure 18. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, (1, 3) is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

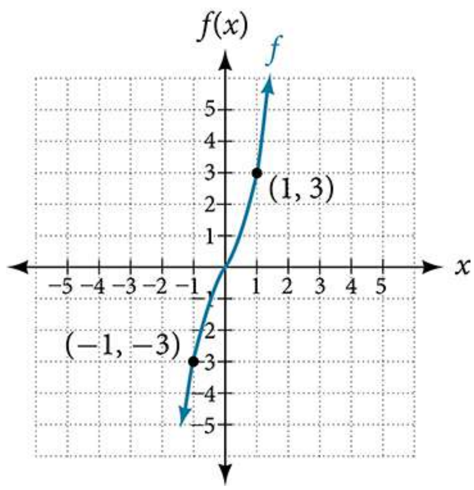


Figure 18

TRY IT #8 Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 19 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

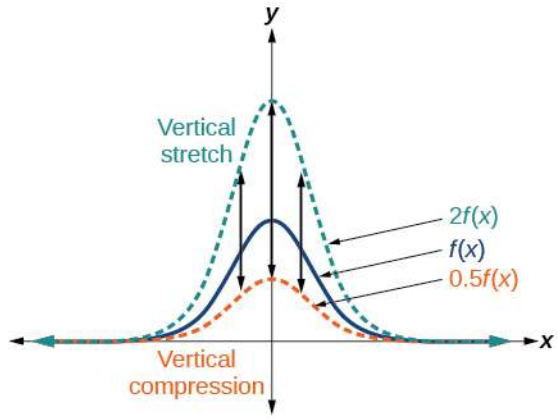


Figure 19 Vertical stretch and compression

Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

HOW TO

Given a function, graph its vertical stretch.

1. Identify the value of a.
2. Multiply all range values by a.
3. If $a > 1$, the graph is stretched by a factor of a.

   If $0 < a < 1$, the graph is compressed by a factor of a.

   If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

EXAMPLE 13

Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure 20.

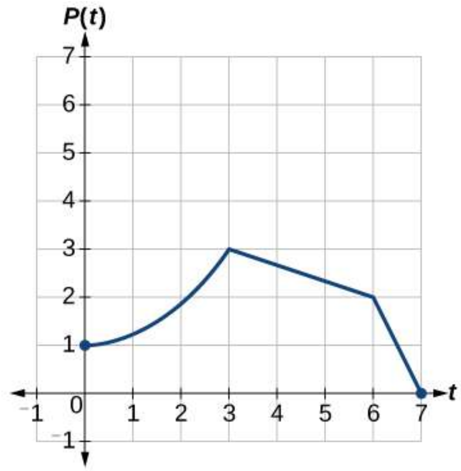


Figure 20

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 21.

If we choose four reference points, (0, 1), (3, 3), (6, 2) and (7, 0) we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$\begin{aligned}(0,\ \ 1) &\to (0,\ \ 2)\\(3,\ \ 3) &\to (3,\ \ 6)\\(6,\ \ 2) &\to (6,\ \ 4)\\(7,\ \ 0) &\to (7,\ \ 0)\end{aligned}$$

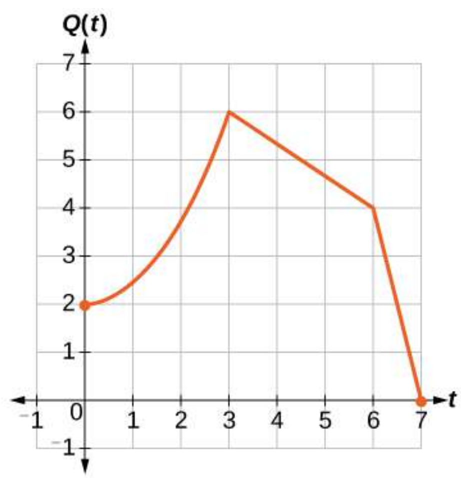


Figure 21

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

HOW TO

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

EXAMPLE 14

Finding a Vertical Compression of a Tabular Function

A function f is given as Table 10. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 10

Solution

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 11.

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Table 11

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.

TRY IT #9 A function f is given as Table 12. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

x	2	4	6	8
$f(x)$	12	16	20	0

Table 12

EXAMPLE 15

Recognizing a Vertical Stretch

The graph in Figure 22 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

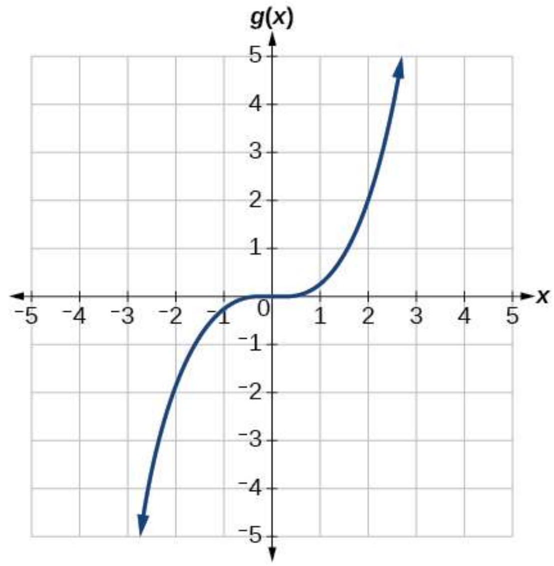


Figure 22

Solution

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3$$

TRY IT #10 Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

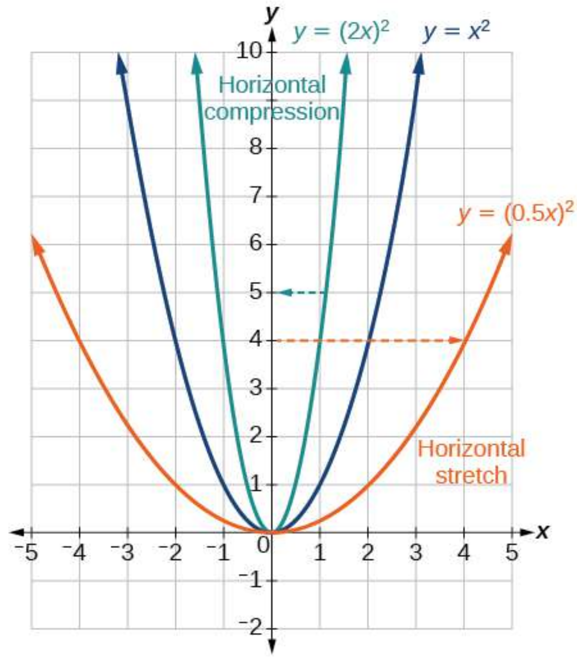


Figure 23

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 23. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of $\frac{1}{2}$.

Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

HOW TO

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

EXAMPLE 16

Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$\begin{aligned} R(1) &= P(2), \\ R(2) &= P(4), \quad \text{and in general,} \\ R(t) &= P(2t). \end{aligned}$$

See Figure 24 for a graphical comparison of the original population and the compressed population.

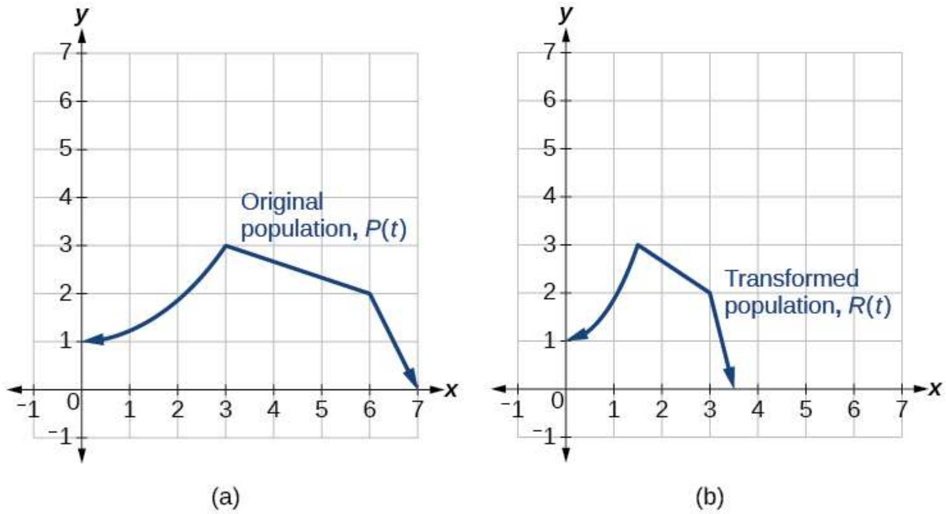


Figure 24 (a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

EXAMPLE 17

Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 13. Create a table for the function $g(x) = f\left(\frac{1}{2}x\right)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 13

Solution

The formula $g(x) = f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f\left(\frac{1}{2} \cdot 2\right) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f\left(\frac{1}{2} \cdot 4\right) = f(2) = 1$$

We do the same for the other values to produce Table 14.

x	4	8	12	16

Table 14

$g(x)$	1	3	7	11

Table 14

Figure 25 shows the graphs of both of these sets of points.

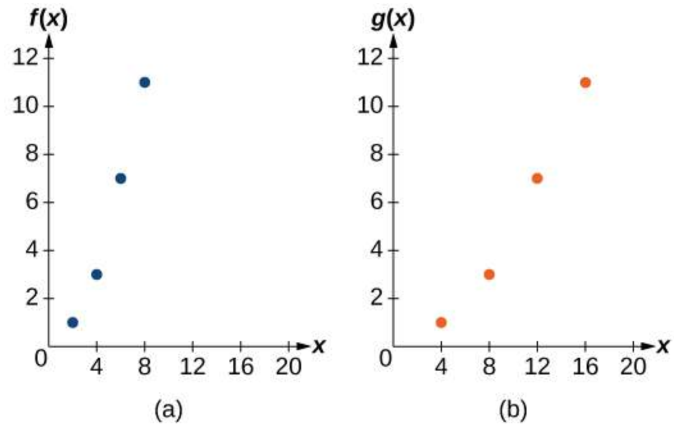


Figure 25

Analysis

Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

EXAMPLE 18

Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 26.

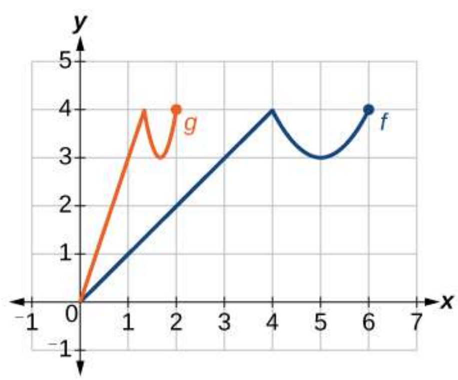


Figure 26

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at $(6, 4)$ and $g(x)$ ends at $(2, 4)$, we can see that the x- values have been compressed by $\frac{1}{3}$, because $6\left(\frac{1}{3}\right) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f\left(\frac{1}{4}x\right)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

TRY IT #11 Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x) + 3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x + 3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7) = 12$. What input to g would produce that output? In other words, what value of x will allow $g(x) = f(2x + 3) = 12$? We would need $2x + 3 = 7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx + p) = f\left(b\left(x + \frac{p}{b}\right)\right)$$

Let’s work through an example.

$$f(x) = (2x + 4)^2$$

We can factor out a 2.

$$f(x) = (2(x + 2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

When combining vertical transformations written in the form $af(x) + k$, first vertically stretch by a and then vertically shift by k.

When combining horizontal transformations written in the form $f(bx - h)$, first horizontally shift by $\frac{h}{b}$ and then horizontally stretch by $\frac{1}{b}$.

When combining horizontal transformations written in the form $f(b(x - h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

EXAMPLE 19

Finding a Triple Transformation of a Tabular Function

Given Table 15 for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x) + 1$.

x	6	12	18	24
$f(x)$	10	14	15	17

Table 15

Solution

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x- value by $\frac{1}{3}$. See Table 16.

x	2	4	6	8
$f(3x)$	10	14	15	17

Table 16

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 17.

x	2	4	6	8
$2f(3x)$	20	28	30	34

Table 17

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 18.

x	2	4	6	8
$g(x) = 2f(3x) + 1$	21	29	31	35

Table 18

EXAMPLE 20

Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 27 to sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$.

