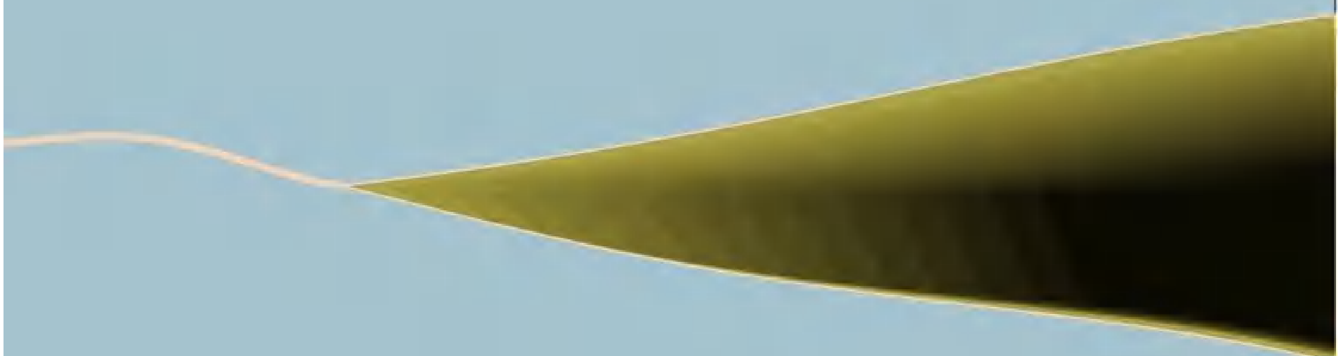
Essential Mechanics - Statics and Strength of Materials with MATLAB and Octave



P. Venkataraman

DESCRIPTION

Essential Mechanics - Statics and Strength of Materials with MATLAB and Octave

P. Venkataraman

Description of Book

Essential Mechanics - Statics and Strength of Materials with MATLAB and Octave combines two core engineering science courses - “Statics” and “Strength of Materials” - in mechanical, civil, and aerospace engineering. It weaves together various essential topics from Statics and Strength of Materials to allow discussing structural design from the very beginning. The traditional content of these courses are reordered to make it convenient to cover rigid body equilibrium and extend it to deformable body mechanics.The e-book covers the most useful topics from both courses with computational support through MATLAB/Octave. The traditional approach for engineering content is emphasized and is rigorously supported through graphics and analysis. Prior knowledge of MATLAB is not necessary. Instructions for its use in context is provided and explained. It takes advantage of the numerical, symbolic, and graphical capability of MATLAB for effective problem solving. This computational ability provides a natural procedure for *What if?* exploration that is important for design. The book also emphasizes graphics to understand, learn, and explore design. The idea for this book, the organization, and the flow of content is original and new. The integration of computation, and the marriage of analytical and computational skills is a new valuable experience provided by this e-book. Most importantly the book is very interactive with respect to the code as it appears along with the analysis.

CONTENTS

(The page numbers maybe approximate since they were determined after compilation)

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Essential Mechanics - Statics and Strength of Materials with MATLAB and Octave

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DEDICATION

This book is dedicated to all of you who have generously made available your resources and ideas so that all of us could learn and improve.
This book is also dedicated to all of you who power, edit and help maintain sites like Wikipedia (Wikimedia Media Foundation), Stack exchange, Quora and other places where you can seek information easily.

I have also to thank Vinayak, Archana, Austin, and Jayanti for their sufferance as this book written on my personal time.

FOREWORD

This book is the result of a long held passion and belief that computation can empower and energize thinking and learning even in basic courses. I am hoping that all of you who use this book will confirm or disagree with this idea by dropping me a line at the email address at the end of this page.

The book took over five years to finish. It was mostly due to the many forks in the development.

Fork 1

It did not begin as a book. It started as a collection of MATLAB code that could accompany traditional instruction in basic engineering (Statics) and structural mechanics (Strength of Materials) courses. It essentially translated standard equilibrium and later structural analysis into code. The emphasis was not writing code but rather using MATLAB as a super calculator. The initial code had detailed explanation but subsequent development and extensions were just copy, paste, and edit with minor new features.

As a background for this effort "Statics" is usually the first engineering science course in most engineering curriculum. The traditional instruction is about introducing the concepts of equilibrium across different physical problem varieties and defining the physical quantities involved. Sadly you cannot design the structure in this course because that information is delivered through another course "Strength of Materials". In many institutions this is a two course sequence. In other institutions it is a single course delivered in sequential fashion. The new information in the second course, which allows for design, is the introduction of stress and strain. This is a simple concept and does not require a significant amount of time to incorporate. Structural design is mostly analysis from Statics and finished with stress and displacement calculations from Strength of Materials.

Fork 2

In Fork 1 it became apparent that you could effortlessly do what if? analysis with this code - the essence of design. You could change the weight, or the length, or the diameter, and obtain the result without much sweat. You could also adapt the code easily to different problems with minor effort. In affecting this transition and making it work in code you have to pay attention to structural analysis. This is a powerful skill to learn in a basic course. In addition the ability to vary parameters of the problem allowed for development of design sensitivity right from the beginning provided it was utilized.

One might ask why not move the learning of stress and strain to "Statics" and then start talking about design from the start? This would actually perfectly merge the two courses and make it one.

I tried to develop content that did just this and started to extend problems in Statics to calculate stress, strain, and displacement so that it appeared a natural integration. This was easier to do than organizing the table of contents to deliver this information formally in the most useful manner. One constraint was that Statics was a freshman course and was calling upon mathematic principles that were being simultaneous introduced in a separate mathematics courses. Including structural design early meant including discussions of fracture early. Fracture even in the most simplest form is a challenging concept. Avoiding fracture is the goal of successful structural design. This took a long time to juggle and incorporate and I do not believe it is perfectly done here. It is a challenge and I am relying on the instructors to get the students through this with discussion and illustration. Its importance in design can be easily understood by the students, but the content necessary for analysis takes some time to learn and understand.

One effective way to understand the concepts was through graphics. MATLAB is a wonderful tool to explore engineering content graphically. Simple graphics are easy to generate in MATLAB. In addition, the classical formulation and solution of the engineering analysis was easily incorporated using symbolic calculations in MATLAB. It was similar to working out analysis on paper and following the same steps. The graphic confirmation of design solutions is also an effective way to understand and learn.

Fork 3

The efforts in Fork 2 extended the current work to include all topics in Strength of Materials. It naturally included the corresponding required topics in Statics. Suddenly, it started taking shape as a complete text book with an emphasis on design that was naturally incorporated. This led me to reflect on applied design.

Structural design in essence is choosing material or materials, determining loads the structure should support, and identifying dimensions that can withstand the loads without failure. In most text book situations the loads are generally known and the choice of material is usually based on experience and practice. The design challenge, particularly in instructional practice, is to select the dimensions of the structure so that it does not fail.

In traditional textbooks and instruction, cross-sectional properties based on dimensions of structures are often presented in Statics, where they are not used and relegated to the appendix in Strength of Materials, where they are necessary and important. In this book we avoid the appendix all together. The cross-sectional properties are introduced in the beginning as they are the essence of design. They also provide an avenue to become familiar with MATLAB code while using it numerically, symbolically, graphically, and sometimes textually.

Fork 4

This happened recently. In early 2019, the book felt bare with only examples and no additional practice problems. I started adding a few problems at the end of the sections.

Fork 5

In July 2019, I retired as an academic. I had always planned to release this book under Creative Commons license. I felt compelled to make the book truly useful under the new license. I resolved to provide support for Octave in this book. At least one MATLAB code in every section will have the Octave support and discuss the behavior of the code in Octave.

I am very comfortable with MATLAB and have been using it for more than a decade. The initial MATLAB code in the book was created using Version 2015a. All of the code in this book is verified with this version. I have not used Octave until now. I expected some learning delay. Since Octave accepts m-files I tried running the same MATLAB in Octave and to my surprise it worked. This made my task easier. I used the MATLAB code in Octave and debugged sufficiently to see that the code executed in Octave with the same results.

In this book, the text is written by me. The code is also original. Most of the figures used in the analysis are drawn by me. You will find some whimsical ones before I got more disciplined. There is a lot of color as it was planned as an e-book.

This is **Version 1** of **Essential Mechanics - Statics and Strength of Materials with MATLAB and Octave.**

Technical Information involved in the development of the book:
The book was written in **Scrivener** from *Literature and Latte*
The figures are created using **Canvas X** from *Canvas GFX*
The mathematical formulas are generated using **MathType** from *Design Science*
MATLAB code using **MATLAB** version 2015a from *Mathworks*
Octave code using **GNU Octave** version 5.1 from *GNU Octave site*
Python and **Sympy** through *Anaconda Python 3 distribution*

The website for the book is at:
https://sites.google.com/site/essentialmechanics/home

All figures used for illustration are from public domain sources. The majority of them from Wikimedia Foundation (Wikipedia). If a particular figure in this book should not be present because of proprietary reason, please notify the author and he will exclude it and update the electronic file.
Some table of properties are also from Wikimedia Foundation (Wikipedia).

I would have welcomed any assistance for editing the manuscript. I have done it twice for content and several times for formatting. Initially I planned to publish this as an epub file where formatting can be lose. However for reaching a bigger audience a pdf file in letter size made more sense. That means compiling and editing images so that they remain complete. Adjusting code so that there is no unintentional wrapping. I am sure there are lots of errors still. Hopefully with your help I can capture it in the website for the book.

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January 2020.
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INTRODUCTION

Essential Mechanics: Statics and Strength of Materials with MATLAB and Octave

Formal Description:
Essential Mechanics - Statics and Strength of Materials with MATLAB and Octave combines two core engineering science courses - "Statics" and "Strength of Materials" - in mechanical, civil, and aerospace engineering. It weaves together various essential topics from Statics and Strength of Materials to allow discussing structural design from the very beginning. The traditional content of these courses are reordered to make it convenient to cover rigid body equilibrium and extend it to deformable body mechanics.The e-book covers the most useful topics from both courses with computational support through MATLAB/Octave. The traditional approach for engineering content is emphasized and is rigorously supported through graphics and analysis. Prior knowledge of MATLAB is not necessary. Instructions for its use in context is provided and explained. It takes advantage of the numerical, symbolic, and graphical capability of MATLAB for effective problem solving. This computational ability provides a natural procedure for *What if?* exploration that is important for design. The book also emphasizes graphics to understand, learn, and explore design. The idea for this book, the organization, and the flow of content is original and new. The integration of computation, and the marriage of analytical and computational skills is a new valuable experience provided by this e-book.

Essential Mechanics - *Statics and Strength of Materials with MATLAB and Octave,* can be used in the earliest core engineering science course in mechanical, civil, and aerospace engineering. It is probably a required service course for most other engineering disciplines. In this book it represents an integration of topics from Statics and Strength of Materials to make it useful for design. If you have already graduated these courses you will have been introduced to the topics through through adjoining courses probably titled "Statics" and "Strength of Mechanics". Many would have had a single course with Strength of Materials content following Statics in sequence.

Statics is about calculating loads (forces and couples) assuming a rigid structure. Strength of materials assumes a deformable structure and uses the loads determined in Statics to study the deformation of structure as well as the stress capacity of the structure. A significant portion of the effort in the Strength of Materials course is applying Statics initially to the problem.

For engineering design, which attempts to design structures that do not fail structurally during their life time, the critical knowledge is acquired through the Strength of Materials course which depends significantly on Statics. The course on Statics merely lays the preliminary calculation of the loads that the individual parts of the structure must carry during the operation. With just "Statics" the students cannot design as they cannot determine if the product or device will fail. "Statics" is just an special applied "Physics" course dealing with equilibrium. Students are forced to recollect their "Statics" when they continue with "Mechanics of Materials" in a two-course sequence. This is a challenge in todays learning environment. It is also a waste of opportunity in design instruction. Design is complete when analyzed for failure. The failure is related to the stresses that the structure must endure. This book attempts to *combine* and *thread* the calculations of the **loads** with the **stresses** to ensure that the structure will not fail and the approach should make intuitive sense.

Another advantage of proceeding in this direction is to make available the extra course to continue with advanced applications of mechanics that could involve numerical approaches to structural design. These are essential for real world structural design - using software that deliver solutions through finite and other discrete element methods.

Let us consider a simple example to illustrate this connection between Statics and Strength of Materials in design:

Figure. A simple design problem

In this problem you seek the answer to the question : *Will he be rudely awakened or the rope will hold the weight?*

You can reformulate the question : *Will the rope fail?*

"Statics" will solve for the load carried by the rope.
"Statics" cannot answer the question by itself.
"Strength of Materials" will use this load to calculate if the stress in the rope will cause failure.
The question can only be answered by the combination of "Statics" and "Strength of Materials"

Calculating the stress is such a *simple* extension that it is a shame to postpone it to another course and make the student wait to understand design.

In summary - **Statics** ignores the deformation of the structure. In **Strength of Materials** we allow the structure to deform. A design goal is to ensure the deformation is elastic, that it the structure returns to the undeformed state once the loads are removed. The actual material properties will determine this behavior. In this book we will combine this analysis.

How to calculate the answer?

The solution in most of structural engineering problems is to use the ***Laws of Physics*** – in this case the simplified Newton's Law for stationary objects – called an equilibrium equation.

We solve this problem by:

- Simplifying the figure by making assumptions

- this is a physical model representing the actual problem

○ Identifying the information that is known
  - like the angle the rope makes with the horizontal line

○ Solving for the unknowns (or missing information) using the natural law or Newton's Law– in this case the **force** in the rope - using algebra and/or calculus and/or geometry
  - this is the mathematical model

These are standard procedure followed by all text books in Statics

○ Calculating the **stress** in the rope and checking that it will not rupture, and sometimes calculate the sag in the hammock

This is the the application of Strength of Materials
This book solves this problem immediately instead of waiting for another course

○ Checking that the number makes sense
  - this comes must become a part of your engineering instinct - through practice and experience

○ Additionally in this book we use the computational tool MATLAB/Octave to solve the problem.

○ The required MATLAB code is integrated into the book. It is not a pre-requisite.

To be effective, you should do all of the above in a

- Simple
- Consistent
- Effective, and *very importantly* through an
- Easily remembered process

Why MATLAB?
Many times engineering calculations can be extensive. Many times the graphical solution help understand the solution and develop an instinct. Often there are several unknown that must be simultaneously solved. Designing is about answering ***What If***? questions. Programming the calculations is a smart way to deal with design. If things go well then programming reinforces engineering knowledge. Programming can be learned by using MATLAB as a highly efficient calculator that can effortlessly combine symbolic, numeric, graphical, and textual computation in an integrated manner.

The MATLAB code in this book will also run on Octave except in specific cases because of syntax and parsing issues. Sometimes it is just a matter of a single line that must be commented out or included or changed. The code uses symbolic variables in most exercises and you will need to include the corresponding toolbox in Octave. The symbolic analysis in Octave is parsed to the Python module *sympy.* Knowledge of Python is not necessary. The Octave code is verified to run for at least one example in each section where MATLAB code appears. The Octave used in the book is GNU Octave, version 5.1.0. Remember it is the same file with the MATLAB code with suggested changes if any. For this book programming in Octave is same as programming in MATLAB. You will type in the MATLAB code in the Octave editor. The MATLAB code runs on Version 2015a. It should run on later versions without problems. This was tested with some random examples from the book. The MATLAB version was 2019a.

Important Note about Materials: Till recently the material meant metals. Today, design is all about new engineered materials. Most new aircraft designs use composite materials, a mixture of metal and plastic that is light and can handle all the stresses that the aluminum alloys can. As a benefit - it is lighter than the metal it will replace. Same performance but at lower weight and hence more efficient. Recently Ford announced that its new trucks will be made of Aluminum instead of steel and thereby saving over 600 lb in weight. This also improves engine performance because of reduced weight. Another extreme example is synthetic biology where you can design organisms through software. Engineering new materials is an exciting new field of rewarding pursuit.Computational multi-physics is developing new models for additive manufacturing. Nano-materials are being embedded in new concrete structures. All of these require new areas today rely of computational skills.

To the Users:
Please consider this as a WARNING. The best way to view this e-book is through a laptop or a desktop screen with at least 90 columns of text data. This will prevent inadvertent code wrapping. Both MATLAB and Octave are not kind when this event occurs. They will flag errors that may take time and effort to fix, apart from being very frustrating. To avoid parsing errors both MATLAB and Octave require to be informed about continuation of code on the next line. They only allow it certain circumstances.

To the Student:
Thank you for for picking up this e-book. I really hope you find this useful. I am sure you might come across some errors and if you find them kudos - pat yourself on the back since you understand the material you are trying to learn. Initially you might find it easy to copy and run these codes and maybe think " this author dude certainly knows his MATLAB". If you always did it you will continue to just confirm my expertise. However, if you played with the code, improved it, made it do additional interesting things in the same context, used it to explore more design issues, then Thank You. I hope you will share you work with others. I wrote this book specially for you. Please drop me a line about what can make it more useful. I will certainly use it in my next book.

To the Instructor:
Thank you for picking up the book, recommending it, or even adopting it. I am hopeful it works for you. You will find that the analysis in this book does not lack substance compared to the other texts out there. It does not discount or shorten the development of the technical topic or the idea behind it. You will see that the development is not weak in graphics either. It might appear that there is a lot going on all the time with the code. Students may be able to run with it if you start it for them. Here, I encourage you to consider coding is just copy, paste, edit, and extend. The words Essential Mechanics at the beginning of the tile is to keep the text as small as possible by focusing on important and necessary topics. I will look forward to your comments. Please also consider assigning students to come up with their problems and share them with all of us. This e-book is released under Creative Commons licensing. You can add, subtract, and change topics and make it your own.

P. Venkataraman
January 2020.
Rochester, NY, USA.

ESSENTIAL FOUNDATIONS IN ENGINEERING SERIES

Essential Foundations in Engineering Series in is a planned wishful collection of e-books in core engineering courses, Each one will address a core area contained in its name and provide an excellent substitute for standard texts on the topic. The initial set of books will target the common core courses from mechanical, aerospace, or civil engineering, primarily with beginning mechanics, strength of materials, and fluid mechanics. They are also useful as service courses for the remaining engineering programs. One singular characteristic of these introductory books is the presentation being *to the point, relevant,* and *sufficient*. The student can always follow up, if necessary, at online resources like "*Wikipedia"* or "*Scholarpedia",* "*The Engineering Toolbox*", "*eFunda*", other websites, and other books. The topics include those covered in any standard course at any university. These e-books will be next generation textbook with emphasis on design thinking woven throughout the book and integrated from the beginning with computation. This will allow instructors to challenge their students with advanced problem solving using open-ended design exercises and encourage visual examination of the solution. It allows students to develop a new, and much required skill set that is easily extensible and professionally valuable.

Each e-book will have more than just formulas. The formulas themselves are developed rigorously through clear examples and graphics. In addition there is MATLAB/Octave code to perform regular, complex, and involved calculations based on these formulas. The code integrates symbolic, numerical, graphical and textual programming as appropriate. These computational exercises can be hands-on and part of lecture in a regular classroom. The software used is readily available for personal laptops. The e-book series has no requirement for prior knowledge of software. Exposure to basic mathematics courses in a standard engineering curriculum will be necessary. The books teaches and incorporates computational code as required and encourages students to use and extend the code at all times. MATLAB/Octave is used an an efficient and high powered calculator to solve very specific problems rather than a software package that requires training and general exposure. The skill learned here can be easily deployed in other courses. The book can be used independently of the MATLAB/Octave content, but there are far more elegant textbooks on the subject that will be more useful and less distracting. The book can still be used as a text formally ignoring the MATLAB/Octave content with students can be encouraged to use MATLAB on their own The coverage is the same as in any standard textbook on the subject.

This book is illustration driven. The calculations are performed both step by step and also use software. Initially you might find this tedious, however sustained use in a short period of time will likely change your mind. The book encourages you to streamline your coding through copy, extend, and reuse. The earlier code you meet are commented in greater detail to guide your understanding. Consider this simple fact: *You need only about 250 MATLAB commands to cover the typical mechanical engineering curriculum - and this includes most of the courses*. This is demonstrated in this course taught by the author: https://sites.google.com/site/mece689specialtopics/home

Having taught over the last 35 years in a department of mechanical engineering the author realizes that both teaching and learning change with each generation. Each generation appears to have different outlook on learning, the effort that they are willing to invest, their ability to comprehend, their motivation to think, and very important today, their ability to handle distractions. Of course other educators may have a very different experience, which also depends on the institution, its branding, and the nature of students it attracts.

- These e-books are meant to communicate the essential knowledge referred in their titles. They may also serve as a reference for prior learning that has been forgotten. Unlike standard text books that are packed with material, the aim here is to to illustrate important topics in a simple and unified manner and get to the applications faster. Topics requiring more detailed knowledge are also available in the books.
- Finally, it matters ***if*** the students can apply what they learn consistently, ***if*** they remember the concepts, ***if*** they recollect the procedure which may have been applied consistently some time ago. For this reason, the e-books contemplated in this series should be a big help.

One important ***question*** still remains: Do we really need another book in these core courses?

There are a lot of excellent books on core courses in engineering. Today, they are also accompanied by enormous publisher support. They

Have excellent color illustrations
Detailed explanation of topics - more details with new edition
Lots of supplementary materials to enable teaching and learning
Supporting websites with more materials that appear with clicks
More pages with each edition
More expensive with each edition
Increased cost of sustaining these features

With Essential Foundations in Engineering you get a very concise version. The guarantee that you will learn will depends on you and not on any of these books, including the one being touted here. The primary motivation for this book is to provide a challenging, productive, skilled, and happy assimilation of your academic and learning experience in engineering problem solving.

The first book in this series is :

Essential Mechanics - ***Statics and Strength of Materials with MATLAB and Octave***

Currently two others are planned by the author to appear within the next five years.

Essential Fluid Mechanics -Fluid Mechanics with MATLAB and Octave
Essential Dynamics - Dynamics and Vibrations with MATLAB and Octave

The remaining is left for others to take the lead..

ACKNOWLEDGMENTS

The book is new in many ways:

- The idea behind the book is original.
- The organization is original.
- The flow of content is original.
- The inclusion of design in many discussions is original.
- The relevant graphics for all analysis in the book are originally drawn.
- The discussion is original.
- The inclusion of over and under-determined problem is original.
- The MATLAB code is original. The required programming is developed in the book.
- Knowledge of MATLAB programming is not a pre-requisite for the book.
- The Octave code is the same MATLAB code (with minor changes highlighted)
- The integration of computation throughout the book is original.
- The marrying of analytical and computational skills is original.

The theoretical content is similar to many outstanding text-books from the area of **Statics** and **Strength of Materials** that have educated so many generations so well for so many years. This book would be impossible without those resources and the author credits his knowledge to many of them, both as a student and as an instructor in mechanical engineering for more than 35 years. During this time the author has accessed many of them to gain understanding of the subject material.The author would like to acknowledge the authors of all the text books in this area, though only a small list is included below.

The books included here are primarily those that include Statics and/or Strength of Materials, followed by some that only deal with structural mechanics. Many of the authors listed here have separate books on Statics, Strength of Materials, and Dynamics. They are not included here separately. For many books listed below later edition are available.

1. Ferdinand Beer, E. Russell Johnston, Jr., John DeWolf, and David Mazurek, ***Statics and Mechanics of Materials***, 2nd Edition, McGraw Hill.

2. Hibbeler, R. C., ***Statics and Strength of Materials***, 5th Edition, Pearson.

3. George F. Limbrunner, Leonard Spiegel, ***Applied Statics and Strength of Materials*** 6th Edition, Pearson.

4. Harold I. Morrow, Robert P. Kokernak, ***Statics and Strength of Materials***, 5th Edition, Pearson.

5. James W Dally, Robert J Bonenberger, ***Statics and Mechanics of Materials: An Integrated Treatment***, College House Books.

6. William F. Riley, Leroy D. Sturges, Don H. Morris, ***Statics and Mechanics of Materials: An Integrated Approach,*** 2nd Edition, Wiley.

7. Fa-Hwa Cheng, ***Statics and Strength of Materials***, 2nd Edition, McGraw Hill.

8. James Ambrose, ***Simplified Mechanics and Strength of Materials***, 6th Edition, Wiley.

9. Anthony M. Bedford, Kenneth M. Liechti, Wallace Fowler, ***Statics and Mechanics of Materials***, Pearson.

10. Douglas W. Hull , ***Mastering Mechanics I, Using MATLAB: A Guide to Statics and Strength of Materials*** 1st Edition.

11. Bichara B. Muvdi, Souhail Elhouar, ***Mechanics of Materials: With Applications in Excel,*** CRC Press.

12. James L. Meriam, L. G. Kraige, Brian D. Harper, ***Solving Statics Problems in MATLAB to accompany Engineering Mechanics Statics*** 6e, Wiley.

13. Megson, T. H. G., ***Aircraft Structures for Engineering Students***, 5th Edition, Butterworth-Heinemann.

14. Surya Patnaik, Dale Hopkins, ***Strength of Materials: A New Unified Theory for the 21st Century***, 1st Edition, Butterworth-Heinemann.

15. Bingen Yang, ***Stress, Strain, and Structural Dynamics - An Interactive Handbook of Formulas, Solutions, and MATLAB Toolboxes,*** 1st Edition, Academic Press.

16. Robert Mott, Joseph A. Untener, ***Applied Strength of Materials,*** 6th Edition, Cengage.

Some images in the book are from sites like Wikipedia, Wikimedia Commons, or the national agencies. Those images did not need permissions. If any image is used without permission it is inadvertent. The author will remove them if identified.

ABOUT THE AUTHOR

P. Venkataraman graduated from I.I.T. Kanpur in 1974 with a B.Tech in Aeronautical Engineering.

He worked in Helicopter Design at Hindustan Aeronautics Ltd., Bangalore between 1974 and 1979.

He graduated from Rice University with a PhD from the Department of Mechanical Engineering and Material Science in 1984.

He was a faculty in the Department of Mechanical Engineering at Rochester Institute of Technology from 1983 to 2019.

Currently he is a retired academic with interests in publishing textbooks, creating mathematics inspired digital design, learning Carnatic music, gardening, and experimenting with vegetarian cuisine.

His other books can be accessed at: https://sites.google.com/site/venkatpan/my-books

1. PRELIMINARIES

Prior to dealing with ideas from mechanics we will attend to some preliminary information that is ***important*** for communication of values involved in engineering calculations. This is the idea of dimensions and units. They are independent of any discussion, of any topic, of any area, from science and applied physics.

This chapter also suggests a way to get started with MATLAB. It provides some comments on the way it will be used in the book. This assumes that MATLAB is available on your laptop or or have access to it. Today it is also possible to access it on line through a browser as long as you have a license for it.

1.1 UNITS

What do you want to know when you see the giant pumpkin? There is a clue in the picture. Like most of us you want to know how much it **weighs**.

I tell you "**it weighs 2500!**". As an engineer you would like to nail down the details and you have a follow up question "**2500 what?**"

If you are reading this book in the US you probably instinctively assume it weighs **2500 [lb]**. However if you are reading this in the rest of the world you will instinctively assume **2500 [kg]**. One of the two is inaccurate since for the rest of the world the *US pumpkin* would be about **5514 [lb]** (this is heavier than your car!)**;**

If I told you that "it weighs 2500 feet per second!"
You definitely know that something is odd – it cannot be "2500 feet per second "

As an engineer, particularly working in global teams, there is a need to communicate technical information accurately– in this example - the weight of the pumpkin. If it weighs 2500 [lb], then it should weigh 11200 [N] (N stands for Newton), or about 1142 [kg] (kg stands for kilogram). The reason for two different numbers for the weight is that it is expressed in two ***different systems of units*** -one used in USA (*US System*) and the other used in most of the world (*SI System*). In addition different quantities are commonly (due to practice) expressed in certain ways even if the units do not appear correct.

http://sunnyday.mit.edu/accidents/mco-oberg.htm

Figure 1.1.1 How much does it weigh?

There are **three** quantities we needed to express this technical information - the weight of the pumpkin - (it is true for all other technical information too) are:

US System

SI System

Weight (*Quantity of interest*)
2500 (*Value*)
Units [lb]

Weight (*Quantity of interest*)
11200(*Value*)
Units [N]

Even in the above expression we are not consistent with practice. The US unit expresses the weight as standard practice, while in the SI system, the standard practice is to describe is the weight through the unit of mass [kg].

Example 1.1

Figure. 1.1.2 Units involved in changing the tire

You are using the tire wrench to remove the lug nuts to change the flat tire. The same torque

Quantity of interest: Torque *Value*: 25 *Newton-meter* : Units
Quantity of interest: Torque *Value*: 18.44 *pound-feet*: Units

1.1.1 Problems

Problem 1.1.1
In what different ways the capacity/performance of a new car is expressed? Record at least 3 and express their information for your favorite car.

Problem 1.1.2
How is the performance of the Jet engine for the Boeing 787 aircraft expressed?

Problem 1.1.3
What are the different ways in which climate change is expressed?

1.2 SYSTEM OF UNITS

A single system is efficient, convenient, economical, and makes sense. The US system units is difficult for the United States to give up because a lot of expert technical knowledge and intuition gained over generations is tied to this particular unit system.

The world will use a single system once the US decides it wants too give up the US system. The intent has been there for decades but the US units are still used vigorously. Living 2 miles from the grocery store is better imagined than 3.21 kilometers for persons living in the US.

There are two popular "Systems of Units" in practice. The first one is the "System International" or SI System of Units – used across the world where the unit for weight is Newtons. The second is the US or the English System – mostly used in USA – where the unit for weight is pound (or pound force to be strict). Any material for sale in the USA will display the specifications in both system of units.

Figure 1.2.1 Widespread use of US Units

-US, Burma, Liberia are three countries that have not adopted the SI as the sole system of units (*from Wikipedia*)

1.2.1 SI - or System International

The International system of units is used globally for commercial purposes as well as in science and engineering. It appeared in 1960 and some of the standards are maintained at the International Bureau of Weights and Measures at Severn, near Paris, France. It is also sometimes referred to as the metric system.

The SI is a more complete system.
In fact the US/English system have no units for many of the physical quantities that have been defined or discovered recently.

The SI is a system of units of measurement defined through seven (7) BASE UNITS

Table 1.1 SI Base Units

Name	Unit Symbol	Quantity	Symbol
metre/meter	m	length	*l*

kilogram	kg	mass	*m*
second	s	time	*t*
ampere	A	electric current	*I*
kelvin	K	thermodynamic temperature	*T*
candela	cd	luminous intensity	I_v
mole	mol	amount of substance	*n*

SI: Base and Derived Units
The meter stick is an instrument that measures one meter between marked ends.

Figure 1.2.2 The meter stick

In SI units the measured length is 1 meter. The information on length can be expressed in several alternate units in the same unit system.

In the same SI units it is also :

- 100 centimeters
- 1000 millimeters
- 10^6 microns
- 10^9 nanometers

It is important to use the appropriate unit defined through context and practice. For example the ***nanometer*** is useful in discussing the wavelength of light. ***Microns*** may be appropriate when discussing products in the semiconductor industry. ***Millimeters*** may be appropriate for discussing blood vessels. The BASIC unit of length is the meter (alternately *metre*)

In addition to **BASIC** units many physical quantity can be expressed through **DERIVED** units. Derived units imply a combination of basic units. For example

The density of water is 1000 kilogram/meter^3 [kg/m^3]. This is a derived unit. Every physical quantity can be associated with a corresponding basic or derived unit in the SI system of units.

Consider another example of the linear elastic spring from physics. We will apply a force (F) to stretch it through a distance (s). Not only is the force equal to spring constant (k) times the deflection (s), the units for the force must equal to the product of the units for the spring constant and the deflection in base units.

$$F = k\,s \qquad \text{or} \qquad [N] = [k]\,[s]$$
$$k = F/s \qquad \text{or} \qquad [k] = [N/m]$$

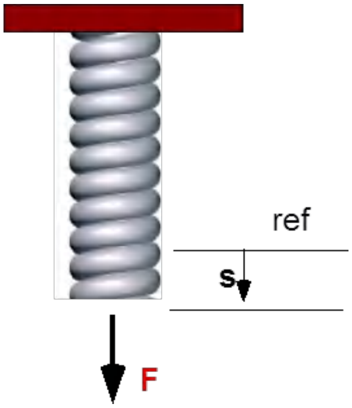


Figure 1.2.3 A linear spring - derived units

Units used in Mechanics

Table 1.2 SI Units in Mechanics

Quantity	Unit	Unit Symbol	Alternate Symbol
length	meter	**m**	
mass	kilogram	**kg**	
time	second	**s**	
acceleration	meter per second squared	**m/s^2**	
angle	radian	**rad**	
area	square meter	**m^2**	
density	kilogram per cubic meter	**kg/m^3**	
energy	Joule	**J**	1 Joule = 1Nm
force	Newton	**N**	1 N: 1 $kg\text{-}m/s^2$
moment of a force (torque)	Newton meter	**Nm**	
pressure	Newton per square meter	**N/m^2**	Pa: Pascal
stress	Newton per square meter	**N/m^2**	Pa: Pascal
volume (solid)	cubic meter	**m^3**	
volume (liquid)	liter	**10^{-3} m^3**	L
work	Joule	**J**	1 Joule = 1Nm

There are two **corollaries** that we will respect:

1. Any law or equation we develop must have the same unit on both side of the equal sign
2. Every term in the equation must also have the same units
(These are usually better expresses using the word dimensions rather than units - but that comes later)

1.2.2 US System of Units

The US customary unit system has some overlap with the British Imperial system. It was defined in modern form in 1959. In 1988 Congress passed a bill for confirming adoption of the SI system but left the timing indefinite. So the adoption is in transition. We will refer to this system as the US system.

We can recognize base units and derived units like we did for SI units. We also need a way to translate information between the two systems. However the US system does not have units available for a lot of physical quantities that have been important recently.

Unlike the SI system where units for the same quantity , for example length, can vary by orders of 10 (millimeter, centimeter, meter, kilometer), the units for the same quantity in the US have no connection. The units for length in the US system of units can be expressed in several ways. This is displayed in Table 1.2.3.

Table 1.3 Various US units for length (source *Wikipedia*)

Unit	Alternate Unit	SI Equivalent
1 inch (in)		2.54 cm
1 foot (ft)	12 in	0.3048 m
1 yard (yd)	3 ft	0.9144 m
1 mile (mi)	1760 yd	1.609344 km
Survey		
1 link (li)	$^{33}/_{50}$ ft or 7.92 in	0.2011684 m
1 (survey) foot (ft)	$^{1200}/_{3937}$ m	0.3048006 m
1 rod (rd)	25 li or 16.5 ft	5.029210 m
1 chain (ch)	4 rd	20.11684 m
1 furlong (fur)	10 ch	201.1684 m
1 survey (or statute) mile (mi)	8 fur	1.609347 km
1 league (lea)	3 mi	4.828042 km
Nautical		
1 fathom (ftm)	2 yd	1.8288 m
1 cable (cb)	120 ftm or 1.091 fur	219.456 m
1 nautical mile (NM or nmi)	8.439 cb or 1.151 mi	1.852 km

Similar tables can be established for area, volume, mass, cooking measures etc.

US: Base and Derived Units

Table 1.2.4 lists some of the base and derived units in the US system. It also shows the corresponding SI Units and the corresponding conversion factors for equivalence.

Table 1.4 US Units in Statics and the corresponding SI Units with conversion factors

Quantity	US Unit	Equivalent SI Unit
length	1 feet [ft]	0.3048 [m]
mass	1 slug [slug]	14.594 [kg]
mass	1 pound mass [lbm]	0.4536 [kg]
time	1 second [s]	1 [s]
acceleration	1 [ft/s^2]	0.3048 [m/s^2]
angle	1 [rad]	1 [rad]
area	1 [ft^2]	0.0929 [m^2]
density	1 [slug/ft^3]	515.38 [kg/m^3]
energy	1 [ft-lb]	1.356 [J]

force	1 pound force [lb]	4.448 [N]
moment of a force (torque)	1 [lb-ft]	1.356 [Nm]
pressure	1 [lb/ft^2]	47.88 [Pa]
pressure	1 [lb/in^2], [psi]	6.895 x10^3 [Pa]
Stress	1 [lb/in^2], [psi]	6.895 x10^3 [Pa]
volume (solid)	1 [ft^3]	0.02832 [m^3]
volume (liquid)	1 gallon [gal]	3.785 Liter [L]
work	1 [ft-lb]	1.356 [J or Nm]

1.2.3 Conversion between Systems of Units

To work in both system of units you need to how to convert between them. It is a simple arithmetic process with both the numbers and the units themselves.

Example 1.2

For example, in USA, the speed at which you are traveling is usually expressed in miles per hour – say 25 [mph]. In basic units you will need to express the speed in feet per second [ft/s]. In the SI system the corresponding base units are meter per second [m/s], while the popular expression for the speed is kilometer per hour [km/hr].

In order to convert from [mph] to feet per second [ft/s] it we need to know that

1 [mile] is 5280 feet [ft]
1 [hour] is 3600 second [s]

$$25\left[\frac{\text{miles}}{\text{hour}}\right]=25\left[\frac{\cancel{\text{miles}}}{\cancel{\text{hour}}}\right]\times\left[\frac{1\ \cancel{\text{hour}}}{3600\ \text{seconds}}\right]\times\left[\frac{5280\ \text{feet}}{1\ \cancel{\text{mile}}}\right]$$
$$=25\times\frac{5280}{3600}\left[\frac{\text{feet}}{\text{seconds}}\right]=36.67\left[\frac{ft}{s}\right]$$

To convert from US units [mph] to SI units [m/s]

$$25\left[\frac{\text{miles}}{\text{hour}}\right]=25\left[\frac{\cancel{\text{miles}}}{\cancel{\text{hour}}}\right]\times\left[\frac{1\ \cancel{\text{hour}}}{3600\ \text{seconds}}\right]\times\left[\frac{5280\ \cancel{\text{feet}}}{1\ \cancel{\text{mile}}}\right]\times\left[\frac{0.3048\ \text{meter}}{1\ \cancel{\text{feet}}}\right]$$
$$=25\times\frac{5280}{3600}\times 0.3048\left[\frac{\text{meter}}{\text{seconds}}\right]=11.176\left[\frac{m}{s}\right]$$

To convert from [m/s] to [km/hr] or [kph] we can use the conversion factor listed in Table 1.4

$$11.176\left[\frac{m}{s}\right]=11.176\left[\frac{\cancel{m}}{\cancel{s}}\right]\times\left[\frac{1\ km}{1000\ \cancel{m}}\right]\times\left[\frac{3600\ \cancel{s}}{1\ hr}\right]=40.234\left[\frac{km}{h}\right]=40.234\ [kph]$$

1.2.4 Accuracy

For the conversion of units and other calculations in this book you are likely to use a calculator and sometimes you will get a long number. Further additional operations with the number you will continue to obtain these long numbers. For example: cos (25) = 0.906307787 and 2*cos(25) = 1.812615574.

Do we need to know the answer to so many places after the decimals? Let us consider that you are expressing the length in centimeters.
If you are building desk and chairs you need one place after the decimal for a well built product.
If you are building cars you need two places after the decimal for a good looking car.
If you are building airplanes you need three places after the decimal for a well functioning aircraft.

The number of decimal points is related to the accuracy of your calculations. In engineering problems the information is usually considered to be accurate to 0.2%. For example if you are using 2500 [N] for the weight of the pumpkin, a 0.2% error in this estimate is expressed as 2500 ± 5 [N]. Your calculations are only as good as the accuracy of your information. It does not make sense to report answers to the ninth place after the decimal. In this eBook we will report our calculations to two places after the decimal – unless otherwise warranted.

You can choose also to round off your value to the second decimal place. If the number in the third place after the decimal is greater than or equal to 5 then the second place number is increased by 1, except if the second place number is 9. You will then have to change the first place number or not round off. You must be consistent in applying round off. Hence cos (25) = 0.91 and 2*cos(25) = 1.81 (and not 1.82).

1.2.5 Dimension

How do we know that the units for acceleration in basic SI units is [m/s^2]? We can always look it up in a table (Table 1.2) , in a book ,or obtain it by searching the Internet. A more fundamental idea is that every physical quantity is associated with a ***Dimension***.
This is very important in the subject of fluid mechanics. Also, this is very different from the concept of Dimension (D) that is geometrically associated with a problem (like 1 D problem, 2D problem , or 3D problem) in science and engineering, that progressively includes more information, more equations, more work, and is more difficult to solve. We will come this path in this book later. A simple way to understand Dimension that is associated with the units of a physical quantity is to accept that are a set of basic dimensions – (Table 1.2.5).

Table 1.5 Basic Dimensions

Quantity	Symbol
length	L
mass	m
time	t
electric current	I
thermodynamic temperature	T
luminous intensity	I_v
amount of substance	n

Let us consider speed as an example. We know that the average speed is computed by dividing the distance traveled by the time for the travel.

$$speed = \frac{\Delta s}{\Delta t} = \frac{distance}{time} = \left[\frac{L}{t}\right]$$

The dimension for speed is [L/t]. The units for speed can be obtained by substituting appropriate units for the dimensions in the definition. Units for speed : [m/s]; [km/h]; [ft/s]; [ft/h]; [miles/h]; [in/s]; etc. Similarly acceleration is

$$acceleration = \frac{speed}{time} = \left[\frac{\frac{L}{t}}{t}\right] = \left[\frac{L}{t^2}\right]$$

Units for acceleration: [m/s^2]; [ft/s^2]; [in/s^2]; etc.

Example 1.3
Using the Newton's law of gravitation , find (a) the dimension of the universal gravitational constant (G); (b) Its basic units in the SI system; (c) Its basic unit in the US system.

The Newton's law of gravitation, where F is the force between the two masses, m1 and m2, and r is the distance between the centers, can be expressed as:

$$F = G\frac{m_1 m_2}{r^2}$$

Assumptions: To proceed we need to know the dimension of the force. We can look up a book or we can obtain it by using Newton's second law, ***F = ma*** (from physics).

Solution:

Dimension of force, F:

$$[F] = [m] \times [a] = \left[\frac{mL}{t^2}\right]$$

(a) Dimension of G :

$$[G] = \frac{[F][r][r]}{[m_1][m_2]} = \frac{\left[\frac{\cancel{m}L}{t^2}\right][L][L]}{[\cancel{m}][m]} = \left[\frac{L^3}{mt^2}\right]$$

(b) [G] = [m^3/kg-s^2] or [m^3 kg^{-1} s^{-2}]

(c) [G] = [f^{t3}/slug-s$^{2]}$ = [ft^3 slug^{-1} s^{-2}]

Note: We work with symbols the same was as we work with numbers.

1.2.6 Additional Problems

Problem 1.2.1
The stress has the units of force/area. The stress in a material is 13500 lb/in^2 (or **psi**). Convert the stress to the basic SI system (N/m^2). This unit is also called a Pascal [**Pa**].

Problem 1.2.2
The power of a jet engine powering the big airplanes is about 60 MW (mega Watts). Calculate its power in basic US units. Then express it in terms of horsepower.

Problem 1.2.3
Express the standard atmospheric pressure at sea level in both basic SI units and US units.

1.3 USING MATLAB/OCTAVE IN THIS BOOK

If you are already comfortable with MATLAB you can skip this section. If you are new maybe this brief introduction may be useful. All coding is hands on in this book. You are expected to type the code and understand what is happening. There are explanation for what you are doing when you are using it for the first time. All code in the book is run on MATLAB version R015a. That was the version when I started writing this book.

If you are planning to use the book with Octave. The code is the same as MATLAB. Installation discussion is in Section Section 1.3.4.

MATLAB is always evolving. Twice a year there are updates to the software and they are newer versions. In general MATLAB is backward compatible until the command is retired. These updates should not affect its use in this book. You can have the latest version of MATLAB if that is accessible to you. The results of running the code should be the same. However sometimes the layout of the directories will have moved. When you first open MATLAB by double clicking the program for the first time you will see a layout in Figure 1.3.1 called the default layout which includes five separate windows clustered together that I have also explicitly identified.

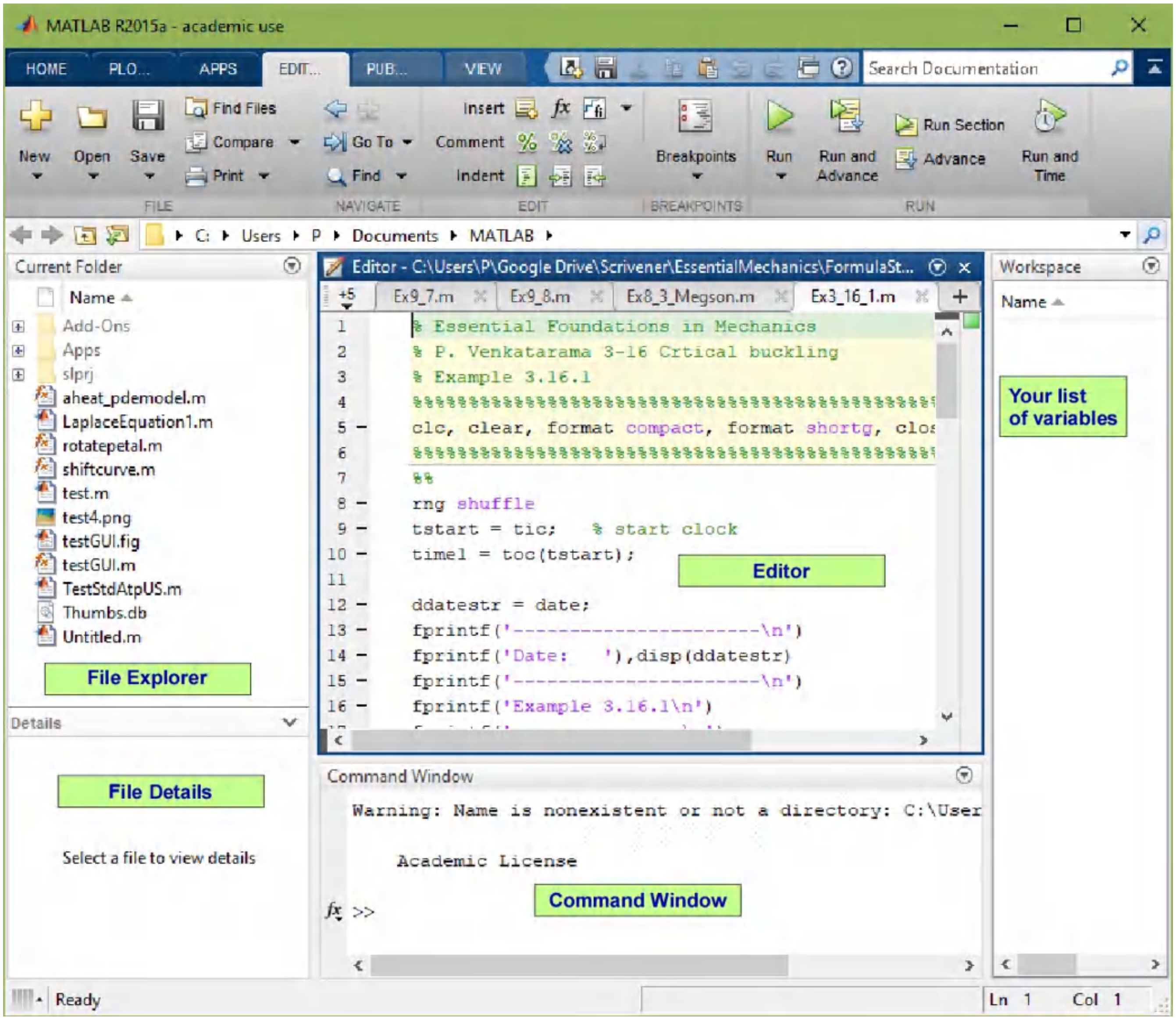


Figure 1.3.1 MATLAB Default Layout

These **five windows** are identified as:

1. Command Window: This is the window with two forward arrows **>>**
The two forward arrows is the MATLAB prompt. The prompt indicates that MATLAB is ready to act on your commands.
This is the window you can interact directly with MATLAB. All results of numerical calculations appear in this window.
You can suppress information in this window by placing a **semi colon** at the end of the sentence. This is the most important **Window** for this book (and for the use of MATLAB)

2. **Editor:** This is the window above Command Window. This is where MATLAB **code** is written. MATLAB code is also called a **script**. MATLAB is a scripting language. It is interpreted instead of being compiled. If you do not follow it is OK. It means that MATLAB will debug your script as it executes the commands in sequence.
Another important thing for this book is that instead of interacting with MATLAB one line at a time, we set up a several lines of code that can be run one at a time automatically and the calculations can be seen in the Command Window. The collection of code is saved in a file before running. This is called a **script file**. The extension for these files is **.m**. You can assign any name to these files but

remember to start with an alphabet and avoid any spaces in the name. You can include numbers and the underscore. You should probably name them so that you can pick them out later to run them again, extend them, or modify them.
This is second most important **Window** for this book. This way all your code is always saved.

MATLAB will debug your script before executing it. If you have errors then you will have to correct them before a successful execution of the script. MATLAB will advise you of the type of error and sometimes suggests way of correcting it. One kind of error is the semantic error because you did not follow the rules of the language. We will be aware of the MATLAB language as we write more and more code and this error will decrease as we solve more problems. The second is the execution error and this is a problem with your calculation. For example you are dividing by a zero accidentally and this will cause NaN - not a number.

Errors are always a source of frustration but resolving them gives you more capability of avoiding them in the future. Tackling errors is part of learning to use any engineering software.

3. **Workspace:** This window will show the number and type of your variables. Is is also identified as **Your List of variables** in Figure 1.3.1. It prevents you from using the same name for the variable for two different quantities. This should not be a problem in this book as our collection of variables will likely be small and you may be able to remember them always - since you are writing the code.
This Window can be **closed** for this book as its use is very limited.

4. **Current Folder**: The current folder window is like a windows **File Explorer** window. It is a display of files in your current folder. You do not need this window. Once you save your file and run the code you will be in placed in the current folder since you will have to select it. You can press the icon Open to see additional files in the folder or in other folders.
This window can be **closed** for the book as its use is very limited.

5. **Details:** This show the type of data in your file.
This Window can be **closed** for this book as its use is very limited.

There is another window in MATLAB that appears only when necessary.

6. **Figure Window:** The Figure Window will appear only if we are are plotting in MATLAB. It is used to display plots and figures. MATLAB produces publication quality graphics and you can control all of the properties through selection or through script.

1.3.1 Two Window Layout

The two-window layout is recommended for this book. This the **Command Window** and the **Editor** organized side by side. You can do this by doing the following two steps.

Step 1. Click the Home Tab at the top level
- (a) Click on the Environment area to drop down a collection of icons
- (b) Click on the Layout Icon to drop down a another menu
- (c) Select Command Window Only

You should see two windows one on top of other as in Figure 1.3.2. During the next step we will disconnect them and arrange them next to each other.

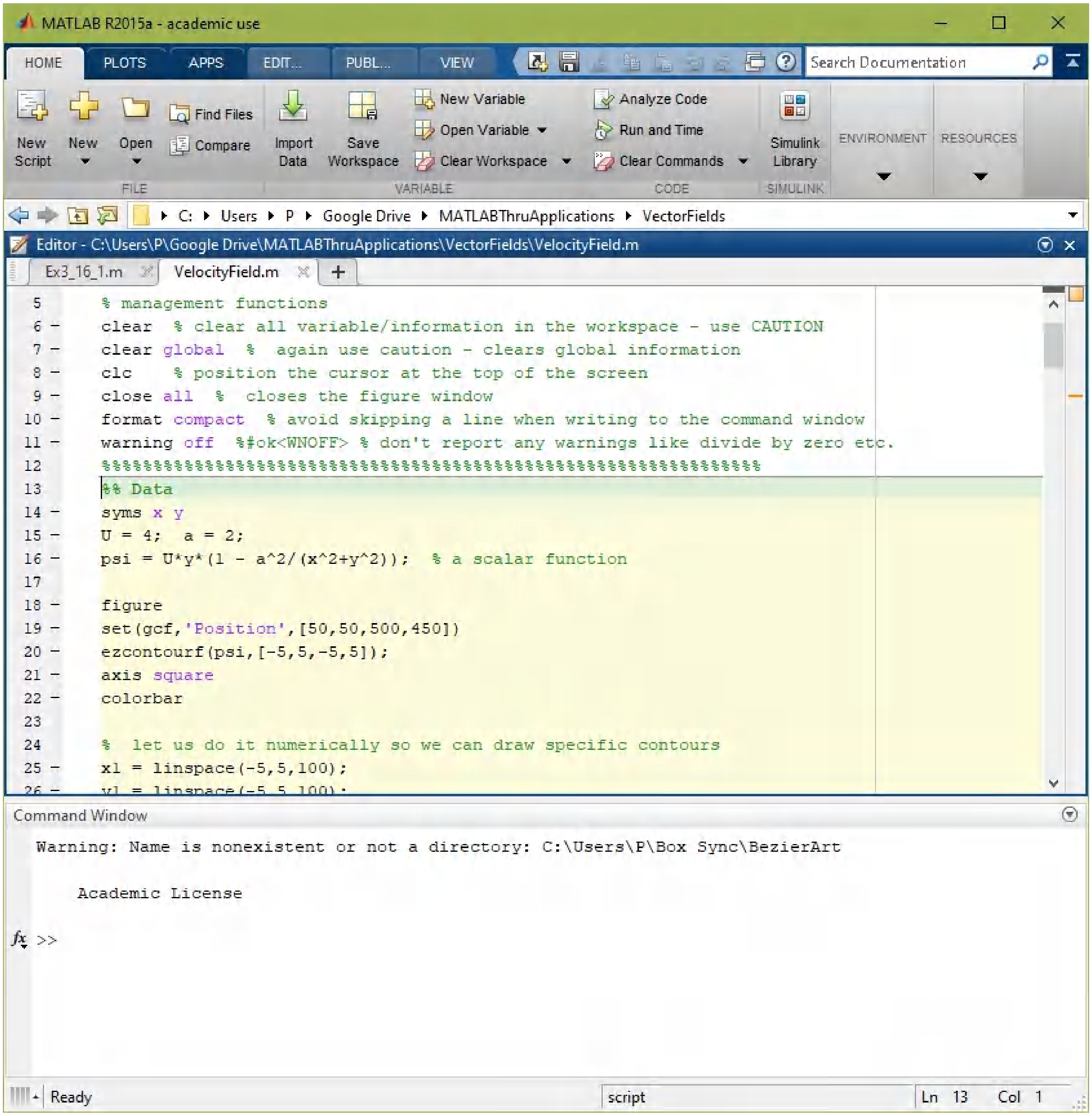


Figure 1.3.2 Two Windows Layout.

Step 2. In the Command Window (lower one) on the right there is a drop down menu. Click on the drop down menu and select **Undock.**
The **Command Window** and the **Editor** are separate. Adjust them so that they are side by side and of the same length

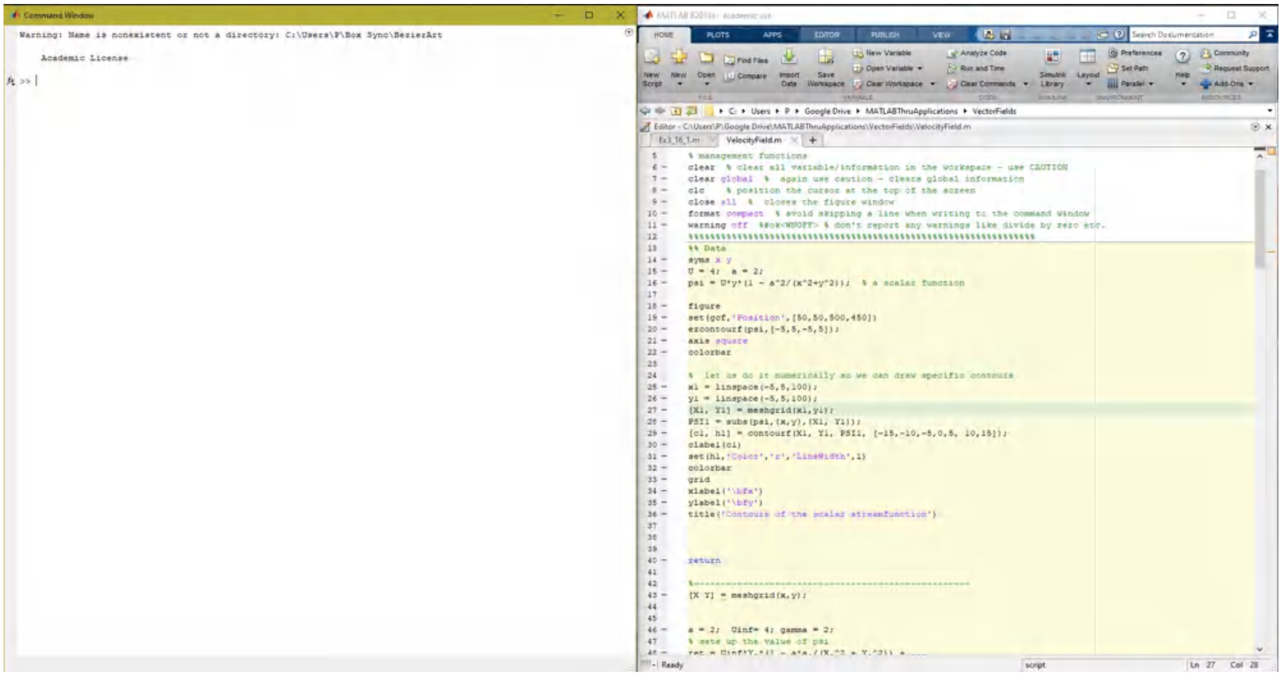


Figure 1.3.3 Command Window and **Editor** Side by Side

1.3.2 Calculating in Command Window

We can do small calculations within the command window. Consider the first part of Example 1.2.1.

We define the variable *mile* as 5280
We define the variable *hour* as 3600

The equal sign in MATLAB (and all other programming languages) is called an assignment statement . It does not mean **equal to**.

mile = 5280 will store the value 5280 at a location in the computer memory addressed by the word **mile**. This is what a **variable** means. It is a placeholder for the number 5280.
hour = 3600 defines another variable storing the value 3600.

25*mile/hour multiplies 25 by the value in the variable mile and divides by the value in the variable hour to calculate to 36.667

This is the rounded value for the value of 25 miles per hour in feet per second.
You have to interpret it this way. MATLAB has no clue about units and dimensions. It is just a calculator. You are responsible for any references to the calculations. The calculations are captured in Figure 1.3.4. You type at the MATLAB prompt. It echos the information you type by default.

MATLAB does all its calculation to 16 decimal places even if it does not display all the digits.

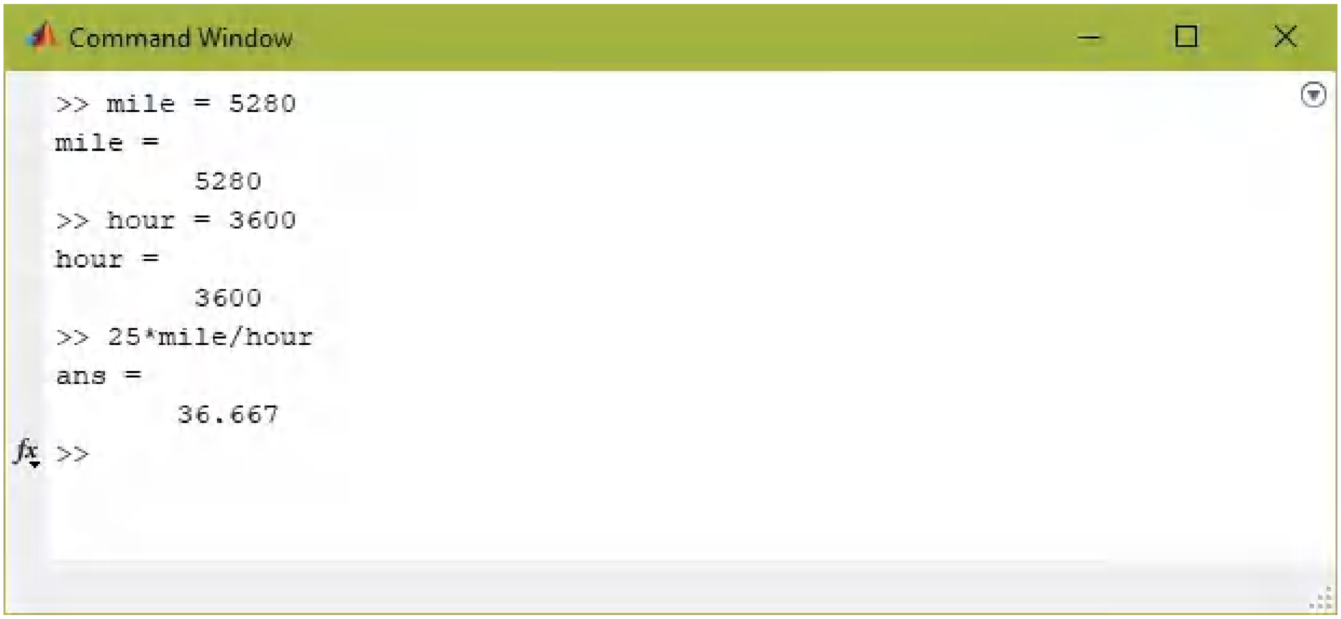


Figure 1.3.4 Direct calculations in the Command Window

1.3.3 Calculation Through the Editor

The same calculation are performed by creating and saving a script file in the editor. The code in the editor and the results in the Command Window are shown below. The code is run in the editor by pressing the **big green filled triangle** named **Run** in the editor.

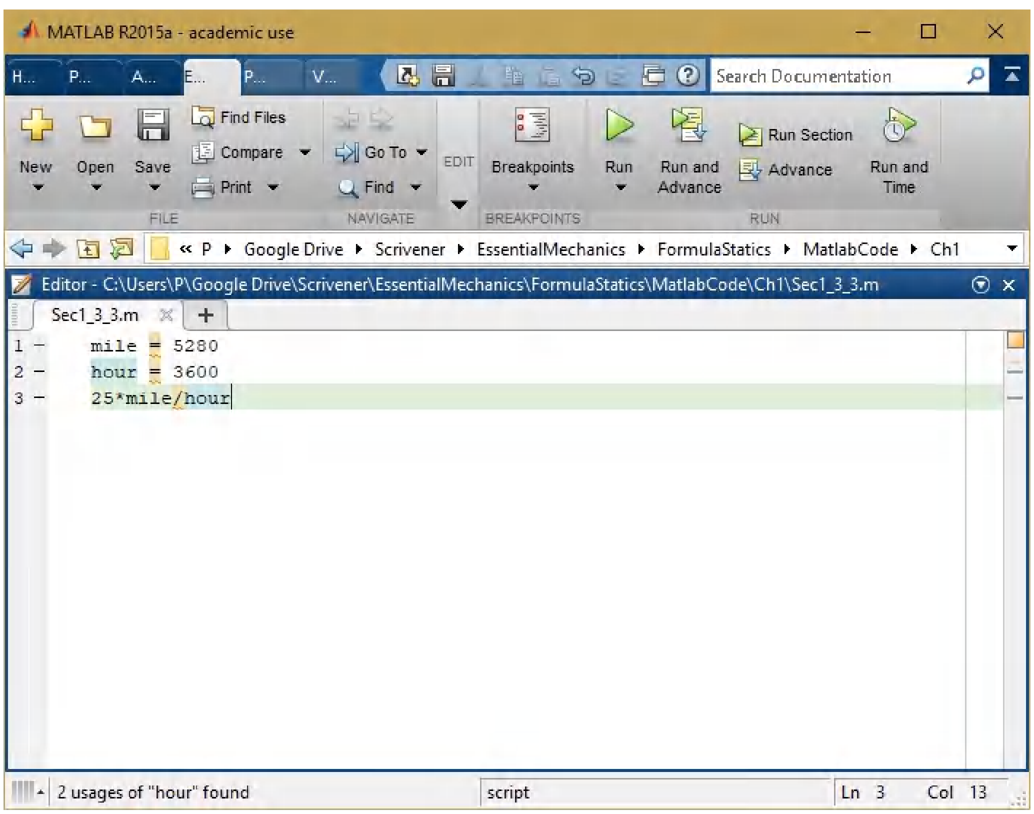


Figure 1.3.5a The code in the editor.

The file name is in the tab. The complete path is in the window banner.

Figure 1.3.5b Results in Command Window

Note that this code is saved in a file until you choose to delete it.
In this book we will collect the code in the **Editor** and run it instead of interacting directly with the **Command Window**.

1.3.4 Octave Program

The Octave program is a free software from GNU foundation that is compatible with MATLAB scripts (https://www.gnu.org/software/octave/0). For the most part your MATLAB script should execute in octave without any change. There is no information about Octave in this book. If you are familiar with it then it should be easy to move between MATLAB and Octave. In this book the MATLAB script is executed in Octave with highlighted changes.

If you are already an Octave user you can skip this section unless you have not used symbolic calculations in Octave. If you a new Octave user you will be typing the same code as MATLAB. So all explanations apply to you too.

The Octave code is edited in the **Octave GUI editor.** The results are available in the **Octave Command Window**. The figure is available in the **Octave Figure Window**.

The following is just provided for illustration. This is using the GNU Octave GUI.

Command Window: Octave

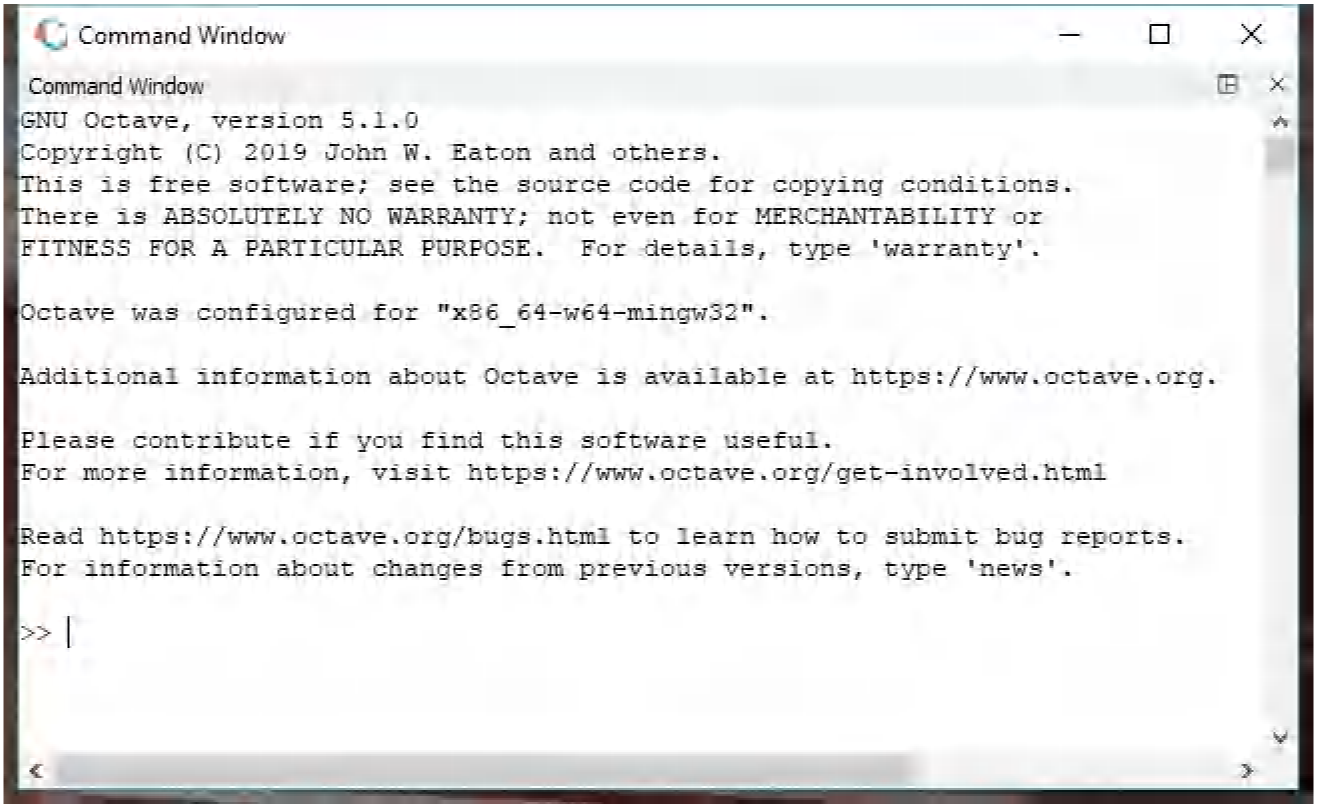


Figure 1.3.6a The Octave Command window

Editor: Octave

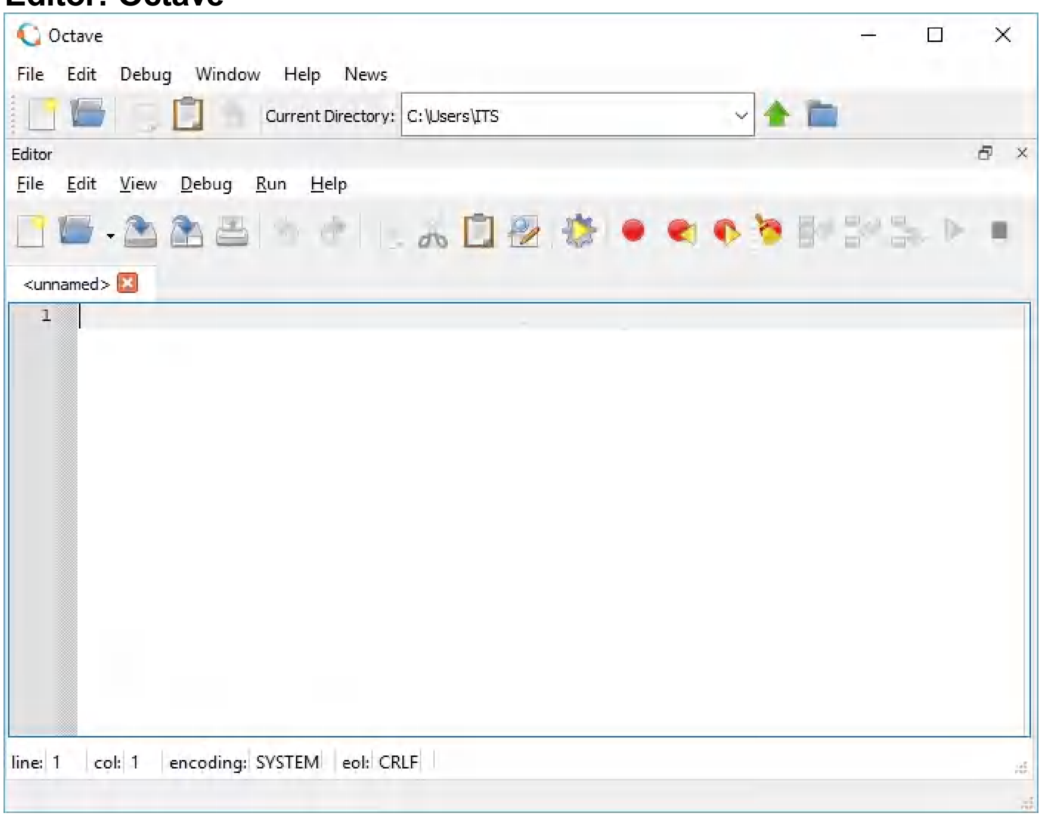


Figure 1.3.6b The Octave Editor

We will use Octave with the same MATLAB m-files that appear in the text and will appropriately edit them.

1.3.5 Octave Installation

In this book we use a lot of symbolic calculations. The symbolic program support is provided by **sympy** - which is the symbolic toolbox for **python**. You really do not need to know sympy and python to run the code in the book. You just need to install the symbolic package as explained in the text at all places where it is required.

I found the instruction available here quite helpful: I hope the link is still available when you are using the book.

https://github.com/cbm755/octsympy/wiki/Notes-on-Windows-installation

I have the Anaconda3 package installed for python and sympy. The GNU Octave version is 5.1.0. Almost all the code worked. I have included the Octave run for the last piece of MATLAB code in each section to show the same code works for MATLAB and Octave.

Note that Octave is quite forgiving of MATLAB code for *comment* even though it uses a different character to signal a comment.

The assumption in this book is that you will be using consistently, either MATLAB or Octave, for all the problems.

1.3.6 Some Problems

Problem 1.3.1

Define a variable that will convert miles per hour and give you kilometer/hour. Therefore if you multiply a value - say 60 - with this variable it will give you the value of 60 miles/hour in terms of kilometers per hour.

Problem 1.3.2

Define a variable that will convert [psi] to [Pa] . Therefore if you multiply a value - say 60 - with this variable it will give you the value of 60 psi in Pascals.

Problem 1.3.3

Define a variable that will convert [kW] (kilo Watts) to [ft-lb/s].

2. HELPFUL CONCEPTS

Any engineering course will be built upon some essential concepts from physics and mathematics. That is an important reason to pay attention to the physics and math service courses that is taught outside the engineering department. The first set of courses in several engineering disciplines are usually the core courses in **Statics**, **Dynamics**, and **Strength of Materials**. These are developed from the basic physics of **Newtonian Mechanics**. In the engineering version of this course (namely *Statics, Dynamics, and Strength of Materials*). Here the topics are embellished, stretched, or applied in a specific manner in a formal and consistent way so that the information is useful for design. This chapter of the book revisits some of those elements from physics and mathematics that are essential in the development of many of the core courses, including this book *Essential Mechanics*. It is an important review and hosts a collection of many formulas that will appear in problem solving throughout the book. The content should be familiar or should be covered concurrently if you are in an undergraduate engineering program and following this book. This chapter should refresh some of your memory. it is also used to get you to solve problems using MATLAB. Some topics may skip some detailed developments

This Chapter includes the following:

- Newton's Third Law - Action and Reaction
- Scalars , Vectors and Matrices
- Vector Multiplication
- Trigonometric relations
- Functions, Derivatives, and Integrals
- Center of Mass, Moment of Areas
- Forces
- Moments and Couples

The list above is quite expansive. Calculations are shown in detail. Please be informed that some of the calculations are performed in MATLAB as it is integrated throughout the book. All MATLAB code appears where they are used. You can ignore it if you do not want to learn to do it in MATLAB. The book still works without using MATLAB. But there are far more, very good books, without the distraction of MATLAB. It is nevertheless a way to gain exposure and confidence in using software to solve engineering problems. MATLAB in this book is used as a high powered calculator rather than a programming platform.

Why MATLAB? Why not Python? Or something else. MATLAB has become very ubiquitous and is now usually available to engineering undergraduates and graduates across the world through the institutional license or through a personal copy of the student edition while you are a student. The author has lots of resources in MATLAB and feels that it is the quickest way to solve, display, explore, and understand engineering problems with a minimum of effort, especially for routine calculations, particularly at the introductory level.

MATLAB is not free. Octave which works like MATLAB is free software from the GNU Foundation. The reason for MATLAB also applies to Octave.

It will be useful to remember that you will likely comeback to reference the information here many times. You might initially want to move on if you understand the topics here as you may have seen them in math and physics courses before. They are usually found in the Appendix in most books. They are here because they are the most important ideas for structural design. They also form an essential foundation for many topics to come.

You must visit the last two sections in this chapter. They describe Forces and Couples. These are the two ways action is delivered to the structure that you are designing. All of Statics, Mechanics , and even Dynamics is how to create or resist or handle these actions on the structures.

2.1 NEWTON'S THIRD LAW - ACTION AND REACTION

It is strange that Newton's third Law appears here before the first and the second Laws. However the application of the first two Laws (most of the book), is based on the concept espoused by the third Law which recognizes the important concepts of ***action*** and ***reaction***. The following illustrations should explain the concept.

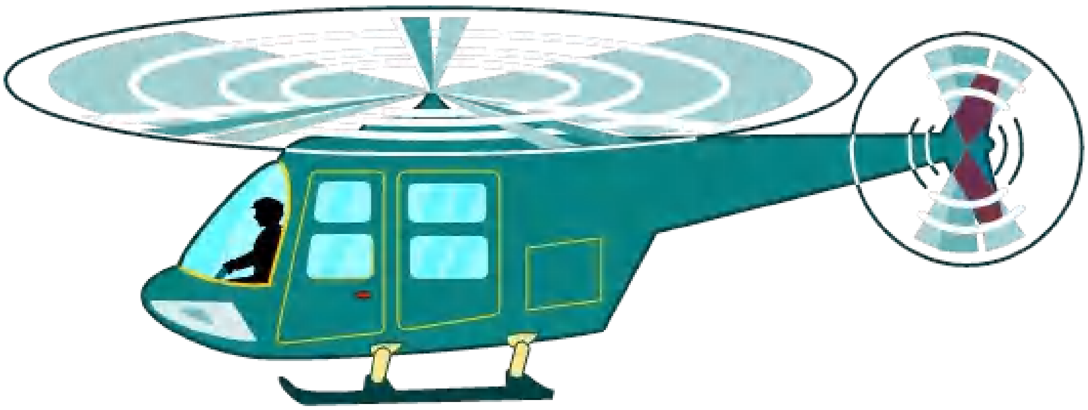


Figure 2.1.1 Newton's third law

The main effort in a helicopter is provided by the rotating big blades through a torque. This lifts the helicopter and causes it to move. The engine provides the power to turn these blades. When the engines turn this rotor, the rotor will attempt to turn the engine on the fuselage where people sit in the opposite direction (making them dizzy). This is a consequence of Newton's third law - ***action and reaction***. In Figure 2.1.1, the tail rotor of the helicopter prevents the helicopter from spinning. The tail rotor will generate a Force, which will create a couple that will balance the torque. The reaction to the force created by the tail rotor is a torque that is easily absorbed by the fuselage. A two rotor design in Figure 2.1.2 will counteract the torque by having two rotors turning in the opposite directions.

Figure 2.1.2 Contra-rotating rotors providing action and reaction

Another example is the recoil from firing a rifle will be absorbed by the shoulder – sometimes called a "kick". Thus the action of sending the bullet flying off creates a reaction that must be absorbed by the shoulder

Figure 2.1.3. Recoil

2.1.1 Action and Reaction

Here is another simple illustration that you can follow along. If an eraser is not convenient you can grab your pen or anything else you find. Experiencing it will make more sense. I am going to grab the pencil eraser by two fingers and push it against the wall. I am doing this horizontally. My fingers or the eraser do not touch the table. Neither the wall or the eraser will move – *a problem in statics.*

Figure 2.1.4 The eraser, grabbing it, and pushing it against a wall

As I push the eraser against the immovable wall, I will feel the eraser pushing against my fingers. We term this as Reaction to the Action of pushing against the wall.

In fact, since the eraser is in contact with the wall, the wall will push back on the eraser – Action and Reaction between the eraser and wall.

But since my fingers are in contact with the eraser, this reaction is transmitted to my fingers through the eraser – Action and Reaction between the eraser and my fingers.

This is Newton's Third Law. Interaction of two bodies will produce an Action and Reaction that will be equal and opposite.

This action, in this example, we will call a **FORCE.** The Reaction is also a **FORCE.** In problems in statics and dynamics very often the reaction is as important as the force in setting up and solving the problem.

2.1.2 Idealization

We will often use or assume certain pattern of action/reaction to simplify the problem. Often, the realistic problem cannot be solved easily. We need to model the reality to use the mathematics we know. In this sense these are physical models - that model the physics, and mathematical models - that model the interactions. Sometimes it is difficult to separate the two. In the early classes the models are simpler and the same topic in graduate school may be handled in a more complex manner to take advantage of the advanced mathematics you have been exposed to. Let us return to the last picture in Figure 2.1.4. This is my action of holding the eraser against the wall through my two fingers. The action and reaction of the eraser and the wall is taking place at the highlighted section

Figure 2.1.5 Action and reaction at the wall

While we accept that there is ***action*** and ***reaction*** between eraser and wall at the highlighted section, however, if we look at them together we cannot *expose* or *see* this action/reaction to calculate its magnitude since the action and reaction will cancel each other. ***We have to look at the wall or the eraser alone***. Now, we look at the eraser only – focusing on the right end where it interacts with the wall. We will *model the effect of the wall on the eraser.* We are also going to ignore the effect of my fingers on the eraser for the time being.

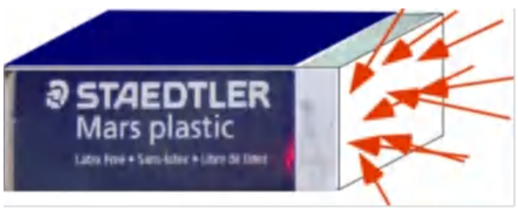


Figure 2.1.6a Effect of the wall on the eraser

The effect of the wall is to push on the eraser through the regions of the wall in contact (only some is shown to avoid cluttering the figure). You will find that the smooth looking end of the eraser has lots of bumps (visible under magnification) and these bumps will contact the wall more than the other areas in the cross-section. These bumps are randomly distributed. The forces at these contact points can be in any direction. To use this model we need to know random distributions, averaging, and many other advanced mathematical topics. Let us look at a simple model.

Figure 2.1.6b Uniformly distributed force

Intuitively, we can expect that it will be easier if we can represent the effect of the wall as a uniformly distributed action. A more simpler model is shown below.

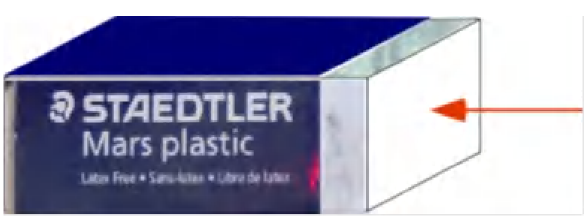


Figure 2.1.6c Concentrated force

It is even easier if we represent the action of the wall by a single arrow at the center of the area of the eraser. This single arrow – representing the **action of the wall** is called the **force** of the **wall** on the **eraser.** It is the ***lumped effect*** of the *distribution of forces* on the eraser **- make the math simpler.** It will be definitely easier to establish than the randomly located randomly directed force distribution in the eraser. This is also called a *concentrated force.* As you can see we have a choice in the way we model the interaction of the wall and the eraser. This will dictated by convenience and your ability to handle the mathematics

Focusing on the eraser *alone* and leaving the wall and the fingers *off the figure* is essential to analyze problems in Mechanics. This is termed as "***Drawing a Free Body Diagram (FBD)***"

2.1.3 The Free Body Diagram (FBD)

Here is another illustration with the eraser in Figure 2.1.7. This time I am just holding up the eraser between my fingers (the leftmost figure). We will assume uniformly distributed forces. On the eraser we can represent the interaction of my fingers as shown (next illustration). In this illustration we need to know the extent of the forces and the location to truly represent the forces. The magnitude of the forces need not be the same. There is equal and opposite reaction on my fingers (next illustration). If we represent the physics through a concentrated force (next two illustrations), we need to know the location only. The forces can only be shown if we consider the ***eraser*** or the ***finger alone***. Together, we cannot expose the forces as in the first figure of the series.

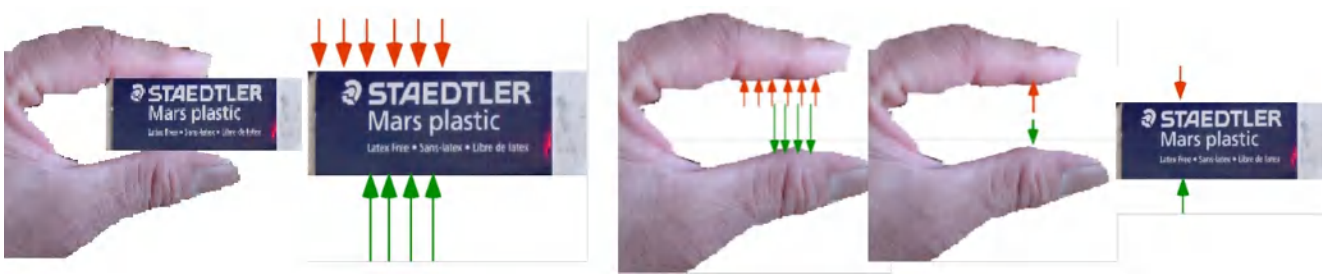


Figure 2.1.7 The interaction between the eraser and the fingers.

Consider the eraser alone in the last illustration. This is called the ***freebody diagram*** (**FBD**) of the eraser. (you have removed (**freed**) the eraser from the problem). **FBD** is the most important concept in Mechanics (no analysis is possible without it).

Additional Examples of FBD

The second example is the man trying to move the heavy dresser. On the left we have him exerting a concentrated force and a distributed force on it. On the right is the FBD of the dresser. Here we have isolated the action on the dresser by the man, the floor, and the weight of the dresser.

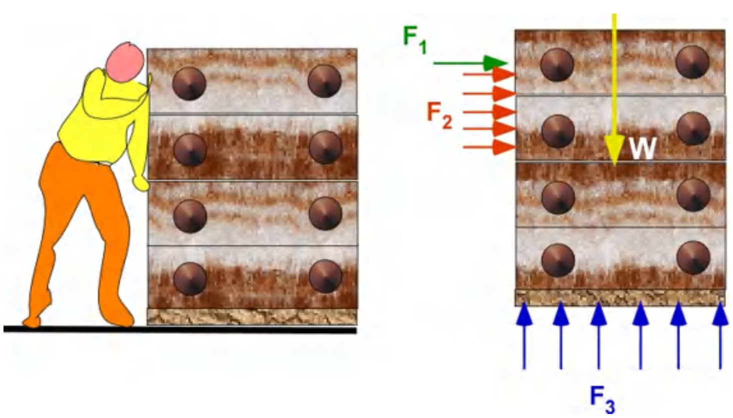


Figure 2.1.8 FBD of the dresser

The third example is you resting after dealing with all this new concepts in mechanics. On the right is the is your FBD with your distributed weight and the action of the mattress on you. If you are wondering why they are equal and opposite, it is because of another Newton's Law - if you are at rest there should be no unbalanced force on you.

Figure 2.1.9 FBD of you resting

If you look at the examples above you should wonder why the dresser and you are two dimensional. Why are the figures representing the problem in two dimensions so convincing? This is another idealization - ***reducing the dimension*** to make the mathematics easy. We are assuming that the action is the same across the object in the direction normal to the diagram shown. Or, more technically, the third dimension will not affect the results of the problem.

Surface and Body Forces

Let us revisit the FBD in Figure 2.1.8. There is another new thing that has suddenly appeared in the figure. You can see the model of the man pushing and being representation by a concentrated and a distributed force. You can also see the action of floor on the dresser as a distributed force. These are called ***surface forces***. They are always placed on the boundary of the body/object. In this case on the left side and the bottom. These are also classified as applied forces or reactions.

There is the third force in the figure called **W**. This is the weight. This acts through the volume of the object/body - which is the dresser in this example. This is called a ***body force***. It is always present in all problems. We can choose to ignore it in the model if it is appropriate and does not influence the problem as much as the applied forces at the surface. It is usually placed at the center of mass - another concept for another day. The center of mass is influenced by the distribution of mass in the object/body. A uniform distribution of mass will locate this point at the center of the geometry.

2.1.4 Statics and Equilibrium

Statics is the study of objects and structures that are ***stationary.*** The objects do not move. Therefore there can be no **unbalanced loads** on the object. According to **Newton's second law** any unbalanced loads will cause the object or the structure move accordingly. If the sum of applied loads on the object is zero then the object is considered to be in ***equilibrium.*** This is best illustrated through the FBD of the object or structure. Every problem in *Statics* and *Strength of Materials* is examined under equilibrium. Consider a simple illustration of an object attached to the wall which is subject to to a tensile force in Figure 2.1.10a. In addition we use the following simplifications:

The wall is considered an immovable object - so the wall and the object do not move either together or separately.
To understand what is happening to the object at the wall a FBD of the object is necessary
In the FBD the object has the applied force on the right end. To prevent it from moving the wall must

exert a reaction force equal and opposite to the applied force - Figure 2.1.10b.

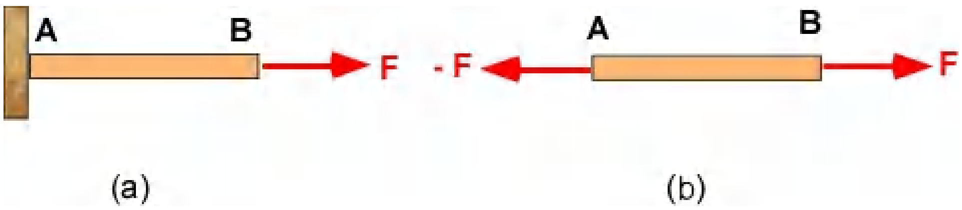


Figure 2.1.10 Original tensile loading and FBD of object AB

In the following we will be using this concept in several examples before actually defining equilibrium formally in Chapter 4.

2.1.5 Additional Problems

Problem 2.1.1
A person is on a diving board. Describe the action and reaction

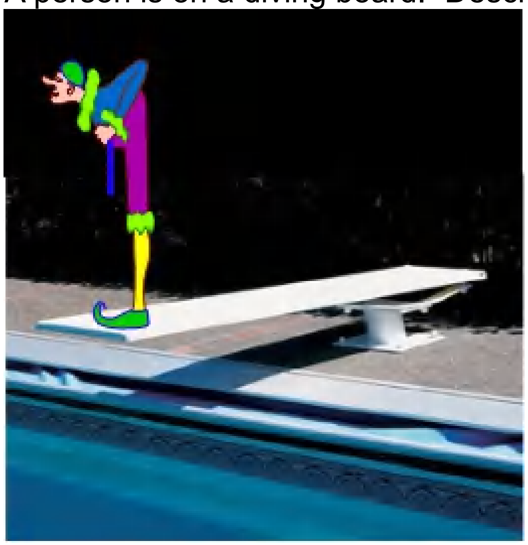


Figure. Problem 2.1.1

Problem 2.1.2.
Changing the lug nut. Describe the action and reaction

Figure. Problem 2.1.2

Problem 2.1.3

One way to bring down a damaged branch is to throw a rope over the limb and pull on either side. Describe the action and reaction

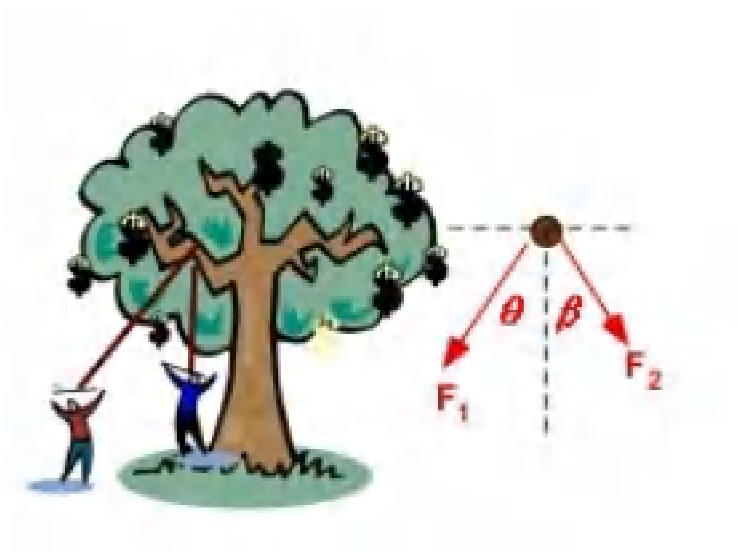


Figure. Problem 2.1.3

2.2 SCALARS AND VECTORS

We have been representing the force as an arrow in the previous discussion. That is deliberate since **Force** is a **Vector**. Every physical quantity in Mechanics and other subjects, which required a value and an unit can be characterized as a (i) s*calar,* (ii) v*ector*, or (iii) t*ensor*.

A *tensor* is usually handled in an advanced courses of mechanics. In the earlier courses, the *tensor* quantities, like stress and strain are just dealt without reference to them being a tensor. It should not detract from your understanding of mechanics. They are extremely useful in expressing the natural laws in a succinct form to make them independent of coordinate systems. We will avoid using the term *tensor* in the book.

2.2.1 Scalar

A ***scalar*** quantity is completely characterized by value and its units. No additional description is required. For example, the length of the table is 1.2 [m] or 3.94 [ft]. The work done while climbing a flight of stairs is 1600 [J] or 1180.1 [ft-lb].

To use a *scalar* in calculations – that is if you are adding to it or multiplying it or dividing with it - you just treat its value like a number – like regular arithmetic.

Some examples of *scalar* are length, mass, time, energy, frequency, power, work, density, and volume.

2.2.2 Vector

A ***vector*** quantity, in addition to value and units , is also characterized by

(a) the direction in which it acts (represented by an arrow).
(b) the point at which it is applied (important for vectors like force) – point of application.

The illustration of a vector is best accompanied by a figure. Figure 2.2.1 shows the weight suspended by two ropes attached to the trees - a model of the person enjoying the hammock. The force in the ropes must be directed along the ropes.

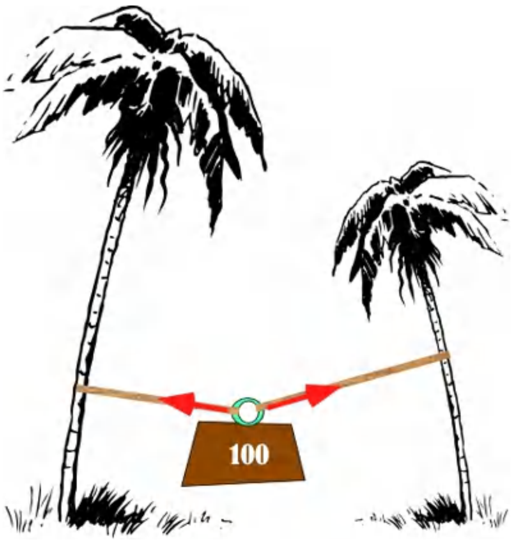


Figure 2.2.1 Vectors

Some examples of *vectors* are: displacement, force, couple, moment of a force, velocity, acceleration, momentum, impulse.
A point force or concentrated force identifies the point of application of the force (through the tail or the head of the arrow).

Vector Arithmetic has a completely different set of rules. Many physical laws are best expressed through vectors..

Representing the two-dimensional (2D) force vector
We look at the example of the weight suspended between the palm trees. The force on the right rope has a value of F_R and it is at an angle of θ_1 with respect to the horizontal. The force on the left rope has a value of F_L and is at an angle of θ_2 with respect to the horizontal (the ropes and the weight must be in a plane). Let us call the force due to the weight (body force - remember) of 100 [kg] as W and it acts straight down (actually towards the center of the earth). Figure 2.2.2 is an idealization of Figure 2.2.1 and is sufficient (and necessary) for engineering calculations. We have shown the vector with a bar over the top in the graphics. In the text on the page it is difficult to place an over bar through the software. We will distinguish the vector from its magnitude by using bold font. Notice, that Figure 2.2.2 represents a FBD of the ring in Figure 2.2.1.

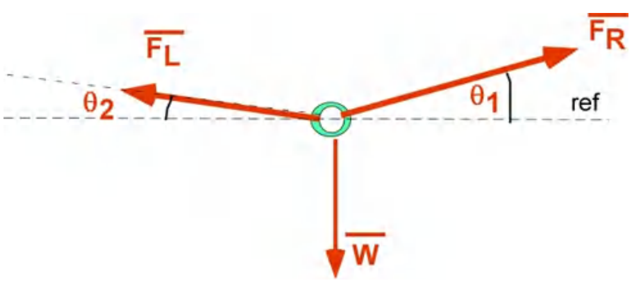
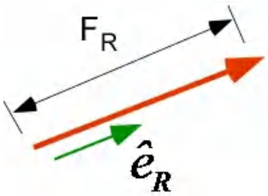


Figure 2.2.2 Idealization of Figure 2.2.1 and vector representation

There are **three ways** to represent a 2D Vector: If the magnitude of FR is 1474.16 [N] and θ1 is 25 [deg]
1. The vector is written as the product of a magnitude of the vector with a unit vector indicating its direction.



$$\bar{F} = F_R \hat{e}_R$$
$$\bar{F} = |F_R| \hat{e}_R$$

Figure 2.2.3a Product of magnitude and unit vector

$$\bar{F}_R = 1474.16\, \hat{e}_R \qquad (2.1)$$

2. The vector is written as the components in a reference coordinate system (default is rectangular/Cartesian system).
In the Cartesian coordinate system, There are two axes (or coordinates or arrows) at 90 degrees drawn with arrows.The tail of the arrows meet at the origin: O. One axis is called the x- axis. It is also identified by the unit vector **i**. The other axis is called the y –axis. It is also identified by the unit vector **j.**
We represent the vector $\mathbf{F_R}$ of Figure 2.2.3b by geometrically drawing an arrow of the proper length at the angle θ_1 with the tail at the origin O.
Note the unit vector $\mathbf{e_R}$ in the same direction as $\mathbf{F_R}$ is also drawn parallel.

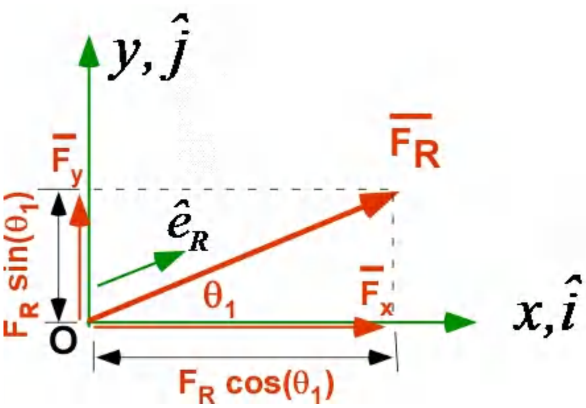


Figure 2.2.3b Vector representation in Cartesian system

$$\begin{aligned}\bar{F}_R &= \bar{F}_x + \bar{F}_y = F_x \hat{i} + F_y \hat{j} \\ &= (F_R \cos\theta_1)\hat{i} + (F_R \sin\theta_1)\hat{j}\end{aligned}$$

$$\begin{aligned}\bar{F}_R &= 1474.16\cos(25)\hat{i} + 1474.16\sin(25)\hat{j} \\ &= 1336.04\,\hat{i} + 623.01\,\hat{j} = \bar{F}_x + \bar{F}_y\end{aligned} \qquad (2.2)$$

As a consequence of the **Cartesian representation** the vector components have the following additional relations. The information between two vertical bars stands for absolute value of F_x. This is a positive value.

$$F_R = |F_R| = \sqrt{F_x^2 + F_y^2} = \sqrt{1336.04^2 + 623.01^2} = 1474.16$$

$$\tan\theta_1 = \frac{F_y}{F_x}; \quad \cos\theta_1 = \frac{F_x}{F_R}; \quad \sin\theta_1 = \frac{F_y}{F_R};$$

$$\theta_1 = \cos^{-1}\frac{1336.04}{1474.16} = \cos^{-1} 0.436 = 25 \qquad (2.3)$$

$$\hat{e}_R = 1\cos(25)\hat{i} + 1\sin(25)\hat{j} = 0.906\hat{i} + 0.423\hat{j}$$

$$\bar{F}_R = 1474.16 \times (0.906\hat{i} + 0.423\hat{j}) = 1336.04\hat{i} + 623.01\hat{j}$$

The advantage of this representation is that we can represent the vector in terms of components. The components of the vector $\mathbf{F_R}$ are $\mathbf{F_x}$ and $\mathbf{F_y}$ in the x and y directions. The magnitude/value of the vector along the x and y directions are F_x and F_y respectively. The component vectors are formed by just multiplying the magnitude with the corresponding unit vectors **i** and **j**. Vector addition is geometric. We lay the first vector ($\mathbf{F_x}$). Then lay the next vector ($\mathbf{F_y}$) maintaining its direction so that it's tail is at the head of the first vector. The new vector (resultant vector) is then the vector from the tail of the first vector to the head of the second vector - $\mathbf{F_R}$. Figure 2.2.3c illustrates the vector addition for the $\mathbf{F_R}$ vector (2.2) expresses the results mathematically. $\mathbf{F_x}$ and $\mathbf{F_y}$ are at right angles to each other.