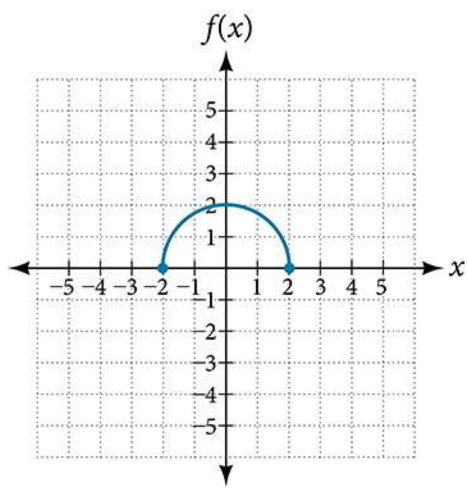


Figure 27

Solution

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x + 1\right) - 3 = f\left(\frac{1}{2}(x + 2)\right) - 3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point (0,2) remains at (0,2) while the point (2,0) will stretch to (4,0). See Figure 28.

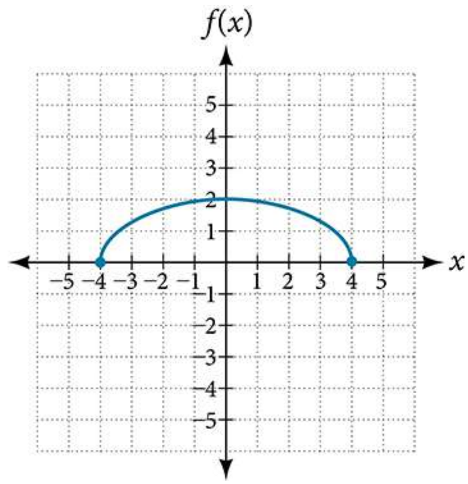


Figure 28

Next, we horizontally shift left by 2 units, as indicated by $x + 2$. See Figure 29.

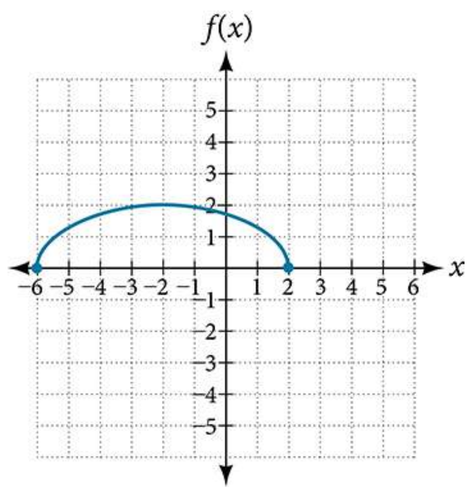


Figure 29

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See Figure 30.

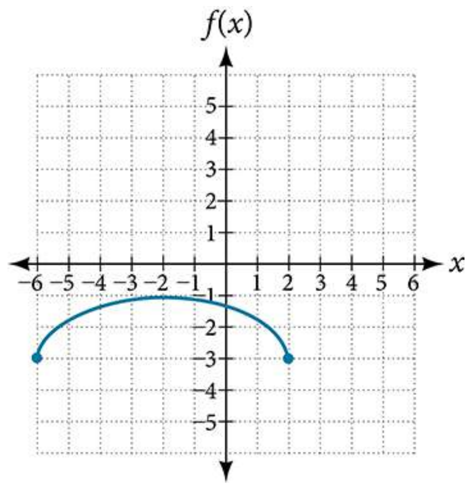


Figure 30

MEDIA

Access this online resource for additional instruction and practice with transformation of functions.

Function Transformations (http://openstax.org/l/functrans)

3.5 SECTION EXERCISES

Verbal

1. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?

2. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?

3. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?

4. When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the *x*-axis from a reflection with respect to the *y*-axis?

5. How can you determine whether a function is odd or even from the formula of the function?

Algebraic

For the following exercises, write a formula for the function obtained when the graph is shifted as described.

6. $f(x) = \sqrt{x}$ is shifted up 1 unit and to the left 2 units.

7. $f(x) = |x|$ is shifted down 3 units and to the right 1 unit.

8. $f(x) = \frac{1}{x}$ is shifted down 4 units and to the right 3 units.

9. $f(x) = \frac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

10. $y = f(x - 49)$

11. $y = f(x + 43)$

12. $y = f(x + 3)$

13. $y = f(x - 4)$

14. $y = f(x) + 5$

15. $y = f(x) + 8$

16. $y = f(x) - 2$

17. $y = f(x) - 7$

18. $y = f(x - 2) + 3$

19. $y = f(x + 4) - 1$

For the following exercises, determine the interval(s) on which the function is increasing and decreasing.

20. $f(x) = 4(x + 1)^2 - 5$

21. $g(x) = 5(x + 3)^2 - 2$

22. $a(x) = \sqrt{-x + 4}$

23. $k(x) = -3\sqrt{x} - 1$

Graphical

For the following exercises, use the graph of $f(x) = 2^x$ shown in Figure 31 to sketch a graph of each transformation of $f(x)$.

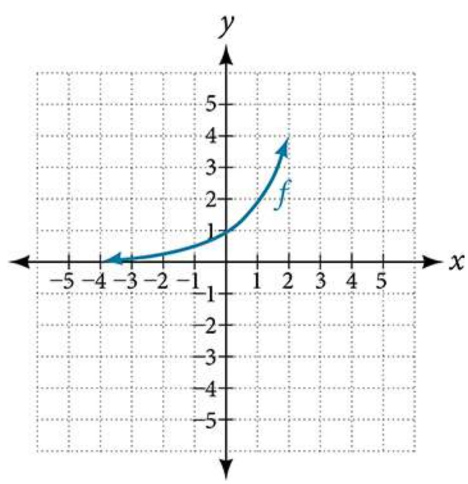


Figure 31

24. $g(x) = 2^x + 1$

25. $h(x) = 2^x - 3$

26. $w(x) = 2^{x-1}$

For the following exercises, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

27. $f(t) = (t + 1)^2 - 3$

28. $h(x) = |x - 1| + 4$

29. $k(x) = (x - 2)^3 - 1$

30. $m(t) = 3 + \sqrt{t + 2}$

Numeric

31. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	−2	−1	0	1	2
$f(x)$	−2	−1	−3	1	2

x	−1	0	1	2	3
$g(x)$	−2	−1	−3	1	2

x	−2	−1	0	1	2
$h(x)$	−1	0	−2	2	3

32. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

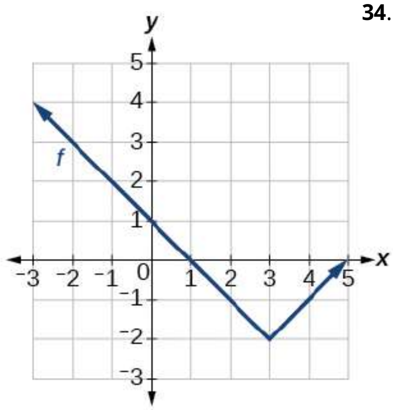
x	−2	−1	0	1	2
$f(x)$	−1	−3	4	2	1

x	−3	−2	−1	0	1
$g(x)$	−1	−3	4	2	1

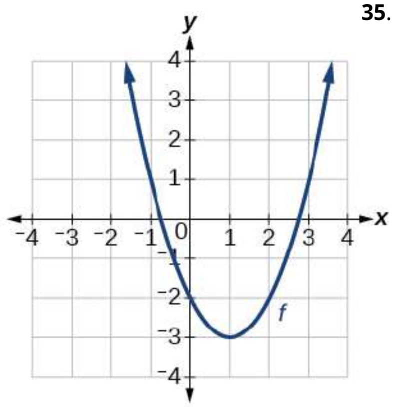
x	−2	−1	0	1	2
$h(x)$	−2	−4	3	1	0

For the following exercises, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

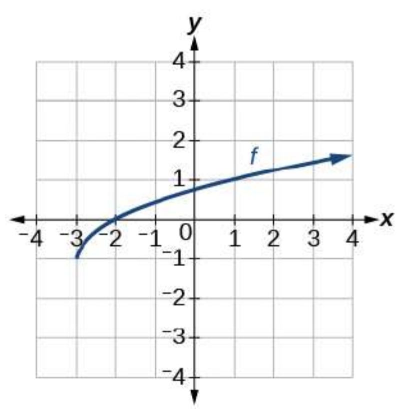
33.



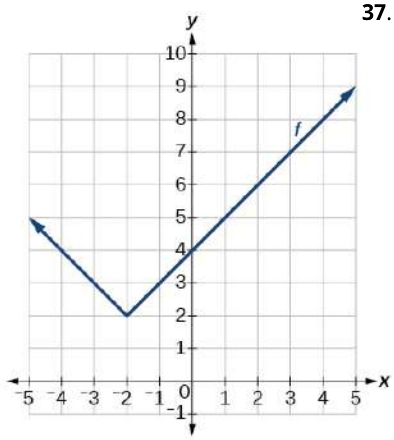
34.



35.



36.



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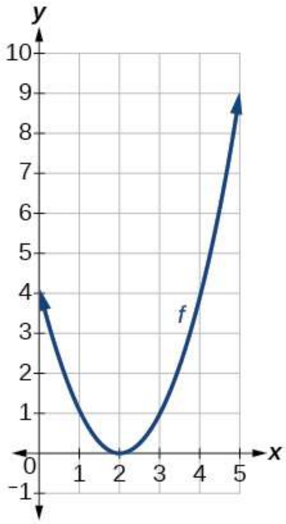

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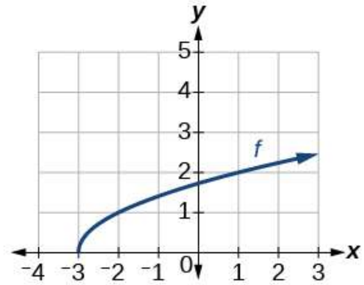
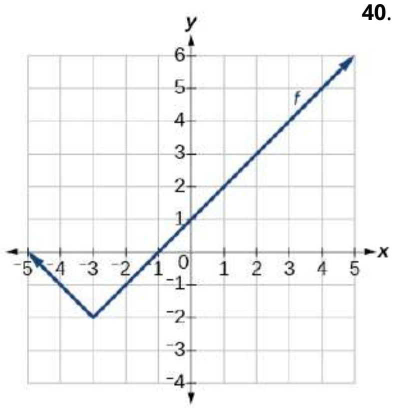


39.



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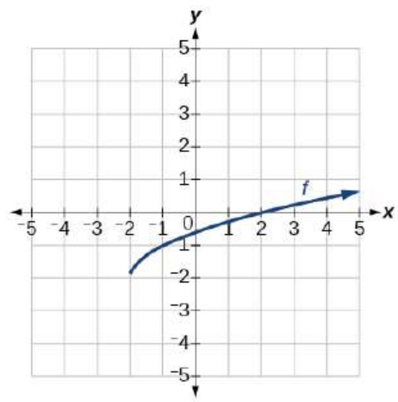
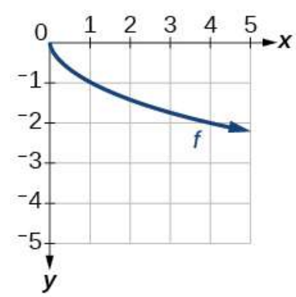
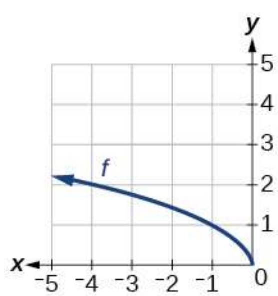


For the following exercises, use the graphs of transformations of the square root function to find a formula for each of the functions.

41.

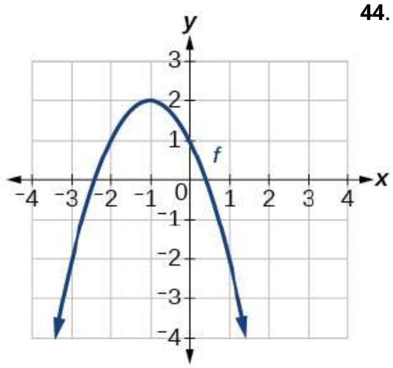


42.



For the following exercises, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

43.



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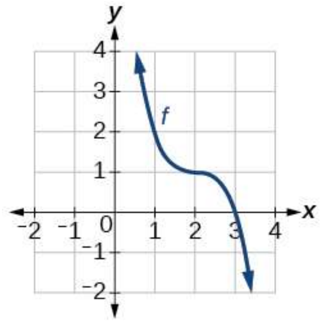

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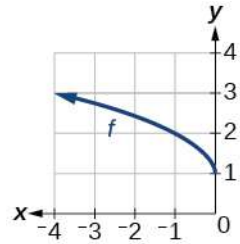
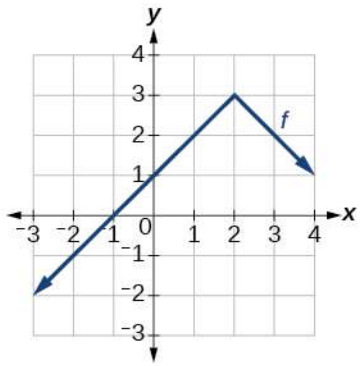


46.



For the following exercises, determine whether the function is odd, even, or neither.

47. $f(x) = 3x^4$

48. $g(x) = \sqrt{x}$

49. $h(x) = \frac{1}{x} + 3x$

50. $f(x) = (x-2)^2$

51. $g(x) = 2x^4$

52. $h(x) = 2x - x^3$

For the following exercises, describe how the graph of each function is a transformation of the graph of the original function f.

53. $g(x) = -f(x)$

54. $g(x) = f(-x)$

55. $g(x) = 4f(x)$

56. $g(x) = 6f(x)$

57. $g(x) = f(5x)$

58. $g(x) = f(2x)$

59. $g(x) = f\left(\frac{1}{3}x\right)$

60. $g(x) = f\left(\frac{1}{5}x\right)$

61. $g(x) = 3f(-x)$

62. $g(x) = -f(3x)$

For the following exercises, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

63. The graph of $f(x) = |x|$ is reflected over the y -axis and horizontally compressed by a factor of $\frac{1}{4}$.

64. The graph of $f(x) = \sqrt{x}$ is reflected over the x -axis and horizontally stretched by a factor of 2.

65. The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.

66. The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

67. The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.

68. The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

For the following exercises, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

69. $g(x) = 4(x+1)^2 - 5$

70. $g(x) = 5(x+3)^2 - 2$

71. $h(x) = -2|x-4| + 3$

72. $k(x) = -3\sqrt{x} - 1$

73. $m(x) = \frac{1}{2}x^3$

74. $n(x) = \frac{1}{3}|x-2|$

75. $p(x) = \left(\frac{1}{3}x\right)^3 - 3$

76. $q(x) = \left(\frac{1}{4}x\right)^3 + 1$

77. $a(x) = \sqrt{-x+4}$

For the following exercises, use the graph in Figure 32 to sketch the given transformations.

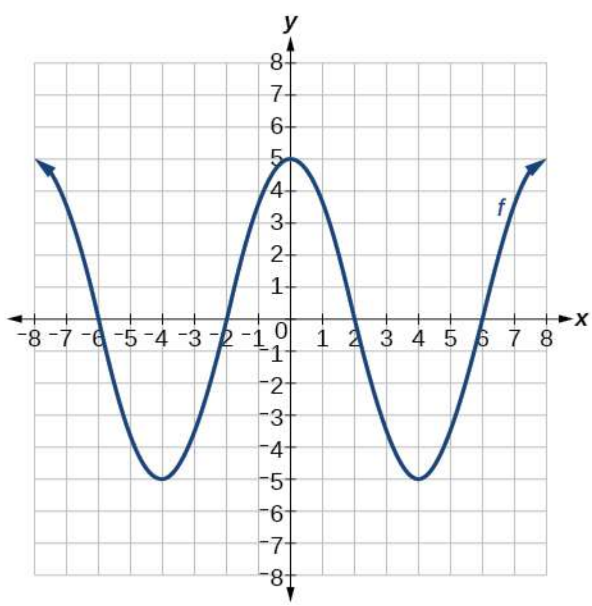


Figure 32

78. $g(x) = f(x) - 2$

79. $g(x) = -f(x)$

80. $g(x) = f(x+1)$

81. $g(x) = f(x-2)$

3.6 Absolute Value Functions

Learning Objectives

In this section, you will:

- Graph an absolute value function.
- Solve an absolute value equation.

COREQUISITE SKILLS

Learning Objectives

1. Solve absolute value equations (IA 2.7.1)
2. Identify graphs of absolute value functions (IA 3.6.2)

Objective 1: Solve absolute value equations (IA 2.7.1)

Recall that in its basic form, $f(x)=|x|$, the absolute value function is one of our toolkit functions. The absolute value function is often thought of as providing the distance the number is from zero on a number line. Numerically, for whatever the input value is, the output is the magnitude of this value.

The absolute value function can be defined as a piecewise function

$f(x) = |x| = -x$, when $x < 0$ or x , when $x \geq 0$

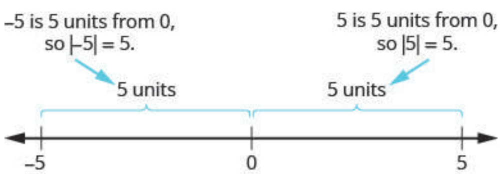
EXAMPLE 1

Solve absolute value equations

ⓐ Solve for x: |x|=5

Solution

Since the absolute value of a number is its distance from 0 on the number line. Notice that both 5 and –5 are 5 units from 0 on the number line.



If |x|=5
x=5, or x=–5

ⓑ Solve for x: |2x–5|=3

Solution

In this case the number represented by 2x–5 is 3 units from zero on the number line. So 2x–5 could either equal 3 or –3.

$2x - 5 = 3$
$2x = 8$
$x = 4$

$2x - 5 = -3$
$2x = 2$
$x = 1$

Practice Makes Perfect

Solve absolute value equations.

1. Solve for x: $|x| = 0$

2. Solve for x: $|x| = 2$

3. Solve for x: $|x – 2| = 6$

4. Solve for x: $|2x + 1| = 7$

5. Solve for t: $|3t – 1| = –2$

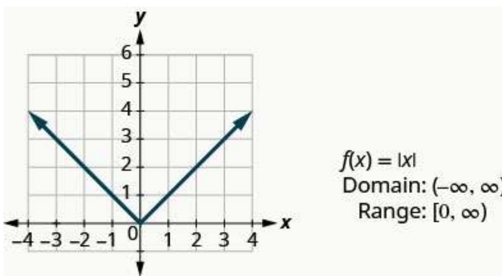
6. Solve for z: $|2z – 3| – 4 = 1$

7. Solve for x: $–2|x – 3| + 8 = –4$

Objective 2: Identify and graph absolute value functions (IA 3.6.2)

Absolute value functions have a “V” shaped graph. If scanning this function from left to right the corner is the point where the graph changes direction.

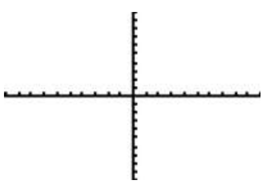
$f(x) = |x| = -x$, when $x < 0$ or x , when $x \geq 0$



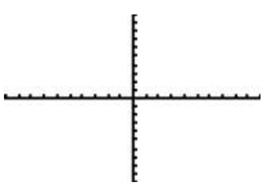
Practice Makes Perfect

Identify and graph absolute value functions. Graph each of the following functions. Label at least one point on your graph.

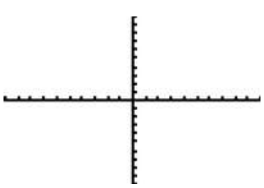
8. $y = |x| – 2$



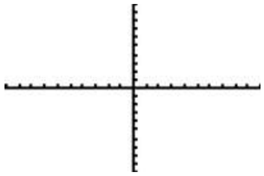
9. $y = |2x| + 3$



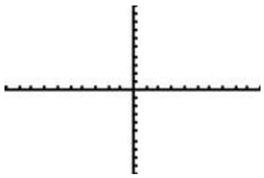
10. $y = |x – 4| + 3$



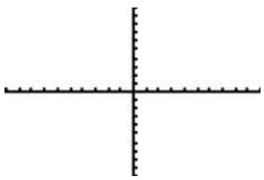
11. $y = -|x+6| - 4$



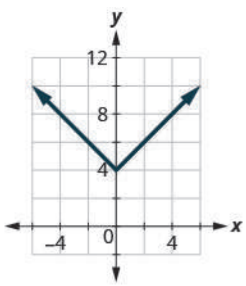
12. $y = -|x-2|$



13. $y = 3|x+1| + 4$



14. Find the domain and range of the following absolute value function.



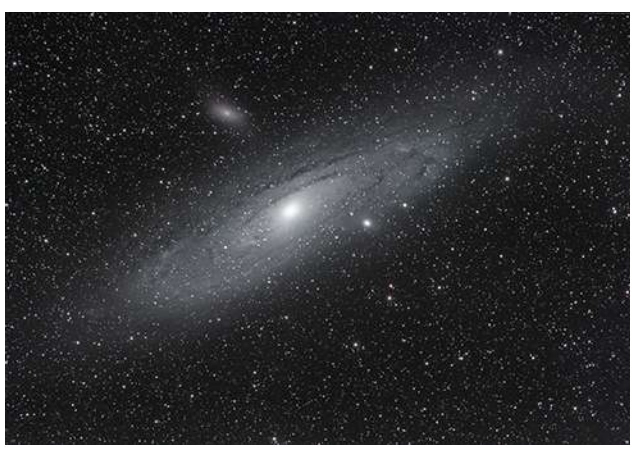


Figure 1 Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will continue our investigation of absolute value functions.

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the absolute value function is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign. Knowing this, we can use absolute value functions to solve some kinds of real-world problems.

Absolute Value Function

The absolute value function can be defined as a piecewise function

$$f(x) = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

EXAMPLE 1

Using Absolute Value to Determine Resistance

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often $\pm1\%$, $\pm 5\%$, or $\pm10\%$.

Suppose we have a resistor rated at 680 ohms, $\pm5\%$. Use the absolute value function to express the range of possible values of the actual resistance.

Solution

We can find that 5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R - 680| \leq 34$$

TRY IT #1 Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure 2.

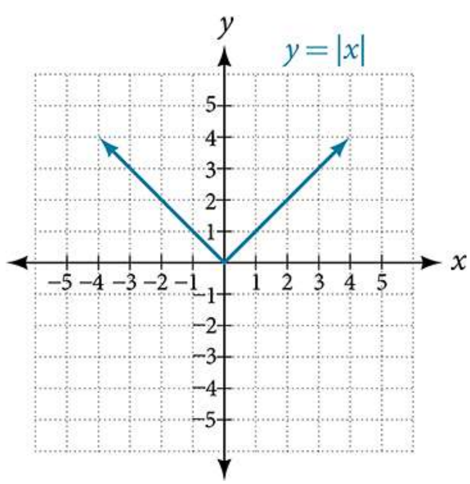


Figure 2

Figure 3 shows the graph of $y = 2\,|x - 3| + 4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

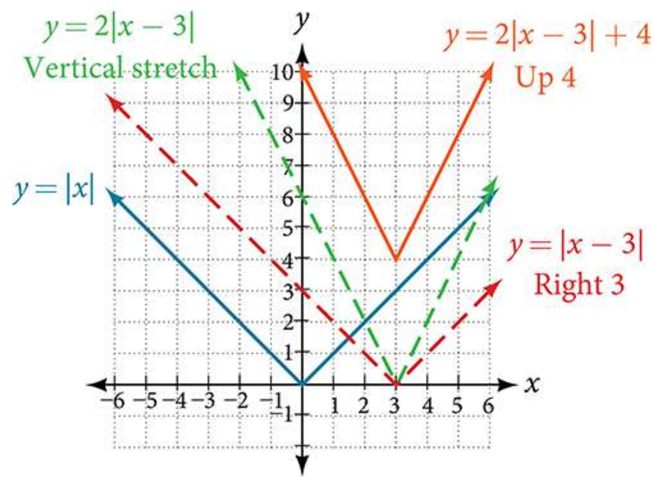


Figure 3

EXAMPLE 2

Writing an Equation for an Absolute Value Function Given a Graph

Write an equation for the function graphed in Figure 4.

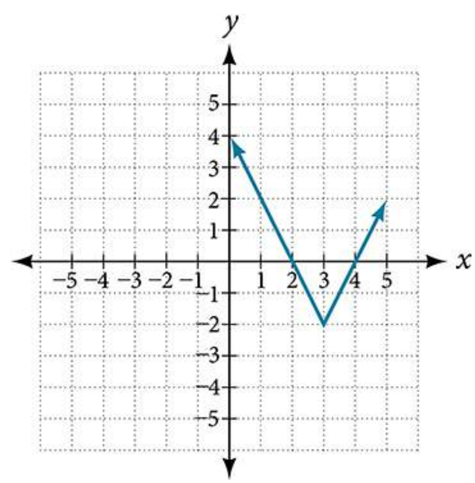


Figure 4

Solution

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure 5.