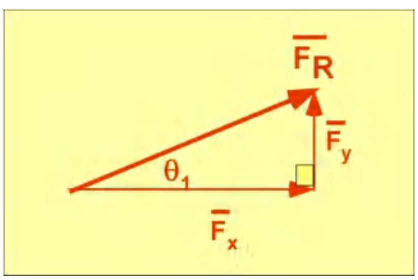


Figure 2.2.3c -Vector addition of components

3. Another set of angles associated with a vector are direction cosines - angle the vector makes with the positive directions of the coordinates. These are angles α and β as shown in Figure 2.2.3d. The sum of α and β is 90 degrees. The definition of the vector this way throws up the set of relations in (2.4)

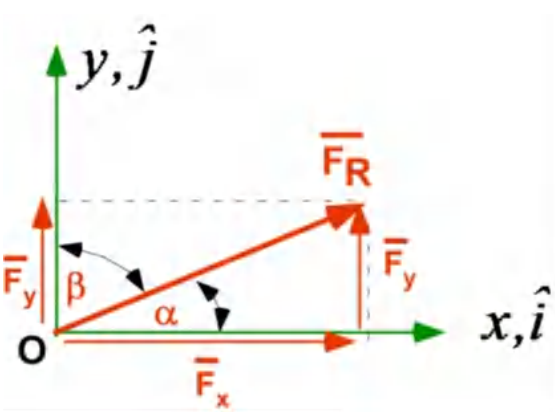


Figure 2.2.3d Direction cosines

$$\bar{F}_R = F_R(\cos\alpha)\hat{i} + F_R(\cos\beta)\hat{j}$$
$$F_R = \sqrt{F_R^2\cos^2\alpha + F_R^2\cos^2\beta} \qquad (2.4)$$
$$\sqrt{\cos^2\alpha + \cos^2\beta} = 1 \quad OR \quad \cos^2\alpha + \cos^2\beta = 1$$

Vector Addition (Components)
We illustrated vector addition previously in defining vectors through their components. Let us choose a value for the force in the left rope in Figure 2.2.2 and add the two vectors. We can add vectors through their components instead of the geometrical addition that was demonstrated in Figure 2.2.3c. Say F_L is 1383.17 [N] at θ_2 = 15 [deg] (as shown in the figure below). Let R be the resultant vector - the new vector that results from vector addition.

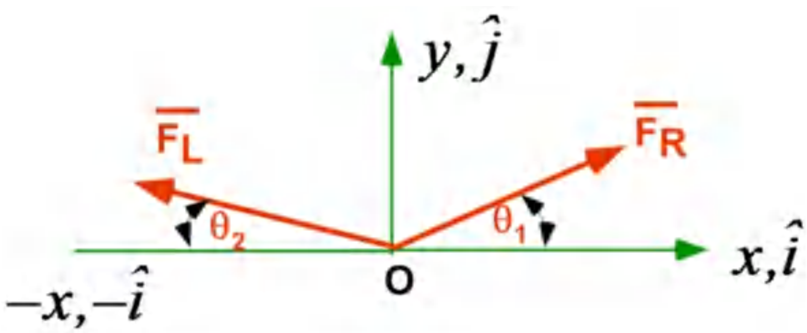


Figure 2.2.4a The two vectors

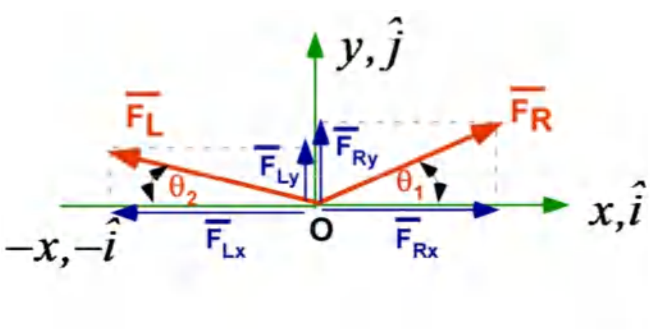


Figure 2.2.4b Their components

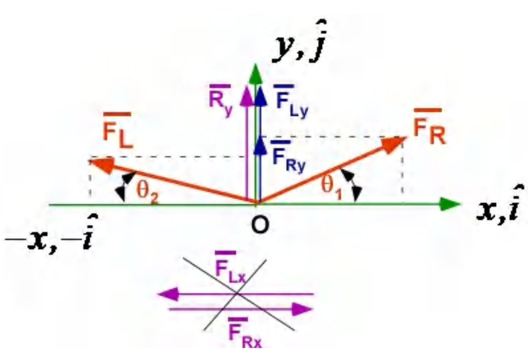


Figure 2.2.4c Their components

We have extended the x-axis on the left and this will indicate negative x-components (of the vector F_L , and negative unit vector **-i**. Also taken for granted is that the arrows indicate the positive directions of the coordinates (and positive directions for unit vectors). The set of calculation in adding vectors using components are shown in Equation (2.5)

$$\overline{R} = \overline{F}_R + \overline{F}_L$$

$$\overline{R} = \overline{R}_x + \overline{R}_y; \quad \overline{F}_R = \overline{F}_{Rx} + \overline{F}_{Ry}; \quad \overline{F}_L = \overline{F}_{Lx} + \overline{F}_{Ly}$$

$$R_x\hat{i} + R_y\hat{j} = \left(F_{Rx} + F_{Lx}\right)\hat{i} + \left(F_{Ry} + F_{Ly}\right)\hat{j}$$

$$\overline{F}_R = 1336.04\,\hat{i} + 623.01\,\hat{j} \qquad (2.5)$$

$$\overline{F}_L = 1383.17\left(\cos 15\right)\left(-\hat{i}\right) + 1383.17\left(\sin 15\right)\left(\hat{j}\right)$$

$$= -1336.04\hat{i} + 357.99\hat{j}$$

$$R_x = 1336.04 - 1336.04 = 0; \quad R_y = 623.01 + 357.99 = 981$$

Vector Addition (Geometric)

Let us add the two vectors geometrically. We start with vector $\mathbf{F_R}$. Move $\mathbf{F_L}$ parallel to itself so that the tail of $\mathbf{F_L}$ is at the head of $\mathbf{F_R}$. The resultant vector **R** is drawn from the tail of the first vector to the head of the second vector. Vector addition of two non-parallel vectors will result in a triangle as shown in Figure 2.2.5a.

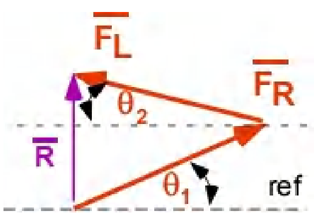


Figure 2.2.5a Vector addition

There are some geometric identities in vector addition that are useful when the vectors form a triangle as in Figure 2.2.5a. They are called the **cosine rule** and **sine rule**. This can be seen in the geometry of vector addition in the set of relations Equations (2.5). These are very useful in establishing angles and magnitudes. Additional angles are introduced for these relations.

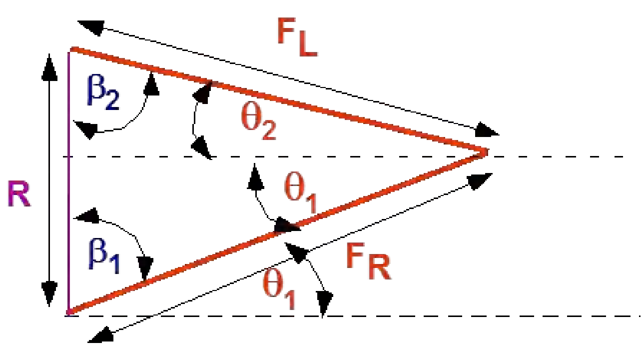


Figure 2.2.5b Triangle identities

$$\text{CosineRule}: R^2 = F_R^2 + F_L^2 - 2F_R F_L \cos(\theta_1 + \theta_2)$$

$$\text{sineRule}: \quad \frac{F_R}{\sin\beta_2} = \frac{F_L}{\sin\beta_1} = \frac{R}{\sin(\theta_1 + \theta_2)}$$

$$R = \sqrt{1474.16^2 + 1383.17^2 - 2\times 1474.16 \times 1383.17 \times \cos(25+15)} = 981 \qquad (2.6)$$

$$\beta_2 = \sin^{-1}\left(\frac{F_R \times \sin(\theta_1 + \theta_2)}{R}\right) = \sin^{-1}\left(\frac{1474.16 \times \sin(25+15)}{981}\right) = 75$$

Addition of Three Vectors

In Figure 2.2.6a there are three vectors. Let us add these three vectors using the previous values for the components of $\mathbf{F_R}$ and $\mathbf{F_L}$. **W** is 981 [N] in the direction **-j**.

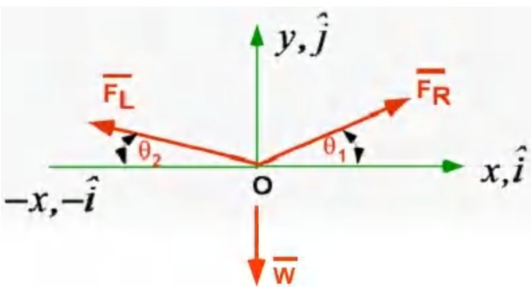


Figure 2.2.6a Three vectors

Adding the vectors geometrically, note that the start of **R** and end of **R** is the same point - a zero vector

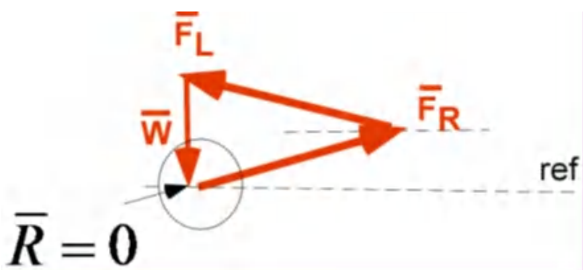


Figure 2.2.6b Geometric addition of three vectors

2.2.3 MATLAB Vector Handling

Vector handling is natural in MATLAB. By default vectors are defined as row vectors until you make them column vectors. Even if we are dealing with only a 2D vector we will represent the same vector as a 3D vector so that regular matrix arithmetic is easily included. In this case the third component is zero. Remember, you can drop the semi-colon at the end of the line to see the calculations corresponding to that line. This is our first code.

The code is very long particularly if this is the first time you are using MATLAB. The code is long because of explanations and the fact that we are printing a lot of information in easily readable format. Then there are code that does the calculations. Also you will see the figure window for the first time. If you pay attention now the next time we will not need these same explanations again - so the code length will be smaller. We are printing everything. For the next code we will copy, paste and edit a lot of statements.

This time:

1. Start MATLAB so that the Command Window (**CW**) and **Editor** are visible
2. Press the **+** button below the icons in the Editor. It should generate a blank space for a new script file.
3. **Copy** and **paste** the code below to that region. Select the code with a **mouse** and use the **right button** to **copy** and **paste**
4. Press the green **run arrow** in the editor
5. You will have to **save the file**. Save the file in a directory where you will collect the code for this chapter. Make sure you do not have a **'space'** in the name of the file
6. If there are no errors (there should be none if you copied all the code) you should see some information in the command window and a plot of the vector addition.

In the code below:

- MATLAB code is **color coded**
- **%** - MATLAB **comment**: MATLAB ignores anything to the right of percent sign - the characters are in green color
- **clc, clear, format compact, close all** are ways to control interaction with MATLAB
- Ignore any underlining in the code below - it is unwanted spell check by the word processor
- Elements in **single quotes** are **text/string** elements which are in violet color
- In general all MATLAB commands are in lower case

```
% Essential Mechanics
% P. Venkataraman
% Section 2.2.3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% clc : stands for place cursor on top of command window
% clear :  clear all variables
%          you are planning to run this code from the beginning each time
%          remove this command if you running multiple script files
% format compact : avoid an extra line feed when writing to command window
% close all : close all plot windows that are open
% WE WILL USE THESE COMMANDS IN ALL FUTURE CODE (without this explanation)
% REMEMBER run without semicolon at the end to see what MATLAB prints to
CW
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% print information about problem - CW stands for command window
fprintf('Section 2.2.3\n')  % prints information  moves to next line
fprintf('Vector addition\n')  % same  and \n makes MATLAB go to next line
fprintf('-----------------------')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data and print
FR = 1474.16;  tht1 = 25; % define FR (magnitude) and tht1 (angle)
fprintf('\nFR = %6.2f  tht1 = %6.2f',FR,tht1)  % print information to CW
FL = 1383.17;  tht2 = 15;  % define FL (magnitude) and tht2 (angle)
fprintf('\nFL = %6.2f  tht2 = %6.2f',FL,tht2)
M = 100; g = 9.81;
fprintf('\nM = %6.2f',M)  % define mass
```

```
%% create vectors - rectangular system
% vector is a 3 element row array with x,y,z components within []
parenthesis
% for 2D vectors define z - component as zero
FRv = [FR*cosd(tht1),FR*sind(tht1),0];  % vector FR
fprintf('\nVector FR = [%6.2f, %6.2f, %6.2f] ',FRv) % print vector FR to
CW
FLv = [-FL*cosd(tht2),FL*sind(tht2),0];  % vector FL
fprintf('\nVector FL = [%6.2f, %6.2f, %6.2f] ',FLv) % print vector FL to
CW
Wv = M*g*[0,-1,0];    % vector weight
fprintf('\nVector W = [%6.2f, %6.2f, %6.2f] ',Wv) % print vector FL to CW

%% add the vectors
R = FRv + FLv + Wv;  % R is the sum of the vector addition
fprintf('\nResultant vector R = [%6.2f, %6.2f, %6.2f] ',R) % print R

%% Unit vectors
eR = FRv/norm(FRv);
fprintf('\nUnit Vector eR= [%6.2f, %6.2f, %6.2f] ',eR) % print unit vector
eR
eL = FLv/norm(FLv);
fprintf('\nUnit Vector eL= [%6.2f, %6.2f, %6.2f] ',eL) % print unit vector
eR
fprintf('\n\n') % skip two lines and then display prompt
%% Drawing the vectors -2D ARROWS using quiver command
quiver(0,0,FRv(1),FRv(2),0,'r');
% first two values  0, 0, are the starting x and y values - TAIL
% second two values FRv(1),FRv(2) - vector components length (x, y) - HEAD
% next value 0 is scaling factor
% 'r is red color

hold on  % adding more stuff to the same figure
quiver(FRv(1),FRv(2),FLv(1),FLv(2),0,'b');  % second vector
% tail for FL vector is the head of first vector - first two values
% second two values FLv(1),FLv(2) - vector components length (x, y) - HEAD
% no scaling
% 'b' blue color

quiver(FRv(1)+FLv(1),FRv(2)+FLv(2),Wv(1),Wv(2),0,'m');  % add third vector
axis image  % fit the vectors
```

In the Command Window

```
Section 2.2.3
Vector addition
----------------------
FR = 1474.16  tht1 =  25.00
FL = 1383.17  tht2 =  15.00
M = 100.00
Vector FR = [1336.04, 623.01,   0.00]
Vector FL = [-1336.04, 357.99,   0.00]
Vector W = [  0.00, -981.00,   0.00]
```

```
Resultant vector R = [  0.00,  -0.00,   0.00]
Unit Vector eR= [  0.91,   0.42,   0.00]
Unit Vector eL= [ -0.97,   0.26,   0.00]
```

In the Figure Window

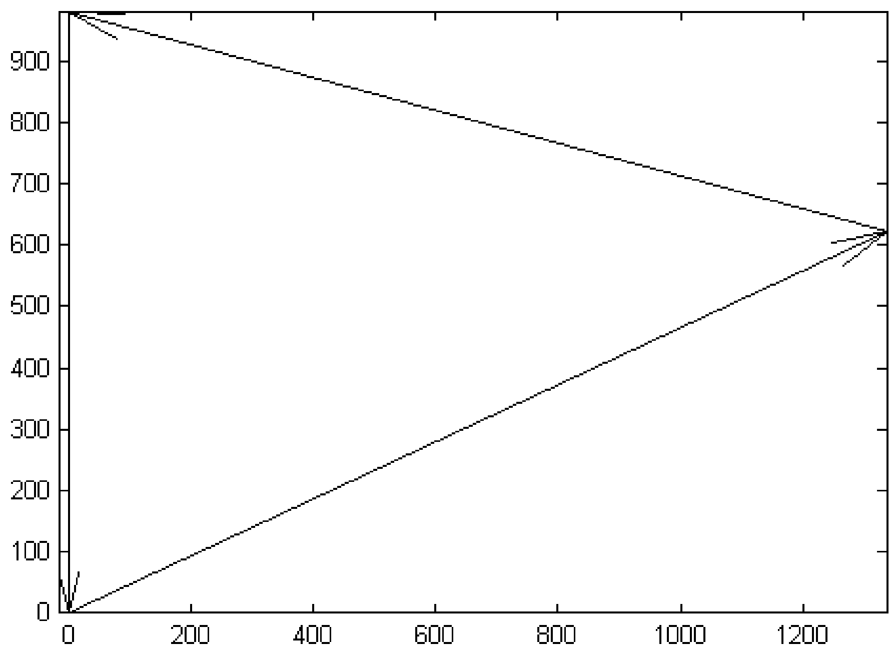


Figure 2.2.7 Geometric addition of three vectors in MATLAB

Now some additional exercises and explanations:

1. First compare the results printed to the command window with the values calculated in the previous section. Do they match?
2. After running the code walk through it line by line make a list of items for which you need more explanations
3. Notice that the unit vectors ***I, j,*** and ***k*** are implicit. We need to work with only the **components**.
4. A physical vector has three values between **square parenthesis**.This is also called an **array**.
5. We work with three components in the array even if this is a two-dimensional example. They are in order of x, y, and z components.
6. Count the number of lines starting with `%.` **These don't do any calculations**
7. Count the number of lines that begin with **`fprintf.`** These are only used to **print** neatly to **CW**.
8. The characters **`%6.2f`** prints a real number with two numbers after decimal and total width of six character spaces. When printing a vector you need to print three numbers. That is why `[%6.2f, %6.2f, %6.2f]`
9. The value between single quotes like `'Section 2.2.3\n'` is a **string** that is printed as written unless it also prints a number.
10. You can define multiple variables or commands on a single line. The must be separated by a **semicolon** or **comma.**
11. FR = 1474.16; This is an **assignmen**t statement. You are storing the value of 1474.16 in the variable called FR. This use of "=" sign is true for all programming languages. Same is true for

other variable declaration. In this sense it is not an “equal to” implication.

12. Here you are assigning an array to the variable FRv. `FRv = [FR*cosd(tht1),FR*sind(tht1),0];`
13. The **quiver** command draws an arrow as explained above. It requires the **tail location** and the **length of the arrow components**.
14. The command **hold on** will allow you to add additional information to the figure. Otherwise it will overwrite the previous information (which will be deleted from the figure).

2.2.4 GNU Octave

The following is the version information for Octave

GNU Octave, version 5.1.0
Copyright (C) 2019 John W. Eaton and others.
This is free software; see the source code for copying conditions.
There is ABSOLUTELY NO WARRANTY; not even for MERCHANTABILITY or
FITNESS FOR A PARTICULAR PURPOSE. For details, type 'warranty'.

Octave was configured for "x86_64-w64-mingw32".

Additional information about Octave is available at https://www.octave.org.

1. Copy and Paste the MATLAB Code in the Octave Editor. Run the code in Octave

Octave Execution:

1. Start **Octave** GUI so that the Command Window (**CW**) and **Editor** are visible
2. Press the **New script** button below the menu bar in the Editor. It should generate a blank space for a new script file.
3. **Copy** and **paste** the code above to that region.
4. Press the yellow **run arrow** in the editor
5. You will have to **save the file**. Save the file in a directory where you will collect the code for this chapter. Use a different directory for Octave files. Make sure you do not have a **‘space’** in the name of the file
6. If there are no errors (there should be none if you copied all the code) you should see some information in the command window and a plot of the vector addition.

In Octave Command Window

```
 Section 2.2.3
Vector addition
----------------------
FR = 1474.16  tht1 =  25.00
FL = 1383.17  tht2 =  15.00
M = 100.00
Vector FR = [1336.04, 623.01,   0.00]
Vector FL = [-1336.04, 357.99,   0.00]
Vector W = [  0.00, -981.00,   0.00]
Resultant vector R = [  0.00,  -0.00,   0.00]
Unit Vector eR= [  0.91,   0.42,   0.00]
```

Unit Vector eL= [-0.97,   0.26,   0.00]

In Octave Figure Window

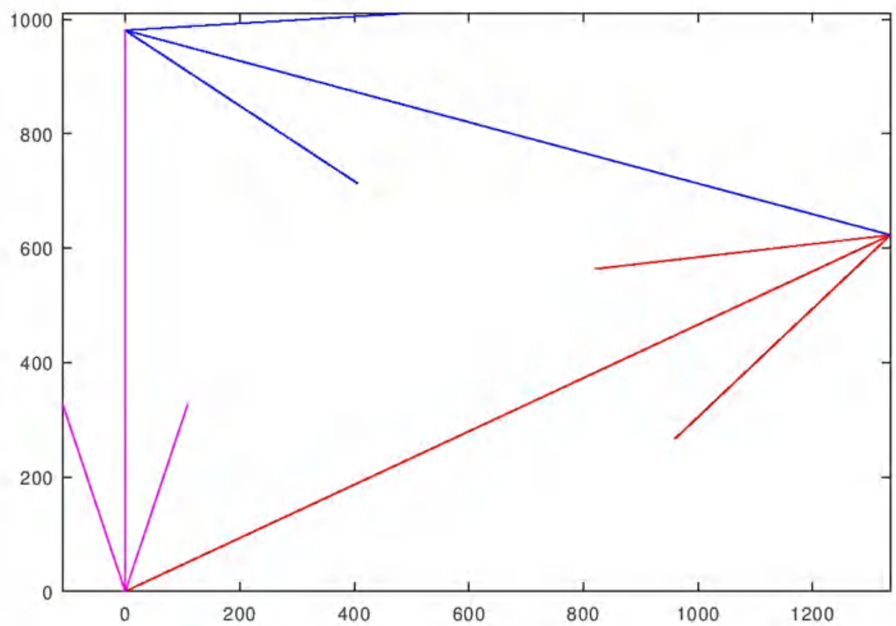


Figure 2.2.8 Geometric addition of three vectors in Octave

The ***same code is run in Octave***. The numerical results are the same while the figure is the same except for the arrow heads.

2.2.5 Additional Problems

In the following problems, first solve the problem by hand and then re-solve by MATLAB/Octave and compare.

To solve the problem in MATLAB:

1. Copy and paste the code in the editor and save it with a different name for each problem.
2. Edit the code to define your own variables and assign it the new values
3. Edit the print statements to reflect the new variables
4. Delete any line that you do not need.
5. Check the calculations have the right variable names in them
6. Delete any calculations that do not apply to the problem.
7. Run the code and make sure you have no programming error - it appears in red and will point you to the line where the error is.
8. If hand calculation and MATLAB calculation do not match - figure out why they do not match. Go forward calculation by calculation

Problem 2.2.1

Add the two vectors **F1** and **F2.** Vector **F1** has a magnitude of 100 and is inclined at a negative angle of θ = 35 degrees (below the positive x direction). Vector **F2** which has components 35 and 50 in the x and y directions respectively (units do not matter for this problem). Calculate the vector which results from this vector addition and sketch the addition in your note book.

Problem 2.2.2
R1, **R2**, **R3**, and **R4** are four vectors with magnitudes of 15, 20, 25, 30 units respectively. The angle with respect to the positive x-direction are 30, 75, 135, and 225 degrees respectively. Find the vector sum of these four vectors and sketch them in your note book.

Problem 2.2.3
Add the vectors in Problem 2.2.1 and Problem 2.2.2 and obtain the resulting vector.

2.3 THREE DIMENSIONAL VECTORS

Three dimensional (3D) vectors, mathematically are no different than 2D vectors. We will use for illustration a popular celebration in India called “Gokulashtami”, or Lord Krishna’s birthday which mostly occurs in the month of August. A human pyramid is formed to smash the pot filled with money and goodies suspended far above the road.

Figure 2.3.1a. The pot of goodies (image from Google)

We will bring the problem down to earth and redefine it as

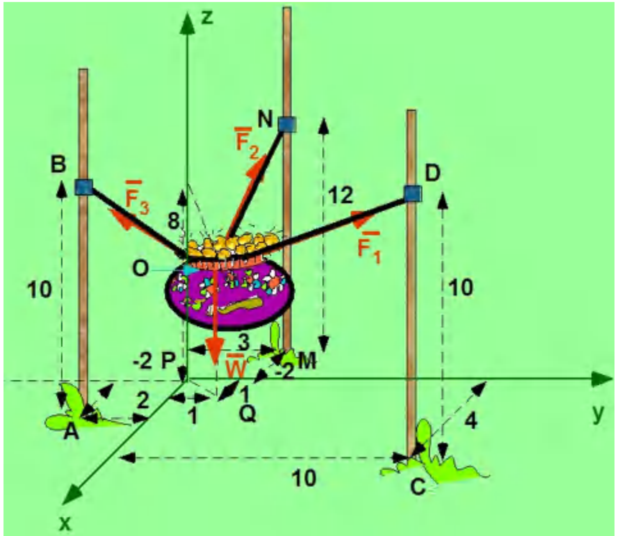


Figure 2.3.1b. The pot of gold

We can represent the pot through a three point suspension. We expect the three ropes to be in tension: there is a force in each rope, along the rope from the pot to the point of suspension. The weight of the gold is downward. We assume that these forces meet at a point to keep it simple. We then have the most unglamorous description of the problem in Figure 2.3.1c. You cannot even distinguish that they are 3D vectors because of the limitation of the 2D illustration. We must embellish the representation to convey the actual information. We introduce some reference, in this case a coordinate system, to make sense of direction.

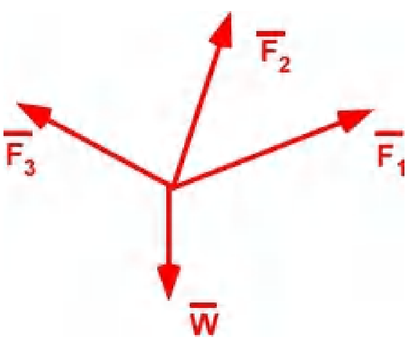


Figure 2.3.1c. A model of the original problem

2.3.1 3D Vector Representation

Like 2D vectors, 3D vectors can also be represented in 3 ways.

Magnitude and unit vector

The vector is written as the product of a magnitude of the vector with a unit vector indicating its direction. Same as the 2D vector. Let us focus on vector $\mathbf{F_1}$.

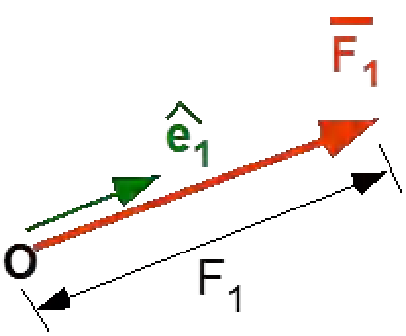


Figure 2.3.2 The 3D vector

$$\bar{F}_1 = F_1 \hat{e}_1 = |F_1| \hat{e}_1 \qquad (2.7)$$

Cartesian Coordinate System

The popular and simple way to deal with vectors is to use the three-dimensional *rectangular* or the ***Cartesian*** system. While there are alternative coordinate systems like *cylindrical* or *spherical* coordinate system, depending on the geometry of the problem, we will limit our vector description to the rectangular system at this phase. The Cartesian coordinate system (3D) has an origin (point O), and three mutually perpendicular axes (or *coordinates* or *arrows*) marked as x, y, z with positive unit vectors **i**, **j,** and **k** respectively.

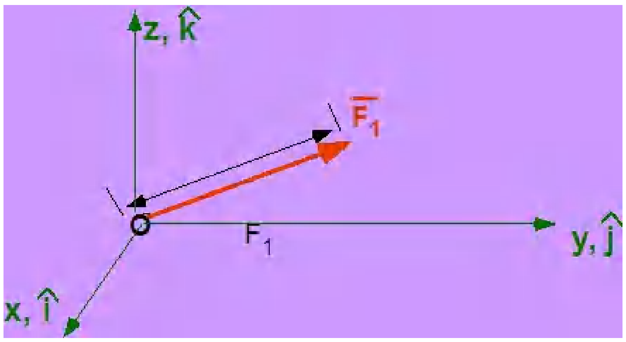


Figure 2.3.3 Vector F1 in the Cartesian system

This coordinate system we use must also be a right-handed system so that all the formulas you know work consistently. The three coordinates must be defined in a certain way. Once the first two coordinate directions (x, y) are defined then the third (z) is established using the right hand rule. Hold out your right hand. Assume x is along the fingers, y along the hand (towards elbow), then z must be in the direction of the thumb, as you rotate from x to y. This is illustrated in Figure 2.3.4.

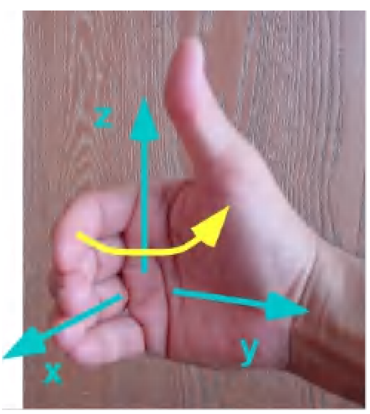


Figure 2.3.4 Right handed system

3D Vector using Components

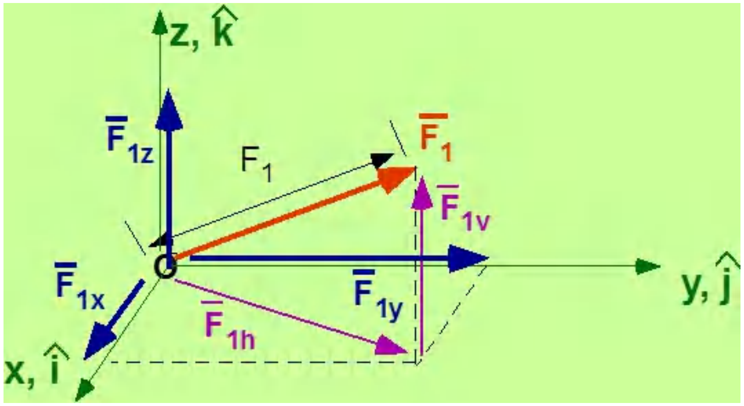


Figure 2.3.5 Vector and components

The vector $\mathbf{F_1}$ can be described by the vector addition of the components $\mathbf{F_{1x}}$, $\mathbf{F_{1y}}$, $\mathbf{F_{1z}}$. You will also notice that several other vector additions can be defined in Figure 2.3.5. $\mathbf{F_{1h}}$ is in horizontal plane. $\mathbf{F_{1v}}$ is normal to this plane.

$$\begin{aligned}
&\bar{F}_1 = \bar{F}_{1x} + \bar{F}_{1y} + \bar{F}_{1z} = F_{1x}\hat{i} + F_{1y}\hat{j} + F_{1z}\hat{k} \\
&\bar{F}_1 = \bar{F}_{1h} + \bar{F}_{1v} \\
&\bar{F}_{1h} = \bar{F}_{1x} + \bar{F}_{1y}; \quad \bar{F}_{1v} = \bar{F}_{1z} \\
&\bar{F}_1 = \bar{F}_{1h} + \bar{F}_{1v} = \bar{F}_{1x} + \bar{F}_{1y} + \bar{F}_{1z}
\end{aligned} \tag{2.8}$$

3D Vector using Magnitude (F_1) and two angles (θ, ϕ)

Once again we will define the $\mathbf{F_1}$ vector. We will use its magnitude (F_1) and angle θ is measured from the x – axis (typically), while the angle φ is measured from the horizontal plane. This is often referred to as a *spherical coordinate system*.

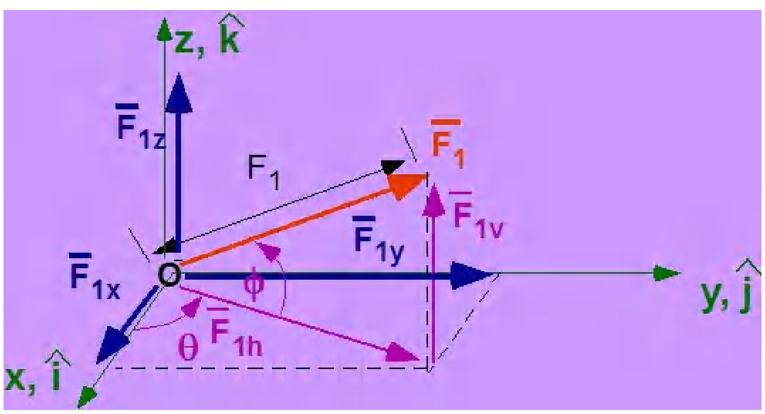


Figure 2.3.6 3D vector using magnitude and two angles

$$\begin{aligned}
&\bar{F}_1 = \bar{F}_{1h} + \bar{F}_{1v} \\
&F_{1h} = F_1 \cos\phi; \quad F_{1v} = F_1 \sin\phi = F_{1z} \\
&F_{1x} = F_1 \cos\phi \cos\theta; \quad F_{1y} = F_1 \cos\phi \sin\theta
\end{aligned} \tag{2.9}$$

3D Vector using Magnitude (F_1) and Direction Cosines (α, β, γ)

We will define the vector $\mathbf{F_1}$ using the vector magnitude (F_1) and three angles (α, β, γ) - called the direction angles - drawn with respect to the positive coordinate directions (x, y, z) and vector $\mathbf{F_1}$

respectively.

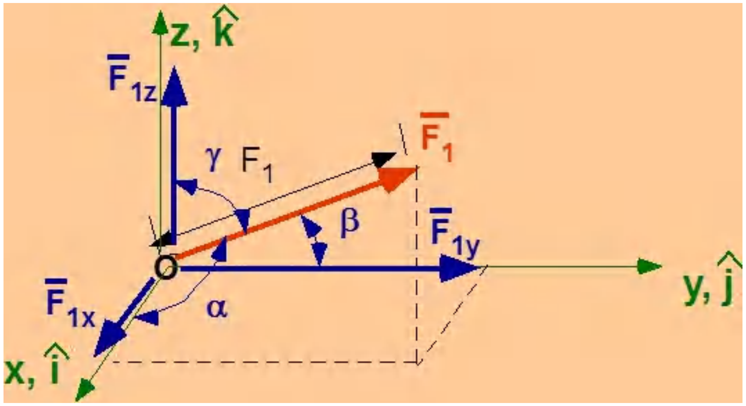


Figure 2.3.7 3D vector using magnitude and direction cosines

$$F_{1x} = F_1 \cos\alpha; \quad F_{1y} = F_1 \cos\beta; \quad F_{1z} = F_1 \cos\gamma; \qquad (2.10)$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Note that there are only two independent angles because of the relation among the cosine of the three angles.

Vectors from Coordinate Locations

Physical vectors can be derived from coordinate locations. Any point in space can be identified by its coordinate location - location with respect to the origin of the coordinate system. This is an important way to obtain **unit vectors**. The vector from point A to point B can be obtained from the coordinates at B minus the coordinates of A, multiplied by the corresponding unit vector along the coordinates. The unit vector in the direction of the vector is the same vector divided by its magnitude. Consider Figure 2.3.1b again.

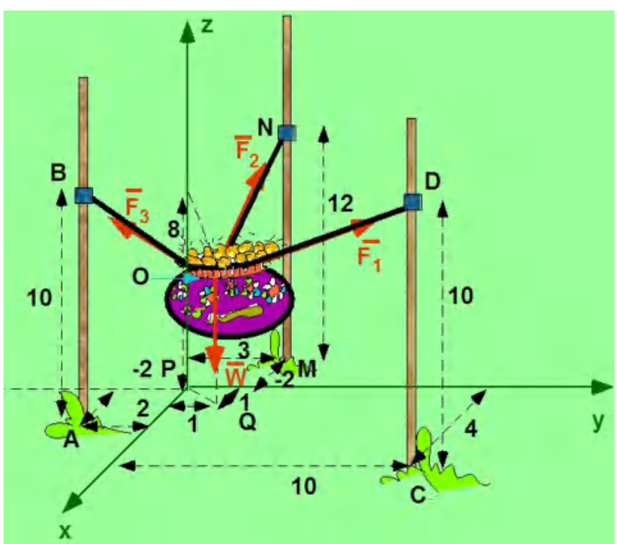


Figure 2.3.1b Pot of gold suspended by three ropes

The point P which is the origin has the value P (0, 0, 0). The coordinates of the other points , with respect to point P, are also provided within a parenthesis in order: (x value, y value, z value). They are

A (2, -2,0); B(2, -2, 10); C (4, 10, 0); D (4, 10, 10); M (-2, 3, 0); N (-2, 3, 12);
Q (1, 1, 0); O (1, 1, 8).

Let us calculate the vector $\mathbf{F_1}$ with some values. Let F_1 have a magnitude of 657.02 [N] or 146.59 [lb]. F_1 is directed along the line OD, where point O is at O (1, 1, 8) [m] and point D is at D (4, 10, 10) [m]. Figure 2.3.8 shows the line joining the points O and D. It also shows the various vectors associated with the vector $\mathbf{F_1}$.

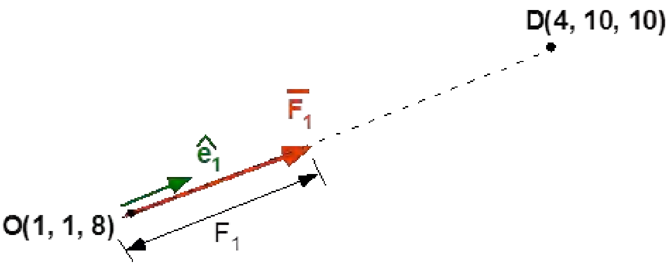


Figure 2.3.8 Vector F_1

If O_x, O_y, and O_z represents the x, y, z coordinates of point O, and D_x , D_y, D_z represents the x, y, z coordinates of point D

$$\hat{e}_1 = \hat{e}_{OD} = \frac{\hat{i}\left(D_x - O_x\right) + \hat{j}\left(D_y - O_y\right) + \hat{k}\left(D_z - O_z\right)}{\sqrt{\left(D_x - O_x\right)^2 + \left(D_y - O_y\right)^2 + \left(D_z - O_z\right)^2}}$$

$$\hat{e}_1 = \hat{e}_{OD} = \frac{\hat{i}(4-1) + \hat{j}(10-1) + \hat{k}(10-8)}{\sqrt{(4-1)^2 + (10-1)^2 + (10-8)^2}} = \frac{\hat{i}(3) + \hat{j}(9) + \hat{k}(2)}{\sqrt{3^2 + 9^2 + 2^2}} \qquad (2.11)$$

$$\hat{e}_1 = \hat{e}_{OD} = 0.31\hat{i} + 0.93\hat{j} + 0.21\hat{k}$$

$$\bar{F}_1 = 657.02\left(0.31\hat{i} + 0.93\hat{j} + 0.21\hat{k}\right)$$

$$= 203.3\hat{i} + 609.9\hat{j} + 135.53\hat{k}\,[N]$$

$$= F_{1x}\hat{i} + F_{1y}\hat{j} + F_{1z}\hat{k}$$

$\mathbf{F_1}$ is shown **scaled** in the figure.

Let us calculate ϕ and θ

$$\phi = \sin^{-1}\frac{F_{1z}}{F_1} = \sin^{-1}\frac{135.53}{657.02} = \sin^{-1}0.2062 = 11.90[\text{deg}]$$

$$\theta = \cos^{-1}\frac{F_{1x}}{F_1\cos\phi} = \cos^{-1}\frac{203.3}{657.02\times\cos(11.9)} = \cos^{-1}0.3162 = 71.57[\text{deg}] \qquad (2.12)$$

Let us calculate α, β, and γ

$$\alpha = \cos^{-1}\frac{F_{1x}}{F_1} = \cos^{-1}(e_{1x}) = \cos^{-1}0.31 = 71.94[\text{deg}]$$

$$\beta = \cos^{-1}\frac{F_{1y}}{F_1} = \cos^{-1}(e_{1y}) = \cos^{-1}0.93 = 21.57[\text{deg}] \qquad (2.13)$$

$$\gamma = \cos^{-1}\frac{F_{1z}}{F_1} = \cos^{-1}(e_{1z}) = \cos^{-1}0.21 = 78.09[\text{deg}]$$

Example 2.1. Compute the vectors F_2 and F_3

Use the Figure 2.3.1b and the coordinates of the various points to identify vectors $\mathbf{F_2}$ and $\mathbf{F_3}$ given that the magnitudes are 938.4 {N] and 1195.35 [N] respectively.

$$\hat{e}_2 = \hat{e}_{ON} = \frac{\hat{i}(-2-1)+\hat{j}(3-1)+\hat{k}(12-8)}{\sqrt{-3^2+2^2+4^2}} = -0.5571\hat{i}+0.3714\hat{j}+0.7428\hat{k}$$

$$\bar{F}_2 = 938.4\times(-0.5571\hat{i}+0.3714\hat{j}+0.7428\hat{k}) = -522.77\hat{i}+348.51\hat{j}+697.03\hat{k}\ [N]$$

$$\hat{e}_3 = \hat{e}_{OB} = \frac{\hat{i}(2-1)+\hat{j}(-2-1)+\hat{k}(10-8)}{\sqrt{1^2+(-3)^2+2^2}} = 0.2673\hat{i}-0.8018\hat{j}+0.5345\hat{k}$$

$$\bar{F}_3 = 1195.35\times(0.2673\hat{i}-0.8018\hat{j}+0.5345\hat{k}) = 319.47\hat{i}+-958.41\hat{j}+638.94\hat{k}\ [N]$$

(2.14)

Handling 3D Vectors in MATLAB

The following code computes the vectors $\mathbf{F_1}$ and all the angles - (Eqn. 2.12 - 2.14) and vectors $\mathbf{F_2}$, and $\mathbf{F_3}$.

Also no formatted printing is involved. All calculations are displayed in the command window (no semicolon at the end of code)

Note that the code is mostly calculation

MATLAB Code

```
% Essential Mechanics
% P. Venkataraman
% Section 2.2.3 and Example 2.1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data
% the points
P =[0,0,0];  A = [2,-2,0]; B =[2,-2,10];  C = [4,10,0]; D = [4,10,10];
M = [-2,3,0];  N = [-2,3,12]; Q = [1,1,0]; O = [1,1,8];

% the magnitude
F1 = 657.02;  F2 = 938.4;  F3 = 1195.35;

%% Section 2.3.1 - Vector F1
eOD = (D-O)/norm(D-O)
```

```
F1v = F1*eOD
phi1 = asind(F1v(3)/F1)
theta1 = acosd(F1v(1)/(F1*cosd(phi1)))
alpha1 = acosd(F1v(1)/F1)
beta1 = acosd(F1v(2)/F1)
gamma1 = acosd(F1v(3)/F1)

%% Example 2.3.1
% the unit vectors F2 F3
eON = (N-O)/norm(N-O)
eOB = (B-O)/norm(B-O)

% the vector components
F2v = F2*eON
F3v =F3*eOB
```

In the Command Window

```
eOD =
    0.3094    0.9283    0.2063
F1v =
  203.2993  609.8979  135.5329
phi1 =
   11.9047
theta1 =
   71.5651
alpha1 =
   71.9753
beta1 =
   21.8319
gamma1 =
   78.0953
eON =
   -0.5571    0.3714    0.7428
eOB =
    0.2673   -0.8018    0.5345
F2v =
 -522.7695  348.5130  697.0260
F3v =
  319.4707 -958.4122  638.9415
```

Observation
The code is mostly calculations and is fairly short. You can introduce formatted printing if you wish. You can borrow from previous code and edit the statements.

2.3.2 Multiplication of Two Vectors

Vector multiplication occurs frequently in Statics and Dynamics and in all areas of mechanical engineering. Given two vectors there are 3 ways to multiply them. The first way multiplies them to yield a ***scalar*** result. The second way multiplies them so the result is a ***vector***. The third way multiplies two vectors and the result is a ***second order tensor***. You come across tensors in advanced courses and research, so we will avoid it in this book. In the undergraduate engineering curriculum you will probably not require it *formally*. We use it without calling it a tensor. In the following will use the two vectors, $\mathbf{V_1}$ and $\mathbf{V_2}$, illustrated in Figure 2.3.9. We have included the coordinate system. We have moved the vectors parallel to themselves so they are placed *tail-to-tail* at R to display the angle θ between them.. You must have learned that we can do that with vectors - that is move them parallel to themselves without changing the problem (unless they are force vectors). It is still okay to do that with force vectors to illustrate multiplication, but force vectors can be moved along their line of action only.

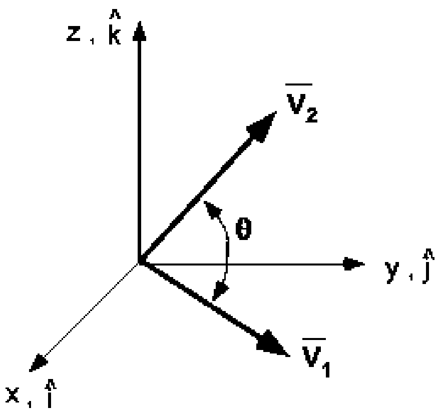


Figure 2.3.9 Two vectors

Scalar Product or Dot Product

Two vectors $\mathbf{V_1}$ and $\mathbf{V_2}$ are multiplied in such a way that the result is a scalar, *S*. Note: we cannot display *S* graphically. This is called the *scalar* product or the *dot* product. This multiplication is implied by placing a ***dot*** between the vectors. The multiplication depends on the included angle θ, between the vectors when their tails meet.

$$\bar{V}_1 \bullet \bar{V}_2 = S$$
$$\bar{V}_1 \bullet \bar{V}_2 = |\bar{V}_1||\bar{V}_2|\cos\theta = V_1 V_2 \cos\theta \qquad (2.15)$$

Consider the dot product among the unit vectors in the right handed Cartesian system. The unit vectors are perpendicular to each other

$$\hat{i} \bullet \hat{j} = 1 \times 1 \times \cos 90 = 0 = \hat{i} \bullet \hat{k} = \hat{j} \bullet \hat{k}$$
$$\hat{i} \bullet \hat{i} = 1 \times 1 \times \cos 0 = 1 = \hat{j} \bullet \hat{j} = \hat{k} \bullet \hat{k} \qquad (2.16)$$
$$\hat{i} \bullet (-\hat{i}) = 1 \times 1 \times \cos 180 = -1$$

If the vectors are expressed as components, and the dot product is only defined between two vectors, then the details of the dot product can be worked out and cleaned up to a simple formula as below. The angle between the vectors can also be identified through the components.

$$\bar{V}_1 = V_{1x}\hat{i} + V_{1y}\hat{j} + V_{1z}\hat{k}; \quad \bar{V}_2 = V_{2x}\hat{i} + V_{2y}\hat{j} + V_{2z}\hat{k}$$

$$\bar{V}_1 \bullet \bar{V}_2 = \left(V_{1x}\hat{i} + V_{1y}\hat{j} + V_{1z}\hat{k}\right) \bullet \left(V_{2x}\hat{i} + V_{2y}\hat{j} + V_{2z}\hat{k}\right)$$

$$\bar{V}_1 \bullet \bar{V}_2 = \left(V_{1x}\hat{i}\right)\bullet\left(V_{2x}\hat{i}\right) + \left(V_{1x}\hat{i}\right)\bullet\left(V_{2y}\hat{j}\right) + \left(V_{1x}\hat{i}\right)\bullet\left(V_{2z}\hat{k}\right)$$
$$+\left(V_{1y}V_{2x}\right)\left(\hat{j}\bullet\hat{i}\right) + \left(V_{1y}V_{2y}\right)\left(\hat{j}\bullet\hat{j}\right) + \left(V_{1y}V_{2z}\right)\left(\hat{j}\bullet\hat{k}\right)$$
$$+\left(V_{1z}V_{2x}\right)\left(\hat{k}\bullet\hat{i}\right) + \left(V_{1z}V_{2y}\right)\left(\hat{k}\bullet\hat{j}\right) + \left(V_{1z}V_{2z}\right)\left(\hat{k}\bullet\hat{k}\right)$$

$$\bar{V}_1 \bullet \bar{V}_2 = V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z}$$

$$\cos\theta = \frac{\bar{V}_1 \bullet \bar{V}_2}{V_1 V_2} = \frac{V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z}}{\sqrt{V_{1x}^2 + V_{1y}^2 + V_{1z}^2}\sqrt{V_{2x}^2 + V_{2y}^2 + V_{2z}^2}} \qquad (2.17)$$

Let us work through an example. We will define **V$_1$** = -2**i** + 2**j** + 3**k** [m/s] and **V$_2$** = 1**i** + 4**j** + -5**k** [m/s]

$$\bar{V}_1 \bullet \bar{V}_2 = (-2)(1) + (2)(4) + (3)(-5) = -9\left[m^2/s^2\right]$$

$$\cos\theta = \frac{\bar{V}_1 \bullet \bar{V}_2}{\left|\bar{V}_1\right|\left|\bar{V}_2\right|} = \frac{-9}{\sqrt{-2^2 + 2^2 + 3^2} \times \sqrt{1^2 + 4^2 + -5^2}} = \frac{-9}{4.12 \times 6.48} = -0.337$$

$$\theta = 109.68$$

Can you find the angle between the **F$_1$** and **F$_2$** vectors of the earlier example (Figure 2.3.1b) ?

Vector Product or Cross Product

Two vectors **V$_1$** and **V$_2$** are multiplied in such a way that the result is a third vector **V$_3$**. Vector **V$_3$** is

- normal to both vectors **V$_1$** and **V$_2$** (also normal to the plane formed by the two vectors)
- directed using the right hand rule as the remaining fingers rotate from the first vector **V$_1$** to the second vector **V$_2$**

Unlike the Scalar product, the *order of the vectors* being multiplied is important in the Vector product

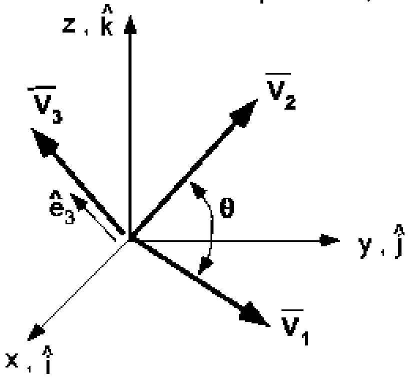


Figure 2.3.10 Vector cross product

$$\bar{V}_1 \times \bar{V}_2 = \bar{V}_3$$
$$\bar{V}_1 \times \bar{V}_2 = \left|\bar{V}_1\right|\left|\bar{V}_2\right| sin\,\theta\,\hat{e}_3 \qquad (2.18)$$

The cross/vector product among the unit vectors

$$\hat{i} \times \hat{j} = 1 \times 1 \times \sin 90 \hat{k} = \hat{k}; \quad \hat{j} \times \hat{i} = -\hat{k}$$
$$\hat{j} \times \hat{k} = (1)(1)\sin 90 \hat{i} = \hat{i}; \quad \hat{k} \times \hat{j} = -\hat{i} \qquad (2.19)$$
$$\hat{k} \times \hat{i} = (1)(1)\sin 90 \hat{j} = \hat{j}; \quad \hat{i} \times \hat{k} = -\hat{j}$$
$$\hat{i} \times \hat{i} = (1)(1)\sin 0 = 0 = \hat{j} \times \hat{j} = \hat{k} \times \hat{k}$$

Therefore, if the two vectors are expressed as components in the Cartesian coordinate system, then

$$\bar{V}_1 \times = V_{1x}\hat{i} + V_{1y}\hat{j} + V_{1z}\hat{k}; \quad \bar{V}_2 = V_{2x}\hat{i} + V_{2y}\hat{j} + V_{2z}\hat{k};$$
$$\bar{V}_1 \times \bar{V}_2 = \left(V_{1x}\hat{i}\right)\times\left(V_{2x}\hat{i}\right) + \left(V_{1x}\hat{i}\right)\times\left(V_{2y}\hat{j}\right) + \left(V_{1x}\hat{i}\right)\times\left(V_{2z}\hat{k}\right)$$
$$+\left(V_{1y}V_{2x}\right)\left(\hat{j}\times\hat{i}\right) + \left(V_{1y}V_{2y}\right)\left(\hat{j}\times\hat{j}\right) + \left(V_{1y}V_{2z}\right)\left(\hat{j}\times\hat{k}\right)$$
$$+\left(V_{1z}V_{2x}\right)\left(\hat{k}\times\hat{i}\right) + \left(V_{1z}V_{2y}\right)\left(\hat{k}\times\hat{j}\right) + \left(V_{1z}V_{2z}\right)\left(\hat{k}\times\hat{k}\right)$$
$$\bar{V}_1 \times \bar{V}_2 = \cancel{\left(V_{1x}\hat{i}\right)\times\left(V_{2x}\hat{i}\right)} + \left(V_{1x}V_{2y}\right)\left(\hat{k}\right) + \left(V_{1x}V_{2z}\right)\left(-\hat{j}\right)$$
$$+\left(V_{1y}V_{2x}\right)\left(-\hat{k}\right) + \cancel{\left(V_{1y}V_{2y}\right)\left(\hat{j}\times\hat{j}\right)} + \left(V_{1y}V_{2z}\right)\left(\hat{i}\right)$$
$$+\left(V_{1z}V_{2x}\right)\left(\hat{j}\right) + \left(V_{1z}V_{2y}\right)\left(-\hat{i}\right) + \cancel{\left(V_{1z}V_{2z}\right)\left(\hat{k}\times\hat{k}\right)}$$
$$\bar{V}_1 \times \bar{V}_2 = \hat{i}\left[V_{1y}V_{2z} - V_{1z}V_{2y}\right] + \hat{j}\left[V_{1z}V_{2x} - V_{1x}V_{2z}\right] + \hat{k}\left[V_{1x}V_{2y} - V_{1y}V_{2x}\right]$$

A simpler and useful way to evaluate the cross product in the Cartesian coordinate system is to set up a determinant as shown below:

$$\bar{V}_1 \times \bar{V}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ V_{1x} & V_{1y} & V_{1z} \\ V_{2x} & V_{2y} & V_{2z} \end{vmatrix} = \hat{i}\begin{vmatrix} V_{1y} & V_{1z} \\ V_{2y} & V_{2z} \end{vmatrix} - \hat{j}\begin{vmatrix} V_{1x} & V_{1z} \\ V_{2x} & V_{2z} \end{vmatrix} + \hat{k}\begin{vmatrix} V_{1x} & V_{1y} \\ V_{2x} & V_{2y} \end{vmatrix} \qquad (2.20)$$
$$\bar{V}_1 \times \bar{V}_2 = \hat{i}\left[V_{1y}V_{2z} - V_{1z}V_{2y}\right] - \hat{j}\left[V_{1x}V_{2z} - V_{1z}V_{2x}\right] + \hat{k}\left[V_{1x}V_{2y} - V_{1y}V_{2x}\right]$$

The two formula are identical while the terms in the j parenthesis are switched to give the negative sign in the second.

Working through an example with $\mathbf{V_1}$ **= -2i + 2j + 3k** [m/s] and $\mathbf{V_2}$ **= 1i + 4j + -5k** [m/s]

$$\bar{V}_3 = \bar{V}_1 \times \bar{V}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 3 \\ 1 & 4 & -5 \end{vmatrix}$$

$$= \hat{i}\left[(2)(-5)-(3)(4)\right] - \hat{j}\left[(-2)(-5)-(3)(1)\right] + \hat{k}\left[(-2)(4)-(2)(1)\right]$$

$$= \hat{i}\left[-22\right] - \hat{j}\left[7\right] + \hat{k}\left[-10\right]$$

$$V_3 = \left|\bar{V}_3\right| = \sqrt{22^2 + 7^2 + 10^2} = 25.16\left[m^2/s^2\right]$$

$$V_3 = V_1 V_2 \sin 109.68 = \sqrt{2^2+2^2+3^2}\sqrt{1+4^2+5^2}\,\sin(109.68) = 25.16\left[m^2/s^2\right]$$

The order of multiplication of the vectors is important. Hence

$$\bar{V}_2 \times \bar{V}_1 = -\bar{V}_3 \tag{2.21}$$

Scalar/Dot Vector and Vector/Cross Product Multiplication in MATLAB

In the MATLAB Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 2.3.2 - Dot and cross product
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------\n')
fprintf('Scalar and Vector Product\n')
fprintf('-------------------------\n')
%% Data
V1 = [-2,2,3]
V2 = [1,4,-5]

%% Scalar or dot product
S = dot(V1,V2)
theta = acosd(S/(norm(V1)*norm(V2)))

%% Vector or cross product
V3 = cross(V1,V2)
V3Mag = norm(V3)
```

In the Command Window

```
-------------------------
Scalar and Vector Product
-------------------------
V1 =
    -2     2     3
V2 =
     1     4    -5
S =
```

```
    -9
theta =
  109.6830
V3 =
   -22    -7   -10
V3Mag =
   25.1595
```

2.3.3 Multiplication of Three Vectors

These special multiplication appear in dynamics but otherwise rare. There are two of them. The first is a scalar triple product and the second is the vector triple product.

Scalar Triple Product

The scalar triple product is defined between three vectors. The final value is a scalar. For vectors $\mathbf{U_1}$, $\mathbf{U_2}$ and $\mathbf{U_3}$,this product is defined as:

$$S = \bar{U}_1 \bullet \left(\bar{U}_2 \times \bar{U}_3\right) \qquad (2.22)$$

This is an ordered multiplication sequence. The cross product is performed first to yield a vector and then the dot product between vectors yields a scalar.
You can easily show that the scalar triple product can be evaluation as a determinant using the vector components in order.

$$\bar{U}_1 = \left[U_{1x}, U_{1y}, U_{1z}\right]; \quad \bar{U}_2 = \left[U_{2x}, U_{2y}, U_{2z}\right]; \quad \bar{U}_3 = \left[U_{3x}, U_{3y}, U_{3z}\right];$$

$$S = \bar{U}_1 \left(\bar{U}_2 \times \bar{U}_3\right)$$

$$= \begin{vmatrix} U_{1x} & U_{1y} & U_{1z} \\ U_{2x} & U_{2y} & U_{2z} \\ U_{3x} & U_{3y} & U_{3z} \end{vmatrix}$$

Vector Triple Product

The vector triple product is defined between three vectors. It results in a vector. It is very useful in the study of dynamics. For vectors U1, U2 and U3, the product will be a fourth vector $\mathbf{U_4}$. This involves two serial applications of the vector product. The order of multiplication is suggested in the definition of the product.

$$\bar{U}_4 = \bar{U}_1 \times \left(\bar{U}_2 \times \bar{U}_3\right) \qquad (2.23)$$

Scalar and Vector Triple Product in MATLAB

In the MATLAB Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 2.3.3 - Triple Product
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------\n')
fprintf('Scalar and Vector Triple Product\n')
fprintf('---------------------------------\n')
%% Data
U1 = [-1,2,3]; U2 = [1,-2,3]; U3 = [1,2,-3];

%% Scalar Triple Product
S1 = dot(U1,cross(U2,U3))  % definition
A = [U1;U2;U3]
S2 = det(A)          % using determinant

%% Vector Triple Product
U4 = cross(U1,cross(U2,U3))
```

In the Command Window

```
-------------------------------
Scalar and Vector Triple Product
---------------------------------
S1 =
    24
A =
    -1     2     3
     1    -2     3
     1     2    -3
S2 =
    24
U4 =
   -10     4    -6
```

Octave Execution

The same code was run in octave editor without change. The results are shown below

In Octave Command Window

```
---------------------------------
Scalar and Vector Triple Product
---------------------------------
S1 =  24
A =
  -1   2   3
   1  -2   3
   1   2  -3

S2 =  24
U4 =
  -10    4   -6
```

Recommendation

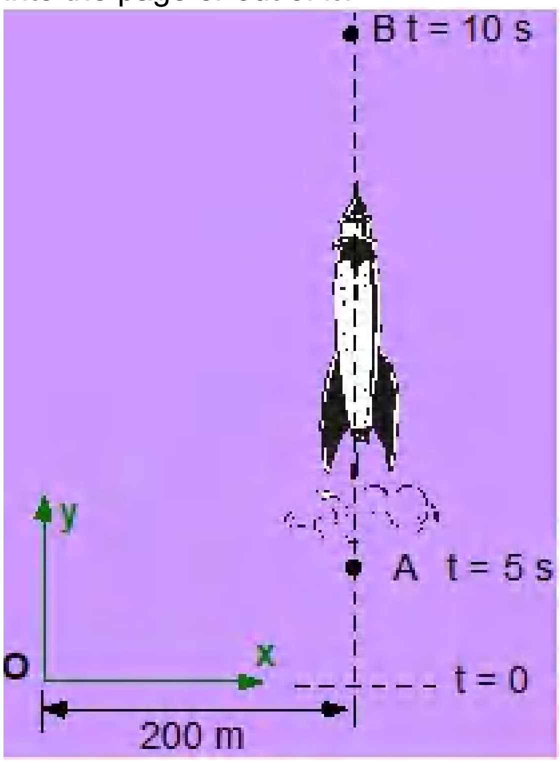
This code tested both the dot and the cross product in Octave. Students are recommended to test the MATLAB code in Sections 2.3.1 and 2.3.2 in Octave to see if the numbers are the same

2.3.4 Additional Problems

In the following problems, first solve the problem by hand and then re-solve by MATLAB/Octave and compare your solutions

Problem 2.3.1

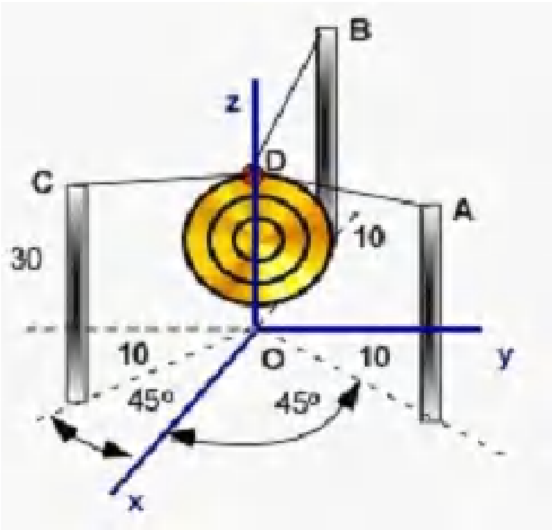
The rocket is moving straight up in the vertical plane. It's location is a function of time and can be expressed as $y = 50\,t^2$ [m], where t is the time in seconds. At t = 0 it starts with y = 0. (a) Find the unit vector along the line OA. (b) Find the unit vector along the line OB. (c) Is the z – axis directed into the page or out of it?



Problem 2.3.1

Problem 2.3.2

The large heavy metal sculpture of mass 500 kg is hung by three cables off tall posts that are 30 m high which intersect at D. The poles are located 10 m from the center O at various angles to the axis shown. The sag in the cables at the center O is 2 m (D is below the point A by 2 m). (a) Find unit vectors along the three cables drawn from point D. (b) What is the dot product of the vector **DB** and **DC**? (c) What is the cross product of the vectors **DA** and **DB**? (d) What is the result of the scalar triple product of the vectors DA,DB, DC expressed as **DA**.(**DB x DC**)? (e) What is the result of the vector triple product **DA x** (**DB x DC**)?



Problem 2.3.2

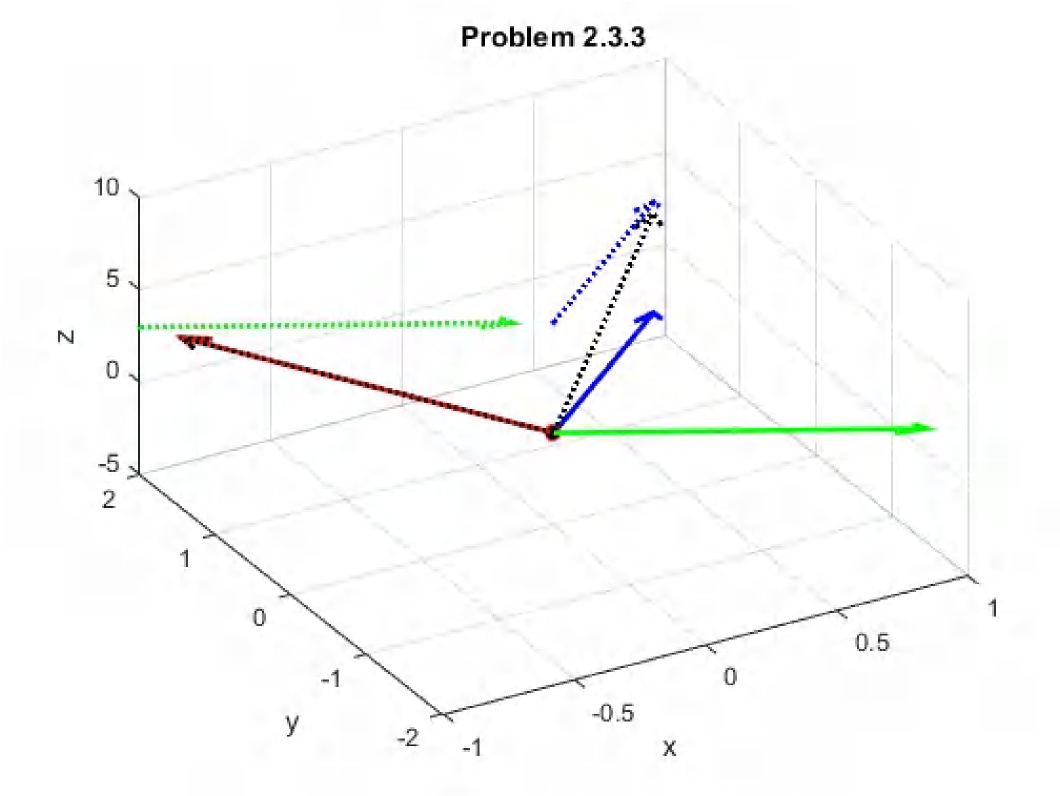
Problem 2.3.3

We can make MATLAB draw 3D vectors, show the result of their addition. It is similar to the code in the previous section except that you have to use quiver3. From MATLAB documentation:

A three-dimensional quiver plot displays vectors with components (u,v,w) at the points (x,y,z), where u, v, w, x, y, and z all have real (non-complex) values.
quiver3(x,y,z,u,v,w) plots vectors with directions determined by components **(u,v,w)** at points determined by **(x,y,z)**.

Given three vectors (through the components) : **U1** = [-1,2,3]; **U2** = [1,-2,3]; **U3** = [1,2,-3];
(a) First draw the three vectors with their tails at the origin.
(b) Display the vector addition of the three vectors with the tail of the first vector (U1) at the origin.

See if you can reproduce the solution. The dotted vectors are part of the vector addition. The black vector is the final vector after the addition of the three vectors.



Problem 2.3.3

In MATLAB you can get help on any command in the extensive MATLAB documentation. For less extensive information you can type the command name in the command window after the prompt:

```
>> help quiver3
```

2.4 MATRICES AND ALGEBRAIC EQUATION

Some of you may not have heard of matrices yet. Some of you may have. They are a special numerical quantity like vectors. They are very important in solving real engineering problems using software. There are usually several courses offered on this subject all the way to graduate school. In this book we are interested in them because they are useful in solving problems in statics. We really do not require its definition, except for knowing how to set it up so that you can use the extensive features of your calculator to easily solve problems involving matrices. Solving by hand takes a lot of time. Matrices are natural to MATLAB and they are easily handled. To begin with the vectors we used in MATLAB in the previous sections are are special matrices. They are a matrix with a single row or a single column. In a matrix information is organized in a number of rows and a number of columns. Every row will have the same number of column elements. A *square* matrix will have equal number of rows and columns. We have used row vectors in MATLAB during previous calculations. Remember that this book just rehashes all the material you have studied or will soon learn in engineering.

In general problem solving in engineering courses mostly involve:

- a *simultaneous* solution for many unknown quantities (or unknowns)
- these unknowns are related through a set of equilibrium equations (or the physical laws)
- these equations will be linear (only these are solvable)
- these equations can also be organized as a *matrix* equation
- we will need as many equations as there are unknowns (a square matrix)

We certainly do not need to know about *matrices* to solve the set of these equations. They can be solved naturally through *substitution* and *elimination.* This takes longer and tedious. On the other hand

Using matrices,

- you can take advantage of your graphics calculator and solve these problems quickly and with little effort.
- you can take advantage of Wolfram (MATHEMATICA) server to solve your matrix problems on line.
- you can take advantage of any engineering software available on your computer (MATLAB, Octave, MAPLE, MATHEMATICA,PYTHON, etc.).
- this is what you will be doing in advanced courses and therefore we can get started early .

2.4.1 Definition of Matrix

Here is a brief introduction to a matrix. We have previously used the vector $\mathbf{V_1}$

$$\bar{V}_1 = \begin{bmatrix} V_{1x} & V_{1y} & V_{1z} \end{bmatrix}$$

This is a row vector, or a matrix with 1 row and 3 columns. The **transpose** of this vector/matrix will be a column vector with the row elements. Its symbol (superscript T) is shown as:

$$\bar{V}_1^T = \begin{bmatrix} V_{1x} \\ V_{1y} \\ V_{1z} \end{bmatrix}$$

The matrix is a set of numbers organized as rows and columns and surrounded by square parenthesis. We will mostly deal with square matrices (same number of rows and column). The size of the matrix is defined by the number of rows and columns. The example of a 4x3 (first number indicates number of rows and the second indicates the number of columns) matrix, in various forms is shown below: The first is a matrix with numbers. The second is a matrix with a symbol ***a*** and two subscripts. The first subscript is the row location. The second subscript indicates which column it belongs to. The third representation is just defines a matrix.

$$[A] = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 5 & 6 \\ 9 & 8 & 7 \\ 1 & 4 & 7 \end{bmatrix}; \quad [A]_{4x3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}; \quad [A]_{4x3} \qquad (2.24)$$

A new matrix [B] can be formed by taking the ***transpose*** of matrix [A] – interchanging the rows and columns (it will have 3 rows and 4 columns). Note the standard representation of a transpose - superscript of ***T*** in capitals.

$$[B]_{3x4} = [A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \end{bmatrix} \qquad (2.25)$$

2.4.2 Multiplying Matrices

You can multiply two matrices (and obtain a third matrix) if and only if (***iff***) the number of columns of the first matrix equals the number of rows of the second matrix. Let us create the matrix [D] by multiplying matrices $[A]_{4x3}$ and $[C]_{3x2}$

$$[C]_{3\times 2} = \begin{bmatrix} 1 & -3 \\ 4 & 2 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$[D]_{(4\times 3)(3\times 2)} = [D]_{4\times 2} = [A]_{4\times 3}[C]_{3\times 2} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \\ d_{41} & d_{42} \end{bmatrix} \qquad (2.26)$$

$$d_{32} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = a_{31}c_{12} + a_{32}c_{22} + a_{33}c_{32}$$

$$[D] = [A][C] = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 5 & 6 \\ 9 & 8 & 7 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 4 & 2 \\ -5 & 6 \end{bmatrix};$$

$$[D] = \begin{bmatrix} (3)(1)+(1)(4)+(2)(-5) & (3)(-3)+(1)(2)+(2)(6) \\ (4)(1)+(5)(4)+(6)(-5) & (4)(-3)+(5)(2)+(6)(6) \\ (9)(1)+(8)(4)+(7)(-5) & (9)(-3)+(8)(2)+(7)(6) \\ (1)(1)+(4)(4)+(7)(-5) & (1)(-3)+(4)(2)+(7)(6) \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ -6 & 34 \\ 6 & 31 \\ -18 & 47 \end{bmatrix} \qquad (2.27)$$

We can use matrix multiplication on vectors also. If $\mathbf{V_1}$ = [-2 2 3] and $\mathbf{V_2}$ = [1 4 -5]

$$[\bar{V}_1][\bar{V}_2]^T = [-2 \quad 2 \quad 3] \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix} = [(-2)(1)+(2)(4)+(3)(-5)] = [-9] = \bar{V}_1 \bullet \bar{V}_2$$

$$(2.28)$$

$$[\bar{V}_1]^T[\bar{V}_2] = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} [1 \quad 4 \quad -5] = \begin{bmatrix} -2 & -8 & 10 \\ 2 & 8 & -10 \\ 3 & 12 & -15 \end{bmatrix}$$

Matrix Multiplication in MATLAB
In MATLAB Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 2.4.2 - Matrix Multiplication
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------\n')
fprintf('Matrix Multiplication\n')
fprintf('-------------------------\n')
%% Data  - matrices are written within square parenthesis
A = [3,1,2;4,5,6;9,8,7;1,4,7]
% a comma (,) separates elements in the same row
% a semicolon (;) separates rows
C = [1,-3;4,2;-5,6]

V1 = [-2,2,3]  % a row vector
V2 = [1,4,-5]  % another row vector
V2T = V2'% transpose of V2
% the transpose is represented by an apostrophe (single quote)

%% Matrix multiplication
D = A*C

%% Vector Multiplication
V3 = V1*V2'

V4 = V1'*V2
```

In the Command Window

```
-------------------------
Matrix Multiplication
-------------------------
A =
     3     1     2
     4     5     6
     9     8     7
     1     4     7
C =
     1    -3
     4     2
    -5     6
V1 =
    -2     2     3
V2 =
     1     4    -5
V2T =
     1
     4
    -5
D =
    -3     5
    -6    34
     6    31
   -18    47
V3 =
```

```
    -9
V4 =
    -2    -8    10
     2     8   -10
     3    12   -15
```

2.4.3 System of Algebraic Equations

System of linear algebraic equations is quite common in Statics. We can set up the system of equations as a matrix equation and there are efficient procedures to solve these equations. You can also grind the solution by substitution and elimination but it is very much less work if you have a graphical calculator or MATLAB as most of you have.

In the following we create a ***system of three equations*** in ***three unknowns*** x, y, and z. We are interested in the value of x, y and z that ***simultaneously solves*** the three equations. We expect them to have unique values. Since we are multiplying each unknown by a constant these are called ***linear equations.*** We will solve them the traditional way in Section 2.4.4. Here we will recognize that the equations can be assembled as a ***linear matrix equation*** using vector/matrix multiplication in Eq. (2.29)

$$2x+2y+z=9$$
$$3x+2y-2z=1$$
$$2x-2y+z=1$$

The first equation can be set up as

$$\begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}$$

The second equation can be set up as

$$\begin{bmatrix} 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

The last equation can be set up as

$$\begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

We can combine the three matrix forms of each equation into a single one by consolidating a matrix equation as

$$[A][x]=[b]; \qquad (2.29)$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & 2 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix} \qquad (2.30)$$

Solving Matrix Equation in MATLAB

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 2.4.2 - System of Algebraic Equations
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
fprintf('--------------------------------\n')
fprintf('System of Algebraic Equations\n')
fprintf('---------------------------------\n')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This Example uses Symbolic Calculations  in the first part
% Symbolic calculation is similar to how you would solve in your
% lecture notes.
% Symbolic variables are a new type of variable
% They are not numbers but symbols
% You have to identify them before using them
% MATLAB treats them differently
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data
syms x y z   % x y and z are three symbolic variables

eq1 = '2*x + 2*y + z = 9'  % define first equation
eq2 = '3*x + 2*y -2*z = 1'  % define second equation
eq3 = '2*x -2*y + z = 1'    % define the third equation

%% solution - first method -(symbolic solution)
sol = solve(eq1,eq2,eq3)
% solve the three equations and store the solution in the variable sol
% sol is also a new kind of variable
% it is a struct variable with values for the solution

sol.x   % the solution for x is recovered by typing sol.x
sol.y   % the solution for y is recovered by typing sol.y
sol.z   % the solution for x is recovered by typing sol.x

%% solution - second method -(using matrix)
A = [2,2,1;3,2,-2;2,-2,1]   % define matrix A
b = [9,1,1]'   % define column vector b
xs = A\b   % solve the matrix equation
```

In the Command Window

```
--------------------------------
System of Algebraic Equations
---------------------------------
eq1 =
2*x + 2*y + z = 9
eq2 =
3*x + 2*y -2*z = 1
eq3 =
2*x -2*y + z = 1
sol =
    x: [1x1 sym]
    y: [1x1 sym]
    z: [1x1 sym]
ans =
1
ans =
2
ans =
3
```

```
A =
     2     2     1
     3     2    -2
     2    -2     1
b =
     9
     1
     1
xs =
    1.0000
    2.0000
    3.0000
```

Execution in Octave

The code is same as in MATLAB above except for the changes shown below

In Octave Editor:

```
% Section 2.4.2 - System of Algebraic Equations
clc, clear, format compact, close all
pkg load symbolic  #  Loads the package for symbolic calculations

eq1 = 2*x + 2*y + z == 9  % define first equation
eq2 = 3*x + 2*y -2*z == 1  % define second equation
eq3 = 2*x -2*y + z == 1   % define the third equation
```

In Octave Command Window

```
-------------------------------
System of Algebraic Equations
--------------------------------
eq1 = (sym) 2*x + 2*y + z = 9
eq2 = (sym) 3*x + 2*y - 2*z = 1
eq3 = (sym) 2*x - 2*y + z = 1
ans = (sym) 1
ans = (sym) 2
ans = (sym) 3
A =
   2   2   1
   3   2  -2
   2  -2   1

b =
   9
   1
   1

xs =
   1
   2
   3
```

There are some formatting differences but the solution is the same

2.4.4 Traditional Solution to System of Algebraic Equations

We know how to solve systems of equations in MATLAB/Octave. The traditional way is to substitute and eliminate unknowns. This is done as follows:

$$2x+2y+z=9 \qquad \text{first}$$

$$3x+2y-2z=1 \qquad \text{second}$$

$$2x-2y+z=1 \qquad \text{third}$$

The equations are identified for reference in the following sequence of calculations. Use the first equation to define z

$$z=9-2x-2y$$

Substitute z in the second equation and define y using the second equation

$$3x+2y-2(9-2x-2y)=1$$

$$7x+6y=1+18=19$$

$$y=\frac{(19-7x)}{6}$$

Now, substitute for y and z in the third equation and solve for x

$$2x-2\frac{(19-7x)}{6}+\left(9-2x-2\left[\frac{(19-7x)}{6}\right]\right)=1$$

$$2x-\frac{19}{3}+\frac{7x}{3}+9-2x-\frac{19}{3}+\frac{7x}{3}=1$$

$$2\,\frac{7x}{3}=1-9+2\frac{19}{3}=-8+2\frac{19}{3}=\frac{-24+38}{3}$$

$$x=\frac{14}{14}=1$$

Now substitute the value of x to calculate y and z

$$y=\frac{(19-7x)}{6}=\frac{12}{6}=2$$

$$z=9-2x-2y=9-2-4=3$$

These should be the same as the values for x, y, and z from MATLAB. The number of calculations significantly increases if you add more unknowns and more equations. In rigid body statics there are six equations that can accommodate six unknowns. In MATLAB the procedure does not change with more unknowns and equations.

2.4.5 Additional Problems

In the following problems, first solve the problem by hand and then re-solve by MATLAB/Octave and compare your solutions

Problem 2.4.1

Find the product of matrices A and B

$$A = \begin{bmatrix} 4 & 4.5 & 1.4 & 4.8 \\ 4.5 & 3.2 & 2.7 & 0.8 \\ 0.6 & 0.5 & 4.8 & 2.9 \end{bmatrix}; \quad B = \begin{bmatrix} 2.9 & 1.3 \\ 1.5 & 2.7 \\ 2.4 & 2.4 \\ 0.4 & 2.9 \end{bmatrix}$$

Problem 2.4.2

Solve the following set of algebraic equations:

$$2x + y + 3z = 7.6 \qquad (e1)$$
$$x + 2y + 2z = 7.5 \qquad (e2)$$
$$x + 1.5y + 2z = 6.4 \qquad (e3)$$

Problem 2.4.3

Solve the following set of algebraic equations:

$$x + 2y + 3z = 8 \qquad (e1)$$
$$2x + y + 1.5z = 7 \qquad (e2)$$
$$1.5x + 3y + 4.5z = 12 \qquad (e3)$$

If you run into a problem - look over the set of equations and see if you can spot an important fact.

2.5 USEFUL MATHEMATICAL RELATIONS

This section reviews some useful mathematics that is taken for granted in engineering courses. Some of these are part of your high school curriculum or the earliest mathematics course you attended as part of the engineering curriculum. These usually appear as part of formula derivation. These are part of your engineering vocabulary. These will include algebra, calculus, and trigonometry. We will review geometry in a separate section because it is very important in Mechanics. This review is similar to the practical coverage of vectors and matrices in earlier sections. These are short subsections and provide instructions on how to solve them using symbolic calculations in MATLAB. In a sense they are provided as a reference.

2.5.1 Polynomials

Polynomials are usually expressed with the variable x. However they can be expressed using any symbol. They are represented as a function $\boldsymbol{p_n(x)}$. The are formed by the addition of terms made up of a constant multiplying $\boldsymbol{x}$ raised to an integral power. The highest power is $\boldsymbol{n}$ - which is the order of the polynomial. The common polynomial is the quadratic where the highest power is 2. Cubic polynomials are sometimes used to smooth data. In the following we will avoid the subscript $\boldsymbol{n}$ since we will be using mostly second or third order polynomial.

$$
\begin{aligned}
p_2(x) &= p(x) = ax^2 + bx + c \\
p_3(x) &= p(x) = ax^3 + bx^2 + cx + d
\end{aligned}
\tag{2.31}
$$

The polynomial is completely described by the the constants. The number of constants describes the order of the polynomial. Therefore

$$p(x) = ax^2 + bx + c \Rightarrow [a; b; c]; \tag{2.32}$$

Quadratic equation: The quadratic equation is a solution to the algebraic equation defined by setting the quadratic polynomial with known constants to zero. The solution to this equation determines the roots of the equation or the zeros of the polynomial. What exactly are roots? They are just the values of x that solve the equation. These are the same values that make the polynomial evaluate to zero - which can be seen in (2.33). The second order polynomial will have two solutions or two roots. Let us denote them as x_1 and x_2.Then the polynomial can also be expressed through the roots as:

$$ax^2 + bx + c = 0 = (x - x_1)(x - x_2) \tag{2.33}$$

The formula for the roots are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{2.34}$$

The roots can be imaginary too. However we will deal with polynomials with real roots and leave imaginary numbers for another time. You can program the calculator to solve for the roots based on the value of the constants a, b, and c. Off course we will use MATLAB as our calculator. Note you can get the formula as well as a numerical solution.

Quadratic Equation in MATLAB
In the Editor:

```
% Essential Mechanics
% P. Venkataraman
% Section 2.5.1 - Polynomials
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------\n')
fprintf('Polynomials: Quadratic equation\n')
fprintf('--------------------------------\n')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Symbolic Calculations
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% quadratic equation
syms x
a = 2; b = -3; c = 1;

sol = solve('a*x^2 + b*x + c == 0') % this will give formula

% here a, b and c are treated as symbolic variables as they are defined
within a string expression

sol = subs(sol)  % substitute known values of a, b, c in solution
```

In the Command Window:

```
-------------------------------
Polynomials: Quadratic equation
--------------------------------
sol =
 -(b + (b^2 - 4*a*c)^(1/2))/(2*a)
 -(b - (b^2 - 4*a*c)^(1/2))/(2*a)
sol =
 1/2
   1
```

Note: you can see that MATLAB reproduces the formula in Eqn. (2.34).

Execution in Octave
Symbolic substitution is more formal in Octave. The MATLAB code does not work as is. The code for Octave is below. The difference in the code is highlighted. The symbolic display by default is natural. This is more readable. If you want to flatten it uncomment the sympref display flat - command. The change in code is due to the following:

1. Solve does not appear to like a string expression.
2. Substitution of defined unknowns can only be scalar substitution - that is only one unknown at a time.

In the Octave Editor

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Essential Mechanics
% P. Venkataraman
% Section 2.5.1 - Polynomial
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all,warning off
pkg load symbolic
##sympref display flat
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------\n')
fprintf('Polynomials: Quadratic equation\n')
fprintf('--------------------------------\n')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Symbolic Calculations
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% quadratic equation
syms x a b c
sol = solve(a*x^2 + b*x + c,x ) % this will give formula

fprintf('\n\n')
% substitute known values of a, b, c in solution
##a = 2; b = -3; c = 1;
% In octave you have to substitute one by one unknown
sol = subs(sol,'a',a);
sol = subs(sol,'b',b);
sol = subs(sol,'c',c)

% or you can substitute a matrix
fprintf('\n\n')

sol = subs(sol,[a,b,c],[2,-3,1])
```

In the Octave Command Window

```
-------------------------------
Polynomials: Quadratic equation
--------------------------------
Symbolic pkg v2.7.1: Python communication link active, SymPy v1.3.
sol = (sym 2x1 matrix)
  [            _____________]
  [           /           2 ]
  [   b     \/   -4*a*c + b  ]
  [- --- - ----------------]
  [  2*a          2*a       ]
  [                         ]
  [            _____________]
  [           /           2 ]
  [   b     \/   -4*a*c + b  ]
  [- --- + ----------------]
  [  2*a          2*a       ]


sol = (sym 2x1 matrix)
```

```
  [              _______________]
  [             /             2 ]
  [   b       \/   -4*a*c + b   ]
  [- --- - ---------------------]
  [  2*a           2*a          ]
  [                             ]
  [              _______________]
  [             /             2 ]
  [   b       \/   -4*a*c + b   ]
  [- --- + ---------------------]
  [  2*a           2*a          ]


sol = (sym 2x1 matrix)
  [1/2]
  [   ]
  [ 1 ]
```

Cubic Equation: It is possible to solve for the roots of a quadratic equation using analytical formulas similar to those for the quadratic equation Eqn.(2.34). They are more complex and long. Here we commit to use MATLAB to handle these equations. MATLAB has efficient built in functions for this. Before we move on to the example, note that a polynomial can be defined by the coefficients of the terms multiplying x and its powers. An important way to assemble these coefficients in MATLAB is to use an array/vector to collect the coefficients from the highest power to the constant term. A coefficient of zero must be used if the polynomial lacks a term corresponding to x raised to a specific power - Eqn. (2.32).

Cubic Polynomial in MATLAB
In the Editor:

```
% Essential Mechanics
% P. Venkataraman
% Section 2.5.1 - Polynomials
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------\n')
fprintf('Polynomials: Cubic polynomial\n')
fprintf('---------------------------------\n')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Symbolic Calculations
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% cubic polynomial with roots at -2, 1.3, and 3
syms x
f = (x+2)*(x-1.3)*(x-3)   % f will be a third order polynomial
                          % with roots -2, 1.3, 3
f = collect(f,x)   % returns the polynomial in decreasing powers of x

p = coeffs(f,x) % extract the polynomial coefficients
              % coeffs orders the polynomial coefficients from low to high

p = fliplr(p)   %  Here we reverse it to make use
```

```
               % of MATLAB program roots to find the roots of polynomial

sol = roots(p)  % should give you the  solution  for polynomial roots
             % and it should check with what w used to create polynomial
```

In the Command Window:

```
-------------------------------
Polynomials: Cubic polynomial
--------------------------------
f =
(x + 2)*(x - 3)*(x - 13/10)
f =
x^3 - (23*x^2)/10 - (47*x)/10 + 39/5
p =
[ 39/5, -47/10, -23/10, 1]
p =
[ 1, -23/10, -47/10, 39/5]
sol =
    -2
 13/10
     3
```

Execution is Octave

There are two changes in code

1. Octave does not appear to have a collect function
2. input to the roots function must be a real number and not a symbolic number

In Octave Editor

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Symbolic Calculations
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% cubic polynomial with roots at -2, 1.3, and 3
syms x
f = (x+2)*(x-1.3)*(x-3)   % f will be a third order polynomial
                          % with roots -2, 1.3, 3
##f = collect(f,x)   % returns the polynomial in decreasing powers of x

p = coeffs(f,x) % extract the polynomial coefficients
              % coeffs orders the polynomial coefficients from low to high
p = double(p)    % the roots function need real numbers instead of symbolic
%                  MATLAB is flexible in this regard

p = fliplr(p)   %  Here we reverse it to make use
               % of MATLAB program roots to find the roots of polynomial

sol = roots(p)  % should give you the  solution  for polynomial roots
             % and it should check with what w used to create polynomial
```

In Octave Command Window

```
-------------------------------
Polynomials: Cubic polynomial
--------------------------------
f = (sym)
          /    13\
```

```
  (x - 3)*|x - --|*(x + 2)
          \    10/
p = (sym 1x4 matrix)
  [      -47   -23    ]
  [39/5  ----  ----  1]
  [       10    10    ]
p =
   7.8000  -4.7000  -2.3000   1.0000

p =
   1.0000  -2.3000  -4.7000   7.8000

sol =
  -2.0000
   3.0000
   1.3000
```

2.5.2 Logarithms

Logarithms are popular in engineering mathematics. Logarithms are defined over a base. The most frequent one you come across is the natural logarithm (***ln***) which is based on the mathematical constant ***e = 2.718282***. Another popular one is the logarithm to the base 10 (***log_{10})***. There are a lot of mathematical solutions that rely on this base. Computer science likes logarithms to the base 2 (***log_2***) and 16 (***log_{16}***) and they are important part of computer programming. Logarithm is defined below.

$$
\begin{aligned}
&b^x = y; \quad x = \log_b y \\
&e^x = y; \quad x = \log_e y = \ln y; \quad \ln e = 1 \\
&10^x = y; \quad x = \log_{10} y
\end{aligned}
\tag{2.35}
$$

In addition the rules for basic arithmetic operations (any base) are

$$
\begin{aligned}
&\log(ab) = \log(a) + \log(b) \\
&\log\left(\frac{a}{b}\right) = \log(a) - \log(b) \\
&\log(a^n) = n\log(a) \\
&\log\left(\frac{1}{n}\right) = -\log(n) \\
&\log(1) = 0
\end{aligned}
\tag{2.36}
$$

Logarithms in MATLAB
In the Editor:

```
% Essential Mechanics
% P. Venkataraman
% Section 2.5.2 - Logarithms
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('------------------------------\n')
fprintf('Logarithms: \n')
fprintf('-------------------------------\n')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Numerical Calculations
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Logarithms and bases
% log is natural logarithm
exp(1)  % should give the value of e
log(exp(1))  %  should give you 1

% log2  is logarithm  with the base 2
log2(8)  %  should yield 3 since (2)^3 = 8

% log10 is logarithm with the base 10
log10(100)  % this should yield 2 since 10^2 = 100

% There are only three logarithm functions in MATLAB
% other log functions can be set up through
% logb(x) = log(x)/log(b)
% For example log to base 16 can be defined as
log16_64 = log(64)/log(16)  % cannot use log16(64) since
                            % MATLAB will create an array with 64 values
```

In the Command Window:

```
-------------------------------
Logarithms:
--------------------------------
ans =
    2.7183
ans =
     1
ans =
     3
ans =
     2
log16_64 =
    1.5000
```

Execution in Octave
The code is the same and so are the results

2.5.3 Trigonometry

The basic trigonometric functions we normally use are centered around the right angled triangle. In addition trigonometric functions repeat after one revolution or 2π radians or 360 degrees.

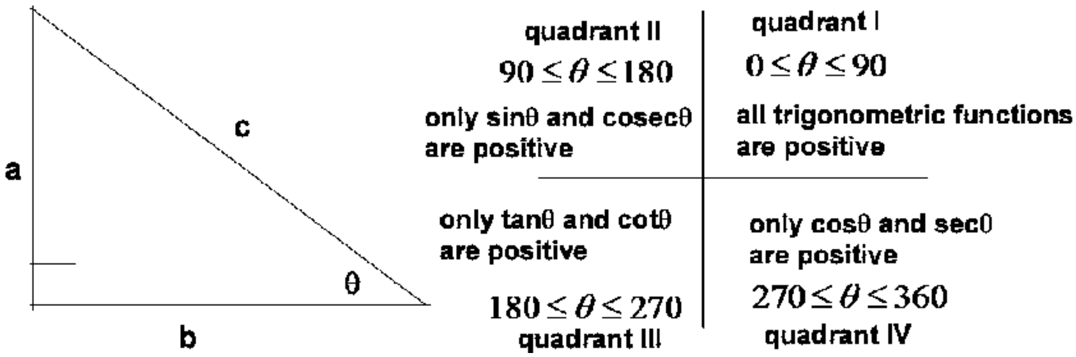


Figure 2.5.1 Basic trigonometry definition

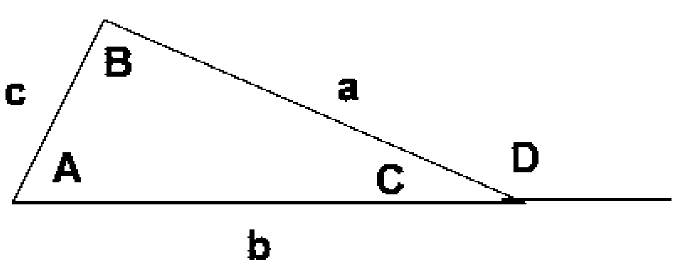
Base definition:

$$\text{sine}\,\theta = \sin\theta = \frac{a}{c};\quad \text{cosine}\,\theta = \cos\theta = \frac{b}{c};\quad \text{tangent}\,\theta = \tan\theta = \frac{a}{b};$$
$$\text{cosecant}\,\theta = \csc\theta = \frac{c}{a};\quad \text{secant}\,\theta = \sec\theta = \frac{c}{b};\quad \text{cotangent}\,\theta = \cot\theta = \frac{b}{a} \tag{2.37}$$

Additional Identities:
In addition the following relations will be useful in derivations and proof.

$$\sin^2\theta + \cos^2\theta = 1;\quad 1 + \tan^2\theta = \sec^2\theta;\quad 1 + \cot^2\theta = \csc^2\theta;$$
$$\sin\frac{\theta}{2} = \sqrt{\frac{1}{2}(1-\cos\theta)};\quad \cos\frac{\theta}{2} = \sqrt{\frac{1}{2}(1+\cos\theta)};$$
$$\sin 2\theta = 2\sin\theta\cos\theta;\quad \cos 2\theta = \cos^2\theta - \sin^2\theta; \tag{2.38}$$
$$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b;$$
$$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b;$$

For a regular triangle the following relations are very useful (the large letter represent angles and the lower case the length of the sides)



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};\quad \text{sine rule}$$
$$c^2 = a^2 + b^2 - 2ab\cos C;\quad \text{cosine rule} \tag{2.39}$$
$$c^2 = a^2 + b^2 + 2ab\cos D;$$

We have previously used many of these relations while working with vectors. This section consolidates the useful information from trigonometry. We have also used MATLAB to process trigonometric information while dealing with vectors.

2.5.4 Derivatives

The usual solution for many engineering problems, especially in the early courses in engineering, are continuous solutions. Very often additional properties are dependent on the derivatives and integrals of these functions. Through intense practice, many of you may remember the formulas associated with differentiation and Integration. There are handbooks that provide the formula for integration of popular functions. However, symbolic calculations through any of the software packages that support them, makes such a resource less necessary. In the following we outline some of the basic formulas for differentiation that appear often in engineering.

It is also important to note the difference between ***derivatives*** and ***differentials***. They are nearly the same except the first provides unit change with respect to a variable, while the latter provides the small changes in the function for small changes in the independent variables, using the derivative information. Often the lower case Greek letter delta δ is used for differentials. Here I will use the lowercase **d** for it. Let us clarify with an example of a function *f(x, y)*. This is a function of two independent variables. You must use partial derivatives.

$$\begin{aligned}
&f(x, y) \quad \text{is a two variable function}\\
&\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y; \quad \text{same as } df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy; \quad \text{is the differential}\\
&\frac{\partial f}{\partial x}: \text{ is the partial derivative of f with respect to x}\\
&\frac{\partial f}{\partial y}: \text{ is the partial derivative of f with respect to y}
\end{aligned} \tag{2.40}$$

What about a function of a single variable? What are the derivative and the differential in that case?