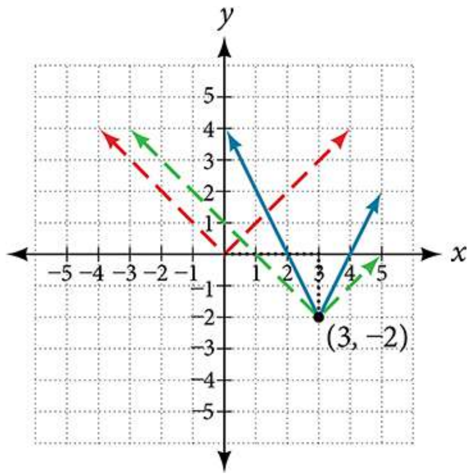


Figure 5

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure 6.

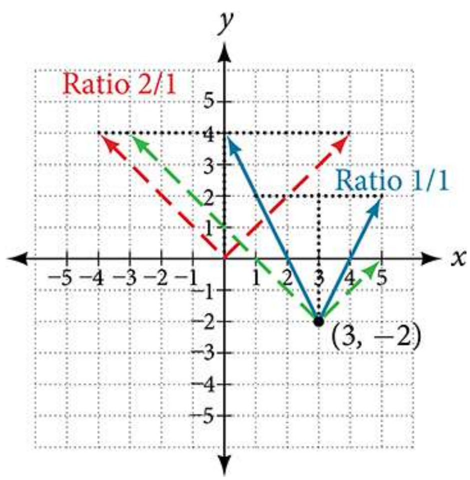


Figure 6

From this information we can write the equation

$$\begin{aligned} f(x) &= 2|x-3|-2, && \text{treating the stretch as } a \text{ vertical stretch,or} \\ f(x) &= |2(x-3)|-2, && \text{treating the stretch as } a \text{ horizontal compression.} \end{aligned}$$

Analysis

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression. Note also that if the vertical stretch factor is negative, there is also a reflection about the x-axis.

Q&A **If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?**

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

$$f(x) = a|x-3|-2$$

Now substituting in the point (1, 2)

$$\begin{aligned} 2 &= a|1-3|-2 \\ 4 &= 2a \\ a &= 2 \end{aligned}$$

TRY IT #2 Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Q&A **Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?**

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (see Figure 7).

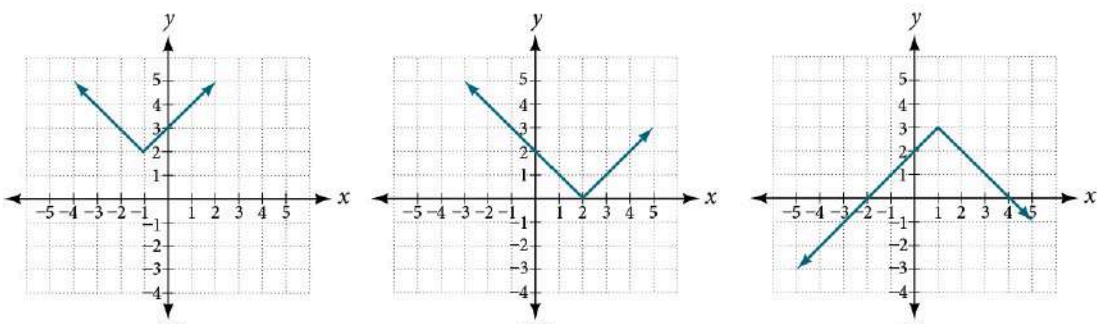


Figure 7 (a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

Solving an Absolute Value Equation

In Other Type of Equations, we touched on the concepts of absolute value equations. Now that we understand a little more about their graphs, we can take another look at these types of equations. Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8=|2x-6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$\begin{aligned} 2x-6 &= 8 \\ 2x &= 14 \\ x &= 7 \end{aligned} \quad \text{or} \quad \begin{aligned} 2x-6 &= -8 \\ 2x &= -2 \\ x &= -1 \end{aligned}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An absolute value equation is an equation in which the unknown variable appears in absolute value bars. For example,

$$\begin{gathered} |x|=4, \\ |2x-1|=3, \text{ or} \\ |5x+2|-4=9 \end{gathered}$$

Solutions to Absolute Value Equations

For real numbers A and B , an equation of the form $|A|=B$, with $B \geq 0$, will have solutions when $A=B$ or $A=-B$. If $B<0$, the equation $|A|=B$ has no solution.

HOW TO

Given the formula for an absolute value function, find the horizontal intercepts of its graph.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $-\mathrm{A} = B$, assuming $B > 0$.
3. Solve for x.

EXAMPLE 3

Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x + 1| - 7$, find the values of x such that $f(x) = 0$.

Solution

$$\begin{array}{rcll} 0 & = & |4x+1| - 7 & \text{Substitute 0 for } f(x). \\ 7 & = & |4x+1| & \text{Isolate the absolute value on one side of the equation.} \end{array}$$

$$\begin{array}{rclcrcll} 7 & = & 4x+1 & \text{or} & -7 & = & 4x+1 & \text{Break into two separate equations and solve.} \\ 6 & = & 4x & & -8 & = & 4x & \\ x & = & \frac{6}{4} = 1.5 & & x & = & \frac{-8}{4} = -2 & \end{array}$$

The function outputs 0 when $x = \frac{3}{2}$ or $x = -2$. See Figure 8.

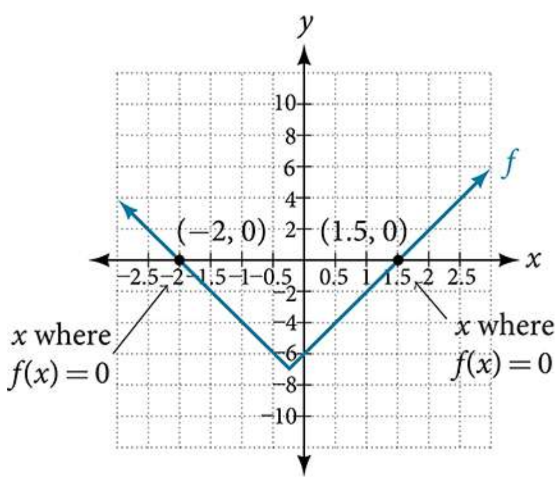


Figure 8

TRY IT #3 For the function $f(x) = |2x - 1| - 3$, find the values of x such that $f(x) = 0$.

Q&A **Should we always expect two answers when solving $|A| = B$?**

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x - 5| = 1$.

MEDIA

Access these online resources for additional instruction and practice with absolute value.

Graphing Absolute Value Functions (http://openstax.org/l/graphabsvalue)

Graphing Absolute Value Functions 2 (http://openstax.org/l/graphabsvalue2)

3.6 SECTION EXERCISES

Verbal

1. How do you solve an absolute value equation?

2. How can you tell whether an absolute value function has two *x*-intercepts without graphing the function?

3. When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?

4. How can you use the graph of an absolute value function to determine the *x*-values for which the function values are negative?

Algebraic

5. Describe all numbers x that are at a distance of 4 from the number 8. Express this set of numbers using absolute value notation.

6. Describe all numbers x that are at a distance of $\frac{1}{2}$ from the number −4. Express this set of numbers using absolute value notation.

7. Describe the situation in which the distance that point x is from 10 is at least 15 units. Express this set of numbers using absolute value notation.

8. Find all function values $f(x)$ such that the distance from $f(x)$ to the value 8 is less than 0.03 units. Express this set of numbers using absolute value notation.

For the following exercises, find the x- and y-intercepts of the graphs of each function.

9. $f(x) = 4\,|x - 3| + 4$

10. $f(x) = -3\,|x - 2| - 1$

11. $f(x) = -2\,|x + 1| + 6$

12. $f(x) = -5|x + 2| + 15$

13. $f(x) = 2|x - 1| - 6$

14. $f(x) = |-2x + 1| - 13$

15. $f(x) = -|x - 9| + 16$

Graphical

For the following exercises, graph the absolute value function. Plot at least five points by hand for each graph.

16. $y = |x - 1|$

17. $y = |x + 1|$

18. $y = |x| + 1$

For the following exercises, graph the given functions by hand.

19. $y = |x| - 2$

20. $y = -|x|$

21. $y = -|x| - 2$

22. $y = -|x - 3| - 2$

23. $f(x) = -|x - 1| - 2$

24. $f(x) = -|x + 3| + 4$

25. $f(x) = 2|x + 3| + 1$

26. $f(x) = 3|x - 2| + 3$

27. $f(x) = |2x - 4| - 3$

28. $f(x) = |3x + 9| + 2$

29. $f(x) = -|x - 1| - 3$

30. $f(x) = -|x + 4| - 3$

31. $f(x) = \frac{1}{2}|x + 4| - 3$

Technology

32. Use a graphing utility to graph $f(x) = 10|x - 2|$ on the viewing window $[0, 4]$. Identify the corresponding range. Show the graph.

33. Use a graphing utility to graph $f(x) = -100|x| + 100$ on the viewing window $[-5, 5]$. Identify the corresponding range. Show the graph.

For the following exercises, graph each function using a graphing utility. Specify the viewing window.

34. $f(x) = -0.1\,|0.1(0.2 - x)| + 0.3$

35. $f(x) = 4 \times 10^9\left|x - \left(5 \times 10^9\right)\right| + 2 \times 10^9$

Extensions

For the following exercises, solve the inequality.

36. If possible, find all values of a such that there are no x- intercepts for $f(x) = 2|x + 1| + a$.

37. If possible, find all values of a such that there are no y-intercepts for $f(x) = 2|x + 1| + a$.

Real-World Applications

38. Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least 100 miles and x represents the distance from city B to city A, express this using absolute value notation.

39. The true proportion p of people who give a favorable rating to Congress is 8% with a margin of error of 1.5%. Describe this statement using an absolute value equation.

40. Students who score within 18 points of the number 82 will pass a particular test. Write this statement using absolute value notation and use the variable x for the score.

41. A machinist must produce a bearing that is within 0.01 inches of the correct diameter of 5.0 inches. Using x as the diameter of the bearing, write this statement using absolute value notation.

42. The tolerance for a ball bearing is 0.01. If the true diameter of the bearing is to be 2.0 inches and the measured value of the diameter is x inches, express the tolerance using absolute value notation.

3.7 Inverse Functions

Learning Objectives

In this section, you will:

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

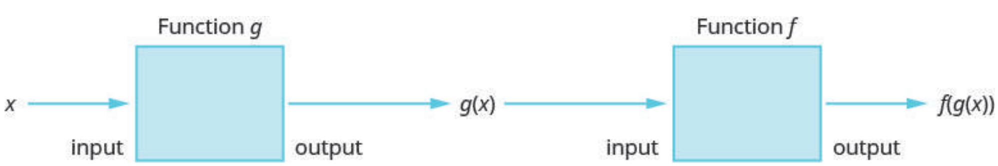
COREQUISITE SKILLS

Learning Objectives

1. Find and evaluate composite functions (IA 10.1.1).
2. Determine whether a function is one-to-one (IA 10.1.2).

Objective 1: Find and evaluate composite functions (IA 10.1.1).

A **composite** function is a two-step function and can have numerical or variable inputs.



$(f \circ g)(x) = f(g(x))$ is read as "f of g of x".

To evaluate a composite function, we always start by evaluating the inner function and then evaluate the outer function in terms of the inner function.

EXAMPLE 1

Find and evaluate composite functions.

For functions $f(x) = 2x - 7$, $g(x) = \frac{x+7}{2}$, find:

ⓐ $g(5)$ ⓑ $f(g(5))$ ⓒ $f(g(x))$

Solution

ⓐ To find $g(5)$, we evaluate $g(x)$ when x is 5.

$g(x) = \frac{x+7}{2}$

$g(5) = \frac{5+7}{2} = \frac{12}{2} = 6$

ⓑ To find $f(g(5))$, we start evaluating the inner function g in terms of 5 (see part a) and then evaluate the outer function f in terms of this value.

$f(g(5)) = f(6) = 2(6) - 7 = 12 - 7 = 5$

ⓒ In parts a and b we had numerical outputs because our inputs were numbers. When we find $f(g(x))$ this will be a function written in terms of the variable x.

$f\left(g(x)\right) = f\left(\frac{x+7}{2}\right) = 2\left(\frac{x+7}{2}\right) - 7 = x + 7 - 7 = x$

This is interesting, notice that the functions $f(x)$ and $g(x)$ have a special relationship in that one undoes the other. We call functions like this, **inverses** of one another. For any one-to-one function $f(x)$, the inverse is a function $f^{-1}\left(x\right)$ such that $f^{-1}\left(f\left(x\right)\right) = x$.

Practice Makes Perfect

Find and evaluate composite functions.

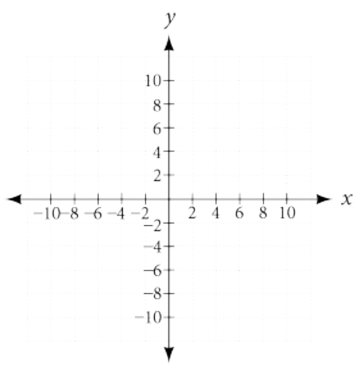
For each of the following function pairs find:

1. $f(x)= \sqrt[3]{x-2}$, $g(x)= x^3 + 2$

ⓐ $f(g(x))$

ⓑ $g(f(x))$

ⓒ Graph the functions f(x) and g(x) on the same coordinate system below



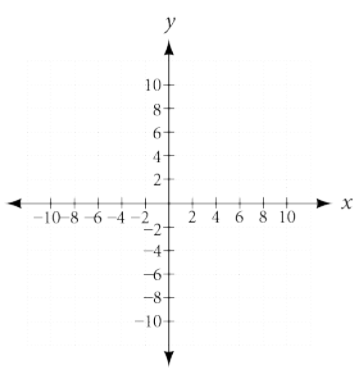
ⓓ What do you notice about the relationship between the graphs of f(x) and g(x)?

2. $f\left(x\right) = \frac{1}{(x+3)}$, $g\left(x\right) = \frac{1}{x} - 3$

ⓐ $f(g(x))$

ⓑ $g(f(x))$

ⓒ Graph the functions f(x) and g(x) on the same coordinate system below



ⓓ What do you notice about the relationship between the graphs of f(x) and g(x)?

Objective 2: Determine whether a function is one-to-one (IA 10.1.2).

In creating a process called a function, f(x), it is often useful to undo this process, or create an inverse to the function, $f^{-1}(x)$. When finding the inverse, we restrict our work to one-to-one functions, this means that the inverse we find should also be one-to-one. Remember that the horizontal line test is a great way to check to see if a graph represents a one-to-one function.

For any one-to-one function f(x), the inverse is a function $f^{-1}(x)$ such that $f^{-1}\left(f(x)\right) = x$ and $f\left(f^{-1}(x)\right) = x$.

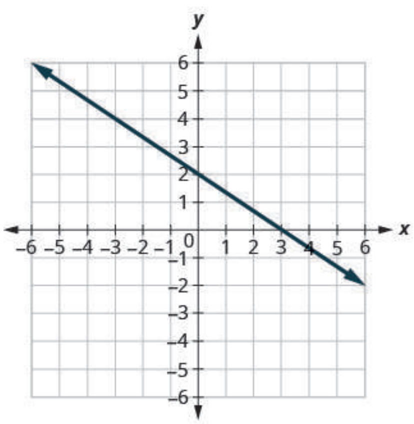
The following key terms will be important to our understanding of functions and their inverses.

Function: a relation in which each input value yields a unique output value.
Vertical line test: a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once.
One-to-one function: a function for which each value of the output is associated with a unique input value.
Horizontal line test: a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once.

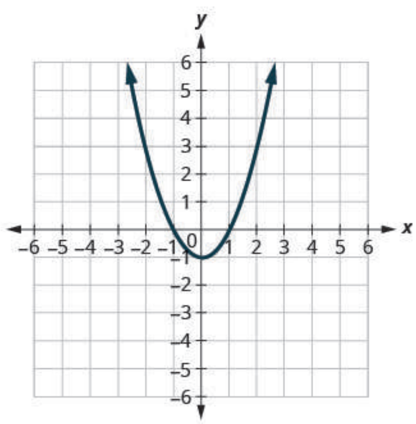
EXAMPLE 2

Determine whether a function is one-to-one.

Determine (a) whether each graph is the graph of a function and, if so, (b) whether it is one-to-one.



(a)



(b)

Solution

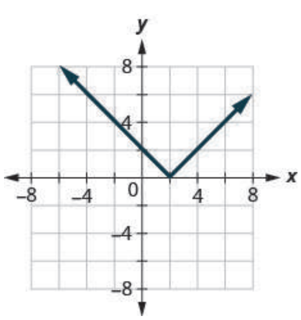
(a) Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. Since any horizontal line intersects the graph in at most one point, the graph is the graph of a one-to-one function.

(b) Since any vertical line intersects the graph in at most one point, the graph is the graph of a function. However, a horizontal line shown on the graph may intersect it in two points. This graph does not represent a one-to-one function.

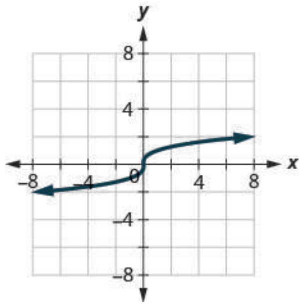
Practice Makes Perfect

Determine whether each graph is the graph of a function and, if so, whether it is one-to-one.

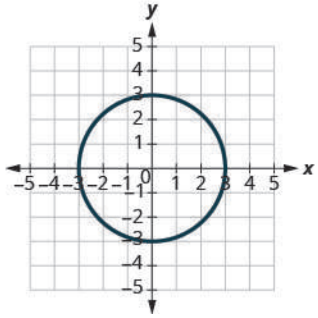
3.



4.



5.



A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function “machines” we have been studying can also run backwards. Figure 1 provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

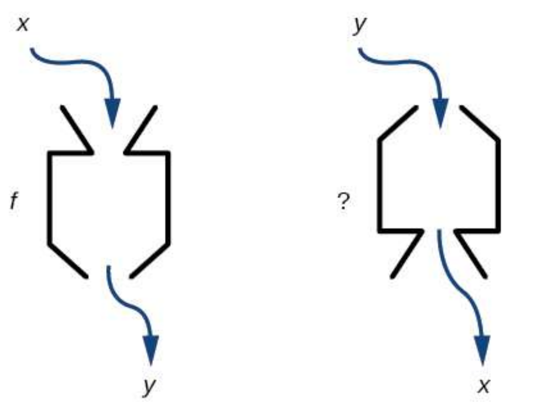


Figure 1 Can a function “machine” operate in reverse?

Verifying That Two Functions Are Inverse Functions

Betty is traveling to Milan for a fashion show and wants to know what the temperature will be. She is not familiar with the Celsius scale. To get an idea of how temperature measurements are related, Betty wants to convert 75 degrees Fahrenheit to degrees Celsius using the formula

$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24°\text{C}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, Betty gets the week’s weather forecast from Figure 2 for Milan, and wants to convert all of the temperatures to degrees Fahrenheit.

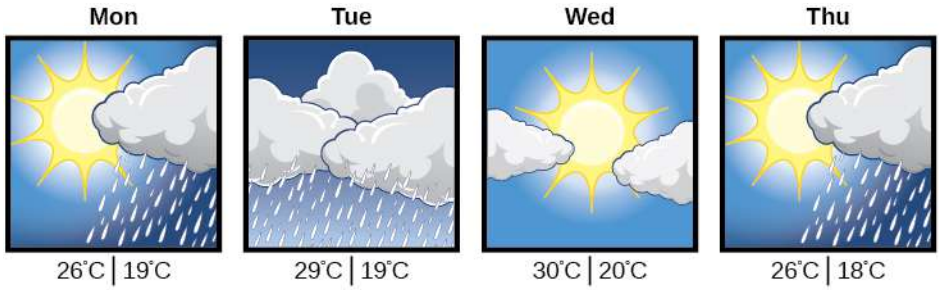


Figure 2

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$\begin{aligned} 26 &= \tfrac{5}{9}(F-32) \\ 26 \cdot \tfrac{9}{5} &= F-32 \\ F &= 26 \cdot \tfrac{9}{5} + 32 \approx 79 \end{aligned}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1} \circ f$ equals the identity function, that is,

$$\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$\left(f^{-1} \circ f\right)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x$$

and

$$\left(f \circ f^{-1}\right)(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x$$

A few coordinate pairs from the graph of the function $y = 4x$ are (−2, −8), (0, 0), and (2, 8). A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are (−8, −2), (0, 0), and (8, 2). If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Inverse Function

For any one-to-one function $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

and not all functions have inverses.

EXAMPLE 1

Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution

The inverse function reverses the input and output quantities, so if

$$\begin{aligned} f(2) &= 4, \quad \text{then } f^{-1}(4) = 2; \\ f(5) &= 12, \quad \text{then } \mathrm{f}^{-1}(12) = 5. \end{aligned}$$

Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table 1.

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Table 1

TRY IT #1 Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

HOW TO

Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.
2. If either statement is true, then both are true, and $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

EXAMPLE 2

Testing Inverse Relationships Algebraically

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

Solution

$$\begin{aligned} g(f(x)) &= \frac{1}{\left(\frac{1}{x+2}\right)} - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

so

$$g = f^{-1} \text{ and } f = g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$\begin{aligned} f(g(x)) &= \frac{1}{\frac{1}{x} - 2 + 2} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

TRY IT #2 If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

EXAMPLE 3

Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution

$$f(g(x)) = \frac{x^3}{27} \neq x$$

No, the functions are not inverses.

Analysis

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

TRY IT #3 If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 3.

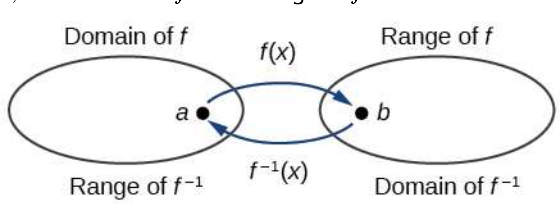


Figure 3 Domain and range of a function and its inverse

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $\left[0, \infty\right)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its domain limited to $\left[0, \infty\right)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x - 1)^2$ on $\left[1, \infty\right)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of f^{-1} = $\left[1, \infty\right)$.
- The domain of f^{-1} = range of f = $\left[0, \infty\right)$.

Q&A **Is it possible for a function to have more than one inverse?**

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

HOW TO

Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

EXAMPLE 4

Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 2. We restrict the domain in such a fashion that the function assumes all *y*-values exactly once.

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	**Cube root**	**Square root**	**Absolute value**	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Table 2

Solution

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no inverse.

The absolute value function can be restricted to the domain $\left[0, \infty\right)$, where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $\left(0, \infty\right)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

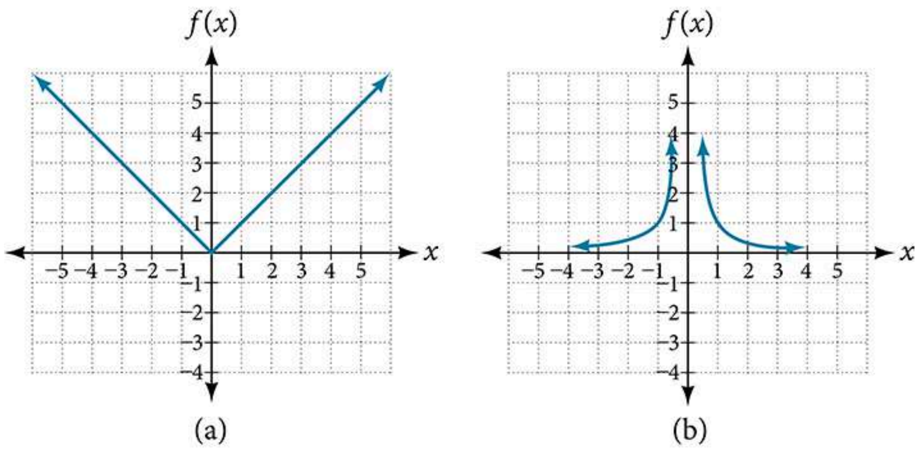


Figure 4 (a) Absolute value (b) Reciprocal square

TRY IT #4 The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

EXAMPLE 5

Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Table 3

Solution

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

TRY IT #5 Using Table 4, find and interpret (a) $f(60)$, and (b) $f^{-1}(60)$.