The popular relations for the derivatives are

$$\begin{aligned}
&\frac{d\,x^n}{dx} = nx^{n-1}; \quad \frac{d\,(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}; \quad \frac{d\left(\dfrac{u}{v}\right)}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2};\\
&\frac{d\,(\sin x)}{dx} = \cos x; \quad \frac{d(\cos x)}{dx} = -\sin x; \quad \frac{d(\tan x)}{dx} = \sec^2 x;\\
&\frac{d\,(\sinh x)}{dx} = \cosh x; \quad \frac{d(\cosh x)}{dx} = \sinh x; \quad \frac{d(\tanh x)}{dx} = \mathrm{sech}^2 x;
\end{aligned} \tag{2.41}$$

Derivatives in MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 2.5.4 - Derivatives
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------\n')
fprintf('Derivatives: \n')
fprintf('--------------------------------\n')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Symbolic Calculations
% Math Handbook for derivatives
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Derivatives
syms x
% power
fprintf('diff(x^3) = '),disp(diff(x^3))
fprintf('diff(x^3,x) = '),disp(diff(x^3,x))
% diff(x^3) == diff(x^3,x)  % good idea to include x to avoid
% interpretation

% product
u = 3*x;  v = 2*exp(x);  % exp(x) == e^x
fprintf('u(x) = '),disp(u)
fprintf('v(x) = '),disp(v)
fprintf('diff(u*v) = '),disp(diff(u*v))

% division
fprintf('diff(u/v) = '),disp(diff(u/v))

% trigonometric
fprintf('diff(sin(x)) = '),disp(diff(sin(x)))

% hyperbolic
fprintf('diff(sinh(x)) = '),disp(diff(sinh(x)))

% all - check if it is right
fprintf('diff(x^3*exp(x)*sin(x)*sinh(x))= \n'), ...
    disp((diff(x^3*exp(x)*sin(x)*sinh(x))))

%% You can also define implicit dependence
fprintf('\nImplicit functions:\n')

% Define implicit functions
u = sym('u(x)'); % u(x)
v = sym('v(x)'); % v(x)
f1 = diff(u*v);  % this should give you the chain rule formula
fprintf('diff(u*v) = '),disp(f1)
```

In the Command Window

```
--------------------------------
Derivatives:
--------------------------------
diff(x^3) = 3*x^2
diff(x^3,x) = 3*x^2
```

```
u(x) = 3*x
v(x) = 2*exp(x)
diff(u*v) = 6*exp(x) + 6*x*exp(x)
diff(u/v) = (3*exp(-x))/2 - (3*x*exp(-x))/2
diff(sin(x)) = cos(x)
diff(sinh(x)) = cosh(x)
diff(x^3*exp(x)*sin(x)*sinh(x))=
x^3*exp(x)*cos(x)*sinh(x) + x^3*exp(x)*cosh(x)*sin(x) +
3*x^2*exp(x)*sin(x)*sinh(x) + x^3*exp(x)*sin(x)*sinh(x)

Implicit functions:
diff(u*v) = u(x)*diff(v(x), x) + v(x)*diff(u(x), x)
```

You do not have to remember the formula but you have make sure that you are not making a coding error.

Execution in Octave
The code is the same and so are the results

2.5.5 Integrals

We can use symbolic calculation in MATLAB to function as an Handbook of Integrals. It may not always be a neat looking function. Another important fact about integrals in MATLAB is that it returns **definite integrals** correctly. However, it does not include an **integration constant** for **indefinite integrals**. If it does return one sometimes it may not be the one you are expecting. It is a good idea to remember that you are responsible for the constant of integration when processing indefinite integrals. In the following please note the the integration constant **c** will not be provided by MATLAB. Only a small number of examples are included in the exercise. You should be able to calculate integrals for any expression you come across. If MATLAB cannot solve the integral it will just return the same integral expression. The following are examples from integral tables:

Integrals of simple functions of x

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c; \quad \int x^{-1} dx = \ln x + c$$
$$\int \sqrt{a+bx}\, dx = \frac{2}{3b}\sqrt{(a+bx)^3} + c; \quad \int \frac{dx}{\sqrt{a+bx}} = \frac{2}{b}\sqrt{a+bx} + c; \qquad (2.42)$$
$$\int x\sqrt{a+bx}\, dx = \frac{2}{15b^2}(3bx-2a)\sqrt{(a+bx)^3} + c$$
$$\int \frac{x dx}{(a+bx)} = \frac{1}{b^2}(bx - a\ln(a+bx)) + c$$

Integrals of basic trigonometric and hyperbolic functions

$$\int \sin x\,dx = -\cos x + c; \quad \int \cos x\,dx = \sin x + c;$$

$$\int \sec x\,dx = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + c;$$

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + c; \quad \int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + c;$$

$$\int \sin x \cos x\,dx = \frac{\sin^2 x}{2} + c;$$

$$\int \sinh x\,dx = \cosh x; \quad \int \cosh x\,dx = \sinh x; \quad \int \tanh x\,dx = \ln(\cosh x);$$

(2.43)

Integrals of exponential functions

$$\int e^{ax}dx = \frac{e^{ax}}{a} + c; \quad \int \ln x\,dx = x\ln x - x + c; \quad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1) + c;$$

$$\int e^{ax}\sin px\,dx = \frac{e^{ax}(a\sin px - p\cos px)}{a^2+p^2} + c;$$

$$\int e^{ax}\cos px\,dx = \frac{e^{ax}(a\cos px + p\sin px)}{a^2+p^2} + c;$$

(2.44)

Definite integrals:

$$\int_2^5 \sqrt{(1+2x)}\,dx = \frac{2}{3\bullet 2}\sqrt{(1+2x)^3}\Big|_2^5 = \frac{1}{3}\left[\sqrt{11^3} - \sqrt{5^3}\right] = 8.43417$$

$$\int_3^4 \tanh(2.5x)\,dx = \frac{2}{5}\ln(\cosh(2.5x))\Big|_3^4 \qquad (2.45)$$

$$\int_2^5 xe^{-2x}\,dx = \frac{e^{-2x}}{4}(-2x-1)\Big|_2^5$$

Integrals in MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 2.5.5 - Integrals
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('------------------------------------\n')
fprintf('Integrals: \n')
```

```
fprintf('--------------------------------\n')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Symbolic Calculations
% Math Handbook for integrals
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Integrals of x
syms x n a b p
% functions of x
fprintf('Integrals of functions of x\n')
fprintf('--------------------------------\n')
fprintf('int(x^n) = '),disp(int(x^n))
fprintf('int(sqrt(a + b*x),x) = '),disp(int(sqrt(a + b*x),x))
fprintf('int(1/sqrt(a + b*x),x) = '),disp(int(1/sqrt(a + b*x),x))
fprintf('int(x*sqrt(a + b*x),x) = '),disp(int(x*sqrt(a + b*x),x))
fprintf('int(x/(a + b*x),x) = '),disp(int(x/(a + b*x),x))

%% basic trignometric functions
fprintf('\n----------------------------------------')
fprintf('\nIntegrals of trignometric functions\n')
fprintf('--------------------------------------\n')
fprintf('int(sin(x)) = '),disp(int(sin(x)))
fprintf('int(sec(x))) = '),disp(int(sec(x)))
fprintf('int(sin(x)^2)) = '),disp(int(sin(x)^2))
fprintf('int(tanh(x))) = '),disp(int(tanh(x)))


%% basic exponential functions
fprintf('\n----------------------------------------')
fprintf('\nIntegrals of exponential functions\n')
fprintf('--------------------------------------\n')
fprintf('int(exp(a*x),x) = '),disp(int(exp(a*x),x))
fprintf('int(log(x)) = '),disp(int(log(x)))
fprintf('int(x*exp(a*x)) = '),disp(int(x*exp(a*x)))
fprintf('int(exp(a*x)*cos(p*x)) = '),disp(int(exp(a*x)*cos(p*x)))

%% Definite Integrals
fprintf('\n--------------------------------')
fprintf('\nDefinite Integrals of functions \n')
fprintf('--------------------------------\n')
fprintf('int(sqrt(1 + 2*x),x) between [2,5]= '),...
    disp(vpa(int(sqrt(1 + 2*x),x,2,5),3))

fprintf('int(tanh(2.5*x)),x,3,4) = '),disp(vpa(int(2.5*tanh(x),x,2,3),5))
fprintf('int(x*exp(-2*x),x,2,5) = '),disp(vpa(int(x*exp(-2*x),x,2,5),5))
```

In the Command Window

```
-------------------------------
Integrals:
--------------------------------
Integrals of functions of x
--------------------------------
int(x^n) = piecewise([n == -1, log(x)], [n ~= -1, x^(n + 1)/(n + 1)])
int(sqrt(a + b*x),x) = (2*(a + b*x)^(3/2))/(3*b)
```

```
int(1/sqrt(a + b*x),x) = (2*(a + b*x)^(1/2))/b
int(x*sqrt(a + b*x),x) = -(10*a*(a + b*x)^(3/2) - 6*(a +
b*x)^(5/2))/(15*b^2)
int(x/(a + b*x),x) = -(a*log(a + b*x) - b*x)/b^2

----------------------------------------
Integrals of trigonometric functions
--------------------------------------
int(sin(x)) = -cos(x)
int(sec(x))) = log(1/cos(x)) + log(sin(x) + 1)
int(sin(x)^2)) = x/2 - sin(2*x)/4
int(tanh(x))) = log(cosh(x))

----------------------------------------
Integrals of exponential functions
--------------------------------------
int(exp(a*x),x) = exp(a*x)/a
int(log(x)) = x*(log(x) - 1)
int(x*exp(a*x)) = (exp(a*x)*(a*x - 1))/a^2
int(exp(a*x)*cos(p*x)) = (exp(a*x)*(a*cos(p*x) + p*sin(p*x)))/(a^2 + p^2)

--------------------------------
Definite Integrals of functions
--------------------------------
int(sqrt(1 + 2*x),x) between [2,5]= 8.43
int(tanh(2.5*x)),x,3,4) = 2.4608
int(x*exp(-2*x),x,2,5) = 0.02277
```

Solution Using Octave

The code is same as in MATLAB above except for the changes shown below

In Octave Editor:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Symbolic Calculations
% Math Handbook for integrals
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
pkg load symbolic # Loads the package for symbolic calculations
sympref display flat   # writes to the command window cleanly
```

In Octave Command Window

```
-------------------------------
Integrals:
--------------------------------
Integrals of functions of x
--------------------------------
int(x^n) =   Piecewise((x**(n + 1)/(n + 1), Ne(n, -1)), (log(x), True))
int(sqrt(a + b*x),x) =   2*(a + b*x)**(3/2)/(3*b)
int(1/sqrt(a + b*x),x) =   2*sqrt(a + b*x)/b
int(x*sqrt(a + b*x),x) =   -4*a**(9/2)*sqrt(1 + b*x/a)/(15*a**2*b**2 +
15*a*b**3*x) + 4*a**(9/2)/(15*a**2*b
**2 + 15*a*b**3*x) - 2*a**(7/2)*b*x*sqrt(1 + b*x/a)/(15*a**2*b**2 +
```

```
15*a*b**3*x) + 4*a**(7/2)*b*x/(15*a**2*
b**2 + 15*a*b**3*x) + 8*a**(5/2)*b**2*x**2*sqrt(1 + b*x/a)/(15*a**2*b**2 +
15*a*b**3*x) + 6*a**(3/2)*b**3*x
**3*sqrt(1 + b*x/a)/(15*a**2*b**2 + 15*a*b**3*x)
int(x/(a + b*x),x) =   -a*log(a + b*x)/b**2 + x/b

----------------------------------------
Integrals of trignometric functions
--------------------------------------
int(sin(x)) =   -cos(x)
int(sec(x))) =   -log(sin(x) - 1)/2 + log(sin(x) + 1)/2
int(sin(x)^2)) =   x/2 - sin(x)*cos(x)/2
int(tanh(x))) =   x - log(tanh(x) + 1)

----------------------------------------
Integrals of exponential functions
--------------------------------------
int(exp(a*x),x) =   Piecewise((exp(a*x)/a, Ne(a, 0)), (x, True))
int(log(x)) =   x*log(x) - x
int(x*exp(a*x)) =   Piecewise(((a*x - 1)*exp(a*x)/a**2, Ne(a**2, 0)),
(x**2/2, True))
int(exp(a*x)*cos(p*x)) =   Piecewise((x, Eq(a, 0) & Eq(p, 0)), (I*x*exp(-
I*p*x)*sin(p*x)/2 + x*exp(-I*p*x)*
cos(p*x)/2 + exp(-I*p*x)*sin(p*x)/(2*p), Eq(a, -I*p)), (-
I*x*exp(I*p*x)*sin(p*x)/2 + x*exp(I*p*x)*cos(p*x)/
2 + exp(I*p*x)*sin(p*x)/(2*p), Eq(a, I*p)), (a*exp(a*x)*cos(p*x)/(a**2 +
p**2) + p*exp(a*x)*sin(p*x)/(a**2
+ p**2), True))

--------------------------------
Definite Integrals of functions
--------------------------------
int(sqrt(1 + 2*x),x) between [2,5]=   8.43
int(tanh(2.5*x)),x,3,4) =   2.4608
int(x*exp(-2*x),x,2,5) =   0.022770
```

2.5.6 Series

The following is a limited collection of some popular series that you will come across in engineering. The series are usually a way to approximate the function using polynomial terms. They are usually infinite series - or infinite terms used in evaluating the function. The more terms the better the approximation. Binomial, Fourier, and Taylor expansions are popular in engineering. Sometimes the series are referred as expansions about a point (a). For Maclaurian series the point a is 0.

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \quad ... \quad [x^2 < 1]; \quad \textbf{binomial}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \quad ... \quad \text{exponential}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \quad ... \quad [\text{x in radian}]$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \quad ... \quad [\text{x in radian}] \tag{2.46}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \quad ...$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \quad ...$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}; \quad where \quad (Fourier\ series)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx; \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

$$f(x) = f(a) + f'(a)*(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \quad ... \quad + \frac{f^n(a)}{n!}(x-a)^n + \quad ...$$

(Taylor series about point a)

$$f(x) = f(0) + f'(0)*(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \quad ... \quad + \frac{f^n(0)}{n!}(x)^n + \quad ...$$

(Maclaurian series about point a) (2.47)

We will discuss series as and when needed.

2.5.7 Additional Problems

In the following problems, first solve the problem by hand and then re-solve by MATLAB and compare your solutions

Problem 2.5.1

Find the roots of the quadratic equation:

$$x^2 + x - 12 = 0$$

Problem 2.5.2

Find the derivative of: $e^{-2x}\sqrt{x}$ OR

$$\frac{d\left(e^{-1x}\sqrt{x}\right)}{dx}$$

Problem 2.5.3
Find the integral:

$$\int e^{-3x}\cos\left(2x\right)dx$$

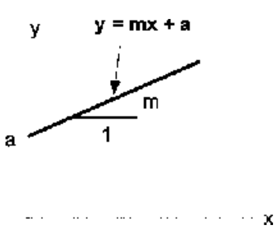
2.6 GEOMETRY

In this section we will review some ideas from geometry. Geometrical form is the important element of design and it is often challenging to choose the right geometry for the best outcome. Many solutions and models depend on geometrical formulation and their various properties. The word geometry incorporates many ideas and is usually tagged to indicate its nature. We will try to capture some essential information from several aspects of geometry.

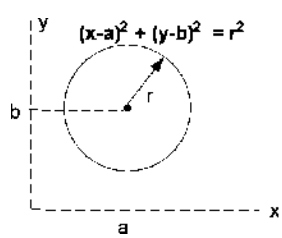
2.6.1 Analytical Geometry

Analytical geometry refers to a convenient description of the physical geometry using a mathematical equation. In many cases this provides a way to define the constraint, or better still, provides an equation to reduce the number of unknowns. The equation for the straight line, the circle, and the ellipse should be familiar. Sections 2.61 and 2.6.2 should be useful for this book. The rest is for fun and allows you to use MATLAB to actually draw shapes that are of correct scale. You can skip those sections if you wish.

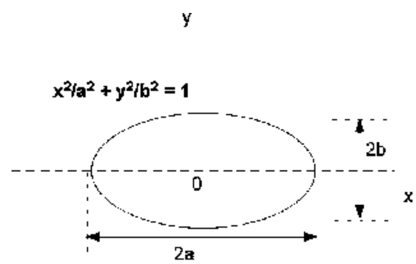
Table 2.1 Popular Objects



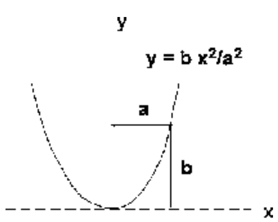
Line



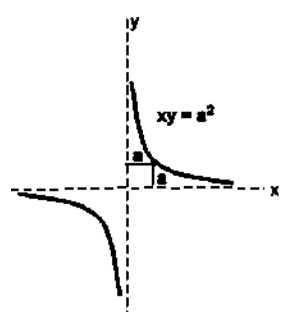
Circle



Ellipse



Parabola



Hyperbola

Let us use MATLAB to create the objects in Table 2.1

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 2.6.1 - Drawing geometric primitives
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------\n')
fprintf('Drawing Geometric Primitives: \n')
fprintf('--------------------------------\n')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Drawing simple geometric primitives
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure %  open a figure window and park it in the lower left corner
set(gcf,'Position',[25,50,400,350]);
hold on  % allow multiple plots in the same figure

%% line
syms x y
a = 2;  m = 0.5;  b = 1; r= 1;
y = a + m*x  % the line
ezplot(y,[-3,3.5])

%% circle
% sqrt will account for negative and positive values giving two solutions
hold on
y1 = sqrt(r^2 - (x-a)^2) + b  % first solution to y
y2 = -sqrt(r^2 - (x-a)^2) + b  % second solution to y
hp1 = ezplot(y1,[1,3]);  % plot half the circle
set(hp1,'Color','r')
hp2 = ezplot(y2,[3,1]);  % plot the second half
set(hp2,'Color','r')

%% ellipse
y3 = b*sqrt(1 - (x/a)^2)  % first solution for y
hp3 = ezplot(y3,[-3,3]);  % plot half the ellipse
set(hp3,'Color','k')
y4 = -b*sqrt(1 - (x/a)^2)  % second solution
hp4 = ezplot(y4,[-3,3]);  % plot the other half
set(hp4,'Color','k')

%% Parabola
y5 = b*x^2/a^2
hp5 = ezplot(y5,[-3,3]);  % plot the parabola
set(hp5,'Color','m')

%% Hyperbola
y6 = a^2/x;
hp6 = ezplot(y6,[-3,0.5]);
set(hp6,'Color','g','LineWidth',2)
hp7 = ezplot(y6,[0.5,3.5]);
set(hp7,'Color','g','LineWidth',2)

%%  label the plot and emphasize x and y axis-
title('The Objects')
axis([-3,3.5,-2.5,3])  % to accommodate all figures
line([0,0],[-2.5,3],'Color','k')  % draw the y - axis
line([-3,3.5],[0,0],'Color','k')  % draw the x - axis
grid on
```

```
xlabel('x')
ylabel('y')
daspect([1,1,1])
```

The functions corresponding to the objects appear in the Command window and are not reproduced here. The objects in the figure window provide confirmation (Figure 2.6.1).

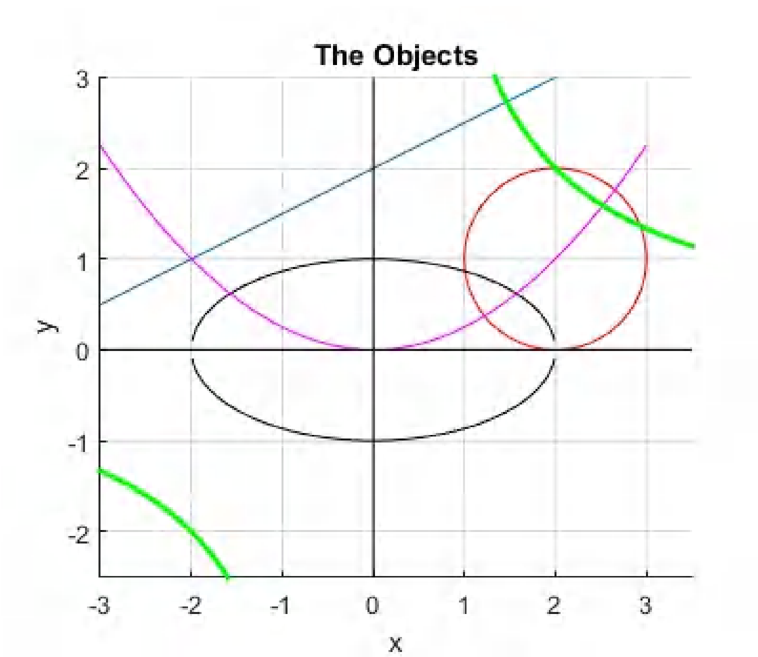


Figure 2.6.1 Basic objects created in MATLAB using mathematics

2.6.2 Just Geometry

In this section we recount the calculation of area, perimeter, volume as appropriate for the geometry. We also include some geometric properties that are exploited in the development of appropriate physics.

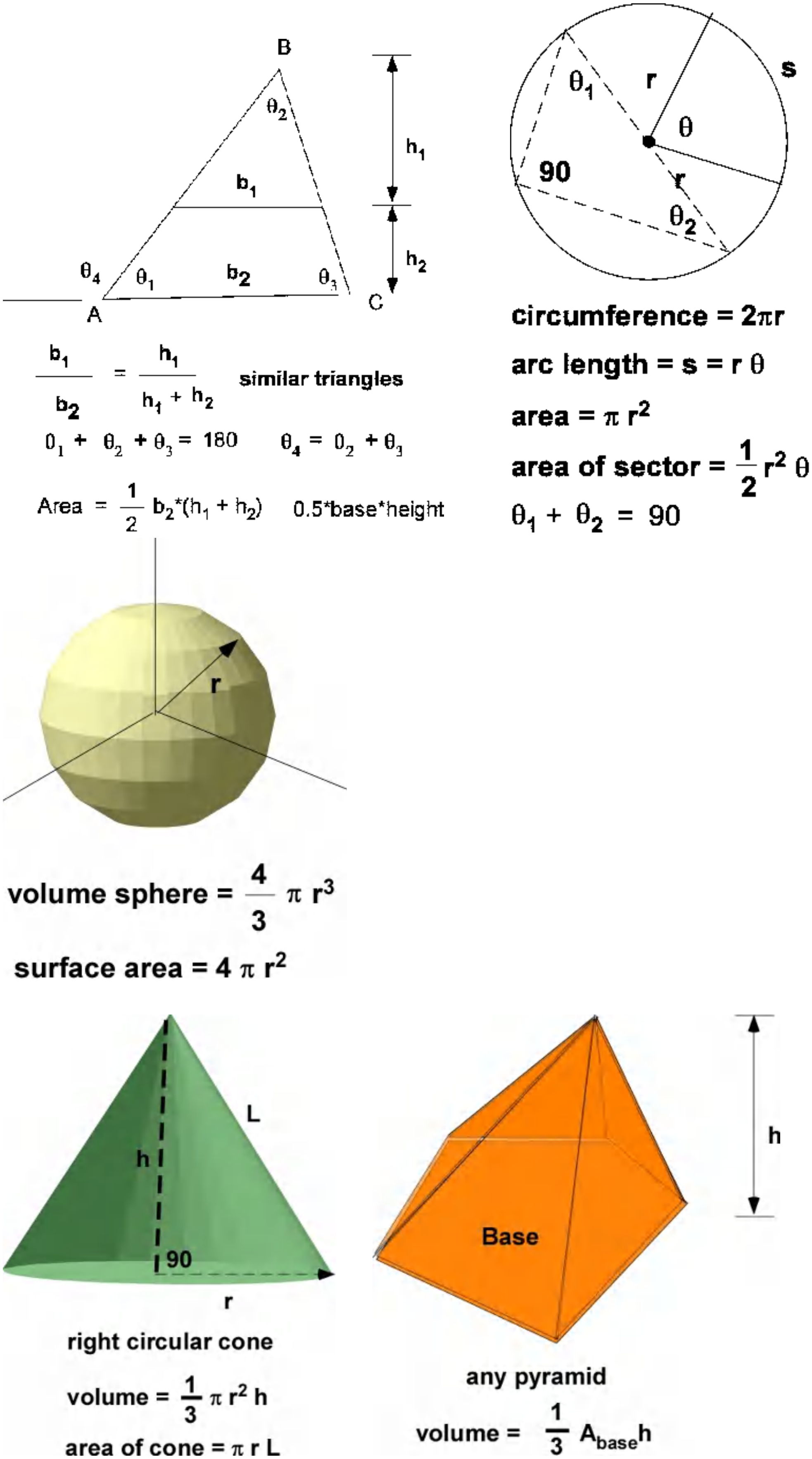


Figure 2.6.2 Basic geometric objects and properties

2.6.3 Just For Fun

Here we will create the various objects with filled color in MATLAB using patch and surface. You can use MATLAB to draw and animate precisely.

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 2.6.3 - Drawing more geometric primitives
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-----------------------------------\n')
fprintf('Drawing More Geometric Primitives: \n')
fprintf('-------------------------------------\n')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Drawing more simple geometric primitives
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure %  open a figure window and park it in the lower left corner
set(gcf,'Position',[25,50,400,350]);
hold on  % allow multiple plots in the same figure


%% triangle (2D object)
xv = [2,4,3]; % the x coordinates of the vertices
yv = [0,0,1];
patch(xv,yv,'y');  % patch fills color between the points
%                    it doe not have to be triangular

%% filled circle - animated - (2D object)
cen = [2,2];  rad = 1;
theta = linspace(0,2*pi,51);
xc = cen(1) + rad*cos(theta);
yc = cen(2) + rad*sin(theta);
% creating an animation
for i = 1:length(theta)-1
   xpatch = [cen(1),xc(i),xc(i+1)];
   ypatch = [cen(2),yc(i),yc(i+1)];
   patch(xpatch,ypatch,'g','EdgeColor','None')
   pause(0.1)  % slow it down so you can see
   daspect([1,1,1]);  % perfect circle
end

%% generate right cone - this object is 3D
r = linspace(0,1.5,31);
theta = linspace(0,2*pi,31);
[R,THETA]=meshgrid(r,theta);
X=R.*cos(THETA);
Y=R.*sin(THETA);
Z=4-R;

surf(X,Y,Z)
view(135,30)
axis tight
xlabel('x-axis')
```

```
ylabel('y-axis')
zlabel('z-axis')
colormap(hsv);

%% a unit sphere
sphere

%%  label the plot and emphasize x and y axis-
title('The Objects')
axis([-4,4,-2.5,3])  % to accommodate all figures
line([0,0],[-2.5,3],'Color','k')  % draw the y - axis
line([-3,3.5],[0,0],'Color','k')  % draw the x - axis
grid on
xlabel('x')
ylabel('y')
daspect([1,1,1])
```

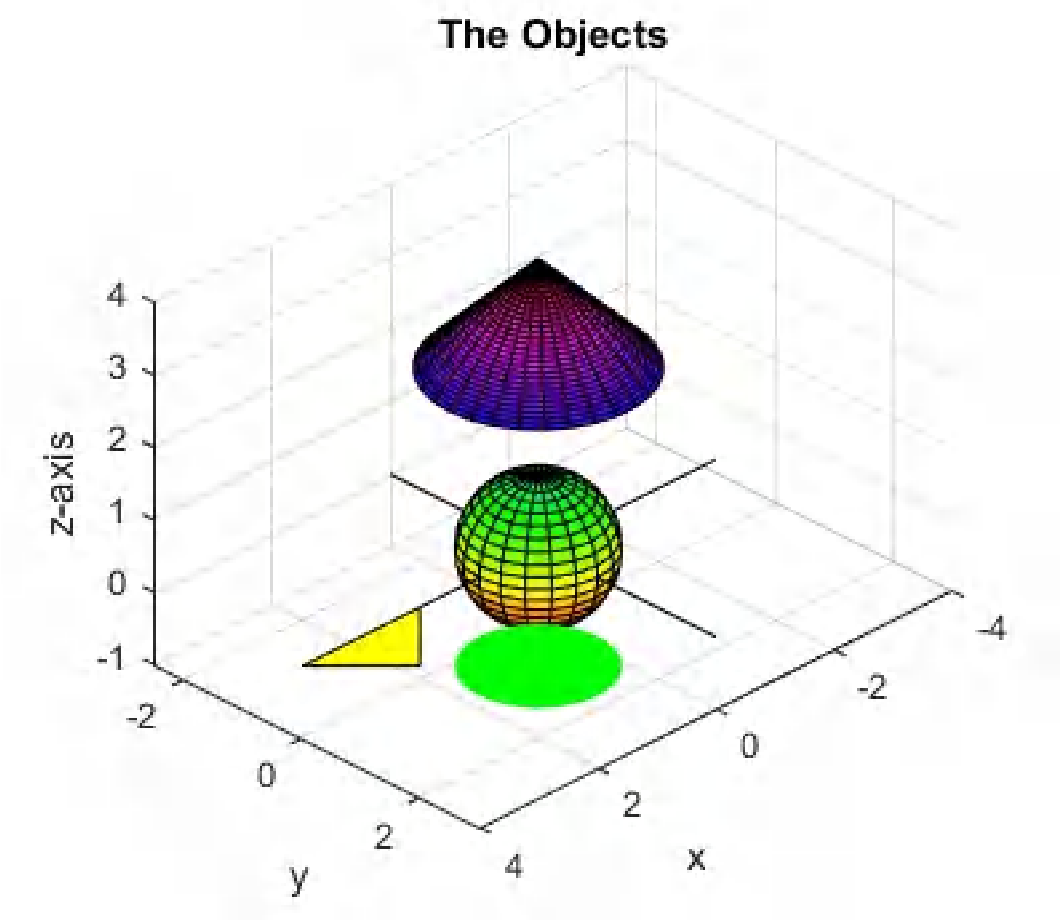


Figure 2.6.3 Additional Objects

Execution in Octave

The code is same as in MATLAB above

In Octave Figure window

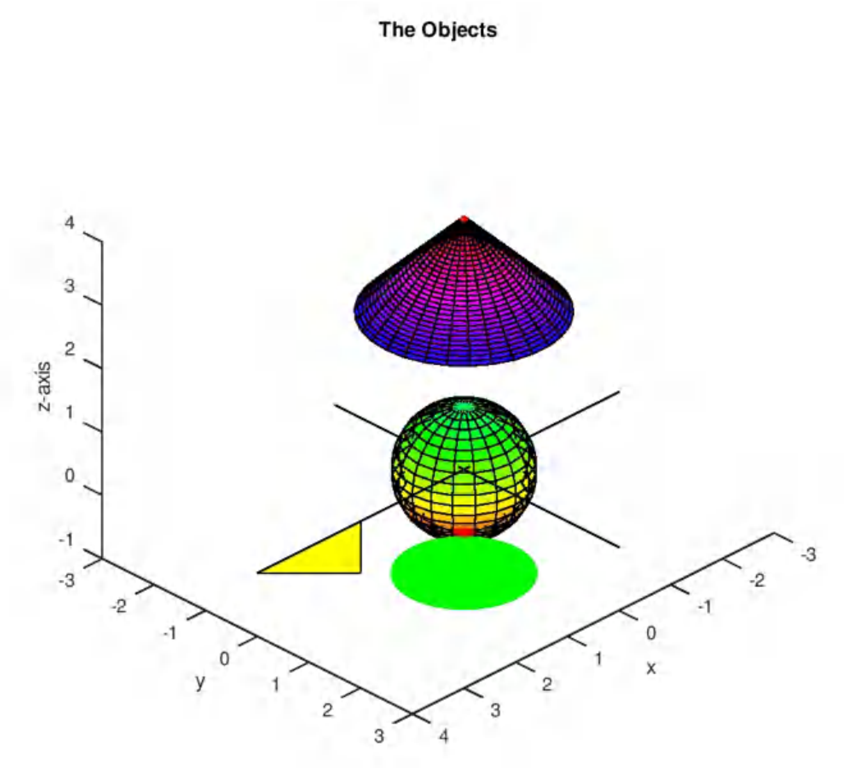


Figure 2.6.4 Figure in Octave

No Additional Problems are prescribed. Students are encouraged to test if the MATLAB code in Section 2.6.1 works

2.7 MASS CENTER AND INERTIA

This is a topic that is usually covered after half way through the subject of statics and usually appears as an appendix in dynamics and mechanics of materials. The ideas present in this topic is quite instinctive and is probably better managed early as it just depends on basic calculus for computation. As far as this book is concerned, these topics are most important for design as they determine the actual physical structure of design. They need to be introduced before you can discuss design. All of the calculation in this section involve geometry.

Mass center, area center, volume center all imply the same thing. Their calculations are similar. Physically it allows us to idealize and simplify the calculations by replacing distributed forces by concentrated forces at the these special points. Consider the human body. Every small volume of the body will have a different density - it may be due to fat, muscle, bone, blood - but most likely a different proportion of all of them at different locations on the body. So your weight, which is the density multiplied by the volume and further multiplied by the gravitational acceleration constant, is actually varying over your body. That will present a problem during quick calculations that require the weight.When you stand on that weighing scale, you are not interested in any particular distribution of weight in your body. You are looking for a single number that is integrated over your body's volume. So the weight that you read is an integrated value of a distribution that is lumped together. A companion question, particularly important if you are a gymnast, is where is this weight located so that you can control it to achieve dizzying rotations. Engineers came up with the idea to locate the weight of the body at the *center of mass*. Off course, you will need to know where the center of mass for the body is. This will require another calculation. With data and experience the point can be approximately located, maybe below your belly button. In this process we have accommodated the distribution of the mass within the body to a point within the body, which will influence the problem in the same way as the distributed weight, but with less calculations. This is much easier than calculations that must include the distribution of mass with every calculation. The center of mass , which is a point on the body, is also the center of weight (uniform density is assumed) and which is also termed as the center of gravity. In many problems we need the center of area and this is termed as the centroid of the area. Similarly we have the centroid of the volume. If you were to study fluid mechanics then the aerodynamic forces are located at the center of pressure for convenience, which are not related to the center of mass or the centroid. We will just term everything as center and let the context identify the particular definition.

In many cases the location of the center will be obvious from geometry. For example the center of the circular area is at the physical center. This is facilitated by the assumption of uniform density or uniform thickness. The assumption of uniformity brings with it an additional simplification. We can reduce the dimension of the problem. Instead of a three dimensional problem it will become a two dimensional problem or even a one dimensional problem. This is usually accompanied by a reduction in the calculations. You will meet very limited number of problems in three-dimensions before you graduate with an engineering degree, because the calculations are not trivial. In addition to the center there are additional properties associated with the mass distribution or area distribution for mechanical design. The real world is mostly three dimensional so you will have to teach yourself these calculations. Many simulation software will provide information on these properties. We will discuss the calculations in a later section.

In addition to the mass center, there are many other properties associated with the area or the volume. One of them is the moment of inertia (MOI). This relates to the distribution of the area about the centroid, particularly along specific axis. The larger the value the better the stress handling property of the cross-section. This appears in the first strength of materials course. For traditional geometry you can usually find them through the Internet, through handbooks, or the text. This is easier than actually computing them using basic calculus. However for one of a kind geometry you must calculate it yourself. That will be our focus in this section. A two-dimensional shape will have three values of inertia that must be computed. A three dimensional object will have six value for the inertia. In the following we have only one example and we will keep it generic so you can extend it to any problem. We will increase the dimensionality of the object and you will notice that the equations used for computation are very similar.

One more thing we will do from this section on, if appropriate, is to break up the MATLAB code in segments so it address the immediate formulas and avoids confusion by creating a long piece of code that does everything. This way you can learn from pieces of code, their objective, and their results. You can then collect all the pieces in a single file and create the front end and the description of what is being implemented.

2.7.1 A Line/Curve

A one-dimensional problem is represented by a rod of constant diameter and density, but whose curve is mathematically defined. If you have a rod and it is not mathematically defined then you can approximate the shape by curve-fitting mathematical functions. We are interested in the mass, the location of the mass center, and its moment of inertia about the coordinate axes. Here is an example:

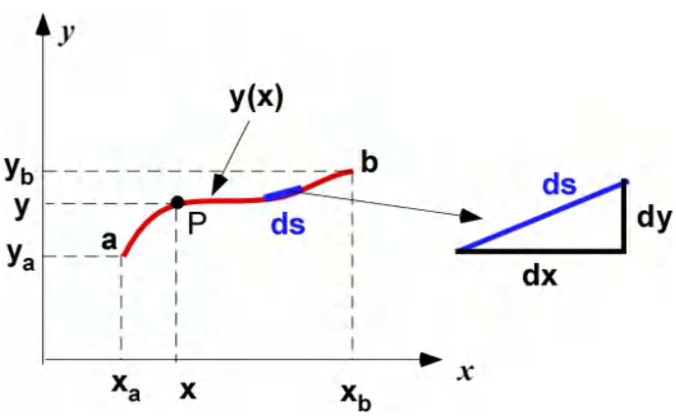


Figure 2.7.1 A line/curve

The length of the line can be obtained by integration of an differential element of the line ***ds***. This is infinitesimal length of the line segment.Since the line is available through x and y we need to convert this information before integration through the geometry shown in Figure 2.7.1.

Mass of Line

The mass of the rod of uniform cross-sectional area (**A**), with uniform density of the material (ρ) is

$$m = LA\rho$$

$$L = \int_0^L ds = \int_{x_a}^{x_b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (2.48)$$

If the area and the density are varying as a function of *x* then you should be easily extend the formulas as:

$$m = \int_{x_a}^{x_b} A(x)\rho(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad (2.49)$$

```
%% line
% circular rod of diameter d and density rho
d = 2.5; % [cm]
rho = 8.96;  % [g/cm^3] - copper
xa = 1;  xb = 3;
syms x
y = 0.5 - 2*x + 4*sqrt(x);
ya = subs(y,xa);
yb = subs(y,xb);
dydx = diff(y,x);
L = double(int(sqrt(1 + dydx^2),x,xa,xb));
mass = (pi*d^2/4)*rho*L;
fprintf('y(x) = '),disp(y)
fprintf('L [cm] = '),disp(L)
fprintf('mass [g] = '),disp(mass)

set(gcf,'Position',[50,50,300,2*300/3])
hp1 = ezplot(y,[xa,xb]);
set(hp1,'Color','r','LineWidth',2)
xlabel('x')
ylabel('y')
grid on
axis([0,4,0,5])
```

In command window

```
y(x) = 4*x^(1/2) - 2*x + 1/2
L [cm] =    2.3078e+00
mass [g] =    1.0150e+02
```

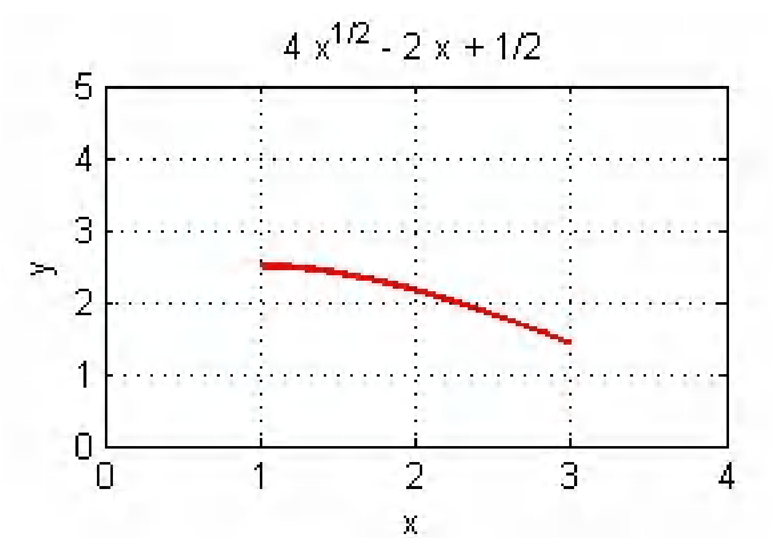


Figure 2.7.2 Actual line

Mass Center of Line

The location of the mass center cab be established with respect to any point - a reference point. If you have a mathematical expression for the line it is better to define it with respect to the origin so you do not have to reestablish the equation about this other reference point. The calculation of the mass center depends on the first moment of the mass of the line about the coordinate axes through the reference point. For the line in two-dimensional space in the previous example there will be first moment of the line about the ***x*** and the ***y*** axis. The location of the mass center from the origin and along the particular axis (our reference point) is the first moment of the mass of the line about that axis divided by the mass of the line. Calculus is used to develop the relations .The first moment of a differential element is defined and then integrated over the line. This is done for all of the coordinate directions/axes. This mass center does not have to be physically on the object. We will use the symbol Q for the first moment. The mass center will be located at the point G (x_G, y_G). In the figure below, the differential element is located at P, which is at the point (***x,y***) form the origin.

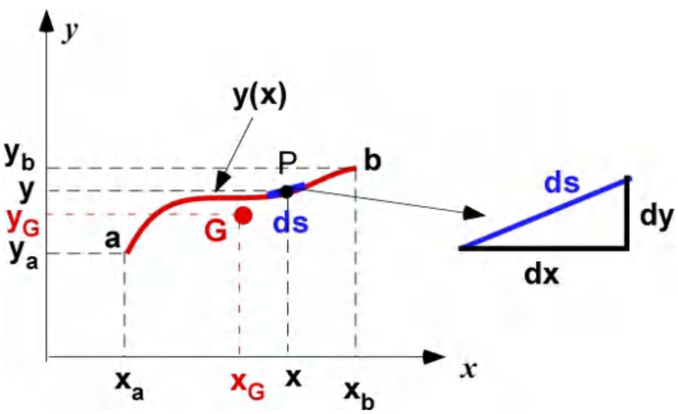


Figure 2.7.3 First moment of the line

In the current example as the density and the diameter are constant we can calculate the location of the point G in terms of the length of the rod. Otherwise we have to locate the point G using the differential mass in the calculations. Q_x and Q_y are the first moment of inertia about the respective axis.

$$x_G = \frac{\int_s y\,ds}{\int_s ds} = \frac{\int_{x_a}^{x_b} y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{L}; \quad x_G = \frac{Q_y}{L}$$

$$y_G = \frac{\int_s x\,ds}{\int_s ds} = \frac{\int_{x_a}^{x_b} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{L}; \quad y_G = \frac{Q_x}{L} \qquad (2.50)$$

$$x_G = \frac{\int_s y(A\rho ds)}{\int_s (A\rho ds)} = \frac{\int_s y(A\rho ds)}{m}$$

$$y_G = \frac{\int_s x(A\rho ds)}{\int_s (A\rho ds)} = \frac{\int_s x(A\rho ds)}{m} \qquad (2.51)$$

(Continued MATLAB code)

```
% mass center
Qy = double(int(y*sqrt(1 + dydx^2),x,xa,xb));
xG = Qy/L;
Qx = double(int(x*sqrt(1 + dydx^2),x,xa,xb));
yG = Qx/L;
fprintf('Qx = '),disp(Qx)
fprintf('Qy = '),disp(Qy)
fprintf('xG  [cm] = '),disp(xG)
fprintf('yG  [cm] = '),disp(yG)
hold on
plot(xG,yG,'ro','MarkerFaceColor','r')
```

In Command Window

```
Qx =    4.7272e+00
Qy =    4.7725e+00
xG  [cm] =    2.0680e+00
yG  [cm] =    2.0484e+00
```

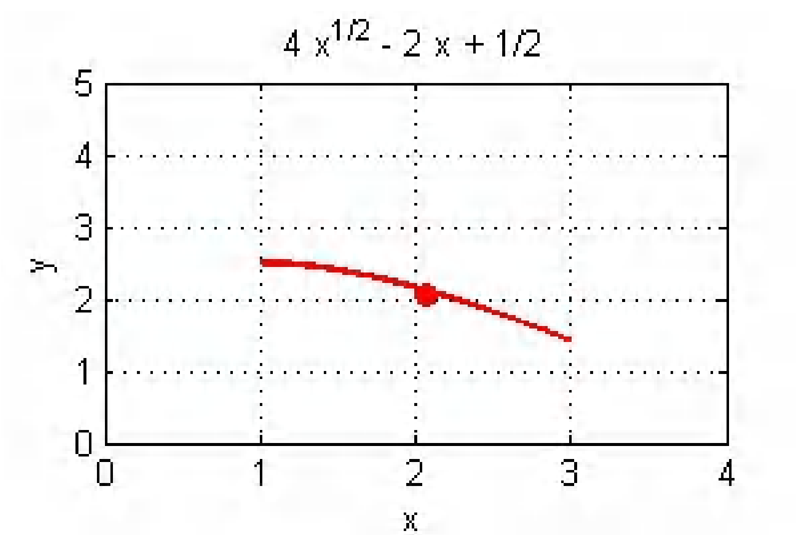


Figure 2.7.4 Mass center

Moment of Inertia of Line

The moment of inertia (MOI) of the line is the second moment of the mass of the line around an axis. It is a simple extension of the first moment of the mass in the previous discussion. It is also possible to calculate the MOI about the z-axis of the line described in the x-y plane. We will use the symbol **I** for moment of inertia. To be specific we will indicate the axis and the point about about which it is calculated through subscripts. $\mathbf{I_{Ox}}$ is the MOI through the x-axis passing through the origin O. Similarly $\mathbf{I_{Gy}}$ is the MOI of the line about the y-axis through the point G. We can also define the MOI of the **line** rather the the MOI of the **mass** of the line. This will remove the area and the density from discussion. In dynamics the **mass** MOI is important. In statics and strengths the line and **area** MOI is important. The modification to the formula is simple and is not done here. The following relations are for the the mass MOI.

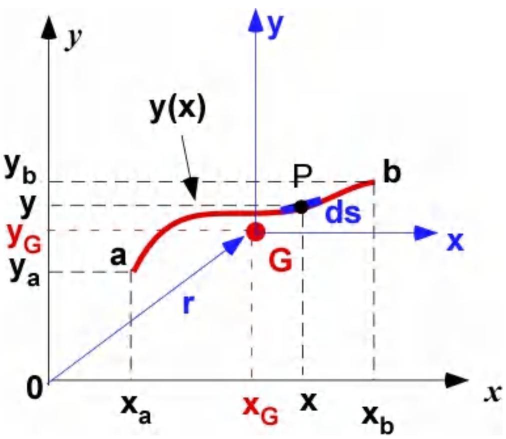


Figure 2.7.5 Moment of inertia (set up)

$$I_{Ox} = \int_m y^2\,dm = \int_m y^2 (A\rho)\,ds = A\rho \int_{x_a}^{x_b} y^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$I_{Oy} = \int_m x^2\,dm = \int_m x^2 (A\rho)\,ds = A\rho \int_{x_a}^{x_b} x^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad (2.52)$$

$$I_{Oz} = \int_m \left(x^2+y^2\right)dm = \int_m \left(x^2+y^2\right)(A\rho)\,ds = A\rho \int_{x_a}^{x_b} \left(x^2+y^2\right) \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

(Continued MATLAB code)

```
IOx = (pi*d^2/4)*rho*int(y^2*(sqrt(1 + dydx^2)),x,xa,xb);
IOx = double(IOx);
IOy = (pi*d^2/4)*rho*int(x^2*(sqrt(1 + dydx^2)),x,xa,xb);
IOy = double(IOy);
IOz = (pi*d^2/4)*rho*int((x^2+y^2)*(sqrt(1 + dydx^2)),x,xa,xb);
IOz = double(IOz);
```

In the Command Window

```
IOx [g-cm^2] =    4.4508e+02
IOy [g-cm^2] =    4.5942e+02
IOz [g-cm^2] =    9.0449e+02
```

Inertia About Mass Center

The MOI computed in the previous section is about the axis through the point O (origin). In many design analysis the MOI about the mass center G is quite significant. This would require describing the equation of the curve about the mass center and then using the integrals described above. There is a however nifty relation that allows us to calculate the MOI about CG knowing the MOI about another point (O in our case). Keep in mind that the MOI about point O is the easiest to compute because of the curve definition. This is called the ***Parallel Axis Theorem.*** This theorem also suggests that the MOI is the least about the center of mass.

$$I_{Ox} = I_{Gx} + m(x_O - x_G)^2; \quad I_{Gx} = I_{Ox} - m(x_O - x_G)^2$$
$$I_{Oy} = I_{Gy} + m(y_O - y_G)^2; \quad I_{Gy} = I_{Oy} - m(y_O - y_G)^2 \qquad (2.53)$$

(Continued MATLAB Code)

```
% parallel axis theorem
x0 = 0; y0 = 0;
IGx = IOx - mass*(x0 - xG)^2;
IGy = IOy - mass*(y0 - yG)^2;
fprintf('IGx [g-cm^2] = '),disp(IGx)
fprintf('IGy [g-cm^2] = '),disp(IGy)
```

In the Command Window

```
xG  [cm] =    2.0680e+00
yG  [cm] =    2.0484e+00
IGx [g-cm^2] =    1.0991e+01
IGy [g-cm^2] =    3.3524e+01
```

In this section we have used an example to show how we can translate the equations to MATLAB and solve for important properties of the line. Note that by simply changing the equation of the line - one line of code - you can generate all of the properties using the same code

Execution in Octave

This MATLAB code was quite difficult to execute in Octave directly. It could be the appearance of the *square root of x* in the expressions. The **int** command in Octave in the documentation is illustrated using a single expression. That is the idea in the changes in the code below. MATLAB did not have a problem integrating the multiplication of several function.

- This code took a lot of debugging. I have included the code that finally worked for me. There is probably a better way to make it work but at this time it is beyond me.
- I have also included the complete information from the command window with warts and all.
- There are lots of printing with semicolons removed for debugging along with the formal fprintf statements
- There are locations where the processing takes a lot of time - indicated by Waiting ………..
- The solution requires some patience

In Octave Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 2.7 1- Properties of a line
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, warning off
##setenv python C:\Users\venka\Anaconda3\python.exe
```

```
pkg load symbolic
sympref display flat
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('--------------------------------\n')
fprintf('Properties of a Line \n')
fprintf('---------------------------------\n')
%% line
% circular rod of diameter d and density rho
d = 2.5; % [cm]
rho = 8.96;  % [g/cm^3] - copper
xa = 1;  xb = 3;
syms x
y = 0.5 - 2*x + 4*sqrt(x)

ya = double(subs(y,xa));
yb = double(subs(y,xb));
dydx = diff(y,x)

exp00 = expand(dydx^2)
exp0 = 1 + exp00
exp1 = sqrt(exp0)
L = int(exp1,x,xa,xb);
##L = (int(sqrt(1 + dydx^2),x,[xa,xb]))
L = double(L);
mass = (pi*d^2/4)*rho*L;
fprintf('y(x) = '),disp(y)
fprintf('L [cm] = '),disp(L)
fprintf('mass [g] = '),disp(mass)

set(gcf,'Position',[50,50,300,2*300/3])
hp1 = ezplot(y,[xa,xb]);
set(hp1,'Color','r','LineWidth',2)
xlabel('x')
ylabel('y')
grid on
axis([0,4,0,5])

%% mass center
exp2 = expand(y*exp1);
Qy = int(exp2,x,[xa,xb]);
Qy =double(Qy);
##Qy = double(int(y*sqrt(1 + dydx^2),x,xa,xb));
xG = Qy/L;

exp3 = expand(x*exp1);
Qx = int(exp3,x,[xa,xb]);
Qx = double(Qx);
##Qx = double(int(x*sqrt(1 + dydx^2),x,xa,xb));
yG = Qx/L;
fprintf('Qx = '),disp(Qx)
fprintf('Qy = '),disp(Qy)
fprintf('xG  [cm] = '),disp(xG)
fprintf('yG  [cm] = '),disp(yG)
hold on
plot(xG,yG,'ro','MarkerFaceColor','r')
```

```
% parallel axis theorem
exp4= expand(y*y*exp1)
IOx = (pi*d^2/4)*rho*int(exp4,x,xa,xb);
IOx = double(IOx);

exp5= expand(x*x*exp1)
IOy = (pi*d^2/4)*rho*int(exp5,x,xa,xb);
IOy = double(IOy);

exp6 =expand((x*x+y*y)*exp1)
IOz = (pi*d^2/4)*rho*int(exp6,x,xa,xb);
IOz = double(IOz);


x0 = 0; y0 = 0;
IGx = IOx - mass*(x0 - xG)^2;
IGy = IOy - mass*(y0 - yG)^2;
fprintf('IGx [g-cm^2] = '),disp(IGx)
fprintf('IGy [g-cm^2] = '),disp(IGy)
```

In Octave Command Window

```
-------------------------------
Properties of a Line
--------------------------------
Symbolic pkg v2.7.1: Python communication link active, SymPy v1.3.
y = (sym) 4*sqrt(x) - 2*x + 1/2
dydx = (sym) -2 + 2/sqrt(x)
exp00 = (sym) 4 + 4/x - 8/sqrt(x)
exp0 = (sym) 5 + 4/x - 8/sqrt(x)
exp1 = (sym) sqrt(5 + 4/x - 8/sqrt(x))
Waiting.........
y(x) =   4*sqrt(x) - 2*x + 1/2
L [cm] =  2.3078
mass [g] =  101.50
Waiting........
Waiting.............
Qx =  4.7272
Qy =  4.7725
xG  [cm] =  2.0680
yG  [cm] =  2.0484
exp4 = (sym) -16*x**(3/2)*sqrt(5 + 4/x - 8/sqrt(x)) + 4*sqrt(x)*sqrt(5 +
4/x - 8/sqrt(x)) + 4*x**2*sqrt(5 + 4/x - 8/sqrt(x
)) + 14*x*sqrt(5 + 4/x - 8/sqrt(x)) + sqrt(5 + 4/x - 8/sqrt(x))/4
Waiting.........
exp5 = (sym) x**2*sqrt(5 + 4/x - 8/sqrt(x))
Waiting..............................
exp6 = (sym) -16*x**(3/2)*sqrt(5 + 4/x - 8/sqrt(x)) + 4*sqrt(x)*sqrt(5 +
4/x - 8/sqrt(x)) + 5*x**2*sqrt(5 + 4/x - 8/sqrt(x
)) + 14*x*sqrt(5 + 4/x - 8/sqrt(x)) + sqrt(5 + 4/x - 8/sqrt(x))/4
Waiting........
IGx [g-cm^2] =  10.991
IGy [g-cm^2] =  33.524
```

The Figure is the same. The results match the one in MATLAB.

2.7.2 An Area

An area is a two dimensional object. Constant thickness makes it three dimensional. The development of the calculations follow the definition for the line object and the equations are quite similar but more terms. For the derivation we chose an area located away from the origin and enclosed by the line y = $\mathbf{y_c}$ on the bottom. The left side is x = $\mathbf{x_a}$ and the right side is defined by x = $\mathbf{x_b}$. The top is the function ***y(x)*** - the same used in the definition of the curve previously. This is the region enclosed by red line in Figure 2.7.6. While this is a specific example it can be easily generalized.

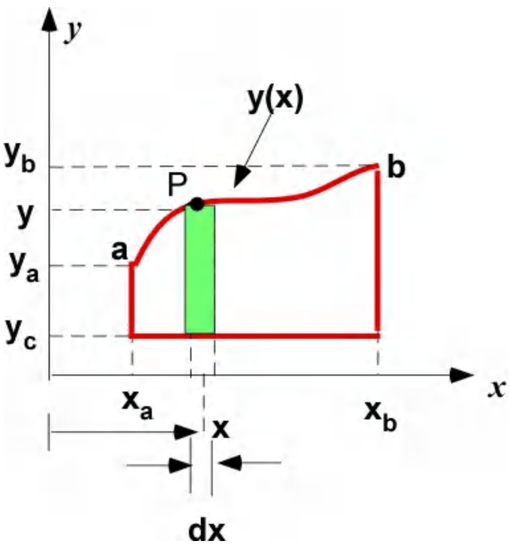


Figure 2.7.6 The area

The Area Calculation

This section deals with plane area in the x-y plane. There are two definitions of area in play here. The Area object and the value for the region covered by the area. The calculation of the area is done through calculus using a differential area (shown in green). It is then integrated over the region.

$$Area = A = \int_A dA = \int_{x_a}^{x_b} (y - y_c)\, dx \tag{2.54}$$

(Start of new set of MATLAB code)

```
xa = 1;  xb = 3;
syms x
y = 0.5 - 2*x + 4*sqrt(x);
ya = double(subs(y,xa));
yb = double(subs(y,xb));
yc = 0.5;
dA = y - yc;
A = double(int((y - yc),x,xa,xb));

fprintf('Position a [xa, ya] = '), disp([xa, ya])
fprintf('Position b [xb, yb] = '), disp([xb, yb])
fprintf('yc  = '),disp(yc);
fprintf('A  = '),disp(A);
```

```
% using patch to draw the object
ezplot(y,[xa,xb])
xx = linspace(xa,xb,11);
yy = double(subs(y,x,xx));
xp = [xa,xx,xb];
yp = [yc,yy,yc];
patch(xp,yp,'y','LineWidth',2)
axis([0,4, 0,4])
grid on
set(gcf,'Position',[50,50,300,250])
xlabel('x')
ylabel('y')
text(xa-0.2,ya,'a','FontWeight','b')
text(xb+0.1,yb,'b','FontWeight','b')
```

In the Command Window

```
Position a [xa, ya] =    1.0000e+00   2.5000e+00
Position b [xb, yb] =    3.0000e+00   1.4282e+00
yc  =     5.0000e-01
A   =    3.1897e+00
```

In the Figure Window

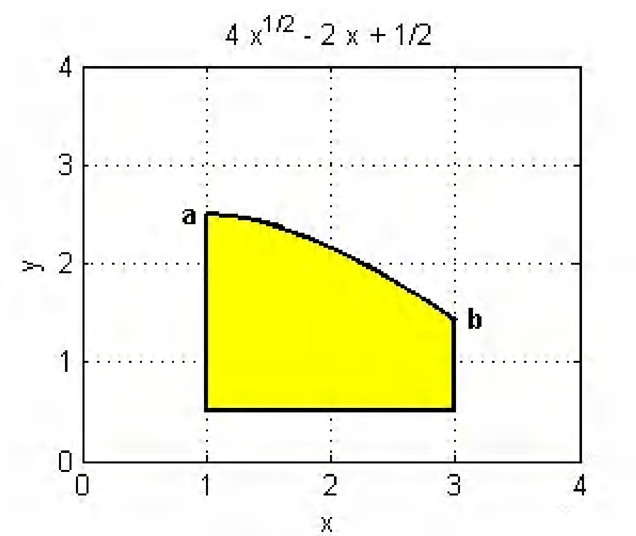


Figure 2.7.7 Area enclosed

The Centroid

The centroid is located with respect to a reference axis. In this example it is the origin. The center of a differential area (green) is located with respect to the reference axis. We then find the moment (first) of this differential area about the reference axis using the center and integrate over the area. You will notice, we have taken advantage of the specific geometry in this example to reduce the area integral to a line integral. For each axis, the location of the centroid is the first moment of area divided by the area. Remember the first moment of area about the x-axis is $\mathbf{Q_x}$ and it determines the **y**-location of the centroid. Here $\mathbf{x_m}$ and $\mathbf{y_m}$ are the center of the differential area.

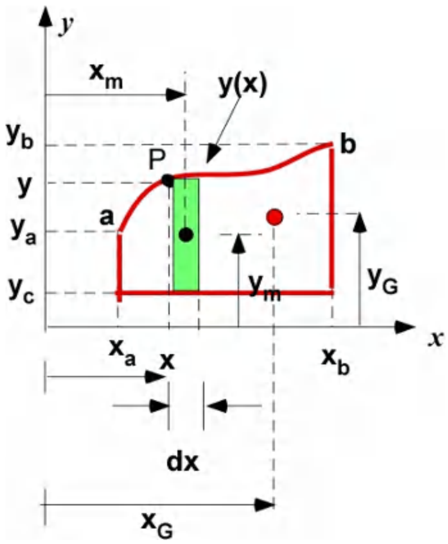


Figure 2.7.8 Calculating the centroid

$$x_G = \frac{\int_A x_m \, dA}{\int_A dA} = \frac{\int_{x_a}^{x_b} x\left(y - y_c\right) dx}{A} = \frac{Q_y}{A}; \quad x_m = x$$

$$y_G = \frac{\int_A y_m \, dA}{\int_A dA} = \frac{\int_{x_a}^{x_b} \left[y_c + \frac{1}{2}\left(y - y_c\right)\right]\left(y - y_c\right) dx}{A} = \frac{Q_x}{A} \qquad (2.55)$$

(Continued MATLAB code)

```
Qy = double(int((y - yc)*x,x,xa,xb));
ym = yc + 0.5*(y-yc);
Qx = double(int((y - yc)*ym,x,xa,xb));
xG = Qy/A;
yG = Qx/A;

fprintf('Qx [cm^3] = '),disp(Qx)
fprintf('Qy [cm^3] = '),disp(Qy)
fprintf('xG  [cm] = '),disp(xG)
fprintf('yG  [cm] = '),disp(yG)
```

In the Command Window

```
Qx =    4.2451e+00
Qy =    6.0082e+00
xG  [cm] =    1.8836e+00
yG  [cm] =    1.3309e+00
```

In the Figure Window

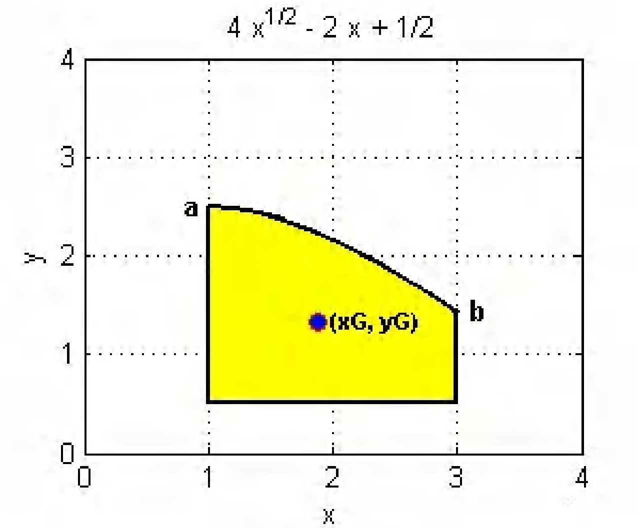


Figure 2.7.9 Centroid of area

Area Moment of Inertia

The moment of inertia (MOI) of the area is the second moment of the area around an axis. It is a direct extension of the first moment of the area. You would just square the distances from the references used in the calculation of Q_y and Q_x. It is also possible to calculate the MOI about the z-axis of the area described in the x-y plane. The symbol $\mathbf{I}$ is continued for moment of inertia. As a reminder, $\mathbf{I_{Ox}}$ is the MOI through the x-axis passing through the origin O. The following relations are for the the area MOI obtained from Figure 11.8. We can then find the MOI about the centroid using the **parallel axis theorem**.

$$x_m = x; \quad y_m = y_c + \frac{1}{2}(y - y_c);$$

$$I_{Oy} = \int_A x_m^2\, dA = \int_{x_a}^{x_b} x^2 (y - y_c)\, dx$$

$$I_{Ox} = \int_A y_m^2\, dA = \int_{x_a}^{x_b} \left[y_c + \frac{1}{2}(y - y_c) \right]^2 (y - y_c)\, dx \qquad (2.56)$$

$$I_{Gy} = I_{Oy} - A(x_G - x_O)^2 = I_{Oy} - A x_G^2$$

$$I_{Gx} = I_{Ox} - A(y_G - y_O)^2 = I_{Ox} - A y_G^2$$

$$I_{Oz} = \int_A \left[x_m^2 + y_m^2 \right] dA = \int_{x_a}^{x_b} \left[x_m^2 + y_m^2 \right](y - y_c)\, dx \qquad (2.57)$$

$$I_{Gz} = I_{Oz} - A\left[(x_G - x_O)^2 + (y_G - y_O)^2 \right] = I_{Oz} - A\left[x_c^2 + y_c^2 \right]$$

(Continued MATLAB code)

```
IOy = double(int((y - yc)*x^2,x,xa,xb));
```

```
IOx = double(int((y - yc)*ym^2,x,xa,xb));
IOz =  double(int((y - yc)*(x^2 + ym^2),x,xa,xb));

IGy = IOy - A*xG^2;
IGx = IOx - A*yG^2;
IGz = IOz - A*(xG^2 + yG^2);
fprintf('IOx [cm^4] = '),disp(IOx)
fprintf('IOy [cm^4] = '),disp(IOy)
fprintf('IOz [cm^2] = '),disp(IOz)

fprintf('IGx [cm^4] = '),disp(IGx)
fprintf('IGy [cm^4] = '),disp(IGy)
fprintf('IGz [cm^2] = '),disp(IGz)
```

In the Command Window

```
IOx [cm^4] =    5.7237e+00
IOy [cm^4] =    1.2303e+01
IOz [cm^4] =    1.8027e+01
IGx [cm^4] =    7.3934e-02
IGy [cm^4] =    9.8623e-01
IGz [cm^4] =    1.0602e+00
```

Product of Inertia
The MOI defined above are the ones you will likely see in the basic courses in engineering. In advanced courses you will require the products of inertia. For two-dimensional objects there is one but for three-dimensional structures there will be three. The product of inertia of the differential area is the area multiplied by the x and y locations from the reference point. Products of inertia can be negative as it is influenced by the location of the reference point. Figure 2.7.8 is used to set up the relations.

$$I_{Oxy} = \int_A x_m y_m dA = \int_{x_a}^{x_b} x\left[y_c + \frac{1}{2}(y - y_c)\right]dx \qquad (2.58)$$

$$I_{Gxy} = I_{Oxy} - A(x_G - x_O)(y_G - y_O)$$

(Continued MATLAB code)

```
IOxy = double(int((y - yc)*x*ym,x,xa,xb));
IGxy = IOxy - A*xG*yG;
```

In the Command Window

```
IOxy [cm^4] =    7.7309e+00
IGxy [cm^4] =   -2.6528e-01
```

Note: Instead of **Area** we could have computed the **Mass** moments and inertia. They will require extensions of the formula not included here but should be straight forward. The sample is available in the previous section dealing with the line.

2.7.3 A Volume

The centroid of the volume or the center of mass of the volume can be similarly defined. These points are located using the appropriate first moments of volume. MOI and products of inertia (second moments of volume) are usually defined with respect to the mass distribution (instead of the area). The volume is a three-dimensional object and therefore there will be a additional calculations because of the additional dimension. An interesting way to create a surface is to revolve a line about an axis. This is called a surface of revolution. Revolving an area would generate a volume of revolution. An example is the simple flower vase which is usually axisymmetric. These are important geometric objects and are very abundant in practice. In this section we will mostly deal with these objects as it is not easy to characterize an arbitrary three-dimensional object, since they will be generated by complex equations. There are many useful volumes that can be generated through revolution and this section focuses on them.

Surface of Revolution

We will start with an arbitrary curve - the solid black line. Let us revolve it about the dotted vertical line for one revolution. The outline in the cross section is shown in black. The area generated by revolving the line is shown below the line in green (was created in a software called Canvas version 10). Also shown on the side is the bottom view of the area so that you can recognize the curve which generated it. The calculation of the surface area is expected and though could have been done in the previous section it is developed here because of the theorem of Pappus-Guldinus. There are two of them. Both of them refer to objects derived by revolving about a line (axis of rotation).

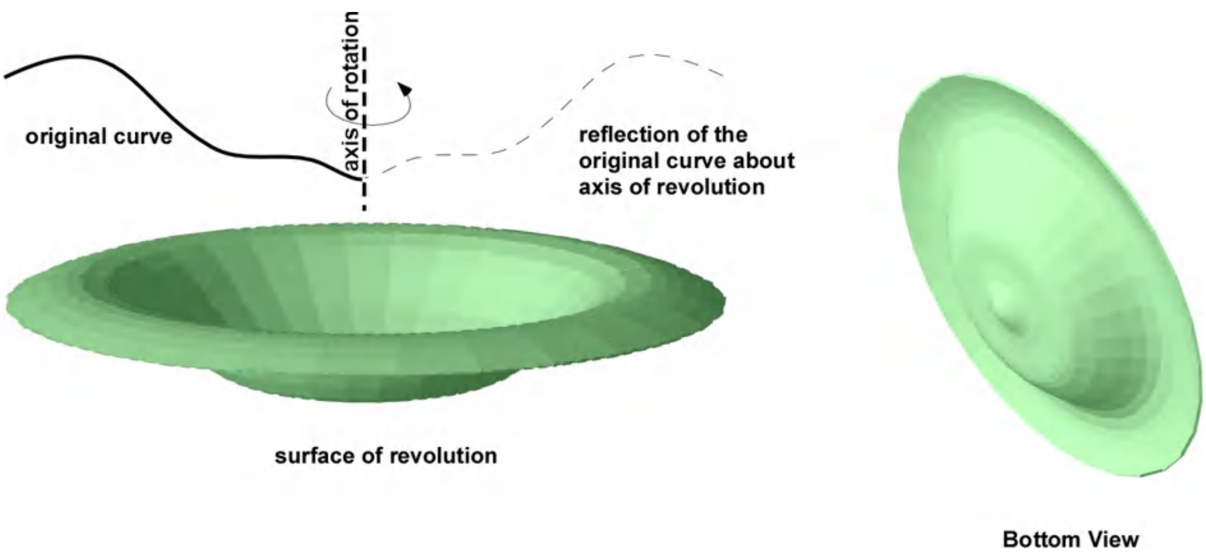


Figure 2.710 Surface of revolution

The First theorem expresses the area obtained by the revolution of a line about an axis. For example the rotation of the line about the x-axis as set up in Figure 2.7.11

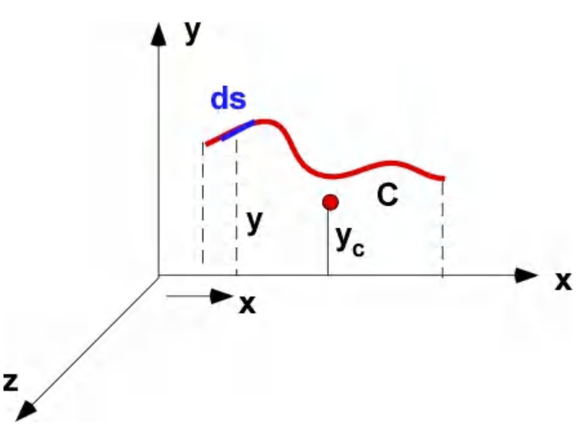


Figure 2.7.11 Surface of revolution

The actual area of the surface of revolution is

$$A = \int_{\theta}^{\theta_2}\int_{x_1}^{x_2} ds\left(y\,d\theta\right) = \int_{\theta}^{\theta_2}\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,d\theta \qquad (2.59)$$

Theorem 1 - Pappus-Guldinus

This area is the product of the length of the line multiplied by the distance traveled by the centroid of the line through one revolution.

$$A = 2\pi y_c L \qquad (2.60)$$

L is the length of the line and $\mathbf{y_c}$ is the distance of the centroid from the axis of rotation. For less than a revolution the factor **2π** in the formula is replaced with the angular travel in radians.

(Start of new set of MATLAB code)

```
syms x th

% let us create the curve that will rotate about x-axis
figure
set(gcf,'Position',[50,500,400,350])
y = 2 + exp(-0.1*x)*cos(2*x);  % the line
ezplot(y,[1,3])  % plot the line
axis([1,3,0,3])

xx = linspace(1,3,31);
tt = linspace(0,2*pi,31);

[X,T] = meshgrid(xx,tt);% mesh to create the surface
Y = double((2 + exp(-0.1*X).*cos(2*X)).*cos(T));
Z = double((2 + exp(-0.1*X).*cos(2*X)).*sin(T));
figure
set(gcf,'Position',[50,60,400,350])
% fancy figure
surf(X,Y,Z, ...
    'FaceColor','interp',...
    'EdgeColor','k',...
    'FaceLighting','phong')
camlight left
view(30,45)
 % label axis so you can see the rotation about x-axis
xlabel('x')
ylabel('y')
zlabel('z')
```

In Figure Window

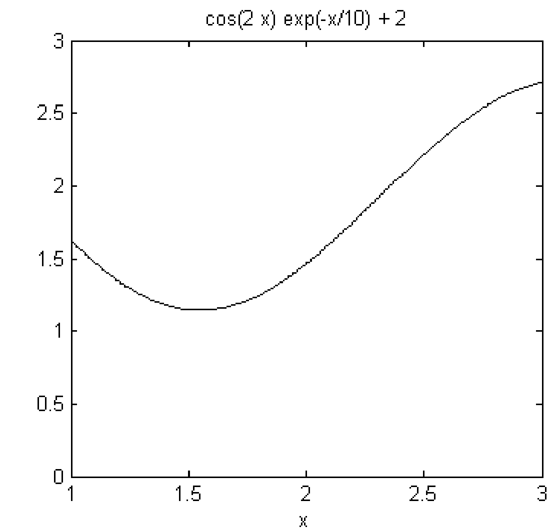


Figure 2.7.12a. The line

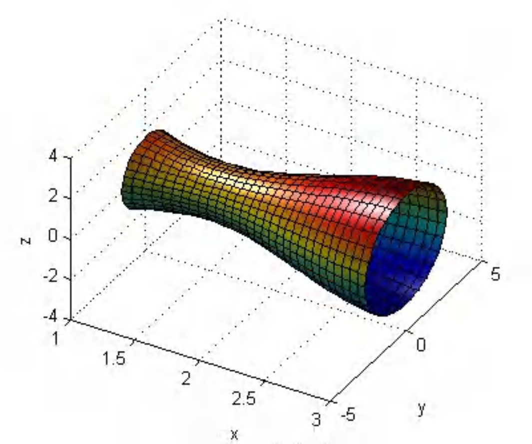


Figure 2.7.12b. The surface of revolution

The curve is rotated about the x-axis. The actual area and the area computed by Theorem 1 is compared below

(Continued MATLAB code)

```
% the centroid of the line (previous code)
dydx = diff(y,x);
L = double(int(sqrt(1 + dydx^2),x,1,3)); % length
fprintf('y(x) = '),disp(y)
fprintf('L = '),disp(L)
% centrod
Qx = double(int(y*sqrt(1 + dydx^2),x,1,3));
Qy = double(int(x*sqrt(1 + dydx^2),x,1,3));
yG = Qx/L;
xG = Qy/L;

fprintf('Qx = '),disp(Qx)
fprintf('Qy = '),disp(Qy)
fprintf('xG  [cm] = '),disp(xG)
fprintf('yG  [cm] = '),disp(yG)
%
Area_int = double(int(int(y*sqrt(1 + dydx^2),x,1,3),th,0,2*pi));
```

```
Area_pap = 2*pi*L*yG;
fprintf('Area - Integrated = '),disp(Area_int)
fprintf('Area - Pappus      = '),disp(Area_pap)
```

In the Command Window

```
y(x)  = cos(2*x)*exp(-x/10) + 2
L =     2.9477e+00
Qx =     5.1820e+00
Qy =     5.9504e+00
xG  =     2.0187e+00
yG  =     1.7580e+00
Area - Integrated =     3.2560e+01
Area - Pappus      =     3.2560e+01
```

Body of Revolution

Revolving an area around an axis will create a volume of revolution. The second theorem of Pappus and Guldinus is associated with computing this volume. To compute the volume by calculus, the differential area shown in green in Figure 2.7.13a is revolved around the x-axis. A differential revolution through the angle **d**θ will generate a pie shaped wedge shown in Figure 2.7.13b.

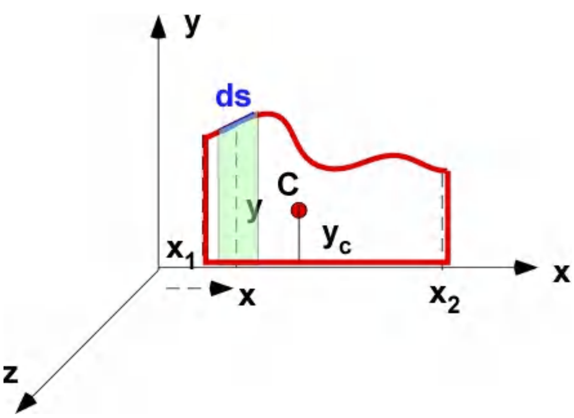


Figure 2.7.13a Volume of revolution

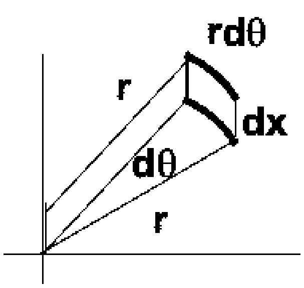


Figure 2.7.13b Pie shaped wedge

Here **r** is the radius of rotation - which is the same as **y** in Figure 2.7.13a. The differential wedge element's volume and the volume of the body of rotation is derived below for a rotation through an angle **d**θ. This leads to the second theorem of Pappus-Guldinus

$$dV = \left(\frac{1}{2} r\, r\, d\theta\right) dx = \left(\frac{1}{2} y\, y\, d\theta\right) dx = \left(\frac{y}{2}\right)(y\,dx)\,d\theta = \left(dQ_x\right) d\theta \qquad (2.61)$$

$$V = \int\int \left(dQ_x\right)\left(d\theta\right) = \int Q_x\, d\theta = Q_x\, \Delta\theta = y_C\, A\, \Delta\theta$$

Theorem 2. Pappus-Guldinus
The volume of the body of rotation is equal to the area generating the volume times the distance traveled by the centroid of the area

We will rotate the area formed by the line in the previous section and the x- axis and revolve it around the x-axis for one revolution.

(Continued MATLAB code)

```
%% volume of revolution
syms x th
y = 2 + exp(-0.1*x)*cos(2*x);  % the line
xx = linspace(1,3,31);
tt = linspace(0,2*pi,31);
y = 2 + exp(-0.1*x)*cos(2*x);  % the line
yy = double(subs(y,x,xx));
patchx = [xx(1) xx xx(end)];
patchy = [0 yy 0];

A = double(int(y,x,1,3));
Qy = double(int(y*x,x,1,3));
ym = 0.5*y;
Qx = double(int(y*ym,x,1,3));
xG = Qy/A;
yG = Qx/A;

V= 2*pi*A*yG;

fprintf('A  = '),disp(A)
fprintf('Qx = '),disp(Qx)
fprintf('Qy = '),disp(Qy)
fprintf('xG = '),disp(xG)
fprintf('yG = '),disp(yG)
fprintf('V  = '),disp(V)

figure
set(gcf,'Position',[50,50,400,350])
% subplot(121)
patch(patchx,patchy,'y','LineWidth',2)
xlabel('x'); ylabel('y');
hold on
plot(xG,yG,'ro','MarkerFaceColor','b')
hold off
text(xG-0.2,yG-0.2,'G')
```

In Command Window

```
A  =    3.4593e+00
Qx =    3.2731e+00
Qy =    7.4616e+00
xG =    2.1570e+00
yG =    9.4618e-01
V  =    2.0566e+01
```

In Figure Window

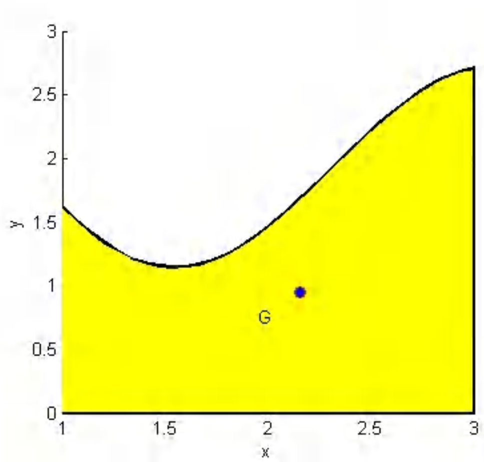


Figure 2.7.14 Area of revolution and centroid

The volume of revolution is the region within the surface in Figure 2.7.12b. At the time of writing MATLAB does not have a way to create a solid volume.

Centroid of Volume

The calculation for the centroid of the volume (mass center), MOI, and product of inertia can be directly extended from the definitions for the line and the area. The differential volume and the total volume are shown in Figure 2.7.15. If the integrals to define the volume can be defined using mathematical functions, then calculus can be used to determine the geometric properties. The differential volume is centered at P and the centroid of the volume is at G. We use the first moment of the volume to establish the centroid and the second moment of volume to calculate the MOI. Note that these are triple integrals. There is very different idea of the *first moment* in this instance. For example, ***xdV***, is the first moment of the differential volume with respect to the *y-z plane*. Previously we required the first moment about a line or an axis. Since this is not about an axis we label this differential moment of volume as $\boldsymbol{Q_{yz}}$. This is very different from $\mathbf{Q_x}$ of the previous sections - where the first moment was about the axis/line. The computation of Qx is shown through Figure 2.7.15 and the equation below.

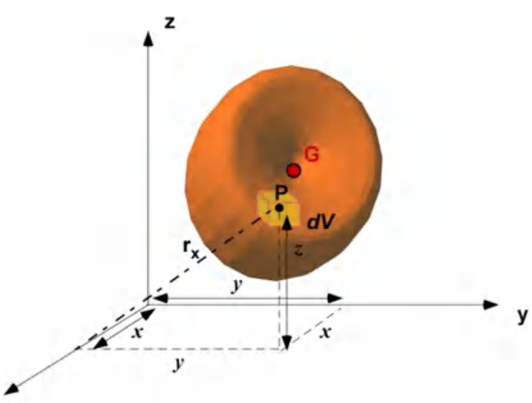


Figure 2.7.15 An arbitrary volume.

$$V = \int dV$$

$$x_G = \frac{\int x dV}{V} = \frac{Q_{yz}}{V}; \quad y_G = \frac{\int y dV}{V} = \frac{Q_{zx}}{V}; \quad z_G = \frac{\int z dV}{V} = \frac{Q_{xy}}{V}; \qquad (2.62)$$

$$Q_x = \int r_x\, dV = \int \sqrt{(y^2 + z^2)}\, dV$$
$$Q_y = \int \sqrt{(x^2 + z^2)}\, dV; \quad Q_z = \int \sqrt{(x^2 + y^2)}\, dV \quad (2.63)$$

Example: Pyramid
The volume and centroid of a pyramid is computed in this illustration. It is a regular pyramid with the base in the yz plane as shown in Figure 2.7.16. The base dimensions are a and b. Symmetry will establish that $y_G = 0$ and $z_G = 0$. Only x_G needs to be calculated. The figure also shows the projection of the pyramid in the xy-plane so that the equations necessary for calculations can be inferred. The rectangular cross-section varies linearly from the base to a value of zero at the tip of the pyramid. The tip lies on the x-axis. At a distance x from the base the sides if the cross-section are a(x) and b(x).

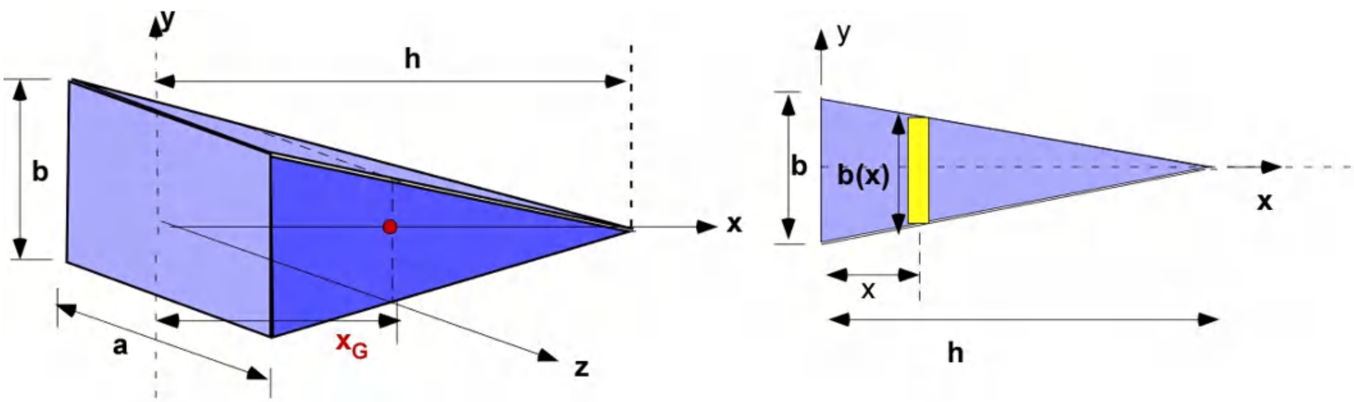


Figure 2.7.16 Regular pyramid and projection

$$\frac{b(x)}{b} = \frac{h-x}{h}; \quad \frac{a(x)}{a} = \frac{h-x}{h};$$
$$V = \int dV = \int_0^h a(x)\, b(x)\, dx \quad (2.64)$$
$$x_G = \frac{\int_0^h a(x)\, b(x)\, x\, dx}{V}$$

(Continued MATLAB code)

```
%% A regular pyramid
syms x a b h
bx = b*(h-x)/h;
ax = a*(h-x)/h;
dV = ax*bx;
V = int(dV,x,0,h);
xG = int(dV*x,x,0,h)/V;
fprintf('V  = '),disp(V)
fprintf('xG = '),disp(xG)
```

In Command Window

```
V  = (a*b*h)/3
xG = h/4
```

The relations above are the formulas for the volume and the location of the centroid for any pyramid (of similar geometry). These are usually available in text books and reference literature.The MATLAB code above is a substitute for a handbook .This is the advantage of symbolic processing. A *fair* knowledge of MATLAB is like having a reference book in many topics.

Moment of Inertia

The mass moment of inertia for a three dimensional object is challenging. There are three MOI about the coordinate axis and three products of inertia about the three planes. This is important in dynamics, particularly the motion of a six degree of freedom system like an aircraft. In statics and mechanics usually the area MOI is sufficient in the previous section is sufficient. To define the various MOI we use the definition in Figure 2.7.15 (*redrawn below*). The differential mass (**dm**) of the differential volume is ρ**dV**.

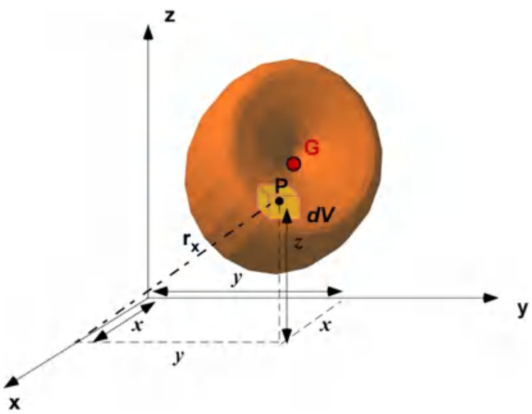


Figure 2.7.15 An arbitrary volume

$$I_x = \int_V (y^2 + z^2)(\rho dV); \quad I_y = \int_V (z^2 + x^2)(\rho dV);$$

$$I_z = \int_V (x^2 + y^2)(\rho dV); \qquad (2.65)$$

$$I_{xy} = \int_V xy(\rho dV); \quad I_{yz} = \int_V yz(\rho dV); \quad I_{zx} = \int_V zx(\rho dV);$$

Returning to the example of the pyramid, the computation of the MOI will be assisted by Figure 2.7.17 that allows the use of the formulas above. The differential volume dV (dx*dy*dz) is located in the positive quadrant at a distance x, y, and z from the origin. The limits of the integration are defined by the geometry and they are critical to the computation of the MOI.

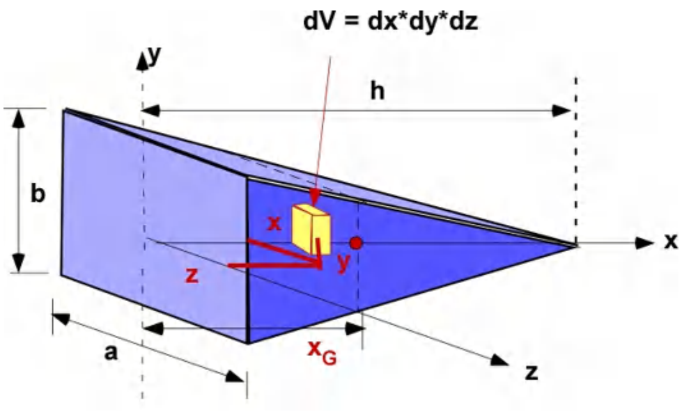


Figure 2.7.17 The differential volume and generic location.

Setting up the computation as a triple integral, I_x can be computed through

$$I_x = \rho \int_{z=0}^{z=h} \int_{y=-b(x)/2}^{y=+b(x)/2} \int_{x=-a(x)/2}^{x=+a(x)/2} \left(y^2 + z^2\right) dz\, dy\, dx \qquad (2.66)$$

(Continued MATLAB code)

```
syms x y z a b h rho
bx = b*(h-x)/h;
ax = a*(h-x)/h;
dV = ax*bx;
V = int(dV,x,0,h);
xG = int(dV*x,x,0,h)/V;

Ix = rho*int(int(int(y^2 + z^2,z,-ax/2,ax/2),y,-bx/2,bx/2),x,0,h);
fprintf('Ix= '),disp(Ix)
```

In the command window

```
Ix= (a*b*h*rho*(a^2 + b^2))/60
```

Once again this is a formula for pyramids whose geometry is defined through values for a, b and h. Other MOI and products of inertia can be similarly set up. The complete set of formulas are obtained below.

(Continued MATLAB code)

```
%% A regular pyramid
syms x y z a b h rho
bx = b*(h-x)/h;
ax = a*(h-x)/h;
dV = ax*bx;
V = int(dV,x,0,h);
xG = int(dV*x,x,0,h)/V;

fprintf('V  = '),disp(V)
fprintf('xG = '),disp(xG)
```

```
Ix = rho*int(int(int(y^2 + z^2,z,-ax/2,ax/2),y,-bx/2,bx/2),x,0,h);
Iy = rho*int(int(int(x^2 + z^2,z,-ax/2,ax/2),y,-bx/2,bx/2),x,0,h);
Iz = rho*int(int(int(x^2 + y^2,z,-ax/2,ax/2),y,-bx/2,bx/2),x,0,h);
Ixy = rho*int(int(int(x*y,z,-ax/2,ax/2),y,-bx/2,bx/2),x,0,h);
Iyz = rho*int(int(int(y*z,z,-ax/2,ax/2),y,-bx/2,bx/2),x,0,h);
Izx = rho*int(int(int(z*x,z,-ax/2,ax/2),y,-bx/2,bx/2),x,0,h);
fprintf('Ix  = '),disp(Ix)
fprintf('Iy  = '),disp(Iy)
fprintf('Iz  = '),disp(Iz)
fprintf('Ixy = '),disp(Ixy)
fprintf('Iyz = '),disp(Iyz)
fprintf('Izx = '),disp(Izx)
```

In the Command Window

```
V  = (a*b*h)/3
xG = h/4
Ix  = (a*b*h*rho*(a^2 + b^2))/60
Iy  = (a*b*h*rho*(a^2 + 2*h^2))/60
Iz  = (a*b*h*rho*(b^2 + 2*h^2))/60
Ixy = 0
Iyz = 0
Izx = 0
```

Execution Using Octave

The code is same as in MATLAB but gathered in a file without duplication. The following statements are included

```
fprintf('-------------------------------------\n')
fprintf('Example Pyramid \n')
fprintf('--------------------------------------\n')
pkg load symbolic
sympref display flat
```

In Octave Command Window

```
------------------------------------
Example Pyramid
-------------------------------------
V  =   a*b*h/3
xG =   h/4

Ix=   rho*(h*(-a**3*b - a*b**3)/4 + h*(a**3*b/12 + a*b**3/12) +
11*h*(a**3*b + a*b**3)/60)

Iy  =   rho*(-a**3*b*h/12 + h*(-a**3*b - 6*a*b*h**2)/12 + h*(a**3*b +
2*a*b*h**2)/6 + h*(a**3*b + 12*a*b*h*
*2)/60)
Iz  =   rho*(-a*b**3*h/12 + h*(-a*b**3 - 6*a*b*h**2)/12 + h*(a*b**3 +
2*a*b*h**2)/6 + h*(a*b**3 + 12*a*b*h*
*2)/60)
Ixy =   0
Iyz =   0
Izx =   0
```

Note: Unlike the code in Section 2.7.1 Octave had no problem executing all the integrations for this example

The products of inertia being zero can be reconciled with instinctive reasoning. If there is a differential volume (dV) that is symmetrically located such that any y,z is balanced by -y,z, then the integration will cancel their contributions to the calculation of the corresponding inertia. A plane of symmetry will yield a zero product of inertia with respect to that plane. You can also determine the MOI about the centroid by using the parallel axis theorem. The axis must be parallel and must pass through the centroid. If *m* is the mass of the object and d_x, d_y and d_z are the distance between the parallel axes, then

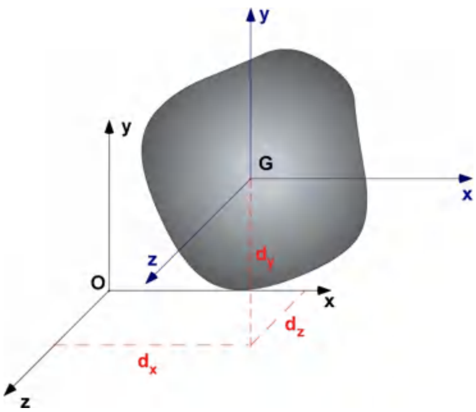


Figure 2.7.18 Parallel axis theorem

$$I_{Ox} = I_{Gx} + m\left(d_y^2 + d_z^2\right); \quad I_{Oy} = I_{Gy} + m\left(d_z^2 + d_x^2\right); \quad I_{Oz} = I_{Gz} + m\left(d_x^2 + d_y^2\right);$$
$$I_{Oxy} = I_{Gxy} + m\,d_x d_y; \quad I_{Oyz} = I_{Gyz} + m\,d_y d_z; \quad I_{Ozx} = I_{Gzx} + m\,d_z d_x;$$
(2.67)

These calculations are possible if there is a continuous distribution of volume. If the volume is made up of discrete components then the integration in all of the formula above is replaced by summation over the discrete volumes involved. This is more practical and is illustrated in the next section.

2.7.4 Additional Problems

There are two suggestions before you attempt these problems.

1. Prior to the attempt of problems it would be useful to evaluate the integral outside of MATLAB to make sure MATLAB is giving you the right results. This will also suggest you set up the computation correctly and programmed the right code. You can use a handful of results to confirm instead of all of them.

2. Collect all of the connected pieces of code in a separate file so that you can run the example by changing a few lines of code.

3. If you are using Octave then please verify other pieces of code in this section.

Problem 2.7.1
Create your own cubic polynomial between x = 0.5 and x = 3. Draw your figure and your coordinate system. Calculate (a) the centroid; (b) the MOI about the coordinate axes; and (c) the MOI about the

centroid. .

Problem 2.7.2
Define your enclosed area using the polynomial in Problem 2.7.1. Draw the figure and the coordinate system. Calculate (a) the centroid; (b) the MOI about the coordinate axes; and (c) the MOI about the centroid.

Problem 2.7.3
Rotate the area in Problem 2.7.2 about the y-axis. Draw your figure and coordinate system. Calculate (a) the volume generated; (b) the total surface area; (c) the centroid of the volume; and (d) the MOI about the axis of rotation.

Problem 2.7.4
Define your own volume. Draw the figure. Calculate (a) the volume; (b) the total surface area; (c) the centroid of the volume; and (d) the MOI of the volume about the coordinate axis.

2.8 INERTIA OF DISCRETE GEOMETRIES

Practical designs often involve complicated objects that are composed of simple geometry. In the previous section we obtained the centroid and inertia using mathematical integration. That was because the geometry was described by continuous functions. In Engineering very often a good estimate of the geometrical properties is sufficient for design. Very often manufacturing constraints dictates simple geometries. In practice, our geometry can be constructed of addition of piecewise simple shapes. In this case we translate the integration to a summation over the various individual geometries that collective give form to the object. This is true of an object comprised of wire frame elements, areas, and volumes.The properties of these simple objects are known or can be easily obtained. In this instance it will be difficult to compute area and mass properties through mathematical integration that was used in the previous section. Using summation rather than integration is an engineering approach to develop geometric information of a composite object. Figure 2.8.1a are simple examples of a wire frame, an two-dimensional object whose area can be broken into simpler geometry, and a volume that can be composed of simpler geometry.

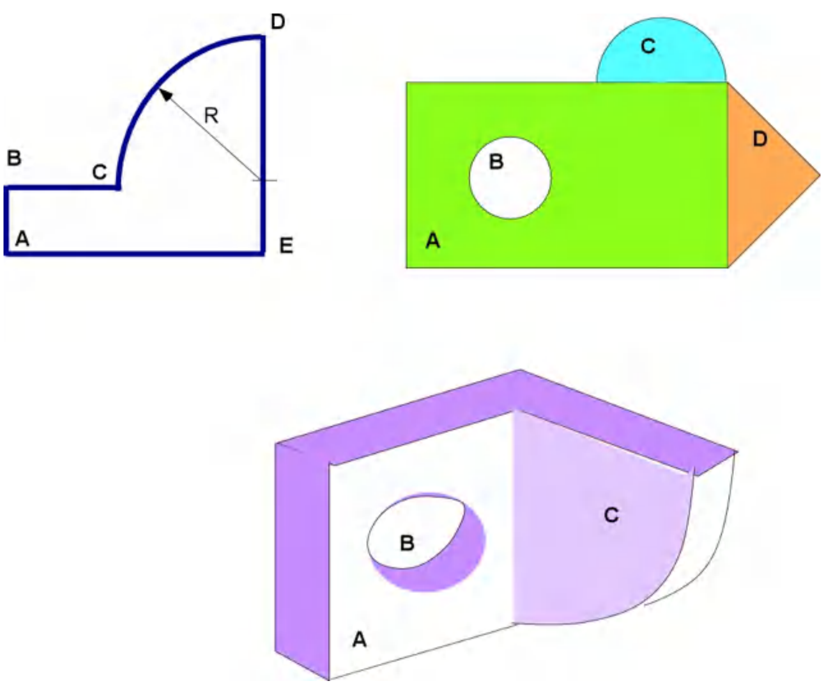


Figure 2.8.1a Examples of composite objects

Figure 2.8.1b is a composite cross-section made of two triangles and a rectangle identified as Objects A, B, and C.The individual areas and the location of the respective centroids are known from some reference - the XY axis. The areas of the elements are A_A, A_B, and A_C. The procedure for obtaining the centroid **(x', y')** of the composite area is illustrated below.

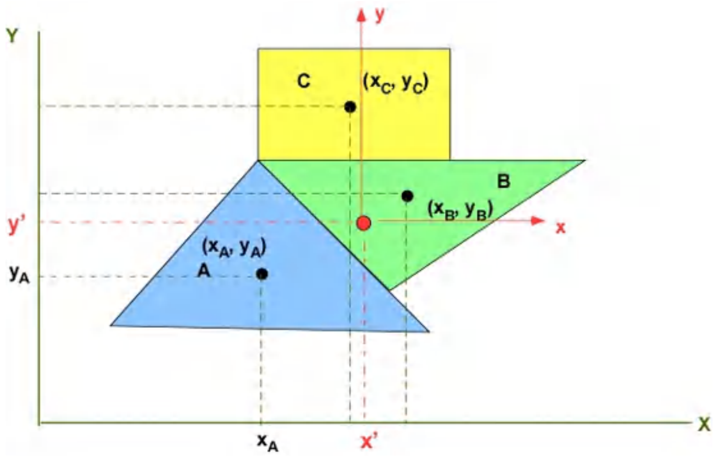


Figure 2.8.1b A composite area object

$$x' = \frac{x_A A_A + x_B A_B + x_C A_C}{A_A + A_B + A_C}$$
$$y' = \frac{y_A A_A + y_B A_B + y_C A_C}{A_A + A_B + A_C} \qquad (2.68)$$

Next, the area MOI of each element is known about its own centroid. For example $\mathbf{I_{xA}}$ is the moment of inertia (MOI) of the area A along the x axis through the centroid of A. Similarly $\mathbf{I_{yA}}$ is the MOI of area A about the y axis passing through the centroid of A. The ***parallel axis theorem*** can be profitably employed to calculate the composite MOI. The relations for applying this theorem to calculate the MOI of the element A about the overall centroid axis **(x', y') is**

$$I_{x'A} = I_{xA} + A_A \left(y' - y_A\right)^2$$
$$I_{y'A} = I_{yA} + A_A \left(x' - x_A\right)^2 \qquad (2.69)$$

Now, the composite MOI over all of the elements about the centroidal axis is:

$$I_{x'x'} = \left[I_{xA} + A_A \left(y' - y_A\right)^2\right] + \left[I_{xB} + A_B \left(y' - y_B\right)^2\right] + \left[I_{xC} + A_C \left(y' - y_C\right)^2\right]$$
$$I_{y'y'} = \left[I_{yA} + A_A \left(x' - x_A\right)^2\right] + \left[I_{yB} + A_B \left(x' - x_B\right)^2\right] + \left[I_{yC} + A_C \left(x' - x_C\right)^2\right] \qquad (2.70)$$

The above calculations were made using the area of the objects. The extension to mass center and mass MOI is direct and no examples are necessary.

In the following we will pursue calculations with respect to the three examples in Figure 2.8.1a. Note that the calculations are strictly problem dependent. It is the procedure that will remain consistent with each class of problems. One more assumption in the following is that the mass center or the centroid are known for the individual elements that make up the composite piece . A hole will be subtracted in the summation. It is the standard practice to tabulate the calculations for consistency and clarity. In the calculations below we will calculate the moment of inertia about the major axes. We will not be calculating the products of inertia. The extension to the products of inertia calculation are left as an exercise.

2.8.1 The Line Object

The wire frame is the simplest object in this section. The overall object is made up of linear or curved elements in two-dimensional space. There are five elements as seen in Figure 2.8.2a. The dimension and the location of the mass center for each element is also marked in the figure. They are at the middle for the straight line elements. We will assume that the area of cross-section is the same for all elements and the density is uniform so that we can work with the mass per unit length. This is represented by the symbol *m*.

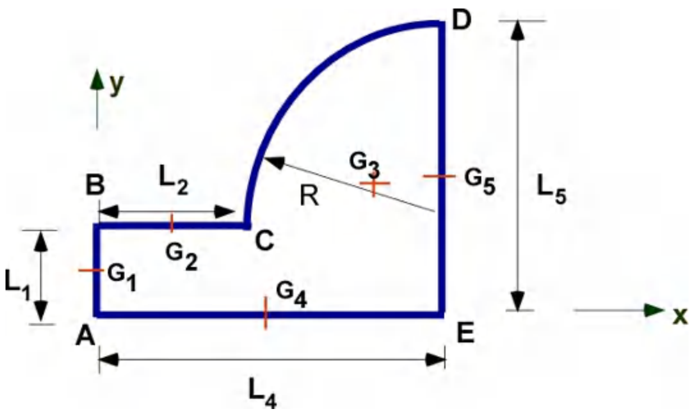


Figure 2.8.2a Composite line object

The 5 line objects are addressed through the the index i that varies from 1 to 5. The center of mass of the element m_i is located at the point shown by the letter G_i. This is at the mid point of the line, except for the quarter circle. For the quarter circle the center of mass is 2R/π from the center of the circular arc along both axis. The uniform mass per unit length is ***m***. The mass of the structure is M. The origin of the coordinate system is at the point A in Figure 2.8.2a. For the calculation of the centroid all locations are from the origin can be calculated. For computing the MOI of the composite structure it is presumed that the MOI of each of the composite elements about its own centroid is known. For the straight segments the MOI of the line about its center is $(1/12)ML^2$, where L is the length of the object.. For the circular arc, the centroid location is easily available through Wikipedia and eFunda, but the MOI is difficult to track down. The formula for the MOI for the quarter circle is obtained below.

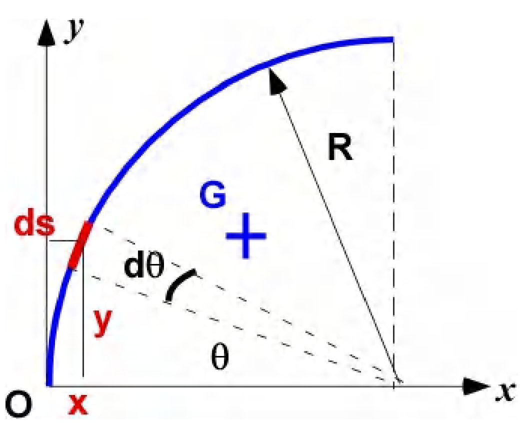


Figure 2.8.2b MOI of an arc

The mass per unit length is **m**. The differential mass is **dm.** Since m is constant we can establish the centroid using length.

$$L=\int ds=\int_0^{\pi/2} R\,d\theta; \quad M=mL;$$

$$x=R-R\cos\theta; \quad y=R\sin\theta$$

$$Q_y=\int x\,ds=\int_0^{\pi/2}(R-R\cos\theta)R\,d\theta; \quad x_G=\frac{Q_y}{L};$$

$$Q_x=\int y\,ds=\int_0^{\pi/2} R\sin\theta\,R\,d\theta; \quad y_G=\frac{Q_x}{L}; \tag{2.71}$$

$$I_{Ox}=\int y^2 m\,ds=\int_0^{\pi/2} y^2 mR\,d\theta; \quad I_{Ox}=I_{Gx}+My_G^2$$

$$I_{Oy}=\int x^2 m\,ds=\int_0^{\pi/2} x^2 mR\,d\theta; \quad I_{Oy}=I_{Gy}+Mx_G^2$$

(Start of new set of MATLAB code)

```
% MOI of an arc

syms R tht m

%% calculate the centroid of the arc
L = int(R,tht,0,pi/2);
x = R - R*cos(tht);   y = R*sin(tht);
Qy = int(x*R,tht,0,pi/2);
xG = Qy/L;
Qx= int(y*R,tht,0,pi/2);
yG = Qx/L;
fprintf('L    = '),disp(L)
fprintf('Qx   = '),disp(Qx)
fprintf('Qy   = '),disp(Qy)
fprintf('xG   = '),disp(xG)
fprintf('yG   = '),disp(yG)

%% MOI of the arc
IOx = int(m*y^2*R,tht,0,pi/2);
IOx = simplify(IOx);
IOy = int(m*x^2*R,tht,0,pi/2);
IOy = simplify(IOy);
IGx = IOx - m*L*yG^2;  % parallel axis
IGx = simplify(IGx);
IGy = IOy - m*L*xG^2;
IGy = simplify(IGy);
fprintf('IOx  = '),disp(IOx)
fprintf('IOy  = '),disp(IOy)
fprintf('IGx = '),disp(IGx)
fprintf('IGy = '),disp(IGy)
```

In the Command Window

```
L    = (pi*R)/2
Qx   = R^2
Qy   = (R^2*(pi - 2))/2
xG   = (R*(pi - 2))/pi
yG   = (2*R)/pi
IOx   = (pi*R^3*m)/4
IOy   = (R^3*m*(3*pi - 8))/4
IGx = (R^3*m*(pi^2 - 8))/(4*pi)
IGy = (R^3*m*(pi^2 - 8))/(4*pi)
```

This information is useful for calculating the MOI for the composite object. Using Figure 2.8.2a

$$M = m_1 + m_2 + m_3 + m_4 + m_5 = \sum_{i=1}^{5} m_i$$
$$= mL_1 + mL_2 + m\left(\frac{1}{4} 2\pi R\right) + mL_3 + mL_4 \qquad (2.72)$$

Th center of mass (located with respect to the point A) is computed through:

$$x_G = \frac{G_{1x}m_1 + G_{2x}m_2 + G_{3x}m_3 + G_{4x}m_4 + G_{5x}m_5}{M} = \frac{\sum_{i=1}^{5} G_{ix} m_i}{M}$$
$$y_G = \frac{G_{1y}m_1 + G_{2y}m_2 + G_{3y}m_3 + G_{4y}m_4 + G_{5y}m_5}{M} = \frac{\sum_{i=1}^{5} G_{ix} m_i}{M} \qquad (2.73)$$

If the MOI about the centroid is known for each line object then

$$I_{Ax} = \sum_{i=1}^{5} \left[I_{Gx_i} + m_i y_{Gi}^2 \right]$$
$$I_{Ay} = \sum_{i=1}^{5} \left[I_{Gy_i} + m_i x_{Gi}^2 \right] \qquad (2.74)$$

The line object in Figure 2.8.2a is assigned values for the five segments for an illustrative calculation. It is available in the code below. The MOI about the axes through the origin are obtained through the following code:

(Continued MATLAB code)

```
% Composite MOI

L(1) = 0.1; L(2)= 0.3; R = 0.5;
L(4) = L(2) + R; L(5) = L(1) + R; L(3) = 2*pi*R/4;
L
m = 2;  me = m*L
```

```
M = sum(me);
xg = [0,0.5*L(2),(L(4)-(2*R/pi)),0.5*L(4),L(4)]
yg = [0.5*L(1),L(1),(L(1) +(2*R/pi)),0,0.5*L(5)]
XG = sum(xg.*me)/M
YG = sum(yg.*me)/M

% MOI of elements about their G
moixg = [me(1).*L(1)^2/12,0,(R^3*m*(pi^2 - 8))/(4*pi), ...
    0,me(5).*L(5)^2/12]
moiyg = [0,me(2).*L(2)^2/12,(R^3*m*(pi^2 - 8))/(4*pi), ...
    me(4).*L(4)^2/12,0]

IAx = sum(moixg + me.*yg.^2)
IAy = sum(moiyg + me.*xg.^2)
```

In the command window

```
L =
    0.1000    0.3000    0.7854    0.8000    0.6000
me =
    0.2000    0.6000    1.5708    1.6000    1.2000
xg =
         0    0.1500    0.4817    0.4000    0.8000
yg =
    0.0500    0.1000    0.4183         0    0.3000
XG =
    0.4732
YG =
    0.2102
moixg =
    0.0002         0    0.0372         0    0.0360
moiyg =
         0    0.0045    0.0372    0.0853         0
IAx =
    0.4627
IAy =
    1.5290
```

2.8.2 Composite Area

The composite area object is the second example. It is an assembly of standard two dimensional shapes whose centroid and moment of inertia about the centroid is available. This information is used to establish the geometrical properties of the composite object. A more detailed geometry of the area object from Figure 2.8.1a is laid out in Figure 2.8.3. There are four standard area objects. The rectangle A with sides **a** and **b**, the hole B of radius $\mathbf{r_B}$, The semicircle C of radius $\mathbf{r_C}$, and the isosceles triangle D of height **h** as shown. The location of the centroids are also marked on the figure with +. The y location of the centroid of B and D are at the mid point f the side **b**.

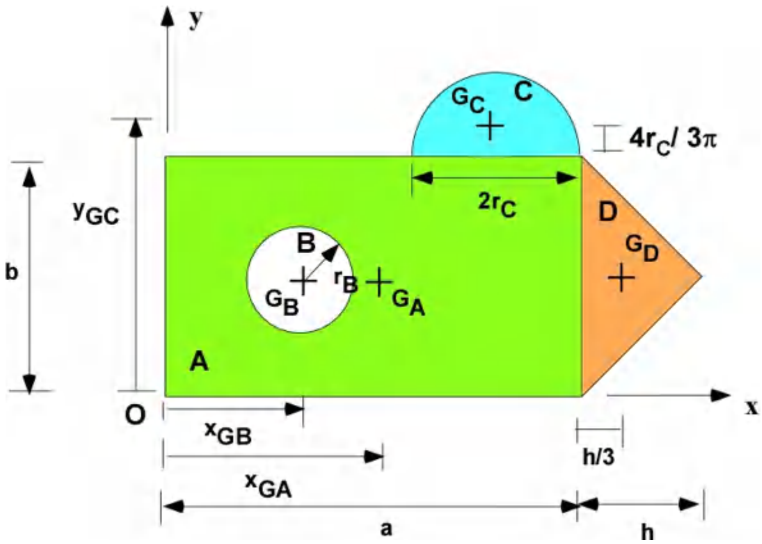


Figure 2.8.3 Composite area object

The calculations of the centroid and MOI are with respect to the area with the assumption of uniform mass per unit area. The area of the hole is negative. In this example the formulas are not recorded separately but are incorporated into the MATLAB code. You can find these formulas through the Internet. The approach is similar to the line object except the calculations are based on area rather than mass. The assumption of unit depth allows direct extension to plate objects. The code below is long but note that it is the same for each object. You could have taken advantage of vector operations but it will be accompanied with some loss of clarity, particularly if the formulas are not explicitly developed or expressed. You can always delete the semi colon to see other values.

(Start of new set of MATLAB code)

```
% MOI of Composite areas
% given values
a = 1; b = 2; rB = 0.5; rC = 0.7; h = 0.8;
fprintf('a  = '),disp(a)
fprintf('b  = '),disp(b)
fprintf('h  = '),disp(h)
fprintf('rB = '),disp(rB)
fprintf('rC = '),disp(rC)
% derived values
AreaA = a*b;  AreaB = -pi*rB^2;  AreaC = pi*rC^2/2;  AreaD = 0.5*b*h;
AT = AreaA + AreaB + AreaC + AreaD;
fprintf('Area A  = '),disp(AreaA)
fprintf('Area B  = '),disp(AreaB)
fprintf('Area C  = '),disp(AreaC)
fprintf('Area D  = '),disp(AreaD)
fprintf('Total Area : '),disp(AT)
% Centroid location
xga = a/2;  yga = b/2;  xgb = a/3;  ygb = b/2;
xgc = a-rC;  ygc = b + (4*rC/3/pi);
xgd = a+ h/3;  ygd = b/2;

% Composite centroid
xG = ((xga*AreaA)+(xgb*AreaB) + (xgc*AreaC) +(xgd*AreaD))/AT;
yG = ((yga*AreaA)+(ygb*AreaB) + (ygc*AreaC) +(ygd*AreaD))/AT;
fprintf('xG  = '),disp(xG)
fprintf('yG  = '),disp(yG)
% MOI of each piece about its centroid
```

```
IAx = a*b^3/12;  IAy = b*a^3/12;  IAz = (a*b/12)*(a^2 + b^2);
IBx = -pi*rB^4/4; IBy = -pi*rB^4/4; IBz = -pi*rB^4/2;
ICx = (pi/8 - (8/9/pi))*rC^4; ICy = pi*rC^4/8;
ICz = (pi*rC^4/4)- (pi*rC^2/2)*(4*rC/3/pi)^2;
IDx =  2*h*(b/2)^3;  IDy = b*h^3/36;  IDz = sqrt(IDx^2 + IDy^2);

% MOI of the Composite area about its centroid
IAxG = IAx + AreaA*(yG - yga)^2;
IBxG = IBx + AreaB*(yG - ygb)^2;
ICxG = ICx + AreaC*(yG - ygc)^2;
IDxG = IDx + AreaD*(yG - ygd)^2;
IGx = IAxG + IBxG + ICxG + IDxG;

IAyG = IAy + AreaA*(xG - xga)^2;
IByG = IBy + AreaB*(xG - xgb)^2;
ICyG = ICy + AreaC*(xG - xgc)^2;
IDyG = IDy + AreaD*(xG - xgd)^2;
IGy = IAyG + IByG + ICyG + IDxG;

IAzG = IAz + AreaA*((yG - yga)^2+(xG - xga)^2);
IBzG = IBz + AreaB*((yG - ygb)^2+(xG - xgb)^2);
ICzG = ICz + AreaC*((yG - ygc)^2+(xG - xgc)^2);
IDzG = IDz + AreaD*((yG - ygd)^2+(xG - xgd)^2);
IGz = IAzG + IBzG + ICzG + IDzG;

fprintf('IGx  = '),disp(IGx)
fprintf('IGy  = '),disp(IGy)
fprintf('IGz  = '),disp(IGz)
```

In the Command Window

```
a   =      1
b   =      2
h   =     0.8000
rB  =     0.5000
rC  =     0.7000
Area A  =      2
Area B  =    -0.7854
Area C  =     0.7697
Area D  =     0.8000
Total Area :     2.7843
xG   =     0.7120
yG   =     1.3586
IGx  =     3.1809
IGy  =     2.0227
IGz  =     3.7471
```

2.8.3 Composite Area - Inclined

This is an example of composite area with an inclined edge. The sides all have the same thickness **t** and the inclination of the two sides is 45 degrees. The cross-section is symmetric. The relationship among the sides and the thickness are shown in the figure. Such connections can also be established for other inclinations and non symmetric composite area. Let t = 5 mm and b = 60 mm.

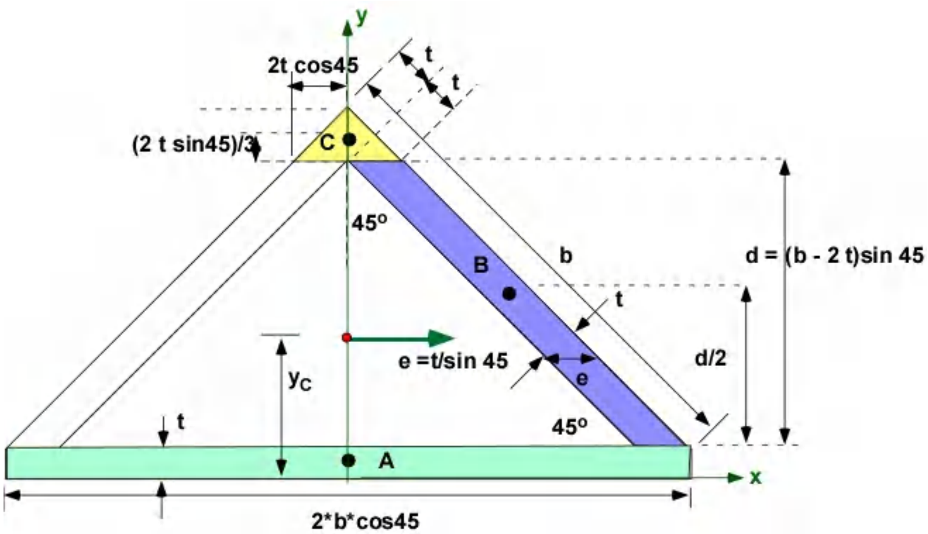


Figure 2.8.4a Inclined composite area

Take a moment to view the various relations about the sides and the thickness. Since the object has an axis of symmetry we know that the x-location of the centroid is at the center of the cross-section. We only need to calculate the y location. The area of the parallelogram is the base multiplied by the height and the area of the triangle is one-half the base multiplied by the height. We will use MATLAB to calculate the centroid and the moment of inertia about the centroid. The values for some of the geometric elements, the areas and the centroid location from the base are reported below. If you calculate different numbers let me know.

Establish the following values

```
t  [mm] = 5.00
b  [mm] = 60.00
d  [mm] = 35.36
e  [mm] = 7.07
ya [mm] = 2.50
yb [mm] = 22.68
yc [mm] = 42.71
Aa [mm2] = 424.26
Ab [mm2] = 250.00
Ac [mm2] = 50.00
```

(Start of new set of MATLAB code)

```
yC = (ya*Aa + 2*yb*Ab + yc*Ac)/(Aa + 2*Ab + Ac)
yC [mm] = 14.92

The MOI of Base
IAx at yC [mm^4] = 66319.522
IAy at yC [mm^4] = 254558.441
```

The MOI for the triangle is available in Section 2.9

```
The MOI of the top triangle
ICx at yC [mm^4] = 38626.133
Iyx at yC [mm^4] = 1666.667
```

The calculation of the MOI of the inclined plate is more involved. Using Figure 2.8.4b

$$I_y = \int_0^d \left[\frac{e^3}{12} + e\left(x(y) + \frac{e}{2} \right)^2 \right] dy \tag{2.75}$$

$$I_{xC} = \frac{e d^3}{12} + e d \left(y_b - y_C \right)^2$$

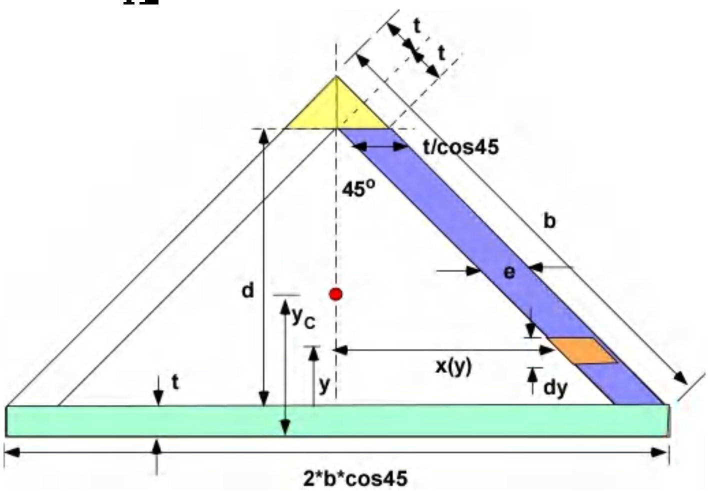


Figure 2.8.4b MOI of inclined plate

```
The MOI of the inclined member
IBx at yC [mm^4] = 82181.273
IBy at yC [mm^4] = 279166.67
```

2.8.4 Composite Volume

The composite volume provided by the third example in Figure 2.8.1 is three dimensional. There should more calculations to compute all of the Inertias. The volume is formed through the addition of a rectangular plate, a hole, and a quarter circular plate. Once again the center of mass of each of these pieces and their MOI about their own center of mass is known through other sources. This is a three dimensional problem. It is almost the same as composite area example except that the circular plate is at right angles to the rectangular plate.The example from Figure 2.8.1 is refurbished with dimensional symbols which will appear in the formula and calculation. The hole is aligned with the center of the rectangular plate. The mass is uniform for the entire object and the mass per unit volume is *m*.

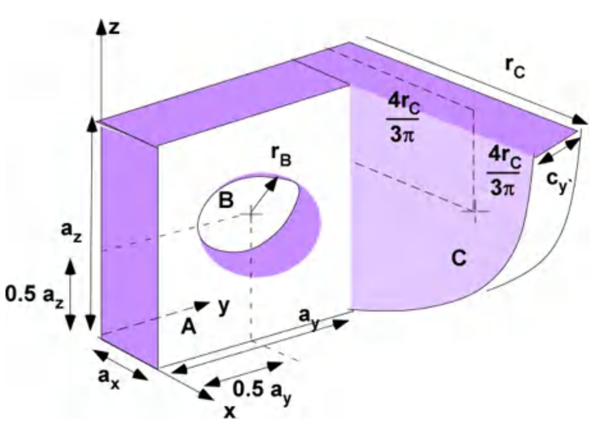


Figure 2.8. 5 Composite Volume

The formulas are again embedded in the MATLAB code as in the previous section. In Figure 2.9.5

the location of the center of mass of each volume is shown. As a reminder it is in the plane through the center of the constant thickness. The center of volume of the composite object is obtained below:

(Start of new set of MATLAB code)

```
% Center of mass and MOI of Composite volumes  -

% given values
ax = 0.5; ay = 1;  az = 1.5; rB = 0.25; rC = 1.5; cy = 0.5;  m = 2;
fprintf('ax  = '),disp(ax)
fprintf('ay  = '),disp(ay)
fprintf('az  = '),disp(az)
fprintf('cy  = '),disp(cy)
fprintf('rB  = '),disp(rB)
fprintf('rC  = '),disp(rC)
fprintf('m   = '),disp(m)

%% derived values
fprintf('\nMass Calculations\n')
massA = m*ax*ay*az;  massB = -m*pi*rB^2*ax;  massC = m*cy*pi*rC^2/4;
mT = massA + massB + massC;
fprintf('Mass A  = '),disp(massA)
fprintf('Mass B  = '),disp(massB)
fprintf('Mass C  = '),disp(massC)
fprintf('Total Mass : '),disp(mT)

%% Centroid location
xga = ax/2;  yga = ay/2;  zga= az/2;
xgb = ax/2;  ygb = ay/2;  zgb = az/2;
xgc = 4*rC/3/pi;  ygc = ay + cy/2;  zgc = rC -4*rC/3/pi;
fprintf('\nLocal Center of mass\n')
fprintf('Centroid A : xga = %4.2f;  yga = %4.2f;  zga = %4.2f;\n',...
    xga,yga,zga)
fprintf('Centroid B : xgb = %4.2f;  ygb = %4.2f;  zgb = %4.2f;\n',...
    xgb,ygb,zgb)
fprintf('Centroid C : xgc = %4.2f;  ygc = %4.2f;  zgc = %4.2f;\n',...
    xgc,ygc,zgc)

fprintf('\nComposite Center of mass\n')
xG = (massA*xga + massB*xgb + massC*xgc)/mT;
yG = (massA*yga + massB*ygb + massC*ygc)/mT;
zG = (massA*zga + massB*zgb + massC*zgc)/mT;
fprintf('Centroid xG = '),disp(xG)
fprintf('Centroid yG = '),disp(yG)
fprintf('Centroid zG = '),disp(zG)
```

In the Command Window

```
ax  =     0.5000
ay  =      1
az  =     1.5000
cy  =     0.5000
rB  =     0.2500
rC  =     1.5000
m   =      2

Mass Calculations
```

```
Mass A   =     1.5000
Mass B   =    -0.1963
Mass C   =     1.7671
Total Mass :      3.0708

Local Center of mass
Centroid A : xga = 0.25;  yga = 0.50;  zga = 0.75;
Centroid B : xgb = 0.25;  ygb = 0.50;  zgb = 0.75;
Centroid C : xgc = 0.64;  ygc = 1.25;  zgc = 0.86;

Composite Center of mass
Centroid xG =     0.4725
Centroid yG =     0.9316
Centroid zG =     0.8152
```

The MOI about the center of mass of the composite object is obtained using the parallel axis theorem. The MOI of the rectangular and circular area, abouts its own center of mass is readily available through a simple search. However the information about the quarter circle cylinder is difficult to track down. The MOI for a quarter circle length is detailed in a previous section. We can extend it to the quarter cylinder. Here the information is explicitly made available through the formula below and Figure 12.6 which is defined to coincide with composite volume under discussion. The mass of the quarter cylinder is $M_c = m*(\pi*r_c^2/4)*c_y$.

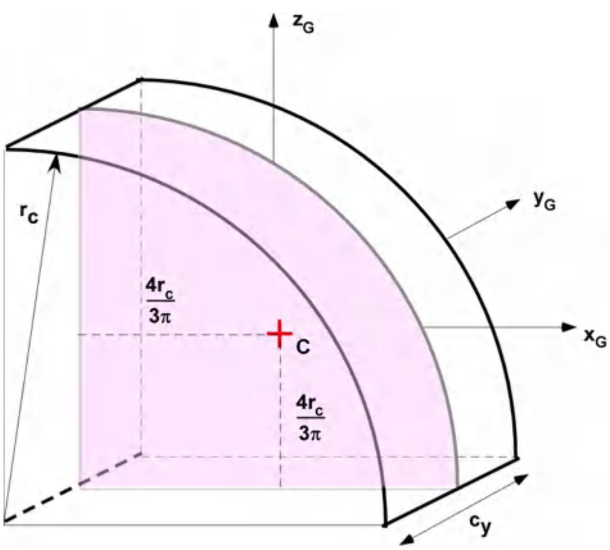


Figure 2.8.6 Quarter cylinder

$$I_{xc} = M_c\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)r_C^2 + \frac{M_c}{12}c_y^2$$

$$I_{zc} = M_c\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)r_C^2 + \frac{M_c}{12}c_y^2 \qquad (2.76)$$

$$I_{yc} = M_c\left(\frac{1}{2} - \frac{32}{9\pi^2}\right)r_C^2$$

(Continued MATLAB code)

```
% square cylinder about its mass center
IAxg = massA*(az^2+ay^2)/12;
IAyg = massA*(ax^2+az^2)/12;
IAzg = massA*(ax^2+ay^2)/12;
fprintf('MOI of A about xga = '),disp(IAxg)
fprintf('MOI of A about yga = '),disp(IAyg)
fprintf('MOI of A about zga = '),disp(IAzg)
% circle about its mass center
IBxg = massB*rB^2/2;
IByg = massB*(rB^2/4) + massB*ax^2/12;
IBzg = massB*(rB^2/4) + massB*ax^2/12;
fprintf('\nMOI of B about xgb = '),disp(IBxg)
fprintf('MOI of B about ygb = '),disp(IByg)
fprintf('MOI of B about zgb = '),disp(IBzg)
% quarter cylinder abouts its mass center
ICxg = massC*((1/4)-(16/9/pi^2))*rC^2 + (massC*cy^2/12);
ICyg = massC*((1/4)-(16/9/pi^2))*rC^2 + (massC*cy^2/12);
ICzg = massC*((1/2)-(32/9/pi^2))*rC^2;
fprintf('\nMOI of C about xgc = '),disp(ICxg)
fprintf('MOI of C about ygc = '),disp(ICyg)
fprintf('MOI of C about zgc = '),disp(ICzg)

%% Composite MOI
IxG = IAxg + massA*((yG-yga)^2+(zG-zga)^2)+ ...
      IBxg + massB*((yG-ygb)^2+(zG-zgb)^2)+ ...
      ICxg + massC*((yG-ygc)^2+(zG-zgc)^2);

IyG = IAyg + massA*((xG-xga)^2+(zG-zga)^2)+ ...
      IByg + massB*((xG-xgb)^2+(zG-zgb)^2)+ ...
      ICyg + massC*((xG-xgc)^2+(zG-zgc)^2);

IzG = IAzg + massA*((xG-xga)^2+(yG-yga)^2)+ ...
      IBzg + massB*((xG-xgb)^2+(yG-ygb)^2)+ ...
      ICzg + massC*((xG-xgc)^2+(yG-ygc)^2);

fprintf('\nMOI of Composite Volume about xG= '),disp(IxG)
fprintf('MOI of Composite Volume about yG= '),disp(IyG)
fprintf('MOI of Composite Volume about zG= '),disp(IzG)
```

In the Command Window

```
MOI of A about xga =     0.4063
MOI of A about yga =     0.3125
MOI of A about zga =     0.1563

MOI of B about xgb =    -0.0061
MOI of B about ygb =    -0.0072
MOI of B about zgb =    -0.0072

MOI of C about xgc =     0.3146
MOI of C about ygc =     0.3146
MOI of C about zgc =     0.5556

MOI of Composite Volume about xG=     1.1464
MOI of Composite Volume about yG=     0.7418
MOI of Composite Volume about zG=     1.2389
```

The products of inertia are not calculated here.

Execution in Octave

The code is same as in MATLAB but consolidated in a file without duplication. The following statements are included

In Octave Editor

```
fprintf('------------------------------------\n')
fprintf('Example Composite Volume\n')
fprintf('-------------------------------------\n')
pkg load symbolic
sympref display flat
```

In Octave Command Window

```
------------------------------------
Example Composite Volume
-------------------------------------
ax  =  0.50000
ay  =  1
az  =  1.5000
cy  =  0.50000
rB  =  0.25000
rC  =  1.5000
m   =  2

Mass Calculations
Mass A  =  1.5000
Mass B  = -0.19635
Mass C  =  1.7671
Total Mass :  3.0708

Local Center of mass
Centroid A : xga = 0.25;  yga = 0.50;  zga = 0.75;
Centroid B : xgb = 0.25;  ygb = 0.50;  zgb = 0.75;
Centroid C : xgc = 0.64;  ygc = 1.25;  zgc = 0.86;

Composite Center of mass
Centroid xG =  0.47249
Centroid yG =  0.93160
Centroid zG =  0.81525
MOI of A about xga =  0.40625
MOI of A about yga =  0.31250
MOI of A about zga =  0.15625

MOI of B about xgb = -0.0061359
MOI of B about ygb = -0.0071586
MOI of B about zgb = -0.0071586

MOI of C about xgc =  0.31464
MOI of C about ygc =  0.31464
MOI of C about zgc =  0.55564
```

```
MOI of Composite Volume about xG=  1.1464
MOI of Composite Volume about yG=  0.74176
MOI of Composite Volume about zG=  1.2389
```

2.8.5 Additional Problems

There are two suggestions before you attempt these problems.

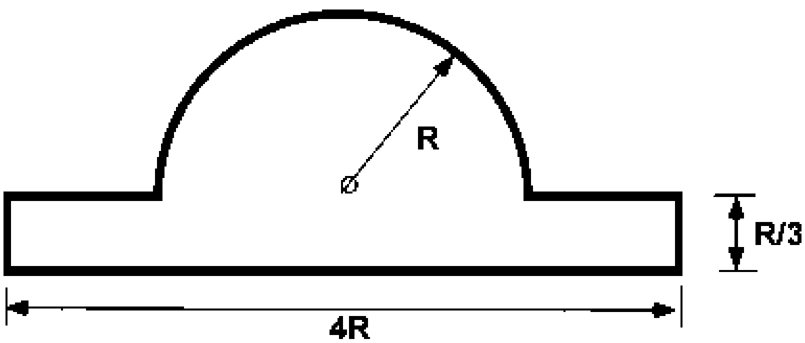
1. Prior to the attempt of problems it would be useful to evaluate the integral outside of MATLAB to make sure MATLAB is giving you the right results. This will also suggest you set up the computation correctly and programmed the right code. You can use a handful of results to confirm instead of all of them.

2. Collect all of the connected pieces of code for the same geometry in a separate file so that you can run the example by changing a few lines of code.

3. If you are using Octave then please verify other pieces of code in this section.

Problem 2.8.1

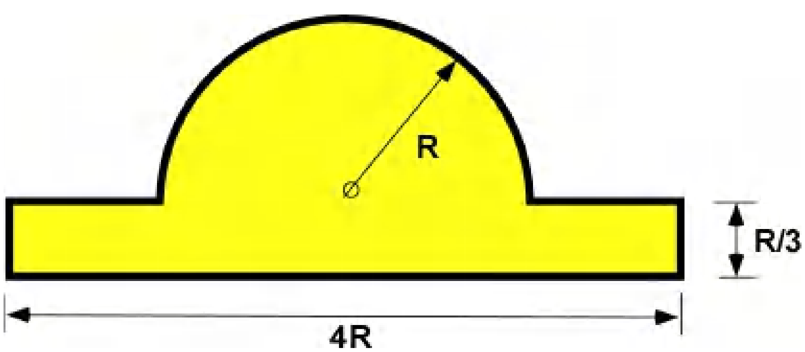
Find the centroid and the MOI about the centroid for the wire object in Problem 2.8.1



Problem 2.8.1

Problem 2.8.2

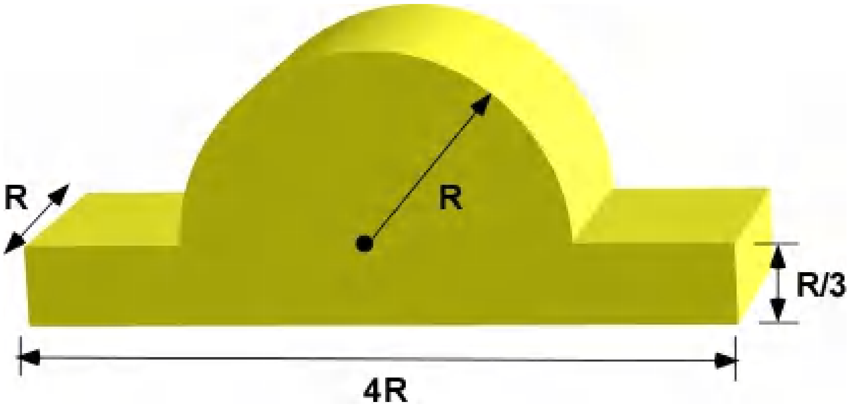
Find the centroid and the MOI about the centroid for the composite area object in Problem 2.8.2



Problem 2.8.2

Problem 2.8.3

Find the centroid and the MOI about the centroid for the composite volume in Problem 2.8.3



Problem 2.8.3

2.9 INERTIA OF POPULAR GEOMETRIES

Instead of calculating the centroid and MOI (moment of inertia) by direct calculations there are available formulas for some popular cross-sections. These formulas can be often leveraged for establishing the geometric properties of complicated geometry using the method of composite areas outlined in the previous section.

Table 2.2 Geometry: Plane Areas

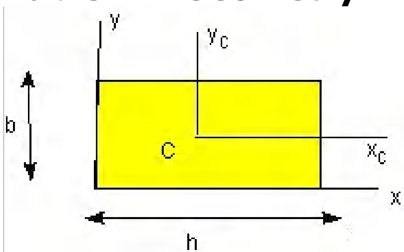
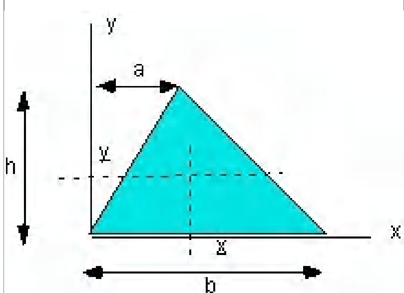
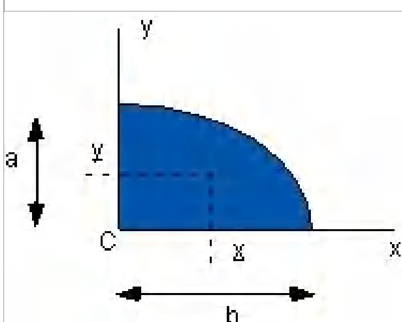
 Rectangle	$I_{x_C} = \frac{bh^3}{12}; \quad I_x = \frac{bh^3}{3}; \quad I_C = \frac{bh}{12}\left(b^2 + h^2\right)$
 Triangle	$I_x = \frac{bh^3}{12}; \quad I_{\underline{x}} = \frac{bh^3}{36};$ $\underline{x} = \frac{4a}{3\pi}; \quad \underline{y} = \frac{h}{3}$
 Quarter Ellipse	$\underline{x} = \frac{4a}{3\pi}; \quad \underline{y} = \frac{4b}{3\pi}$ $I_x = \frac{\pi ab^3}{16}; \quad I_y = \frac{\pi a^3 b}{16}; \quad I_C = \frac{\pi ab}{16}(a^2 + b^2)$ $I_{\underline{x}} = (\frac{\pi}{16} - \frac{4}{9\pi})ab^3; \quad I_{\underline{y}} = (\frac{\pi}{16} - \frac{4}{9\pi})a^3 b$

Table 2.3 Geometry: Circle

	$I_x = I_y = \frac{\pi r^4}{4}; I_C = \frac{\pi r^4}{2};$

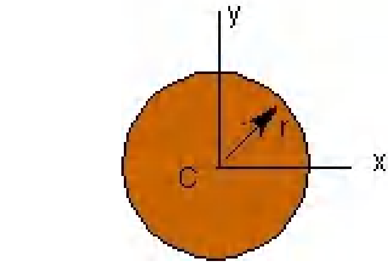
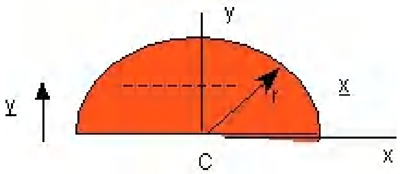
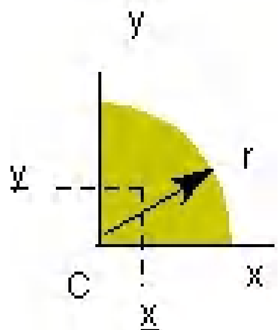
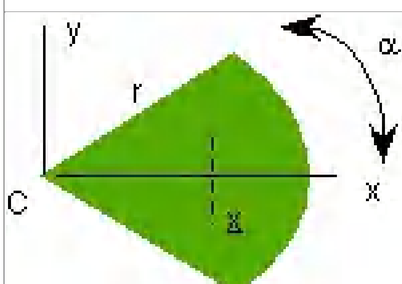
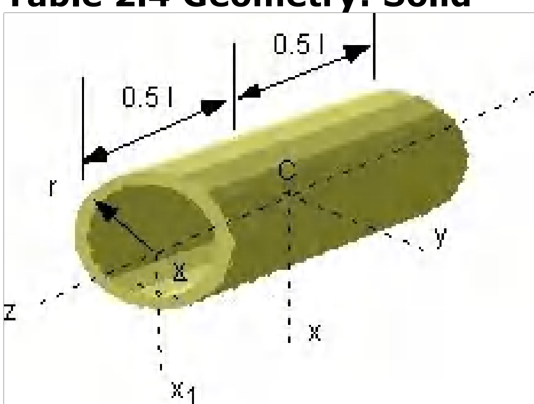
 Circle	
 Semicircle	$I_x = I_y = \frac{\pi r^4}{8}; I_C = \frac{\pi r^4}{4}$ $\underline{y} = \frac{4r}{3\pi}; \quad I_{\underline{x}} = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$
 Quarter Circle	$I_x = I_y = \frac{\pi r^4}{16}; I_C = \frac{\pi r^4}{8}$ $\underline{x} = \underline{y} = \frac{4r}{3\pi}; \quad I_{\underline{x}} = I_{\underline{y}} = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$
 A Sector	$I_x = \frac{r^4}{4}(\alpha - \frac{1}{2}\sin\alpha); I_y = \frac{r^4}{4}(\alpha + \frac{1}{2}\sin\alpha),$ $\underline{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha}; \quad I_C = \frac{1}{2} r^4 \alpha$

Table 2.4 Geometry: Solid

	$I_{xx} = \frac{1}{2} mr^2 + \frac{1}{12} ml^2; \quad I_{zz} = mr^2;$ $I_{x_1x_1} = \frac{1}{2} mr^2 + \frac{1}{3} ml^2;$ $half \quad shell: \underline{x} = \frac{2r}{\pi}; \quad I_{zz_{\underline{x}}} = (1 - \frac{4}{\pi^2}) mr^2$

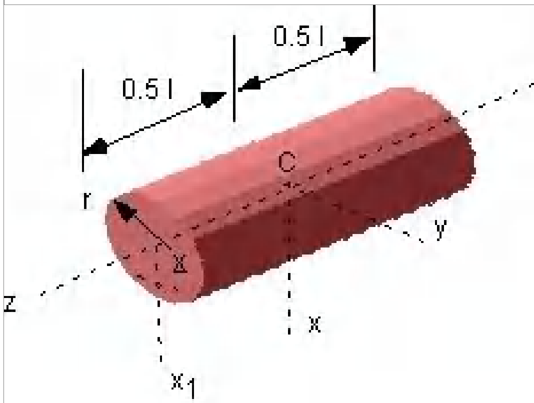
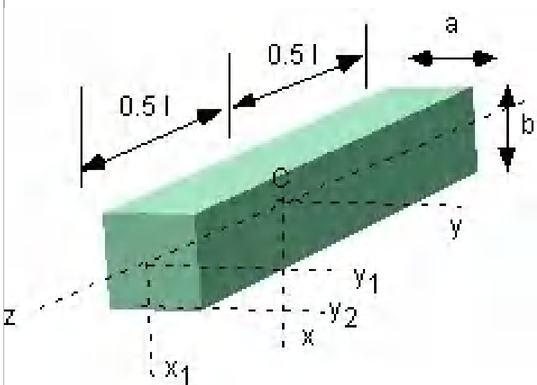
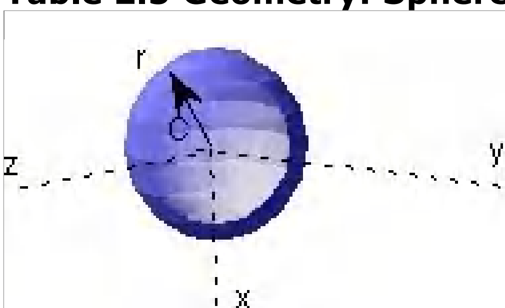
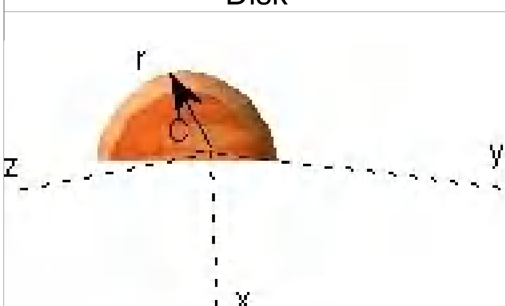
Hollow cylinder	
 Solid Cylinder	$I_{xx}=\frac{1}{4}mr^2+\frac{1}{12}ml^2;\quad I_{zz}=\frac{1}{2}mr^2;$ $I_{x_1x_1}=\frac{1}{4}mr^2+\frac{1}{3}ml^2;$ $half\quad cyl:\underline{x}=\frac{3r}{3\pi};\quad I_{zz_{\underline{x}}}=(\frac{1}{2}-\frac{16}{9\pi^2})mr^2$
 Rectangular Rod	$I_{xx}=\frac{1}{12}m(a^2+l^2);\quad I_{yy}=\frac{1}{12}m(b^2+l^2);$ $I_{zz}=\frac{1}{12}m(a^2+b^2);$ $I_{y_1y_1}=\frac{1}{12}mb^2+\frac{1}{3}ml^2;\quad I_{y_2y_2}=\frac{1}{3}m(l^2+b^2);$

Table 2.5 Geometry: Sphere

 Disk	$I_{zz}=\frac{2}{3}mr^2$
 Half Disk	$\lvert\underline{x}\rvert=\frac{r}{2}$ $I_{xx}=I_{yy}=I_{zz}=\frac{2}{3}mr^2$ $I_{\underline{yy}}=I_{\underline{zz}}=\frac{5}{12}mr^2$

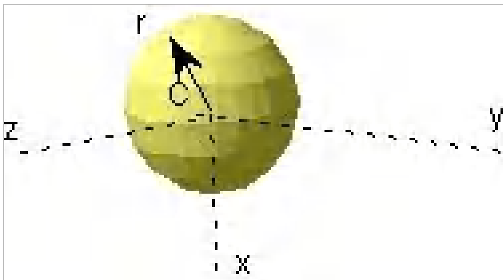
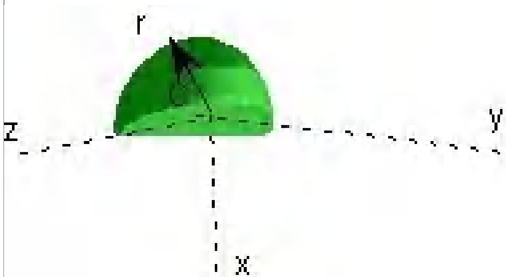
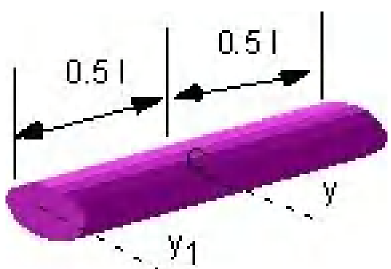
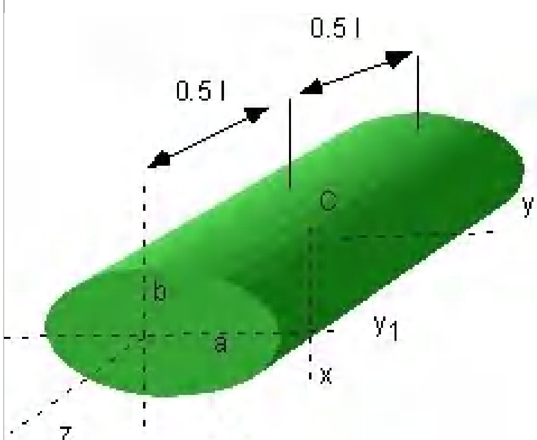
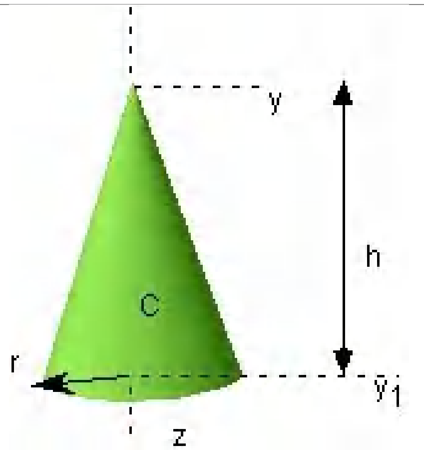
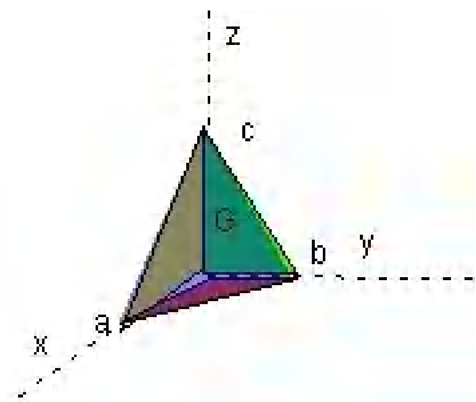
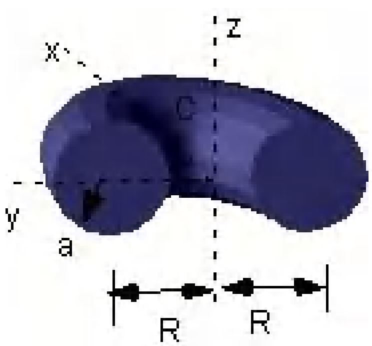
 Sphere	$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$
 Half Sphere	$\lvert\underline{x}\rvert = \frac{3r}{8}$ $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$ $I_{\underline{yy}} = I_{\underline{zz}} = \frac{83}{326}mr^2$

Table 2.6 Geometry: Other

 Slender Rod	$I_{yy} = \frac{1}{12}ml^2; \quad I_{y_1y_1} = \frac{1}{3}ml^2$
 Semi Ellipsoid	$I_{xx} = \frac{1}{4}ma^2 + \frac{1}{12}ml^2; \quad I_{yy} = \frac{1}{4}mb^2 + \frac{1}{12}ml^2$ $I_{zz} = \frac{1}{4}m(a^2 + b^2); \quad I_{y_1y_1} = \frac{1}{4}mb^2 + \frac{1}{3}ml^2$

 Right Cone	$\underline{z} = \frac{3h}{4}$; $I_{yy_c} = \frac{3}{20}mr^2 + \frac{3}{80}mh^2$ $I_{zz} = \frac{3}{10}mr^2$ $I_{yy} = \frac{3}{20}mr^2 + \frac{3}{5}mh^2$; $I_{y_1y_1} = \frac{3}{20}mr^2 + \frac{1}{10}mh^2$
 Prism	$\underline{x} = \frac{a}{4}$; $\underline{y} = \frac{b}{4}$; $\underline{z} = \frac{c}{4}$; $I_{xx} = \frac{1}{10}m(b^2 + c^2)$; I_{yy} *and* I_{zz} *similar* $I_{\underline{xx}} = \frac{3}{80}m(b^2 + c^2)$; *similar*
 Half Torus	$\underline{x} = \frac{a^2 + 4R^2}{2\pi R}$; $I_{xx} = I_{yy} = \frac{1}{2}mr^2 + \frac{5}{8}ma^2$ $I_{zz} = mR^2 + \frac{3}{4}ma^2$

2.10 FORCES

This chapter has been long and had a great variety of information that will be useful later. Before we slip into the study of mechanics it will be useful to refocus our ideas about the reasons that we study mechanics - to understand and resist the actions imposed on a structure - like the lamppost, the building, the bridge, the ship, and even yourself. Mechanics is finally about designing structures that will withstand the action it encounters. There are only two kinds of actions and they can be applied either singly or together. The simpler one is the action that tends to move the structure in a certain direction. This action is called the **Force** and it has an important connection with Newton's second law. The second kind is the **Couple**, which tends to rotate the structure. In this section we focus on the Force and its properties. Force is a vector and we manipulated and calculated it in Section 2.2 without knowing a lot about it. Here we will fill in some important properties of the force in addition to being a vector.

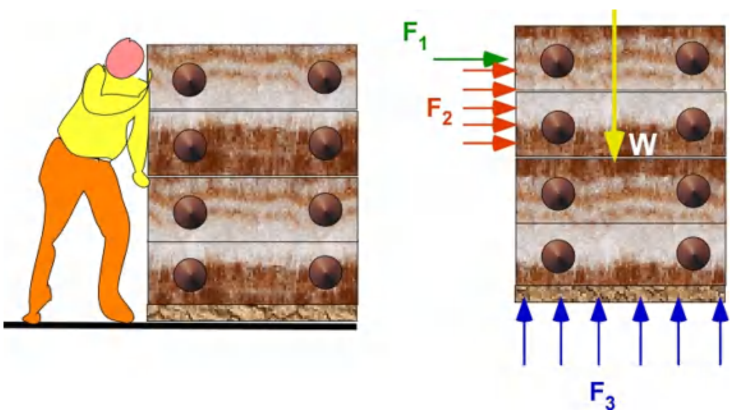


Figure 2.10.1 Forces

Consider Figure 2.10.1. The figure on the left captures the effort of trying to move the dresser to the right. The figure on the right is a simpler illustration of the action taking place. The action of the forces on the right are drawn on a *Free Body Diagram* (**FBD**). We encountered this figure earlier. These forces represent the action on the *dresser* - which is the *free body* for this problem. The *free body* has been isolated from the problem (the the influence of the surroundings). The action of the man, the floor, and the earth are represented by corresponding forces. Such figures are the backbone of the calculations that follow.

The figure also shows two kinds of forces based on how they interact with the object are shown in Figure 2.10.1. They can be further identified as lumped or *concentrated* force (**F_1 and W**) and *distributed* forces (**F_2 and F_3**). We can replace the latter though an equivalent concentrated force for convenience of calculations. So in this section we will focus on concentrated forces. We have learned that Forces are **vectors**. That is why we show them with an arrow indicating its direction.

The two concentrated forces **F_1** and **W** differ in several ways:

- **F_1** is considered a **surface force** as it is usually applied at the boundary of the object - in this case on the left side of the object. **W** represents the weight of the object and acts through the volume. For an object with uniform mass density it is usually at the centroid of the volume. That is why you calculated mass center earlier. For two dimensions it is the centroid of the area.It is called a **body force**.

- **F_1** can be applied due to any reason or direction. It must physically contact the object. **W** is the result of Newton's law of gravitation. It acts at a distance. There is no contact necessary for this force. It is directed towards the center of the earth - or in the vertical direction.
- There are many kinds of surface forces created through different actions. They include friction, tension, air resistance, spring action, normal and shear. Body forces can be created through magnetic or electric fields.

2.10.1 Line of Action of a Force

All forces are vectors. They have a magnitude and a direction. In addition to being vectors the **force** has an additional property that is very important. It also has a ***line of action*** - that is an imaginary line aligned with it as illustrated in Figure 2.11.2. This line is parallel to the force and contains it. Here concentrated forces F_1, W and F_4 are shown with their line of action - as a dotted line.

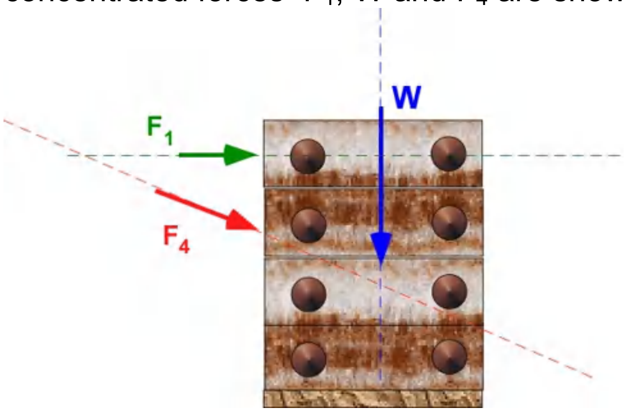


Figure 2.10.2 Line of Action of a Force

Principle 1: You can only move a force along the line of action or parallel to itself (parallel line of action).
Principle 2: You can move a force anywhere along the line of action and its effect on the structure will be the same. This is true even for a surface force. The reason you will be doing this is for the convenience of calculation.
Principle 3: If you move a force parallel to itself it has to be accompanied by an appropriate ***couple/moment.*** Couples are introduced in the next section and this is discussed there.

2.10.2 Idealizations and Simplification

In text books and lectures we rarely solve a problem in mechanics as they are initially identified or postulated. One reason is that solution can become very complex and require mathematical knowledge beyond our current scope. For example the course *Statics* is usually seen at the *Freshman* level. Students posses basic mathematic skills that include algebra, geometry, and introductory calculus. It is therefore necessary we reduced the complexity through idealizations and simplifications. These idealizations however, should determine a solution that should be very close to the exact one if available. This is the nature of engineering calculations.

Let us return to the problem expressed in Figure 2.10.1 and the simplifications made:

- Notice the problem looks flat and two-dimensional even though the man and the dresser are three-dimensional. Three dimensional problems require lot more effort but the magnitude of the forces calculated should not change with dimensions in this case as we can lump the effect in a central plane of the dresser.
- Another idealization is that we have assumed that the man pushes uniformly on the dresser, or

the ground reacts to the dresser uniformly on the dresser. It would really be difficult to replace the uniformly distributed **F**$_2$ with an actual nonlinear distributed force **F**$_2$ without doing a lot of specific testing on how a particular person generated the distributed force.

- The third idealization is uniform mass distribution. The dresser could have different kinds of garments with different densities in each drawer. Even if these details are available it would be difficult to estimate the location of the concentrated force.
- If the distribution of weight in the dresser is non-uniform it is likely that the force distribution due to the ground **F**$_3$ will not be uniform
- Friction is neglected in this problem.

Figure 2.10.3 represents another another example with simple idealization. In the original problem, the figure on the left, the weight of the man on the hammock is seriously distributed, The shape of the hammock is a curve in three dimensions. The interaction of the man with the hammock is usually normal to the plane of contact at the point of the hammock while the weight is vertical. This needs to be defined all across the hammock. The actual support has two forces in the ropes at each end increasing the number of unknowns to be solved for. The idealized problem on the right has made the problem easy to solve. Can you list all the idealizations that have been made?

Figure 2.10.3 Original representation and idealized representation.

The two examples also illustrate a system of forces. A force system can also have additional properties as shown in the following sections.

2.10.3 Concurrent Force System

In a concurrent force system all forces pass through a single point. The advantage of this idealization is that the structure can be considered a particle. There is also no need to compute moments. Figure 2.10.4 shows two examples of concurrent system in three and two dimensions. The first is a pot of gold suspended from three poles. The second is the force on an aircraft in the vertical plane that is flying at constant speed and at a constant altitude. The aircraft is idealized as a particle and the forces are drawn at the center of gravity.

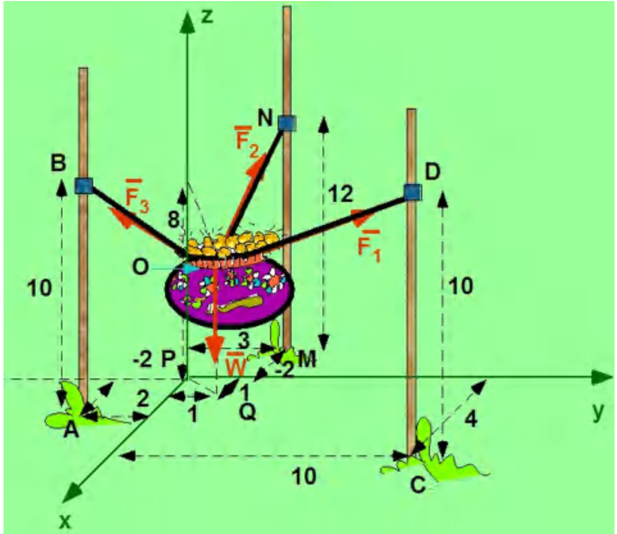


Figure 2.10.4. A concurrent system of forces.

Example 2.2 - Concurrent Forces

A sample calculation of determining the force is revisited in this example. One way to bring down a damaged branch is to throw a rope over the limb and pull on either side. The first person pulls with a force of 600 N, at an angle (θ) of 50 deg to the vertical. Meanwhile, the second person pulls with a force of 850 N, at an angle (β) of 25 deg to the vertical. Both the forces are in the same plane. Use your own coordinate system. (a) Express both forces as components in the coordinate system. (b) Express both forces as a product of the magnitude and a unit vector. (c) Find the net force in the horizontal-direction. (d) Find the net force in the vertical-direction. (e) Express the sum of the forces as a new force in magnitude and unit vector (f) Will the falling branch hit either person?

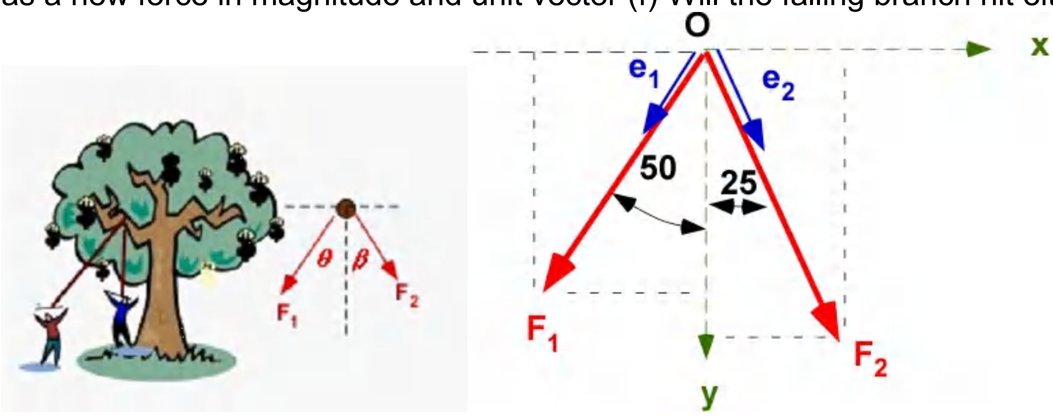


Figure 2.10.5 Example 2.2

Data: F_1 = 600 N; θ = 50 degrees; F_2 = 850 N; β = 25 degrees.   Both vectors in same plane.

Assumptions: none

Solution: Coordinates system shown.

(a)

$$\overline{F}_1 = -600\sin(50)\hat{i} + 600\cos(50)\hat{j} = -459.63\hat{i} + 385.67\hat{j}\,[N]$$
$$\overline{F}_2 = 850\sin(25)\hat{i} + 850\cos(25)\hat{j} = 359.22\hat{i} + 770.36\,\hat{j}\,[N]$$

(b)

$$\bar{F}_1 = 600\,\hat{e}_1 = 600\left[-\sin(50)\,\hat{\imath} + \cos(50)\,\hat{\jmath}\right] = 600\left[-0.77\,\hat{\imath} + 0.64\,\hat{\jmath}\right][N]$$

$$\bar{F}_2 = 850\,\hat{e}_2 = 850\left[\sin(25)\,\hat{\imath} + \cos(25)\,\hat{\jmath}\right] \quad = 850\left[0.42\,\hat{\imath} + 0.91\,\hat{\jmath}\right][N]$$

(c, d)

$$\sum F_x = -459.63 + 359.22 = -100.41[N]$$

$$\sum F_y = 385.67 + 770.36 \quad = 1156.05[N]$$

(e)

$$\bar{F}_{sum} = 1160.40\left(-0.086\,\hat{\imath} + 0.996\,\hat{\jmath}\right)$$

(f) Not enough information

2.10.4 Non Concurrent Force Systems

Parallel Force System

All forces are parallel to each other. It can be in one plane as shown in Figure 2.10.6a This is referred too as coplanar system of forces. It can be three dimensional if you are looking at the weight of the students as they are at their desks in a class room or the forces carried by the columns in the parking garage shown below. All forces are along only one direction.

Figure 2.10.6a A parallel system of forces

Non-Parallel Force System

The forces are not parallel. We further assume that the force are not concurrent too. A planar system has all forces in a plane - making it a 2Dimensional system. A non-planar system will be three dimensional (3D). Such systems are shown in Figure 2.10.6b. In these problems all degrees of freedom must be considered. All equations in all directions must be considered.

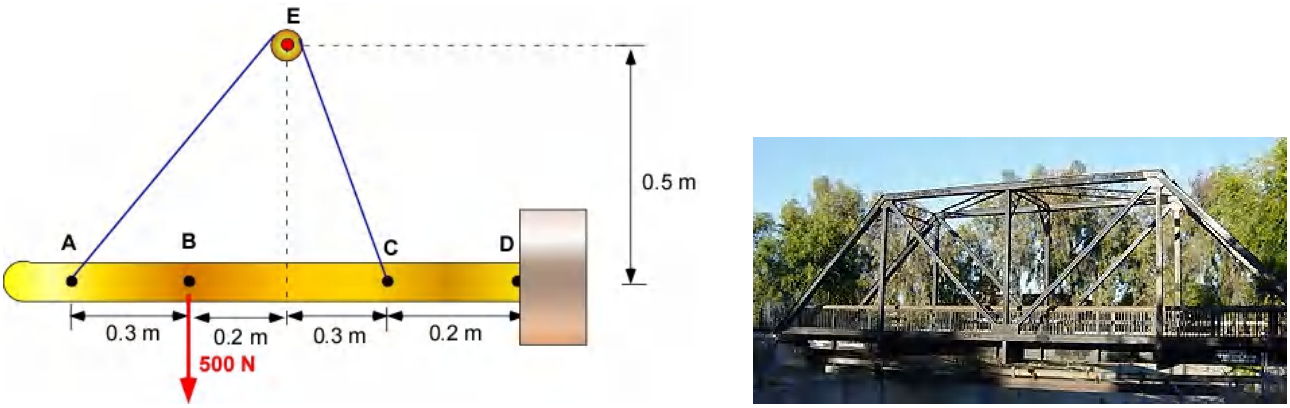


Figure 2.10.6b Non-concurrent and non-parallel system of forces

Distributed Force System

The examples so far mostly involved concentrated loads. Many structures are designed to carry distributed loads. In this case we will have to use calculus and integration to arrive at their equivalent concentrated effects. This is important in the chapter on beams and bending.. Here is an example from Rocky Mountain News in Montana. There have been roofs of buildings and stadiums that collapse under the weight of unexpected excessive snow. Adjacent is the distributed weight of books on a shelf.

Figure 2.10.6c Distributed Loads

Systems of forces regardless of their advantages must obey the same equations of statics. In this way there is only a matter of some convenience rather than any change is our approach to the problem. Working with these categories require the calculation of moments and are therefore postponed to the next section where *moment* is introduced. Working with them will be presented through additional examples.

Resultant of a System of Forces:

If several forces are acting on the object the laws of mechanics react to the vector sum of these forces instead of each single one. Very often we add the vector sum of these forces to express it as a single force acting on the object. This simplifies the calculation. If the object is idealized as a particle there will be only forces acting on it. If the object is a rigid body then there will also be a couple acting on it. We will discuss such examples later after introducing the moment in the next section. The resultant is the sum of the forces. For example the system of the parallel forces in Figure 2.10.6d can be replaced by a single force **R** located at a distance d_R. Computation of d_R requires knowledge of moment.

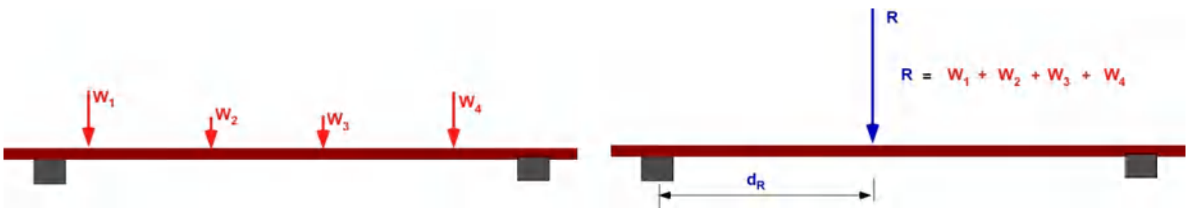


Figure 2.10.6d Resultant of a system of parallel forces.

We can do the same for the planar distributed force in Figure 2.10.6e

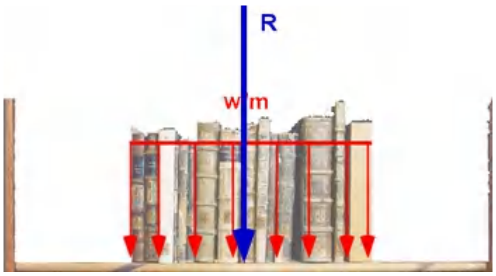


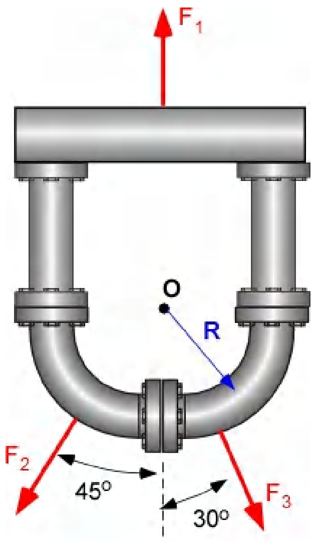
Figure 2.10.6e Resultant of a distributed force

2.10.5 Additional Problems

Use Simplifying assumptions to reduce problem complexity.

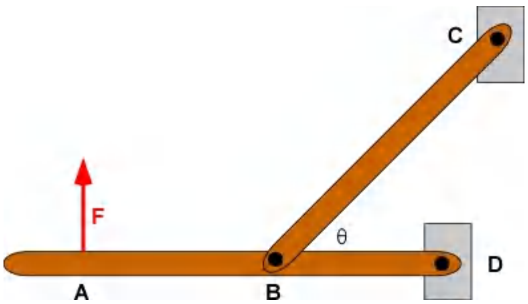
Problem 2.10.1

Three forces are acting on the problem. List all of the simplifications that can be used in the problem. What is the simplest problem?



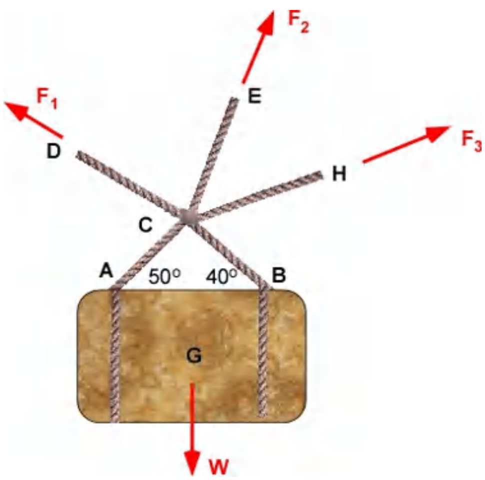
Problem 2.10.1

Problem 2.10.2
There are two links in the problem connected by a pin at B. There is a single applied force on the link at A in the vertical direction as shown.The end C and D are held fixed. One important assumption in this problem is that the force in link BC will be along the link. This is called a two-force link. (a) If you drew the FBD of link AD what is the nature of the force at the end D. These forces are termed as reactions. (b) In the FBD of link BC, what is the nature of the force at C - this is also a reaction.



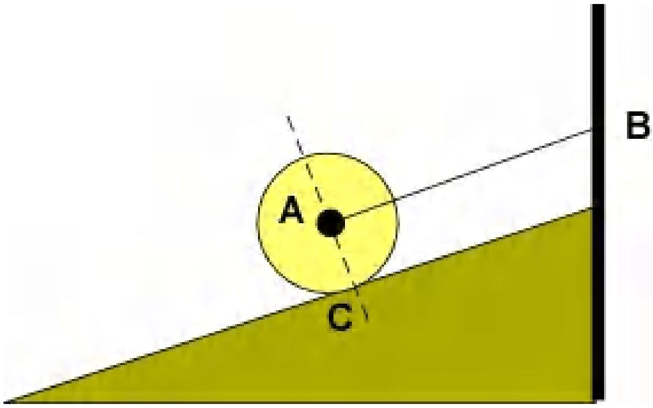
Problem 2.10.2

Problem 2.10.3
Four forces are shown acting on the figure. List all of the simplifications that can be used in the problem. What is the simplest problem?



Problem 2.10.3

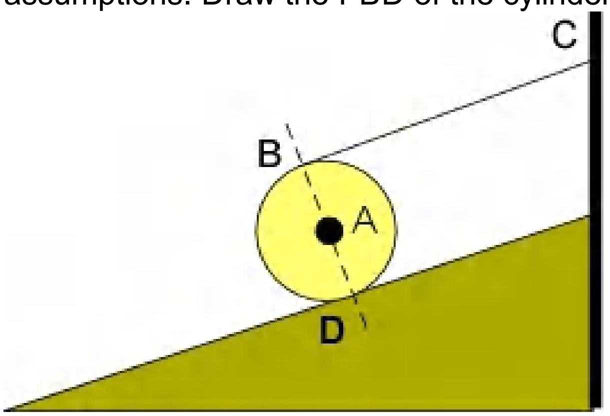
Problem 2.10.4
The cylinder is held in place by the cable AB which is parallel to the inclined plane. List all the assumptions. Draw the FBD of the cylinder.



Problem 2.10.4

Problem 2.10.5

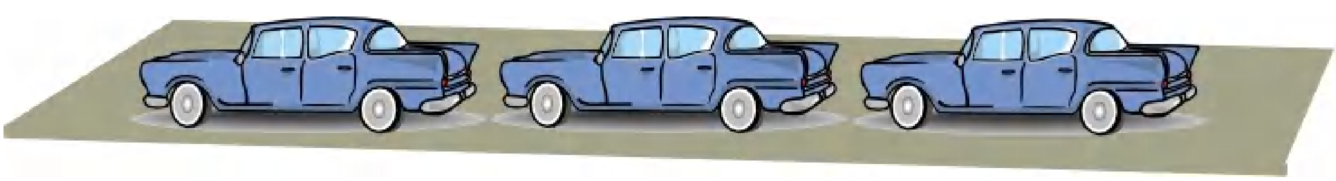
The cylinder is held in place by the cable BC which is parallel to the inclined plane. List all the assumptions. Draw the FBD of the cylinder.



Problem 2.10.5

Problem 2.10.6

Three identical cars are placed in a row on a carrier and are stationary. List all the assumptions and arrive at the simplest model.



Problem 2.10.6

2.11 MOMENTS AND COUPLES

The moment is the action of a force acting at a distance from a point about which the object wants to rotate about. It is a **vector** quantity. Usually moment is associated with a rigid body. Particles by definition cannot distinguish a moment because they do not have size. Let us use a simplified representation of a previous illustration of a man pushing against a chest of drawers. We will replace the chest of drawers by a rectangle and the action due to the person by a horizontal force acting at C as shown in Figure 2.11.1. The chest of drawers has a mass center at G. Here we have simplified the problem by considering it as a two-dimensional one and it appears instinctively acceptable as we can lump the forces in the center plane of the chest of drawers. Also by instinct we know that if we applied P lower, it is likely that the object will move without rotation. The size of the object matters and therefore the object must be a **rigid body.** A and B are the corners of the chest in contact with the floor. The chest is likely to rotate about the point B.

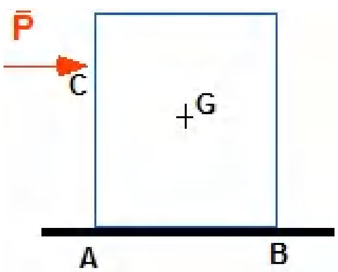


Figure2.11.1 Force on the chest

The **moment**, or the physical quantity that causes the object to rotate, is usually caused by the **force *acting at a distance***. It is important that the line of action of this point not pass through the point of rotation to produce a moment. This formal distance lacks clarity as you can see in Figure 2.11.2 that the distances can be referred with respect to three points G, A, or B and are shown using the letter d with appropriate subscripts. While we can define moment of the force about any point, the most interesting one is the point B about which the object wants to rotate.

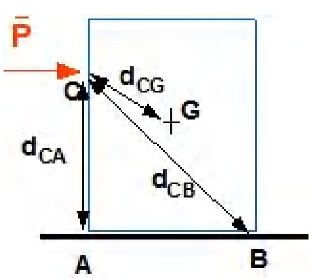


Figure 2.11.2 Force acting at a distance

We can modify the definition by considering that the **moment of the force** is about a **point.** The value of this moment is the product of the force and the **shortest distance between the point and the force.** This brings in the concept of - **the line of action** (LOA) of the force, visited in the last section. This is the line that is parallel to the force and contains the force. The force can be placed any where on this line and cause the same action on the object. This also means that the moment caused by the force will not change as it is moved along the line of action.Figure 2.11.3 illustrates the line of action. It does not matter where the force is placed on this line as far the calculations are concerned. We can move the force along this line to take advantage of the geometry.

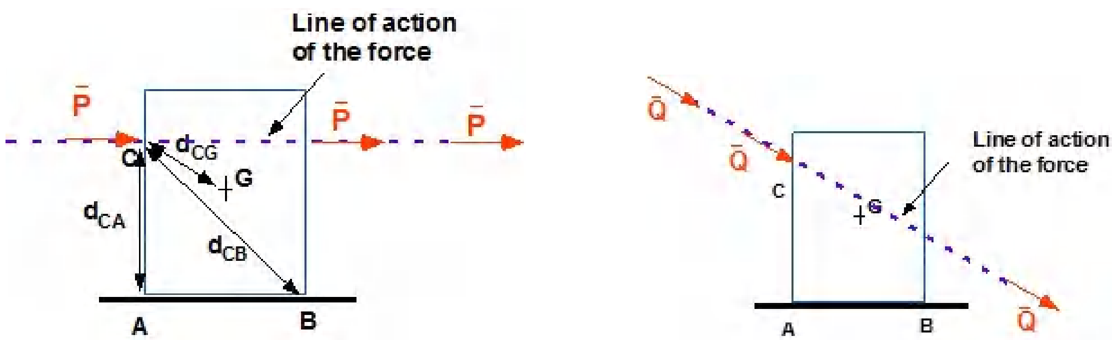


Figure 2.11.3 Line of action

Since the chest tries to rotate about the point B, let us calculate the moment of the force P about the point B. We start with the idealized figure

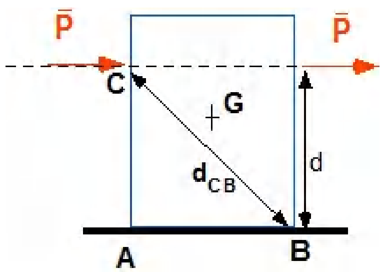


Figure 2.11.4 Calculating Moment

Sliding the force along its line of action the shortest distance appears as d. We notice it is the perpendicular distance between the point B and the line of action (LOA)

$$M = P\,d \qquad (2.77)$$

The value (magnitude) of M is product of the magnitude of the force P and the distance d. M is magnitude of the moment of P about B. The dimension for [M] is the product of force and distance. The basic units for the Moment M is [N-m] or [lb-ft]. But moment **M** is a **vector**!

The direction of M is normal to the plane formed by the force P and d (it is a plane that holds the P vector and the distance d) . Since there are two choices here, the correct one is determined using a right handed system as follows: Roll the fingers of the right hand in the same sense as the rotation induced by this moment while resting on this plane: the thumb will point in the direction of M

Therefore, the **M** vector is directed **into the plane of the screen** you are watching – while **P** and **d** are **in the plane of the screen**.

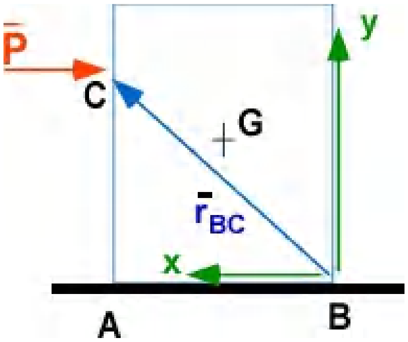


Figure 2.11.5 Moment vector

The more formal calculation for the vector moment **M** is the cross product of the position vector $\mathbf{r}_{BC}$ [from B (point about which you are computing the vector) to C (the point where the force is applied or passes through)] and the force vector **P.** Therefore

$$\bar{M} = \bar{r}_{BC} \times \bar{P} \qquad (2.78)$$

This is valid for calculating **M** in both 2D and 3D problems. This gives you the vector **M.** You have to place the vectors tail to tail for the angle θ in the calculations. The moment for this example is in the +k- direction - normal to the plane determined by the two vectors $\mathbf{r}_{BC}$ and **P**. Even though the moment is a vector it is important to distinguish it from other vectors because it creates a different action on the body. There are three ways to illustrate the moment vector **M** in Figure 2.11.6. The first is a straight arrow and a curved arrow. The second involves a double arrow. This is favored in the book. The third is curved arrow in the plane of the force and distance.

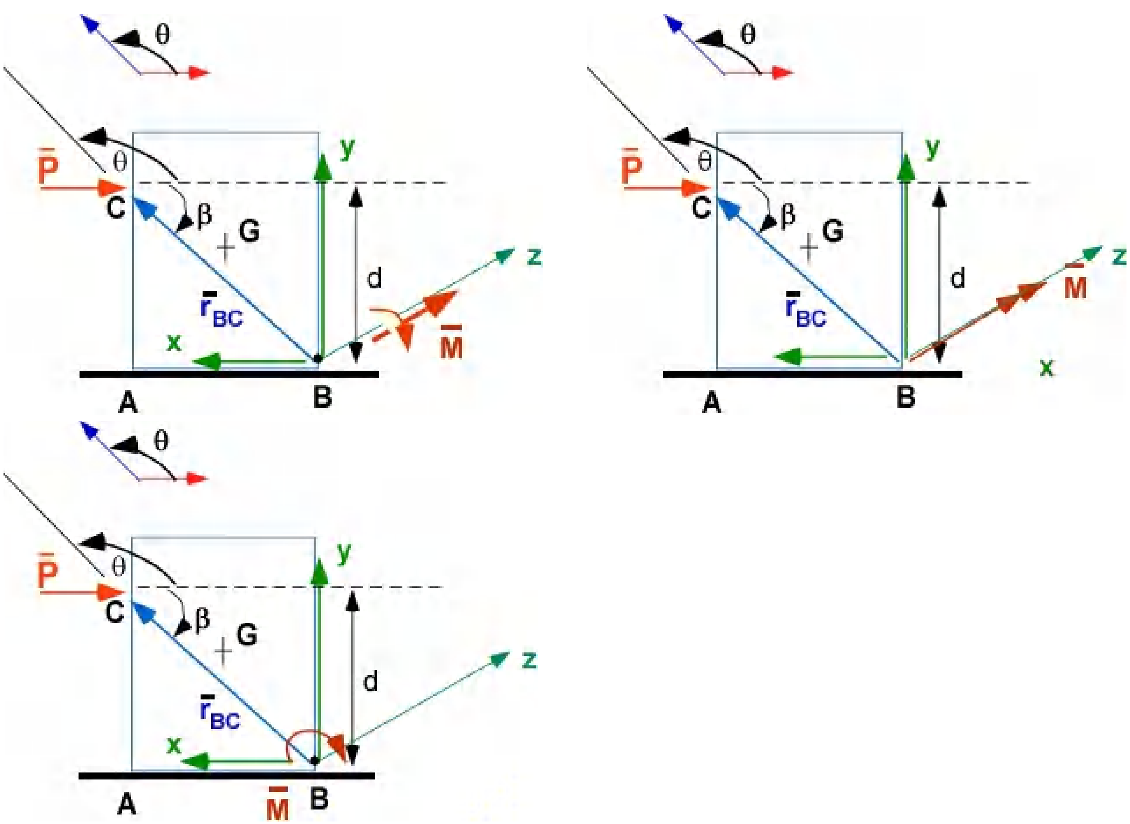


Figure 2.11.6 Three ways to indicate a moment.

We have two vector quantities that create action on a body. The first is the force **P** which attempts to move the body in its direction. The second is the moment **M** that likes to rotate the body. They also have different dimensions and we must be cautious that ***we do not add them together***. We must have a way of distinguishing them in the figures we draw since our analysis will use such figures extensively.

The calculation of the vector **M** and the magnitude M can be summarized as:

$$\bar{M} = \bar{r}_{BC} \times \bar{P} = \hat{e}_n \left|\bar{r}_{BC}\right| \left|\bar{P}\right| \sin\theta$$
$$M = Pd = P\left(r_{BC}\sin\theta\right) = P\left(r_{BC}\sin\left(180-\theta\right)\right) = P\left(r_{BC}\sin\beta\right) \qquad (2.79)$$

e $\mathbf{e}_n$ is the unit vector normal to the plane determined by $\mathbf{r}_{BC}$ and **P** using the right hand. The moment is the same if another point D is chosen along the line of action as in Figure 2.11.7

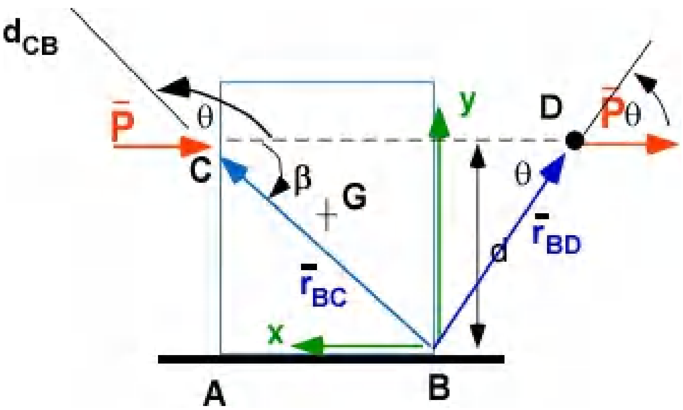


Figure 2.11.7 Same Moment by the force along line of action

$$\bar{M} = \bar{r}_{BD} \times \bar{P}$$
$$M = r_{BD} P \sin\theta = P\left(r_{BD}\sin\theta\right) = Pd \qquad (2.80)$$

Another way to calculate the moment

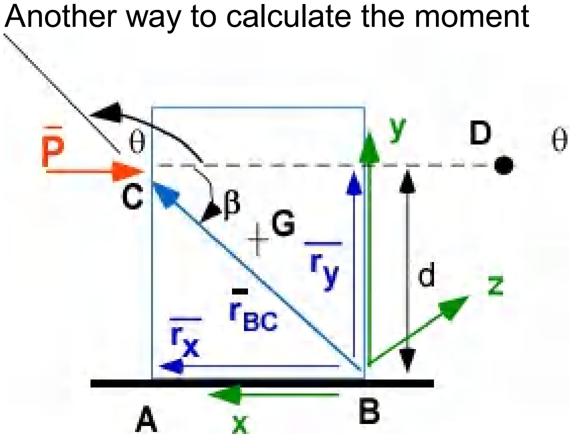


Figure 2.11.8 Cross product using determinant

$$\bar{M} = \begin{vmatrix} i & j & k \\ r_x & r_y & 0 \\ -P & 0 & 0 \end{vmatrix} = i\left(0-0\right) + j\left(0-0\right) + k\left(0 + r_y P\right) = k\left(Pd\right) \qquad (2.81)$$

2.11.1 Example 2.3

You do not like the angle bracket attached to the wall. You are going to pull on the vertical edge at the point shown, with a force of 120 [N], at an angle θ of 20 [deg], to see if you can break it at the wall (in strength of materials you will learn that it is easier to break stuff by applying moments). Find the moment of the force at the wall.

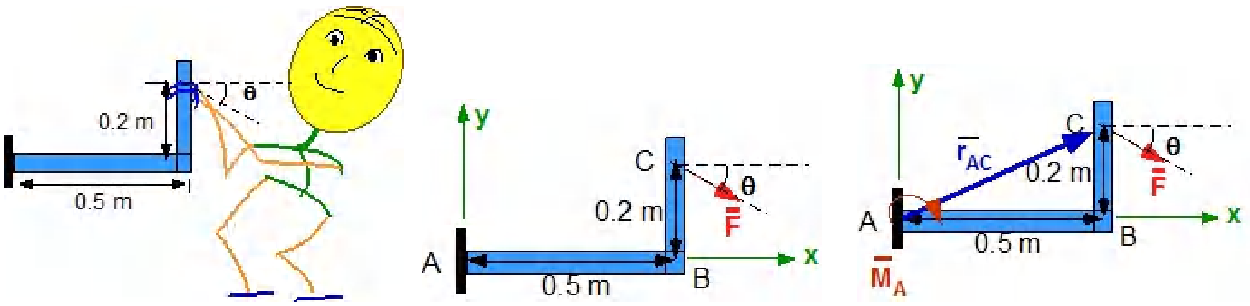


Figure 2.11.9 Example 2.3 (original, simplified, useful)

Data: F = 120 [N]; θ = 20 [deg];
Find: Moment of F at the point A (M_A)
Solution: Using the last figure above

$$\bar{M}_A = \bar{r}_{AC} \times \bar{F}$$

$$\bar{r}_{AC} = 0.5\hat{i} + 0.2\hat{j}\,[m];$$

$$\bar{F} = 120\cos(20)\hat{i} - 120\sin(20)\hat{j} = 112.76\hat{i} - 41.04\hat{j}\,[N]$$

$$\bar{M}_A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.5 & 0.2 & 0 \\ 112.76 & -41.04 & 0 \end{vmatrix}$$

$$= \hat{k}\left[(0.5)(-41.04) - (0.2)(112.76)\right] = \hat{k}(-43.07)\,[Nm]$$

The moment is directed in the **-k** direction (into the page).

Using MATLAB

```
% Essential Mechanics
% P. Venkataraman
% Section 2.11.1 - Example 2.3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('------------------------------\n')
fprintf('Example 2.3\n')
fprintf('-------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data
F = 120;
theta = 20;  thetr = theta*pi/180; % in radians
```

```
%% calculations
A = [0,0,0];  B = [0.5,0,0];  C = [0.5,0.2,0];
rAC= C - A; % position vector from a to C
Fvect = [F*cos(thetr),-F*sin(thetr),0];
MA1= cross(rAC,Fvect);
fprintf('Position vector AC [m]   = '),disp(rAC)
fprintf('Force vector at C [N]    = '),disp(Fvect)
fprintf('Moment of F about A [Nm] = '),disp(MA1)
```

In the Command Window

```
-------------------------------
Example 2.3
--------------------------------
Position vector AC [m]   =     0.5000    0.2000         0
Force vector at C [N]    =   112.7631  -41.0424         0
Moment of F about A [Nm] =          0         0  -43.0738
```

Example 2.3a - Moving force along line of action

Solve Example 2.3 by moving the force along the line of action to D (see Figure).

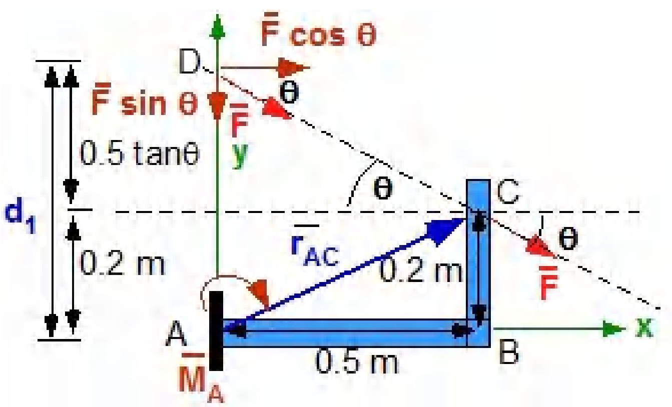


Figure 2.11.10 Force at D

Move the force F along the line of action till it intersects the y axis at D. The y-component of the force at D passes through A and will not create a moment about A (because there is no moment arm). The x-component of the force at D is perpendicular to the line (or the moment arm) AD. From the geometry in the figure the magnitude of the moment is

$$M_A = d_1 (F\cos\theta) = (0.2 + 0.5\tan 20)(120\cos 20) = 43.074[Nm]$$

The moment causes the fingers of the right hand roll from D to B about A and the thumb points into the screen that is in the **– k** direction.

Example 2.3b - Moving force along line of action

Solve Example 2.3 by moving the force along the line of action to E (see Figure).

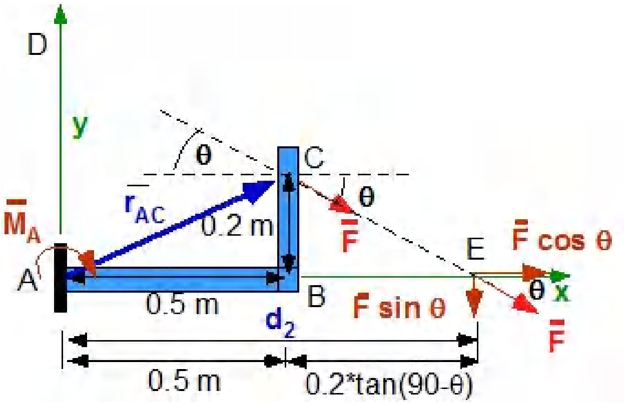


Figure 2.11.11 Force at E

Move the force F along the line of action till it intersects the x axis at E. The x-component of the force at E passes through A and will not create a moment about A (because there is no moment arm). The y-component of the force at E is perpendicular to the line (or the moment arm) AE. From the geometry in the figure the magnitude of the moment is

$$M_A = d_2\, F \sin(90 - \theta) = (0.5 + 0.2\ \tan 70)(120 \sin 20) = 43.074\,[Nm]$$

$$\bar{M}_A = 43.074\,(-\hat{k})\,[Nm]$$

Example 2.3c - Moment using shortest distance

Calculate the moment using the shortest distance between A and the line of action of the force.

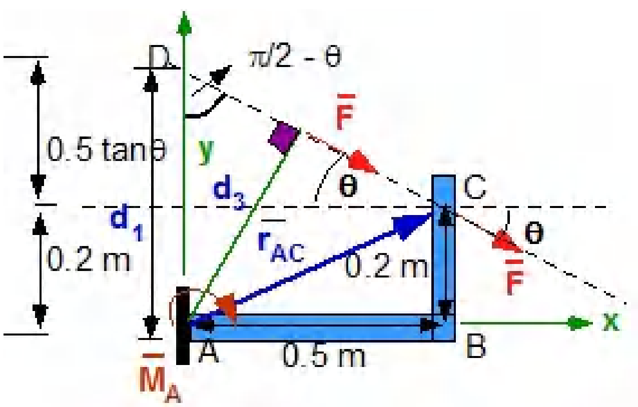


Figure 2.11.12 Shortest distance

From the geometry in the figure the magnitude of the moment can be obtained as:

$$M_A = d_3\, F = d_1 \sin(90 - \theta)\, F = (0.2 + 0.5 \tan 20)\sin(90 - 20)\, 120 = 43.074\,[Nm]$$

The direction is obtained using the right hand.

Example 2.3d - Scalar implementation of vector multiplication

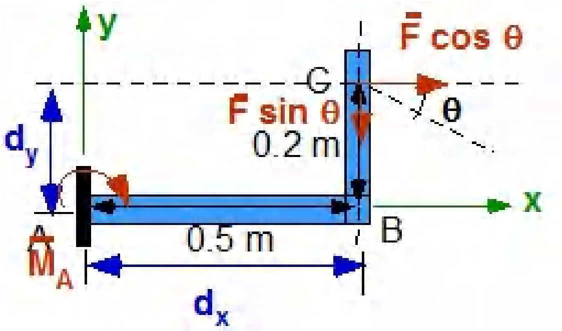


Figure 2.11.13 Vector multiplication scalar implementation

First resolve the force components. Calculate moment by each component by the multiplication of force and its shortest distance from A. If they produce roll in the same direction they will add. Moment is positive if it is the positive coordinate direction.

$$M_A = -d_x\, F \sin\theta - d_y\, F \cos\theta = -0.5*120*\sin 20 - 0.2*120*\cos(20) = -43.074[Nm$$

Summary of these various methods:

Method 1 (Example 2.3) is the most direct and straight forward. It works for both 2D and 3D problems. It provides both the magnitude and the direction of the moment - In other words it calculates the Vector. It has a simple mathematical basis of applying the cross-product

Method 2, 3, 4 (Example 2.3 a, b, c) require construction, geometry, and trigonometry. It challenges your visualization skills but will develop intuition. It is easy in 2D problems but not so easy for 3D problems. 3D visualization is a challenge for all of us.

Methods 2 through 5 (Example 2.3 a, b, c) are useful in determining the magnitude. The vector direction requires use of the right hand. It is difficult in 3D problems.

Method 4 is challenging as it requires visualization skills and an ease with geometrical reasoning

Method 5 is very useful for 2D problems. In fact it is equivalent to Method 1 for such problems

2.11.2 Couple Moment

We now understand that to rotate an object we need to apply a moment that is typically produced by a force that is located at some distance from a reference point (maybe the center of mass). Place an eraser on the table and apply a concentrated force from the center of mass in the horizontal plane. The top view illustrates that the eraser will move and turn

Figure 2.11.14 Applying a moment

Consider the eraser on the table again. This time apply a pair of equal and opposite concentrated forces away from the center of mass – in the horizontal plane (a moment?). If the experiment went right you should see the eraser turning in place and not displacing. This special moment is a **couple**. In the case of the couple, the net force in the horizontal plane is zero, so the eraser is in force equilibrium.

Figure 2.11.15 A couple moment

Definition: A couple moment is produced by a pair of equal but oppositely directed forces separated by a distance.

Properties:

- The couple moment is normal to the plane formed by the forces
- The sum of these forces is zero since they are equal and opposite
- The object experiences a **moment** only. This is a pure moment.
- It causes the object to rotate in space
- Couple moment can be moved parallel to itself without changing its effect on the object
- Couple and couple moment refer to the same thing

2.11.3 Calculating the Couple Moment

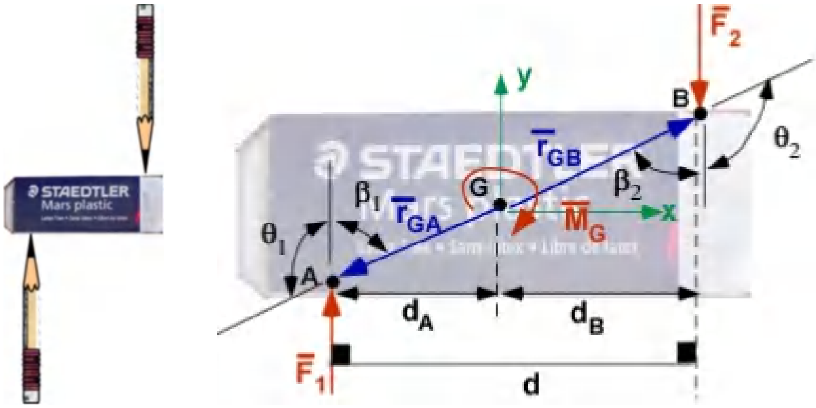


Figure 2.11.16 Calculating couple moment

We will represent the pencils by concentrated forces on a rigid body at points A and B. The geometry is shown in Figure 2.11.16. $\mathbf{F_1}$ and $\mathbf{F_2}$ have the same magnitude and oppositely directed ($F_1 = F_2$). G is a reference point (say center of mass), $\mathbf{r_{GA}}$ and $\mathbf{r_{GB}}$ are the position vectors from G to the points A and B (these points are on the line of action of the two forces). The moment of the couple can be calculated as the moment about G of the two forces $\mathbf{F_1}$ and $\mathbf{F_2}$. Remember that the angle for the cross product is between the two vectors with their tails in contact. Using the right hand we can determine that the magnitude of the couple (couple moment) is F*d - Force multiplied by the shortest distance between them.

$$\begin{aligned}
&\bar{M}_G = \left(\bar{r}_{GA} \times \bar{F}_1\right) + \left(\bar{r}_{GB} \times \bar{F}_2\right) = F_1 r_{GA} \sin\theta_1 \left(-\hat{k}\right) + F_2 r_{GB} \sin\theta_2 \left(-\hat{k}\right)\\
&\theta_1 = \pi - \beta_1; \quad \theta_2 = \pi - \beta_2; \quad \sin\left(\pi - \beta_1\right) = \sin\beta_1; \quad \sin\left(\pi - \beta_2\right) = \sin\beta_2\\
&r_{GA} \sin\beta_1 = d_A; \quad r_{GB} \sin\beta_2 = d_B\\
&\bar{M}_G = \left(-\hat{k}\right)\left(F_1 d_A + F_2 d_B\right) = \left(-\hat{k}\right) F_1 d \quad = \quad \left(-\hat{k}\right) F_2 d
\end{aligned} \qquad (2.82)$$

Figure 2.11.17 illustrates the forces applied at a different orientation. The couple is still F*d, though d is different in this case. It is still the shortest distance between the forces.

$$\bar{M}_G = \left(-\hat{k}\right)\left(F_1 d_A + F_2 d_B\right) = \left(-\hat{k}\right) F_1 d \quad = \quad \left(-\hat{k}\right) F_2 d \qquad (2.83)$$

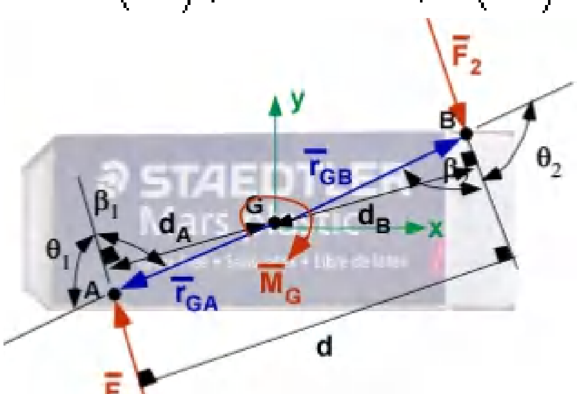


Figure 2.11.17 Couple moment

Point G does not have to be a special point (like the center of mass). We can choose an arbitrary point E on the body, see Figure 2.12.18.

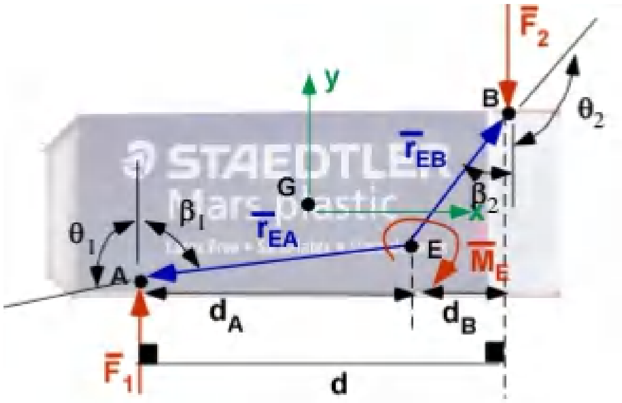


Figure 2.11.18. Couple Moment (again)

$$\bar{M}_E = \left(\bar{r}_{EA} \times \bar{F}_1\right) + \left(\bar{r}_{EB} \times \bar{F}_2\right) = F_1 r_{EA} \sin\theta_1\left(-\hat{k}\right) + F_2 r_{EB} \sin\theta_2\left(-\hat{k}\right)$$

$$\theta_1 = \pi - \beta_1; \quad \theta_2 = \pi - \beta_2; \quad \sin\left(\pi - \beta_1\right) = \sin\beta_1; \quad \sin\left(\pi - \beta_2\right) = \sin\beta_2 \qquad (2.84)$$

$$r_{EA}\sin\beta_1 = d_A; \quad r_{EB}\sin\beta_2 = d_B$$

$$\bar{M}_E = \left(-\hat{k}\right)\left(F_1 d_A + F_2 d_B\right) = \left(-\hat{k}\right)F_1 d \quad = \quad \left(-\hat{k}\right)F_2 d$$

The point does not have to be on the body, see Figure 2.11.19. Consider point D

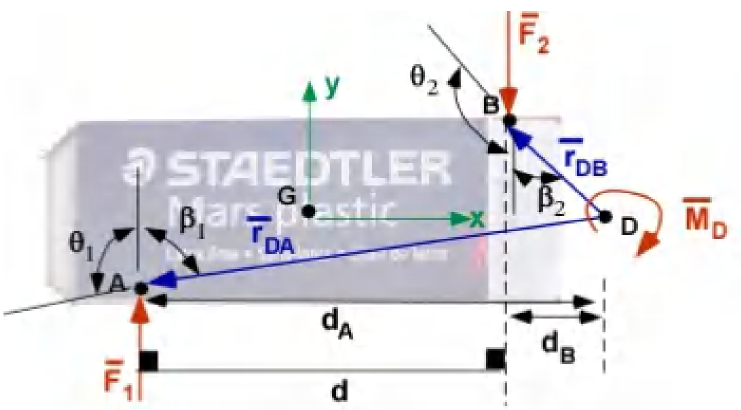


Figure 2.11.19 Couple Moment (one more time)

$$\bar{M}_D = \left(\bar{r}_{DA} \times \bar{F}_1\right) + \left(\bar{r}_{DB} \times \bar{F}_2\right) = F_1 r_{DA} \sin\theta_1\left(-\hat{k}\right) + F_2 r_{DB} \sin\theta_2\left(+\hat{k}\right)$$

$$\theta_1 = \pi - \beta_1; \quad \theta_2 = \pi - \beta_2; \quad \sin\left(\pi - \beta_1\right) = \sin\beta_1; \quad \sin\left(\pi - \beta_2\right) = \sin\beta_2$$

$$r_{DA}\sin\beta_1 = d_A; \quad r_{DB}\sin\beta_2 = d_B$$

$$\bar{M}_D = \left(-\hat{k}\right)\left(F_1 d_A - F_2 d_B\right) = \left(-\hat{k}\right)F_1\left(d_A - d_B\right) = \left(-\hat{k}\right)F_1 d \qquad (2.85)$$

The couple moment is a **free vector.** You can move it to G, D, E and it will have the same effect on the body.