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Table 4

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in Functions and Function Notation that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

HOW TO

Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the *y*-axis of the given graph.
2. Read the inverse function's output from the *x*-axis of the given graph.

EXAMPLE 6

Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 5. Find $g(3)$ and $g^{-1}(3)$.

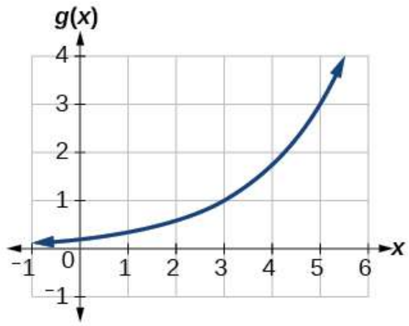


Figure 5

Solution

To evaluate $g(3)$, we find 3 on the *x*-axis and find the corresponding output value on the *y*-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of *x* for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 6.

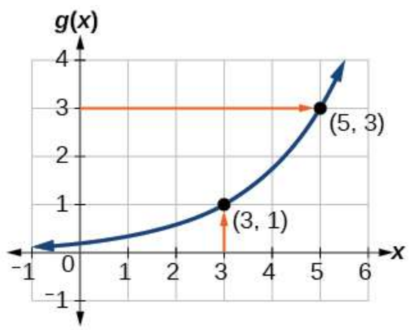


Figure 6

TRY IT #6 Using the graph in Figure 6, ⓐ find $g^{-1}(1)$, and ⓑ estimate $g^{-1}(4)$.

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula—for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

HOW TO

Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x.
3. Interchange x and y.

EXAMPLE 7

Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32)$$

Solution

$$\begin{aligned} C &= \tfrac{5}{9}(F - 32) \\ C \cdot \tfrac{9}{5} &= F - 32 \\ F &= \tfrac{9}{5}C + 32 \end{aligned}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32),$$

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

TRY IT #7 Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$.

EXAMPLE 8

Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

Solution

$$\begin{aligned} y &= \tfrac{2}{x-3} + 4 && \text{Set up an equation.} \\ y - 4 &= \tfrac{2}{x-3} && \text{Subtract 4 from both sides.} \\ x - 3 &= \tfrac{2}{y-4} && \text{Multiply both sides by } x - 3 \text{ and divide by } y - 4. \\ x &= \tfrac{2}{y-4} + 3 && \text{Add 3 to both sides.} \end{aligned}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 5.

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Table 5

EXAMPLE 9

Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x - 4}$.

Solution

$$\begin{aligned} y &= 2+\sqrt{x-4} \\ (y-2)^2 &= x-4 \\ x &= (y-2)^2+4 \end{aligned}$$

So $f^{-1}(x) = (x-2)^2 + 4.$

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$.

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

TRY IT #8 What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure 7.

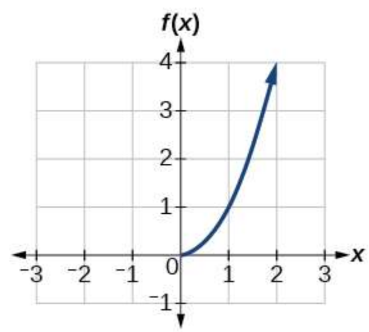


Figure 7 Quadratic function with domain restricted to [0, ∞).

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x- axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 8.

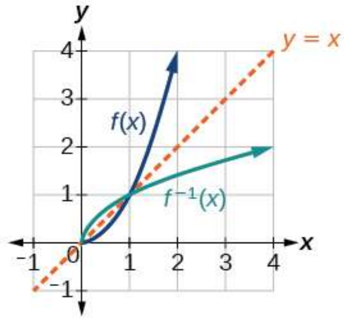


Figure 8 Square and square-root functions on the non-negative domain

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

EXAMPLE 10

Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 9, sketch a graph of $f^{-1}(x)$.

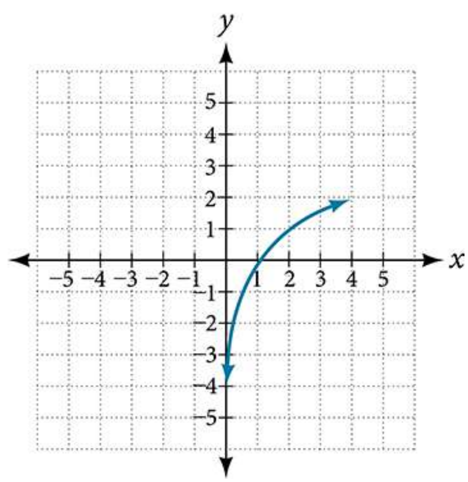


Figure 9

Solution

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $\left(0, \infty\right)$ and range of $\left(-\infty, \infty\right)$, so the inverse will have a domain of $\left(-\infty, \infty\right)$ and range of $\left(0, \infty\right)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 10.

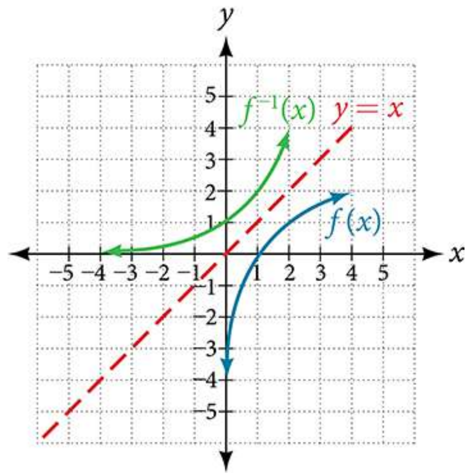


Figure 10 The function and its inverse, showing reflection about the identity line

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TRY IT #9 Draw graphs of the functions f and f^{-1} from Example 8.

Q&A **Is there any function that is equal to its own inverse?**

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

MEDIA

Access these online resources for additional instruction and practice with inverse functions.

Inverse Functions (http://openstax.org/l/inversefunction)
One-to-one Functions (http://openstax.org/l/onetoone)
Inverse Function Values Using Graph (http://openstax.org/l/inversfuncgraph)
Restricting the Domain and Finding the Inverse (http://openstax.org/l/restrictdomain)

3.7 SECTION EXERCISES

Verbal

1. Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

2. Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?

3. Can a function be its own inverse? Explain.

4. Are one-to-one functions either always increasing or always decreasing? Why or why not?

5. How do you find the inverse of a function algebraically?

Algebraic

6. Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

7. $f(x) = x + 3$

8. $f(x) = x + 5$

9. $f(x) = 2 - x$

10. $f(x) = 3 - x$

11. $f(x) = \frac{x}{x+2}$

12. $f(x) = \frac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

13. $f(x) = (x + 7)^2$

14. $f(x) = (x - 6)^2$

15. $f(x) = x^2 - 5$

16. Given $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$:

ⓐ Find $f(g(x))$ and $g(f(x))$.
ⓑ What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

17. $f(x) = \sqrt[3]{x - 1}$ and $g(x) = x^3 + 1$

18. $f(x) = -3x + 5$ and $g(x) = \frac{x-5}{-3}$

Graphical

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

19. $f(x) = \sqrt{x}$

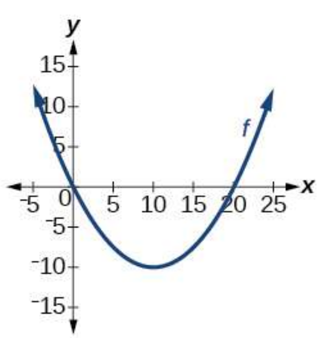
20. $f(x) = \sqrt[3]{3x + 1}$

21. $f(x) = -5x + 1$

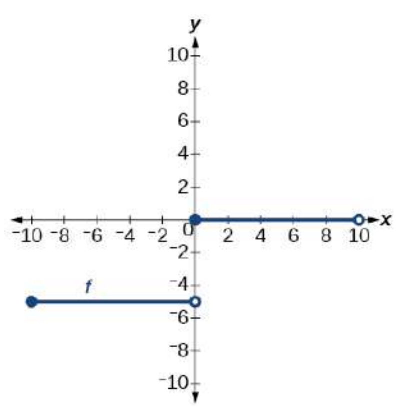
22. $f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

23.



24.



For the following exercises, use the graph of f shown in Figure 11.

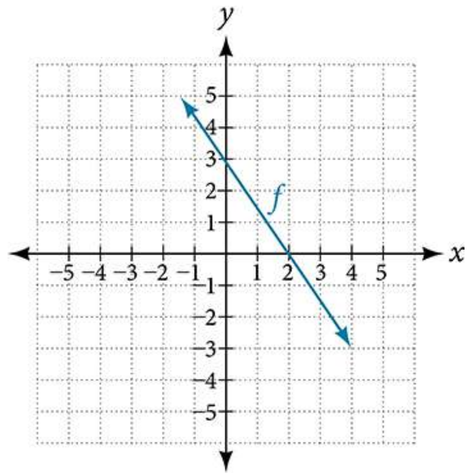


Figure 11

25. Find $f(0)$.

26. Solve $f(x) = 0$.

27. Find $f^{-1}(0)$.

28. Solve $f^{-1}(x) = 0$.

For the following exercises, use the graph of the one-to-one function shown in Figure 12.

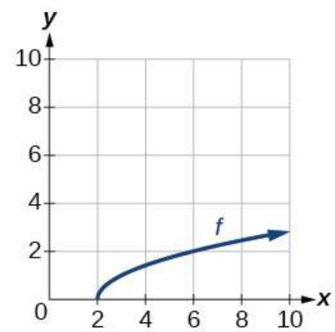


Figure 12

29. Sketch the graph of f^{-1}.

30. Find $f(6)$ and $f^{-1}(2)$.

31. If the complete graph of f is shown, find the domain of f.

32. If the complete graph of f is shown, find the range of f.

Numeric

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

33. If $f(6) = 7$, find $f^{-1}(7)$.

34. If $f(3) = 2$, find $f^{-1}(2)$.

35. If $f^{-1}(-4) = -8$, find $f(-8)$.

36. If $f^{-1}(-2) = -1$, find $f(-1)$.

For the following exercises, use the values listed in Table 6 to evaluate or solve.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	0	7	4	2	6	5	3	9	1

Table 6

37. Find $f(1)$.

38. Solve $f(x) = 3$.

39. Find $f^{-1}(0)$.

40. Solve $f^{-1}(x) = 7$.

41. Use the tabular representation of f in Table 7 to create a table for $f^{-1}(x)$.

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Table 7

Technology

For the following exercises, find the inverse function. Then, graph the function and its inverse.

42. $f(x) = \frac{3}{x-2}$

43. $f(x) = x^3 - 1$

44. Find the inverse function of $f(x) = \frac{1}{x-1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

Real-World Applications

45. To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \frac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.

46. The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.

47. A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

Chapter Review

Key Terms

absolute maximum the greatest value of a function over an interval

absolute minimum the lowest value of a function over an interval

average rate of change the difference in the output values of a function found for two values of the input divided by the difference between the inputs

composite function the new function formed by function composition, when the output of one function is used as the input of another

decreasing function a function is decreasing in some open interval if $f(b) < f(a)$ for any two input values a and b in the given interval where $b > a$

dependent variable an output variable

domain the set of all possible input values for a relation

even function a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y- axis

function a relation in which each input value yields a unique output value

horizontal compression a transformation that compresses a function’s graph horizontally, by multiplying the input by a constant $b > 1$

horizontal line test a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

horizontal reflection a transformation that reflects a function’s graph across the y-axis by multiplying the input by -1

horizontal shift a transformation that shifts a function’s graph left or right by adding a positive or negative constant to the input

horizontal stretch a transformation that stretches a function’s graph horizontally by multiplying the input by a constant $0 < b < 1$

increasing function a function is increasing in some open interval if $f(b) > f(a)$ for any two input values a and b in the given interval where $b > a$

independent variable an input variable

input each object or value in a domain that relates to another object or value by a relationship known as a function

interval notation a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

inverse function for any one-to-one function $f(x)$, the inverse is a function $f^{-1}(x)$ such that $f^{-1}(f(x)) = x$ for all x in the domain of f; this also implies that $f\left(f^{-1}(x)\right) = x$ for all x in the domain of f^{-1}

local extrema collectively, all of a function's local maxima and minima

local maximum a value of the input where a function changes from increasing to decreasing as the input value increases.

local minimum a value of the input where a function changes from decreasing to increasing as the input value increases.