Also understand that the ***same couple*** is obtained by doubling the distance and reducing the magnitude of the forces by half. Hence different force systems can produce the same couple moment. These are called **equivalent** systems.

Example 2.4

Three couples are shown acting on the triangular prismatic body shown. The magnitude of forces F_1, F_2, F_3, acting in x, y, and z direction respectively are 10, 12, and 15 [N]. (a) Calculate the moment due to the three couples; (b) Identify the resultant couple.

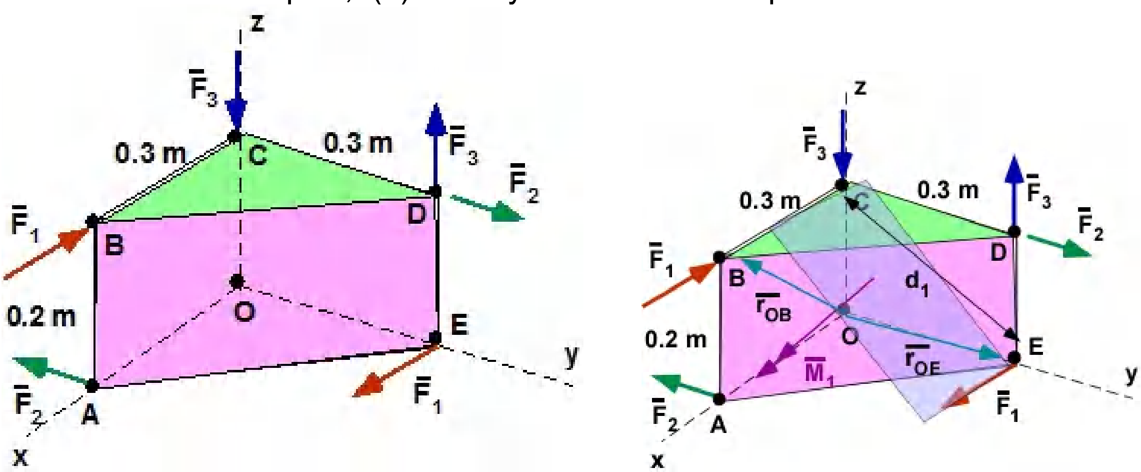


Figure 2.11.20 Example 2.4 and the plane of the couples

Data: F1 = 10 [N], F_2 = 12 [N], F_3 = 15 [N]. Location shown on figure. [note that they are couples since the forces (i) have the same magnitude; (ii) have opposite direction; and (iii) are separated].

Find: (a) the individual couples (**M_1, M_2, M_3**), and (b) sum of the couples

Solution:

$$\left|\bar{M}_1\right| = M_1 = F_1 d_1$$

$$\bar{M}_1 = (\bar{r}_{OB}) \times \left(-F_1 \hat{i}\right) + (\bar{r}_{OE}) \times \left(F_1 \hat{i}\right)$$

$$\bar{M}_1 = \left(0.3\hat{i} + 0.2\hat{j}\right) \times \left(-10\hat{i}\right) + \left(0.3\hat{j}\right) \times \left(10\hat{i}\right) = -2\hat{j} - 3\hat{k}$$

$$\left|\bar{M}_1\right| = 3.6056 [Nm]$$

$$\left|\bar{M}_2\right| = M_2 = F_2 d_2$$

$$\bar{M}_2 = \left(0.3\hat{i}\right) \times \left(-12\hat{j}\right) + \left(0.3\hat{j} + 0.2\hat{k}\right) \times \left(12\hat{j}\right) = -3.6\hat{k} - 2.4\hat{i}; M_2 = 4.3267 [Nm]$$

$$\bar{M}_3 = (15)(0.3)\hat{i} = 4.5\hat{i} \, [Nm]$$

$$\sum \bar{M} = \left[-2\hat{j} - 3\hat{k}\right] + \left[-3.6\hat{k} - 2.4\hat{i}\right] + \left[4.5\hat{i}\right] = 2.1\hat{i} - 2\hat{j} - 6.6\hat{k} \, [Nm]$$

2.11.4 Moving a Force Parallel to Itself

A particular force on the body effects it in a unique way. You can only move the location of the force on the body if you do not change its effect. The direction of the force cannot be changed. . You can move the force on the object in two ways without changing its effect.

1. Move the force along the line of action. This does not change its effect on the object. We saw this earlier when we computed the moment.
2. Move the force parallel to itself. To deliver the same effect on the body you also need to add moment or a couple to the problem.

This is demonstrated though the example we used Figure 2.11.9 which is reproduced on the left in Figure 2.11.21. The force **F** is initially applied at C. We wish to move it to the point A.

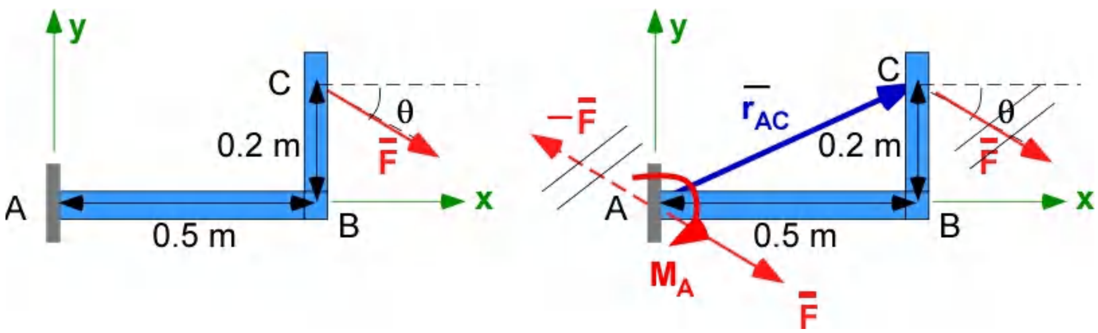


Figure 2.11.21 Moving a force parallel to itself

The construction follows the procedure for illustrating a couple moment.

We place the force **F** and **-F** at A. Because they cancel the structure is still experiences the force **F** at C. We combine the force **F** at C with the force **-F** at A to define a couple **M_A** as shown in the figure. This leaves the force **F** at A and the couple moment **M_A**. The resultant of a single force **F** at C is the force **F** at A and a couple **M_A** at A.

The couple **M_A** is calculated as a moment due to the force F at the point A.

$$\bar{M}_A = \bar{r}_{AC} \times \bar{F}$$

Example 2.5 Moving a force parallel to itself

Consider the force **P** of magnitude 100 N with directional cosine angles 55, 45, 60 respectively applied at the point **A** as shown. Point **B** is at a distance d_1 = 0.3 m along the negative z direction from the origin O. Point **A** is at a distance d_2 = 0.1 m along the positive x-direction from **B**. Figure 2.11.22 shows the layout of the points and the Force. (a) Move the force P to the point B. (b) Move the force at A to the origin at O.

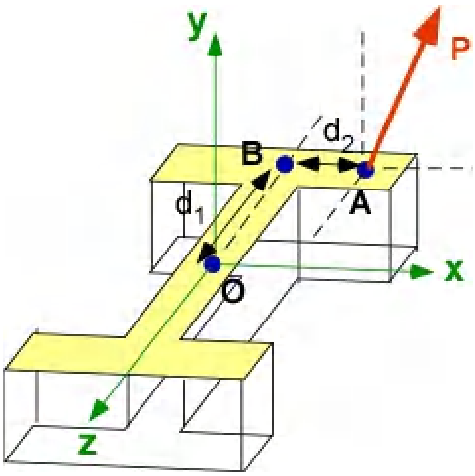


Figure 2.11.22a Example 2.5

Data: P = 100*(cos(55) ***I*** + cos(45) ***j*** + cos(60) ***k***] [N]. Point O(0, 0, 0). Point A(0.1 , 0, -0.3) [m]. Point B(0.0.-03) [m].

Find: (a) Move **P** to point B from point A; (b) Move **P** from point A to point O

Solution: At B and O there will be the force **P** and a Moment. The force is the same as the original force in magnitude and direction.

(a)

$$\bar{P} = [57.36\hat{i} + 70.71\hat{j} + 50\hat{k}][N]$$

$$\bar{r}_{BA} = 0.1\hat{i}\,[m]$$

$$\bar{M}_B = \bar{r}_{BA} \times \bar{P} = (0.1 \times 70.71)(\hat{i} \times \hat{j}) + (0.1 \times 50)(\hat{i} \times \hat{k})$$

$$= -5\hat{j} + 7.07\hat{k} \quad [Nm]$$

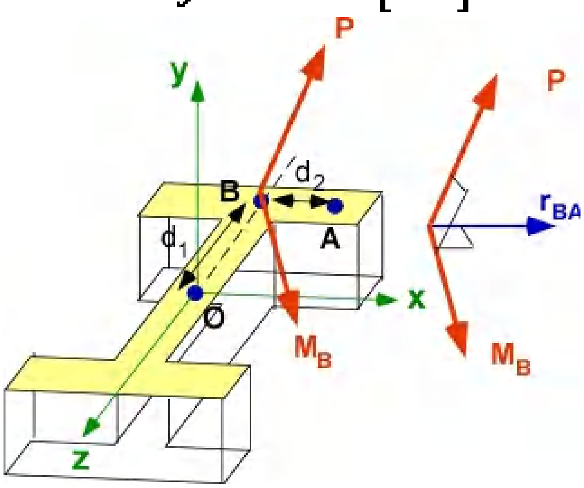


Figure 2.11.22b Solution (a)

(b)

$$\bar{P} = [57.36\hat{i} + 70.71\hat{j} + 50\hat{k}][N]$$

$$\bar{r}_{OA} = 0.1\hat{i} - 0.3\hat{k}[m]$$

$$\bar{M}_O = \bar{r}_{OA} \times \bar{P} = (0.1 \times 70.71)(\hat{i} \times \hat{j}) + (0.1 \times 50)(\hat{i} \times \hat{k}) +$$

$$(-0.3 \times 57.36)(\hat{k} \times \hat{i}) + (-0.3 \times 70.71)(\hat{k} \times \hat{j})$$

$$= 21.21\hat{i} - 22.21\hat{j} + 7.07\hat{k} \quad [Nm]$$

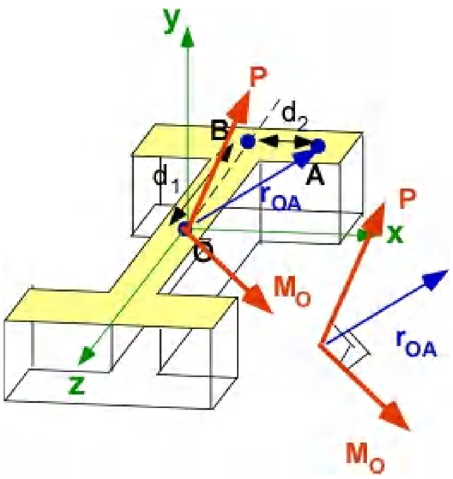


Figure 2.11.22c Solution (b)

2.11.5 Moment of a Force About a Line

In some problems we will need the moment of a force along a line. This is important if the object is constrained to only be able to revolve around the line or a hinge. For example when you open your laptop you may be applying a force at the corner of the lid in any direction upward. This force creates a moment about the hinge of the lid which opens the laptop. Note that this a dynamic problem. Here we are calculating the component of the moment about a line or the hinge. The result is a scalar and with a value of the component of the moment along the line. This can be set up as a triple vector product as shown in the development.

Consider you are opening the door of the refrigerator using the force **F**. The hinges are aligned with the line OB. The moment about the hinge that opens the door is the moment about the line OB. We calculate the moment about the point O (**M$_O$)** and then take the component of this moment along the hinge line by calculating the dot product of this moment and the unit vector **e$_{OB}$**.

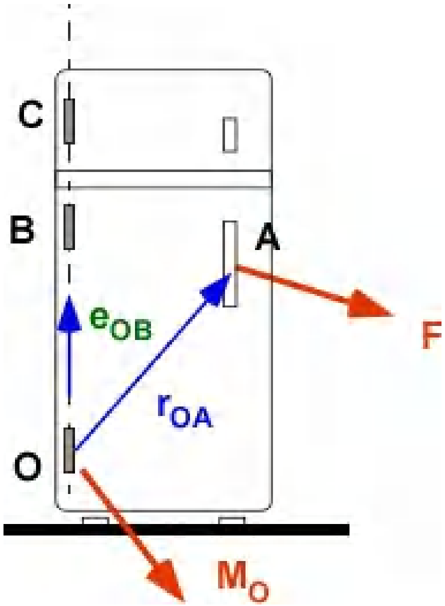


Figure 2.11.23a Moment about a line

$$\bar{M}_O = \left(\bar{r}_{OA} \times \bar{F}\right)$$

$$M_{OB} = \hat{e}_{OB} \cdot \left(\bar{r}_{OA} \times \bar{F}\right) = \left(\bar{r}_{OA} \times \bar{F}\right) \cdot \hat{e}_{OB}$$

$$= \begin{vmatrix} e_{OBx} & e_{OBy} & e_{OBz} \\ r_{OAx} & r_{OAy} & r_{OAz} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= e_{OBx}\left(r_{OAy}F_z - r_{OAz}F_y\right) + e_{OBy}\left(r_{OAz}F_x - r_{OAx}F_z\right) + e_{OBz}\left(r_{OAx}F_y - r_{OAy}F_x\right)$$

Example 2.6 Moment of a force about a line

A force of 250 N is directed along the line AB as in Figure 2.11.23b. Find the moment of this force about the line CD.

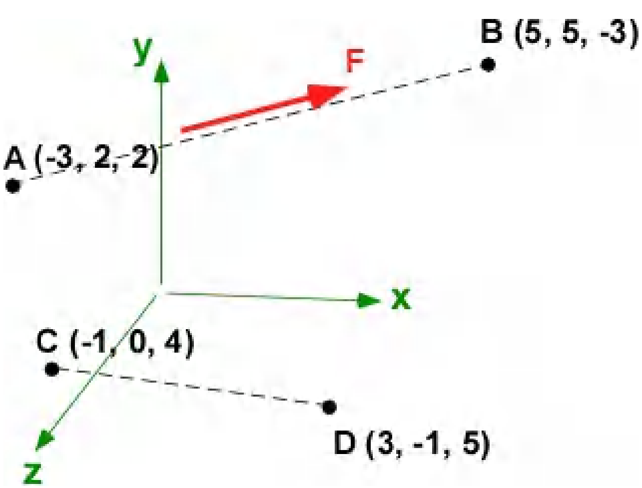


Figure 2.11.23b Moment about a line

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 2.11.5 - Example 2.6
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('--------------------------------\n')
fprintf('Example 2.6 \n')
fprintf('---------------------------------\n')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data
F = 250;
A= [-3, 2, 2];   B = [5, 5, -3];  C = [-1, 0, 4]; D = [3, -1, 5];

%% calculations
AB = B- A;
eAB = AB/norm(AB);
Fv = F*eAB;

rCA = A - C;
MC = cross(rCA,Fv);

CD = D-C;
eCD = CD/norm(CD);
Mcd = dot(MC,eCD);

fprintf('Fv[N]    = '),disp(Fv)
fprintf('rCA [m]     = '),disp(rCA)
fprintf('Moment of F about C [Nm] = '),disp(MC)

fprintf('\neCD [m]     = '),disp(eCD)
fprintf('Moment of F about CD [Nm] = '),disp(Mcd)
```

In Command Window

```
-------------------------------
Example 2.6
--------------------------------
Fv[N]    =   202.0305   75.7614 -126.2691
rCA [m]     =      -2      2     -2
Moment of F about C [Nm] =  -101.0153 -656.5992 -555.5839

eCD [m]     =      0.9428   -0.2357    0.2357
Moment of F about CD [Nm] =   -71.4286
```

Execution in Octave

The code is same as in MATLAB

In Octave Command Window

```
-------------------------------
Example 2.6
```

```
--------------------------------
Fv[N]    =    202.031    75.761  -126.269
rCA [m]     =   -2   2  -2
Moment of F about C [Nm] =   -101.02  -656.60  -555.58

eCD [m]     =    0.94281  -0.23570   0.23570
Moment of F about CD [Nm] = -71.429
```

2.11.6 The Resultant

If several forces are acting on the object the laws of mechanics react the same to the vector sum of these forces instead of each single one. Very often we add the vector sum of these forces to express it as a single force acting on the object. This sometimes simplifies the calculation. It also provides ideas of how to constrain the object from moving - both translational and rotational. If the object is idealized as a particle there will be only forces acting on it. If the object is a rigid body then there will also be a couple acting on it unless the forces acting on the body have a special structure like in the following example. For example the system of the parallel forces in Figure 2.11.24 can be replaced by a single force **R** located at a distance d_R. It can also be replaced by the force **R** and a couple at the left end.

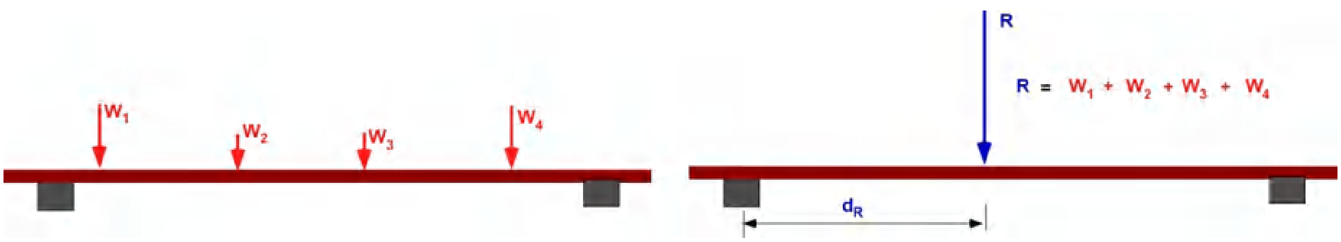


Figure 2.11.24 Resultant of a system of parallel forces

In general a structure subject to a system of forces and couples can be reduced to a (i) ***single*** force or (ii) a ***single*** force and a ***single*** couple (depends on the system of the applied loads). This is called a **resultant** of the system of loads**.** It is important that both the original system of loading and the resultant produce the same effect on the structure. They are also considered **equivalent.** Once again this is usually done for convenience. It does not add or subtract from the laws of mechanics. We can still design structures without using this concept but it does help in developing an instinct about reactions of structures to applied loads. The idea is based on moving a force parallel to itself that we explored in defining the couple moment above. In Example 2.11.1 a single force was moved. In general a system subject to several forces all of the forces can be similarly moved. The couples can be moved without any modification.

Example 2.7 Resultant of a Set of Parallel Forces

All of the forces are parallel to each other. We create this example from Figure 2.11.24 by providing values for the forces and their locations. W_1 = 200 N, W_2 = 300 N, W_3 = 100 N, W_4 = 250 N. The locations are shown in the diagram. (a) Reduce the system of forces to a single force at G and finds it location. (b) Reduce the system of forces to a single force and a single couple at A.

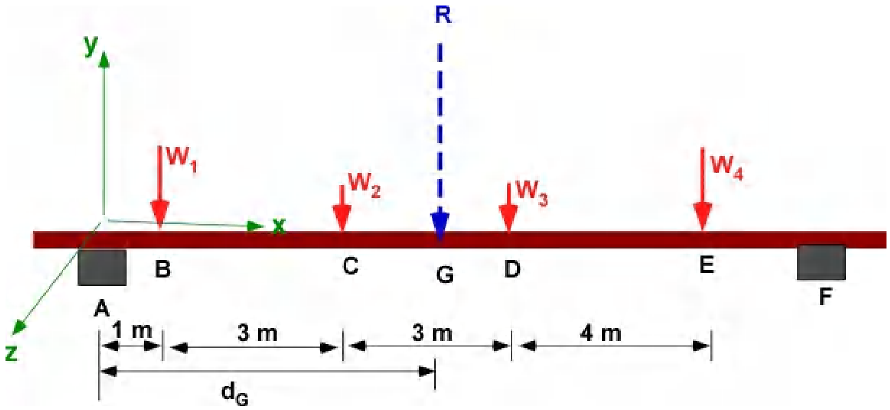


Figure 2.11.25a. Single resultant force for Example 2.7

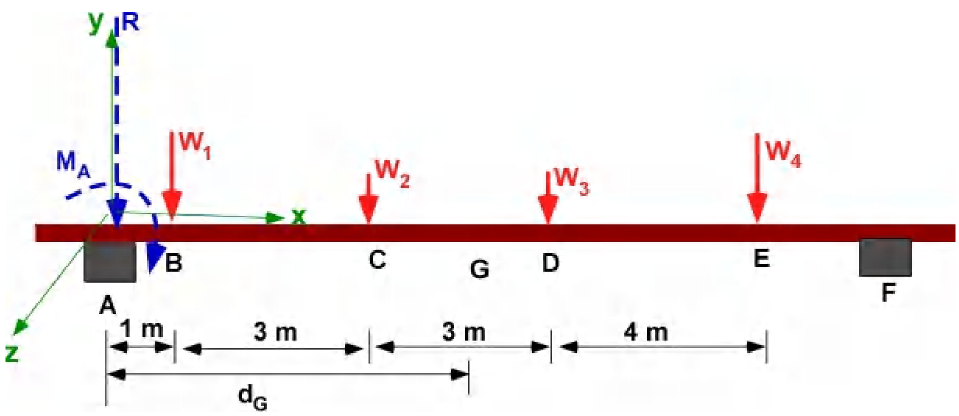


Figure 2.11.25b. Single resultant force and single couple for Example 2.7

Data: W_1 = 200 N, W_2 = 300 N, W_3 = 100 N, W_4 = 250 N. Since the force and distance are at right hand we can calculate moments using the product Fd instead of calculating it using a vector product. The directions are assigned using the right hand.

Find: (a) Resultant **R** and $\mathbf{d_G}$ (b) Resultant **R** and $\mathbf{M_A}$.

Solution:
(a)

$$\bar{R} = \sum \bar{F} = W_1 + W_2 + W_3 + W_4$$
$$= (-\hat{j})[200 + 300 + 100 + 250] = 850[N]$$
$$(-\hat{k})[d_G R] = (-\hat{k})[d_{AB} W_1 + d_{AC} W_2 + d_{AD} W_3 + d_{AE} W_4]$$
$$d_G = \frac{[1\times 200 + 4\times 300 + 7\times 100 + 11\times 250]}{850} = 5.706[m]$$

(b)

$$\bar{R} = \sum \bar{F} = W_1 + W_2 + W_3 + W_4$$
$$= (-\hat{j})[200 + 300 + 100 + 250] = 850[N]$$
$$\left(-\hat{k}\right)[M_A] = \left(-\hat{k}\right)[d_{AB} W_1 + d_{AC} W_2 + d_{AD} W_3 + d_{AE} W_4]$$
$$M_A = [1 \times 200 + 4 \times 300 + 7 \times 100 + 11 \times 250] = 4850[Nm]$$

Example 2.8 Resultant of a Set of Distributed Forces
We can do the same for the distributed force in Figure 2.11.26. The distributed force is usually represented as **w(x)** with basic units of [N/m]. The distributed force in this example is a constant. Here we wish to determine the value of the resultant force **R** and its location **d**.

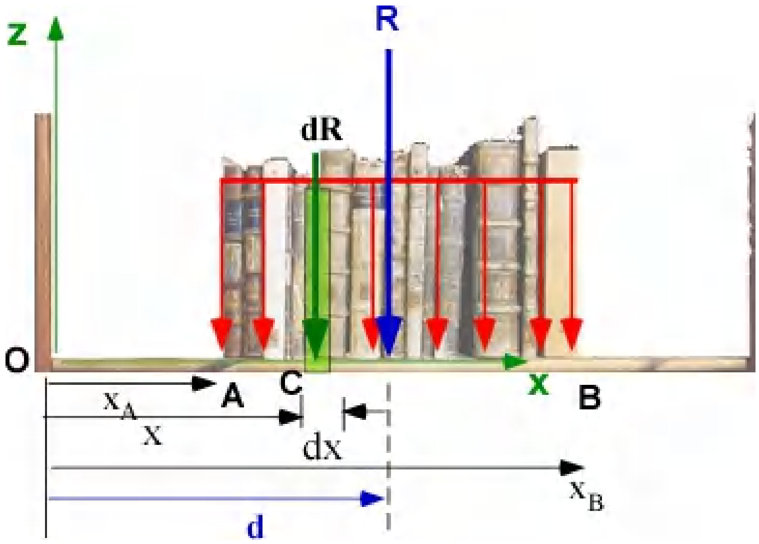


Figure 2.11.26. Example 2.8

The setup uses calculus to calculate the resultant and its location. It uses a differential element of the resultant **dR** and integrates it over the range of *x* to obtain **R**. It then calculates the location **d** by requiring that the moment at any point (say O) by the resultant **R** must be equal to the the distributed moment about the same point. The application is the same for all manner of distributed loads.

Data: The load distribution w(x) is constant. Let us say w(x) = k. It ranges from A to B along the x axis.

Find: (a) Resultant **R** (b) its location d

Solution: We will calculate the values symbolically. We will calculate the magnitude only since the direction is known to be in the negative z-direction.

(a) The force vector **R** is in the ***-k*** direction

$$dR = w(x)\,dx = k\,dx$$

$$R = \int_{x_B}^{x_A} dR = \int_{x_B}^{x_A} w(x)\,dx = \int_{x_B}^{x_A} k\,dx = k(x_B - x_A)$$

$$= \textit{area under the load distribution}$$

(b) The moment is into the board (***+j***). The moment is computed at O.

$$dM_O = x\,w(x)\,dx = k\,x\,dx$$

$$M_O = \int_{x_B}^{x_A} dM_O = \int_{x_B}^{x_A} x\,w(x)\,dx = \int_{x_B}^{x_A} k\,x\,dx = \frac{k}{2}\left(x_B^2 - x_A^2\right) = k(x_B - x_A)\left(\frac{x_B + x_A}{2}\right)$$

$$M_O = k(x_B - x_A)\left(\frac{x_B + x_A}{2}\right) = R\,d = k(x_B - x_A)\,d$$

$$d = \left(\frac{x_B + x_A}{2}\right)$$

$$= \textit{center of the distributed load}$$

2.11.7 Example 2.9

The 2D beam is loaded with concentrated forces, a distributed force and a couple. Find the resultant force $\mathbf{F_R}$ and its location **d** from A.

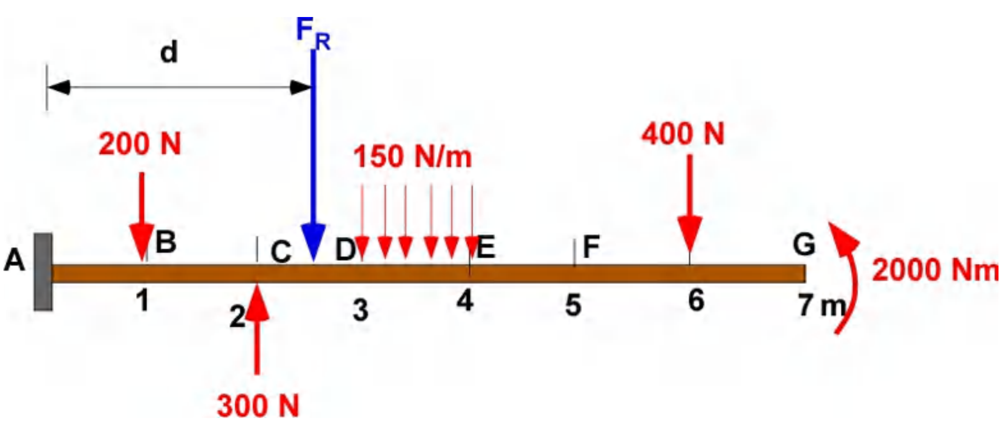


Figure 2.11.27 Example 2.9

Data: Beam and applied loading shown in red. $\mathbf{F_R}$ is the resultant force.
The uniform distributed force is replaced by the concentrated force of (150x1) [N] located at 0.5*(3 + 4) = 3.5 m from the edge A.
While it is not recommended you can solve this using positive directions for force and couple without using a coordinate system. Force is positive downward and couple is positive into the page.

Solution: $\mathbf{F_R}$ at a distance **d** from A produces the same effect on the structure as all the applied loads. Unlike the previous example here we have a single force as the resultant.

$\mathbf{F_R}$ = sum of all the **forces** applied irrespective of their location.

The distance **d** is the moment about A due to all applied loads divided by $\mathbf{F_R}$.

$$F_R = 200 - 300 + (150 \times 1) + 400 = 450\,N$$

$$d = \frac{(200 \times 1) - (300 \times 2) + (150 \times 3.5) + (400 \times 6) - 2000}{450} = 1.17\,m$$

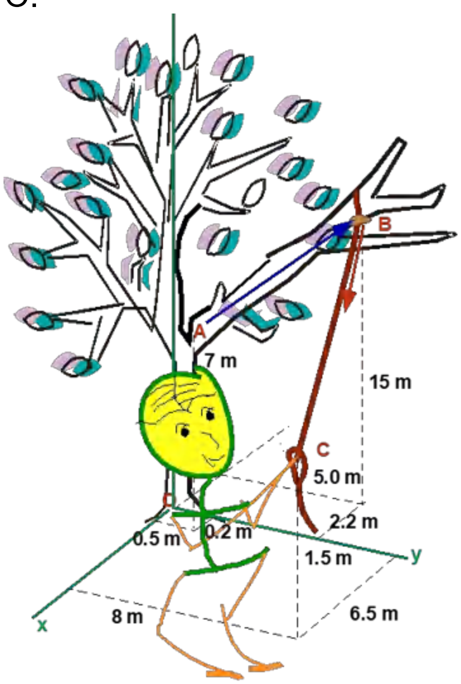
Note: $\mathbf{F_R}$ can be moved to A and in that case there will also be a couple at A. Can you work that out?

2.11.8 Additional Problems

Use Simplifying assumptions to reduce problem complexity if required. Compare answers by solving the problem again in MATLAB.

Problems 2.11.1

In your garden there is an old tree whose branch is starting to decay. You decide to see if you can break of this branch by applying a moment about the point A, after having notched the branch at the point A with a saw. You lasso the branch at the point B (as far from A as possible) using a rope and pull on it (along the direction BC) applying a force of magnitude 350 [N]. What is the moment of this force about the point A? The location of the various points are shown on the figure using the origin at O.

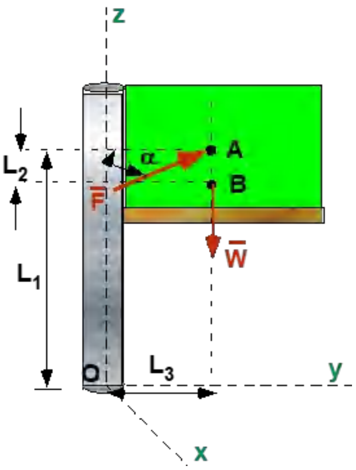


Problem 2.11.1

In case the figure is unclear the various coordinates are O (0, 0, 0); A (0.2, 0.5, 7); B (-2.2, 5.0, 15); C (6.5, 8, 1.5). All dimensions in [m]. F is along BC and has a magnitude of 350 [N].

Problem 2.11.2

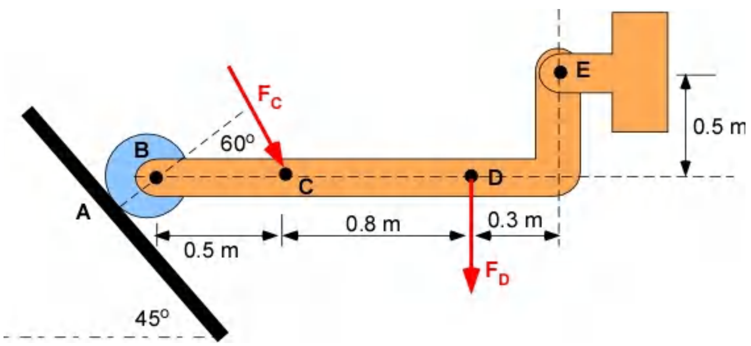
The sign in front of the establishment is being battered by the wind. You are concerned that it will be uprooted. The effect of the wind can be reproduced by a concentrated force F, located at A, of magnitude 1500 N, in the horizontal plane, and at an angle of α of 55 degrees to the plane of the sign which is in the yz plane. The stiffened sign weighs 980 N and the center of mass of the sign is at B. The lengths L1, L2, and L3 are 6.5 m, 0.6m, and 1.2 m respectively. What is the net moment (resultant moment) at the point O?



Problem 2.11.2

Problem 2.11.3

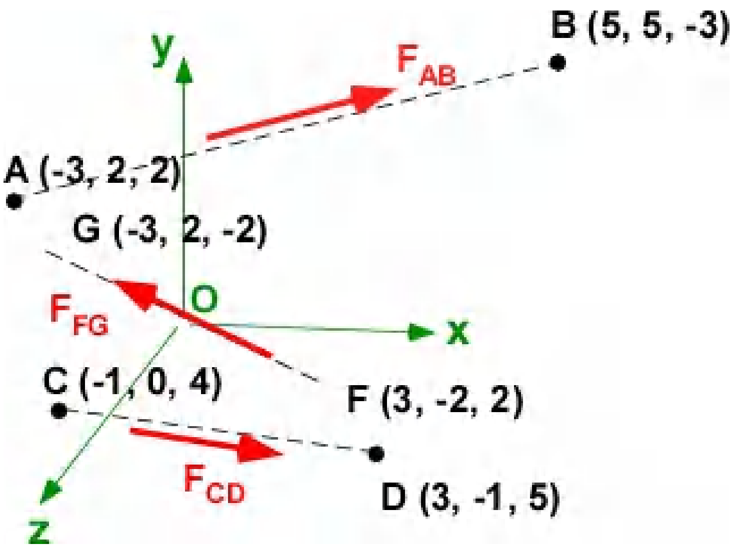
The L shaped link is acted on by two forces $\mathbf{F_C}$ and $\mathbf{F_D}$. Their magnitudes are 500 N each. The contact at A generates a magnitude of 750 N along AB. (a) Find the resultant of this force system at E. (b) If the resultant can be further reduced to a single force only, calculate the location of this resultant force.



Problem 2.11.3

Problem 2.11.4

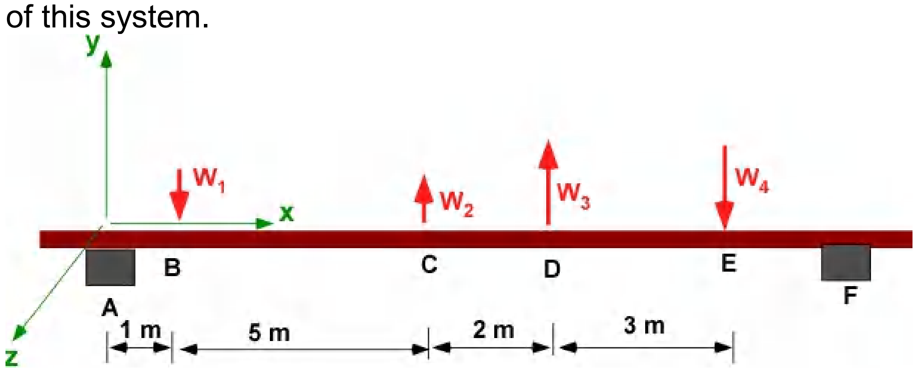
The three force system is shown in the figure. $\mathbf{F_{AB}}$ carries a force of 150 N. $\mathbf{F_{CD}}$ is a force of 250 N. $\mathbf{F_{FG}}$ is a force of 200 N. Find the resultant of the force system about the origin O.



Problem 2.11.4

Problem 2.11.5

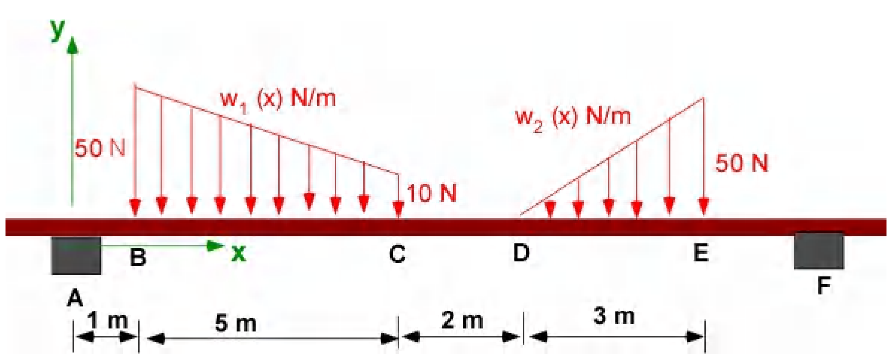
The parallel force system is shown in the figure. The magnitude of $\mathbf{W_1}$ and $\mathbf{W_2}$ are the same at 300 N. Similarly the magnitude of the weights $\mathbf{W_3}$ and $\mathbf{W_4}$ are the same at 500 N. Find the simplest resultant of this system.



Problem 2.11.5

Problem 2.11.6

Two linearly distributed loads are shown in the figure. For each of them (a) determine the expression that describes the loading function w(x); (b) The resultant (or equivalent concentrated) force; and (c) the location of the resultant force.



Problem 2.11.6

3. WHY DESIGNS FAIL?

Structural design is the design of objects, devices, components to perform their task without failure. These designs are withstanding loads (forces and moments) as they serve a purpose. You do not want the car to fall apart while riding on the expressway. Maybe if it bumps into another vehicle the fender breaks and fall off without harming the passengers. The airplane traveling between cities should not fall out of the sky while dealing with the pressure distribution around it. The roof of your house should not fall over your head when there is nothing disturbing outside. In all of these examples we are talking structural failure.

Any structure like a bridge, car, aircraft, building, the computer, or most anything else will be subject to loads (forces, moments, couples etc.) and wear and tear as part of their existence. They must handle/carry these loads without falling apart. This must be ensured when they are designed the first time. We do this by playing with the size, using a different material, changing the design to decrease the load in a particular member, or other remedies suggested by engineering practice and ethics. It is imperative for the design to carry the loads without failing. There are tremendous consequences if failure takes place. This is important for safety, reliability, convenience, and the economic well-being of your company.

In many designs failure may not imply that the structure breaks. It may mean that the structure does not recover the original shape after the loads have been removed. This is called elastic failure. For example when you sit on a chair during the lecture the chair probably deforms. This deformation may go unnoticed as it is designed to be small. Once you raise yourself, this deformation should disappears. It is like a spring. It stretches when you pull on it but returns to its original position when the force is removed.

While the loads are responsible for failure the actual cause depends on how they are internally distributed within the structure. Therefore the load per unit area of the cross section provides a more important determination of failure. This is called **stress**. The discussion of failure centers around the stresses the design can bear and the corresponding change in shape of the object or design. The relative change in shape is related to **strain**. For a given material, the stress and strain are are related and must be established experimentally as the property of the material. This is recognized as material behavior under loads. This chart of material behavior is a useful design tool for designing against failure.

Failure can also be more than stress and strain. It can be due to fatigue or it can be to corrosion. It can also be due to the misfortune of an earthquake or a tornado. In academic discussions stress and strain are usually the culprit for failure and that is the focus in this book.

This therefore brings a big difference to the problems until now where we considered that the object subject to loads is stationary - **statics**. Under the application of loads stresses are developed in the material and the structure experiences strain. This will cause the structure to elongate or change shape as it resists these loads. This resistance depends on the property of the structural material -

strength of materials. In many curriculum, *Strength of Materials* which allows for the exploration of design of structures is usually a follow up course to *Statics* . It is a natural extension. In fact most of the work in applying the strength of materials to a design is the solution of the corresponding statics problem. Studying *Statics* alone we cannot discuss design. In that case any discussion of design must be delayed to the follow up course. It makes sense to integrate *Statics* and *Strength of Materials* so that students are engaged in discussing design all the time. What makes it easy is that the concepts from the *Strength of Materials* course is only a small extension to the calculations but make a big difference in design thinking. The book will attempt to integrate the combination of *Statics* and *Strength of Materials* through application in the succeeding chapters after introducing concepts from *Strength of Materials* here.

3.1 NORMAL STRESS AND STRAIN

Stress is the way the materials internally handle the loads. Consider a uniform rod of a given length that is carrying a pair of equal and opposite forces (concentrated) at the end as shown in Figure 3.1.1. Note that the rod cannot just have one of the forces. In that case according to Newton's Law it will accelerate in the direction of the force. In statics our analysis starts with the idea of equilibrium. We must balance the force by having an equal and opposite one at the other end. This rod is in equilibrium - sum of forces on it must add to zero.This is a one-dimensional (1D) problem shown by the coordinate x.

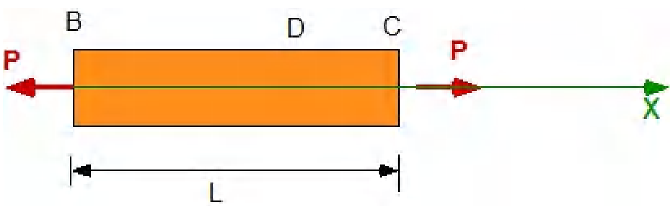


Figure 3.1.1 A rod in tension

Consider the cross section an an interior point in the rod, say at D. To expose what is happening at the cross section, we cut the section at D and draw the free body diagram of the length BD of the rod. We can argue the same with the piece CD. While P is a concentrated force applied load at B, in the cross section at D we need a force P for equilibrium. This will be resisted by all of the cross section. If we assume that P is *uniformly distributed* over the cross section, then this distribution of P over the cross section is called ***stress***. It is represented by the Greek letter **σ**. It is called a **normal stress**. The units are [N/m^2 or Pa]. Note the direction of the force and the normal to the area on which it acts. They are both parallel. This is the reason it is called a **normal stress**. In this illustration the applied force P attempts to stretch the rod and cause it to be in tension. This is a *tensile stress* and is considered *positive*. If the direction of P were reversed then the stress would be compressive and negative by convention.

Figure 3.1.2 Normal stress

$$\sigma_x = \frac{P}{A}; \quad A = area\ of\ cross\ section \tag{3.1}$$

This can be considered an internal reaction - or a reaction to force P by the interior of the material. Under a microscope the cross-section will appear to be composed of individual fibers of material with lots of gaps. It makes sense that the material may fail because some fiber carried more stress than it is was capable of. Distributing the stress uniformly makes for less calculations as the exact distribution of fiber thickness is difficult to establish.

We are looking at **deformable materials** so it is likely that the material will be pulled or increase its length in the direction of this applied load. The study of "Strength of Materials" assumes that materials will deform (or change shape) when subject to forces and moments. In the simplest analysis it is assumed that these deformations are small and will not affect the geometry of the loading. In the illustration of the uniaxial stress above, we expect that there will be a change of length along the direction of the load. For a tensile force there will be a tensile stress and an extension of the rod in the direction of the force. If we further assume that the density of the material is unaltered by this deflection, there will be a shortening of the length in the direction normal to the applied load (transverse direction). Figure 3.1.3 captures the deformation associated with this tensile load. The orange is the original material while the yellow in the deformed material due to the applied force P.

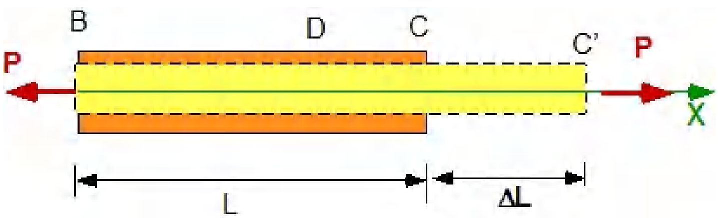


Figure 3.1.3. Definition of strain

This extension, ΔL, when divided by the original length L is defined as strain.

$$\varepsilon_x = \frac{\Delta L}{L} \qquad (3.2)$$

For one dimensional stress we ignore the lateral contraction. This contraction is called the **Poisson's effect**. We will introduce it later. For one-dimensional problem the subscripts on stress and strain are not necessary.

To summarize, the load carried by a structure is resisted in the cross section as a **stress**. This stress is associated with a strain in the cross section which can cause the structure to change shape and even break and therefore fail. The relation between the stress and strain depends on the material used in the structure. These stress and strain are called engineering stress and strain. They are based on the original area of cross section and original length of the member. The counterpart is true stress and strain based on the current area of cross section (reduced) and current length (longer for tension). Most design decisions are based on the engineering stress and strain.

Example 3.1.

A short rod of diameter 2 cm and length 10 cm is subject to a load P of 1500 N. It appears to deflect by 0.01cm under the load.(a) Find the stress and strain on the rod. (b) If the maximum allowable stress in the rod is 12 MPa what must be the diameter of the rod for the same load.

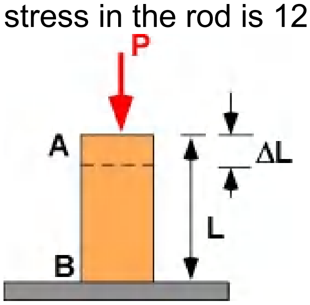


Figure 3.1.4. Example 3.1

Data: d = 2/100 m; L = 10/100 m; P = 1500 [N]; ΔL = 0.01/100 m; σ_{max}=12 MPa.

Assumptions: Uniform Stress

Find: (a) σ and ε ; (b) d_n

Solution:
(a)

$$area = \pi\frac{d^2}{4} = \frac{\pi}{4}0.02^2 = 3.142\times10^{-4}\left[m^2\right]$$

$$\sigma = \frac{P}{\pi d^2/4} = \frac{1500}{3.142\times10^{-4}} = 4.77\times10^6\left[Pa\right]$$

$$\varepsilon = \frac{\Delta L}{L} = \frac{0.01}{10} = 0.001$$

(b)

$$area_n = \frac{P}{\sigma_{max}} = \frac{1500}{12\times10^6} = 1.25\times10^{-4}\left[m^2\right]$$

$$d_n = \sqrt{\frac{4\,area_n}{\pi}} = 0.0126\left[m\right]$$

3.1.1 Point Stress

The assumption of uniform stress is convenient for discussion and calculations. It works well for initial engineering design and estimates. The actual cross-section of the bar may have a distribution of material fibers and empty spaces when viewed under a microscope. There could be a stress distribution on the surface rather than a uniform stress. In this case we can define stress at a point by using the limit from calculus.In Figure 3.1.5 the stress in the cross-section varies with every point.

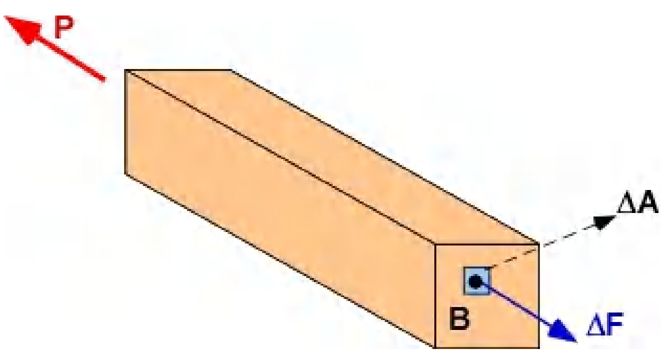


Figure 3.1.5 Point Stress

At any representative point B in the cross-section it can be defined as :

$$\sigma_B = \lim_{\Delta A\to 0}\frac{\Delta F}{\Delta A} \qquad (3.3)$$

With the constraint that (σ_B varies in the cross-section and is used as a place holder)

$$\int_A \sigma_x \, dA = P \qquad (3.4)$$

This makes the stress and load statically equivalent. Point loading as used in Figure 3.1.5 is a problem. They cause a local distortion of stress and strain that cannot be easily described as uniform stress. To uniformly distribute the stress edge plates are used to distribute the point load over the cross-section. This is illustrated in two-dimensions in Figure 3.1.6.

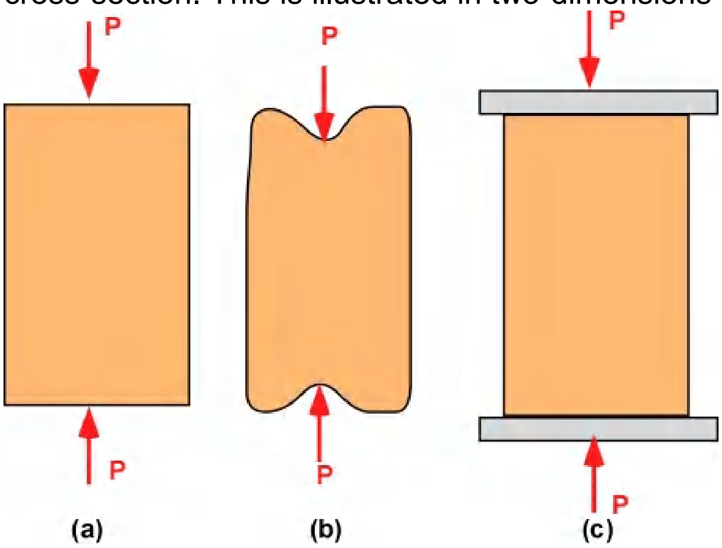


Figure 3.1.6 Applying uniform stress

Placing end plates does not completely solve the problem. It reduces crinkling at the ends but the stress distribution becomes uniform at a finite distance from the end. This is referred to as the *Saint-Venant effect.* It takes about a distance of the width into the member for the stress distribution to become uniform. This is true for any type of loads and stress and is not restricted to axial stress alone. This is illustrated in Figure 3.1.7.

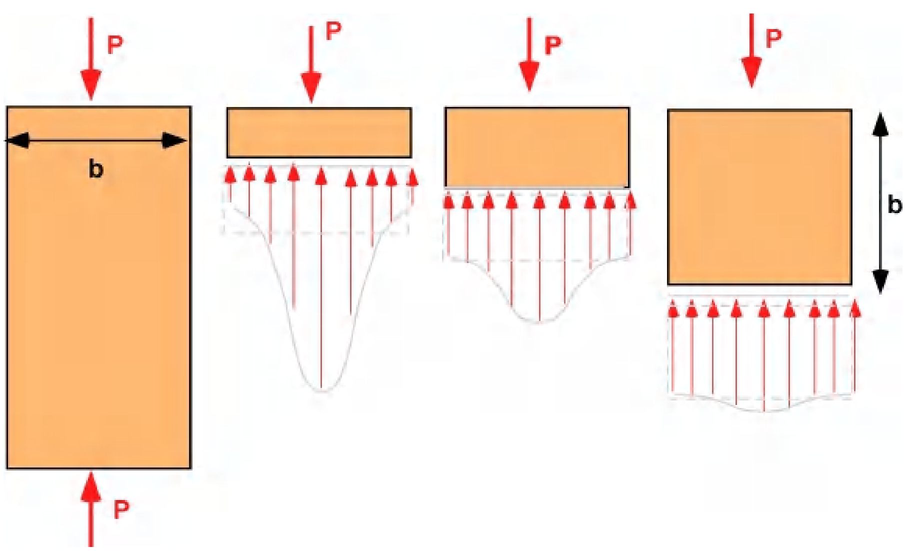


Figure 3.1.7 Saint Venant's principle

In general the load and the stress distribution must satisfy the following:

- The load and the stress distribution are statically equivalent.
- In the immediate region of the applied load, the stress cannot be determined analytically. It

must be established through experiments or simulation.
- In design the variation in stress is often ignored everywhere.

3.1.2 True Stress and True Strain

True stress and true strain are based on actual area of cross-section and the accumulated strain over the loading. It is more accurate in identifying material behavior. For design purposes the common reference to stress and strain are engineering stress and strain based on the original area and original length (before any load was applied. Let $\mathbf{A_o}$ be the original area and $\mathbf{L_o}$ be the original length. Let **A** and **L** be the current area and length respectively. Let ΔL be the deflection measured from original length for the current load P:

Engineering stress and strain : $\sigma = \dfrac{P}{A_o}; \quad \varepsilon = \dfrac{\Delta L}{L_o}$ (3.1, 3.2)

True stress and strain: $\sigma_t = \dfrac{P}{A}; \quad \varepsilon_t = \ln \dfrac{L}{L_o}$ (3.5)

The equations are based on experimental measurements and therefore difficult to illustrate through an example.

3.1.3 Example 3.2

A composite object is made of two different materials is fully attached at the intersection. The bottom (A) is held fixed. At the intersection (B) a compressive force of 5000 N is applied. At the free end C a tensile force of 3500 N is applied. The diameter of AB is 50 mm while the diameter of BC is is 30 mm. Their lengths are shown in the figure. (a) If the magnitude of the strain in each segment has the same value of 0.01 what is the net change in the length of the object. (b) What is the stress in each material. Figure 3.1.8(a) illustrates the problem.

The forces on each piece of the composite object is shown in Figure 3.1.8(b) and (c). In Figure 3.1.8(a) the body is stationary and any portion of the object is also stationary. Figure 3.1.8(b) is the FBD of BC just before the application of the compressive load of 5000 N. Every section in BC sees a tensile load of 3500 N since the net force must be zero as the section is stationary (this is called **equilibrium)**. Figure 3.1.8(c) is the FBD of BA just after the compressive load is applied. The net load on material AB is a compressive load of 1500 N and this portion must also be stationary. Therefore every piece of section AB is seeing a compressive load of 1500 N.

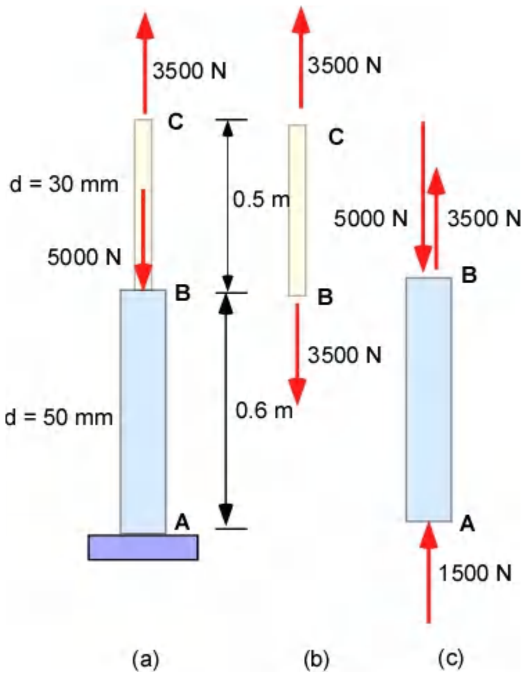


Figure 3.1.8 Example 3.2

Data: d_{AB} = 50/1000 m; L_{AB} = 0.6 m; d_{BC} = 30/1000 m; L_{BC} = 0.5 m.
F_{AB} (compression) = 1500 N; F_{BC} (tension) = 3500 N. $\varepsilon_{AB} = \varepsilon_{BC.}$= 0.01 (magnitude)

Assumptions: Uniform stress

Find: (a) Δ_{AC}; (b) σ_{AB} ; σ_{BC}

Solution:
(a)

$$\Delta_{AC} = \Delta_{AB} + \Delta_{BC} = -\varepsilon L_{AB} + \varepsilon L_{BC}$$
$$= -0.006 + 0.005 = -0.001[m]$$

(b)

$$A_{AB} = \pi \frac{d_{AB}^2}{4} = 0.00196\left[m^2\right]$$

$$A_{BC} = \pi \frac{d_{BC}^2}{4} = 0.00071\left[m^2\right]$$

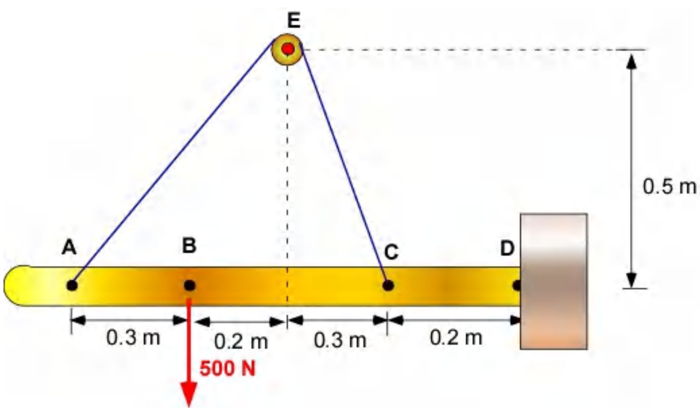
$$\sigma_{AB} = \frac{F_{AB}}{A_{AB}} = \frac{-1500}{0.00196} = -0.764\times10^6[Pa] = -0.764[MPa]$$

$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{3500}{0.00071} = 4.95\times10^6[Pa] = 4.95[MPa]$$

3.1.3 Additional Problems

Problem 3.1.1

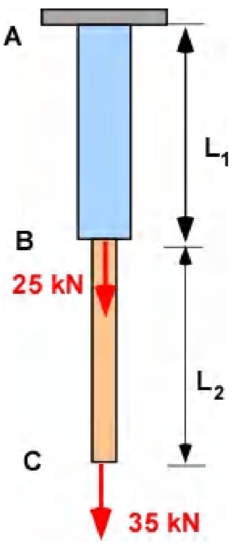
The following figure was used as an example for non concurrent force system in Section 2.10. Assume the force in the cable 200 N. What should be the diameter of the cable so that the stress does not exceed 2 MPa?



Problem 3.1.1

Problem 3.1.2

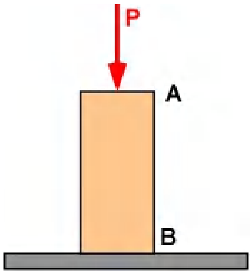
The composite rods are made of square cross-section and welded together. The rod AB is made steel and the maximum tensile stress in it cannot exceed 120 MPa. The rod BC is made of brass and the maximum stress in it cannot exceed 60 MPa. The length of the rods L_1 and L_2 are 25 cm and 30 cm. (a) Determine the dimensions of the cross-section of the rods.



Problem 3.1.2

Problem 3.1.3

A compressive force of 200 kN is applied on the rod with a square cross-section of side 4 cm. It is decided to replace the rod with circular cross-section of the same length and carrying the same stress. What should be the diameter of the rod.



Problem 3.1.3

3.2 HOOKE'S LAW

The Hooke's is one of the important relations used in structural design. It references the elastic behavior of materials where stress is proportional to strain. The constant of proportionality is a material constant that must be established by experiments. The law is used to ensure that the structure returns to undeformed state after the load is removed. Consider the bar of material used in the definition of normal stress and strain - reproduced here as Figure 3.2.1. You are going to test the material by increasing the load in tension progressively. You should also expect the material to progressively extend.

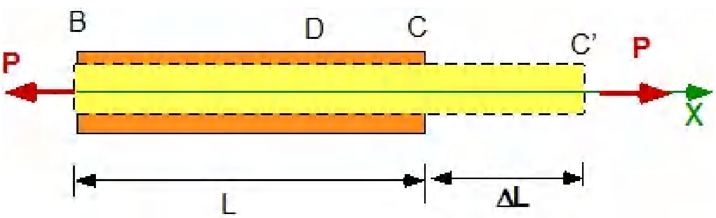


Figure 3.2.1 Force and elongation

A machine that can applied such a load is called a tensile test machine and is shown in Figure 3.2.2.

Figure 3.2.2 Tensile test
(image from from: Wikimedia Commons)

You can record the force and the corresponding elongation undergone by the material. You start from zero force and then increase it till you are able to break the specimen. You can choose to plot the applied force against the elongation. If you did that, for the same material, you will get a different curve for each specimen that had a different initial length or area of cross-section. If however you plotted the engineering stress and the corresponding strain then you will record a single curve for the same material even if your specimen changed. For a ductile metal like steel you are likely to see the red curve in Figure 3.2.3.

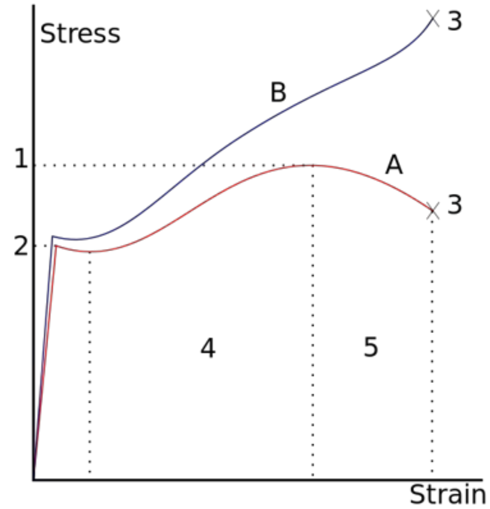


Figure 3.2.3 Stress Vs Strain for a ductile material
(image from from: Wikimedia Commons)

Stress-Strain Curve

Figure 3.2.3 is obtained experimentally. The red curve is the engineering stress strain curve. The blue is the true stress strain curve. The engineering curve is the one used for design. The machine in Figure 3.2.2 can actually break a two inch steel specimen. The point **3** is called ***rupture*** - where the material breaks. The stress corresponding to point **1** is the maximum stress or the ***ultimate stress***. The design limit for stress is indicated by the stress at point **2**, which represents the ***elastic limit***. The corresponding stress is noted as the ***proportional limit*** of the material. The stress-strain behavior is nonlinear after point **2**. The region **3** is called the ***strain hardening*** region - stress increase with strain. The region **5** is the ***necking*** region, where the strain increase with a reduction in stress. You can use the same data to plot the true stress and true strain curve. You can access the actual values for the properties though handbooks or the Internet. The American Society of Testing and Materials (ASTM) is the primary organization that publishes properties of materials through testing. It also provides a numbering system characterizing materials and their properties. For some materials the proportional limit also doubles as the yield stress (σ_y) or yield strength. Often stress and strength are used to describe the same quantity.

The structural steel, ASTM A36, has an yield strength of 250 MPa and an ultimate strength of 400-550 MPa.

True stress and strain are based on current area of cross section and current length of the specimen. Most design and analysis are based on the engineering stress-strain behavior.

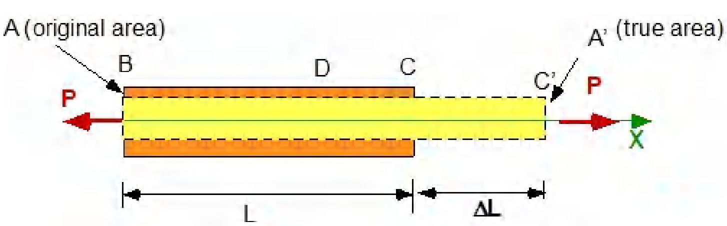


Figure 3.2.4 True stress and true strain

$$\sigma_t = \frac{P}{A'}; \quad true\,stress$$

$$\varepsilon_t = \int_L^{L+\Delta L} \frac{dL}{L} = \ln \frac{L+\Delta L}{L}; \quad true\,strain$$

3.2.1 Elastic Limit

Revisiting the stress strain curve in 3.2.3 it appears that the stress and strain are proportional until point 2. This is called the proportional limit. If this point is unique then it is also the elastic limit. If $\boldsymbol{\sigma_y}$ is the stress corresponding to this point, then for the stress below this value the material will behave like a linear spring. If you pull on it it will stretch but once you remove the pull then the material will get rid of its strain and is unstressed. This is a critical design value and is also a design limit - since you do not want the structure to have a different geometry every time it experiences a high stress value. This is where the stress starts decreasing with strain and the material starts yielding. Usually the elastic limit and the yield strength differ little for ductile materials. Unless otherwise noted the yield stress is also the elastic limit. For an elastic material then on can express the relationship between stress and strain as

$$\sigma_x = E\varepsilon_x \qquad (3.6)$$

Where E is the constant of proportionality, or the modulus of elasticity, or the Young's modulus. This is **Hooke's Law**. This is an experimentally determined property. Table 3.2.1 lists the value of E for several materials.

Table. 3.1 Approximate Young's modulus (E) for various materials

Material	**GPa**	**lbf/in² (psi)**
Aluminum	69	10.0×10^6
Aramid	70.5–112.4	$10.2\times10^6 – 16.3\times10^6$
Aromatic peptide nanospheres	230–275	$33.4\times10^6 – 39.9\times10^6$
Aromatic peptide nanotubes	19–27	$2.76\times10^6–3.92\times10^6$
Bacteriophage capsids	1–3	150,000–435,000
Beryllium (Be)	287	41.6×10^6
Brass	100–125	$14.5\times10^6 – 18.1\times10^6$
Bronze	96–120	$13.9\times10^6 – 17.4\times10^6$
Carbon fiber reinforced plastic (50/50 fibre/matrix, biaxial fabric)	30–50	$4.35\times10^6 – 7.25\times10^6$
Carbon fiber reinforced plastic (70/30 fibre/matrix, unidirectional, along grain)	181	26.3×10^6
Carbyne (C	32,10	4.66×10^9
Copper (Cu)	117	17.0×10^6
Diamond (C)	1,050 - 1210	$152\times10^6–175\times10^6$
Diatom frustules (largely silicic acid)	0.35–2.77	50,000–400,000

Flax fiber	58	8.41×10^6
Glass (see chart)	50–90	$7.25 \times 10^6 - 13.1 \times 10^6$
Glass-reinforced polyester matrix	17.2	2.49×10^6
Graphene	1,050	152×10^6
HDPE	0.8	116,000
Hemp fiber	35	5.08×10^6
High-strength concrete	30	4.35×10^6
Human Cortical Bone	14	2.03×10^6
Low density polyethylene	0.11–0.45	16,000–65,000
Magnesium metal (Mg)	45	6.53×10^6
Medium-density fiberboard (MDF	4	580,000
Molybdenum (Mo)	329 - 330	$47.7 \times 10^6 – 47.9 \times 10^6$
Mother-of-pearl (nacre, largely calcium carbonate)	70	10.2×10^6
Nylon	2–4	290,000–580,000
Oak wood (along grain)	11	1.60×10^6
Osmium (Os)	525 - 562	$76.1 \times 10^6 – 81.5 \times 10^6$
polycrystalline Yttrium iron garnet (YIG)	193	28.0×10^6
Polyethylene terephthalate (PET)	2–2.7	290,000–390,000
Polypropylene	1.5–2	218,000–290,000
Polystyrene	3–3.5	440,000–510,000
PTFE (Teflon)	0.5	75,000
Rubber (small strain)	0.01–0.1	1,450–14,503
Silicon carbide (SiC)	450	65×10^6
Silicon Single crystal, different directions	130–185	$18.9 \times 10^6 - 26.8 \times 10^6$
single-crystal Yttrium iron garnet (YIG)	200	29.0×10^6
Single-walled carbon nanotube	1,000	150×10^6 +
Steel (ASTM-A36)	200	29.0×10^6
Stinging nettle fiber	87	12.6×10^6
Titanium (Ti)	110.3	16.0×10
Titanium alloys	105–120	$15.0 \times 10^6 - 17.5 \times 10^6$
Tooth enamel (largely calcium phosphate)	83	12.0×10^6
Tungsten (W)	400 – 410	$58 \times 10^6 – 59 \times 10^6$
Tungsten carbide (WC)	450 – 650	$65 \times 10^6 – 94 \times 10^6$
Wrought iron	190–210	$27.6 \times 10^6 - 30.5 \times 10$

(from ***wikipedia***)

3.2.2 Example 3.3

Material properties are obtained by testing a specimen of a prescribed geometry in a testing machine. The material is silicon. The specimen has a uniform width of 1.6 mm. The geometry is shown on the left. These are standard specimens whose dimensions are prescribed through standards.(a) What is the change in length between A and B? (b) What is the true strain at this point?

$$\varepsilon_t = \ln \frac{\Delta L}{L} = \ln \frac{L\varepsilon_x}{L} = 1.0081\times 10^{-04}$$

Figure 3.2.5 Example 3.3

Data:
P = 250 N; L = 50/1000 m
w = 12/1000 m; d = 1.6/1000 m
E = 1.29*1011 Pa;

Assumptions:
Uniform one dimensional stress and strain
Stress within elastic limit (use Hooke's Law)

Find: (a) the change in length between A and B, (b) true stress, and true strain.

Solution:
(a)

$$A = w * d = 1.92\times 10^{-05}\, m^2$$

$$\sigma = \frac{P}{A} = 1.302\times 10^{07}\, Pa$$

$$\varepsilon_x = \frac{\sigma}{E} = 1.0082\times 10^{-04}; \quad \Delta L = L * \varepsilon_x = 5.041\times 10^{-06}\, m$$

(b)

$$\varepsilon_t = \ln \frac{L_{new}}{L} = \ln \frac{L\varepsilon_x}{L} = 1.0081\times 10^{-04}$$

3.2.3 Displacement in Elastic region

The displacement or the change in the length of the specimen under normal stress can be calculated using the definitions of stress and strain and Hooke's Law. Referencing Figure 3.2.1

$$\sigma = \frac{P}{A}; \quad \varepsilon = \frac{\Delta L}{L}; \quad \sigma = E\varepsilon$$

$$\frac{P}{A} = E\frac{\Delta L}{L}; \qquad (3.7)$$

$$\Delta L = \frac{PL}{EA}$$

The last equation above is very useful.

Example 3.4
The composite rod is made of solid circular cross-section and welded together. The rod AB is made steel with E = 200 GPa and the maximum tensile stress in it cannot exceed 120 MPa. The rod BC is made of brass with a E of 105 GPa and the maximum stress in it cannot exceed 60 MPa. The length of the rods L_1 and L_2 are 25 cm and 30 cm. Determine the final length of the rod under the action of the applied loads.

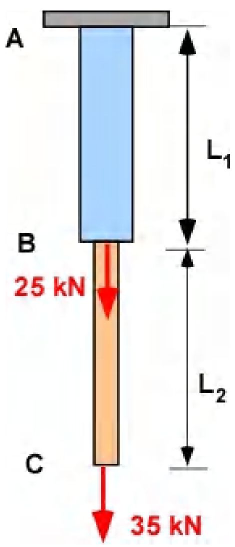


Figure 3.2.6a Example 3.4

Data:
E_{AB} = 200 GPa; σ_{AB} = 120 MPa; L_{AB} = 25/100 m; P_{AB} = 25 kN
E_{BC} = 105 GPa; σ_{BC} = 60 MPa; L_{BC} = 30/100 m; P_{BC} = 35 kN

Assumptions:
Uniform one dimensional stress and strain
Stress within elastic limit (use Hooke's Law)

Find: (a) Final length of the rod.

Solution:
The stress in the rods are distributed as in Figure 3.2.6b. Every section in BC will resist a force of 35 kN in the cross-section by developing a corresponding stress.. Similarly every section of rod AB will resist a force of 60 kN.

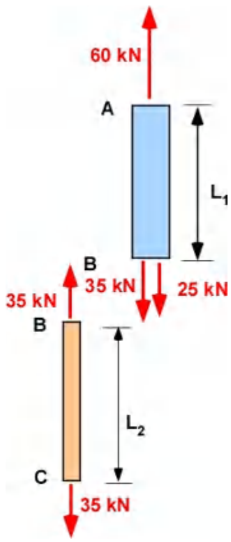


Figure 3.2.6b FBD - Example 3.4 - force in the rods

$$A_{AB} = \frac{P_{AB}}{\sigma_{AB}} = \frac{60000}{120\times1^{6}} = 0.0005 \quad \left[m^{2}\right]$$

$$A_{BC} = \frac{P_{AB}}{\sigma_{AB}} = 15\frac{60000}{60\times1^{6}} = 0.0005833 \quad \left[m^{2}\right]$$

$$\Delta_{AB} = \frac{P_{AB}L_{AB}}{E_{AB}A_{AB}} = \frac{60000\times0.25}{200\times1^{9}\times0.0005} = 0.00015\left[m\right]$$

$$\Delta_{BC} = \frac{P_{BC}L_{BC}}{E_{BC}A_{BC}} = \frac{35000\times0.3}{105\times1^{9}\times0.0005833} = 0.00017143\left[m\right]$$

$$\Delta_{AC} = \Delta_{AB} + \Delta_{BC} = 0.00032143\left[m\right]$$

$$L = L_{AB} + L_{BC} + \Delta_{AC} = 0.25 + 0.3 + 0.00032143 = 0.55032\left[m\right]$$

3.2.4 Displacement and Statically Indeterminate Problem

A principal idea from mathematics is that if you are establishing a value for an unknown quantity (or unknown) then you need an algebraic relation that involves the unknown quantity. This relation is traditionally known as an *equation*. The relation is set up with an equal sign and therefore the name *equation*. In general, if you have several unknowns to establish then you will need the same number of equations, each containing one or more unknowns, to identify the unknown quantities. These equations must be different from each other to work.

A statically indeterminate problem is one where equations from *statics* alone cannot solve for all of the unknowns. That is there are more unknowns than the number of equations. Therefore to obtain a solution additional relations must be established. *Strength of materials* provides these additional equations if constraints are involved. In Figure 3.2.7a,describes a composite rod that is held between the walls. It is subject to a force **P** at the junction B as shown as shown. The rod can have different properties in segment AB and BC which we will indicate by subscripts. The wall will induce reactions on the rod AC due to the applied load P at the ends A and C. There are two reactions. What are the values for these reactions?

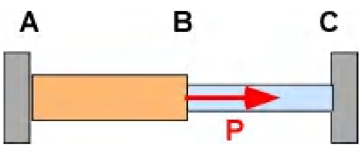


Figure 3.2.7a Composite rod with force

To expose the reactions at the wall a FBD of the rod is required. This is shown in Figure 3.2.7b. This is a one-dimensional problem and the coordinate used is x. The two unknown reactions are A_x and C_x.

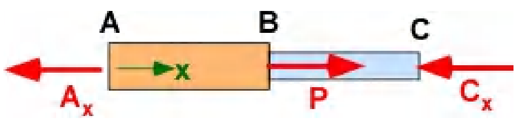


Figure 3.2.7b FBD of rod

Statics: Rod is stationary or in equilibrium

$$-A_x + P - C_x = 0;$$
$$A_x + C_x = P \quad (i)$$

There are two unknowns and one equation>

Strength of Materials: The overall length of the rod cannot change as it is held between two walls. However The segments will change length because of the load in each of them.
The portion AB is in tension and will expand. The force in AB is $\mathbf{A_x}$.
The portion BC is in compression and will contract. The force in BC is $\mathbf{C_x}$.

$$\Delta_{AB} + \Delta_{BC} = 0$$
$$\frac{A_x L_{AB}}{E_{AB} A_{AB}} + \frac{C_x L_{BC}}{E_{BC} A_{BC}} = 0 \qquad (ii)$$

The two equations (i) and (ii) can be used to solve for $\mathbf{A_x}$ and $\mathbf{C_x}$ which is done in the example below. The solution for $\mathbf{C_x}$ must be negative since it is in compression.

Example 3.5 (Modified Example 3.4)
The composite rod is made of solid circular cross-section and welded together. The rod AB is made steel with E = 200 GPa with a diameter of 50 mm. . The rod BC is made of brass with a E of 105 GPa with a diameter of 35 mm. The length of the rods L_1 and L_2 are 25 cm and 30 cm. Determine the reactions at the ends of the rod.

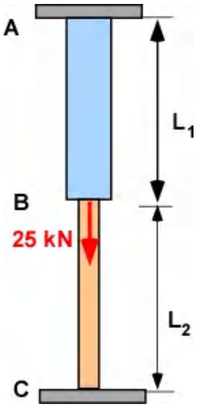


Figure 3.2.8a Example 3.5

Data:
E_{AB} = 200 GPa; d_{AB} = 50/1000 m; L_{AB} = 25/100 m; P = 25 kN
E_{BC} = 105 GPa; d_{BC} = 35/1000 m; L_{BC} = 30/100 m;

Assumptions:
Ignore weight of the rods
Uniform one dimensional stress and strain
Stress within elastic limit (use Hooke's Law)

Find: (a) Final length of the rod.

Solution:
The portion AB is in tension and will expand. The force in AB is $\mathbf{A_x}$.
The portion BC is in compression and will contract. The force in BC is $\mathbf{C_x}$.
The overall length of the rod cannot change

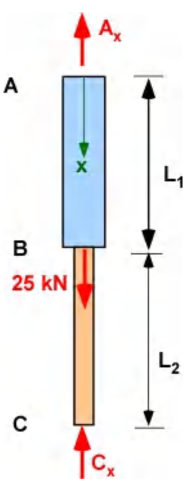


Figure 3.2.8b FBD **-** Example 3.5

Solution Using MATLAB
In MATLAB Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 3.2.4 - Example 3.5
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short g
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------\n')
fprintf('Example 3.5 \n')
fprintf('--------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data

EAB = 200e9; dAB = 50/1000;  LAB = 25/100;  P = 25e03;
EBC = 105e9; dBC = 35/1000;  LBC= 30/100;  PBC = 35e03;

fprintf('--------  Weights ----------\n')
fprintf('EAB [GPa] = '),disp(EAB/1e09)
fprintf('EBC [GPa] = '),disp(EBC/1e9)
fprintf('dAB [m] = '),disp(dAB)
fprintf('dBC [m] = '),disp(dBC)
fprintf('LAB [m] = '),disp(LAB)
fprintf('LBC [m] = '),disp(LBC)
fprintf('P [N] = '),disp(P)


%% calculations
% Part (a)
fprintf('---------------------------\n')
fprintf('---     Part (a)     ------\n')
fprintf('---------------------------\n')

syms Ax Cx
areaAB = pi*dAB^2/4;
areaBC = pi*dBC^2/4;
eq(1)= Ax + Cx - P;

delAB = Ax*LAB/EAB/areaAB;
delBC = Cx*LBC/EBC/areaBC;
eq(2) = delAB + delBC;

sol = solve(eq);
Ax = double(sol.Ax);
Cx = double(sol.Cx);
fprintf('areaAB [m^2] = '),disp(areaAB)
fprintf('areaBC [m^2] = '),disp(areaBC)
fprintf('Ax [N] = '),disp(Ax)
fprintf('Cx [N] = '),disp(Cx)
```

In the Command Window

```
-------------------------------
Example 3.5
--------------------------------
--------  Data ----------
EAB [GPa] =    200
EBC [GPa] =    105
dAB [m] =         0.05
dBC [m] =        0.035
LAB [m] =         0.25
LBC [m] =          0.3
P [N] =        25000
---------------------------
```

```
---      Part (a)      ------
----------------------------
areaAB [m^2] =      0.0019635
areaBC [m^2] =    0.00096211
Ax [N] =         31822
Cx [N] =       -6821.8
```

Please Check these solution using a calculator.

Execution In Octave

The code is the same as in MATLAB except for the additional statements below . The changes are highlighted. You must include the symbolic package.

```
pkg load symbolic
sympref display flat
%% Data
```

In Octave Command Window

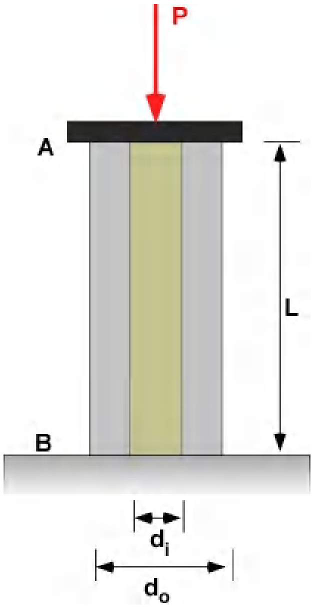
```
--------------------------------
Example 3.5
---------------------------------
--------  Weights ----------
EAB [GPa] = 200
EBC [GPa] = 105
dAB [m] = 0.05
dBC [m] = 0.035
LAB [m] = 0.25
LBC [m] = 0.3
P [N] = 25000
----------------------------
---      Part (a)      ------
----------------------------
areaAB [m^2] = 0.0019635
areaBC [m^2] = 0.00096211
Ax [N] = 31822
Cx [N] = -6821.8
```

3.2.5 Additional Problems

For the problems below use Table 3.1 for values of the modulus of elasticity. Also solve the problems by hand and using MATLAB/Octave. Ensure the solutions match.

Problem 3.2.1

The object AB is made of a hollow aluminum cylinder with a brass insert. The outside diameter of the brass rod is the same as the inside diameter of the aluminum cylinder. A rigid end plate ensures that the deflection in the two materials are the same and the load is shared between the aluminum and brass material. The length of the object is L = 0.6 m. The outside diameter of aluminum cylinder (d_o) is 55 mm. The inside diameter (d_i) is 30 mm. The applied load is 20 kN. Find: (a) The load in the aluminum and the brass materials. (b) The stresses in the material. Do the stresses exceed the yield stress of the material. This will require looking up information. (c) The deflection of the object due to the applied force?

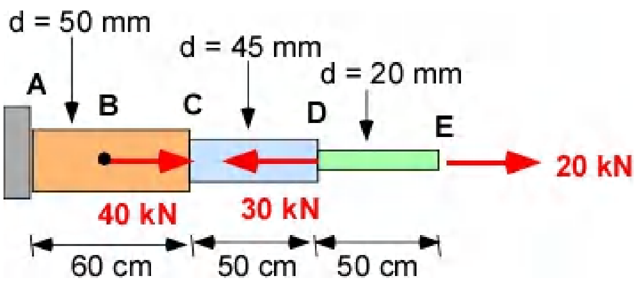


Problem 3.2.1

Problem 3.2.2

Three different materials are joined together and subject to the forces shown. The material for segment AC is Copper. Point B, where the 40 kN force is applied, is located midway AC. The material for segment CD is brass. The force of 30 kN is applied at the junction D. The material for segment DE is steel.The lengths and the diameter of the segments are shown on the figure. (a) Calculate the change in the length of AE after calculating the change in the lengths of each segment. (b) Calculate the stress in the materials and check if they are below the yield stress.

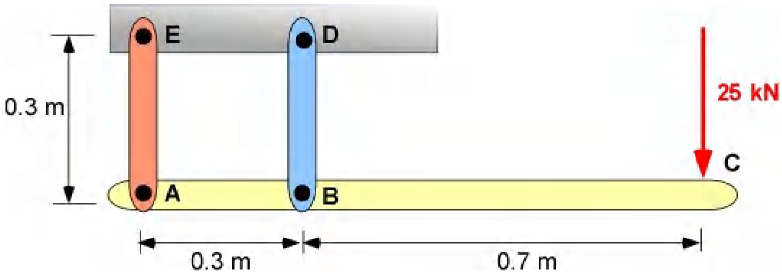
Remember to draw the FBD of each portion of the composite rod needed for calculations.



Problem 3.2.2

Problem 3.2.3

The structure experiences a force of 25 kN applied at C on the bar AC as shown. AE is a brass rod and BD is aluminum. There are pinned to the walls and to the bar AC. The bar AC is rigid.The area of cross-section of AE and BD is the same and is 600 mm^2. (a) Find the forces in the brass and aluminum rods. (b) Find the deflection in the brass and aluminum rods



Problem 3.2.3

3.3 THERMAL STRAIN

When structural materials are subject to a change in temperature they experience a strain or change in the length. It can increase if the temperature increases above the normal operating temperature and decreases if the temperature falls below the operating temperature. It's effect is three dimensional though it is included in the design of long members - like railroad tracks. Generally there is a gap between two sections of the railroad to accommodate this change in length for elevated temperature. As the world's average temperature increases these designs must be changed at significant cost. Figure 3.3.1 is an image of buckled railroad tracks where trains have to slow down. Tracks buckle when there is no space to expand and tremendous compressive forces are created due to this constraint.

Figure 3.3.1 Buckling of railroad tracks - thermal strain
(Courtesy of US Department of transportation - Volpe Center)

There are also other changes in structural materials due to change in temperature whether natural or deliberate. These changes are defined through the following:

- Thermal strain – change in the length of structure with temperature
- Change in material properties due to temperature
- Temperature cycling – for example the aircraft operating between Florida and Buffalo (NY) in the winter will have its structural properties degraded because it is constantly operating in the warm and cold temperatures alternatively in a predictable pattern. This is associated with reduced product life.
- Thermal stresses – if thermal strain is constrained or their is a mismatch in thermal properties,

there will be new stress the material has to withstand. This may cause unexpected forces in the structures

- Decrease in structural capacity due to creep. Creep is a time-dependent deformation under an applied load which is enhanced with high temperature.

You can see the design for thermal strains in rail tracks, bridge structures, and other structures with metals where large increase in temperatures are expected. There is usually a gap between long metal members so that they do not butt each other and develop thermal stresses under change in temperature. A formal definition of one dimensional thermal strain is shown in Figure 3.3.2.

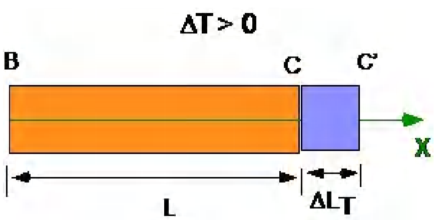


Figure 3.3.2 Thermal strain

A bar of length L is subject to an increase in temperature ΔT will increase in length by ΔL_T. A decrease in temperature will cause a corresponding decrease.This change in length as a fraction of the original length is the thermal strain.

$$\varepsilon_T = \frac{\Delta L_T}{L} = \alpha \Delta T; \quad \textit{thermal strain} \qquad (3.8)$$

This increase in length can be computed as

$$\Delta L_T = \alpha \Delta T L; \qquad (3.9)$$

The force associated with the strain is:

$$P_T = A(\sigma_T) = AE\varepsilon_T = AE\frac{\Delta L_T}{L} = AE\alpha\Delta T \qquad (3.10)$$

Where **α** is the coefficient of thermal expansion of the material. This is a material property that is established through experiments and published by standards body like ASTM.

3.3.1 Example 3.6

The copper board of width 4 inches fits snugly between rigid walls of the box. If the temperature increases by 50 ^oF, (a) what is the stress developed in the material? (b) What is the change in the thickness of the sheet?

Note: This is a rare example in US units. The unit conversion is implemented in MATLAB.

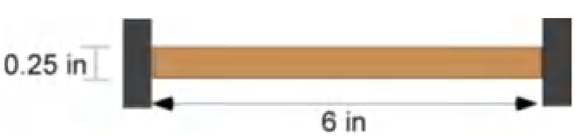


Figure 3.3.3a Example 3.6

Given Data:
L = 6 in, w = 0.25 in, d = 4 in
E =17x10^6 psi, v = 0.33 and
α = 9.6x10^{-6}/^{o}F
Properties for copper: E =17x10^6 psi, v = 0.33 and α = 9.6x10-6/^{o}F

Assumptions:
Uniform stress and strain
Stress within elastic limit (use Hooke's Law)

Solution:

Figure 3.3.3b Example 3.6

Let us remove the right rigid end. Increase in temperature will cause the dimensions to increase.

$$\Delta L = \alpha L \Delta T = 2.8800 \times 10^{-3}\ in$$

$$\Delta w = \alpha w \Delta T = 1.2000 \times 10^{-4}\ in$$ 1/oF

$$\Delta d = \alpha d \Delta T = 1.9200 \times 10^{-3}\ in$$

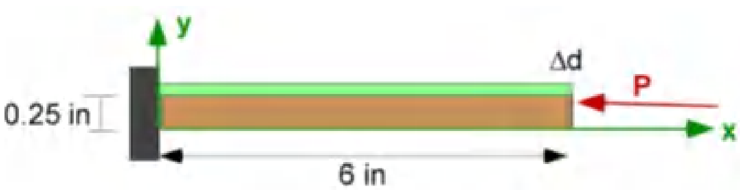


Figure 3.3.3c Example 3.6

The right side cannot expand. Therefore a compressive force P is developed to nullify the thermal expansion.

$$P = \frac{AE\Delta L}{L} = 8.1600 \times 10^3\ lb$$

$$\sigma = -\frac{P}{A} = 8.1600 \times 10^3\ psi$$

The change in w will be the increase due to temperature

$$\Delta w|_{\Delta T} = 1.2000 \times 10^{-4}\ in$$

The above example was different. It was in the **US unit system**. However we can use MATLAB to translate the results to the **SI system**. Note the tabbed and formatted printing

Solution in MATLAB
In Editor Window

```
% Essential Mechanics
% P. Venkataraman
% Section 3.3.1 - Example 3.6
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short g
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('--------------------------------\n')
fprintf('Example 3.6 \n')
fprintf('---------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Conversion factors used
% 1 ft = 12 in
% 1 ft = 0.3048 m
% (DT) C = (DT)F *5/9;
% 1 lb   = 4.448 N
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data  (US units)
in = 0.3048/12;
w = 0.25;  wsi = w*0.3048/12;
L = 6;  Lsi = L*0.3048/12;
d = 4;  dsi = d*0.3048/12;
A = w*d;   Asi = wsi*dsi;
DT = 50;  DTsi = DT*5/9;
aL = 9.6e-6;   aLsi = aL*9/5;
E = 17e6;  Esi = E*4.448/in^2;


dL = aL*L*DT;  dLsi = aLsi*Lsi*DTsi;
dw = aL*w*DT;   dwsi = aLsi*wsi*DTsi;
dd = aL*d*DT;   ddsi = aLsi*dsi*DTsi;
P = A*E*aL*DT;
Psi = Asi*Esi*aLsi*DTsi;
sigma = P/A;
sigmasi = Psi/Asi;

fprintf('---------  Data -------------------------------\n')
fprintf('----------------------------------------------\n')
fprintf('\td [in] = %4.2f ;  d [m] = %4.4f \n',d,dsi)
fprintf('\tw [in] = %4.2f ;  w [m] = %4.4f\n ',w,wsi)
fprintf('\tL [in] = %4.2f ;  L [m] = %4.4f\n ',L,Lsi)
fprintf('\tA [in^2] = %4.2f ;  A [m^2] = %4.4f\n\n ',A,Asi)

fprintf('\tDT [oF] = %4.2f ;  DT [oC] = %4.4f\n\n ',DT,DTsi)

fprintf('\talpha [1/oF] = %4.7f ;  alpha [1/oC] = %4.7f\n ',aL,aLsi)
fprintf('\tE [kpsi] = %4.3f ;  E[GPa] = %4.2f\n ',...
    E/1000,Esi/1000000000)

fprintf('\n---------  Results  --------------------------------\n')
fprintf('----------------------------------------------\n')
```

```
fprintf('\tP [lb] = %4.1f ;  P [N] = %4.1f \n',P,round(Psi))
fprintf('\tsigma [psi] = %4.1f ;  sigma [Pa] = %4.1f \n',...
sigma,round(sigmasi))
fprintf('\tDw [in] = %4.6f ;  Dw [mm] = %4.4f \n',dw,1000*dwsi)
fprintf('\tDd [in] = %4.6f ;  Dd [mm] = %4.4f \n',dd,1000*ddsi)
fprintf('\tDL [in] = %4.6f ;  DL [mm] = %4.4f \n',dL,1000*dLsi)
```

In Command Window

```
-------------------------------
Example 3.6
--------------------------------
---------  Data -------------------------------
-----------------------------------------------
           d [in] = 4.00 ;  d [m] = 0.1016
           w [in] = 0.25 ;  w [m] = 0.0064
           L [in] = 6.00 ;  L [m] = 0.1524
           A [in^2] = 1.00 ;  A [m^2] = 0.0006

           DT [oF] = 50.00 ;  DT [oC] = 27.7778

           alpha [1/oF] = 0.0000096 ;  alpha [1/oC] = 0.0000173
           E [kpsi] = 17000.000 ;  E[GPa] = 117.21

---------  Results  --------------------------------
-----------------------------------------------
           P [lb] = 8160.0 ;  P [N] = 36296.0
           sigma [psi] = 8160.0 ;  sigma [Pa] = 56258417.0
           Dw [in] = 0.000120 ;  Dw [mm] = 0.0030
           Dd [in] = 0.001920 ;  Dd [mm] = 0.0488
           DL [in] = 0.002880 ;  DL [mm] = 0.0732
```

Execution in Octave
The code is same as in MATLAB above.

In Octave Command Window

```
-------------------------------
Example 3.6
--------------------------------
---------  Data -------------------------------
-----------------------------------------------
        d [in] = 4.00 ;  d [m] = 0.1016
        w [in] = 0.25 ;  w [m] = 0.0064
        L [in] = 6.00 ;  L [m] = 0.1524
        A [in^2] = 1.00 ;  A [m^2] = 0.0006

        DT [oF] = 50.00 ;  DT [oC] = 27.7778

        alpha [1/oF] = 0.0000096 ;  alpha [1/oC] = 0.0000173
        E [kpsi] = 17000.000 ;  E[GPa] = 117.21

---------  Results  --------------------------------
-----------------------------------------------
```

```
P [lb] = 8160.0 ;  P [N] = 36296.0
sigma [psi] = 8160.0 ;  sigma [Pa] = 56258417.0
Dw [in] = 0.000120 ;  Dw [mm] = 0.0030
Dd [in] = 0.001920 ;  Dd [mm] = 0.0488
DL [in] = 0.002880 ;  DL [mm] = 0.0732
```

The results are the same. No coding necessary.

3.3.2 Thermal expansion coefficients of materials

(From : Wikipedia)

The following are the coefficient of linear thermal expansion (CLTE) of some popular materials

Table 3.2 Coefficient of linear thermal expansion (CLTE)

Material	**Linear coefficient CLTE α at 20 °C (10^{-6} /°K)**	
Aluminium	23.1	
Aluminium nitride		
Brass	19	
Carbon steel		
CFRP	– 0.8	
Concrete	12	
Copper	17	
Diamond	1	
Douglas-fir 27		
Glass	8.5	
Gold	14	
Ice	51	
Iron	11.8	
Lead	29	
Magnesium	26	
Mercury	61	
Molybdenum	4.8	
Nickel	13	
Oak	54	Perpendicular to the grain
Platinum	9	
Silicon	2.56	
Silver	18	
Stainless steel	10.1 ~ 17.3	
Steel	11.0 ~ 13.0	
Titanium	8.6	
Tungsten	4.5	
Water	69	

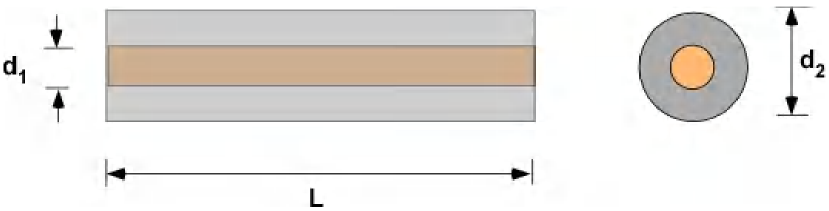
3.3.3 Additional Problems

Solve the following problems and compare the solution with MATLAB/Octave

Problem 3.3.1

The aluminum shell and copper core are fully bonded. The diameter d_1 is 30 mm. The diameter d_2 is

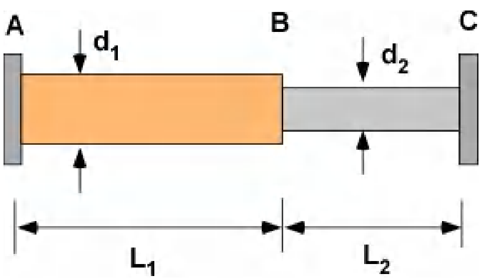
60 mm. The length of the structure is 0.5 m. The modulus of elasticity of aluminum and copper are 70 GPa and 120 GPa respectively. The CLTE of aluminum and copper are 23.1×10^{-6} and 17×10^{-6} /°C respectively. There is no stress in the material at room temperature. The composite structure sees a change in temperature of 150 °C. Determine (a) final length of the composite structure; (b) stress in aluminum; (c) stress in copper.



Problem 3.3.1

Problem 3.3.2

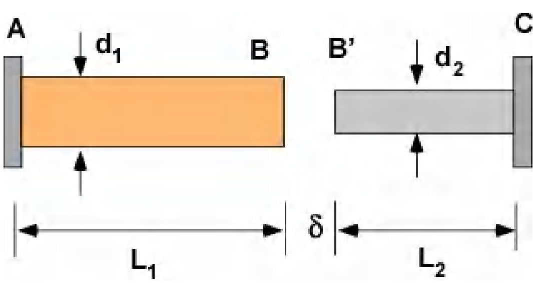
The composite structure is made of a brass and an aluminum rod.The structure is restrained at both ends. The diameter d_1 is 30 mm. The diameter d_2 is 60 mm. The lengths L_1 and L_2 are 250 mm and 300 mm. The modulus of elasticity of brass and aluminum are 105 GPa and 70 GPa respectively. The CLTE of brass and aluminum are 19×10^{-6} and 23.1×10^{-6} /°C respectively. Initially the structure is stress free and the structure is subject to a temperature drop of -75 °C. (a) Find the force in the structure; (b) Find the stress in brass; (c) Find the change in length of the brass rod.



Problem 3.3.2

Problem 3.3.3

The structure is made of the same materials and same dimensions as in Problem 3.3.2 with the same constraints. However initially the structure has a gap δ of 0.4 mm at room temperature and is unstressed. It is subject to a temperature increase of 175 °C. Find (a) normal stress in aluminum; (b) Change in length of the aluminum rod.



Problem 3.3.3

3.4 SHEAR STRESS AND STRAIN

The definition of shear stress requires two-dimensions. The previous sections involving normal stress and strain were set up as problems in one dimensional alone. To make this clear Figure 3.4.1 describes the force and the area on which it acts in the case of normal force and normal stress -*part (a)* as well as shear force and shear stress - *part (b).* The cross-sectional area vector is normal to the area (outward normal) by definition.

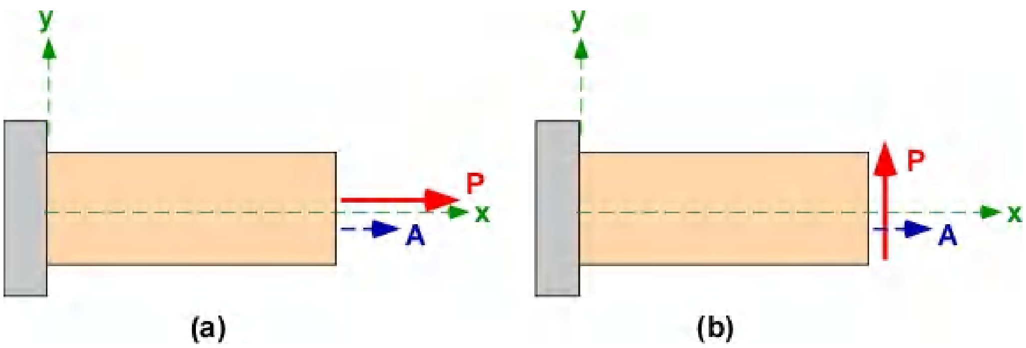


Figure 3.4.1 (a) Normal stress, (b) Shear stress.

The direction of **P** vector and the direction of the area vector **A** on which it acts in part (a) for normal stress are parallel. Or the force is normal to the area on which it acts. In part (b) the force vector is at right angles to the area vector on which it acts. This is the definition of **shear force**. The force is in the plane of the area. The corresponding stress is called ***shear stress*** For the definition of shear stress we see that the force is in the y-direction and the area in the x-direction. Hence shear force and stress require two dimensions and are frequently expressed using two subscripts.

$$\tau = \tau_{xy} = \frac{P}{A} \qquad (3.11)$$

The shear stress (τ) , Greek letter *tau*, is the force divided by the area The first subscript is the vector direction for the *area*. The second subscript is the vector direction for the *force*. In this definition we have idealized that the stress is *uniform*.

Shear force and shear stress are more common than you realize. In previous sections we designed structures where normal stress is below the yield stress to preserve the elastic behavior of the structure. We also noted that it is the stress that causes the material to fail Consider you bite into a carrot for lunch and chop a piece into your mouth. You have basically caused the material of the carrot to fail by applying a shear stress with your incisors. This shear force caused a shear stress that made the material of the carrot yield and break.The reason you can do this mostly ith your front teeth is that they are shaper and behave like a knife. The edge of the knife provides the area for the stress calculations.

You are slicing a softer vegetable like zucchini with a knife. It requires less shear force than a carrot. Figure 3.4.2 demonstrates slicing a zucchini. You can see the knife applying a shear force and the material yielding by failure. More force is used initially in part (a) than part (b) because of skin resistance to shear.

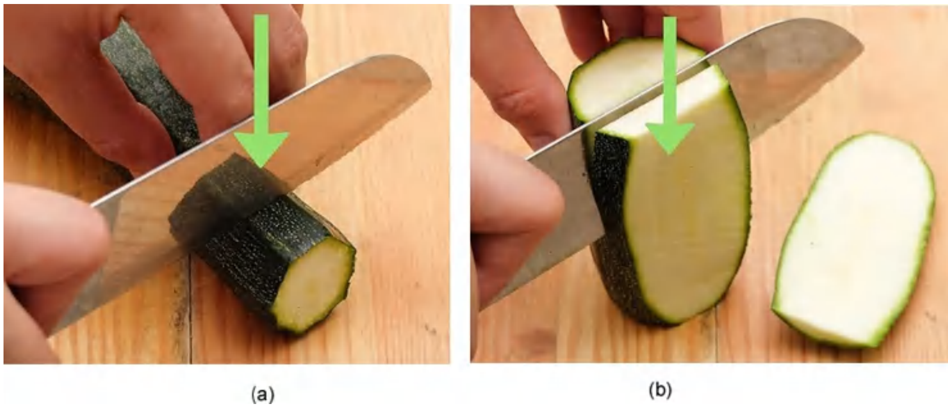


Figure 3.4.2 Shear force and shear stress causing failure

The force exerted by the knife is in the plane of the area. The stress is based on the area of the knife edge. The sharper the knife the less force needed to cause failure in the zucchini because more stress is generated. Note the material is not uniform. The properties of the skin layer make it require (slightly) more force than the central layer. Cutting a tomato with a flat knife is a skill. You can do better by employing a sawing motion. Shear force is easily illustrated with Figure 3.4.2. The same things happen with structural materials. In most designs you want to avoid failure by keeping the shear stress with a certain limit. A lot of wood working involves causing material to fail in shear. This can happen by sawing, sanding, shaving, planing, chiseling. You also keep material from shear failure by gluing.

3.4.1 Shear Strain

The definition of shear strain is a little more complex than normal strain.

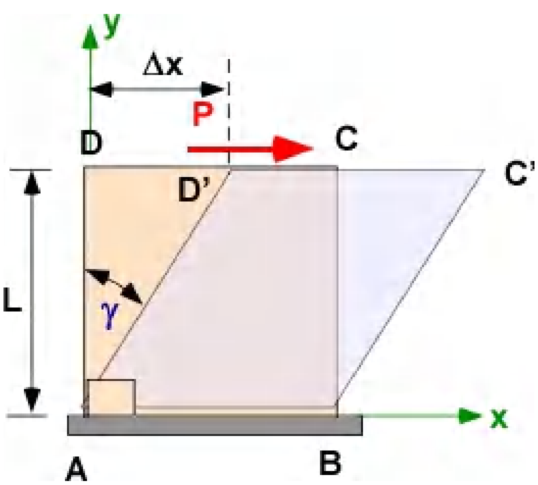


Figure 3.4.3 Definition of shear

To understand shear stress and strain we must conduct a thought experiment. Let us imagine you were able to hold firm a two-dimensional square material at the bottom and apply a force **P** on the top along the surface as shown below in Figure 3.4.3. To make it easy you can imagine the material is rubber. For small deformation the material is likely to deform like a rhombus. The original angle, before P was applied, between the sides AB and AD was 90°. After the application of P, the sides are no longer at 90°. Furthermore, consider the force P applied over an area A on the top. This causes a shear stress **τ.**

The shear strain is the angle (in radians) γ which is the ***decrease in the original right angle*** in the presence of the *shear stress* τ. Normally the shear stress and strain will have two subscripts.The first subscript denotes the normal to the area, while the second describes the direction of the force. The particular shear strain in the illustration is γ_{yx}. Also, for small shear strain

$$\gamma = \gamma_{yx} = \angle BAD - \angle BAD' \qquad (3.12)$$

$$\tan\gamma = \gamma = \frac{\Delta x}{L} \qquad (3.13)$$

The actual representation of Figure 3.4.3 should be in three-dimensional as in Figure 3.4.4

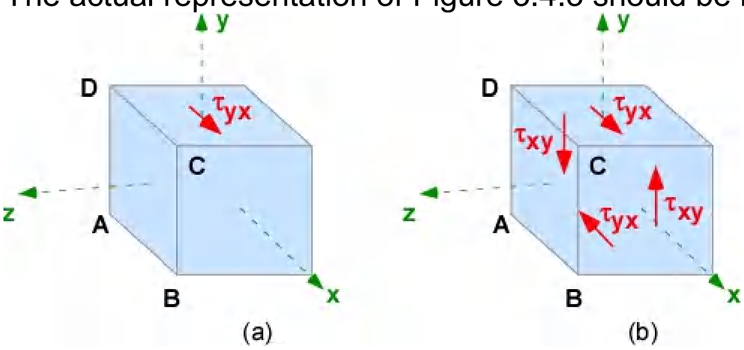


Figure 3.4.4 3D representation of Figure 3.4.3

In Figure 3.4.4 part (a) we have the FBD of the cube and the applied shear stress due to P. We can discuss the effects of shear force and shear stress equivalently. In part (a) the cube is not in equilibrium. In statics every structure must be in equilibrium. Part (b) is in equilibrium. This is justified as follows:

- First we add equal and opposite shear to the bottom because of the constraint.
- This cancels the force but there will now be a rotation about the point A or B due to the shear on top.
- To cancel this moment we must add a shear to the right face in the + y direction.
- This now causes a force unbalance in the y-direction which must be balanced by a negative shear in the negative y-direction on the opposite face.

Note that the subscripts for the pair of shear stress are different. In the limit there is only one shear stress and we can show that

$$\tau_{yx} = \tau_{xy} \qquad (3.14)$$

Instructural problems in three dimensions there are at most three shear stresses with different pairs of subscripts. The subscripts can be interchanged as in Eqn. (3.14). Finally, the FBD of the element in Figure 3.4.3 can be shown (without subscripts) as

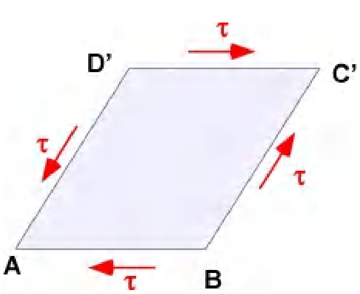


Figure 3.4.5 Shear equilibrium fro an element

3.4.2 Hooke's Law for Shear

In Section 3.2 the material property E was obtained experimentally using a massive machine called the tensile testing machine. There are similar machines that can subject standard specimens to shear stress. One way is to apply torsion and measure twist angle. These can be related to shear stress and shear strain respectively. If the material is elastic,or within the elastic region, then the corresponding Hooke's Law for shear, where G is the modulus of rigidity is:

$$\tau_{yx} = G\gamma_{yx};\qquad (3.15)$$

This can be used for design of structures to avoid shear failure

Table 3.3 Shear Modulus of Materials

Material	**Typical values for shear modulus (GPa)** (at room temperature)
Diamond	478.0
Steel	79.3
Iron	52.5
Copper	44.7
Titanium	41.4
Glass	26.2
Aluminium	25.5
Polyethylene	0.117
Rubber	0.0006

(from Wikipedia)

Example 3.7

To calculate the shear modulus of an unknown material the following experiment is performed. A rectangular brick of the material (Figure 3.4.6) is bonded to rigid end plates. The bottom plate is fixed. The top plate is subject to a horizontal force of 120 kN. The height of the block (h) is 50 mm. The width of the brick (w) is 60 mm. The length of the brick is (L) is 200 mm. The upper plate moves through a distance (ΔL) of 2 mm. (a) Calculate the shear strain. (b) Calculate the modulus of rigidity of the material.

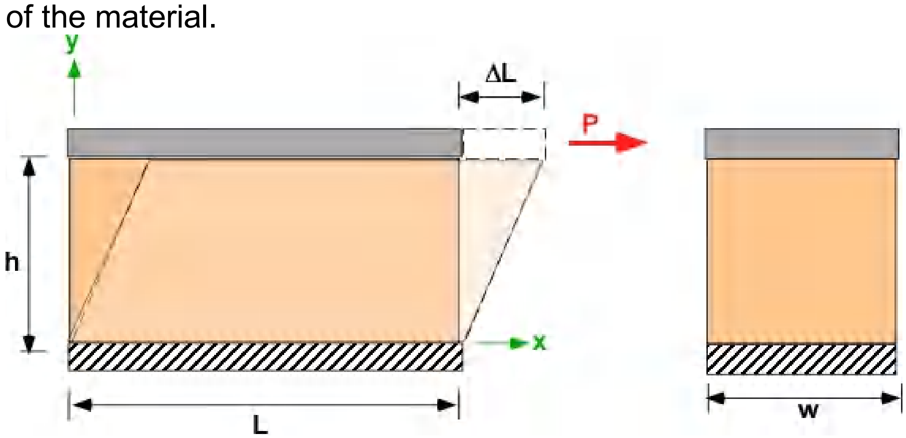


Figure 3.4.6 Example 3.7

Given Data:

L = 200 mm; h = 50 mm; w = 60 mm; ΔL = 2 mm

P = 120 kN

Assumptions:
Uniform stress and strain
Stress is within proportional limit (use Hooke's Law)

Solution: (Calculation based on definition - FBD not necessary)

$$(a) \quad \gamma = \frac{\Delta L}{h} = \frac{2}{50} = 0.04[rad]$$

$$\tau = \frac{P}{A} = \frac{P}{Lw} = \frac{120000}{0.2 \times 0.06} = 10[MPa]$$

$$(b) \quad G = \frac{\tau}{\gamma} = \frac{10^7}{0.04} = 0.25[GPa]$$

3.4.3 Poisson's Effect or Ratio

This is a material property associated with normal strain. One important effect of axial normal stress and corresponding normal strain is a transverse effect that is not negligible. This affects normal strain only and its inclusion in discussion of shear is prompted by two considerations. First, we are considering two dimensional effects in this section. The previous sections discussed only one-dimensional effects. Second the modulus of elasticity E, and the modulus of rigidity G, are related through the value of the Poisson's ratio ν (Greek letter nu) which is a measure of this transverse effect of normal strain. Consider **direct stress** applied in the x-direction through the force P as shown below. The original dimensions of the material are L_x and L_y.

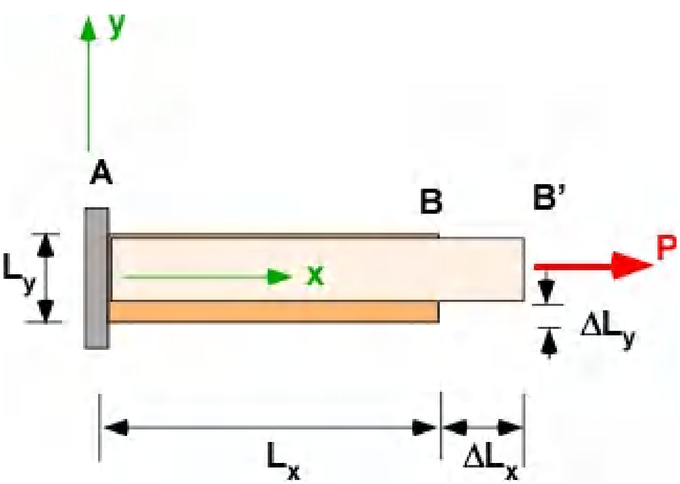


Figure 3.4.7 Direct stress and Poisson's effect

The direct stress in x-direction will elongate the material in the x-direction causing the direct normal strain ε_x. However the density of the material is constrained to be the same. The material must therefore must reduce the lateral dimension in the y-direction. This lateral displacement and the corresponding normal strain, ε_y is called the **Poisson's effect** . It is incorporated through the Poisson's **ratio** (ν) - which is an another material property as

$$\varepsilon_x = \frac{\Delta L_x}{L_x}; \quad \sigma_x = \frac{P}{A}$$

$$\varepsilon_y = \frac{\Delta L_y}{L_y}$$

$$\varepsilon_y = -\nu \varepsilon_x \qquad (3.16)$$

The transverse effect is opposite to the direct effect. That is the reason for the negative sign in Equation 3.16.

Table 3.4 Poisson's ratio of some materials

Material	Poisson's ratio
Aluminum-alloy	0.32
cast iron	0.21–0.26
clay	0.30–0.45
concrete	0.1–0.2
copper	0.33
cork	0.0
foam	0.10–0.50
glass	0.18–0.3
gold	0.42–0.44
magnesium	0.252–0.289
metallic glasses	0.276–0.409
rubber	0.4999
sand	0.20–0.455
saturated clay	0.40–0.49
stainless steel	0.30–0.31
steel	0.27–0.30
titanium	0.265-0.34

(*from Wikipedia*)

A few material called auxetic materials have a negative Poisson's ratio. That is the sign in Equation (3.16) is positive. Also from continuum mechanics it can be established that the Poisson's ratio must be less than 0.5. Most engineering materials have a value of Poisson's ratio between 0.2 and 0.5. Cork has a Poisson's ratio of zero. That is one of the reasons it is used as a stopper for wine bottles.

Example 3.8

A rod made of aluminum alloy of rectangular cross-section is subject to an axial load P (Figure 3.4.8) of value 120 kN. The height of the rod (h) is 50 mm. The width of the rod (w) is 60 mm. The length of the rod (L) is 2 m. The change in length of the rod (ΔL) is 1.2 mm. (a) Calculate the direct strain and the transverse strains in the rod (b) Calculate the final dimension of the rod.

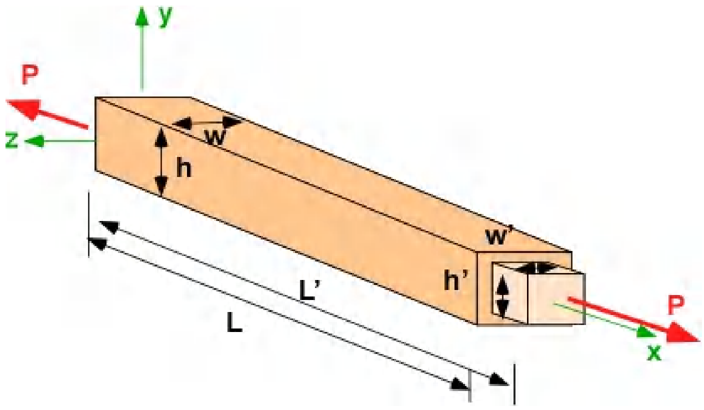


Figure 3.4.8 Example 3.8

Given Data:
L = 2 m; h = 50 mm; w = 60 mm; ΔL = 1.2 mm
P = 120 kN
Material : Aluminum; $\nu = 0.32$

Assumptions:
Uniform normal stress and strain
Stress is within proportional limit (use Hooke's Law)

Solution: (Calculation based on definition - FBD not necessary)

(a)

$$\varepsilon_x = \frac{\Delta L}{L} = \frac{1.2}{2000} = 0.0006$$

$$\varepsilon_y = -\nu\varepsilon_x = -0.32 \times 0.0006 = -0.000192$$

$$\varepsilon_z = -\nu\varepsilon_x = -0.32 \times 0.0006 = -0.000192$$

(b)

$$L' = (1+\varepsilon_x)L = (1+0.0006) \times 2 = 2.0012[m]$$

$$h' = (1+\varepsilon_y)h = (1-0.000192) \times 0.05 = 0.04999[m]$$

$$w' = (1+\varepsilon_z)w = (1-0.000192) \times 0.06 = 0.059988[m]$$

3.4.4 Relation between E, G, and ν

The modulus of elasticity E, the modulus of rigidity G, and the Poisson's ratio are three material properties that we have introduced so far. Only two of them are independent. It can be established that for materials that have the same material properties in all directions

$$G = \frac{E}{2(1+\nu)} \qquad (3.17)$$

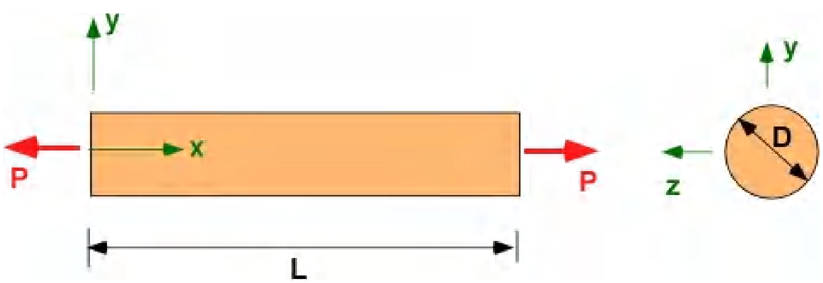
This is related to the idea that an axial load can produce both normal strain and shear strains which can be related. This requires more understanding of the transformation of strain and is beyond the scope of the simple definitions we have advanced until now. We will accept Eqn. (3.17) and use it if necessary.

3.4.5 Additional Problems

Solve the following problems on paper and in MATLAB if applicable.Use material properties for the problem from any resource.

Problem 3.4.1

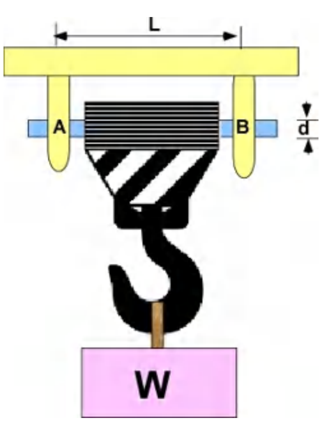
A force of P = 10 kN is applied to a rod of initial length 200 mm. The original diameter is 20 mm. The change in length of the rod is 15 mm and the change in diameter is 1 mm after the application of the force. (a) Calculate the modulus of elasticity of the material. (b) Calculate the Poisson's ratio of the material. (c) What is the modulus of rigidity of the material?



Problem 3.4.1

Problem 3.4.2

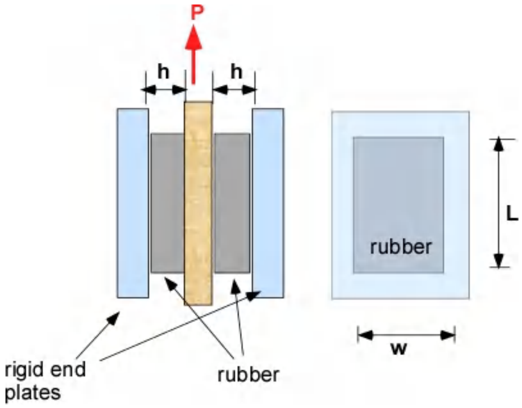
A massive weight of 100 kN is hanging from the large hook. A steel rod of diameter d = 75 mm supports the load at the bearings A and B. (a) What is the stress in the rod? (b) Is it likely to fail in shear?



Problem 3.4.2

Problem 3.4.3

The two rectangular blocks of rubber are bonded to the rigid end plates as well as the central plate. Two dimensions of the rubber block are L = 200 mm, w = 150 mm. The maximum shearing stress in rubber is 2 MPa. The maximum deflection of the plate is 10 mm. (a) Calculate the dimension h of the rubber block for maximum deflection. (b) What is the load applied to the plate to cause this deflection?



Problem 3.4.3

3.5 CLASSIFICATION OF MATERIALS

A design or structural engineer has thousands of engineering material that can be used for design. Most of the time the choice is usually suggested by knowledge, history, and prior practice. One of the impressive changes in current engineering activity is the explosive development and use of new materials. There are so many of these new materials coming on line. These materials, largely represented by composites can actually be designed for particular applications. In addition, the manufacturing is also seeing a radical change through custom three-D printing and additive manufacturing. As you can expect these developments will certainly affect the way we approach design, and particular instruction, such as this book. They have yet to be formalized as they are largely experimental. Another fascinating area of development is the area of bio-materials, where molecular changes can be incorporated at the molecular level. New compounds can be synthesized on the computer before resulting in a liquid medium. This section is largely the domain of courses like material science or metallurgy.

The material in this section is not original. It is a summary based on the references mentioned below. In this book we just need material properties to analyze and solve engineering problems. A brief introduction is useful for understanding applications.

The reference for this page is NDT resource center(https://www.nde-ed.org/EducationResources/CommunityCollege/Materials/Introduction/classifications.htm) and Wikipedia

3.5.1 The Material Classification

The standard classification of materials currently involves four groups

1. Metals. This is divided into Ferrous metals and alloys (irons, carbon steels, stainless steels), and Nonferrous metals and alloys (aluminum, copper, nickel, titanium, noble materials, refractory materials, super alloys)
2. Ceramics (Glasses, Glass ceramics, Graphite, Diamond)
3. Polymers (Thermoplastic plastics, thermoset plastics, elastomers)
4. Composites (reinforced plastics, metal-matrix composites, ceramic-matrix, sandwich structures, concrete)

In order to use these materials in design one would expect information about its material properties, like stress-strain behaviors, thermal behavior, that we saw in the previous section. In some cases additional properties can be necessary like its behavior subject to cyclic loads, or exposure to radiation. Within each group. Materials are further classified by their composition and physical properties. A brief discussion of each group follows.

3.5.2 Metals

Most introductory textbooks on mechanics of materials deal with this category. Until recently metals was the significant material used in engineering applications. Low mass design, computers controlling structures, 3D printing, and other novel manufacturing techniques have caused all categories of materials to be equally important in engineering applications. For example the the modern commercial aircraft structure has gone from almost one hundred percent aluminum to include about 40 % composite materials. Metals are chosen for engineering applications because they can tolerate stress, are ductile, have good thermal and electrical properties, are tough, and have high melting points. Metals are typically hard, opaque, shiny, and malleable. 91 out of 118 elements of the periodic table are metals. The common metals are:

a. Iron/Steel : are used where strength is needed. Used in bridges, buildings, ships
b. Aluminum: Light weight design, easy to form, corrosion resistance, easily available, and recyclable
c. Copper: high ductility, corrosion resistance, good electrical conductivity
d. Titanium: high temperature applications, corrosion resistance, light weight - high speed aircrafts have a lot of titanium. It is also expensive to extract
e. Nickel: high temperatures, good corrosion
f. Refractory materials: high temperature applications like turbine blades

3.5.3 Ceramics

A ceramic is a non-metallic solid that starts in powdered form and is made into products through processing. They usually involve a combination of metallic and nonmetallic elements, are crystalline in nature. They are usually good thermal and electric insulators. They can be hard and tough (high moduli of elasticity) and have high melting temperatures and good chemical resistance. They are brittle (low ductile) by nature. They are useful in lots of applications. The category continuous to evolve with advanced ceramics. They are now important in engineering and are termed as technical ceramics and are classified as:

- Oxides (alumina, beryllia, ceria, zirconia
- Non oxides: carbide, boride, nitride, silicide
- Composite materials: particulate reinforced and fiber reinforced

You can find them in knife blades, brake disks, armored vehicles, ballistic armored vests, ball bearings, gas turbine engines, tissue engineering, biomedical implants and watch making.

The traditional classification of ceramics is:

a. Structural Clay: sewer pipe, wall tiles
b. Whitewares: dinnerware, electric porcelain
c. Refractories:used in petroleum and chemical industries
d. Glasses:flat glass, glass fibers, optical fibers
e. Abrasives: natural diamond, silicon carbide
f. Cements: roads, buildings
g. Advanced Ceramics:
    - Structural : bioceramics, engine components
    - Electrical: piezoelectrics, magnets, superconductors
    - Coatings: cutting tools, industrial wear
    - Environmental: filters, membranes, catalysts

3.5.4 Polymers

A Polymer is a large molecule consisting of repeated sub molecules forming a chain and a solid. There are natural and synthetic polymers and they have a broad range of properties. They are valued for toughness and viscoelasticity. The basic classification refers to natural polymers and synthetic polymers. Natural polymers include shellac, amber, wool, silk, rubber and cellulose. Synthetic polymers include synthetic rubber, phenol formaldehyde, neoprene, nylon, polyvinyl chloride, polystyrene, polyethylene, polypropylene. Silicone, and many more.

The advantage of polymers are low density, resists atmospheric and other forms of corrosion, is bio compatible, and a good conductor of electric current. They are strong too. Kevlar is used for bullet proof vests and is about twenty times stronger than steel and much lighter.

For engineering materials the two categories are plastics and elastomers. Plastic are usually obtained through forming or molding. They are further classified as thermoplastic or thermoset polymers. Elastomers are used for elastic load bearing and suppression of vibrations.

The thermoplastic polymers are polyethylene, polypropylene, polystyrene, and polyvinyl chloride. The thermosetting polymers are alkyds, amino and phenolic resins, epoxies, polyurethanes, and unsaturated polyesters.

3.5.5 Composites

A composite is a composite material is made from two or more different constituent material with different physical and chemical properties and results in a material with very different characteristic than the material it is created from. In fact today engineering properties can be customized through composites. The materials forming the composite are of two types. One of them is considered the reinforcer and the other is called the matrix. The reinforcing material is usually used in particulate form, as fibers, or as sheets and is usually a smaller percentage of the composite. They are folded into the matrix in different ways. The constituents materials can be metal, ceramic , or polymers. Natural composites are palms and bamboo. Bricks, made from straw and mud were known from the beginning. Concrete, plywood, pappier-mache are examples of composite materials.The reinforcing material is of low density and higher strength. The matrix is ductile or tough. Composites can be

- Reinforced plastics
- Metal-matrix composites
- Ceramic-matrix composites
- Sandwich structures
- Concrete

The strength property of the composite depends if it is dispersion strengthened, particle reinforced, or fiber reinforced. Metal-matrix are of the first category. For particle strengthening the volume fraction of the reinforcer s large. The strength of the composite is shared between the matrix and the reinforcer. In fiber reinforced composite the fiber carried the load.

The reference for this page is NDT resource center(https://www.nde-ed.org/EducationResources/CommunityCollege/Materials/Introduction/classifications.htm) and Wikipedia

3.6 STRESS ON AN INCLINED PLANE

If you apply a tensile force to a rod of homogeneous material you expect to see simple tensile stresses in the cross section along the rod. Today engineering materials, like composites, are specially designed for efficient handling of stress in prescribed directions. That is, It provides better strength in certain directions. A natural example is wood with fibers that improve its structural performance in those directions. If you applied a tensile force to wood in a certain direction, it is likely that it may fail along a very different direction then the one in which the force is applied. This is decided by the orientation of the wood fibers to the direction of the applied force. This suggests that we must be concerned by the stresses in cross sections that are not normal to the applied force. This is illustrated b the following development. This is important in designs involving composite materials.

Let us begin with a a component in simple tension. The cross section normal to the load experiences a uniformly distributed stress σ.

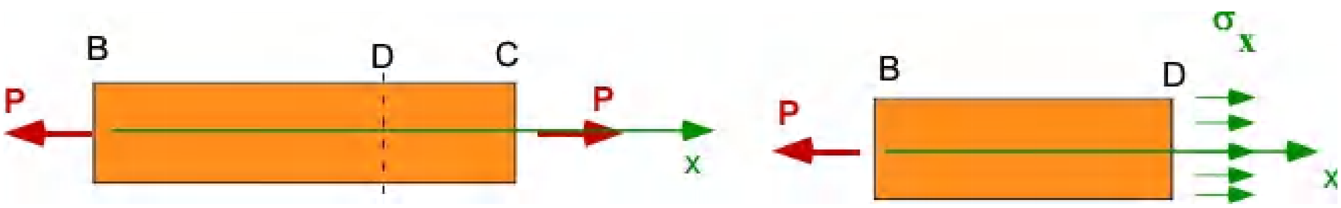


Figure 3.6.1 Simple tension

Let us look at a cross section that is inclined at an angle θ to the vertical as shown (Figure 3.6.2a). We cut along this plane to expose the forces/stresses on this inclined plane. Since the original structure was in equilibrium the two sections cut must also be in equilibrium. Consider the equilibrium of the piece to the of the cu.t It will have force components normal to and in the plane of the cut (Figure 3.6.2b).

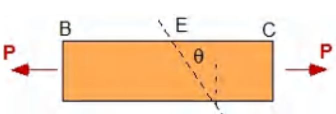


Figure 3.6.2a Inclined plane

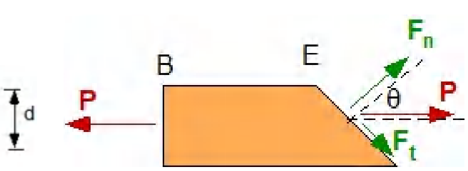


Figure 3.6.2b FBD on inclined plane

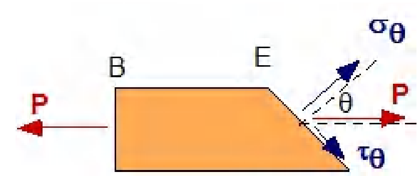


Figure 3.6.2c Stresses - inclined plane

The force P can be resolved into a component (**F_n**) normal to the inclined plane which will have an area of A_θ producing normal stress.It will also have the component (**F_t**) in the inclined plane producing shear stress as shown in Figure 3.6.2c:

$$\sigma(\theta) = \frac{F_n}{A_\theta} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta = \sigma_x \cos^2\theta$$

$$\tau(\theta) = \frac{F_t}{A_\theta} = \frac{P\sin\theta}{A/\cos\theta} = \frac{P}{A}\sin\theta\cos\theta = \sigma_x \sin\theta\cos\theta \qquad (3.18)$$

The location of E and the angle θ are arbitrary and can be located anywhere on the beam. Therefore, even if the original loading suggests normal stress the beam can experience different values of normal and shear stress along different planes in the material. This analysis is important for materials that are glued or made up of fibers, like the PCB, fiber reinforced plastic etc. This idea is used to discuss principal stresses and Mohr's circle in a later section. Meanwhile the variation of the stresses for different values of θ can be easily generated by MATLAB as shown in Figure 3.6.3.

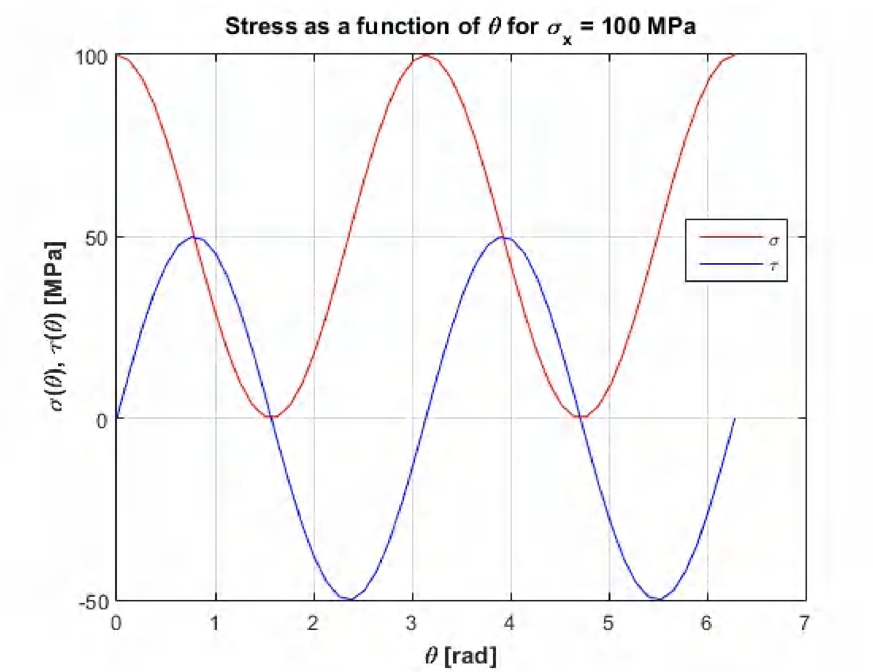


Figure 3.6.3 Stress on inclined plane

Example 3.9

The block AB is glued together at the section CD - Figure 3.6.4a. The dimensions of h and w are 60 and 120 mm respectively.The maximum allowable tensile stress in the glue is 500 kPa. (a) What is the maximum load P than can applied? (b) What is the corresponding shear stress?

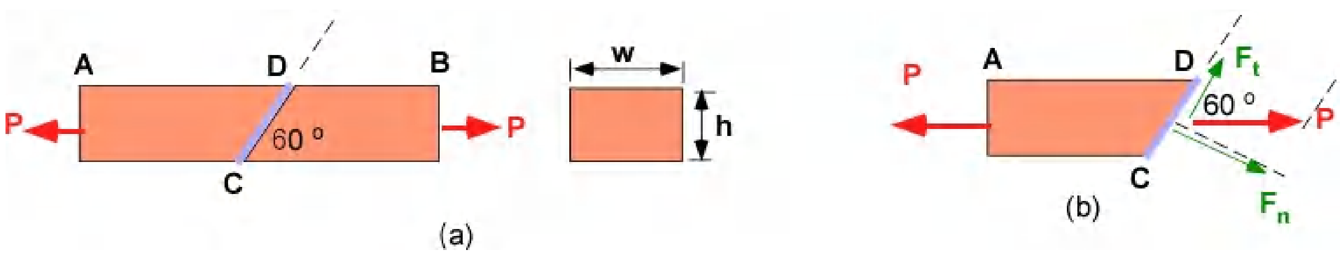


Figure 3.6.4 Example 3.9

Given Data:

h = 60 mm; w = 120 mm; θ = 60°

σ_{max} = 500 kPa

Assumptions:

Stress on inclined plane

Solution: (Calculation based on Figure 3.6.4b)

$F_n = P\sin(60);$

$F_t = P\cos(60);$

$$A_n = \frac{wh}{\sin(60)} = \frac{(0.06)(0.12)}{\sin(60)} = 0.0083$$

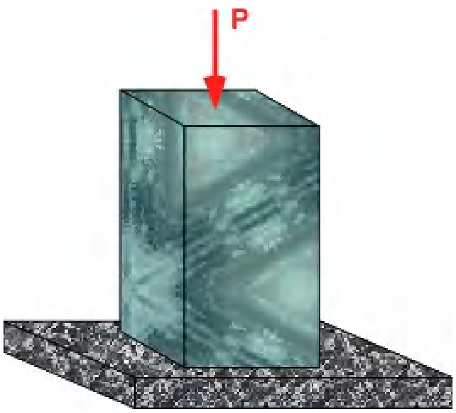
$$\frac{F_n}{A_n} = 500000 = \frac{P\sin(60)}{0.0083};$$

$P = 4800\,N$

$$\tau = \frac{P\cos(60)}{A_n} = \frac{4800\cos(60)}{0.0083} = 194kPa$$

3.6.1 Additional Problems

Problem 3.6.1
The glass block of cross-section 125 mm x 150 mm is subject to a compressive force of 1000 kN. (a) What is the maximum normal stress in the material? (b) What is the maximum shear stress in the material?



Problem 3.6.1

Problem 3.6.2
The composite wood of square cross-section is subject to a tensile load of 20 kN. The cross-section is square and the ply angle is 45 degrees as shown. The maximum tensile stress is 2 MPa and the maximum shearing stress is 1.2 MPa. (a) What is the largest cross-section of the piece?

Problem 3.6.2

3.7 MULTIDIMENSIONAL NORMAL STRESS AND STRAIN

So far our discussion of normal stresses were restricted to one dimension. Shear stress by definition acts on a plane (or two dimensions). Consider normal force/stress only. Let us visit a situation when forces in the x and y directions are simultaneous applied. This is best illustrated by force on a 2D rectangular element shown in Figure 3.7.1 below.

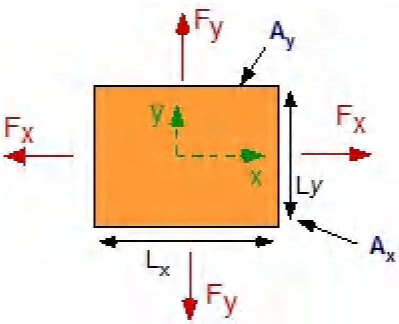


Figure 3.7.1 two normal forces

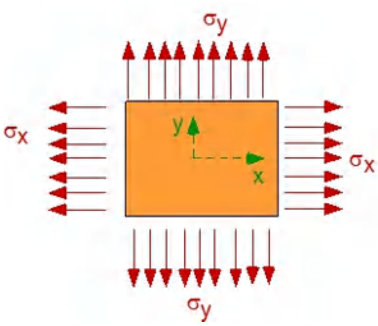


Figure 3.7.2 Direct Normal stresses

The forces will cause direct stresses (Figure 3.7.2)

$$\sigma_x = \frac{F_x}{A_x}; \sigma_y = \frac{F_y}{A_y}$$

This defines a problem in **Plane stress.**
Each direct stress will cause **strain** in the other direction due to Poisson's effect.
The strain will cause a change in the size (dimensions) of the rectangle.

3.7.1 Plane Stress

The direct stress in x-direction will elongate the material in the x-direction causing the direct normal strain ε_x. This will cause the normal strain, ε_y and ε_z through Poisson's effect . It is incorporated through the Poisson's ratio (ν). A stress in the y-direction must do the same and create a Poisson's effect in the x and z-direction. As a result the we can write the sum of the direct strain and the Poisson strain for an object subject to simultaneous stress in two directions as:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right]$$
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right] \qquad (3.19)$$
$$\varepsilon_z = \frac{1}{E}\left[-\nu\sigma_x - \nu\sigma_y\right]$$

We can also express the relation in (3.19) in terms of stresses and noting that there is no stress in z-direction

$$\sigma_x = \frac{E}{\left(1-\nu^2\right)}\left(\varepsilon_x + \nu\varepsilon_y\right)$$
$$\sigma_y = \frac{E}{\left(1-\nu^2\right)}\left(\varepsilon_y + \nu\varepsilon_x\right) \qquad (3.20)$$

The strains in Eqn. (3.19) will cause a change in the dimension of the object as shown in Figure 3.7.3.

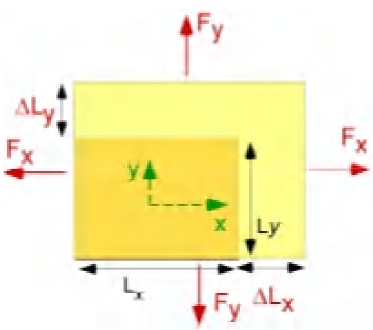


Figure 3.7.3 Displacements due to the forces

The new dimensions of the object in the x and y directions can be calculated as:

$$L'_x = L_x + \Delta L_x = L_x + L_x\varepsilon_x = L_x\left(1+\varepsilon_x\right)$$
$$L'_y = L_y + \Delta L_y = L_y + L_y\varepsilon_y = L_y\left(1+\varepsilon_y\right) \qquad (3.21)$$

Even if there is no stress in the z-direction (plane stress) there will be a change in the dimensions of the z-direction as:

$$\varepsilon_z = -\nu\varepsilon_x - \nu\varepsilon_y$$
$$L'_z = L_z + \Delta L_z = L_z + L_z\varepsilon_z = L_z\left(1+\varepsilon_z\right) \qquad (3.22)$$

The change in the volume of the material is

$$\Delta(Vol) = L_x L_y L_z (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - L_x L_y L_z$$

$$\textit{For small strains} \tag{3.23}$$

$$\Delta(Vol) = L_x L_y L_z (\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

Example 3.10
The copper plate is subject to two-dimensional loading. The compressive stress in the x-direction is 50 MPa while the tensile stress in the y-direction is 75 MPa. The length L_x is 50 mm while the length L_y is 75 mm. The thickness t of the plate is 10 mm. (a) Calculate the change in the dimensions of the plate.

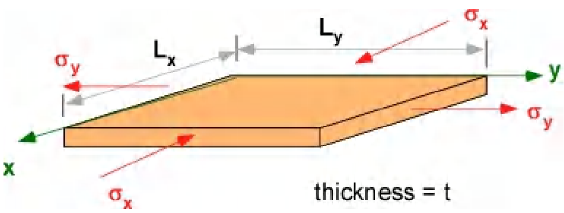


Figure 3.7.4 Example 3.10

Given Data:
L_x = 50 mm; L_y = 75 mm; t = 10 mm;
E = 117 GPa; ν = 0.33;
σ_x =--50 MPa; σ_y =- 75 MPa

Assumptions:
Two-axis loading
Plane stress

Solution: (Do not need a FBD)

$$\varepsilon_x = \frac{1}{E}\{\sigma_x - \nu\sigma_y\} = \left(\frac{1}{117\times10^9}\right)\left[-50\times10^6 - 0.33(75\times10^6)\right] = -0.000639$$

$$\varepsilon_y = \frac{1}{E}\{\sigma_y - \nu\sigma_x\} = \left(\frac{1}{117\times10^9}\right)\left[75\times10^6 - 0.33(-50\times10^6)\right] = 0.00782$$

$$\varepsilon_z = \frac{1}{E}\{-\nu\sigma_y - \nu\sigma_x\} = \left(\frac{1}{117\times10^9}\right)\left[-0.33(-50\times10^6) - 0.33(75\times10^6)\right]$$
$$= -7.05\times10^{-5}$$

The change in each dimension is the strain multiplied by the corresponding length. The new dimensions are:

$$L'_x = L_x(1+\varepsilon_x) = 0.05(1-0.000639) = 0.0499\,m$$

$$L'_y = L_y(1+\varepsilon_x) = 0.075(1+0.00782) = 0.075059\,m$$

$$t' = t(1+\varepsilon_z) = 0.01(1-7.05\times10^{-5}) = 0.00999\,m$$

3.7.2 Three-Dimensional Stress and Strain

One can similarly extend this to a 3D state of stress. The corresponding Hooke's law is termed as the generalized Hooke's law. This is also referred to as multi-axis loading.

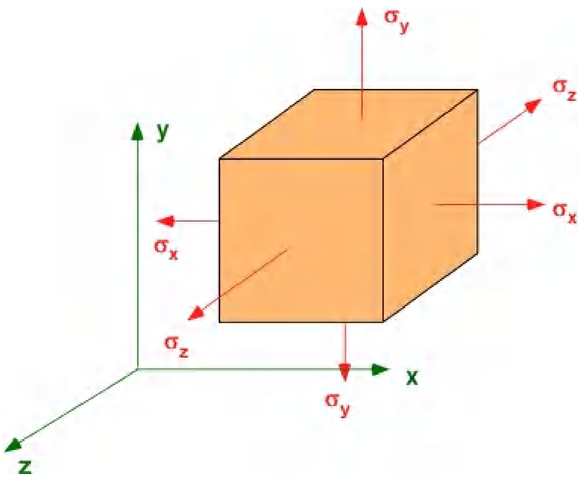


Figure 3.7.5 Three dimensional state of stress.

The generalized Hooke's law

$$\varepsilon_x = \left(\frac{1}{E}\right)\left[\sigma_x - \nu\sigma_y - \nu\sigma_z\right]$$

$$\varepsilon_y = \left(\frac{1}{E}\right)\left[\sigma_y - \nu\sigma_z - \nu\sigma_x\right] \qquad (3.24)$$

$$\varepsilon_x = \left(\frac{1}{E}\right)\left[\sigma_z - \nu\sigma_x - \nu\sigma_y\right]$$

This can be written conveniently in matrix form as:

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\\ -\nu & -\nu & 1\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix} \qquad (3.25)$$

The equations (3.19) and (3.20) assume that the material properties are the same in all directions. This is called an **isotropic** material. If properties change in different directions it is an **anisotropic** material. Metals are usually isotropic while wood and composites are anisotropic. The relations must be modified using direction specific material properties.

3.7.3 Plane Strain

The plane stress problem introduced earlier had no stress in the z-direction ($\sigma_z = 0$) but there was a strain in the z-direction due to Poisson's effect. A plane strain problem will require a constraint to eliminate this strain in the z-direction. This is done by employing a counteracting stress in the z-direction to cancel this strain ε_z. One way to generate this is to use rigid end plates to hold the element in plane stress - Figure 3.7.6

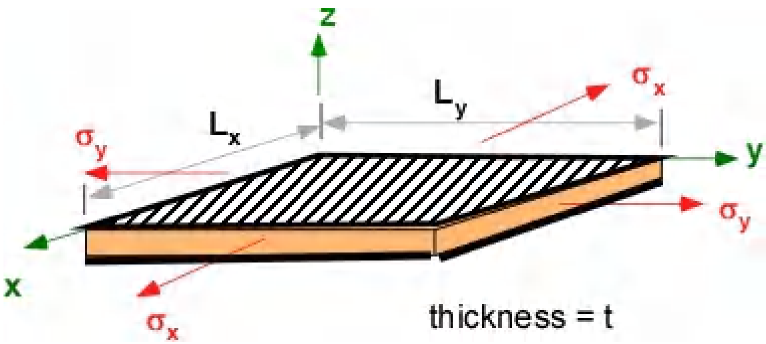


Figure 3.7.6 Plane Strain

The stress required for zero strain in the y-direction can be calculated as:

$$\varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) = \frac{\sigma_z}{E};$$
$$\textit{Applied} \quad \sigma_z = -\left[-\nu\left(\sigma_x + \sigma_y\right)\right] = \nu\left(\sigma_x + \sigma_y\right) \tag{3.26}$$

The equations for the strain in the plane stress direction must incorporate this stress in the z-direction

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right) = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\left(\nu\left[\sigma_x + \sigma_y\right]\right)\right)$$
$$\varepsilon_x = \frac{1}{E}\left[\left(1-\nu^2\right)\sigma_x - \nu\left(1+\nu\right)\sigma_y\right] \tag{3.27}$$
$$\varepsilon_y = \frac{1}{E}\left[\left(1-\nu^2\right)\sigma_y - \nu\left(1+\nu\right)\sigma_x\right]$$

3.7.4 Additional Problems

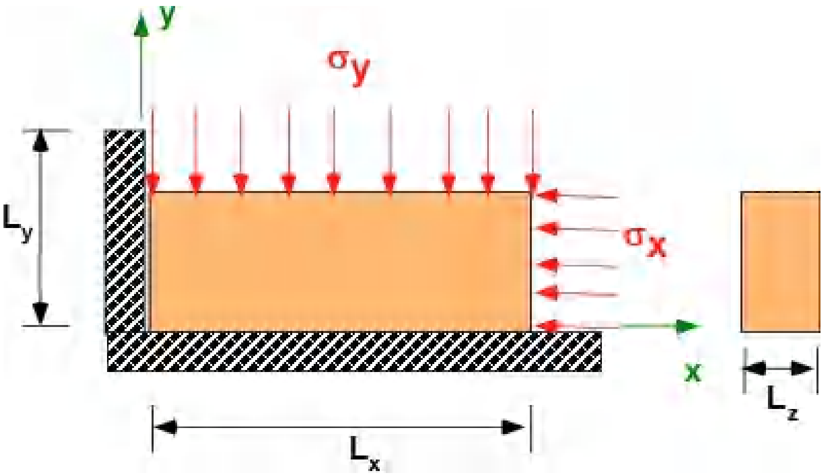
Solve the following problems on paper and in MATLAB if applicable.Use material properties for the problem from any resource.

Problem 3.7.1

Obtain the matrix relation between stress and strain by solving the matrix problem in Eqn. (3.25)

Problem 3.7.2

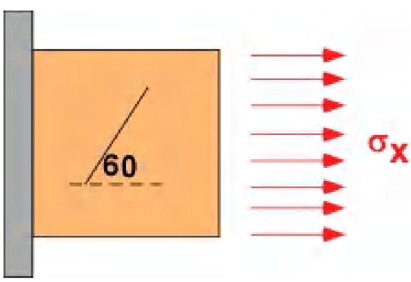
The aluminum block is set up in a wise to be subject to a compressive stress in the x-direction of 150 MPa and a compressive stress in the y direction of 120 MPa. The initial length of the specimen is 150 mm, 30 mm and 15 mm in the x, y and z directions respectively. (a) Calculate the change in the dimension of the object; (b) What is the change in the density of the object ?
(c) What is the change in the area of the cross-section?



Problem 3.7.2

Problem 3.7.3

A line at an angle 60 degrees is inscribed in an unstressed square copper specimen as shown. A stress of 150 MPa is applied to the specimen in the x-direction. (a) What is the new angle the line makes with the horizontal?



Problem 3.7.3

3.8 PRINCIPAL STRESSES

An important approach to structural design in recent years is the use of composite materials. This provides the structure with strength in preferred directions. You can find them on aircrafts (current Boeing transport aircraft are 40 % composite skin), automobiles, sports equipment (all sports racquets, golf clubs), laptop chassis, and most everywhere else. Many are made of carbon/graphite fibers in resin and cured. They provide very high strength to weight ratio. One material, Kevlar is used in body armor. This provides the ability to orient fibers in the direction of maximum stresses.

This idea of principal stresses also references the previous section on *stresses on an inclined plane*. Even if the loading is uniaxial you can expect to find shear stress along different planes in the material. Another important consideration is that stress can vary from point to point in a material continuously. This is an important assumption from *continuum mechanics*.All of this together allows us to define a *state of stress* at a point. This is defined in detail for the two-dimensional case and we will generalize it to the three dimensional case through extrapolation.

3.8.1 Two-dimensional; State of Stress at a Point

The two-dimensional object shown in Figure 3.8.1a is subject to several loads but is in equilibrium. We expect every point in the object to be experiencing stress that varies continuously. What is the **state of stress** at a typical point P in the body? To expose the point P and the stress at point P we cut the section arbitrarily along a plane that includes point P. At P we expect to see a force vector F and the area vector A on which it acts - Figure 3.8.1b. These vectors are independent. We can resolve the force in components in the coordinate directions. We can similarly resolve the area into two components. Stress is defined as force divided by area. There are four such elements. We find stress at a point by using the limit on the area.

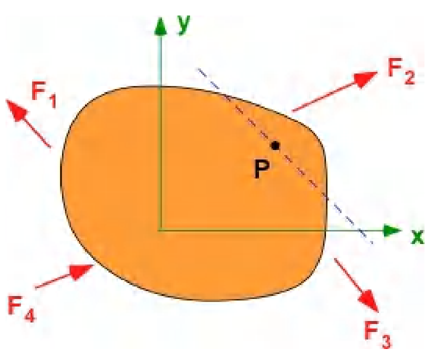


Figure 3.8.1a. Force system on Object

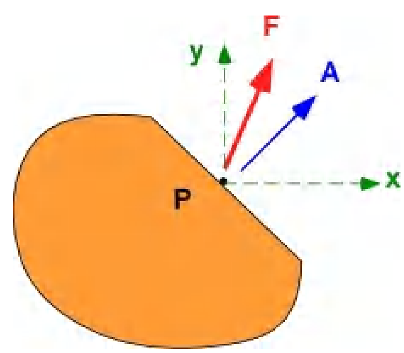


Figure 3.8.1b. Force at point P

$$\bar{F} = F_x\hat{i} + F_y\hat{j}; \quad \bar{A} = A_x\hat{i} + A_y\hat{j}$$

$$\sigma_{xx} = \lim_{A_x \to 0}\frac{F_x}{A_x}; \quad \sigma_{xy} = \lim_{A_x \to 0}\frac{F_y}{A_x}; \quad \sigma_{yx} = \lim_{A_y \to 0}\frac{F_x}{A_y}; \quad \sigma_{yy} = \lim_{A_y \to 0}\frac{F_y}{A_y} \qquad (3.28)$$

The stresses in Eqn. (3.28) are defined so the first subscript references the area component and the second the force component. We recognize that repeated subscripts represent normal stress while the mixed subscripts represent shear stress. The limiting process allows us to represent Figure 3.8.1 in a different way. To define stresses at the point we inscribe an elemental rectangle of infinitesimal size centered around the point - Figure 3.8.2 (a). The sides of the rectangle are of infinitesimal lengths dx and **dy** and the definitions for state of stress is shown in Figure 3.8.2 (b).

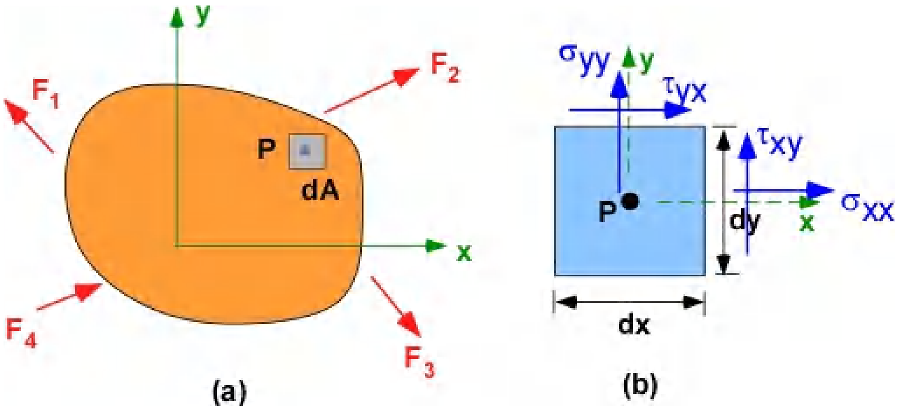


Figure 3.8.2 State of Stress at a point

Since the original object is in equilibrium, the point P with the differential area must also be in equilibrium. This is shown in Figure 3.8.3 where the dimensions of the rectangle are not important since it represents the region around a point.

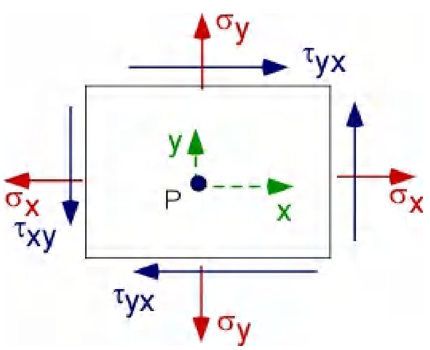


Figure 3.8.3 State of stress at a point

3.8.2 Principal stresses

The discussion here parallels the discussion for stress in an inclined plane. The state of stress at point P given by Figure 3.8.3. For equilibrium τ_{xy} and τ_{yx} are the same.
Consider a rotation of the the element by the angle θ. This is a coordinate rotation by the angle θ. The new coordinate directions are x' and y'. The stress and strain along new coordinate, in terms of the original stresses, is calculated through the relations in Eqn. (3.29). This can be derived through equilibrium and simple rotation of the coordinate system often introduced in math courses. We will avoid the details and apply the relations as given in Eqn. (3.29)

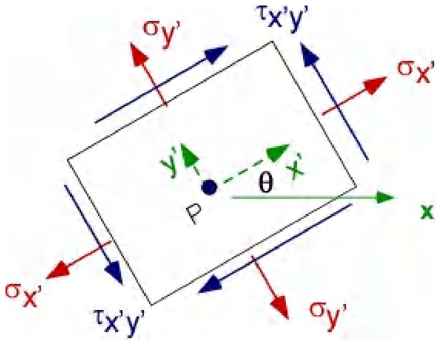


Figure 3.8.4 Coordinate transformation.

The normal and shear stress along the new directions can be obtained as:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} cos\,2\theta + \tau_{xy}\, sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} cos\,2\theta - \tau_{xy}\, sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} sin\,2\theta + \tau_{xy}\, cos\,2\theta \qquad (3.29)$$

This is also referred to as the transformation of stress and strain. For the special case of θ which leads to zero shear stress (θ_p) , called principal direction, there will be only normal stresses on the element. These are the maximum and minimum normal stresses at the point (P).

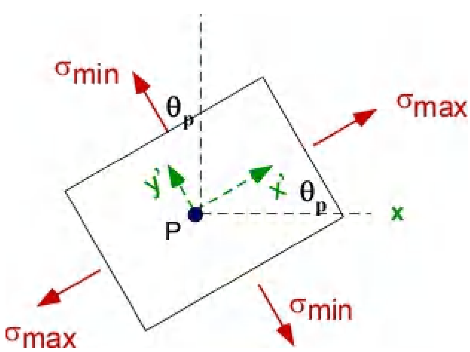


Figure 3.8.5 Principal Stresses

The angle θ_p and the stresses are calculated through Eqn. 3.29

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$\sigma_{min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} \qquad (3.30)$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

The rotation for maximum shear stress is θ_s. The normal stresses are the same for this case.

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right) + \tau_{xy}^2} \qquad (3.31)$$

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

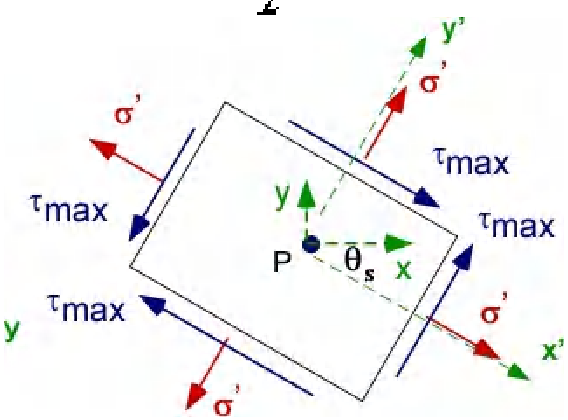


Figure 3.8.6 Plane for maximum shear stress

It can be verified that

$$\sigma'_x + \sigma'_y = \sigma_x + \sigma_y \qquad (3.32)$$

3.8.3 Mohr's Circle

For different values of θ the normal and shear stresses in Eqn (3.30) will generate points that lie on a circle. This is called the Mohr's circle. This is a special circle that can be constructed through the formula or just through geometry. It is based on the current state stress - that is the values for the of the normal and shear stress. The shear stresses are plotted with opposite sign to make the rotation angle θ_p match the physical direction of rotation. The circle is constructed using the nominal values of the σ_x and σ_y and τ_{xy}. Starting with the nominal values in Figure 3.8.3:

- We draw the axis using a convenient scale. The x-axis represents normal stress and the y-axis represent the shear stress.The origin is at (0, 0).
- Plot two points P [σ_x, -τ_{xy}] and Q [σ_x, -τ_{xy}]
- Join points P and Q which intersects the horizontal axis at S.
- With S as center, and radius SQ (or SP) Draw a circle. This is the Mohr's circle

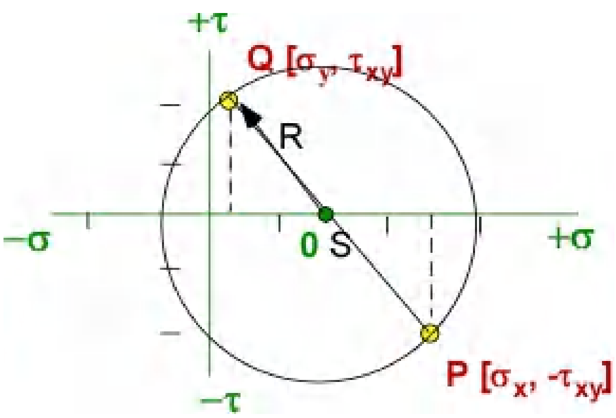


Figure 3.8.7 Mohr's Circle

For any rotation 2θ the $\sigma_x'(\theta)$, $\sigma_y'(\theta)$, and $\tau_{x'y'}(\theta)$ can be read off the axis.

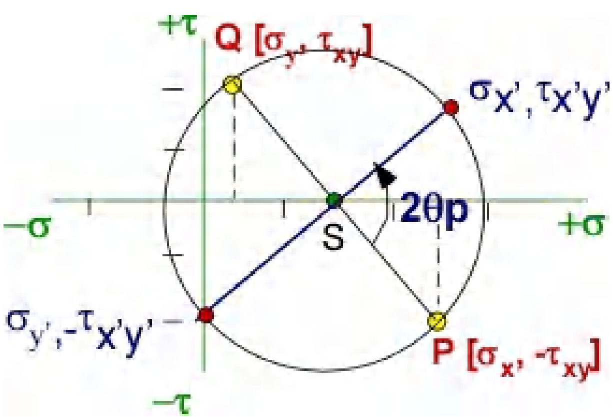


Figure 3.8.8 Transformation of stress

Example 3.11

The state of stress at a point in the specimen is given by the stress diagram shown below. Note $\tau_{xy} = \tau_{yx}$. (a) Find the principal stresses; (b) the rotation of the principal plane; (c) the maximum shear stress for this state of stress; (d) represent the stresses on the Mohr's circle

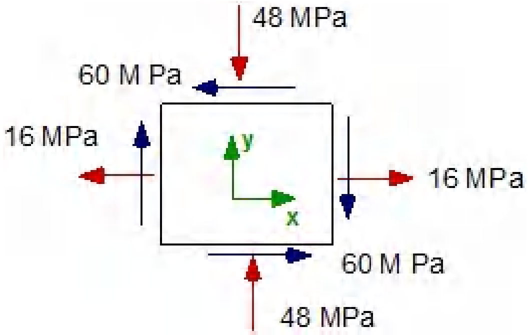


Figure 3.8.9 Example 3.11

Given Data: (sign of stresses are important)
σ_x =-16 MPa; σ_y =- 48 MPa; τ_{xy} = -60 MPa

Assumptions:
2D state of stress

Solution: (Do not need a FBD**)**

(a) Principal stresses

$$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$= 0.5(16-48) + \sqrt{\left[0.5(16-(-48))\right]^2 + (-60)^2} = 52\,MPa$$

$$\sigma_{min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$= 0.5(16-48) - \sqrt{\left[0.5(16-(-48))\right]^2 + (-60)^2} = -84\,MPa$$

(b) The rotation of the principal plane

$$\theta_p = \left(\frac{1}{2}\right)\tan^{-1}\left(\frac{2(-60)}{16-(-48)}\right) = -30.96^\circ$$

(c) The maximum shear stress

$$\tau_{max} = \sqrt{0.5\left[16-(-48)\right]^2 + (-48)^2} = 68\,MPa$$

(d) The Mohr's circle

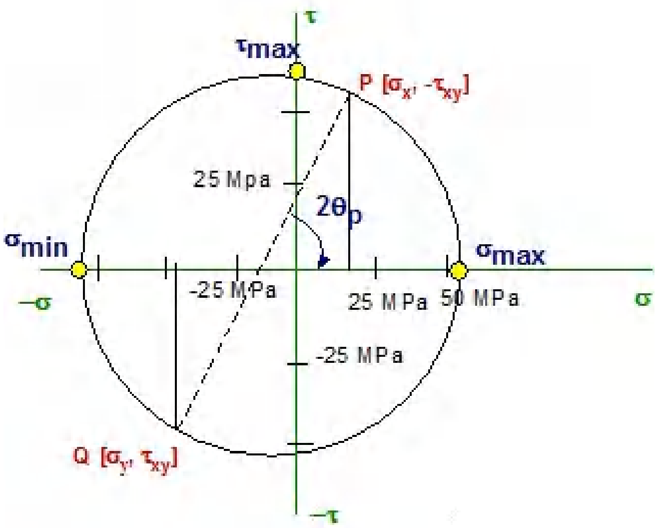


Figure 3.8.10 Mohr's Circle

3.8.4 Calculating and Drawing Mohr's Circle

We will use MATLAB to calculate the principal stresses and draw the Mohr's circle

Solution Using MATLAB

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 3.8 - Example 3.11
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------\n')
fprintf('Example 3.11 \n')
fprintf('--------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Given State of Stress
sigx = 16;   sigy = -48;  txy = -60;
fprintf('--------------------\n')
fprintf('Current State of Stess     \n')
fprintf('--------------------\n')

fprintf('sigx = '),disp(sigx)
fprintf('sigy = '),disp(sigy)
fprintf('txy = '),disp(txy)

%% Calculate Principal Stress
sigav = 0.5*(sigx + sigy);
R = sqrt((0.5*(sigx-sigy))^2 + txy^2);
siga = sigav+R;   sigb = sigav-R;
thtp = 0.5*atan2(2*txy,(sigx-sigy));
fprintf('\n--------------------\n')
fprintf('Principal stresses \n')
fprintf('--------------------\n')
fprintf('siga   [Pa] :'),disp(siga)
```

```
fprintf('sigb   [Pa] :'),disp(sigb)
fprintf('taumax [Pa] :'),disp(R)
fprintf('thtp (deg):'),disp(thtp*180/pi)

%% Draw Mohr's Circle (Graphics)
tht = linspace(0,2*pi,101);
x = sigav + R.*cos(tht);
y = R.*sin(tht);

plot(x,y,'r-','LineWidth',2)
xlabel('Normal stress [MPa]')
ylabel('Shear Stress [MPa]')
title('Mohrs Circle')
grid
axis square
axis tight
hold on
text(sigx,-1.05*txy,'A','FontWeight','b');
plot(sigx,-txy,'ro','MarkerFaceColor','y');
text(sigy,1.05*txy,'B','FontWeight','b');
plot(sigy,txy,'ro','MarkerFaceColor','y');
text(sigav,0,'C','FontWeight','b');
text(siga,5,'\sigma_a','FontWeight','b');
text(sigb,5,'\sigma_b','FontWeight','b');
line([sigx sigy],[-txy txy],'Color','b','LineWidth',2)
line([sigav sigav],[0 R],'Color','b','LineWidth',2)
line([-100 60],[0 0],'Color','k','LineWidth',2)
line([0 0],[-1.1*R 1.1*R],'Color','k','LineWidth',2)

text(sigav,1.1*R,'\tau_{max}','FontWeight','b');
plot(sigav,R,'ro','MarkerFaceColor','y');
plot(sigx,-txy,'ro','MarkerFaceColor','y');
```

In the Command Window

```
-------------------------------
Example 3.11
--------------------------------
-------------------
Current State of Stess
-------------------
sigx =     16
sigy =    -48
txy =    -60

-------------------
Principal stresses
-------------------
siga   [Pa] :    52
sigb   [Pa] :   -84
taumax [Pa] :    68
thtp (deg):  -3.0964e+01
```

In the Figure Window

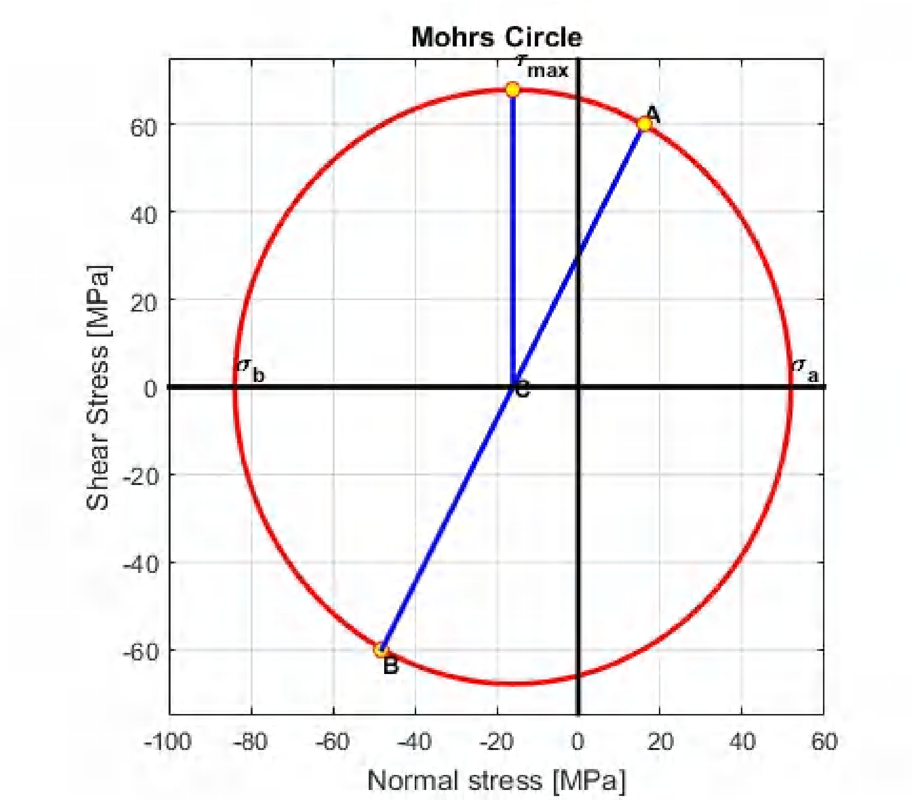


Figure 3.8.11 Mohr's Circle by MATLAB

Execution in Octave

The code is same as in MATLAB above

In Octave Command Window

```
--------------------------------
Example 3.11
---------------------------------
--------------------
Current State of Stess
--------------------
sigx = 16
sigy = -48
txy = -60

--------------------
Principal stresses
--------------------
siga    [Pa] :52
sigb    [Pa] :-84
taumax [Pa] :68
thtp (deg):-30.964
```

In Octave Figure Window

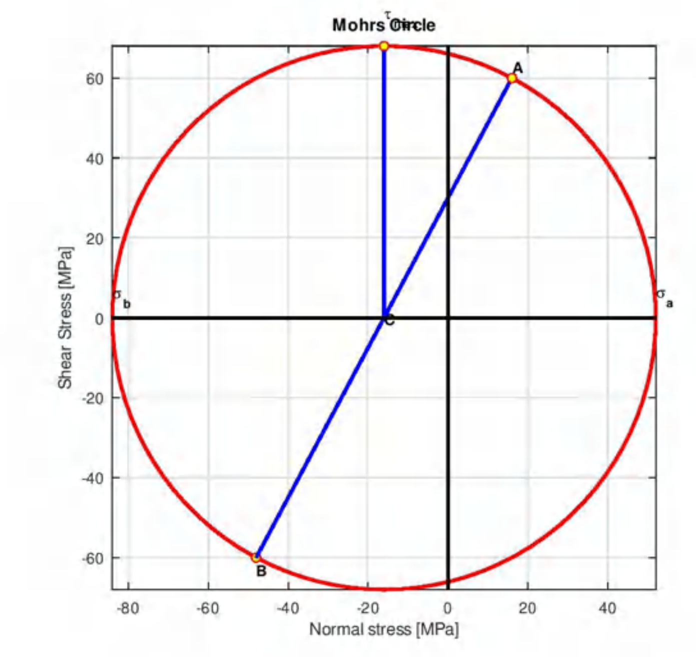


Figure 3.8.12 Mohr's Circle by Octave

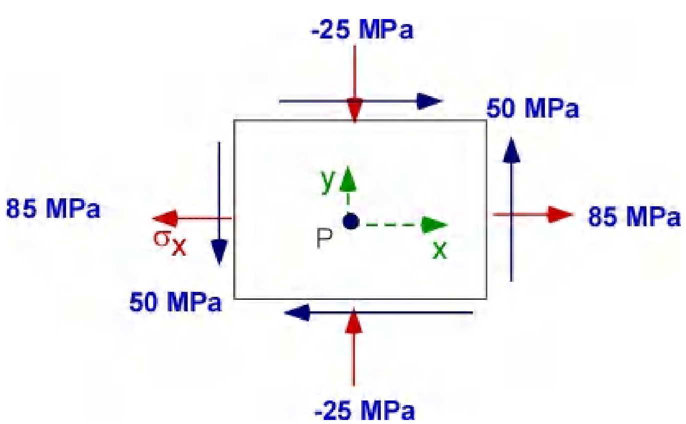
The figures are slightly different and the text location must be fine tuned.

3.8.5 Additional Problems

Solve the following problems on paper and in MATLAB

Problem 3.8.1

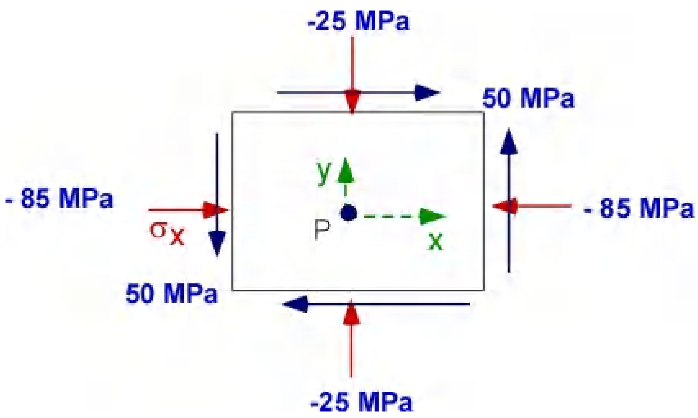
(a) Find the principal stresses; (b) the rotation of the principal plane; (c) the maximum shear stress for this state of stress; (d) represent the stresses on the Mohr's circle



Problem 3.8.1

Problem 3.8.2

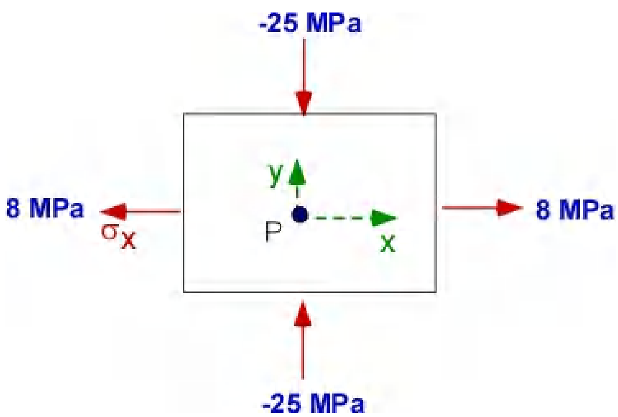
(a) Find the principal stresses; (b) the rotation of the principal plane; (c) the maximum shear stress for this state of stress; (d) represent the stresses on the Mohr's circle



Problem 3.8.2

Problem 3.8.3

(a) Find the principal stresses; (b) the rotation of the principal plane; (c) the maximum shear stress for this state of stress; (d) represent the stresses on the Mohr's circle



Problem 3.8.3

3.9 FAILURE CRITERIA

An engineer will be engaged in designing engineering structures. The *design of an engineering structure* involves:

- Understanding the purpose of the structure
- The dimensions and geometry of the structure
- The loads - that is the forces and moments that the structure will experience
- The restrictions and constraints on the structure and the way it is supported
- The material the structure will be made off.
- Ensuring the structure will not fail
- The structure performs reliably over its lifetime.

Today it is also important that *the design* is:

- Sustainable
- Economical
- Optimal
- Easily replaceable with improved designs
- Disposable
- Environment friendly
- Accessible

The design of a structure that is subject to forces and moments *experiences* the following:

- Two kinds of stresses - normal and shear stress
- The stresses may change from point to point within the structure
- A state of stress can be defined for each point
- Each point will have a maximum normal stresses and maximum shear stresses - principal stresses and the maximum shear stresses
- For failure and reliability these maximum stresses must be below the allowable stresses permitted by the structural material.
- For further safety the maximum stresses must be less than the permitted stresses by the factor of safety.

In summary, any device or object that is designed and manufactured is expected to operate as advertised over a stated length of time. If the product does not function as expected then it is considered a failure. Failure can have many reasons. Failure is usually associated with reliability - the expression of confidence that the product will deliver on its expectation.

Failure also has a practical side effect which is best attributed to Taguchi :"*When a product fails, you must replace it or fix it. In either case, you must track it, transport it, and apologize for it. Losses will be much greater than the costs of manufacture, and none of this expense will necessarily recoup the loss to your reputation"*.

Failure is serious business and designing for actual failure is impossible because of so many variables. Instead we try and ensure that the design meets the **Failure Criteria.** There is no unique criteria and the designer usually satisfies the failure criteria that is appropriate for the type of the product and its underlying design.

Finally, an engineering structure is considered *deformable* - it will deform under loads. If the bridge does not return back to its original state it is likely to cause additional problems in many ways. It is expected to behave elastically: that is when the loads are removed the structure returns to its undeformed state. If the design is stretched beyond the elastic domain then the residual strain on the structure changes the design forever. Many failure criteria are based on principal stresses rather than the standard engineering stress and strain. Since we can calculate the principal stress form the value of the engineering stress and strain at every point, we examine some of the popular failure criteria below.

3.9.1 Mechanical Structural Failures

For most designs we can investigate four types of failures.

a. Failure by elastic structural deflection ($\delta > \delta_{max}$)
b. Failure by structural yielding ($\sigma_{max} < \sigma$)
c. Failure by Fracture ($\sigma = \sigma_{ult}$)
d. Progressive Failure (failure is built slowly during life - creep and fatigue)

There are failures that are difficult to quantify and difficult to investigate.

a. Operating environment (moisture, dirt, dust, corrosion)
b. Aging and shelf life
c. Unanticipated operating current and voltage levels
d. Unintended chemical reactions
e. Electromagnetic interference
f. Material properties may unexpectedly vary due to production and finish

The design/structural engineer must determine the possible modes of failure and establish criteria to predict these failures. In order to account for these unpredictable variations a ***factor of safety*** (FS) is usually adopted. For many structural and machine applications recommended FS are available. They vary by state and country.

$$FS = \frac{\text{Permissible stress in material}}{\text{Maximum stress in the structure}} \qquad (3.33)$$

Figure 3.9.1 Failure n tension (necking)

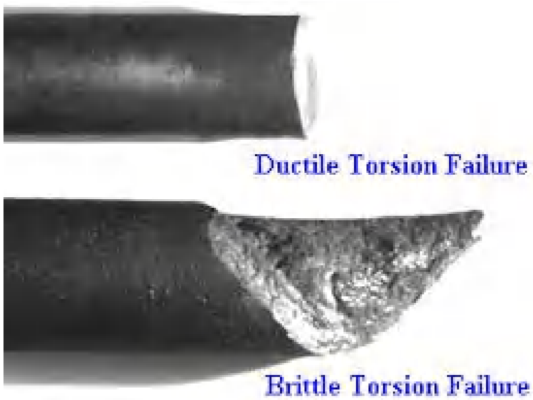


Figure 3.9.2 Failure in torsion

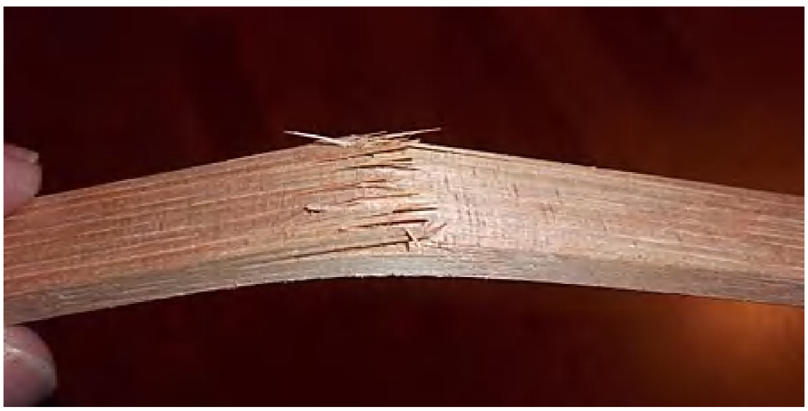



Figure 3.9.3 Failure in bending

(The following images are obtained from Wikimedia commons)

3.9.2 Simple Description of Mechanical Failures

The illustration below is a simple beam that is fixed at one end. When there is no load at the end the entire structure remains horizontal. When a load P is applied at the end the beam will bend/deflect along the length with the maximum deflection at the end as shown. If the beam is elastic then when the load is removed it should be straight and undeformed again. The maximum deflection will depend on the value of the load P. As P increase so does the maximum deflection. However when P exceeds a certain value the beam will not return to its horizontal initial position again. This is considered failure.

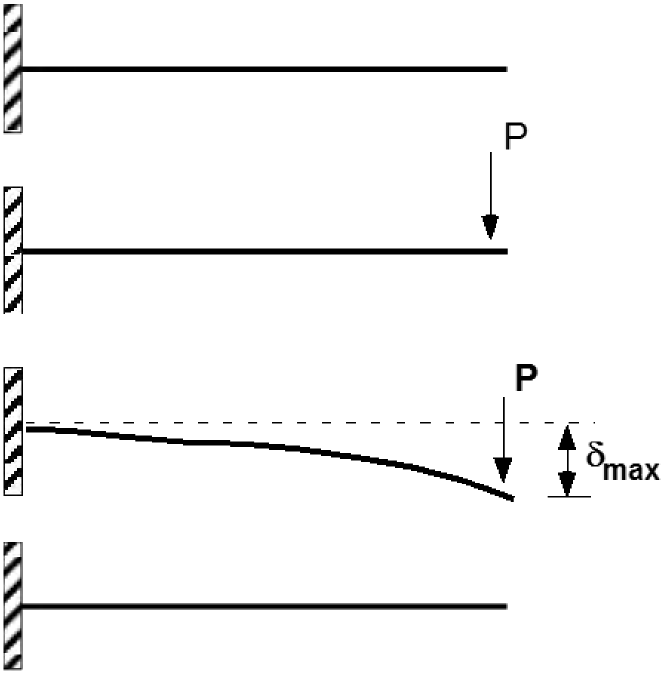


Figure 3.9.4 Illustration of maximum structural deflection

This type of failure is prevalent in vibration when the amplitude is large that parts collide. Also beams and shell may buckle under load. The failure criteria is applied on the maximum deflection of the component that includes buckling load too. A factor of safety is assumed and the deflection must remain less than the maximum elastic deflection.

(i) Maximum Yield Stress
In most structural design, the maximum stress is kept below the yield stress (or proportional limit). Instead of the deflection the stress is monitored. It is assumed that the stress at maximum elastic deflection and maximum yield stress are pretty close by. This type of failure is used for simple structures and simple loading in beams, shells, and plates.

For failure criteria, the designer must calculate the maximum stress of the component that include buckling and an appropriate factor of safety to ensure the actual maximum stress will not exceed the yield stress of the material.. Instead of the maximum stress, the failure criteria have evolved t include principal stress like Tresca and Von Mises failure criteria.

(ii) Failure by Fracture
Failure by fracture is usually associated with brittle materials since ductile materials will have already yielded and will suffer plastic deformation prior to fracture. Fracture will also depend on existing cracks as these cause stress concentrations where local stresses will exceed any stress limit under consideration.

For this type of failure the maximum principal stress must be calculated with an appropriate factor of safety to ensure the stress is still within the yield limit.

(iii) Progressive Failure
During progressive failure, a small failure or small changes is added on to the component during routine operations. This builds up to a sudden failure at a later date even it it is not apparent at the current time. A micro crack is usually the culprit. As noted before this failure can happen due to creep, due to fatigue, and due to changes in material property leading to a change in the stress-strain

behavior, due to small changes in the component.

During Creep failure there is usually

- Increase in strain without increase in stress
- Enhanced strain at high temperature operation

During fatigue failure there is usually

- Change in stress level because of the change in the frequency of the load cycle
- Repeated changes in the direction of load causing change in local material properties

(iv) Failure Criteria

The mechanical failures outline above are actually implemented by verifying that the component satisfies some formal failure criteria. Now with software used for structural analysis it becomes easy to obtain a detailed picture of stress, strain, displacement everywhere through structural simulation software. The software can also report on multiple failure criteria as part of the solution. We look at four popular criteria

Maximum Shear Stress Criterion (Tresca's Hexagon)
Maximum Distortion Energy Criterion (Von Mises)
Maximum Normal Stress Criterion (Coulomb's Criteria)
Maximum Normal Stran Criterion (St. Venants Criteria)

A component will be safe under given loading if the stress at all critical points, including areas of stress concentration, is less that than that recommended by one or more of the failure criteria indicated above.

3.9.3 Maximum Shear Stress Criterion

This criteria is based on plane stress and useful for ductile materials.

It is believed that ductile materials will fail in slippage along an oblique surface due to shear stress

Criteria:

$$\tau_{max}(\text{in the component}) \leq \tau_y\ (\text{shear yieldin test specimen basedon samematerial})$$

Consider a cantilever beam with end load. The principal stress at every point can be evaluated as illustrated below. The state of stress at every point can be translated through the principal stress. σ_a is the maximum principal stress and σ_b is the minimum principal stress:

$$\tau_{max} = \frac{1}{2}\left|\sigma_{max}\right| \quad OR \quad \frac{1}{2}\left(\sigma_a - \sigma_b\right) \qquad (3.34)$$

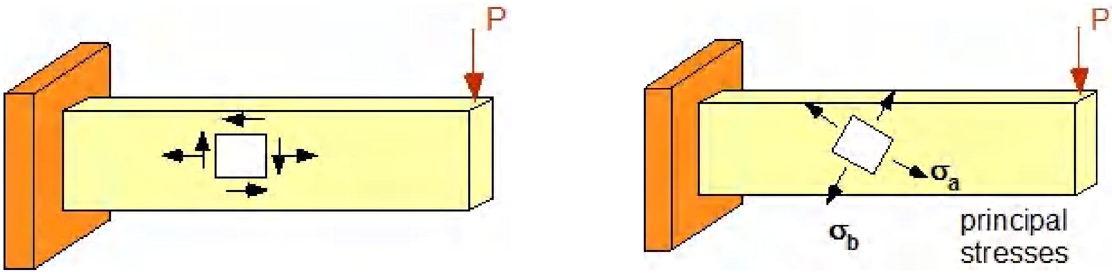


Figure 3.9.5 Regular and principal stresses at the same point

Alternate Criteria:

$$|\sigma_a| < \sigma_Y; \quad |\sigma_b| < \sigma_Y; \quad |\sigma_a - \sigma_b| < \sigma_Y \ (if\ \sigma_a\ and\ \sigma_b\ are\ of\ opposite\ sign) \qquad (3..35)$$

This leads to **Tresca's Hexagon** with the criteria that the principal stresses at every point must lie within the area of the Hexagon shown in Figure 3.9.6 for the component to be safe. σ_Y is the yield strength of the material

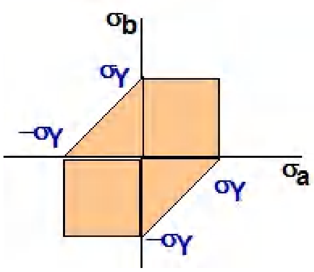


Figure 3.9.6 Tresca's Hexagon

3.9.4 Maximum Distortion Energy Criterion (Von Mises)

A given structural component is safe

If the maximum value of the distortion energy per unit volume of the material is less than the distortion energy per unit volume required to cause yield in a tensile-test specimen f the same material

The distortion energy **(u_d)** , which is the energy associated with the change of shape, is different from the strain energy , which is the change in the volume of the material. These concepts are to early for this first course in mechanics. However we can tie the distortion energy to the principal stresses. For an isotropic material under plane stress with a modulus of rigidity G

$$u_d = \frac{1}{6G}\left(\sigma_a{}^2 - \sigma_a\sigma_b + \sigma_b{}^2\right) \qquad (3.36)$$

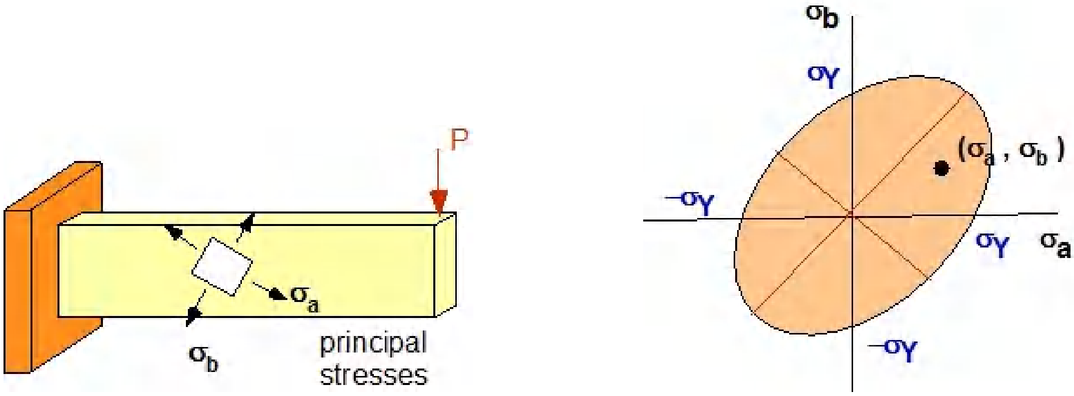


Figure 3.9.7 Principal stress and alternate criteria for distortion energy

Alternate criteria

$$\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 < \sigma_Y^2 \qquad (3.37)$$

The structural component is safe if the principal stresses at every point is within the area enclosed by the ellipse. This is also regarded as the **Von Mises** Criteria.

3.9.5 Maximum Normal Stress Criterion

This failure criteria is applied to brittle materials where failure is expected through rupture or fracture. It will be sudden with no yielding before failure. It would certainly apply to products created using ceramic materials.

Criteria:

A given structural component is safe if the maximum normal stress in the component reaches the ultimate stress in a tensile test specimen made of the same material

If the ultimate stress in tension and compression are the same the criteria is regarded as the Coulomb's criteria and can be summarized as follows:

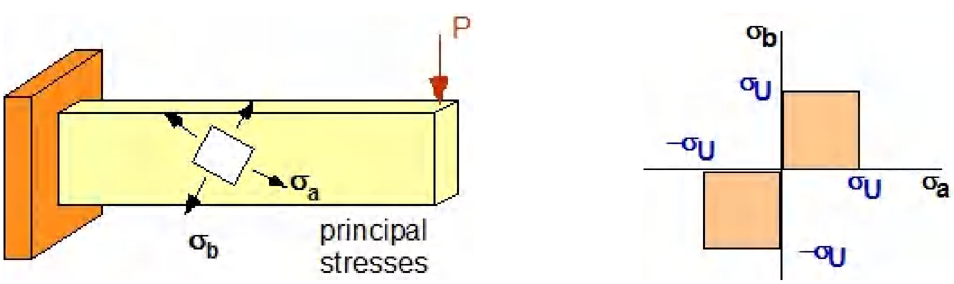


Figure 3.9.8 Coulomb's normal stress criteria

$$|\sigma_a| < \sigma_U; \quad |\sigma_b| < \sigma_U \qquad (3.38)$$

If the ultimate stress in tension and compression are different the criteria is regarded as the Mohr's criteria and can be summarized as follows:

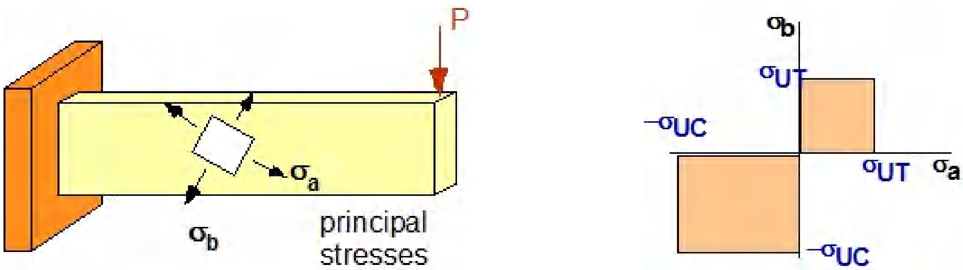


Figure 3.9.9 Mohr's normal stress criteria

$$\sigma_a < \sigma_{UT}; \quad \sigma_b < \sigma_{UT} \ (\sigma_a, \sigma_b > 0)$$

$$|\sigma_a| < |\sigma_{UC}|; \quad [\sigma_b) < |\sigma_{UC}| \ (\sigma_a, \sigma_b < 0) \quad (3.39)$$

3.9.6 Maximum Normal Strain Criterion

This failure criteria is applied to brittle materials where failure is expected through rupture or fracture. It will be sudden with no yielding before failure.

Criteria:

A given structural component is safe if the maximum normal strain in the component remains smaller than the ultimate normal strain in a tensile test specimen made of the same material

For material with the same strain in tension and compression the criteria is called the Saint Venant's criteria and can be summarized as:

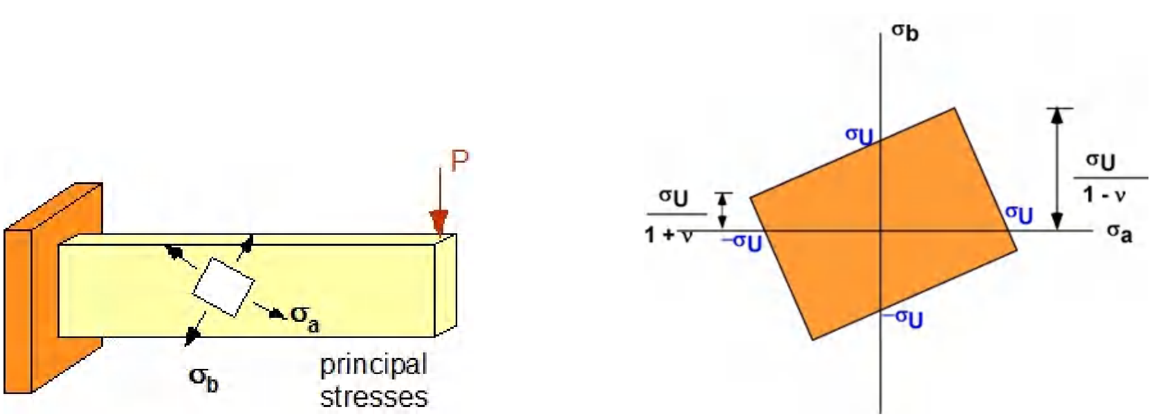


Figure 3.9.10 St. Venant's normal strain criteria

$$|\varepsilon_a| < \varepsilon_U; \quad |\varepsilon_b| < \varepsilon_U \quad (3.40)$$

Alternate Criteria:

A structural component is safe as long as the principal stresses remain within the area of the plot shown above

3.9.7 Example 3.12

The state of stress, at a critical point on the PCB guide, due to warping is a concern. The result of tensile stress tests of the same material establishes the yield stress as 250 MPa (σ_Y). Since this not a brittle material find the factor of safety with respect to yield, using (a) the maximum-shearing-stress criterion; (b) the maximum-distortion-energy criterion

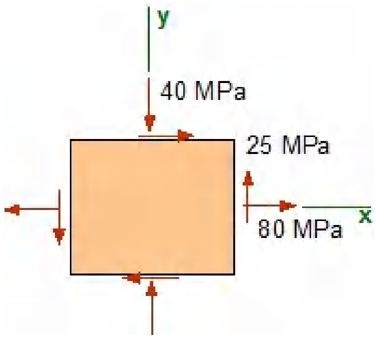


Figure 3.9.11 Example 3.12

Data: $\sigma_x = 80$ MPa; $\sigma_y = -40$ MPa; $\tau_{xy} = 25$ MPa; $\sigma_Y = 250$ MPa; $\tau_Y = 250/2 = 125$ MPa

Assumption: None (not brittle material)

Solution:

$$\sigma_{ave} = \frac{1}{2}\left(\sigma_x + \sigma_y\right) = \frac{1}{2}(80 - 40) = 20\,MPa$$

$$R = \tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{80-(-40)}{2}\right)^2 + 25^2} = 65\,MPa$$

$$\sigma_a = \sigma_{ave} + R = 20 + 65 = 85\,MPa$$

$$\sigma_b = \sigma_{ave} - R = 20 - 65 = -45\,MPa$$

(a) the maximum-shearing-stress criterion

$$FS = \frac{\tau_Y}{\tau_{max}} = \frac{125}{65} = 1.92$$

(b) the maximum-distortion-energy criterion

$$\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 = \left(\frac{\sigma_Y}{FS}\right)^2$$

$$85^2 - 85(-45) + 45^2 = \left(\frac{250}{FS}\right)^2$$

$$114.3 = \frac{250}{FS}; \quad FS = 2.19$$

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 3.9 - Example 3.12
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------\n')
fprintf('Example 3.12 \n')
fprintf('--------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Given State of Stress
sigx = 80;   sigy = -40;  txy = 25;  sigY = 250;  tauY = sigY/2;
fprintf('-------------------\n')
fprintf('Current State of Stess     \n')
fprintf('-------------------\n')

fprintf('sigx = '),disp(sigx)
fprintf('sigy = '),disp(sigy)
fprintf('txy = '),disp(txy)
fprintf('sig (yield) = '),disp(sigY)
fprintf('tau (yield) = '),disp(tauY)
%% Calculate Principal Stress
sigav = 0.5*(sigx + sigy);
R = sqrt((0.5*(sigx-sigy))^2 + txy^2);
taumax = R;
siga = sigav+R;   sigb = sigav-R;
% thtp = 0.5*atan2(2*txy,(sigx-sigy));
fprintf('\n-------------------\n')
fprintf('Principal stresses \n')
fprintf('-------------------\n')
fprintf('siga   [Pa] :'),disp(siga)
fprintf('sigb   [Pa] :'),disp(sigb)
fprintf('taumax [Pa] :'),disp(R)
% fprintf('thtp (deg):'),disp(thtp*180/pi)

%% (a) Failure criteria
FS1 = tauY/taumax;
fprintf('\nFS based on Maximum shear Criteria :'),disp(FS1)

%% (b) Failure criteria
FS2 = sqrt(sigY^2/(siga^2 - siga*sigb + sigb^2));
fprintf('FS based on Von Mises Criteria     :'),disp(FS2)
```

In the Command Window

```
--------------------------------
Example 3.12
---------------------------------
-------------------
Current State of Stess
-------------------
sigx =     80
sigy =    -40
txy =     25
sig (yield) =    250
tau (yield) =    125

-------------------
Principal stresses
-------------------
siga   [Pa] :    85
sigb   [Pa] :   -45
taumax [Pa] :    65

FS based on Maximum shear Criteria :        1.9231
FS based on Von Mises Criteria     :        2.1863
```

Execution in Octave

The code is same as in MATLAB above

In Octave Command Window

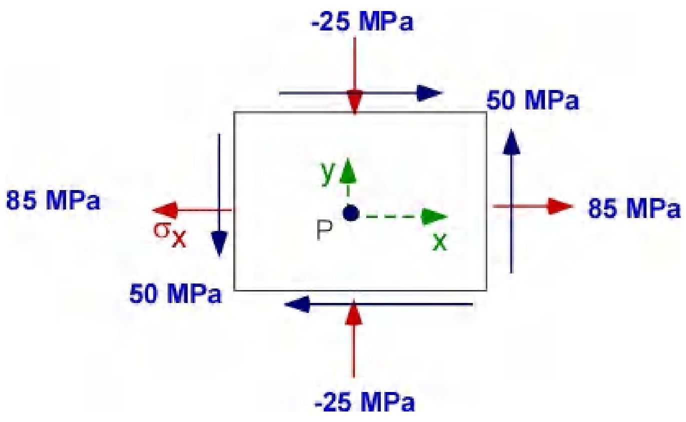
```
-------------------------------
Example 3.12
---------------------------------
-------------------
Current State of Stess
-------------------
sigx = 80
sigy = -40
txy = 25
sig (yield) = 250
tau (yield) = 125

-------------------
Principal stresses
-------------------
siga   [Pa] :85
sigb   [Pa] :-45
taumax [Pa] :65

FS based on Maximum shear Criteria :1.9231
FS based on Von Mises Criteria     :2.1863
```

3.9.8 Additional Problems

Solve the following problems on paper and in MATLAB/Octave. These problems appeared in the section on principal stresses.

Problem 3.9.1

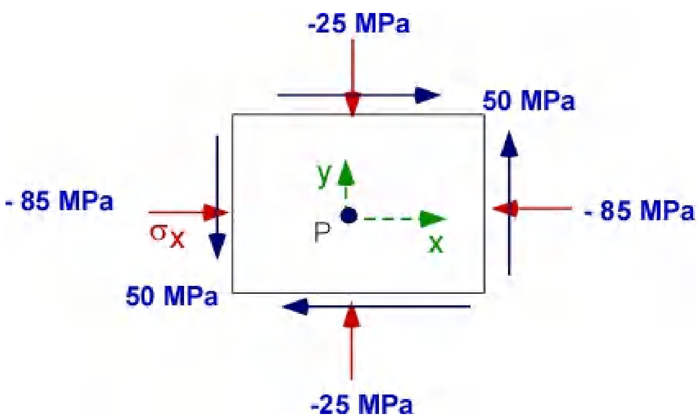
The state of stress is in a ductile material of your choice. Apply, (a) the maximum-shearing-stress criterion; (b) the maximum-distortion-energy criterion



Problem 3.9.1

Problem 3.9.2

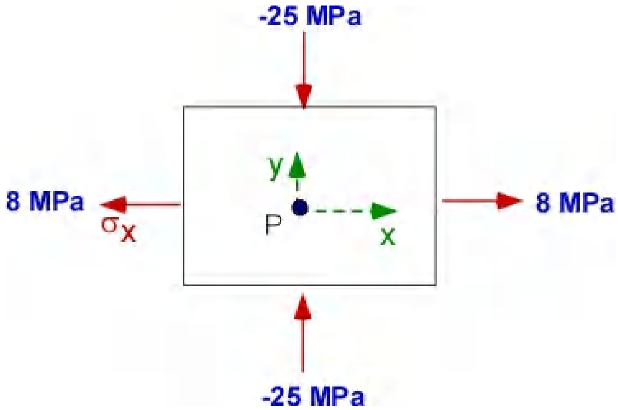
This state of stress is in a brittle material of your choice. Apply (a) the normal maximum stress criteria; (b) the maximum normal strain criteria.