odd function a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

one-to-one function a function for which each value of the output is associated with a unique input value

output each object or value in the range that is produced when an input value is entered into a function

piecewise function a function in which more than one formula is used to define the output

range the set of output values that result from the input values in a relation

rate of change the change of an output quantity relative to the change of the input quantity

relation a set of ordered pairs

set-builder notation a method of describing a set by a rule that all of its members obey; it takes the form $\{x \mid \text{statement about } x\}$

vertical compression a function transformation that compresses the function’s graph vertically by multiplying the output by a constant $0 < a < 1$

vertical line test a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

vertical reflection a transformation that reflects a function’s graph across the x-axis by multiplying the output by -1

vertical shift a transformation that shifts a function’s graph up or down by adding a positive or negative constant to the output

vertical stretch a transformation that stretches a function’s graph vertically by multiplying the output by a constant $a > 1$

Key Equations

Constant function	$f(x) = c$, where c is a constant
Identity function	$f(x) = x$
Absolute value function	$f(x) = \|x\|$
Quadratic function	$f(x) = x^2$
Cubic function	$f(x) = x^3$
Reciprocal function	$f(x) = \frac{1}{x}$
Reciprocal squared function	$f(x) = \frac{1}{x^2}$
Square root function	$f(x) = \sqrt{x}$
Cube root function	$f(x) = \sqrt[3]{x}$

Average rate of change	$\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

Composite function	$(f \circ g)(x) = f(g(x))$

Vertical shift	$g(x) = f(x) + k$ (up for $k > 0$)
Horizontal shift	$g(x) = f(x - h)$ (right for $h > 0$)
Vertical reflection	$g(x) = -f(x)$
Horizontal reflection	$g(x) = f(-x)$
Vertical stretch	$g(x) = af(x)$ ($a > 0$)
Vertical compression	$g(x) = af(x)$ $(0 < a < 1)$
Horizontal stretch	$g(x) = f(bx)$ $(0 < b < 1)$
Horizontal compression.	$g(x) = f(bx)$ ($b > 1$)

Key Concepts

3.1 Functions and Function Notation

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output. See Example 1 and Example 2.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$. See Example 3 and Example 4.
- In tabular form, a function can be represented by rows or columns that relate to input and output values. See

Example 5.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value. See Example 6 and Example 7.
- To solve for a specific function value, we determine the input values that yield the specific output value. See Example 8.
- An algebraic form of a function can be written from an equation. See Example 9 and Example 10.
- Input and output values of a function can be identified from a table. See Example 11.
- Relating input values to output values on a graph is another way to evaluate a function. See Example 12.
- A function is one-to-one if each output value corresponds to only one input value. See Example 13.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point. See Example 14.
- The graph of a one-to-one function passes the horizontal line test. See Example 15.

3.2 Domain and Range

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs. See Example 1.
- The domain of a function can also be determined by identifying the input values of a function written as an equation. See Example 2, Example 3, and Example 4.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation. See Example 5.
- For many functions, the domain and range can be determined from a graph. See Example 6 and Example 7.
- An understanding of toolkit functions can be used to find the domain and range of related functions. See Example 8, Example 9, and Example 10.
- A piecewise function is described by more than one formula. See Example 11 and Example 12.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain. See Example 13.

3.3 Rates of Change and Behavior of Graphs

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See Example 1.
- Identifying points that mark the interval on a graph can be used to find the average rate of change. See Example 2.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See Example 3.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See Example 4 and Example 5.
- The average rate of change can sometimes be determined as an expression. See Example 6.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See Example 7.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See Example 8 and Example 9.
- The highest and lowest points on a graph indicate the maxima and minima. See Example 10.

3.4 Composition of Functions

- We can perform algebraic operations on functions. See Example 1.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example 2 and Example 3.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example 4.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example 5.
- A composite function can be evaluated from a graph. See Example 6.
- A composite function can be evaluated from a formula. See Example 7.

- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Example 8 and Example 9.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See Example 10.

3.5 Transformation of Functions

- A function can be shifted vertically by adding a constant to the output. See Example 1 and Example 2.
- A function can be shifted horizontally by adding a constant to the input. See Example 3, Example 4, and Example 5.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts. See Example 6.
- Vertical and horizontal shifts are often combined. See Example 7 and Example 8.
- A vertical reflection reflects a graph about the x- axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph about the y- axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph. See Example 9.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly. See Example 10.
- A function presented as an equation can be reflected by applying transformations one at a time. See Example 11.
- Even functions are symmetric about the y- axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither. See Example 12.
- A function can be compressed or stretched vertically by multiplying the output by a constant. See Example 13, Example 14, and Example 15.
- A function can be compressed or stretched horizontally by multiplying the input by a constant. See Example 16, Example 17, and Example 18.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order. See Example 19 and Example 20.

3.6 Absolute Value Functions

- Applied problems, such as ranges of possible values, can also be solved using the absolute value function. See Example 1.
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction. See Example 2.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable. See Example 3.

3.7 Inverse Functions

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$. See Example 1, Example 2, and Example 3.
- Only some of the toolkit functions have an inverse. See Example 4.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse. See Example 5.
- The inverse of a function can be determined at specific points on its graph. See Example 6.
- To find the inverse of a formula, solve the equation $y = f(x)$ for x as a function of y. Then exchange the labels x and y. See Example 7, Example 8, and Example 9.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$. See Example 10.

Exercises

Review Exercises

Functions and Function Notation

For the following exercises, determine whether the relation is a function.

1. $\{(a,b),(c,d),(e,d)\}$

2. $\{(5,2),(6,1),(6,2),(4,8)\}$

3. $y^2+4=x$, for x the independent variable and y the dependent variable

4. Is the graph in Figure 1 a function?

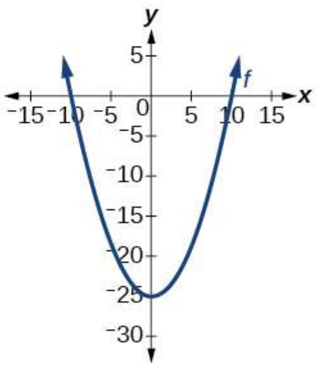


Figure 1

For the following exercises, evaluate $f(-3)$; $f(2)$; $f(-a)$; $-f(a)$; $f(a+h)$.

5. $f(x)=-2x^2+3x$

6. $f(x)=2\,|3x-1|$

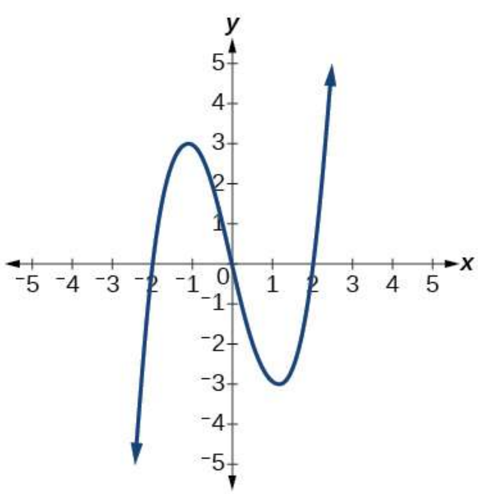
For the following exercises, determine whether the functions are one-to-one.

7. $f(x)=-3x+5$

8. $f(x)=|x-3|$

For the following exercises, use the vertical line test to determine if the relation whose graph is provided is a function.

9.



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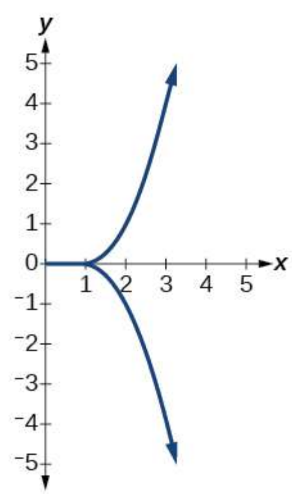

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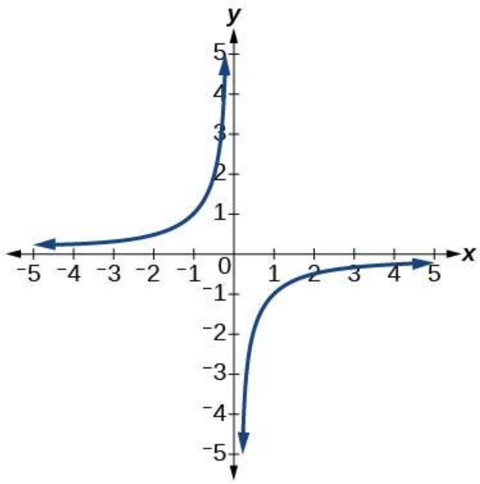


For the following exercises, graph the functions.

12. $f(x) = |x + 1|$

13. $f(x) = x^2 - 2$

For the following exercises, use Figure 2 to approximate the values.

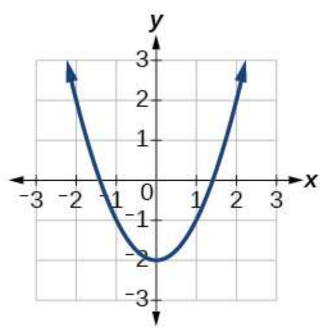


Figure 2

14. $f(2)$

15. $f(-2)$

16. If $f(x) = -2$, then solve for x.

17. If $f(x) = 1$, then solve for x.

For the following exercises, use the function $h(t) = -16t^2 + 80t$ to find the values in simplest form.

18. $\frac{h(2)-h(1)}{2-1}$

19. $\frac{h(a)-h(1)}{a-1}$

Domain and Range

For the following exercises, find the domain of each function, expressing answers using interval notation.

20. $f(x) = \frac{2}{3x+2}$

21. $f(x) = \frac{x-3}{x^2-4x-12}$

22. $f(x) = \frac{\sqrt{x-6}}{\sqrt{x-4}}$

23. Graph this piecewise function:
$$f(x) = \begin{cases} x+1 & x < -2 \\ -2x-3 & x \geq -2 \end{cases}$$

Rates of Change and Behavior of Graphs

For the following exercises, find the average rate of change of the functions from $x = 1$ to $x = 2$.

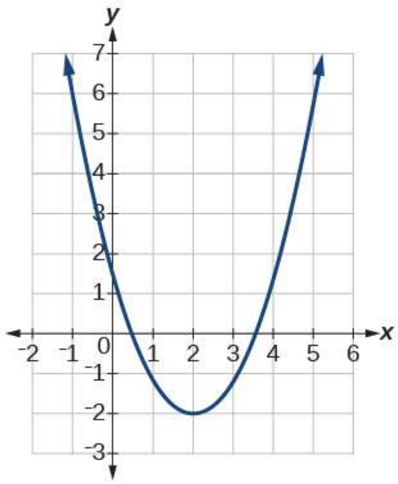
24. $f(x) = 4x - 3$

25. $f(x) = 10x^2 + x$

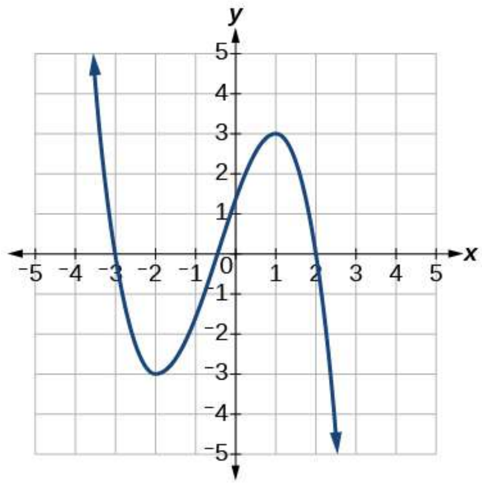
26. $f(x) = -\frac{2}{x^2}$

For the following exercises, use the graphs to determine the intervals on which the functions are increasing, decreasing, or constant.

27.



28.



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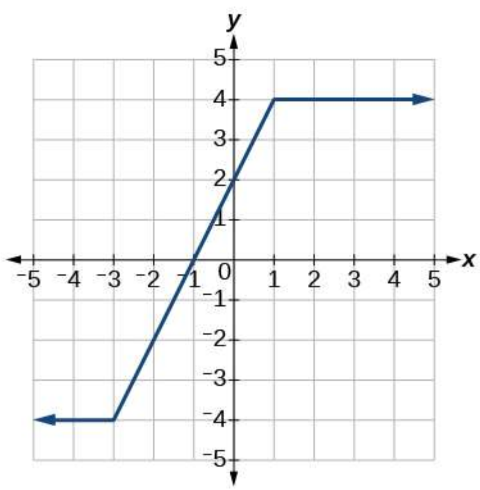


30. Find the local minimum of the function graphed in Exercise 3.27.

31. Find the local extrema for the function graphed in Exercise 3.28.

32. For the graph in Figure 3, the domain of the function is $[-3, 3]$. The range is $[-10, 10]$. Find the absolute minimum of the function on this interval.