Problem 3.9.2

Problem 3.9.3

Chose the material and the failure criteria you will apply.



Problem 3.9.3

3.10 TORSION - AN INTRODUCTION

This material will appear again in Chapter 8. You can skip it if you choose. The essential information in this section are the following:

- Torsion introduces shear stress and shear strain in the object
- Torsional shear stress are distributed across the cross-section
- Designs involving torsion prefer circular cross-sections
- Torsion causes a twisting deflection

In the previous sections we have discussed the stresses, strain, and displacement resulting from pushing and pulling on the object. The concern was to avoid failure. You can also break an object by twisting it - applying **torsion**. An example was included in the previous section. In this section we briefly introduce torsion and then revisit the topic to study it in detail in a later chapter.

While we talked a lot about stresses in the previous sections you must realize that there are only two types of **stresses**. They are the **normal stress** and the **shear stress**. In **normal stress** the *force is normal to the area on which it is acting.* In the case of **shear stress** the *force is over the area over which it is acting*. Whatever the action of the force on the body it can only produce either or both of these stresses. Pulling and pushing normal to a member of the structure produces *normal stresses* as we defined in the previous sections. **Pure torsion** will produce *shear stresses*. **Pure bending** produces *normal stresses*.

3.10.1 Pure Torsion

Torsion is just moment applied along an axis. It is important in power transmission, both during generation and consumption. In Figure 3.10.1.1the vertical wind turbines generate power through the rotational motion created by aerodynamic forces. The turbine generates power by harnessing the rotational motion created along an axis through aerodynamics again. The hydraulic electric generator creates a rotational motion by converting the linear momentum of the fluid stream to angular momentum and torque. The automobile is driven by delivering torque to the wheel axle. You open a wine bottle by twisting or applying torque.

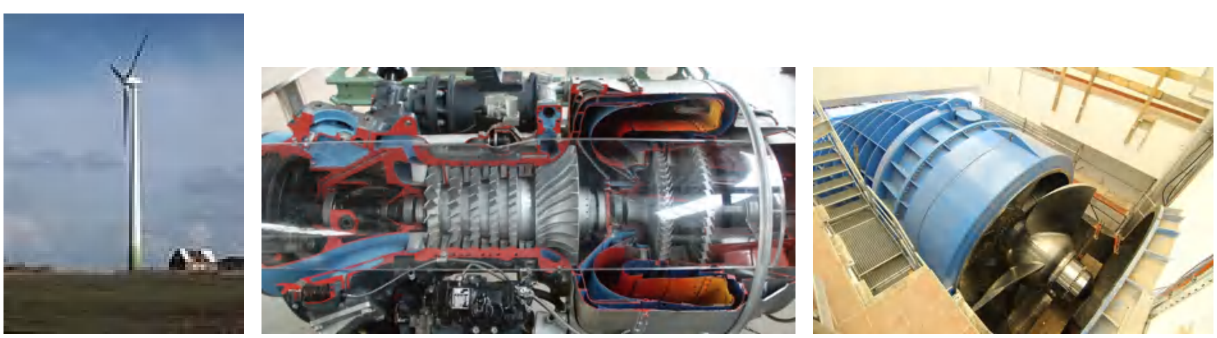


Figure 3.10.1 Examples of useful torsion applications (figures courtesy of Wikimedia commons)

One of the design features in objects that deal with torsion is that the cross-sectional area is usually circular. Transmission shafts are usually solid circular or annular. This has evolved through practice and the fact that circular cross-sections provide rotational symmetry. When you are executing the Biellmann spin on the ice you are not really symmetrical or have a rotational cross-section. We will restrict the cross-sections to be circular for mechanical designs. Non circular sections can also carry torque. Currently the design insight is to use circular geometry for carrying torque. To start our discussion we start with a solid circular shaft of radius R in equilibrium carrying a pair of torque.

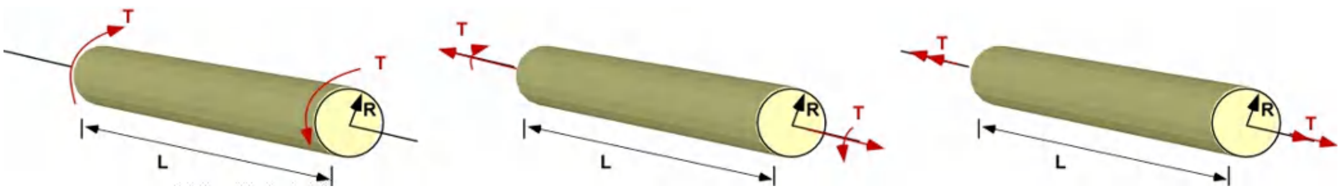


Figure 3.10.2a Shaft in equilibrium - three equivalent representations

Figure 3.10.2a shows a shaft carrying torque T in equilibrium and is represented in three ways. We will use the last representation with the torque represented by double arrows because it is direct and simple. The torsional momnt is given by the curl of the fingers of the right hand as the thumb is in the direction of the double arrows. Since the torsional displacement is relative to the ends of the shaft we will introduce a further simplification that the left end is fixed and it is the right end that will deform due to the application of torque. Figure 3.10.2b represents this depiction along with another illustration of the torque T on Face C. This will assist in following the discussion below.

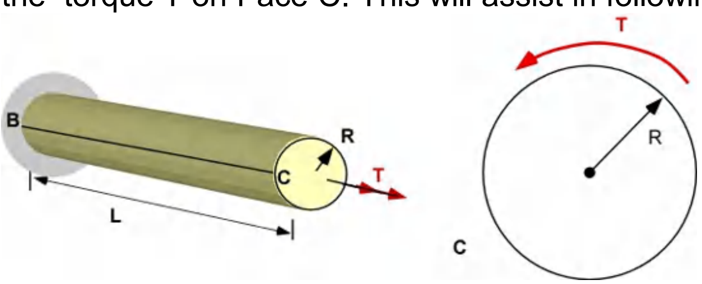


Figure 3.10.2b Representation in the cross section

In our representation the external torque **T** is exerted at the ends of the shaft of length **L** at B and C. What about the cross-sections in between? A simple FBD will require that every section of the shaft along the length to support the same torque **T**. But unlike the ends they must react by developing a force distribution on the area of the cross-section that results in the torque **T**. This is what we term the resultant force in the cross-section. This force will naturally define a stress distribution in the cross-section. Since we have axial/circular symmetry the quantities will be at most a function of the radial location - **r**. Let us list the following ideas for development of relations among various entities in torsion:

1. There is only a single **equivalent** load in each internal cross-section and it is the torque or a moment **T**
2. The stress distribution in the cross-section must integrate over the area of the cross-section to produce this torque **T**.
3. Axial symmetry suggests that this stress may be only a function of the radial distance from the center - **r.**
4. The best way to generate a torque or a moment is to have a force acting at a distance. In this illustration the force is located away from the center of the cross-section **F(r)**. This force must be in the plane of the cross-section and therefore will produce a **shear stress**. We will identify this as $\tau(r)$.
5. The force and the stress around the cross-section will have the same value at the same radial location **r**. An annular dA is the best choice to establish the force at the location **r**. The value of

dA is 2 π **r**dr.

6. Finally we assume deflections are small and we are in the elastic range where distributions can be assumed linear. Angles are small so that sometimes curvature can be neglected

We illustrate this at any cross-section D on the shaft as:

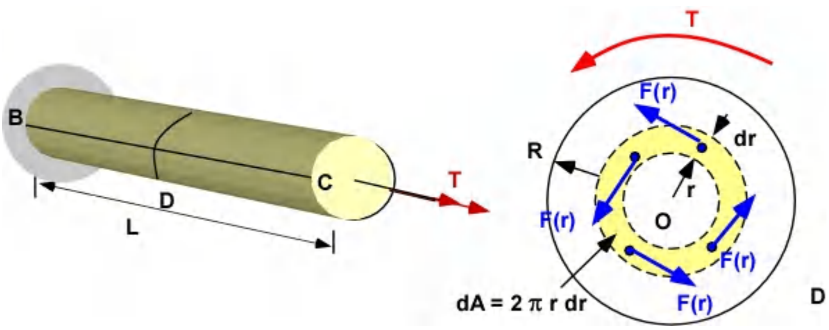


Figure 3.10.3 Force in cross-section leading to T

We can develop the following relation between the torque **T** and the stress τ(r)

$$F(r) = \tau(r) dA$$
$$dT(r) = r \times F(r) = r\,F(r) = \tau(r) r\, dA \qquad (3.41)$$
$$T = \int_0^R dT(r) = \int_0^R \tau(r) r\, dA$$

This is all we can establish using **statics** and **equilibrium**. Since we cannot establish the nature of τ**(r)** it is a statically indeterminate problem. To develop the relationship further we have to include displacement and deformation of the shaft. Let us explore the relative shaft deflection of the end C relative to end B which is held fixed. We illustrate this in Figure 3.10.4 which is shown enlarged. It has two depictions to detail several ideas in the following discussion.

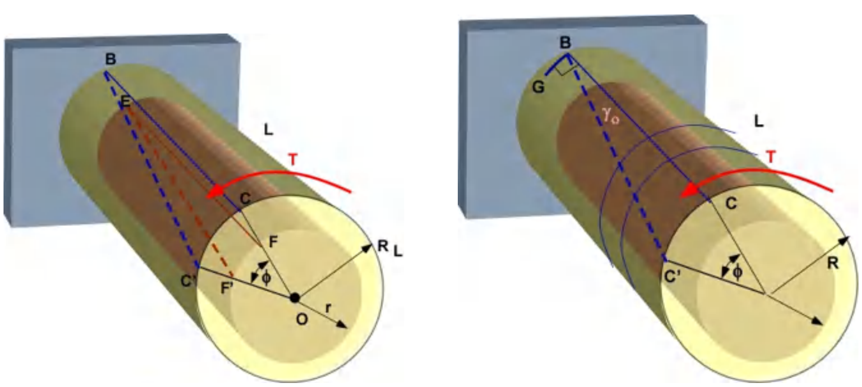


Figure 3.10.4 Shaft deformation or deflection

In the figure on the left above we have shown two surfaces at different radius. The inner surface is at r = r (arbitrary) and the outer surface at r = **R**. In our experience the shaft will twist due to the torque. This is the angle shown as ϕ. We have drawn a line on the outer surface BC and a corresponding line EF on the inner surface. The twist deforms the shaft such that the points C on the outer surface moves to C' and the point F on the inner surface moves to F'. The angle of twist is the same for both surfaces. We can deduce that the length CC' is Rϕ and the length of FF' is rϕ.

In the figure on the right we have the same figure but include another angle γ_o capturing the deflection of the outer surface. Before the application of the torque the angle between BG and BC is 90 degrees. The application of torque causes the angular deflection γ_o, which is the reduction in the original right angle. This is the classic definition of **shear strain**. We can similar define the shear strain at r = r as γ. We can accept that the maximum strain is at the outer surface as it relates the the displacement CC'. The shear strain on the inner surface will be related to displacement FF' as the length of the shaft is the same. We now can relate

$$\tan\gamma_o \simeq \gamma_o = \frac{CC'}{L} = \frac{R\phi}{L}$$
$$\tan\gamma \simeq \gamma = \frac{FF'}{L} = \frac{r\phi}{L} \qquad (3.42)$$
$$\gamma = \gamma_o \frac{r}{R}$$

We now exploit the elasticity of the shaft (shear modulus G) to relate the shear stress to the shear strain and include this in the integration of the torque

$$\tau(r) = G\gamma = \frac{G\gamma_o}{R} r = \tau_o \frac{r}{R}; \quad \tau_o = \tau_{max}$$
$$T = \int_0^R \tau_{max}\, r\, r\, dA = \frac{\tau_{max}}{R}\int_0^R r^2 dA = \frac{\tau_{max} J}{R}; \quad J = \int_0^R r^2 dA \qquad (3.43)$$

J is the polar moment of inertia and is the property of the cross-section. For a solid shaft as in this example with radius R or diameter D

$$J = \int_0^R r^2 dA = \int_0^R r^2 (2\pi r\, dr) = \frac{2\pi}{4} r^4 \Big|_0^R = \frac{\pi}{2} R^4 = \frac{\pi}{32} D^4$$

Let us recompile the information above to express the stress and the twist angle in terms of the the applied torque and the geometry of the shaft.

$$\tau_{max} = \frac{TR}{J} = G\gamma_{max} = G\frac{R\phi}{L}$$

$$\phi = \frac{TL}{GJ}; \quad is\ not\ a\ function\ of\ r \qquad (3.44)$$

$$\gamma_{max} = \frac{R\phi}{L}$$

$$\tau(r) = \frac{Tr}{J} = G\gamma(r)$$

The distribution of the shear stress is illustrated in Figure 3.10.5. The shear stress is linear in the cross-section with zero at the center (for a solid cross-section) and a maximum value at the outer surface. It is same for any radius drawn in the cross-section (or any azimuthal angle θ).

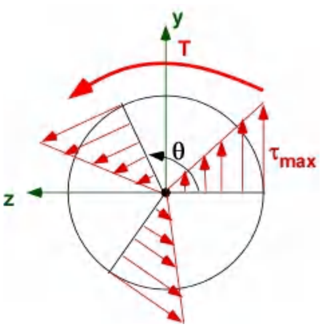


Figure 3.10.5 Shear stress distribution

You will note that we have avoided placing the two subscripts required to define any shear stress - the first subscript identifies the direction normal to the plane of the area on which the force acts and the second the direction of the force. If we use the coordinate **x** to define the axis of the shaft then the cross-section is defined by the coordinates **y** and **z** as shown. For any radius (r) and azimuth (θ) there are two components of the shear stress τ_{xy} and τ_{xz}.

Stress Distribution for Annular Cross-section

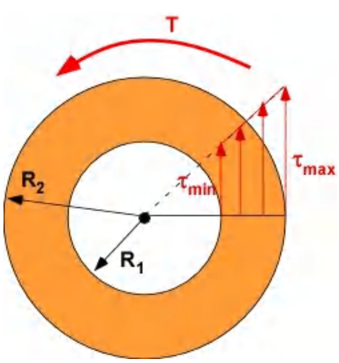


Figure 3.10.6 Stress in annular cross-section.

The relations are the same but the polar moment of inertia id defined similarly but has a different expression for this cross-section. You should confirm the following equations are useful for the annular cross-section with a length L and shear modulus G.

$$J=\frac{\pi}{2}\left(R_2^4-R_1^4\right)$$

$$\tau(r)=\frac{Tr}{J};\quad \tau_{\min}=\frac{TR_1}{J};\quad \tau_{\max}=\frac{TR_2}{J} \qquad (3.45)$$

$$\phi=\frac{TL}{GJ}$$

3.10.2 Example 3.13

Let us complete a set of calculations using the relations we have established. Consider a single shaft of annular cross-section, of length L = 1.2 m, inner diameter D_1 = 50 mm, outer diameter D_2 = 72 mm, made of Aluminum 2014- T6 (τ_{ult} = 275 MPa, τ_y = 230 MPa. G = 27 GPa) subject to torque T along its axis. What is the maximum torque that can be handled by the shaft for a factor of safety of 3 with respect to shear yield?

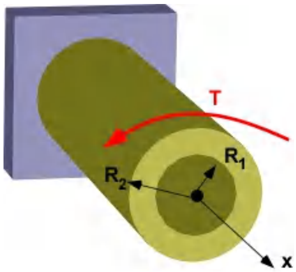


Figure 3.10.7 Example 3.14.1

The calculations are simple and a calculator will do. We will confirm it with MATLAB.

$$J=\left(\frac{\pi}{32}\right)\left(0.072^4-0.05^4\right)=2.025\times10^{-6}\left[m^4\right]$$

$$\tau_{allow}=\frac{\tau_{yield}}{FOS}=\frac{230\times10^6}{3}=76.67\times10^6\left[Pa\right]$$

$$T_{allow}=\frac{\tau_{allow}\times J}{D_2/2}=4312\left[Nm\right]$$

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 3.10 - Example 3.13
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------\n')
fprintf('Example 3.13 \n')
fprintf('---------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%% Data (in meters)
D1 = 50/1000;  R1 = D1/2;  D2 = 72/1000;  R2 = D2/2; L = 1.2;
FOS = 3;  tauy = 230e06;  tallow = tauy/FOS;

%% calculations -
% We can plug data into formula (numerical)
% or solve as an unknown (symbolic)

syms T % torque unknown
J = (pi/32)*(D2^4 - D1^4);
eq1 = T -tallow*J/R2;
T = double(solve(eq1));

fprintf('Length [m]                          = '),disp(L)
fprintf('Inner diameter [m]                  = '),disp(D1)
fprintf('Outer diameter [m]                  = '),disp(D2)
fprintf('J [m^4]                             = '),disp(J)
fprintf('FOS                                 = '),disp(FOS)
fprintf('Max allowable shear stress [Mpa] = '),disp(tallow/1000000)
fprintf('Maximum Torque [Nm]                 = '),disp(T
```

Output in Command Window

```
-------------------------------
Example 3.13
--------------------------------
Length [m]                       =          1.2
Inner diameter [m]               =         0.05
Outer diameter [m]               =        0.072
J [m^4]                          =   2.0247e-06
FOS                              =      3
Max allowable shear stress [Mpa] =       76.667
Maximum Torque [Nm]              =         4312
```

Execution in Octave

The code is the same as in MATLAB except for the additional statements below . The changes are highlighted. You must include the symbolic package and if you do not wish to see warnings you include the command warning off as shown

```
clc, clear, format compact, close all, format short G, warning off

pkg load symbolic
sympref display flat
%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data (in meters)
```

In Octave Command Window

```
-------------------------------
Example 3.13
--------------------------------
Length [m]                       = 1.2
Inner diameter [m]               = 0.05
Outer diameter [m]               = 0.072
J [m^4]                          = 2.0247E-06
FOS                              = 3
```

```
Max allowable shear stress [Mpa] = 76.667
Maximum Torque [Nm]              = 4312
```

We did not use the length in the calculations. We can change the example to calculate design information like the maximum radius for a given torque. We can calculate both the radii if we constrain the twist angle. We will look at Torsion more extensively in a later chapter.

3.10.3 Failure in Torsion

In Example 3.13 we related failure to the maximum yield stress for pure torsion. The failure in torsion can be related to the type of material as shown in Figure 3.10.8.

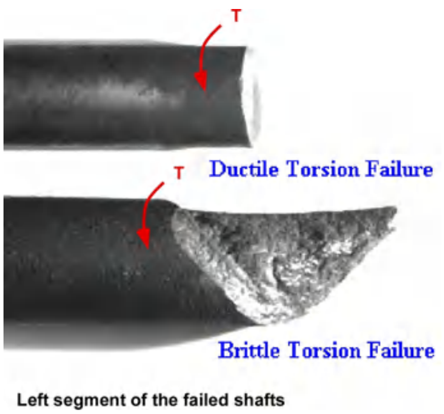


Figure 3.10.8 Torsional Failure (photo by Jeff Thomas, 1997)

To understand the two types of failure let us explore the principal stresses in this case using Mohr's circle of a point on the top surface of a shaft subjected to torsion. To coincide with the failure in the image the direction of torque is changed from earlier figures. This only changes the direction of the shear stress.

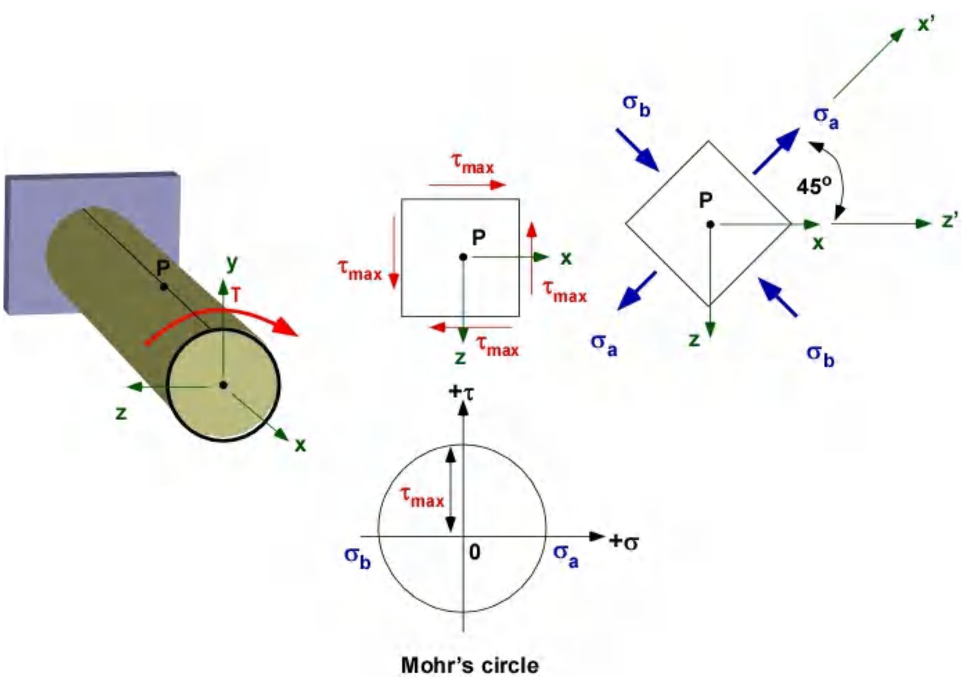


Figure 3.10.9 Torsion at a point on the shaft

Consider point P in Figure 3.10.9. It's state of stress is shown on a rectangular element in the top figure in the second column. There is shear stress in the right side because of the applied torque. The shear stress on the remaining sides can be filled in by considering equilibrium of the element. The lower figure in the second column is the Mohr's circle corresponding to the point P. The principal stresses σ_a and σ_b can be directly established by the circle of radius τ_{max}. The figure in the third column is the orientation of the planes at the point P on which the principal stresses appear. For the applied torque there are two kinds of failure. One type of failure can occur due to the **maximum shear stress** (shear failure) which explains the failure in ductile materials in the top part of Figure 3.10.8. The other type of failure is due to the **normal principal stresses** (normal failure) as seen in the case of the brittle materials in the lower part of Figure 3.10.8. In this case the material has given way in tension even though the primary applied stress is shear. In essence failure can occur along planes that are different from those that handle the direct application of the loads. This is particularly true for composite materials where different planes are designed to handle different stresses.

This also suggests that there is a need for different failure criteria for different types of materials. Failure theories are continuously evolving. Materials are usually non-homogeneous and therefore also need for a factor of safety for the design.

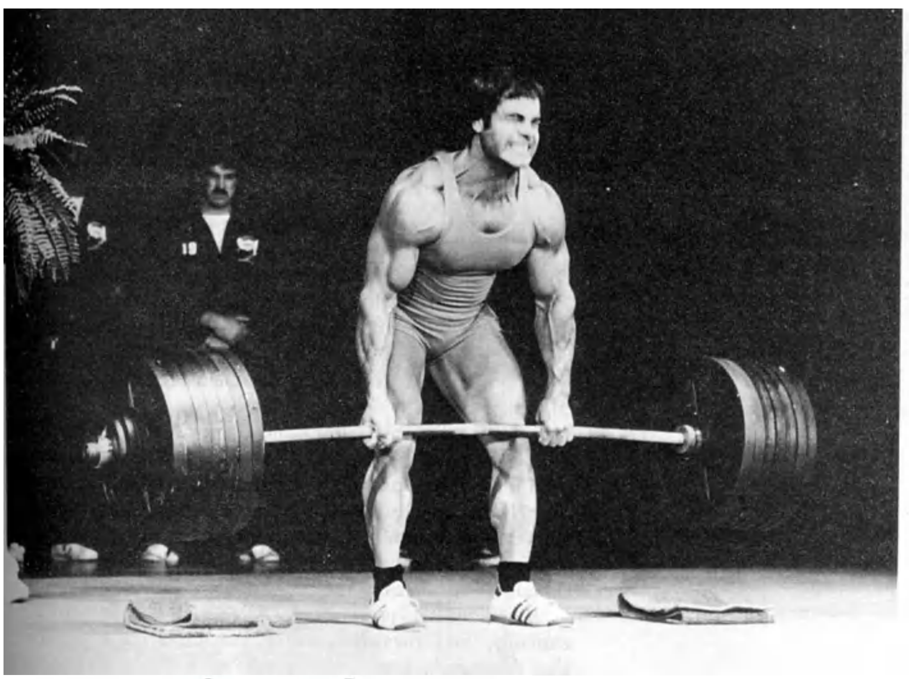
We will explore Torsion in greater detail Chapter 8. Problems are deferred until then.

3.11 BENDING - AN INTRODUCTION

This material will appear again in Chapter 7 and Section 7.3. You can skip it if you choose. The essential information in this section are the following:

- Bending introduces normal stress and normal strain in the object
- Bending stress are distributed across the cross-section
- Designs involving bending have large moment of inertia with respect to bending axis
- Bending causes real bending in beams
- Bending can cause large normal stresses

Let us start with Figure 3.11.1 describing bar bending - a term that describes the ability to bend the barbell by lifting significant amount of weights. The FBD alongside suggests that the bar between CD is in pure bending - only a bending moment is carried/resisted in the section. Also note that it appears to bend in an arc of a circle (larger the radius less the bending?). What are some of the idealizations involved in this problem. If this is a design problem what information are we trying to establish ? - the maximum stress, the elastic deformation (we want the bar to be straight after the weights are removed), etc.



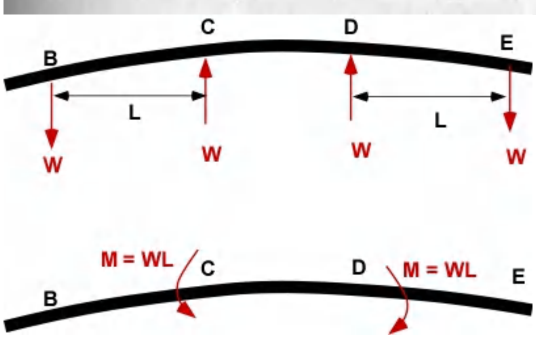


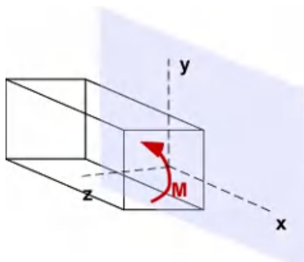
Figure 3.11.1 Pure Bending (Wikimedia Commons) and FBD

Idealizations

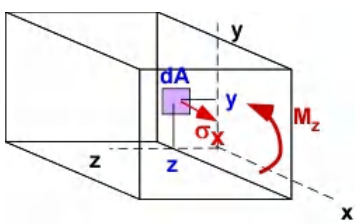
(The helper figures used for illustration and explanation below are not numbered)

1. First we will regard the problem as two-dimensional - the beam was only loaded along a single line. It's depth is ignored in the primary discussion. In this illustration the barbell is round but the FBD alongside ignores the three-dimensionality of the barbell. We have replaced the couple by the moment moments at C and D. Where exactly are these moments located? It is likely they are applied through the vertical plane through the center of the barbell. In the development of the mathematical relations we will ignore the three dimensionality of the problem. This is helpful (initially) if the vertical plane through the center of the beam is also a plane of symmetry.

2. We like our positive x-axis to flow right and the y-axis to go up. The z-axis is out of the screen/page. A positive bending moment is shown in the plane of symmetry and is directed along the positive z-axis. Figure 3.11.1 demonstrates negative bending moment.



3. Now what kind of stresses will be created by this moment in the cross-section (or what kind of stress distribution in the cross-section will result in a moment?)? Let us consider a small area **dA** in the cross-section in the positive quadrant. It is located at a distance y and z (the cross-section is in the yz plane). The stresses on this **dA** will determine the elemental force **dF**. This force can create elemental moments about the axis. These elementals values can be integrated over the cross-section **A**.



Since there only a bending moment in the cross-section there must be a distribution of normal stress to create the bending moment Mz.
We can set

$$M_z = \int_A -y\sigma_x \, dA$$

The negative sign is to relate the direction of the bending moment and the moment produced by the normal stress to be the same.

In order for the integrated moment not to cancel over the area of the cross-section the sign of σ_x must change above and below the z-axis to create the bending moment $\mathbf{M}_z$. To support this assertion further, we know that integral of σ_x over the cross-section must equal a normal force which must be zero since there is only $\mathbf{M_z}$ in the cross-section. This also suggest that the normal stress must change sign over the cross-section to cancel .

$$F_x = \int_A \sigma_x \, dA = 0$$

The same σ_x can also cause $\mathbf{M_y}$ which does not exist in this illustration and so must be zero.

$$M_y = \int_A z\, \sigma_x \, dA = 0$$

The normal stress distribution in the cross-section must satisfy these three equations. This is an overdetermined system.

4. We have arrived at a statically indeterminate system. We cannot solve for the stress unless we can understand the deflection or the deformation of the beam.

5. Since every cross-section in the beam sees the bending moment, every cross-section will bend the same way.

3.11.1 Bending Deformation (Pure Bending)

Pure bending will deform the beam in an arc of a circle. In Figure 3.11.2 we see that the straight bar subject to bending can bend in at least two ways.

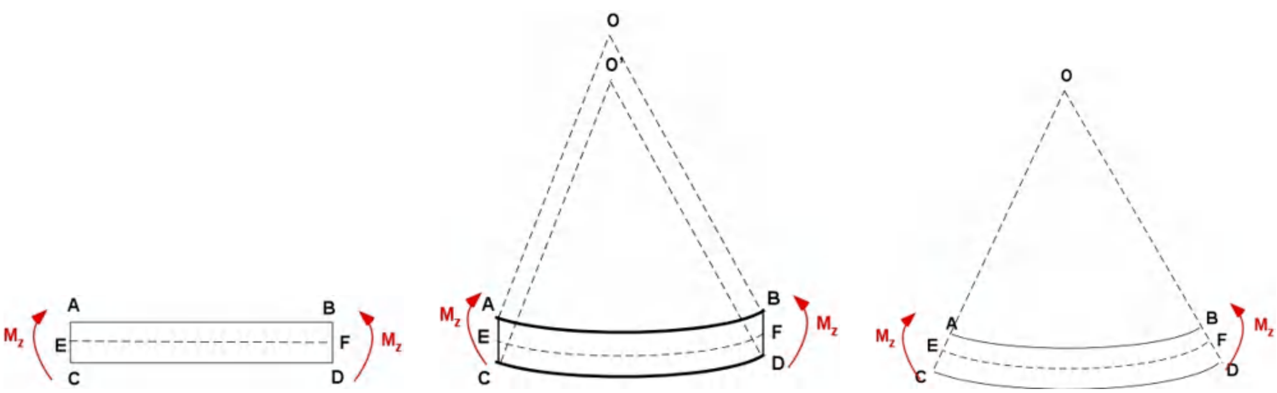


Figure 3.11.2 Original beam, Bending - Case 1, Bending - Case 2

Case 1: the top surface AB and the bottom surface CD can bend with the same radius, have the same length after bending but with the center of the circle at different points. This suggests that the strains might be the same everywhere - and therefore the stresses. That is not what we were expecting in our previous idealizations of bending expressed earlier.

Case 2: In the second case the center of the circle is the same for both surfaces, but AB will shorten and CD will elongate compared to EF. Therefore, Case 2 requires that there will be strains and deformation in bending will vary in the cross-section. Case 2 agrees with our expectation in idealization item 3 above.

Therefore Case 2 describes what happens in pure bending. This leads to another idealization which suggests that the points in the cross-sections will be in the same plane before and after bending moment is applied - *plane sections remain plane*. It is a cleaner description of bending than allowing multiple centers for bending deformation in Case 1. In Figure 3.11.3 we look at Case 2 in more detail to derive the relations between strain, stress, and bending moment.

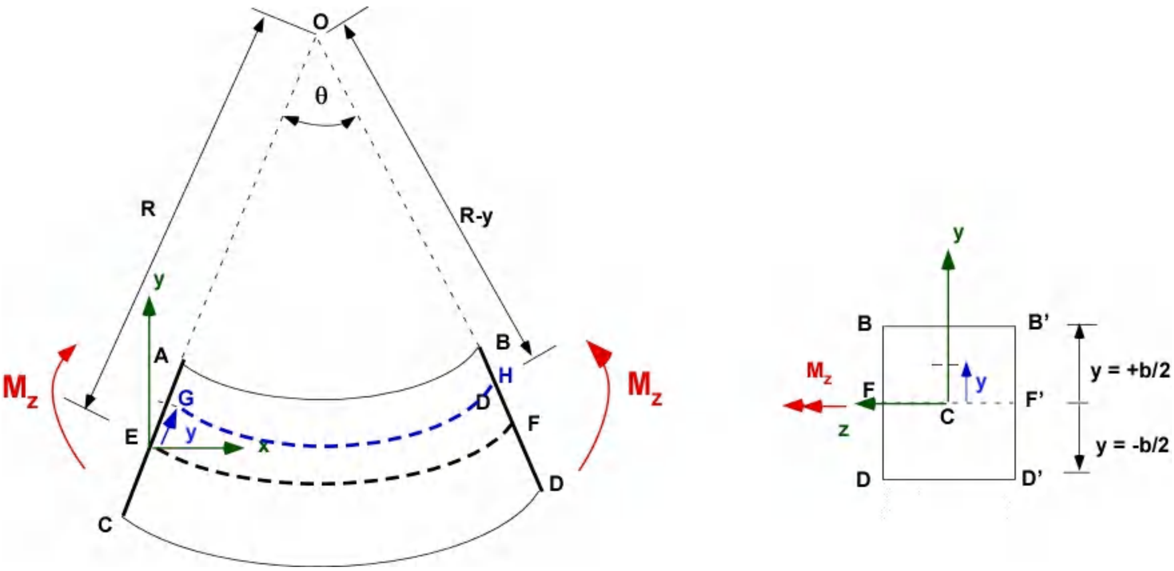


Figure 3.11.3 Pure Bending and deformation

Figure 3.11.3 is explained further here. The pure bending moment $\mathbf{M_z}$ bends an original **straight bar** ABCD into an **arc of a circle of radius R**. Instead of the round cross-section in the first illustration we will assume that the cross-section is rectangular. O is the **center of curvature** and the **radius** is measured from O to a **special line/surface EF**. Plane sections remain plane. In the straight bar the lengths AB, CD, and EF all have the same length (L). Once the bending is applied AB will **shorten** (compression) and CD will **expand** (elongation/tension) since plane sections remain plane. These are normal strains as they are normal to the cross-section. **EF** has the same length before and after bending. **EF** has a special name and is called the **neutral axis (NA)**. **EF** suffers **no strain** and therefore **no stress.** Since the cross-section is **doubly symmetric** it is easy to locate the neutral axis passing through a special point - the centroid. Later on we will show that this is also the case for non-symmetric sections too. The angle subtended by the bar at the center is θ. The radius of curvature is **R** and is drawn from O to the neutral axis **EF** as shown. The line **GH** (or the surface GH) is at a distance **+y** measured from the **NA**.

One of the assumptions we make is that the angle θ is small. Since the bending of the bar is represented by an arc of a circle we can calculate

EF = L =R θ
GH = (R - y) θ
AB = (R-b/2) θ; CD = (R + b/2) θ;

Change in length at the distance y from the neutral axis and the subsequent strain is calculated as

$$\delta = GH - EF = (R-y)\theta - R\theta = -y\theta$$

$$\varepsilon(y) = \frac{\delta}{L} = \frac{-y\theta}{R\theta} = \frac{-y}{R} \qquad (3.46)$$

$$\varepsilon_B = \frac{-b}{2R}; \quad \varepsilon_D = \frac{+b}{2R}$$

The negative sign indicates shortening above the neutral axis and also indicates lengthening below the neutral axis for negative values of y. Another thing to note is that the strain is linear and is zero at the neutral axis. A very important fact in this relation is that the linear strain relation is independent of the type of the cross-section as long as the distance is measured from the NA. It therefore follows the maximum strains are felt by the fibers on the top or the bottom. In our illustration they are the same but of opposite sign because the cross-section is symmetric. The maximum strain depends on the furtherest distance furtherest from the NA and must include the sign. Finally, **every cross-section** of the bar is subject to the same strain distribution.

We have related the bending deformation to normal strains. This is due to the resistance of the beam to bending alone.

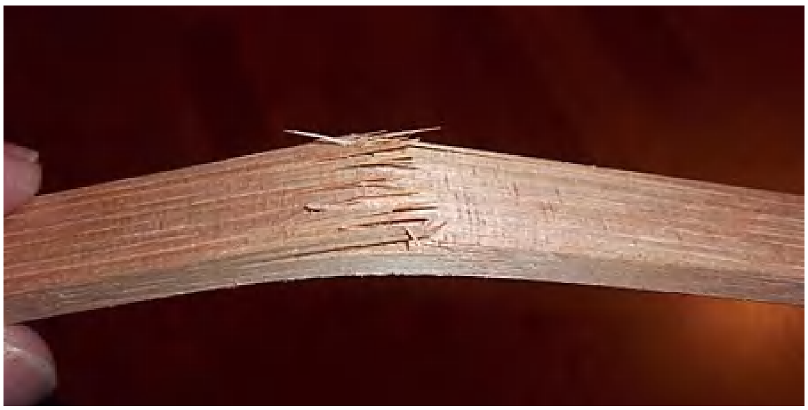



Figure 3.11.4 Failure in bending

3.11.2 Normal Bending Stresses

Pure bending introduces a linear normal stress distribution in the cross-section of the beam. Both tensile and compressive stresses are created in the same cross-section. The maximum stresses are on the outer fibers of the beam. Hence failure is likely to occur either at the top or the bottom of the beam, whichever is higher in magnitude. Since our design will usually be limited to the elastic range, the stress distribution can be related to the strain in pure bending using the modulus of elasticity of the material and normal strain established above. . Using this stress distribution we can then relate the stress and the applied bending moment through the integral we established in the previous section. Note the bending radius R is the same for the entire beam.

$$\begin{aligned}
&\sigma_x(y) = E\varepsilon_x(y) = -E\frac{y}{R} = -\frac{E}{R}y \\
&M_z = \int_A -y\sigma_x(y)\,dA = \int_A -y\left(-\frac{E}{R}y\right)dA = \frac{E}{R}\int_A y^2\,dA = \frac{E}{R}I_{zz} \\
&\frac{E}{R} = \frac{M_z}{I_{zz}} \\
&\sigma_x(y) = -\frac{M_z}{I_{zz}}y
\end{aligned} \qquad (3.47)$$

Since y is measured with respect the NA, $\boldsymbol{I_{zz}}$ is the second moment of area (or the moment of inertia - MOI) about the NA. What about the NA itself? Let us look at the equation for $\mathbf{F_x}$.

$$F_x = 0 = \int_A \sigma_x(y)\,dA = \int_A \left(-\frac{M_z}{I_{zz}}\,y\right) dA = -\frac{M_z}{I_{zz}} \int_A y\,dA$$

$$\int_A y\,dA = 0$$

The NA passes through the centroid for the area of cross-section. Or, the z-centroidal axis of the cross-section is the NA.

3.11.3 Bending Deflection

We will go back to the beginning of the section and examine Figure 3.11.1 again. We really would like to know how is displacement of the beam ***y(x)*** connected to the bending moment moment $\mathbf{M_z}$**.** For pure bending this is connected to the radius of curvature **R** that appears in the equations above. From calculus the radius **R** and **y(x)** are related through the expression:

$$\frac{1}{R} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} = \frac{d^2y}{dx^2} \quad \left(\text{for small } \frac{dy}{dx}\right)$$

We know that the slope of the deflections will be small. From the previous relation we can then write:

$$\frac{1}{R} = \frac{d^2y}{dx^2} = \frac{M_z}{EI_{zz}} \qquad (3.48)$$

The latter defines a second order *differential equation* for **y(x)**. We know that the bending moment $\mathbf{M_z}$ could be a function of length of the beam, **x,** and $\mathbf{I_{zz}}$ could also change with **x** if the cross-section varies. The differential equation can be integrated twice with appropriate boundary conditions to establish **y(x).**

3.11.4 Example 3.14 - Bar bending

Let us calculate the stress and deflection associated with Figure 3.11.1. Figure 3.11.5a is the description and specification of a barbell from Rogue barbell from their website. It gives some of the properties. Let us calculate the stresses and deflection for the highest world record for dead lift. It was Class: 308; Weight Lifted: 939 lb; Lifter: K Konstantinovs; Country: Latvia ; Year: 2009; Federation: AWPC.

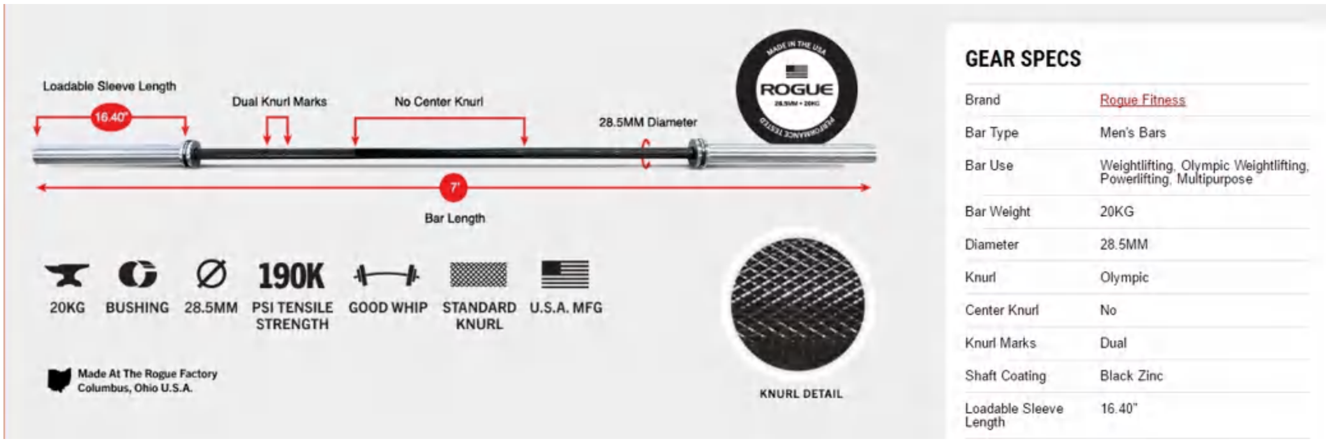


Figure 3.11.5a The barbell specification

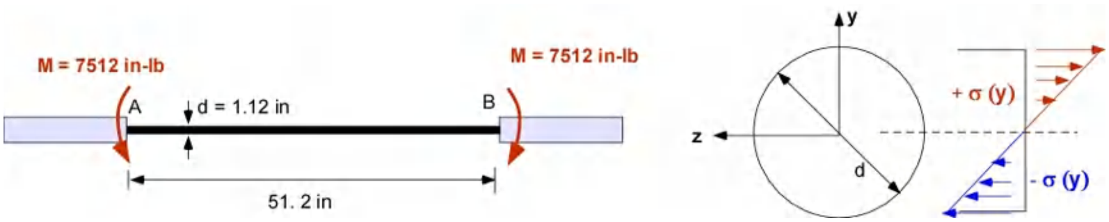


Figure 3.11.5b The bending moment on the barbell and the stress in the cross-section.

We will perform the calculations in US units for this example. The modulus of elasticity for the material is difficult to find. We will choose it the same as a zinc alloy. The yield strength of the material is associated with the tensile strength given above. The calculations are straight forward and involves direct substitution.

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 3.11 - Example 3.14
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('------------------------------\n')
fprintf('Example 3.14 \n')
fprintf('--------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data (in meters)
D1 = 1.12;  sigy = 190000; L = 51.2;  E = 12.6e06;
Mz = -7512;

%% calculations -
% We can plug data into formula (numerical)
% or solve as an unknown (symbolic)

syms y % torque unknown
Izz= (pi/62)*(D1^4);
sigxy = -Mz*y/Izz;
sigm = -Mz*(D1/2)/Izz;
R = E*Izz/abs(Mz);
```

```
fprintf('Length [in]                        = '),disp(L)
fprintf('Diameter [in]                      = '),disp(D1)
fprintf('Moment  [in-lb]                    = '),disp(Mz)
fprintf('Izz [in^4]                         = '),disp(Izz)
fprintf('Max normal stress [lb/in/in]       = '),disp(sigm)
fprintf('Yield strength [lb/in/in]          = '),disp(sigy)
fprintf('FOS                                = '),disp(sigy/sigm)
fprintf('Modulus of Elasticity [lb/in/in]   = '),disp(E)
fprintf('Radius of curvature  [in]          = '),disp(R)
```

In the Command Window

```
-------------------------------
Example 3.14
--------------------------------
Length [in]                        =          51.2
Diameter [in]                      =          1.12
Moment  [in-lb]                    =         -7512
Izz [in^4]                         =      0.079732
Max normal stress [lb/in/in]       =         52761
Yield strength [lb/in/in]                =        190000
FOS                                =        3.6011
Modulus of Elasticity [lb/in/in]   =      12600000
Radius of curvature  [in]          =        133.74
```

We will explore Bending in greater detail Chapter 7.

Execution in Octave

The code is the same as in MATLAB except for the additional statements below . The changes are highlighted. You must include the symbolic package and if you do not wish to see warnings you include the command warning off as shown

```
clc, clear, format compact, close all, format short G, warning off

pkg load symbolic
sympref display flat
%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

In Octave Command Window

```
-------------------------------
Example 3.14
--------------------------------
Length [in]                        = 51.2
Diameter [in]                      = 1.12 as highlighted
Moment  [in-lb]                    = -7512
Izz [in^4]                         = 0.079732
Max normal stress [lb/in/in]       = 52761
Yield strength [lb/in/in]                = 1.9E+05
FOS                                = 3.6011
Modulus of Elasticity [lb/in/in]   = 1.26E+07
Radius of curvature  [in]          = 133.74
```

We will explore Bending in greater detail Chapter 7. Problems are deferred until then.

3.12 BUCKLING - AN INTRODUCTION

This material will appear again in Chapter 9. You can skip it if you choose. The essential information in this section are the following:

- Buckling takes place when compressive axial loads are applied to a tall column
- Bucking causes bending of the column
- Buckling deflection depends on end conditions
- Buckling causes normal compressive stresses
- Buckling deflection is solved as a second order linear differential equation

Design of structures involves calculating stresses due to the loads experienced by the structures and deciding that the stresses will not cause failure. The previous sections considered many ways that the structure can fail. You can push or pull on it to break the structure. You can twist it to cause rupture. You can bend the structure to break it. Buckling is another mode of failure. This happens with tall or long structures that are carrying a compressive load. This section provides a simple introduction to buckling so that it is part of design awareness as knowledge of mechanics of structures are gained in the following chapters.

The simple deflection of the structure causes a bending load due to the eccentricity of the load because of the deflection. Very often this is also termed as column buckling. Figure 3.12.1 illustrates buckling of columns in testing performed at NIST (National Institute of Standards) in the NIST Building and Fire Research Laboratory, particularly the column identified as Col 79. This column is bent under the application of a compressive load. Note Col 80 also bends but has a different shape. Both of the columns are deflected sideways.

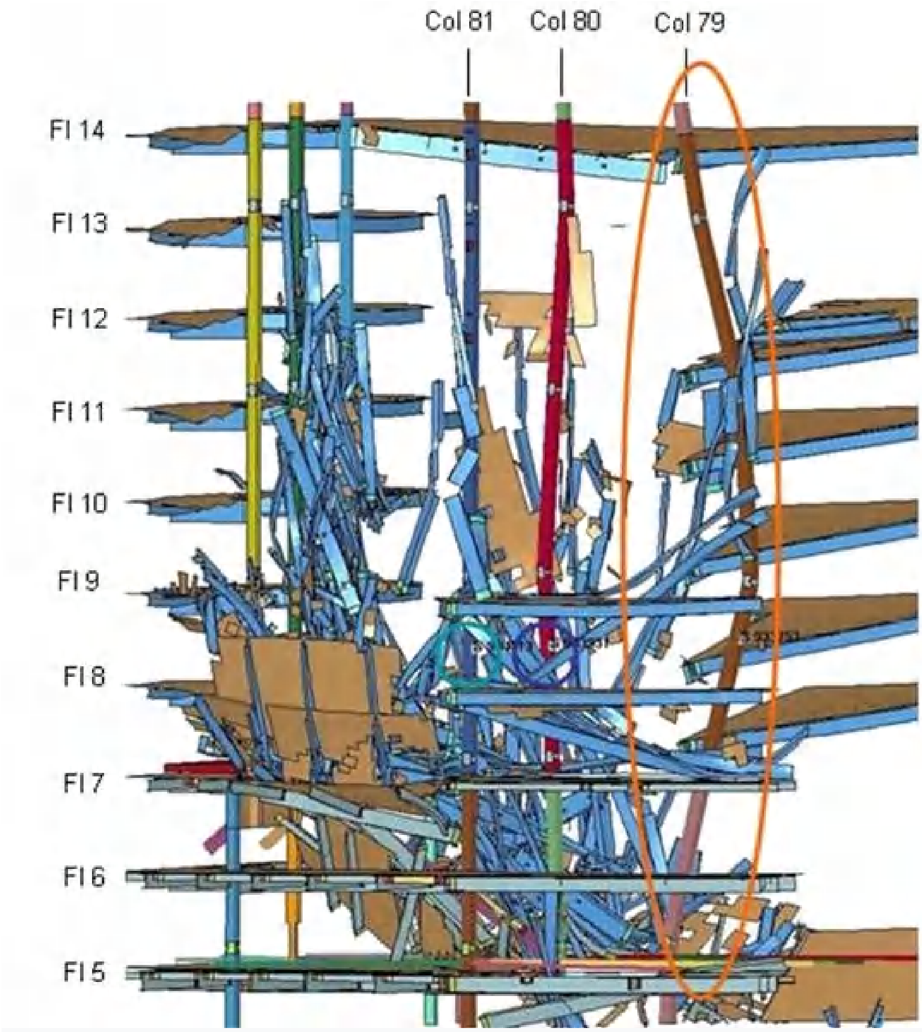


Figure 3.12.1 Column buckling (NIST Building and Fire Research Laboratory).

Buckling can occur at a load well below the direct compressive stress that lead to compressive failure. For this reason bucking is considered a *structural* ***instability***. For a structure carrying a load the simple idea of instability is that if the load exceeds a certain value it will no longer be able to support the load and may fail catastrophically. Instability is bad in all structural designs because you cannot guarantee the safety or performance of the structure. Apart from the failure criteria that was presented earlier it is important that the design will not experience the *critical buckling stress* or the *critical buckling* load. If loading increases beyond the critical buckling load the structure will respond unpredictably leading to the loss of any capacity to carry the load.

Nevertheless *buckling* is a special case of loading. The structure is long and is subject to compression. Buckling can be related to the length of the column as can be derived for the basic model of buckling. Compressive loads can develop on a structure due to thermal strains. An example of this is the gap between between adjacent sections of railway tracks. If the outside temperature exceeds the design maximum the tracks will expand and press against each other creating enough compressive load to detach from the foundation, Figure 3.12.2. There are other types of buckling like multiple column buckling (bicycle wheels), plate buckling (extension of column buckling), surface buckling (creation of potholes in winter). They will always cause failure if they occur.

Figure 3.12.2 Buckling of tracks (Wikipedia)

Column buckling is usually discussed in a later course on mechanics of solids because the simple model of buckling requires knowledge of differential equations and their solutions. In a standard engineering curriculum this usually happen after a course on statics - which means that this concept may be a bit early for discussion here or in Chapter 9. The model itself should be easy to follow but the solutions are borrowed directly and their interpretation are taken for granted.

3.12.1 Euler Buckling Model

The simplest model for column buckling is the Euler formula for columns having *pin connections* at the ends. The buckling shape or deformation is influenced by the type of end connections as shown in Figure 3.12.3. This is explored in Chapter 9.

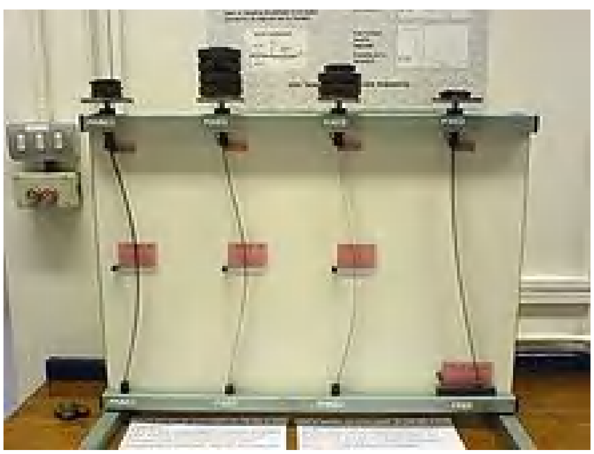


Figure 3.12.3 Types of buckling deformation (Wikipedia)

Since buckling deformation is a transverse deformation the applied load must be able travel so the model has the compressive load mounted on a rail as shown in Figure 3.12.4.

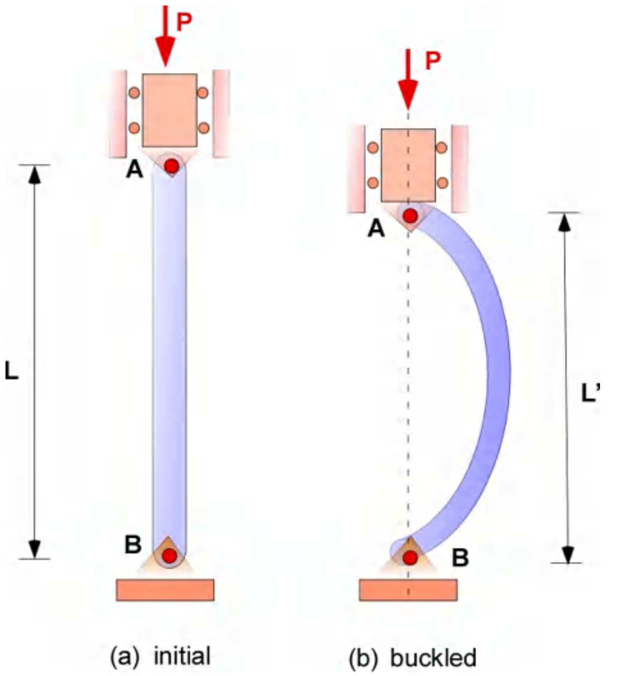


Figure 3.12.4 Illustration of pinned column buckling

The Euler buckling model is developed using a free body diagram of the part of the buckling deformation shown in Figure 3.12.4b. In the model the pin supports cannot deflect. These conditions are translated to the boundary conditions on the differential equations describing the model. Small deflections are also assumed so that the relations from pure bending are incorporated into the development. One anomaly is that the original length L of the column and the deflected length L' are considered the same even though they are visually different - even in the experimental setups above. It is necessary for the completeness of mathematical model since two boundary conditions are required. The model development is shown in Figure 3.12.5.

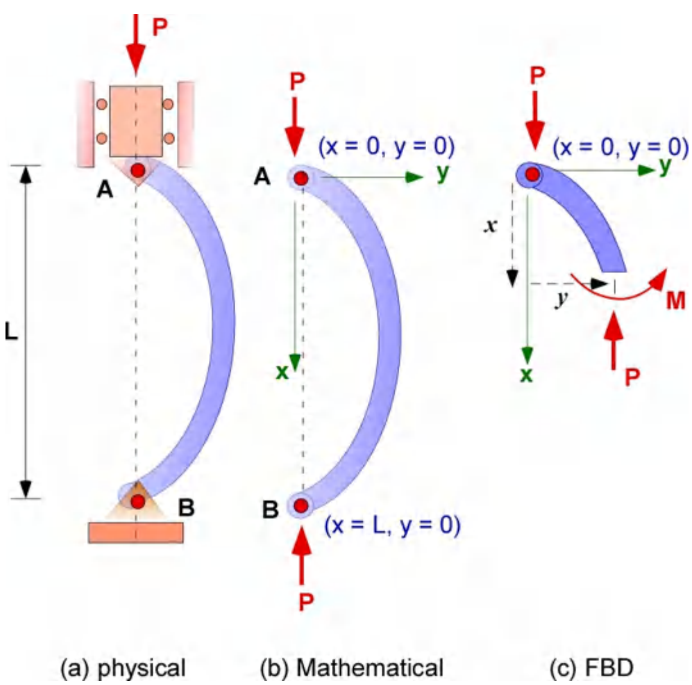


Figure 3.12.5 Euler buckling model

First apply the equations of equilibrium:

$$\sum F_x = 0 = P - P$$
$$\sum M_z = 0 = M + Py$$

Now replace the bending moment M through its relationship to the radius of curvature in pure bending (3.48) as:

$$M = EI\frac{d^2y}{dx^2} = -Py; \quad OR$$
$$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = 0; \quad OR \qquad (3.49)$$
$$\frac{d^2y}{dx^2} + \lambda^2 y = 0; \quad \lambda^2 = \frac{P}{EI}$$

In the above equation ***x*** is considered the *independent* variable and ***y*** is the *dependent* variable. The solution being sought is y as a function of x or ***y(x)***. The final equation is called a *differential equation* since it contains *derivatives* and there is an *equal to* sign in the expression. It is usually expressed by grouping terms in the dependent variable on the *left side* of the equal sign.

The equation is considered *homogeneous* if there is no terms on the right of the equal to sign. The *order* of the equation is based on the highest derivative in the expression. If the powers of the terms involving the dependent variable and the derivatives is *one* then it is considered *linear.* If the coefficients of the terms on the left are constant it is considered a *constant coefficient* differential equation.

The differential equation in (3.49) is a homogeneous, second order, linear, differential equation with constant coefficients. The general solution for this equations is:

$$y(x) = C\sin\lambda x + D\cos\lambda x$$

C and D are two constants that must be determined to describe the solution - the buckling deformation. The differential equation in (3.49) by itself is incomplete. A solution can only be found if the boundary conditions are given. A second order equation must be accompanied by two boundary conditions and these are used to determine the constants C and D. The natural boundary conditions for the pinned support at A (x = 0) and B (x = L) are that the displacement y is zero at these locations.

At x = 0 :

$$y(0) = D = 0$$

At x = L:

$$y(L) = C\sin\lambda L = 0$$

If C = 0 then there is no buckling (simple compression). If C is not zero - there is lateral displacement - then the ***sine*** term must be zero.Therefore

$$\sin(\lambda L) = 0 \quad OR \quad \lambda L = \pm n\pi \quad OR \quad \lambda^2 L^2 = n\pi^2$$

$$\lambda^2 = \frac{n\pi^2}{L^2}; \quad P = EI\,\lambda^2 = \frac{n\pi^2 EI}{L^2}; \tag{3.50}$$

$$y(x) = C\sin\left(\frac{\pm n\pi x}{L}\right)$$

The smallest value of P, the critical buckling load, corresponds to n = 1 and I therefore

$$P_{cr} = \frac{\pi^2 EI}{L^2};$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL^2} \tag{3.51}$$

The relations above are the Euler's formula for buckling. The critical buckling stress is also obtained above. If the MOI can be expressed using the cross-sectional area and the radius of gyration (k) - in most handbooks for common cross-sections then

$$I = Ak^2;$$

$$\sigma_{cr} = \frac{\pi^2 E}{(L/k)^2} \tag{3.52}$$

The ratio ***L/k*** is called the ***slenderness ratio*** of the column. Since buckling is a response to compressive load the structural design considerations is a combined response to simple compressive stress and critical buckling stress. The latter is usually lower but kicks in at a higher value of the slenderness ratio. For structural steel with E = 200 GPa and σ_y = 250 MPa the design limits for compressive loads are seen in Figure 3.12.6. The buckling stresses above the yield strength can be ignored since the material will fail in simple compression. The critical slenderness ration for the plot is 88.86.

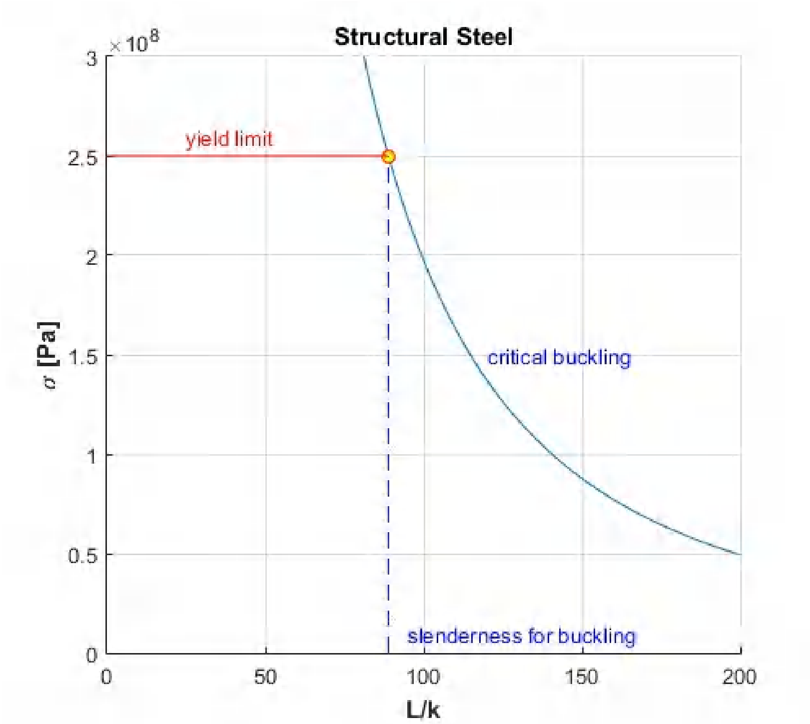


Figure 3.12.6 Column design for compressive stress

Buckling Deflection

When the column has buckled its displacement , or the elastic curve, is given by

$$y(x) = C \sin\left(\frac{\pm n\pi x}{L}\right)$$

This is a sine plot between x = 0 and x = L which is half a wave length. We still have to determine C. We have used the boundary conditions to establish the value of the critical buckling load instead of C. The solution will remain indeterminate unless additional information is available. The value of C is also the maximum displacement. From a design point of view we wish to stay below the critical stress and prevent the initiation of buckling. All of these relations assume that everything is neat and centered. A small force on the side of the column while carrying a load may be able to buckle the column below the critical load.. Hence the idea of instability.

An example is included here to use the relations established above for completeness.

3.12.2 Example 3.15

A 6 m column made of steel , E = 200 GPa, is subject to a compressive load. Calculate the critical buckling load and stress if the cross=section is annular with an outside diameter of 100 mm and a wall thickness of 20 mm.

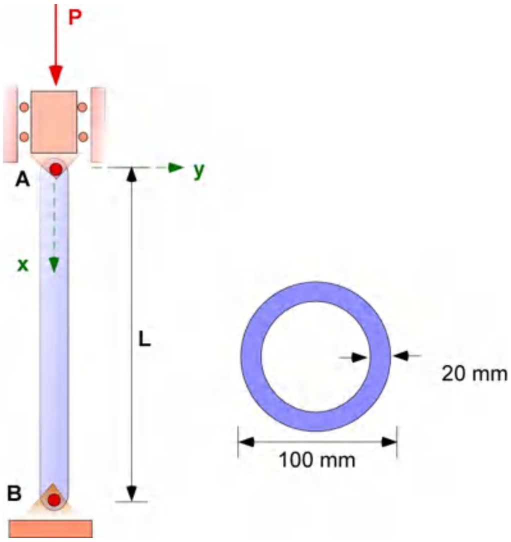


Figure 3.12.7 Example 3.15

Data: L = 6 m; r_o = 50 mm; r_i = 30 mm; E = 200 GPa; (σ_Y = 250 MPa);

Find: Critical buckling load and stress

Assumption: The formulas are available in (3.50)

Solution: The calculations are direct.

$$A = \pi\left(r_o^2 - r_i^2\right) = 0.00503\left[m^2\right]$$

$$I = \frac{\pi}{4}\left(r_o^4 - r_i^4\right) = 4.273\times10^{-6}\left[m^4\right]$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 234.27\left[kN\right]$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = 46.61\left[MPa\right]$$

$$k = \sqrt{\frac{I}{A}} = 0.0292\left[m\right]$$

$$\frac{L}{k} = 205.8$$

$$P_{comp} = \sigma_Y A = 1257\left[kN\right]$$

The last calculation is the compressive load required to cause simple compressible yield. The column would start buckling long before this value.

What is the factor of safety for this design. This is not a simple question in light of previous discussion of FOS. The column has buckled and this is an instability and this must be prevented We can then ask a different design question.

What must be the length of the column to just avoid the critical buckling load?

$$\left(\frac{L}{k}\right)_{max} = \sqrt{\frac{\pi^2 E}{\sigma_y}} = 88.86;$$

$$L_m = k\left(\frac{L}{k}\right)_{max} = 2.59[m]$$

We will explore Buckling in greater detail Chapter 9. Problems are deferred until then.

4. PARTICLES AND EQUILIBRIUM

We will start this chapter by revisiting some of the concepts from Chapter 2 where we introduced forces, reaction, and vectors.. Let us look at two figures we saw earlier.

Figure 4.1 Suspended weight in two and three dimensions

Our interest in looking at these problems at that time was to introduce you to two and three dimensional vectors and how to deal with them. Now, let us associate some ***design issue*** with them.

In both of these examples:

- ○ The cables should not break while carrying the respective weights.
- ○ You can also expect that the cables may not always have the expected properties.
- ○ You know that that the cost of cable is proportional to the diameter.

Your job as a **structural designer** is to choose the proper cable for the job that is most inexpensive. You will recognize that you have essentially two important design activities. The first is to choose a cable that will not fail. Second, ensure that the cable is of the smallest diameter that will do the job adequately. Your activities as a **designer** will involve several activities:

- ■ Problem definition - what is the maximum weight the cable will support?   Is the geometry important.? Are you interested in the sag under the weight - so it does not contact the floor?
- ■ Do you have to choose from different types of cable materials? You will need their **material properties**.
- ■ You will also need to choose a **factor of safety** if you have less confidence in the quality of the cables.
- ■ **You will have to calculate the force in the cables. You will need the Newton' law for this.**

- You will calculate the stress **(normal stress**) in the cables due to the forces in the cables. You will probably assume that the cable will behave **elastically** as otherwise you will have to replace it often. Once you decide the maximum stress in the cables (**based on the yield stress of the material**) you can calculate the diameter of the cable you need.
- By including a **factor of safety** you can ensure that the design will hold under a bad cable or unanticipated weight.
- You can then calculate the **strain** in the cables (**Hooke's law**) and the extension of the cable (**displacement**) to ensure that the stretched cable does not hit the ground. This will allow you to position the unloaded cable above the ground

You will realize in all of your courses in applied math, science, and engineering you will often find values of some quantities (**unknowns**) given the values for another bunch of quantities (**knowns**). Traditionally this is done by exploiting relations about the quantities. In general, the problem is solvable if there are as many independent relations as there are unknowns. Most often these relations are developed from the application of natural laws that everything must obey. The law used in this book is the **Newtons' Law**.

The first important calculation is the **force in the cable** and the use of Newton's law in its determination. In simple problems of the type above we use the idealization of a **particle**. A **particle** is an object with no size, no shape, but has mass (and weight). This is an important **idealization** and primarily suggests that

1. the object on which the forces act can be represented by a point
2. the object and its effect does not change with rotation

We then represent represent the external forces as forces acting at a point on the particle. This is referred to as a **Free body diagram (FBD).** We apply the Newton's second law to this free body diagram.

Newton's Second Law:

The net force on a body in motion is equal to its change in momentum :

$$\sum \bar{F} = \frac{d\left(m\bar{v}\right)}{dt} \qquad (4.1)$$

Since the mass is constant for our problems: $\sum \bar{F} = m\frac{d\bar{v}}{dt} = m\bar{a}$ (4.2)

If the mass is not moving (stationary - STATICS) : $\sum \bar{F} = 0$ (4.3)

This equation or Physical Law represents the **Equation of Equilibrium**

Free body diagram

The consequence of the particle idealization and the free body diagram has the following advantages:

- The free body diagram (**FBD**) is drawn around a point

- The forces all pass through this point
- There is no **Moment** on the free body
- Only **force equilibrium** defines the Physical Law that needs to be satisfied for this problem in Statics
- Force equilibrium for two dimensional problems (or planar problems) implies two equations - one along each coordinate - and therefore only two unknowns - in the problem
- Force equilibrium for three dimensional problems (or space problems) implies three equations - one along each coordinate - and therefore only three unknowns- in the problem

4.1 TWO DIMENSIONAL EQUILIBRIUM AND DESIGN

We will look at equilibrium and design through several examples. The equations of equilibrium is fairly simple and is best understood through solving problems. Let us introduce a context to the weight hung from the palm tree shown earlier and consider this as our first example in this chapter.

Formal Problem Statement: During my recent physical my doctor asked me to increase my levels of Vitamin D. I took it very seriously and dusted of my old hammock as I planned to spend a lot of time on it. My constant worry was that the ropes will break under my mass of 85 [kg]. We will break up the process of solution into subsections.

Figure 4.1.1 Doctor recommendation

4.1.1 Example 4.1: Problem Definition

Why do we need to make some idealization? Consider the realistic description of the problem above.

We do not want to look at a curved hammock as the curve will make the problem non linear (and difficult to model unless you have some advanced mathematics courses under your belt)

We don't want to deal with the distribution of my weight on the hammock. This will likely be a three dimensional distribution and nonlinear. Another reason we do not want to solve this as a three dimensional problem as it will require a lot more calculations than a two dimensional problem. This will make the problem very difficult to solve. Imagine how easy it would be if I can pretend to lump my weight at a single point. Can we assume that the hammock and the trees will form a plane (this is a reasonable assumption)?.

Figure 4.1.2 Simpler representations of the problem

We will regard me as a **particle**. My weight will provide a concentrated force. Remember I have weight but no size.

The hammock is replaced by a single rope suspended between the trees that has two straight parts - one on the right and one on the left of the particle.

The ropes or cables can only function in tension and it is known that the force in the cable can only be along the cable.

If we look along the cables then the tensile force will be along the cable. This is a one-dimensional problem. However, my weight and the force in two cables will make the problem two-dimensional. The last figure in Figure 4.1.2 is a physical model of the problem.

Do we need the palm trees, the hammock, or me ? No, we can just focus on the particle and the rope around it as in Figure 4.1.3. We can still make it simpler and consider that both ropes ate tied at the same height on the trees.

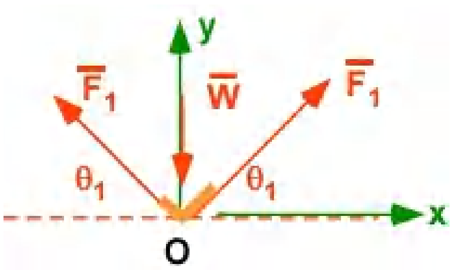


Figure 4.1.3. A mathematical model of the problem

Here we are exploiting another feature called **symmetry**. We assume that the hammock is symmetrically placed between the trees and the forces in the cable segments have the same magnitude and the cables have the same angular deflection with respect to the horizontal. On the application of the weight the hammock will displace down because of elasticity. This also means that when there is no weight on the hammock the hammock is completely horizontal.

This is called the model for the problem. This represents the problem in Figure 4.1.1. This problem can be solved by simple algebra whereas the original problem (non linear) in very difficult to solve.

The inclusion of idealizations and simplifications is a necessary part of problem definition. We will refer to them by the broad term **assumptions** from now on.

4.1.2 Example 4.1 : Applying Equilibrium

Once we have the model of the problem in Figure 4.1.3 we recognize that there are two quantities we must solve $\mathbf{F_1}$ and θ_1. From basic algebra we will need two equations to solve for these values. Since the problem must obey the Newton's law (**in fact every problem in this course must - as it is a law of nature**) there are two equilibrium equations available to solve this two dimensional problem from Eqn. (4.3). We have anticipated this and included the coordinate system in the figure as this is important part of the solution process. The equilibrium equations for this problem, from Figure 4.1.3, summing the forces along the positive coordinate directions are

$$\sum F_x = F_1 \cos\theta_1 - F_1 \cos\theta_1 \equiv 0 \qquad (4.4a)$$

This equation is not useful as it is identically zero. This equation however justifies **symmetry**. Which means the assumption of symmetry uses this equation and therefore is no longer available. If you did not make an assumption of symmetry you would find the equation useful for obtaining the solution. Let us look at the second equation:

$$\sum F_y = F_1 \sin\theta_1 + F_1 \sin\theta_1 - W = 0 \quad OR \quad 2F_1 \sin\theta_1 - W = 0 \qquad (4.4b)$$

This equation is useful but we have **two** unknown and **one** equation. We can only solve for F_1 for known values of θ_1 or the other way around. We need another independent equation to solve explicitly for F_1 and θ_1. We are basically **stuck** at this point based on our problem definition. This is called a **statically indeterminate problem**.

Statically indeterminate problem: This does not imply the problem cannot be solved. It just establishes that you cannot solve the problem from the equations of ***statics*** alone - in this case the equations for *equilibrium*. We have exhausted the knowledge from statics and will need additional information to solve the problem. Usually this brings in the coverage from *strength of materials* - the properties of the material of the rope for example or the maximum sag permitted. - to solve the problem. This also means the problem is now flexible and will be allowed to deform. These ideas land us immediately in the area of ***design***.

We will add one more assumption to the list in section 4.1.1. We will **assume that the deformations are small** and will not change the original geometry present in the problem in any significant manner.

Solution 1

If you have been designing hammocks for a long time, or you have researched your brand of hammocks, you have discovered that people are most comfortable when the deflection θ_1 is 10 degrees or less. Note that this was not part of our original problem definition, but it has now a great significance in the solution. This is the **true nature of design**. Sometimes we need to discover **additional information** that makes sense and enables us to **move forward** with the design. The problem definition will only provide a start. We can now calculate the force F_1 in the rope to be

$$F_1 = \frac{85 * 9.81}{2 * \sin(10)} = 2401[N]$$

We have only determined the force but we have not designed anything yet. We need the hammock to be safe (the rope should not break). To complete our design we have to bring in our knowledge of **stress, strain**, **deformation,** and **failure.** We still have not selected the rope.

4.1.3 Example 4.1: Additional Considerations (for design):

We will assume that

- the rope is made **natural braided coir** with an **Elastic modulus** (**E**) of 110 MPa and with an **yield stress/strength** (σ_y) of 20 MPa.
- The cable weight is ignored.
- In addition we will assume the **distance** between the trees is 4 m. Remember the hammock deflects symmetrically and without the weight the cable is horizontal before I sat on it..
- The **area** (A) is unknown.
- We will use a **simple failure criteria** in that the **stress in the hammock must be significantly less than the yield stress** of the rope material.
- We will assume **uniform stresses** in the rope.

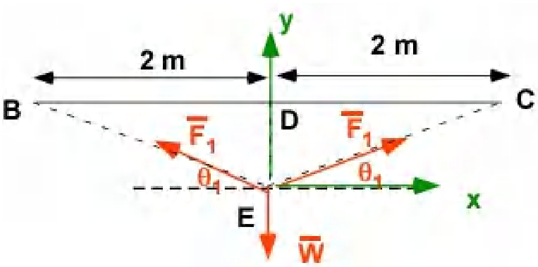


Figure 4.1.4 Hammock Deflection

Figure 4.1.4 is the same as Figure 4.1.3 enhanced with the new geometrical information in the problem. The hammock is horizontal with no weight. The weight causes the deflection θ_1. In the cable itself the stress is one dimensional as illustrated in Figure 4.1.5 looking along the cable

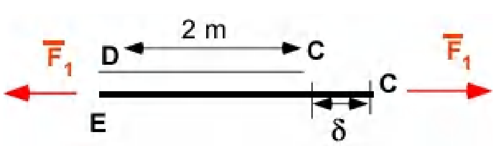


Figure 4.1.5 Deflection in cable

We can now calculate the stress, strain, deflection, and area of the cable as:

$$L_{CD} = 2[m]; \quad L_{EC} = \frac{L_{DC}}{\cos(10)} = 2.0309$$

$$\delta = L_{EC} - L_{CD} = 0.0309[m]$$

$$\sigma = \frac{F_1}{A} = \frac{2401}{A} = E\varepsilon = E\frac{\delta}{L_{CD}} = 110\times10^6\frac{\delta}{2} = \frac{2401}{A}$$

$$\delta = \frac{2*2401}{A*110\times10^6} = 0.0309[m]$$

$$A = 0.0014\left[m^2\right]$$

Another piece of information of interest to design is the sag in the hammock - the hammock deflection - or the distance DE. Since θ_1 is known this should be easy to calculate as

$$L_{DE} = L_{CD}\tan(10) = 0.35\,m$$

Maybe we could have used the sag to determine θ_1**.** This is the reason that design problems are considered **open-ended.** We can calculate different values by focusing on different elements in the same problems. These are usually regarded as **constraints.**

All of these numbers are based on the value of θ_1 of 10 degrees and the choice of the cable material. If these values change, then all the numbers will also change. We have one more calculation. Let us calculate the factor of safety (FS) with this value of A

$$\sigma = \frac{2401}{0.0041} = 1.7\times10^6[Pa]$$

$$FS = \frac{\sigma_Y}{\sigma} = 11.79$$

This factor of safety is about three times the value you would use in this category of problems. This makes the rope heavier than it needs to be and therefor expensive. Now a ***business decision*** is required for design.

There are so many parameters in this problem that it makes sense to use MATLAB.

4.1.4 Example 4.1: Solution 1 using MATLAB

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 4.1 - Example 4.1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
fprintf('-----------------------------------------\n')
fprintf('Example 4.1 - Hammock Design -Solution 1\n')
fprintf('------------------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Data (in meters)
W = 85*9.81;  tht1 = 10; L = 4;
Lcd = L/2; E = 110e06; sigy = 20e06;

%% calculations
F1 = W/(2*sind(tht1));
Lec = 2/cosd(10);
del = Lec - Lcd;
Area = F1*Lcd/del/E;
sigma = F1/Area;
FS = sigy/sigma;

%% solution

fprintf('W[N]                  : '),disp(W)
fprintf('theta [deg]           : '),disp(sigy)
fprintf('Length [m]            : '),disp(L)
fprintf('Yield stress [MPa] : '),disp(sigy)
fprintf('Elasticity   [MPa] : '),disp(E)
fprintf('\n\n')
fprintf('F1 [N]                  : '),disp(F1)
fprintf('Stretched cable [m]     : '),disp(Lec)
fprintf('cable deflection [m]    : '),disp(del)
fprintf('Area [m^2]              : '),disp(Area)
fprintf('Stress in cable [MPa] : '),disp(sigy)
fprintf('Factor of Safety        : '),disp(FS)
fprintf('\n\n')
```

In the command window

```
-----------------------------------------
Example 4.1 - Hammock Design -Solution 1
------------------------------------------
W[N]                :       833.85
theta [deg]         :     20000000
Length [m]          :      4
Yield stress [MPa] :     20000000
Elasticity   [MPa] :    110000000


F1 [N]                 :          2401
Stretched cable [m]    :        2.0309
cable deflection [m]   :      0.030853
Area [m^2]             :     0.0014149
Stress in cable [MPa] :     20000000
Factor of Safety       :        11.786
```

4.1.5 Another Solution to Example 4.1

Let us look at Example 4.1 again but this time without any knowledge of the angle θ_1. We will be using the same equations and the same approach but with the angle θ_1 as an unknown. We will call this solution 2.

$$F_1 = \frac{W}{2\sin\theta_1} = \frac{85\times 9.81}{2\sin\theta_1}$$

$$L_{CD} = 2[m]; \quad L_{EC} = \frac{L_{DC}}{\cos(\theta_1)} = \frac{2}{\cos(\theta_1)}$$

$$\delta = L_{EC} - L_{CD} = \frac{2}{\cos(\theta_1)} - 2$$

Now, we have equate this deflection to the one produced by the stress in the material used. We will use a FS of 6. The maximum stress will be σ_Y/FS. We can then calculate the strain and the deflection. This must be the same as the one calculated through the geometry above.

$$\sigma = \frac{\sigma_Y}{FS} = \frac{20\times 10^6}{6} = 3.33\times 10^6\,[Pa]$$

$$\varepsilon = \frac{\sigma}{E} = \frac{3.33\times 10^6}{110\times 10^6} = 0.0303$$

$$\delta = L_{CD} * \varepsilon = 0.0606 = \frac{2}{\cos(\theta_1)} - 2$$

$$\theta_1 = 13.93[\deg]$$

Not bad! We has assumed θ_1 to be 10 degrees in Solution 1. We had to make a different assumption this time - the **FS** is 6. ***This is the nature of design.*** Let us complete the calculations

$$F_1 = \frac{W}{2\sin(13.93)} = 1731[N]$$

$$A = \frac{F_1}{\sigma} = 0.00051\left[m^2\right]$$

We have a different value for the area because the force has changed.

In Solution 2 we will be using symbolic calculations.

4.1.6 Example 4.1: Solution 2 using MATLAB

In the Editor

```
% Essential Mechanics
% P. Venkataraman
```

```
% Section 4.1 - Example 4.1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 4.1 - Hammock Design -Solution 2\n')
fprintf('-----------------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data (in meters)
W = 85*9.81;  L = 4;
Lcd = L/2; E = 110e06; sigy = 20e06;
syms tht

%% Calculations
F1 = W/(2*sin(tht));
Lec = Lcd/cos(tht);
del1 = Lec - Lcd;
FS = 6;
sigma = sigy/FS;
strain = sigma/E;
del2 = Lcd*strain;
fun = del1 - del2;  % this is the function to solve for theta

%% solving for theta
% there are two solutions which you can write to CW by removing the
% semicolon.  Choose the positive value
thtv = solve(fun);  % solve for theta
tht = max(double(thtv));  % choose the maximum value (positive)
                        %and convert to decimal
thtd = tht*180/pi;
fprintf('theta [deg]         : '),disp(thtd)
F1 = double(subs(F1));
Lec = double(subs(Lec));
del1 = double(subs(del1));
Area = double(subs(F1)/sigma);

%% Printing
fprintf('W[N]                : '),disp(W)
fprintf('Length [m]          : '),disp(L)
fprintf('Yield stress [MPa]  : '),disp(sigy)
fprintf('Elasticity   [MPa]  : '),disp(E)
fprintf('\n\n')
fprintf('F1 [N]                 : '),disp(F1)
fprintf('Stretched cable [m]    : '),disp(Lec)
fprintf('cable deflection [m]   : '),disp(del1)
fprintf('Area [m^2]             : '),disp(Area)
fprintf('Stress in cable [MPa]  : '),disp(sigma)
fprintf('Factor of Safety       : '),disp(FS)
fprintf('\n\n')
```

In the Command Window

```
----------------------------------------
Example 4.1 - Hammock Design -Solution 2
-----------------------------------------
```

```
theta [deg]          :       13.931
W[N]                 :      833.85
Length [m]           :    4
Yield stress [MPa] :     20000000
Elasticity   [MPa] :    110000000


F1 [N]                    :        1731.8
Stretched cable [m]     :        2.0606
cable deflection [m]   :      0.060606
Area [m^2]               :    0.00051954
Stress in cable [MPa] :    3.3333e+06
Factor of Safety       :      6
```

Execution in Octave

The code is the same as in MATLAB except for the additional and highlighted statements below . The changes are highlighted. You must include the symbolic package and if you do not wish to see warnings you include the command warning off as shown. The subs command appears to work differently in Octave. It does not automatically recognize the variables in the workspace. The variables must be specified explicitly.

```
clc, clear, format compact, close all, format short G, warning off

pkg load symbolic
sympref display flat
%%%%%%%%%%%%%%%%%%%%%%%%%%%

Furthermore the subs command needs to be more specific in Octave

##F1 = double(subs(F1));  Original MATLAB Code
##Lec = double(subs(Lec));
##del1 = double(subs(del1));
##Area = double(subs(F1)/sigma);

F1 = double(subs(F1,thtd)); % Modified Octave Code
Lec = double(subs(Lec,thtd));
del1 = double(subs(del1,thtd));
Area = double(F1/sigma);
```

In Octave Command Window

Example 4.1 - Hammock Design -Solution 2

```
theta [deg]          : 346.07
W[N]                 : 833.85
Length [m]           : 4
Yield stress [MPa] : 2E+07
Elasticity   [MPa] : 1.1E+08


F1 [N]                    : 879.04
Stretched cable [m]     : 2.2718
cable deflection [m]   : 0.27179
Area [m^2]               : 0.00026371
```

```
Stress in cable [MPa] : 3.3333E+06
Factor of Safety      : 6
```

The solution from MATLAB and Octave are the same

We now have **two different designs** and both will work. To resolve the business decision of the most inexpensive rope we need more information. We will consider it beyond the scope of this text. You may consider the following: The area in Solution 2 is smaller and most likely will be cheaper. However, coir rope will only be available in certain area of cross sections, unless you can make it to your specifications cheaply. So, finally you will probably be choosing a value between closer to Solution 1 because you want the customer to be comfortable as otherwise the sales might slip because of discomfort.

4.1.7 Example 4.2 : Problem in Statics

Just being aware of stress, strain, and Hooke's law we could introduce design ideas in Example 4.1. Without this knowledge Example 4.1 can only be solved using equilibrium equations alone. That means that the problem must be **statically determinate**. In the traditional statics course you will probably solve for forces in the rope given the geometry information. We will illustrate this in Example 4.2 by removing the symmetry requirement and defining the problem as in Figure 4.1.6

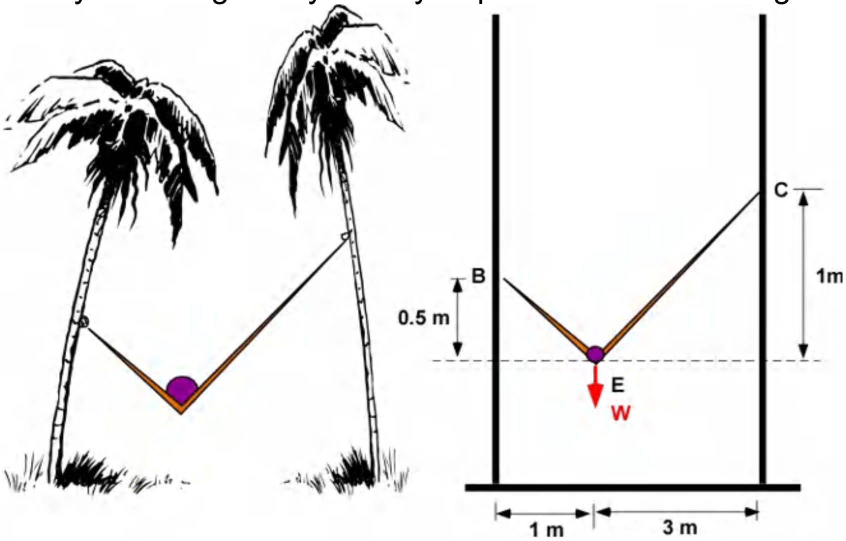


Figure 4.1.6 Example 4.2

Data: The geometry as shown in Figure 4.1.6. Mass = 85 kg

Find: The forces in the ropes

Assumption: Static Equilibrium

Solution: (Draw FBD with coordinate system)

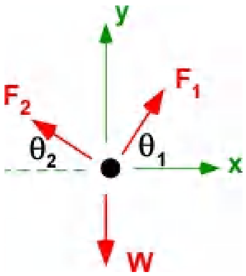


Figure 4.1.7 FBD of Example 4.2

$$\theta_1 = \tan^{-1}\frac{1}{3} = 18.4(\deg)$$

$$\theta_2 = \tan^{-1}\frac{0.5}{1} = 26.6(\deg)$$

Equilibrium:

$$\sum F_x = F_1\cos(18.4) - F_2\cos(26.6) = 0 \qquad (i)$$

$$\sum F_y = F_1\sin(18.4) + F_2\sin(26.6) - 85\times 9.81 = 0 \qquad (ii)$$

Matrix form:

$$\begin{bmatrix}\cos(18.4) & -\cos(26.6)\\ \sin(18.4) & \sin(26.6)\end{bmatrix}\begin{bmatrix}F_1\\ F_2\end{bmatrix} = \begin{bmatrix}0\\ 833.85\end{bmatrix}$$

$$\begin{bmatrix}0.95 & -0.89\\ 0.32 & 0.45\end{bmatrix}\begin{bmatrix}F_1\\ F_2\end{bmatrix} = \begin{bmatrix}0\\ 833.85\end{bmatrix}$$

$$\begin{bmatrix}F_1\\ F_2\end{bmatrix} = \begin{bmatrix}1055\\ 1119\end{bmatrix} \quad [N]$$

Elimination:

$$F_1 = F_2\frac{\cos(26.6)}{\cos(18.4)}$$

$$F_2\frac{\cos(26.6)}{\cos(18.4)}\sin(18.4) + F_2\sin(26.6) = 833.85$$

$$F_2(0.3 + 0.45) = 833.85;$$

$$F_2 = 1118.7[N]$$

$$F_1 = 1054.7[N]$$

Solution appears reasonable.This is all that can be solved by statics alone.

Note : These solutions are different but it is a different problem because of the new geometry. Selection of FS, and material of the rope can help us continue the design of the rope with knowledge of stresses and yield strength.

4.1.8 Example 4.3

A roller of mas 50 kg being pulled across the grass is stuck at a bump of height 10 cm. The diameter of the roller is 1 m. Find the force P required so that the roller just loses contact with the ground. The force P is at an angle of 30 degrees to the horizontal and passes through the center of the roller as shown in Figure 4.1.8a

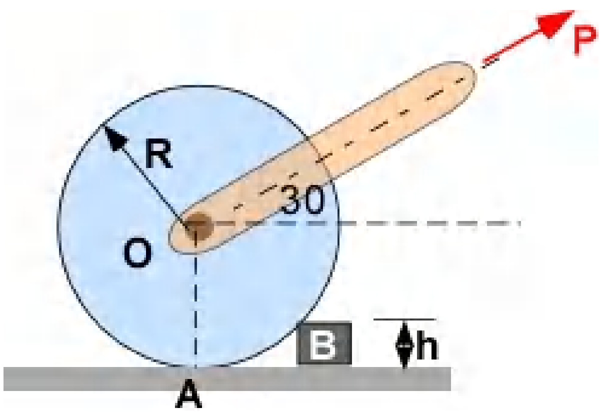


Figure 4.1.8a Example 4.3

The problem does not look like a particle but since the forces are *concurrent* at the center of the roller we can resort to a particle assumption since there are no moments applied in the problem.

Data: The geometry as shown in Figure 4.1.8. Mass = 50 kg; R = 0.5 m; h = 0.1 m

Find: The force P for the roller to just loose contact at point A and maintain contact at point B. Solving for this instance alone

Assumption: (i) Static Equilibrium; (ii) Particle, (iii)Normal force passes through O

Solution: (Draw FBD with coordinate system)

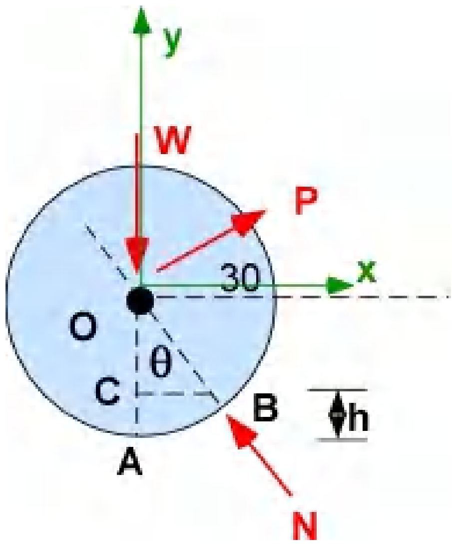


Figure 4.1.8b Example 4.3 - FBD

Geometry:

$$\theta = \cos^{-1}\left(\frac{OC}{OB}\right) = \cos^{-1}\left(\frac{0.5-0.1}{0.5}\right) = 36.87$$

Equilibrium:

$$\sum F_x = P\cos(30) - N\sin(36.87) = 0; \qquad (i)$$
$$\sum F_y = P\sin(30) + N\cos(36.87) - (50\times 9.81) = 0; \qquad (ii)$$

Solution:

$$N = P\frac{\cos(30)}{\sin(36.87)};$$

$$P\sin(30) + P\frac{\cos(30)}{\sin(36.87)}\cos(36.87) = 50\times 9.81$$

$$P = 296.43[N]$$

Solution appears reasonable. We can further find stresses in the rod that applies P if the area is known.

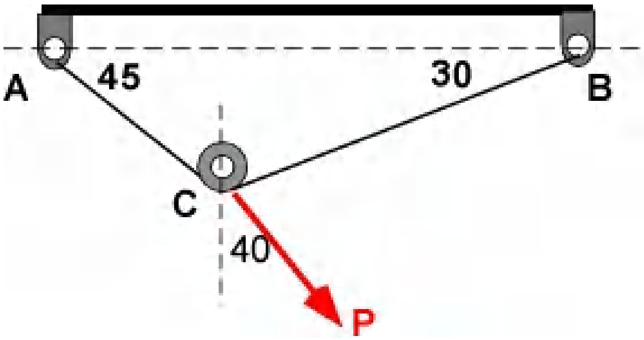
Question: The net force at O is zero. Does this mean that there is no stress on a pin at O?

4.1.9 Additional Problems

Solve the following problems on paper and using MATLAB. For each problem you must draw the FBD with the coordinate system.

Problem 4.1.1

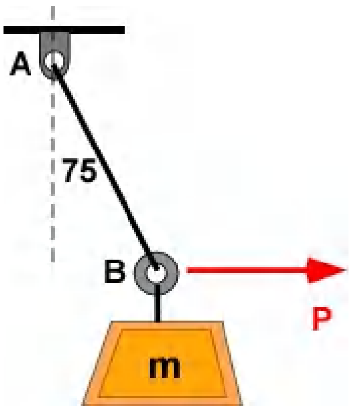
A smooth roller is moved along the cord between A and B. The force P is acting in the direction shown. For this particular geometry calculate the ratio of the force P and the tension in the cord. Note that the tension in c and AB must be the same.



Problem 4.1.1

Problem 4.1.2

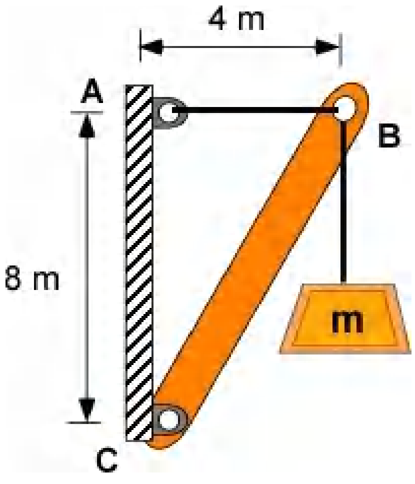
The mass weighs 100 kg. Find the horizontal force P and the tension in the cord for equilibrium. What is the stress in the cord if its diameter is 5 mm?



Problem 4.1.2

Problem 4.1.3

The heavy mass of 1500 kg is supported by an aluminum rod of rectangular cross-section. Design the cross-section area for a factor of safety of 4.



Problem 4.1.3

4.2 PARTICLE EQUILIBRIUM WITH FEATURES

This section continues discussing two-dimensional equilibrium with additional problems that include additional mechanisms, specifically linear elastic springs, frictionless pulleys, and dry friction. The particle assumptions are valid as these examples have forces that are concurrent.

Model for Elastic Spring:
The spring has an unstretched length of $\mathbf{L_0}$. The applied force $\mathbf{F_k}$ will cause the spring to extend to the length **L**. The force in the spring, $\mathbf{F_k}$, is proportional to the spring deflection, $\mathbf{L - L_0}$. We introduce the constant of proportionality, **k**, and calculate the force as

$$F_k = k\left(L - L_0\right) \tag{4.5}$$

The spring constant is basic units is [force/length]. Also known is L_0 with a value of 8. We should be able estimate L from the geometry.

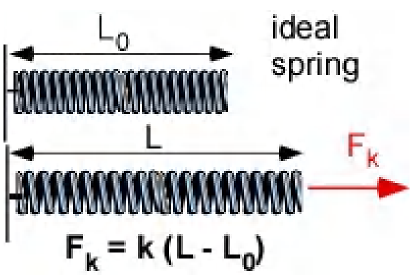


Figure 4.2.1 Elastic Spring

Model for Frictionless Pulley:
The pulley works only in tension. In statics, the pulley is not rotating or moving the tension in the cable on either side of the pulley is the same. The angle the cable make do not matter. The direction of a cable in tension must be as shown. The the direction of the cable forces is given by the geometry of the arrangement.

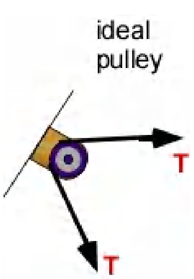


Figure 4.2.2 Frictionless pulley

Model for Dry/Coulomb Friction
A force P acts on a particle but is unable to move it. The particle is in equilibrium. We represent the particle by a box so we can use the edges to show the forces acting on it. This means that the force P is balanced by an equal and opposite force due to the reaction of the floor on the particle. While

this reaction takes place at the juncture of the floor and the box, because of the particle assumption we place it at the center. The force N is the normal reaction of the box and the floor.

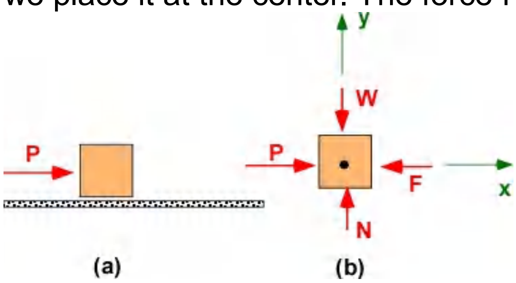


Figure 4.2.3 Dry Friction

Applying equilibrium:

$$\sum F_x = P - F = 0; \quad P = F$$
$$\sum F_y = N - W = 0; \quad N = W \qquad (4.6)$$
$$F = \mu N \quad \text{: model for dry friction}$$

The coefficient of friction, μ, is usually constant and has no dimension. It depends on the two material in contact. For this example the floor and the underside of the box. It is obtained empirically and must be specified in the problem.

4.2.1 Example 4.4 Problem Definition

The Engineering House is building a party room and wants to precisely locate the heavy chandelier, weighing 200 [lb], over different spots in the vertical plane at the center of the room. They rigged up a simple system using dead weights and the elastic spring from B as shown. What are the values for the weights W_A and W_C if the chandelier is to be located at D. The elastic spring constant is 40 [lb/ft] and its unstretched length is 8 ft

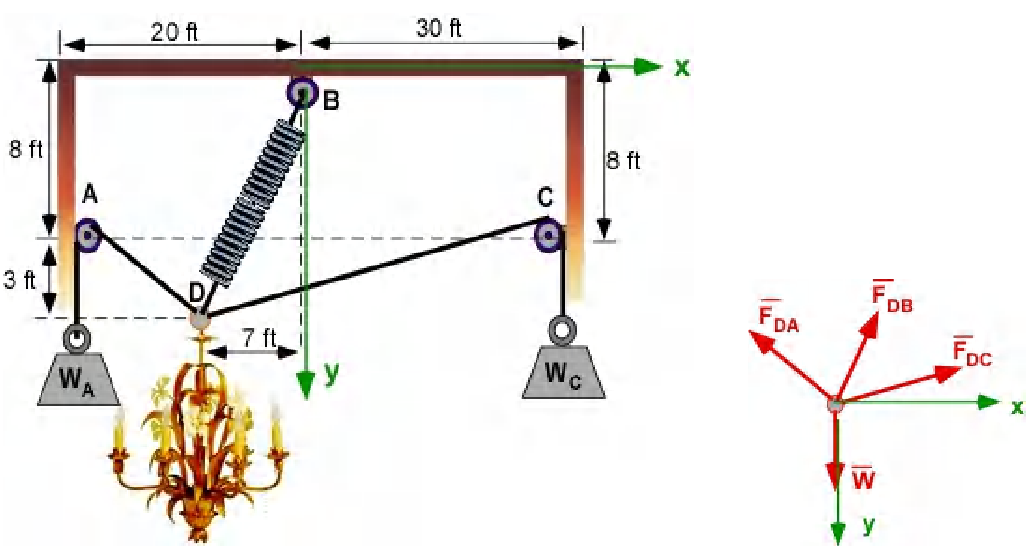


Figure 4.2.4 Example 4.4 and FBD

In this problem the force in the cable DC is the same as the weight W_C. Similarly the force in the cable DA is the same as the weight W_A

We can consider this a 2D problem, with a particle at D, and that the spring behaves linearly. We have two obvious unknowns, $\mathbf{W_A}$ and $\mathbf{W_C}$. We also have two new elements in the problem, a spring and several pulleys. If the spring or the pulleys do not introduce any additional unknowns then we should be able to solve the problem through equilibrium alone. Let us present the models for the spring and the pulley before proceeding with the solution.