33. Find the absolute maximum of the function graphed in Figure 3.

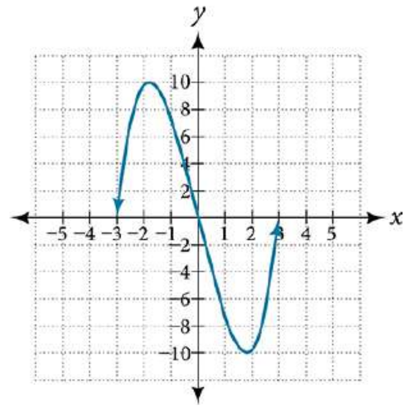


Figure 3

Composition of Functions

For the following exercises, find $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions.

34. $f(x) = 4 - x, \quad g(x) = -4x$

35. $f(x) = 3x + 2, \quad g(x) = 5 - 6x$

36. $f(x) = x^2 + 2x, \quad g(x) = 5x + 1$

37. $f(x) = \sqrt{x + 2}, \quad g(x) = \frac{1}{x}$

38. $f(x) = \frac{x+3}{2}, \quad g(x) = \sqrt{1 - x}$

For the following exercises, find $(f \circ g)$ and the domain for $(f \circ g)(x)$ for each pair of functions.

39. $f(x) = \frac{x+1}{x+4}, \quad g(x) = \frac{1}{x}$

40. $f(x) = \frac{1}{x+3}, \quad g(x) = \frac{1}{x-9}$

41. $f(x) = \frac{1}{x}, \quad g(x) = \sqrt{x}$

42. $f(x) = \frac{1}{x^2-1}, \quad g(x) = \sqrt{x + 1}$

For the following exercises, express each function H as a composition of two functions f and g where $H(x) = (f \circ g)(x)$.

43. $H(x) = \sqrt{\frac{2x-1}{3x+4}}$

44. $H(x) = \frac{1}{(3x^2-4)^{-3}}$

Transformation of Functions

For the following exercises, sketch a graph of the given function.

45. $f(x) = (x - 3)^2$

46. $f(x) = (x + 4)^3$

47. $f(x) = \sqrt{x} + 5$

48. $f(x) = -x^3$

49. $f(x) = \sqrt[3]{-x}$

50. $f(x) = 5\sqrt{-x} - 4$

51. $f(x) = 4\,[|x - 2| - 6]$

52. $f(x) = -(x + 2)^2 - 1$

For the following exercises, sketch the graph of the function g if the graph of the function f is shown in Figure 4.

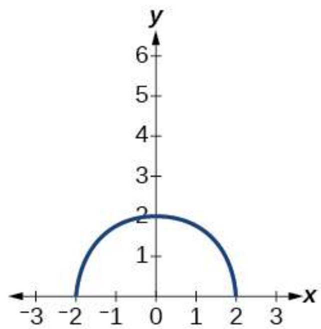


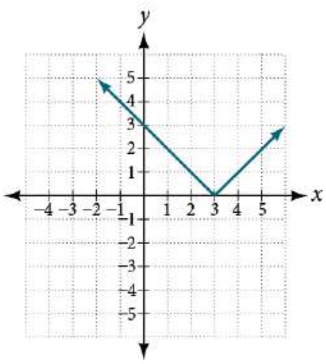
Figure 4

53. $g(x) = f(x - 1)$

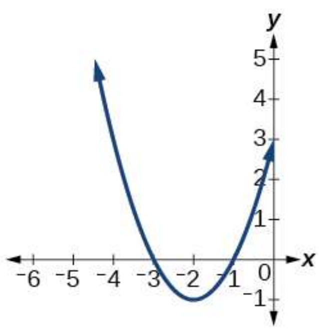
54. $g(x) = 3f(x)$

For the following exercises, write the equation for the standard function represented by each of the graphs below.

55.



56.



For the following exercises, determine whether each function below is even, odd, or neither.

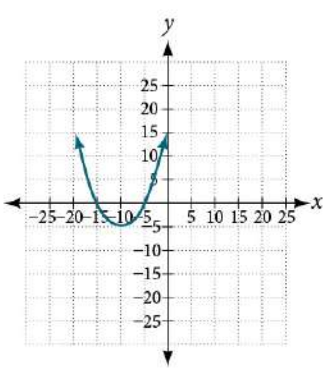
57. $f(x) = 3x^4$

58. $g(x) = \sqrt{x}$

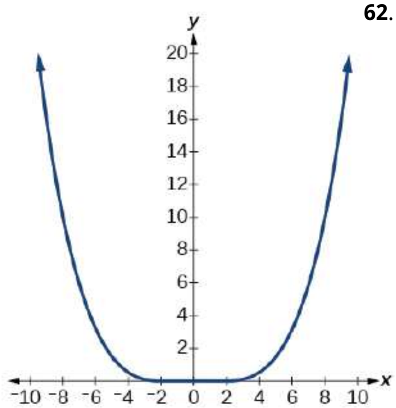
59. $h(x) = \frac{1}{x} + 3x$

For the following exercises, analyze the graph and determine whether the graphed function is even, odd, or neither.

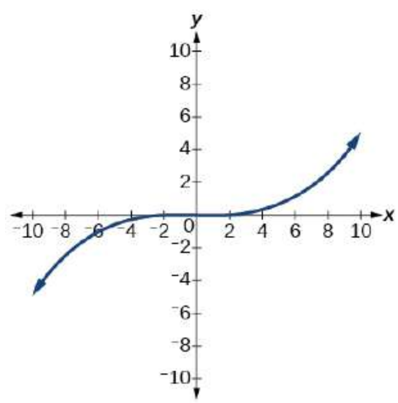
60.



61.



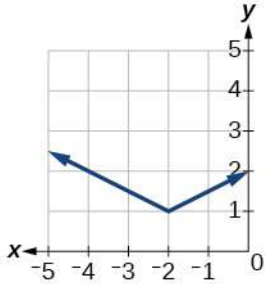
62.



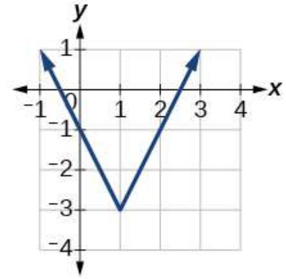
Absolute Value Functions

For the following exercises, write an equation for the transformation of $f(x) = |x|$.

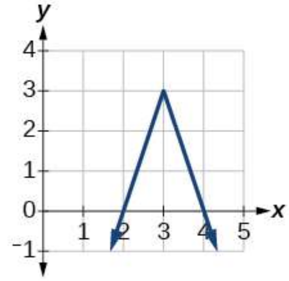
63.



64.



65.



For the following exercises, graph the absolute value function.

66. $f(x) = |x - 5|$

67. $f(x) = -|x - 3|$

68. $f(x) = |2x - 4|$

Inverse Functions

For the following exercises, find $f^{-1}(x)$ for each function.

69. $f(x) = 9 + 10x$

70. $f(x) = \frac{x}{x+2}$

For the following exercise, find a domain on which the function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

71. $f(x) = x^2 + 1$

72. Given $f(x) = x^3 - 5$ and $g(x) = \sqrt[3]{x+5}$:

ⓐ Find $f(g(x))$ and $g(f(x))$.
ⓑ What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

73. $f(x) = \frac{1}{x}$

74. $f(x) = -3x^2 + x$

75. If $f(5) = 2$, find $f^{-1}(2)$.

76. If $f(1) = 4$, find $f^{-1}(4)$.

Practice Test

For the following exercises, determine whether each of the following relations is a function.

1. $y = 2x + 8$

2. $\{(2, 1), (3, 2), (-1, 1), (0, -2)\}$

For the following exercises, evaluate the function $f(x) = -3x^2 + 2x$ at the given input.

3. $f(-2)$

4. $f(a)$

5. Show that the function $f(x) = -2(x-1)^2 + 3$ is not one-to-one.

6. Write the domain of the function $f(x) = \sqrt{3-x}$ in interval notation.

7. Given $f(x) = 2x^2 - 5x$, find $f(a+1) - f(1)$ in simplest form.

8. Graph the function $f(x) = \begin{cases} x+1 & \text{if} & -2 < x < 3 \\ -x & \text{if} & x \geq 3 \end{cases}$

9. Find the average rate of change of the function $f(x) = 3 - 2x^2 + x$ by finding $\frac{f(b)-f(a)}{b-a}$ in simplest form.

For the following exercises, use the functions $f(x) = 3 - 2x^2 + x$ and $g(x) = \sqrt{x}$ to find the composite functions.

10. $(g \circ f)(x)$

11. $(g \circ f)(1)$

12. Express $H(x) = \sqrt[3]{5x^2 - 3x}$ as a composition of two functions, f and g, where $(f \circ g)(x) = H(x)$.

For the following exercises, graph the functions by translating, stretching, and/or compressing a toolkit function.

13. $f(x) = \sqrt{x+6} - 1$

14. $f(x) = \frac{1}{x+2} - 1$

For the following exercises, determine whether the functions are even, odd, or neither.

15. $f(x) = -\frac{5}{x^2} + 9x^6$

16. $f(x) = -\frac{5}{x^3} + 9x^5$

17. $f(x) = \frac{1}{x}$

18. Graph the absolute value function $f(x) = -2|x-1| + 3$.

For the following exercises, find the inverse of the function.

19. $f(x) = 3x - 5$

20. $f(x) = \frac{4}{x+7}$

For the following exercises, use the graph of g shown in Figure 1.

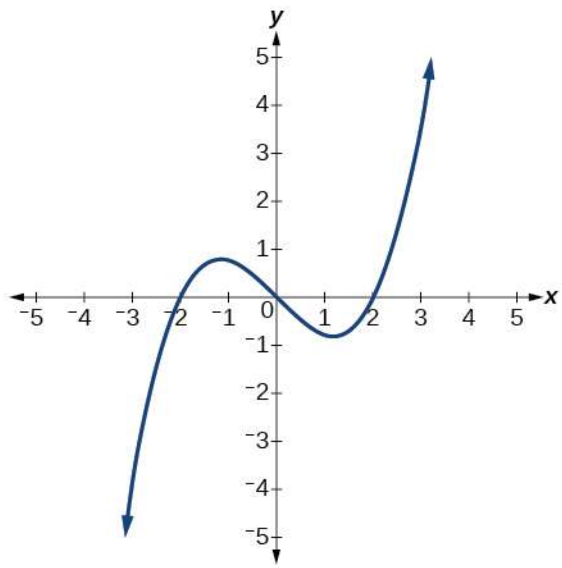


Figure 1

21. On what intervals is the function increasing?

22. On what intervals is the function decreasing?

23. Approximate the local minimum of the function. Express the answer as an ordered pair.

24. Approximate the local maximum of the function. Express the answer as an ordered pair.

For the following exercises, use the graph of the piecewise function shown in Figure 2.

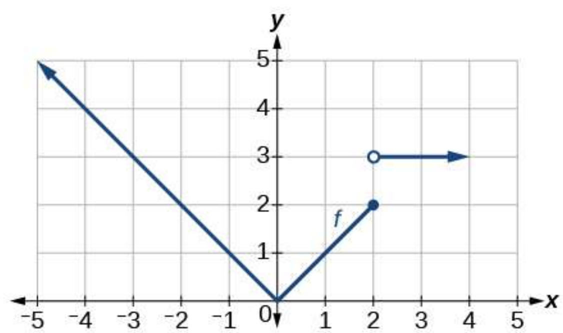


Figure 2

25. Find $f(2)$.

26. Find $f(-2)$.

27. Write an equation for the piecewise function.

For the following exercises, use the values listed in Table 1.

x	0	1	2	3	4	5	6	7	8
$F(x)$	1	3	5	7	9	11	13	15	17

Table 1

28. Find $F(6)$.

29. Solve the equation $F(x) = 5$.

30. Is the graph increasing or decreasing on its domain?

31. Is the function represented by the graph one-to-one?

32. Find $F^{-1}(15)$.

33. Given $f(x) = -2x + 11$, find $f^{-1}(x)$.