Data :
W = 200 [lb];
Coordinates of the points: B(0, 0); A(-20, 8); C(30, 8); D(-7, 11); with respect to the coordinate system shown. Locations are in feet. L_0 = 8;

Find:
W_A and W_B

Assumptions:
Particle at D, 2D problem
Ideal/frictionless pulley
Elastic spring

Solution:
Applying Equilibrium
The magnitude of F_{DA} can be related to the W_A and the magnitude of F_{DC} can be related to W_C. Magnitude of F_{DB} can be calculated using spring geometry.

$$F_{DA} = W_A; \quad F_{DC} = W_C;$$

$$F_{DB} = k\left(\sqrt{7^2 + 11^2} - 8\right) = 40(13.038 - 8) = 201.54$$

$$\hat{e}_{DA} = \frac{i(-20-(-7) + j(8-11)}{\sqrt{13^2 + 3^2}} = i(-0.9744) + j(-0.2248)$$

$$\hat{e}_{DB} = \frac{i(0-(-7) + j(0-11)}{\sqrt{7^2 + 11^2}} = i(0.5369) + j(-0.8437)$$

$$\hat{e}_{DC} = \frac{i(30-(-7) + j(8-11)}{\sqrt{37^2 + 3^2}} = i(0.9967) + j(-0.0808)$$

$$\sum \bar{F} = F_{DA}\hat{e}_{DA} + F_{DB}\hat{e}_{DB} + F_{DC}\hat{e}_{DC} + \bar{W} = 0$$

$$\sum F_x = -0.9744 F_{DA} + 0.5669(201.54) + 0.9967 F_{DC} = 0 \quad (1)$$

$$\sum F_y = -0.2248 F_{DA} - 0.8437(201.54) - 0.0808 F_{DC} + 200 = 0 \quad (2)$$

Solving the equations:
Using (1)

$$F_{DA} = \frac{(0.5369)(201.54)+).9967\,F_{DC}}{0.9744} = 111.05 + 1.0229\,F_{DC} \qquad (3)$$

Substituting (3) in (2)

$$-0.2248\left[111.05+1.0229F_{DC}\right]-(0.8437)(201.54)-0.0808F_{DC}+200=0$$

$$F_{DC} = \frac{-200+(0.2248)(111.05)+(0.8437)(201.54)}{-0.0808-(0.2248)(1.0229)} = 16.08\left[lb\right] = W_C = 71.52\left[N\right]$$

$$F_{DA} = 111.05+(1.0229)(16.08) = 127.5\left[lb\right] = W_A = 567.12\left[N\right]$$

Another approach is to set up the equations in matrix form:

$$\begin{bmatrix} -0.9744 & 0.9967 \\ -0.2248 & -0.0808 \end{bmatrix} \begin{bmatrix} F_{DA} \\ F_{DC} \end{bmatrix} = \begin{bmatrix} -0.5669(201.54) \\ 0.8437(201.54)-200 \end{bmatrix}$$

You should get the same previous solution.

4.2.2 Example 4.4 Using MATLAB

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 4.4.2- Example 4.4
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-----------------------------------------\n')
fprintf('Example 4.4 \n')
fprintf('------------------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data (in meters)
w = 200;
A = [-20 8 0];  B = [0 0 0];  C = [30 8 0]; D = [-7 11 0];
L0 = 8; k = 40;

%% Unknowns
syms WA WC

%% Solution
% unit vectors
DA = A - D;  eDA = DA/norm(DA);
DB = B - D;  L = norm(DB);  eDB = DB/norm(DB);
DC = C - D;  eDC = DC/norm(DC);

Fk = k*(L-L0);
W = w*[0 1 0];
```

```
% Equilibrium
sumF = WA*eDA + WC*eDC + Fk*eDB +W;

sol1 = solve(sumF(1),sumF(2));

WAv = double(sol1.WA);
WCv = double(sol1.WC);

%% Printing
fprintf('W [lb]                 : '),disp(w)
fprintf('unstretched L0 [ft]    : '),disp(L0)
fprintf('stretched L [ft]       : '),disp(L)
fprintf('k [lb/ft]              : '),disp(k)
fprintf('Fk [lb]                : '),disp(Fk)
fprintf('\n')
fprintf('Unit vector: eDA : '),disp(eDA)
fprintf('Unit vector: eDB : '),disp(eDB)
fprintf('Unit vector: eDC : '),disp(eDC)

fprintf('----------------------------------------\n')
fprintf('Equilibrium\n')
fprintf('-----------------------------------------\n')
fprintf('eq(1)  : '),disp(vpa(sumF(1),3))
fprintf('eq(2)  : '),disp(vpa(sumF(2),3))
fprintf('WA [lb])              : '),disp(WA)
fprintf('WC [lb])              : '),disp(WC)
fprintf('\n')
```

In the Command Window

```
----------------------------------------
Example 4.4
-----------------------------------------
W [lb]                 :    200
unstretched L0 [ft]    :      8
stretched L [ft]       :      13.038
k [lb/ft]              :     40
Fk [lb]                :        201.54

Unit vector: eDA :      -0.97439     -0.22486            0
Unit vector: eDB :       0.53688     -0.84366            0
Unit vector: eDC :       0.99673    -0.080816            0
----------------------------------------
Equilibrium
-----------------------------------------
eq(1) : 0.997*WC - 0.974*WA + 108.0
eq(2) : 30.0 - 0.0808*WC - 0.225*WA
WA [lb])              :        127.51
WC [lb])              :        16.094
```

4.2.3 Example 4.5 - Inverse Example 4

Let us define the inverse problem of Example 4.4. Let the weights W_A and W_C be known and we have to locate the point D for equilibrium. To make the problem simple let us use the numbers for the weight from the solution to Example 4.4 and see if we can arrive at the same values for the location of D. This is to illustrate an example where the *force is not the unknown*. This is now a different problem in design. The approach is still the same. We need as many equations as the number of unknowns. The equilibrium equations are still the same.

Problem Definition

The Club Mechanica party house wants to precisely locate the heavy chandelier, weighing 200 [lb], over different spots in the vertical plane at the center of the room. They rigged up a simple system using dead weights and the elastic suspension from B as shown. What is the location of the chandelier if W_A = 127 [lb] and W_C = 16 [lb]? The elastic spring constant is 40 [lb/ft] and its unstretched length is 8 ft. We can verify if we obtain the location information in Example 4.4.

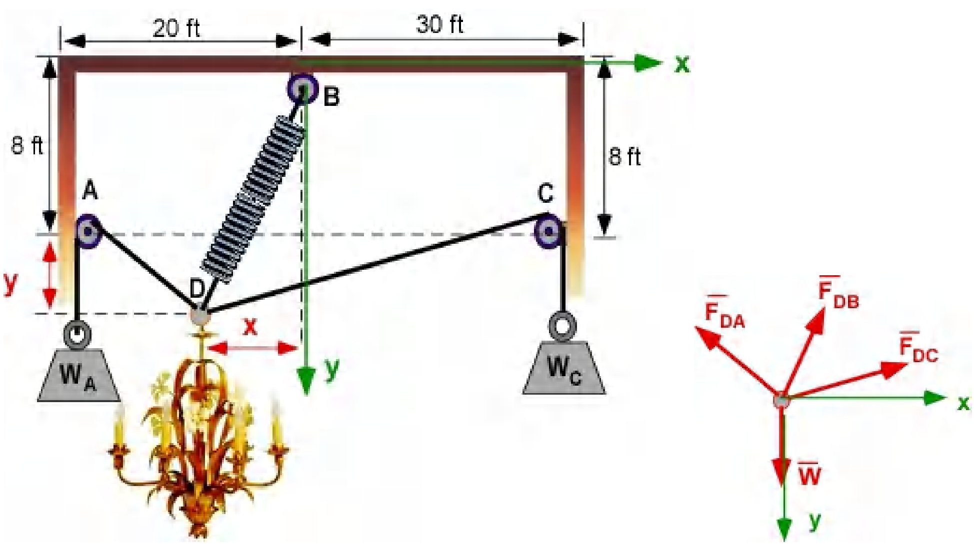


Figure 4.2.5 The problem and the FBD

Data:

W = 200 [lb]; W_A = 127 [lb]; W_B = 16 [lb]

B(0, 0); A(-20, 8); C(30, 8); D(-x, 8+y); with respect to the coordinate system shown. Locations are in [ft].

Find: x = ? y = ? (We have two equations so we should be able to solve for the two unknowns)

Assumptions:

Particle at D, 2D problem

Ideal/frictionless pulley

Elastic suspensions behaves like a linear spring

Solution:

Pulley relations: $F_{DA} = W_A = 127; \quad F_{DC} = W_C = 16$

Spring Force: $F_{DB} = k(\sqrt{x^2 + (8+y)^2} - 8) = 40 * (\sqrt{x^2 + (8+y)^2} - 8)[lb]$

Unit Vectors:

$$\hat{e}_{DA} = \frac{i(-20-(-x)+j(8-(8+y))}{\sqrt{(x-20)^2+y^2}} = \frac{i(x-20)+j(-y)}{\sqrt{(x-20)^2+y^2}}$$

$$\hat{e}_{DB} = \frac{i(0-(-x)+j(0-(8+y))}{\sqrt{x^2+(8+y)^2}} = \frac{i(x)+j(-(8+y))}{\sqrt{x^2+(8+y)^2}}$$

$$\hat{e}_{DC} = \frac{i(30-(-x)+j(8-(8+y))}{\sqrt{(30+x)^2+y^2}} = \frac{i(30+x)+j(-y)}{\sqrt{(30+x)^2+y^2}}$$

Equilibrium: $\sum \overline{F} = F_{DA}\hat{e}_{DA} + F_{DB}\hat{e}_{DB} + F_{DC}\hat{e}_{DC} + \overline{W} = 0$

$$\sum F_x = 127\frac{(x-20)}{\sqrt{(x-20)^2+y^2}} + 40(\sqrt{x^2+(8+y)^2}-8)\frac{x}{\sqrt{x^2+(8+y)^2}}$$

$$+16\frac{(30+x)}{\sqrt{(30+x)^2+y^2}} = 0 \quad (1)$$

$$\sum F_y = 127\frac{(-y)}{\sqrt{(x-20)^2+y^2}} + 40(\sqrt{x^2+(8+y)^2}-8)\frac{(-8-y)}{\sqrt{x^2+(8+y)^2}}$$

$$+16\frac{(-y)}{\sqrt{(30+x)^2+y^2}} + 200 = 0 \quad (2)$$

These are nonlinear equations. They are solved using numerical techniques. You will probably learn to solve them in the third year or so. Here we can attempt to solve them using MATLAB symbolically and see if it works. We can also use graphics to identify the solution. We plot the functions corresponding to the Eqn.(1) and Eqn.(2) and the solution is at the intersection. Both methods are included below.

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 4.4.3- Example 4.5
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 4.5 \n')
fprintf('-----------------------------------------\n')
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data (in meters)
syms x y real

WA = 127; WC = 16; We = 200;
A = [-20 8 0];  B = [0 0 0];  C = [30 8 0];  D = [-x 8+y 0];
L0 = 8;  k = 40;

%% Set up unit vectors
DA = A - D; eDA = DA/norm(DA);
DB = B - D; L = sqrt((DB*DB'));  eDB = DB/norm(DB);
DC = C - D;  eDC = DC/sqrt((DC*DC'));
%% Set up equilibrium equations
Fk = k*(L-L0);
W = We*[0 1 0];
sumF = WA*eDA + WC*eDC + Fk*eDB +W;

%% solve equations
sol1 = solve(sumF(1),sumF(2));
%% solution
xv = double(sol1.x);
yv = double(sol1.y);

%% Printing
fprintf('We [lb]                : '),disp(We)
fprintf('WA [lb]                : '),disp(WA)
fprintf('WC [lb]                : '),disp(WC)
fprintf('unstretched L0 [ft]    : '),disp(L0)
fprintf('stretched L [ft]       : '),disp(L)
fprintf('k [lb/ft]              : '),disp(k)
fprintf('Fk [lb]                : '),disp(Fk)
fprintf('\n')
fprintf('Unit vector: eDA : '),disp(eDA)
fprintf('Unit vector: eDB : '),disp(eDB)
fprintf('Unit vector: eDC : '),disp(eDC)

fprintf('----------------------------------------\n')
fprintf('Equilibrium\n')
fprintf('-----------------------------------------\n')
% fprintf('eq(1) : '),disp(vpa(sumF(1),3))
% fprintf('eq(2) : '),disp(vpa(sumF(2),3))
fprintf('x [ft])              : '),disp(xv)
fprintf('WC [ft])              : '),disp(yv)
fprintf('\n')
```

In the Command Window

```
----------------------------------------
Example 4.5
-----------------------------------------
We [lb]              :    200
WA [lb]              :    127
WC [lb]              :     16
unstretched L0 [ft]  :      8
stretched L [ft]     : ((y + 8)^2 + x^2)^(1/2)
k [lb/ft]            :     40
```

```
Fk [lb]               : 40*((y + 8)^2 + x^2)^(1/2) - 320

Unit vector: eDA : [ (x - 20)/(abs(x - 20)^2 + abs(y)^2)^(1/2), -y/(abs(x
- 20)^2 + abs(y)^2)^(1/2), 0]
Unit vector: eDB : [ x/(abs(y + 8)^2 + abs(x)^2)^(1/2), -(y + 8)/(abs(y +
8)^2 + abs(x)^2)^(1/2), 0]
Unit vector: eDC : [ (x + 30)/((x + 30)^2 + y^2)^(1/2), -y/((x + 30)^2 +
y^2)^(1/2), 0]
----------------------------------------
Equilibrium
-----------------------------------------
x [ft])             :        6.9769
WC [ft])             :          3.01
```

A numerical solution is returned by the solver as the symbolic solution cannot be obtained. Therefore MATLAB is able to determine the solution of this nonlinear problem. . This is an important achievement. We can do more knowing how to use MATLAB for this course and we do not have to limit ourselves to classical linear problems.
If MATLAB did not determine the solution then we should always be able to solve by the next method.

Solution using Graphics
We will now use MATLAB to solve Example 4.5 using graphics. We plot the functions corresponding to the Eqn. (1) and Eqn. (2) . These are functions of two variables and hence we need draw contour plots. We only need to plot the contours of zero value. MATLAB requires you to use a vector of two identical values for a single contour plot as highlighted in the code below. Here we are plotting contour level values of zero.

Extension to the previous script/code

```
%% Draw plots and find solution at their intersection
% sumF(1) = 0; and sumF(2) = 0;
xx = -20:0.5:30;
yy = 0:0.5:15;
[X Y] = meshgrid(xx,yy);
FB = k*(sqrt(X.^2 + (8+Y).^2 )-8);
fun1 = (WA.*(X-20)./sqrt((X-20).^2 + Y.^2)) + ...
    (FB.*X./sqrt(X.^2  +(8+Y).^2)) + ...
    (30 + X).*WC./sqrt((30 + X).^2+Y.^2);
fun2 = (-WA.*Y./sqrt((X-20).^2 + Y.^2)) + ...
    (-FB.*(8+Y)./sqrt(X.^2  +(8+Y).^2)) + ...
    (-Y.*WC./sqrt((30 + X).^2+Y.^2))+200;

[c1 h1] =contour(xx,yy,fun1,[0,0],'r-');
clabel(c1);
hold on
[c2 h2] = contour(xx,yy,fun2,[0 0],'k-');
clabel(c2);
hold off
grid
xlabel('x')
ylabel('y')
title('Example 4.5')
legend('\SigmaFx = 0', '\SigmaFy = 0')
```

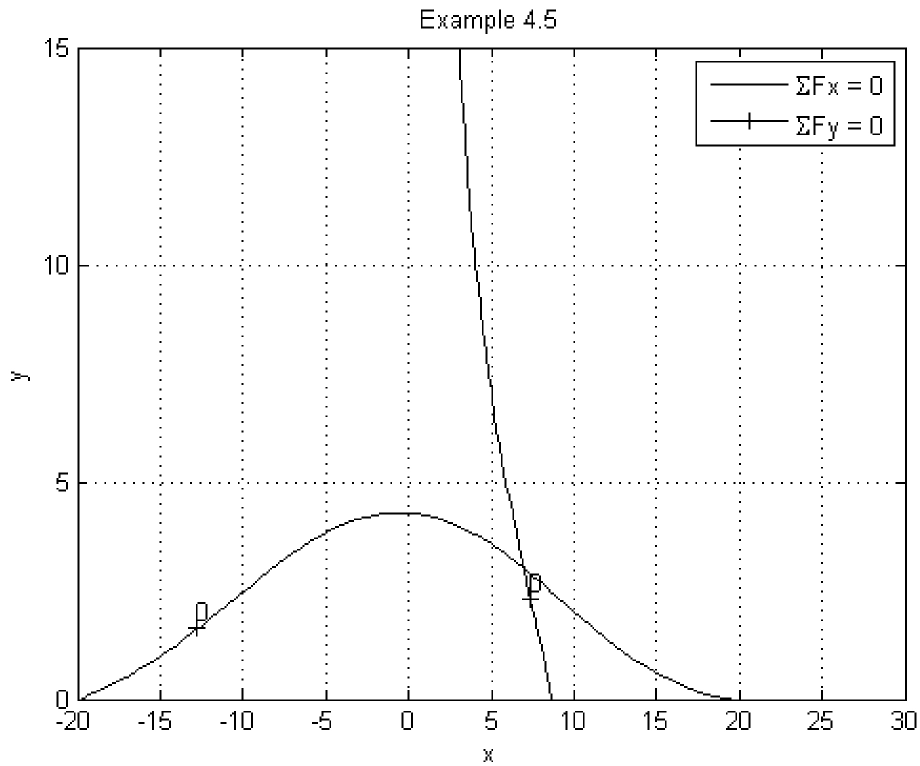


Figure 4.2.6a Solution at the intersection of the plots.

You can zoom into the plot and determine the solution more precisely. Both the approaches have yielded the same values. This is the advantage of including MATLAB in assisting with this course. We can address more complex problems than standard text books.

Execution in Octave

The code is the same as in MATLAB except for the additional statements below . The changes are highlighted. You must include the symbolic package and if you do not wish to see warnings you include the command warning off as shown

```
clc, clear, format compact, close all, format short G, warning off

pkg load symbolic
sympref display flat
```

MATLAB solves this example by switching o a numerical solver automatically if symbolic solution cannot found. Octave attempts to solve the symbolic problem and is unsuccessful. Empty cell is returned as solution. At this time understanding and solving nonlinear equations is well beyond the scope of this book. Nevertheless nonlinear problems in two variables can be solved graphically. Here the graphical solution is obtained graphically by including the same MATLAB code

In Octave Command Window (No solution - but problem information)

```
----------------------------------------
Example 4.5
-----------------------------------------
We [lb]                 : 200
WA [lb]                 : 127
WC [lb]                 : 16
unstretched L0 [ft]     : 8
```

```
stretched L [ft]       :   sqrt(x**2 + (-y - 8)**2)
k [lb/ft]              : 40
Fk [lb]                :   40*sqrt(x**2 + (-y - 8)**2) - 320

Unit vector: eDA :   Matrix([[(x - 20)/sqrt(y**2 + (x - 20)**2), -
y/sqrt(y**2 + (x - 20)**2), 0]])
Unit vector: eDB :   Matrix([[x/sqrt(x**2 + (y + 8)**2), (-y -
8)/sqrt(x**2 + (y + 8)**2), 0]])
Unit vector: eDC :   Matrix([[(x + 30)/sqrt(y**2 + (x + 30)**2), -
y/sqrt(y**2 + (x + 30)**2), 0]])
```

In Octave Figure Window

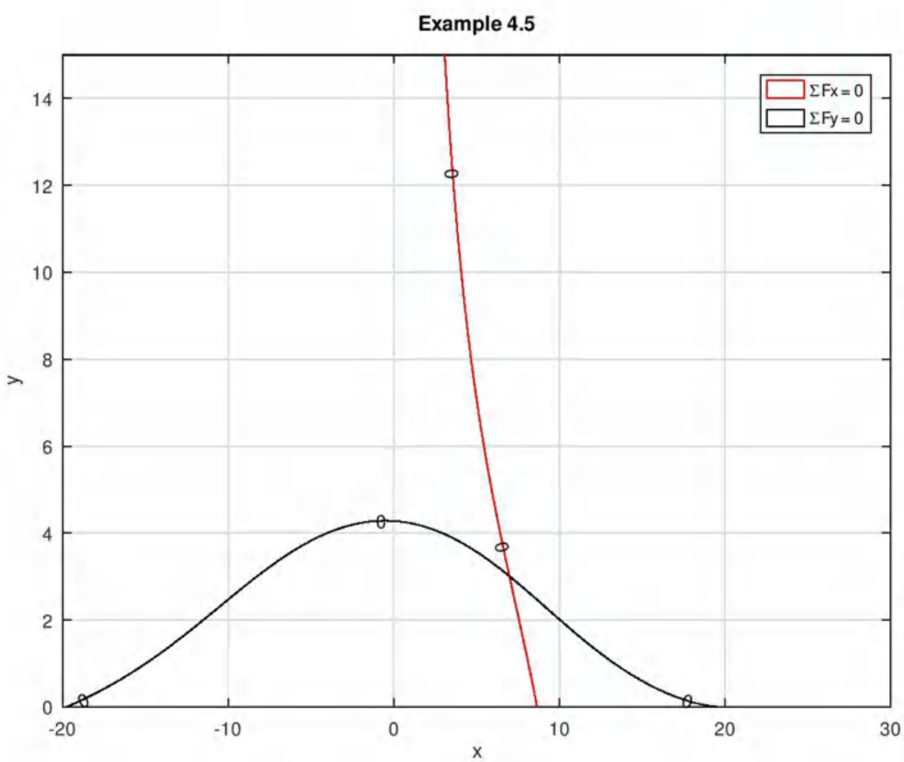


Figure 4.2.6b Solution at the intersection of the plot (Octave).

4.2.4 Example 4.6

We look at an example with friction. In this example we will try and estimate the tension/force in the rope required to move the stone up the slope for constructing the left half of the pyramid through the pulley arrangement. We will assume that the coefficient of friction is 0.2 between the block and the incline and the pulleys are frictionless. The stone has a mass of 1000 kg. This problem is shown in Figure 4.2.7.

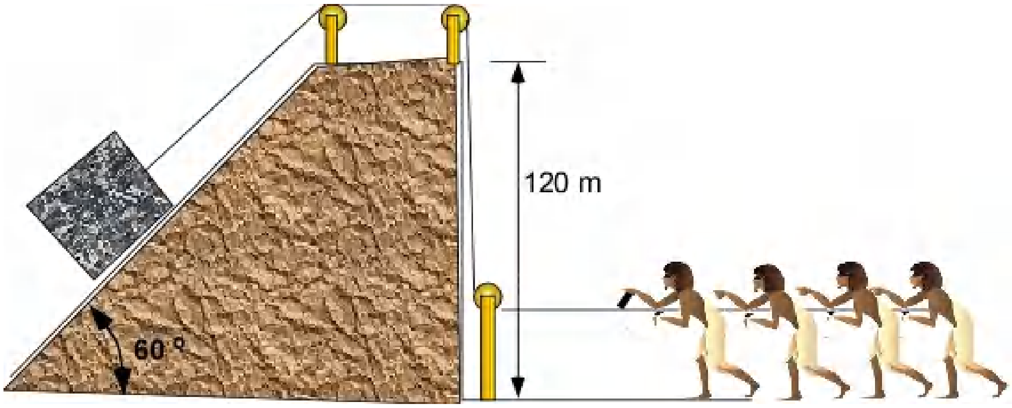


Figure 4.2.7 Problem definition

We begin with the FBD. We will assume it as a particle. These problems are referred to as incline plane problems and it is useful to set up our coordinates parallel and normal to the incline. Even though we have assumed the stone as a particle we draw the FBD as a rectangle to accommodate the forces neatly. We ignore any moments that the representation may suggest. We include the incline for clarifying the geometry of the forces. The force **F** is the tension and **F_f** is the force due to friction.

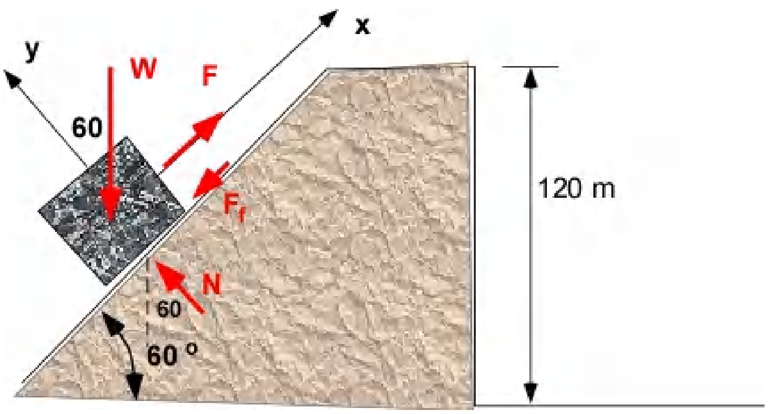


Figure 4.2.8 FBD of Block

Data: M = 1000 kg; μ = 0.2; θ = 60 degrees

Find: F

Assumption: The stone is a particle
Static dry Friction - the stone is about to move
$F_f = \mu N$

Solution:

$$\sum F_x = F - F_f - W \sin 60 = 0$$

$$\sum F_y = N - W \cos 60 = 0$$

$$N = W \cos 60 = 1000 * 9.81 * 0.5 = 4905\, N$$

$$F = F_f + W \sin 60 = 0.2 * 4905 + 1000 * 9.81 * 0.866 = 9476\, N$$

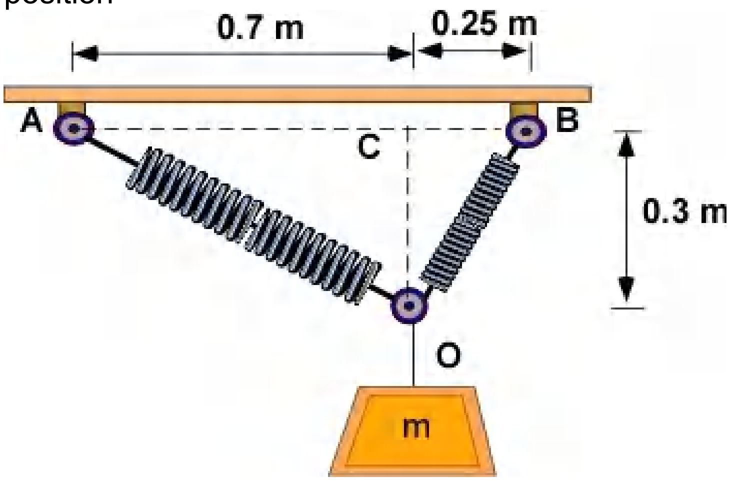
Only slightly less than the weight of the stone.

4.2.5 Additional Problems

Solve the following problems on paper and using MATLAB. For each problem you must draw the FBD with the coordinate system.

Problem 4.2.1

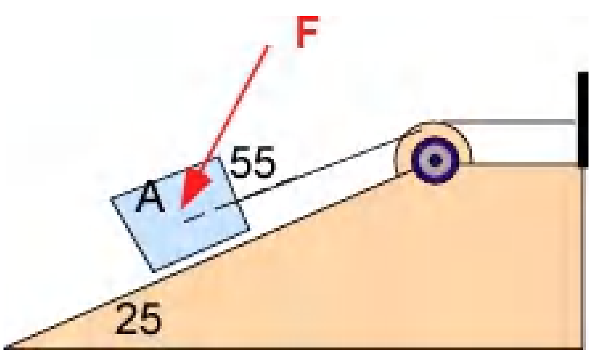
The two elastic springs have the same spring constant k. They are horizontal and unstretched and meet at C with the mass removed. Calculate the mass m and the spring constant k for this deflected position



Problem 4.2.1

Problem 4.2.2

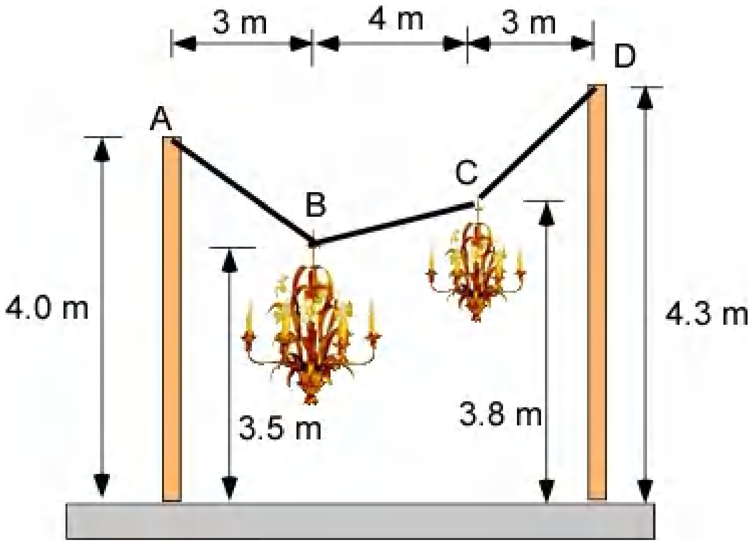
The force F of magnitude 500 N pushes against the mass of 100 kg at A which is connected to an inelastic cord that provides tension aligned with the inclined plane. The coefficient of friction is 0.32. What is the tension in the cord?



Problem 4.2.2

Problem 4.2.3

An arrangement for hanging two chandeliers is shown in Problem 4.2.3. The mass of the chandelier at B is 50 kg. What is the mass of the chandelier at C?



Problem 4.2.3

4.3 PARTICLE EQUILIBRIUM IN THREE DIMENSIONS

Particle equilibrium in three dimensional is not significantly different than equilibrium in two dimensions. In fact the primary equation is Eqn. (4.3) reproduced below.

$$\sum \overline{F} = 0 \qquad (4.3)$$

This equation is independent of dimension or coordinate system. If there are no moments in the problem then it is equivalent to particle equilibrium. This usually will involve forces passing through a point - or a concurrent force system. There will be more work since we will be handling three dimensional vectors. We still need to set up the free body diagram (FBD) before attempting to solve the problem. We will solve Example 4.7 through several sub-sections.

4.3.1 Example 4.7: Problem Definition

A central multi-sided traffic light is being considered to replace the four separate sets lights that are common today. A three point suspension will be customized for the intersection. The layout of the light is shown in Figure 4.3.1. The coordinate system is included so that the distances can be mapped easily. The mass of the new light is 350 [kg]. The various dimensions of the lanes and the poles are shown in the first figure in Figure 4.3.1. Figure 4.3.1 captures the various stages of problem formulation, starting with the geometrical description of the problem in the first figure. In preparation for the application of the equilibrium equations the focus is on the light suspension as shown in the second figure. The final figure provides us with the FBD and here the light is a particle with the three cables in tension and the weight. Note that the FBD is very similar to the one in Example 4.1, except Example 4.7 is in three dimensions. The FBD suggests that the light can be regarded as a particle with a concurrent system of forces. In this problem the deflected geometry is provided , unlike Example 4.1 , where we had to assume the angle of deflection.

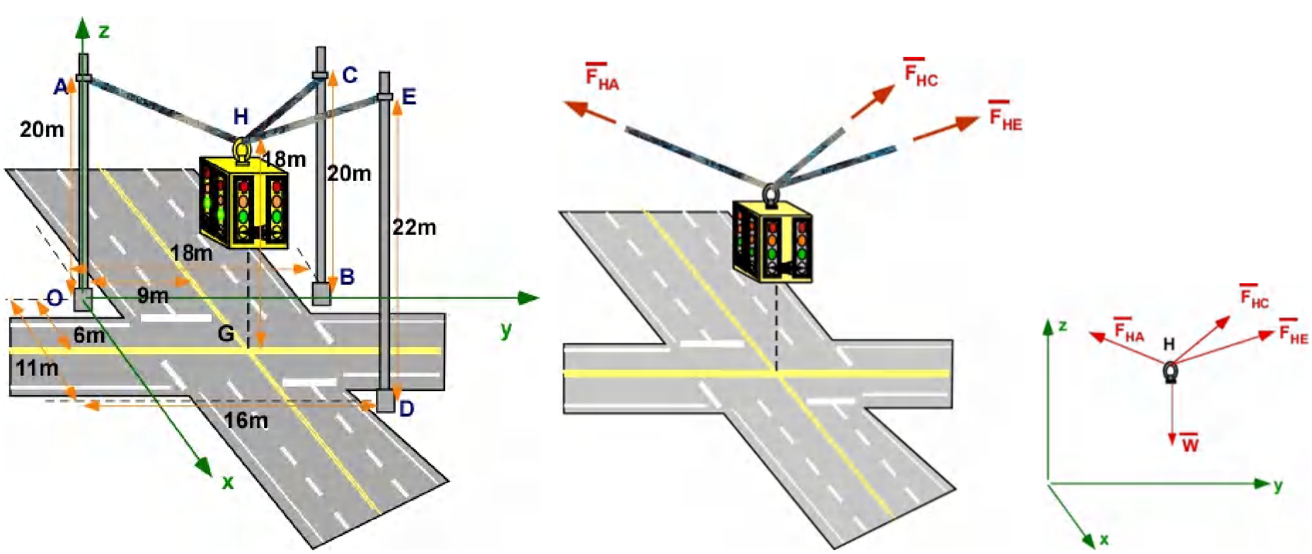


Figure 4.3.1 Original Problem with focus on the light and the FBD

In this problem we are interested in only finding the forces in the cable. There are three unknown forces and we should be able to find them directly using the equations of equilibrium. We can then finish the design by sizing the cable given the choice of material for the cable. Instead of stress, cables are sometimes sold with information of their breaking load or force. In this case our design will stop with the determination that all of the forces are less than the specific breaking strength of the cable. This assumes that the cable is already selected.

We can extend the problem by determining the sag in the light or its final position due to the loads in the cable.

Data:
m = 350 [kg];
The coordinates of the various points of interest are
O(0, 0, 0); A(0, 0, 20); B(0, 18, 0); C(0, 18, 20); D(11, 16, 0); E (11, 16, 22); G(6, 9, 0); H(6, 9, 18); (all in [m])

4.3.2 Example 4.7 Applying Equilibrium

$$\sum \bar{F} = \bar{F}_{HA} + \bar{F}_{HC} + \bar{F}_{HE} + \bar{W} = 0$$

The magnitude of the forces in the cable are the **unknowns,** F_{HA}, F_{HC}, F_{HE}. We can express the force vectors as magnitude multiplied by the respective unit vectors. We have three unknowns and three equations , one for each coordinate direction. This is a deterministic problem and we can solve for the three unknowns through algebra.

The computations of the unit vectors is straight forward.

$$\hat{e}_{HA} = \frac{i(0-6)+j(0-9)+k(20-18)}{\sqrt{6^2+9^2+2^2}} = i(-0.5454)+j(-0.8181)+k(0.1818)$$

$$\hat{e}_{HC} = \frac{i(0-6)+j(18-9)+k(20-18)}{\sqrt{6^2+9^2+2^2}} = i(-0.5454)+j(0.8181)+k(0.1818)$$

$$\hat{e}_{HE} = \frac{i(11-6)+j(16-9)+k(22-18)}{\sqrt{5^2+7^2+4^2}} = i(0.5270)+j(0.7379)+k(0.4216)$$

The weight expressed as a vector is

$$\bar{W} = i(0)+j(0)+k(-350*9.81)$$

Applying the equilibrium equations and assembling the component equations:

$$\sum F_x = 0 = F_{HA}(-0.5454)+F_{HC}(-0.5454)+F_{HE}(0.5270) = 0 \quad (1)$$

$$\sum F_y = 0 = F_{HA}(-0.8181) + F_{HC}(0.8181) + F_{HE}(0.7379) = 0 \quad (2)$$

$$\sum F_z = 0 = F_{HA}(0.1818) + F_{HC}(0.1818) + F_{HE}(0.4216) - 3433.5 = 0 \quad (3)$$

The unknowns are solved through substitution and reduction

From equation (1)

$$F_{HA} = \frac{F_{HC}(-0.5454) + F_{HE}(0.5270)}{0.5454} = -F_{HC} + F_{HE}(0.9662) \quad (4)$$

Substituting this in equation (2)

$$\left[-F_{HC} + F_{HE}(0.9662)\right](-0.8181) + F_{HC}(0.8181) + F_{HE}(0.7379) = 0$$

$$F_{HC}(0.8181) - (0.7904)F_{HE} + F_{HC}(0.8181) + F_{HE}(0.7379) = 0$$

$$F_{HC}(1.6362) - F_{HE}(0.0525) = 0; \quad F_{HC} = \frac{0.0525}{1.6362} F_{HE} = 0.0321 F_{HE} \quad (5)$$

And substituting in (3)

$$F_{HA} = -(0.0321 F_{HE}) + 0.9662 F_{HE} = 0.9341 F_{HE} \quad (6)$$

$$(0.9341 F_{HE})(0.1818) + (0.0321 F_{HE})(0.1818) + F_{HE}(0.4216) = 3433.5$$

Finally leading to the solution

F_{HE} = 5748.8 [N]; F_{HA} = 5369.5 [N]; F_{HC} = 184.54 [N] (7)

One important activity we must undertake after every problem we solve is to question if the numbers make sense.. All the magnitudes must be positive as they represent tensions. Since point E is the highest , the cable connected to E can expect to have the maximum value.

Before we continue further let us understand that we can express the equations (1) - (3) in **matrix form** and use a calculator or MATLAB to solve the matrix equation:

$$\begin{bmatrix} -0.5454 & -0.5454 & 0.5270 \\ -0.8181 & 0.8181 & 0.7379 \\ 0.1818 & 0.1818 & 0.4216 \end{bmatrix} \begin{bmatrix} F_{HA} \\ F_{HC} \\ F_{HE} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3433.5 \end{bmatrix}$$

MATLAB Code for matrix equation:

```
AA = [-0.5454 -0.5454 0.5270;
0.8181 0.8181 0.7379;
```

```
0.1818 0.1818 0.4216]
BB = [0;0;3433.5]
sol2 = AA\BB
```

You should get the same solution as in (7)

4.3.3 Example 4.7 Solution using MATLAB

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 4.3.3- Example 4.7
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 4.7 \n')
fprintf('-----------------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data (in meters)
syms FHA FHC FHE

m = 350;
% coordinates
O = [0 0 0];  A = [0 0 20];  B = [0 18 0];  C = [0 18 20];
D = [11 16 0];  E = [11 16 22];  G = [6 9 0]; H = [6 9 18];

%% Unknowns
syms FHA FHC FHE real

%% Calculations
% Unit vectors
HA = A - H;  eHA = HA/norm(HA);
HC = C - H;  eHC = HC/norm(HC);
HE = E - H;  eHE = HE/norm(HE);

W = m*9.81*[0 0 -1];

%% equilibrium
sumF = (FHA*eHA + FHC*eHC + FHE*eHE+W);
sol1 = solve(sumF);

FHAv = double(sol1.FHA);
FHCv = double(sol1.FHC);
FHEv = double(sol1.FHE);
%% Printing
fprintf('m [kg]              : '),disp(m)


fprintf('\nPoint O [m] : '),disp(O)
fprintf('Point A [m] : '),disp(A)
fprintf('Point B [m] : '),disp(B)
fprintf('Point C [m] : '),disp(C)
```

```
fprintf('Point D [m] : '),disp(D)
fprintf('Point E [m] : '),disp(E)
fprintf('Point G [m] : '),disp(G)
fprintf('Point H [m] : '),disp(H)

fprintf('\nUnit vector: eHA : '),disp(eHA)
fprintf('Unit vector: eHC : '),disp(eHC)
fprintf('Unit vector: eHE : '),disp(eHE)

fprintf('----------------------------------------\n')
fprintf('Equilibrium\n')
fprintf('----------------------------------------\n')
fprintf('eq(1) : '),disp(vpa(sumF(1),3))
fprintf('eq(2) : '),disp(vpa(sumF(2),3))
fprintf('eq(3) : '),disp(vpa(sumF(3),3))

fprintf('\nForce (HA) [N])            : '),disp(FHAv)
fprintf('Force (HC) [N])            : '),disp(FHCv)
fprintf('Force (HE) [N])            : '),disp(FHEv)
fprintf('\n')
```

In the Command Window

```
----------------------------------------
Example 4.7
-----------------------------------------
m [kg]              :    350

Point O [m] :      0     0     0
Point A [m] :      0     0    20
Point B [m] :      0    18     0
Point C [m] :      0    18    20
Point D [m] :     11    16     0
Point E [m] :     11    16    22
Point G [m] :      6     9     0
Point H [m] :      6     9    18

Unit vector: eHA :      -0.54545     -0.81818      0.18182
Unit vector: eHC :      -0.54545      0.81818      0.18182
Unit vector: eHE :       0.52705      0.73786      0.42164
----------------------------------------
Equilibrium
-----------------------------------------
eq(1) : 0.527*FHE - 0.545*FHC - 0.545*FHA
eq(2) : 0.818*FHC - 0.818*FHA + 0.738*FHE
eq(3) : 0.182*FHA + 0.182*FHC + 0.422*FHE - 3433.0

Force (HA) [N])            :        5369.1
Force (HC) [N])            :        185.14
Force (HE) [N])            :        5748.2
```

If we already have a cable with a breaking strength of 10,000 N then we can conclude that we have FS of 10,000/5748.2 which is above 1.74. This is typically low for this type of problem We can recommend that a cable of larger cross sectional area (or diameter) be used as it will increase the breaking strength. If we have a choice of materials and dimensions , then we can decide between one of them if it meets our FS requirement and is justified economically.

To complete the problem we will assume a FS of 5. A commercial 8 mm braided steel wire rope of six strands has a minimum breaking strength of 38[kN]. The maximum load in the cable with a FS of 5 will be 7.6 kN. The maximum load in the cable for Example 4.2 is 5.7 kN. This cable will work fine for our **design**.

4.3.4 Example 4.7 - Final Location of the Light.

An important information in this problem is the final location of the light if the cables are going to stretch. We can generate one more piece of information about the design. A simple way to determine the final location of the centroid of the light is to use superposition of the displacement in each of the cables/wires. To do this we assume that the displacement is small and will not affect change the original geometry. We will use Equation 3.7 to calculate the change the length in each cable. The direction will be given by the corresponding unit vector. We calculate the displacement in each cable and add them as vectors to calculate the final location of H as in Figure 4.3.2. This example invokes invokes the assumption of deformable body - and uses information from the strength of materials which is available in the data. We have chosen a new cable. Note that this does not change the solution obtained through **statics** in the previous example.

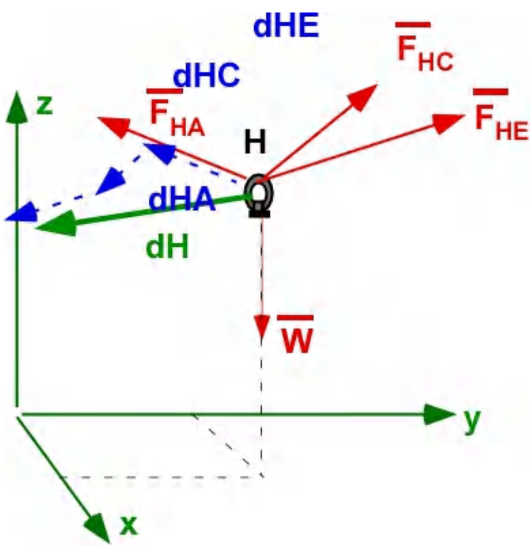


Figure 4.3.2. Displacement - Example 4.7

$$\delta_H = \delta_{HA} + \delta_{HC} + \delta_{HE}$$
$$= \hat{e}_{HA}\left(\frac{F_{HA}\,L_{HA}}{E_{HA}\,A_{HA}}\right) + \hat{e}_{HC}\left(\frac{F_{HC}\,L_{HC}}{E_{HC}\,A_{HC}}\right) + \hat{e}_{HE}\left(\frac{F_{HE}\,L_{HE}}{E_{HE}\,A_{HE}}\right)$$

Data: From Example 4.7 above including unit vectors and the the tension in the wires
Material: Steel with Phosphor bronze coiled wire - with E = 206 GPa
Diameter: 1.166 cm.

The diameter and the modulus of elasticity is the same for all cables.

$$L_{HA} = 11; \quad L_{HC} = 11; \quad L_{HE} = 9.487; \quad [m]$$

$$\delta_{HA} = 0.00267; \quad \bar{\delta}_{HA} = -0.00146\hat{i} - 0.00218\hat{j} + 0.00048\hat{k}\,[m]$$

$$\delta_{HC} = 9.22\times10^{-5}; \quad \bar{\delta}_{HC} = -5.03\times10^{-5}\hat{i} + 7.54\times10^{-5}\hat{j} + 1.677\times10^{-05}\hat{k}\,[m]$$

$$\delta_{HE} = 0.00247; \quad \bar{\delta}_{HE} = 0.00130\hat{i} + 0.00182\hat{j} + 0.00104\hat{k}\,[m]$$

$$\bar{\delta}_{H} = -0.000208\hat{i} - 0.000291\hat{j} + 0.00154\hat{k}\,[m]$$

These are small displacements. What we have done is to use the solution from statics and use strength of materials to complete design information for the problem. Now extend the code in Section 4.3.3 to include the displacement computation.

4.3.5 Example 4.8 - A Different Design

After the prototype was installed it was found that when the wind blew from the left, the wind drag would cause a loss of tension in the cable HC and excessive swings in the traffic light. A simple idea was to install a fourth pole on the fourth corner and use a another cable - HK. The figure below describes the additional pole and its location. The remaining information is the same as in Example 4.7. Do you anticipate a difficulty in obtaining the solution. Your instinct should suggest that there are four unknowns and only three equilibrium equations. Our procedure is the same except the we will include the extra force F_{HK}.

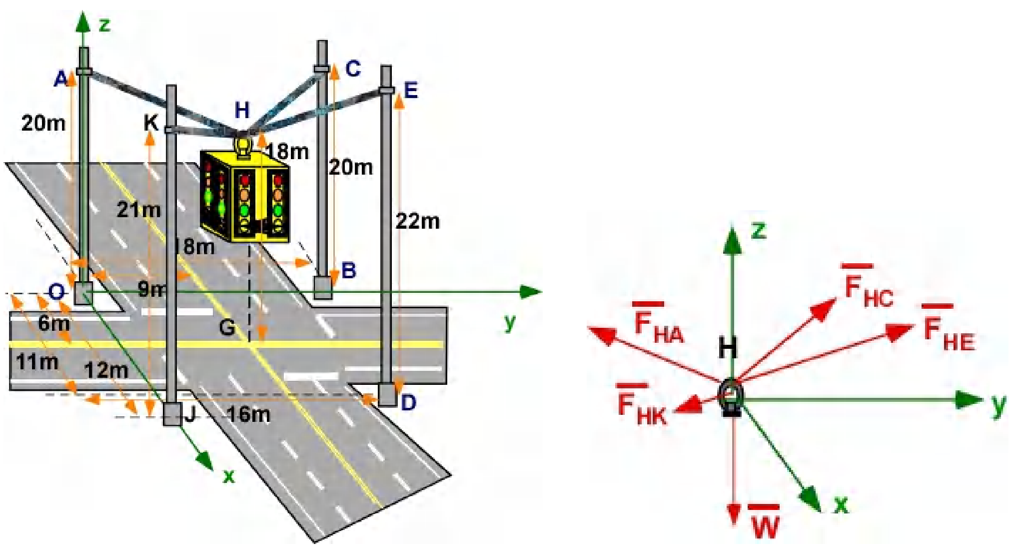


Figure 4.3.3 Example 4.8 and FBD

Data:

m = 350 [kg]; The coordinates of the various points of interest are
O(0, 0, 0); A(0, 0, 20); B(0, 18, 0); C(0, 18, 20); D(11, 16, 0); E (11, 16, 22); G(6, 9, 0);
H(6, 9, 18); J(12, 0,0), **K(12,0,21)** (all in [m])

Find: The forces in the four cables.

Assumptions: Particle Equilibrium

Solution:

Calculating additional unit vector:

$$\hat{e}_{HK} = \frac{i(12-6)+j(0-9)+k(21-18)}{\sqrt{6^2+9^2+3^2}} = i(-0.5345)+j(-0.8018)+k(0.2673)$$

Applying Equilibrium:

$$\sum \bar{F} = \bar{F}_{HA} + \bar{F}_{HC} + \bar{F}_{HE} + \bar{F}_{HK} + \bar{W} = 0; \quad \sum F_x = 0 = \sum F_y = \sum F_z$$

$$\sum F_x = 0 = F_{HA}(-0.5454) + F_{HC}(-0.5454) + F_{HE}(0.5270) + F_{HK}(0.5345) = 0 \quad (1)$$

$$\sum F_y = 0 = F_{HA}(-0.8181) + F_{HC}(0.8181) + F_{HE}(0.7379) + F_{HK}(-0.8018) = 0 \quad (2)$$

$$\sum F_z = 0 = F_{HA}(0.1818) + F_{HC}(0.1818) + F_{HE}(0.4216) + F_{HK}(0.2763) - 3433.5 = 0 \quad (3)$$

Some of you would have recognized that we would reach a dead end before we started to analyze the problem. The equilibrium equation for PARTICLES gives us only - two equations for a 2D problems – and three equations for a 3D problem. We need another equation or look to a **design requirement** to come up with a solution.

New Design Requirement: The maximum force in the cables should not exceed 5000 [N].

Solution continued:

Let us express the equations with the columns containing $\mathbf{F_{HE}}$ to the right and then express the equations in matrix form:

$$F_{HA}(-0.5454) + F_{HC}(-0.5454) + F_{HE}(0.5270) = -F_{HK}(0.5345)$$

$$F_{HA}(-0.8181) + F_{HC}(0.8181) + F_{HE}(0.7379) = -F_{HK}(-0.8018)$$

$$F_{HA}(0.1818) + F_{HC}(0.1818) + F_{HE}(0.4216) = -F_{HK}(0.2763) + 3433.5$$

$$\begin{bmatrix} -0.5454 & -0.5454 & 0.5270 \\ -0.8181 & 0.8181 & 0.7379 \\ 0.1818 & 0.1818 & 0.4216 \end{bmatrix} \begin{bmatrix} F_{HA} \\ F_{HC} \\ F_{HE} \end{bmatrix} = \begin{bmatrix} -F_{HK}(0.5345) \\ F_{HK}(0.8018) \\ 3433.5 - F_{HK}(0.2763) \end{bmatrix}$$

We can solve for F_{HA}, F_{HC}, F_{HE} for different values F_{HK}. We can generate the data for different values of $\mathbf{F_{HK}}$.

F_{HK} [N]	F_{HA} [N]	F_{HC} [N]	F_{HE} [N]
0	5369.05	185.14	5748.18
100	5299.40	280.73	5673.61
500	5020.51	663.11	5375.32

1000	4672.51	1141.08	5002.46

Such calculations are easily done through MATLAB.

4.3.6 Example 4.8 using MATLAB

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 4.3.4- Example 4.8
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 4.8 \n')
fprintf('-----------------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data (in meters)
syms FHA FHC FHE

m = 350;
% coordinates
O = [0 0 0];  A = [0 0 20];  B = [0 18 0];  C = [0 18 20];
D = [11 16 0];  E = [11 16 22];  G = [6 9 0]; H = [6 9 18];
K = [12 0 21];
%% Unknowns
syms FHA FHC FHE FHK real

%% Calculations
% Unit vectors
HA = A - H;  eHA = HA/norm(HA);
HC = C - H;  eHC = HC/norm(HC);
HE = E - H;  eHE = HE/norm(HE);
HK = K - H;  eHK = HK/norm(HK);

W = m*9.81*[0 0 -1];

%% equilibrium
sumF = FHA*eHA + FHC*eHC + FHE*eHE + FHK*eHK + W;
%% solve for FHA, FHC, FHE in terms of FHK
sol = solve(sumF,FHA,FHC,FHE);

FHAv = vpa(sol.FHA,3);
FHCv = vpa(sol.FHC,3);
FHEv = vpa(sol.FHE,3);
%% Printing
fprintf('m [kg]              : '),disp(m)

fprintf('\nPoint O [m] : '),disp(O)
fprintf('Point A [m] : '),disp(A)
fprintf('Point B [m] : '),disp(B)
fprintf('Point C [m] : '),disp(C)
fprintf('Point D [m] : '),disp(D)
fprintf('Point E [m] : '),disp(E)
fprintf('Point G [m] : '),disp(G)
```

```
fprintf('Point H [m] : '),disp(H)
fprintf('Point K [m] : '),disp(K)

fprintf('\nUnit vector: eHA : '),disp(eHA)
fprintf('Unit vector: eHC : '),disp(eHC)
fprintf('Unit vector: eHE : '),disp(eHE)

fprintf('----------------------------------------\n')
fprintf('Equilibrium\n')
fprintf('-----------------------------------------\n')
fprintf('eq(1) : '),disp(vpa(sumF(1),3))
fprintf('eq(2) : '),disp(vpa(sumF(2),3))
fprintf('eq(3) : '),disp(vpa(sumF(3),3))

fprintf('\n----------------------------------------\n')
fprintf('Solution in terms of FKK\n')
fprintf('-----------------------------------------\n')
fprintf('\nForce (HA) [N])             : '),disp(FHAv)
fprintf('Force (HC) [N])             : '),disp(FHCv)
fprintf('Force (HE) [N])             : '),disp(FHEv)

%% Assume values for Fhe and obtain the other values
Fhk = [0,100,500,1000];

for i = 1:length(Fhk);
    Fha(i) = double(subs(sol.FHA,Fhk(i)));
    Fhc(i) = double(subs(sol.FHC,Fhk(i)));
    Fhe(i) = double(subs(sol.FHE,Fhk(i)));
end

fprintf('\n       Fhk [N]      Fha [N]      Fhc [N]      Fhe {N]\n')
disp([Fhk',Fha',Fhc',Fhe'])

fprintf('\n')
```

In the Command Window

```
----------------------------------------
Example 4.8
-----------------------------------------
m [kg]               :    350

Point O [m] :      0      0      0
Point A [m] :      0      0     20
Point B [m] :      0     18      0
Point C [m] :      0     18     20
Point D [m] :     11     16      0
Point E [m] :     11     16     22
Point G [m] :      6      9      0
Point H [m] :      6      9     18
Point K [m] :     12      0     21

Unit vector: eHA :      -0.54545     -0.81818      0.18182
Unit vector: eHC :      -0.54545      0.81818      0.18182
Unit vector: eHE :       0.52705      0.73786      0.42164
----------------------------------------
```

```
Equilibrium
-----------------------------------------
eq(1) : 0.527*FHE - 0.545*FHC - 0.545*FHA + 0.535*FHK
eq(2) : 0.818*FHC - 0.818*FHA + 0.738*FHE - 0.802*FHK
eq(3) : 0.182*FHA + 0.182*FHC + 0.422*FHE + 0.267*FHK - 3433.0

-----------------------------------------
Solution in terms of FKK
-----------------------------------------

Force (HA) [N])            : 5377.0 - 0.697*FHK
Force (HC) [N])            : 0.956*FHK + 185.0
Force (HE) [N])            : 5755.0 - 0.746*FHK

       Fhk [N]      Fha [N]      Fhc [N]      Fhe {N]
           0        5369.1       185.14       5748.2
         100        5299.4       280.73       5673.6
         500        5020.8       663.11       5375.3
        1000        4672.5       1141.1       5002.5
```

Execution in Octave

The code is the same as in MATLAB except for the additional statements below . The changes are highlighted. You must include the symbolic package and if you do not wish to see warnings you include the command warning off as shown

```
clc, clear, format compact, close all, format short G, warning off

pkg load symbolic
sympref display flat
```

In Octave Command Window

```
-----------------------------------------
Example 4.8
-----------------------------------------
Symbolic pkg v2.7.1: Python communication link active, SymPy v1.3.
m [kg]                : 350

Point O [m] :    0   0   0
Point A [m] :    0   0   20
Point B [m] :    0   18   0
Point C [m] :    0   18   20
Point D [m] :    11   16   0
Point E [m] :    11   16   22
Point G [m] :    6   9   0
Point H [m] :    6   9   18
Point K [m] :    12   0   21

Unit vector: eHA :   -0.54545  -0.81818  0.18182
Unit vector: eHC :   -0.54545  0.81818  0.18182
Unit vector: eHE :   0.52705  0.73786  0.42164
-----------------------------------------
Equilibrium
-----------------------------------------
eq(1) :   -0.545*FHA - 0.545*FHC + 0.527*FHE + 0.535*FHK
```

```
eq(2) :   -0.818*FHA + 0.818*FHC + 0.738*FHE - 0.802*FHK
eq(3) :   0.182*FHA + 0.182*FHC + 0.422*FHE + 0.267*FHK - 3.43e+3

----------------------------------------
Solution in terms of FKK
-----------------------------------------

Force (HA) [N])            :   -0.697*FHK + 5.37e+3
Force (HC) [N])            :   0.956*FHK + 185.0
Force (HE) [N])            :   -0.746*FHK + 5.75e+3

       Fhk [N]      Fha [N]      Fhc [N]      Fhe {N]
  0  5369.1  185.14  5748.2
  100  5299.4  280.73  5673.6
  500  5020.8  663.11  5375.3
  1000  4672.5  1141.1  5002.5
```

The results are the same as in MATLAB.

Summary:
The two problems we have solved makes the case that particle equilibrium requires you to:

- Draw the FBD of the Particle
- Apply the Equations of Equilibrium
- Solve for the Unknowns using the Equations of Equilibrium
- If there are more Unknowns than Equations then use Design requirements to create additional Equations
- Ensure that the Solution makes sense and is reasonable
- The unknowns do not have to be forces. It can be distance, other properties, areas etc.

4.3.7 Example 4.9 - Another Three Point Suspension

The large circular sculpture weighing 1500 N is suspended from the ceiling using three nylon cables. The sculpture is horizontal and is of uniform mass. The radius of the structure **r** is 1.2 m. The vertical suspension **d** is 1.5 m. The cables are attached at A, B, and C. Their locations are measured counterclockwise from the x-axis and are 45, 150, and 240 degrees respectively. Calculate the maximum diameter of the nylon cable to be used with a factor of safety (FOS) of 10. The properties of the rope are in Table 4.1

Table 4.1 Strength of nylon rope

Rope diameter (mm)	Breaking Strength (kN)
6	6.61
8	10.2
10	14.4
12	25.2
14	32.0
16	39.5
18	56.8

Courtesy: The Engineering Toolbox

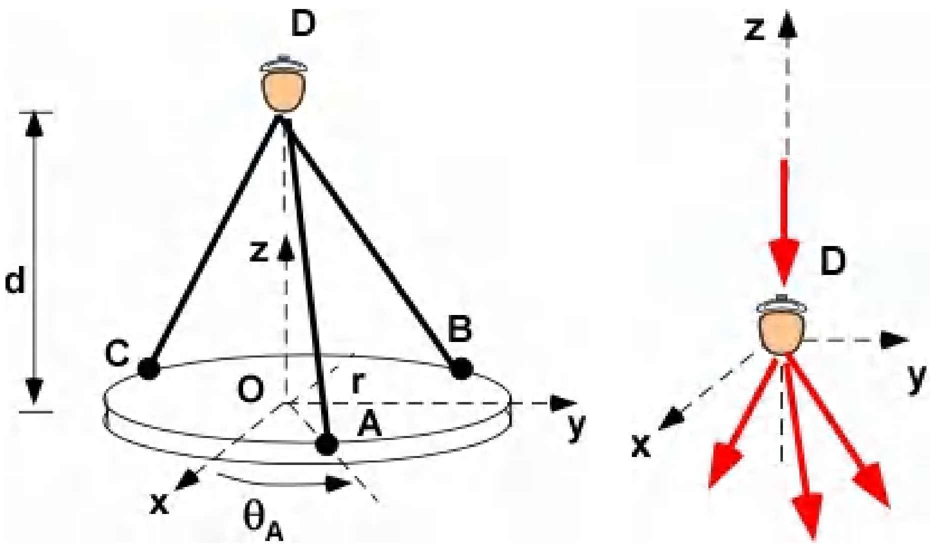


Figure 4.3.4 Example 4.8 and FBD

Data: W = 1500 N, r =1.2 m; d = 1.5 m; θ_A = 45 deg; θ_B = 150 deg; θ_C = 240 deg; FOS = 10;
O : [0,0,0]; A = [1.2*cos(45), 1.2*sin(45),0]; B=[1.2*cos(150), 1.2*sin(150),0];
C = [1.2*cos(240), 1.2*sin(240),0]; D = [0,0,1.5];

Assumption: The weight of the sculpture is at O and passes through D making this a concurrent force example.

Solution:
Calculating unit vectors:

$$e_{AD} = \frac{(-0.848)i + (-0.848)j + 1.5k}{\sqrt{0.848^2 + 0.848^2 + 1.5^2}} = -0.442i - 0.442j + 0.781k$$

$$e_{BD} = \frac{(1.039)i + (-0.6)j + 1.5k}{\sqrt{1.309^2 + 0.6^2 + 1.5^2}} = 0.541i - 0.312j + 0.781k$$

$$e_{CD} = \frac{(0.6)i + (1.309)j + 1.5k}{\sqrt{1.309^2 + 0.6^2 + 1.5^2}} = 0.312i + 0.541j + 0.781k$$

Equilibrium:

$$\sum \bar{F} = F_{ad}\,\hat{e}_{AD} + F_{bd}\,\hat{e}_{BD} + F_{cd}\,\hat{e}_{CD} - W\hat{k} = 0$$

$$\sum F_x = 0.541 * Fbd \;-\; 0.442 * Fad \;+\; 0.312 * Fcd = 0$$

$$\sum F_y = 0.541 * Fcd \;-\; 0.312 * Fbd \;-\; 0.442 * Fad = 0$$

$$\sum F_z = 0.781 * Fad \;+\; 0.781 * Fbd \;+\; 0.781 * Fcd \;-\; 1500.0 = 0$$

Matrix form:

$$\begin{bmatrix} -0.442 & 0.541 & 0.312 \\ -0.442 & -0.312 & 0.541 \\ 0.781 & 0.781 & 0.781 \end{bmatrix} \begin{bmatrix} F_{ad} \\ F_{bd} \\ F_{cd} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1500 \end{bmatrix}$$

Solution for forces:

$$F_{ad} = 863.44[N]; \quad F_{bd} = 223.48[N]; \quad F_{cd} = 834[N]$$

Design for cable:

Maximum breaking force F_{max}

$$F_{max} = FOS \times 863.44 = 8634.4[N]$$

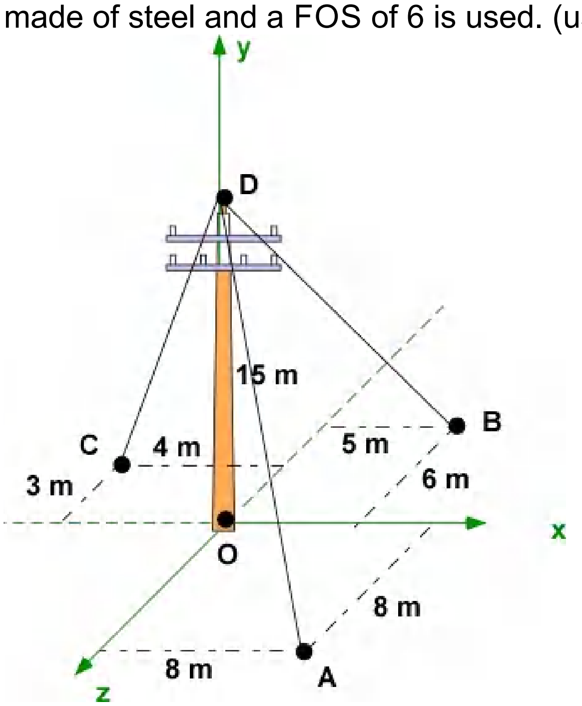
From Table 4.1 **diameter of rope = 8 mm.**

4.3.8 Additional Problems

Solve the following problems on paper and using MATLAB/Octave. For each problem you must draw the FBD with the coordinate system

Problem 4.3.1

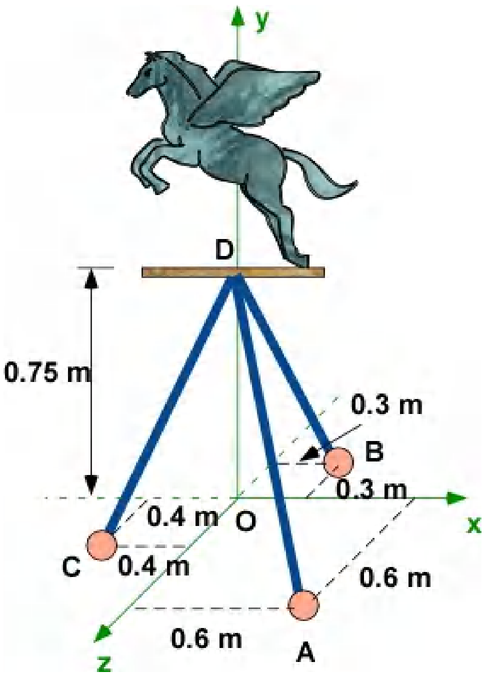
A transmission pole is held down by three cables attached as shown. The pole sees a compressive load of 1500 N. Calculate the force in the cables. What should be their diameter of the cables are made of steel and a FOS of 6 is used. (use your own research for cable strength).



Problem 4.3.1

Problem 4.3.2

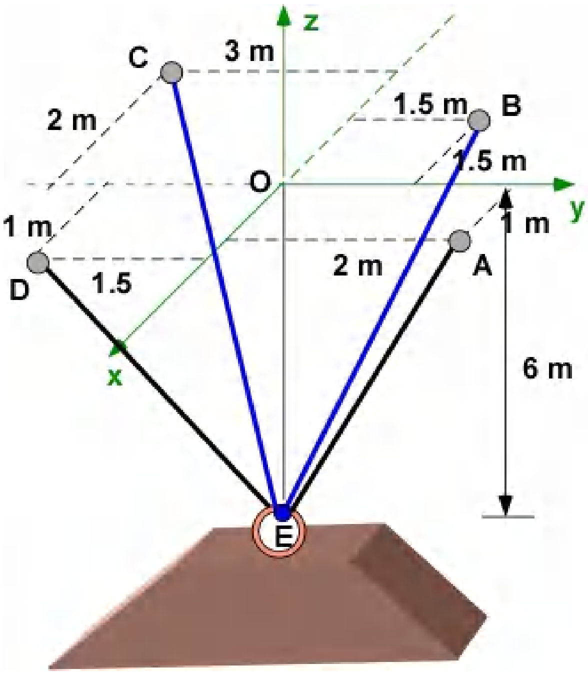
The stone statue of the horse weighs 450 N. It is mounted so that the weight passes through D. The statue is attached to the minimalist three legged stool shown. Calculate the stress in the stool's legs if all the rods have the same diameter of 25 mm.



Problem 4.3.2

Problem 4.3.3

A heavy structure is suspended by three cables mounted on the ceiling. The segment CE and BE belong to a single cable that wraps around the pulley at E. DE and AE are independent cables. The force in the cable AE is 2500 N. Calculate the force in the other cables and determine the weight of the structure.



Problem 4.3.3

5. RIGID BODY

The rigid body is different than a particle that we have seen in the last chapter. Consider the following illustration where you are exerting a force

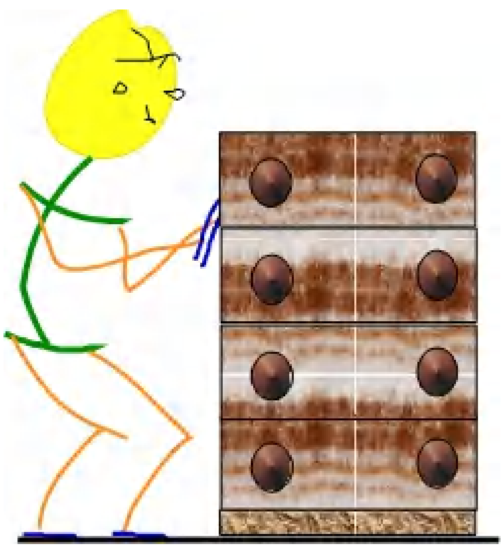


Figure 5.1 Pushing a tall heavy chest of drawers

Take a deep breadth and push harder

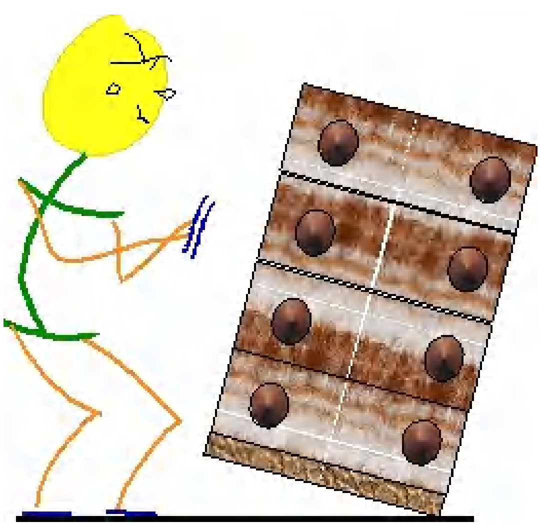


Figure 5.2 Chest wants to roll

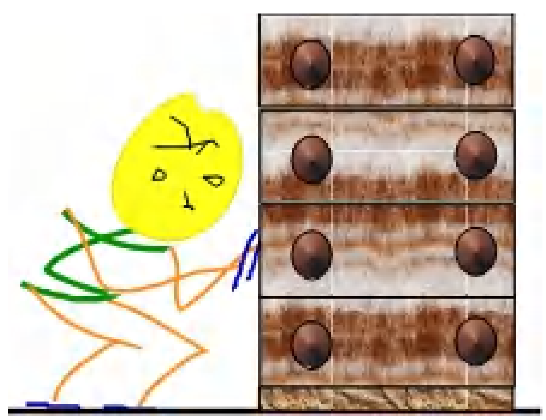


Figure 5.3 Push at a lower point and it moves

The force, or the line of action of the force, on the body matters. It can create translation or rotation. If this was idealized as a particle, you cannot create a rolling moment because the object has no size.

A rigid body is therefore an idealization that has ***size*** and ***shape*** and they matter.

The idealization of the rigid body allows us to calculate the change in the orientation of the object - or the **rotation** of the object. This change is brought about by a quantity called the **Moment.** The handling of forces on the rigid body is the same dealing with forces on a particle. In the next few sections we focus on the **moment**.

5.1 MOMENT IN TWO DIMENSIONS

(This material appeared earlier in Section 2.11 and is reproduced here for convenience and reviewing important information. We will also renumber the equations for this chapter for convenience and consistency)

The moment is the action of a force acting at a distance from a point on the rigid body. Consider the illustration below where we have a horizontal force P acting at C. The object has a mass center at G. It is a **rigid body** with size and shape (rectangular). A and B are the corners of the object in contact with the floor.

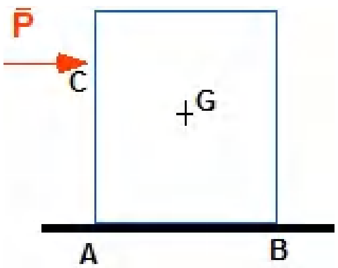


Figure 5.1.1 Force on the chest

The definition of a moment is the physical quantity that causes the object to rotate. It is associated by a **force *acting at a distance.*** This formal distance lacks clarity as you can see in Figure 5.1.2. Several distances can be referred with respect to three points G, A, or B and are shown using the letter d with appropriate subscripts.

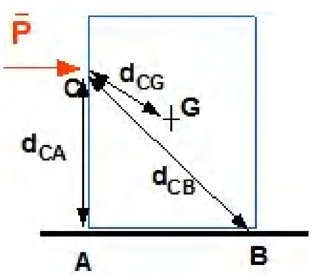


Figure 5.1.2 Force acting at a distance

We can modify the definition by considering that the **Moment of the force** about a **point** is the product of the force and the **shortest distance between the point and the force.** This brings forth another idea - **the line of action** (LOA) of the force. This is the line that is parallel to the force. The forced can be placed any where on this line and cause the same action on the object. This also means that the Moment caused by the force will not change as it is moved along the line of action.Figure 5.1.3 illustrates the line of action. It does not matter where the force is placed as far the calculations are concerned. We can move the force along this line to take advantage of the geometry.

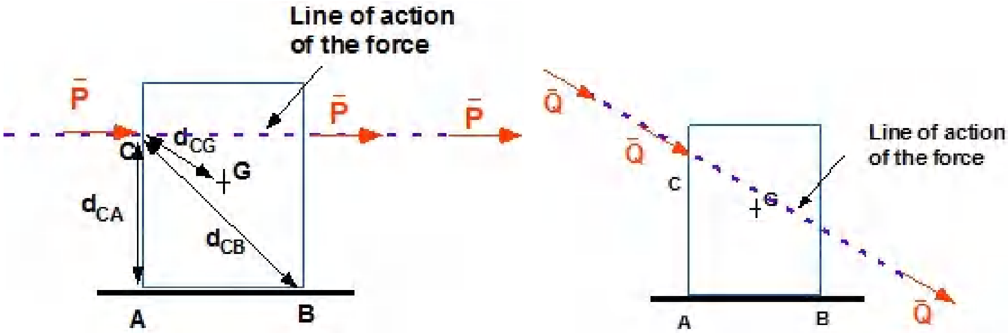


Figure 5.1.3 Line of action

Since the object wants to rotate about the point B, let us calculate the moment of the force P about the point B. We start with the idealized figure

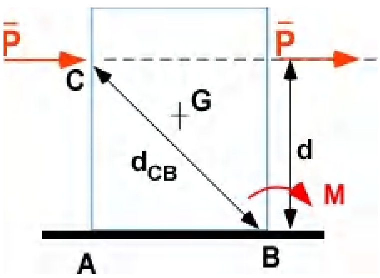


Figure 5.1.4 Calculating Moment

Sliding the force along its line of action the shortest distance appears as d. We notice it is the perpendicular distance between the point B and the line of action (LOA)

M = P d (5.1)

The value (magnitude) of M is product of the magnitude of the force P and the distance d. M is magnitude of the moment of P about B. The dimension for [M] is the product of force and distance. The basic units for the Moment M is [N-m] or [lb-ft]. But Moment **M** is a **vector**!

The direction of M is normal to the plane formed by the force **P** and **d** (it is a plane that holds the P vector and the distance d) . Since there are two choices here, the correct one is determined as follows: Roll the fingers of the right hand in the same sense as the rotation induced by this moment while resting on this plane: the thumb will point in the direction of **M.**

Therefore, the M vector is directed into the plane of the screen you are watching – while P and d are in the plane of the screen. This is best described using a coordinate system even though the moment is independent of the coordinate system.

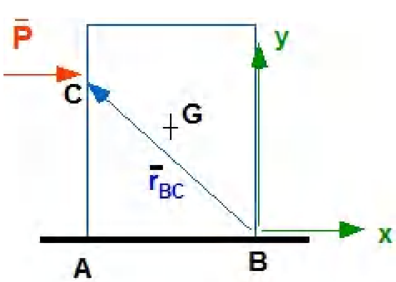


Figure 5.1.5 Moment vector

The more formal calculation for the vector moment **M** is the cross product of the position vector $\mathbf{r_{BC}}$

[from B (point about which you are computing the vector moment) to C (the point where the force is applied or passes through)] and the force vector **P.** Therefore

$$\bar{M} = \bar{r}_{BC} \times \bar{P} \qquad (5.2)$$

This is a vector cross-product and the result is a vector. This is valid for calculating **M** in both 2D and 3D problems. This gives you the vector **M.** The illustration for the vector product is in the figure below. You have to place the vectors tail to tail for the angle θ in the calculations. The moment for this example is in the - **k** direction - normal to the plane determined by the two vectors $\mathbf{r_{BC}}$ and **P**. Even though the moment is a vector it is important to distinguish it from other vectors because it creates a different action on the body.

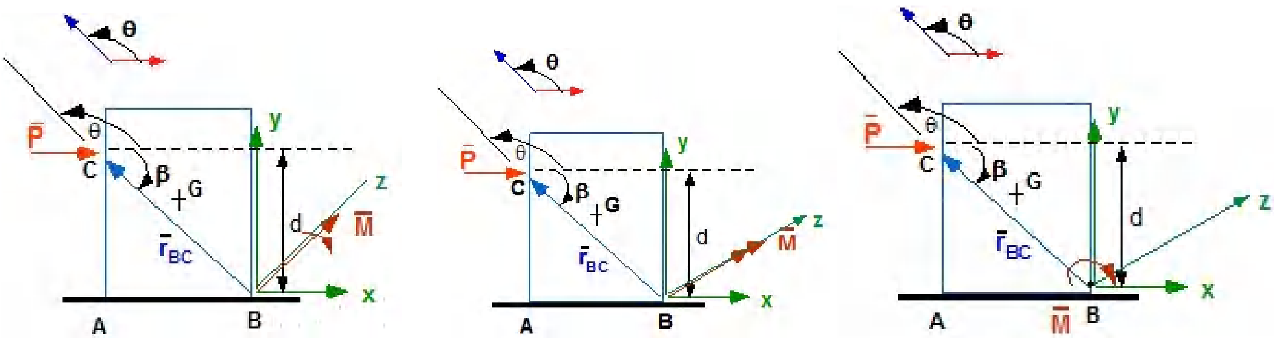


Figure 5.1.6 Three ways to indicate a moment.

We have two vector quantities that create action on a body. **P**, a force likes to move the body. **M**, a moment likes to cause rotation of the body. They also have different dimensions and we must be cautious that ***we do not add them together***. We must have a way of distinguishing them in the figures we draw since our analysis will be based on these figures.

The calculation of the vector **M** and the magnitude M can be summarized as:

$$\bar{M} = \bar{r}_{BC} \times \bar{P} = |\bar{r}_{BC}||\bar{P}|\sin\theta$$
$$M = Pd = P(r_{BC}\sin\theta) = P(r_{BC}\sin(180-\theta)) = P(r_{BC}\sin\beta) \qquad (5.3)$$

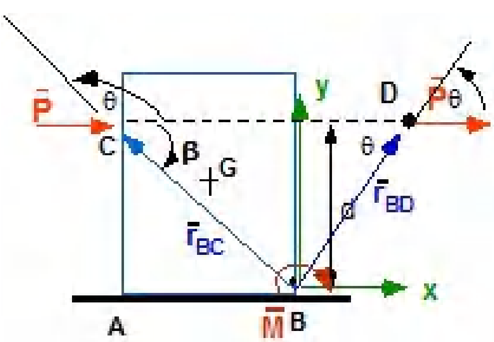


Figure 5.1.7 Same Moment along line of action

$$\bar{M} = \bar{r}_{BD} \times \bar{P}$$
$$M = r_{BD} P \sin\theta = P(r_{BD}\sin\theta) = Pd \qquad (5.4)$$

Another way to calculate the moment

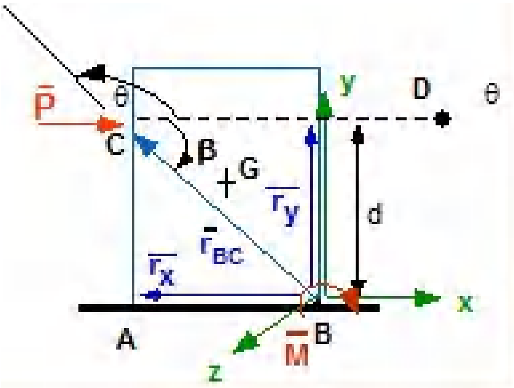


Figure 5.1.8 Cross product using determinant

$$\bar{M} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -r_x & r_y & 0 \\ P & 0 & 0 \end{vmatrix} = \boldsymbol{i}(0-0)+\boldsymbol{j}(0-0)+\boldsymbol{k}(0-r_y P) = -\boldsymbol{k}(Pd)$$

(5.5)

5.1.1 Example 5.1 (also Example 2.3)

You do not like the angle bracket attached to the wall. You are going to pull on the vertical edge at the point shown, with a force of 120 [N], at an angle θ of 20 [deg], to see if you can break it at the wall (in mechanics you will learn that it is easy to break stuff through moments). Find the moment of the force at the wall.

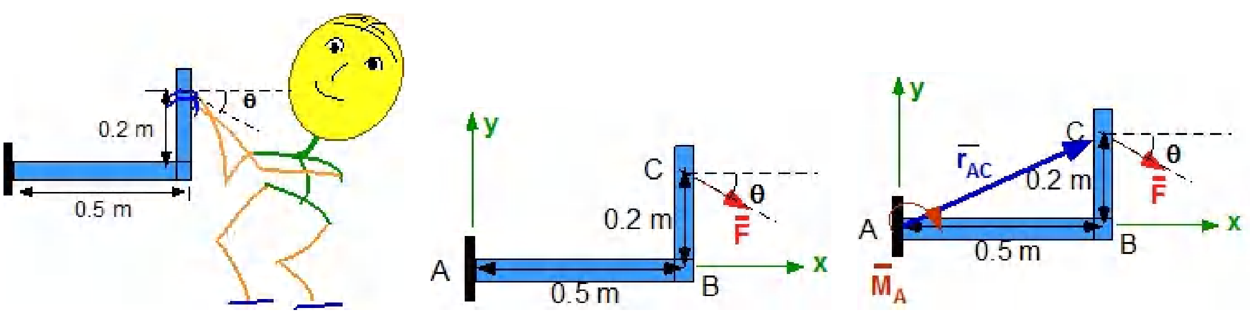


Figure 5.1.9 Example 5.1 (original, simplified, useful)

Data: F = 120 [N]; θ = 20 [deg];
Find: Moment of F at the point A (M_A)
Solution: Using the last figure above

$$\bar{M}_A = \bar{r}_{AC} \times \bar{F}$$

$$\bar{r}_{AC} = 0.5\hat{i} + 0.2\hat{j}\,[m];$$

$$\bar{F} = 120\cos(20)\hat{i} - 120\sin(20)\hat{j} = 112.76\hat{i} - 41.04\hat{j}\,[N]$$

$$\bar{M}_A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.5 & 0.2 & 0 \\ 112.76 & -41.04 & 0 \end{vmatrix}$$

$$= \hat{k}[(0.5)(-41.04) - (0.2)(112.76)] = \hat{k}(-43.07)[Nm]$$

The moment is directed in the **-k** direction (into the page).

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.1.1- Example 5.1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 5.1\n')
fprintf('-----------------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data (in meters)
%% Data
F = 120;
theta = 20;  thetr = theta*pi/180; % in radians
A = [0,0,0];  B = [0.5,0,0];  C = [0.5,0.2,0];
%% calculations
rAC= C - A; % position vector from A to C
Fvect = [F*cos(thetr),-F*sin(thetr),0];
MA1= cross(rAC,Fvect);

%% Printing
fprintf('Point A [m] : '),disp(A)
fprintf('Point B [m] : '),disp(B)
fprintf('Point C [m] : '),disp(C)

fprintf('Position vector AC [m]   = '),disp(rAC)
fprintf('Force vector at C [N]    = '),disp(Fvect)
fprintf('Moment of F about A [Nm] = '),disp(MA1)
```

In the Command Window

```
----------------------------------------
Example 5.1
-----------------------------------------
Point A [m] :       0      0      0
Point B [m] :            0.5             0             0
```

```
Point C [m] :           0.5           0.2           0
Position vector AC [m]    =           0.5           0.2           0
Force vector at C [N]     =        112.76       -41.042           0
Moment of F about A [Nm] =            0             0       -43.074
```

Execution in Octave

The code is the same as in MATLAB.

In Octave Command Window

```
----------------------------------------
Example 5.1
-----------------------------------------
Point A [m] :   0  0  0
Point B [m] :   0.5  0  0
Point C [m] :   0.5  0.2  0
Position vector AC [m]    =   0.5  0.2  0
Force vector at C [N]     =   112.76  -41.042  0
Moment of F about A [Nm] =   0  0  -43.074
```

The solutions are the same as in MATLAB

5.1.2 Example 5.1 - Moving force along line of action

Solve Example 5.1 by moving the force along the line of action to D (see Figure).

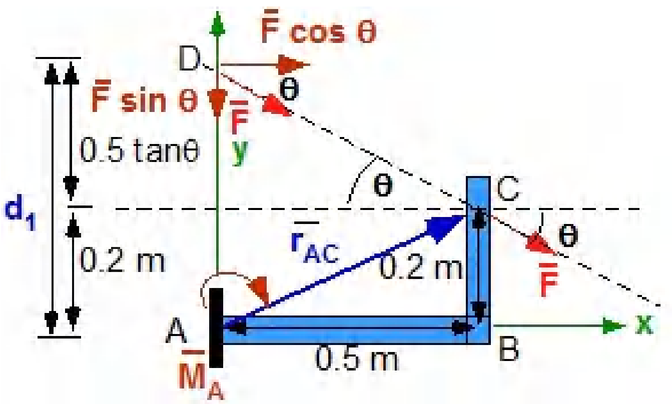


Figure 5.1.10 Force at D

Move the force F along the line of action till it intersects the y axis at D. The y-component of the force at D passes through A and will not create a moment about A (because there is no moment arm). The x-component of the force at D is perpendicular to the line (or the moment arm) AD. From the geometry in the figure the magnitude of the moment is

$$M_A = d_1(F\cos\theta) = (0.2 + 0.5\tan 20)(120\cos 20) = 43.074[Nm]$$

The moment causes the fingers of the right hand roll from D to B about A and the thumb points into the screen that is in the **– k** direction.

5.1.3 Example 5.1 - Moving force to another point along line of action

Solve Example 5.1 by moving the force along the line of action to E (see Figure).

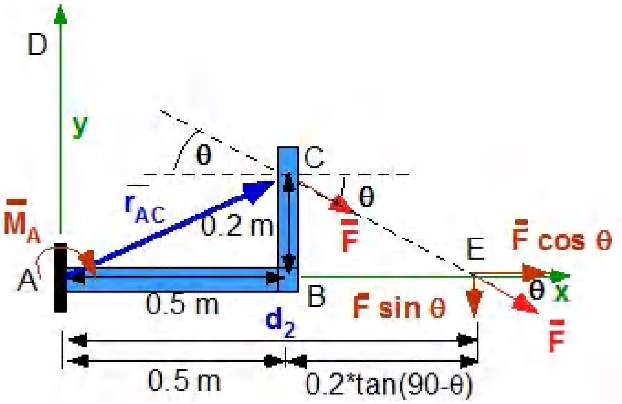


Figure 5.1.11 Force at E

Move the force F along the line of action till it intersects the x axis at E. The x-component of the force at E passes through A and will not create a moment about A (because there is no moment arm). The y-component of the force at E is perpendicular to the line (or the moment arm) AE. From the geometry in the figure the magnitude of the moment is

$$M_A = d_1 F \sin(90-\theta) = (05 + 0.2\tan 70)(120\sin 20) = 43.074[Nm]$$

$$\bar{M}_A = 43.074(-\hat{k})[Nm]$$

5.1.4 Example 5.1 - Moment using shortest distance

Calculate the moment using the shortest distance between A and the line of action of the force.

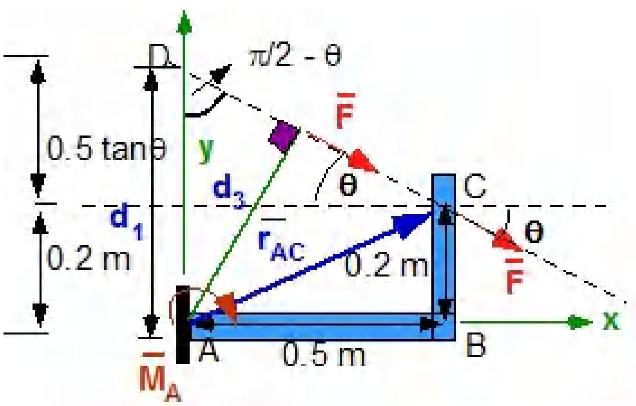


Figure 5.1.12 Shortest distance

From the geometry in the figure the magnitude of the moment can be obtained as:

$$M_A = d_3 F = d_1 \sin(90-\theta) F = (0.2 + 0.5\tan 20)\sin(90-20)\,120 = 43.074[Nm]$$

The direction is obtained using the right hand.

5.1.5 Example 5.1 - Scalar implementation of vector multiplication

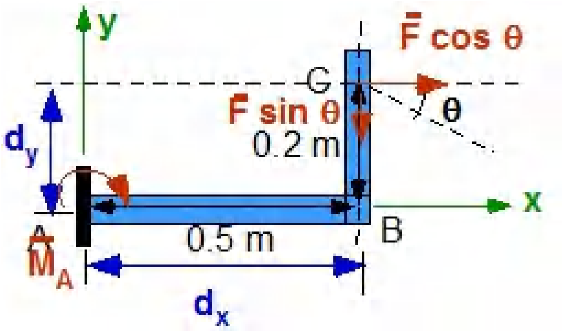


Figure 5.1.13 Vector multiplication scalar implementation

First resolve the force components. Calculate moment by each component by the multiplication of force and its shortest distance from A. If they produce roll in the same direction they will add. Moment is positive if it is the positive coordinate direction.

$$M_A = -d_x F\sin\theta - d_y F\cos\theta = -0.5*120*\sin 20 - 0.2*120*\cos(20) = -43.074\,[Nm]$$

Summary of these various methods:

Method 1 is the most direct and straight forward. It works for both 2D and 3D problems. It provides both the magnitude and the direction of the moment - In other words it calculates the Vector. It has a simple mathematical basis of applying the cross-product

Method 2, 3, 4 require construction, geometry, and trigonometry. It challenges your visualization skills but will develop intuition. It is easy in 2D problems but not so easy for 3D problems. 3D visualization is a challenge for all of us.

Methods 2 through 5 are useful in determining the magnitude. The vector direction requires use of the right hand. It is difficult in 3D problems.

Method 4 is challenging as it requires visualization skills and an ease with geometrical reasoning

Method 5 is very useful for 2D problems. In fact it is equivalent to Method 1 for such problems

5.1.6 Moving a Force parallel to itself

A force can only be moved parallel to itself. Hence a force on a rigid body can only be moved in one of two ways. You can move it along the line of action or you can move it to another point on the rigid body.

Moving a force parallel to itself was done in Sections 5.1.2 and 5.1.3 above. It was done to calculate the moment in a simpler way. By moving the force along the line of action the problem remains the same. The effect on the object did not change.

Moving the force parallel to itself to another point on the rigid body was presented in Section 2.11.4. It is reviewed below using an example. This will change the problem unless a moment also accompanies the force at the new location.

Example 5.2: Example 5.2 is the same as Example 5.1 but the force acting at C must be moved to G producing the same effect on the problem.

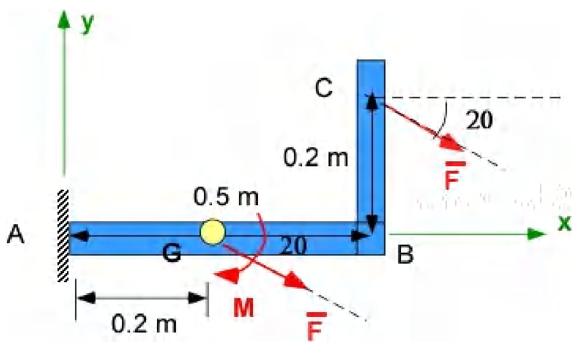


Figure 5.1.14 Example 5.2

Data: F = 120 [N]; θ = 20 [deg]; F is applied at C

Find: Move F to G. The force F at G must be the same in magnitude and direction but now includes a moment **M.** Find moment **M**

Solution:

$$\bar{r}_{GC} = 0.3\hat{i} + 0.2\hat{j}\ [m]$$

$$\bar{F} = 112.76\hat{i} - 41.04\hat{j}\quad [N]$$

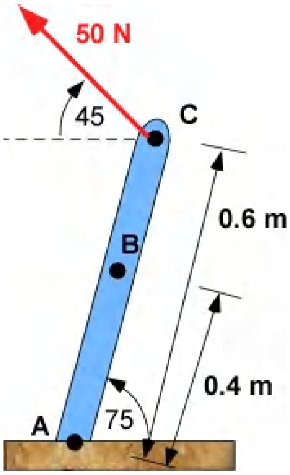
$$\begin{aligned}\bar{M} &= \bar{r}_{GC} \times \bar{F} \\ &= \left[0.3\hat{i} + 0.2\hat{j}\right] \times \left[112.76\hat{i} - 41.04\hat{j}\right] \\ &= (0.3 \times 41.04)\left(-\hat{k}\right) + (0.2 \times 112.76)\left(-\hat{k}\right) \\ &= 34.86\left(-\hat{k}\right)\quad [Nm]\end{aligned}$$

5.1.7 Additional Problems

Solve the following problems on paper and using MATLAB/Octave. For each problem you must draw the position vector and force vector along with the coordinate system

Problem 5.1.1

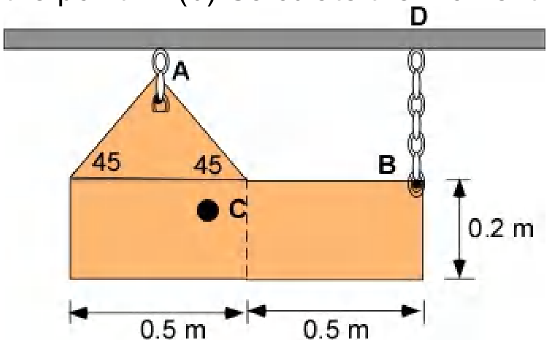
The 50 N force exerts a force of 50 N in the direction shown. (a) Find the moment of this force at A. (b) Move this force from point C to point B. (c) Calculate the moment of the force (now at B) at A. What is the relation between this moment, the moment you computed in part (b), and the moment you computed in part (a)?



Problem 5.1.1

Problem 5.1.2

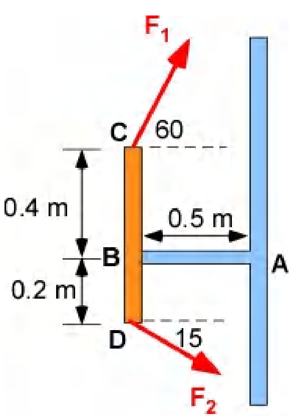
The metal plate is suspended by chains as shown. The mass/area of the metal plate is the same over the area. (a) Calculate the center of mass of the plate. (b) Calculate the moment of this weight about the point D. (b) Calculate the moment of the weight of the plate about the point A.



Problem 5.1.2

Problem 5.1.3

The device has two forces with location and orientation as shown in the figure. $\mathbf{F_1}$ is 100 N and the magnitude of $\mathbf{F_2}$ is 120 N. (a) Move $\mathbf{F_1}$ to A. (b) Move $\mathbf{F_2}$ to A. (c) Calculate the net force and moment at A after the moves.



Problem 5.1.3

5.2 MOMENT IN THREE DIMENSIONS

(This material appeared earlier in Section 2.11 and is reproduced here for convenience and reviewing important information. We will also renumber the equations for this chapter for convenience and consistency)

Moment is calculated by Equation (5.2):

$$\bar{M} = \bar{r}_{BC} \times \bar{P} \qquad (5.2)$$

This is also true for 3D. However here we can expect to see three components of moment instead of only one in the previous section. The application is simple so let us explore this topic through examples.

5.2.1 Example 5.3

In your garden there is an old tree whose branch is starting to decay. You decide to see if you can break of this branch by applying a moment about the point A, after having notched the branch at the point A with a saw. You lasso the branch at the point B (as far from A as possible) using a rope and pull on it (along the direction BC) applying a force of magnitude 350 [N]. What is the moment of this force about the point A? The location of the various points are shown on the figure using the origin at O.

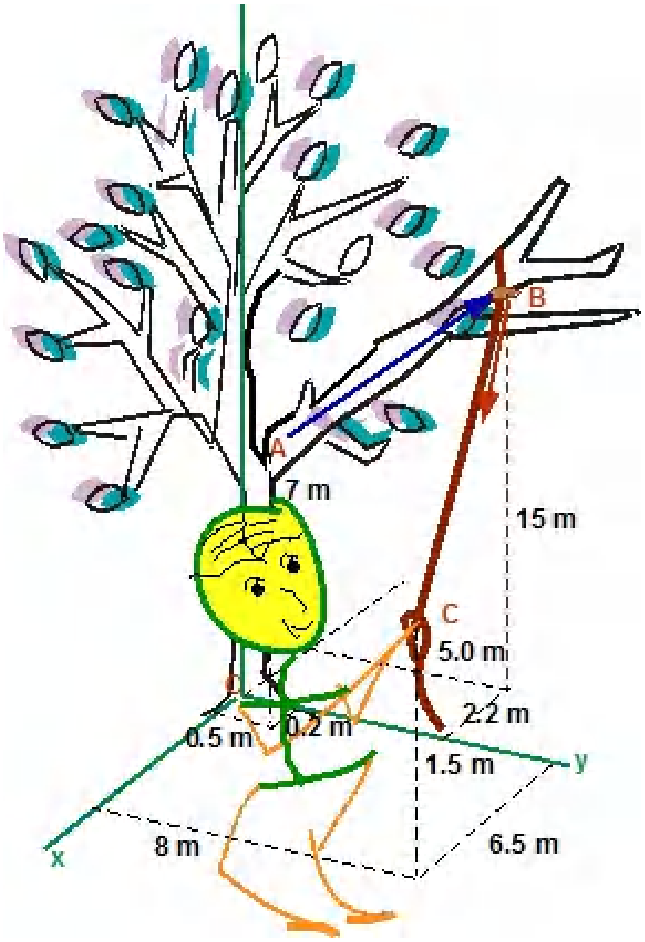


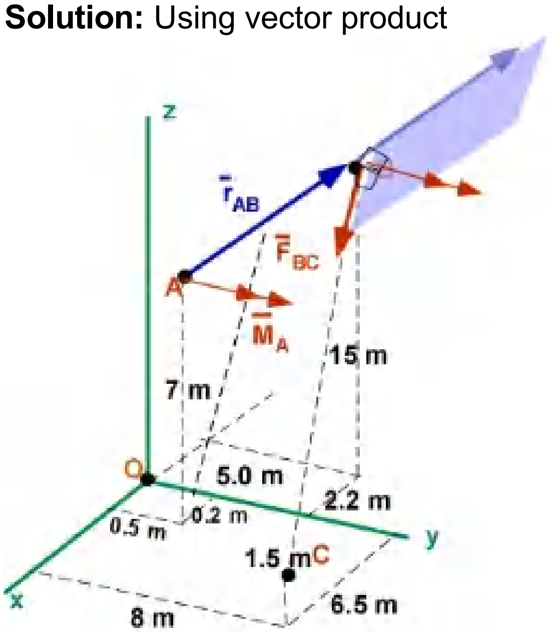
Figure 5.2.1 Example 5.3

Data: Coordinates: O (0, 0, 0); A (0.2, 0.5, 7); B (-2.2, 5.0, 15); C (6.5, 8, 1.5). All dimensions in [m]. F is along BC and has a magnitude of 350 [N]

Find: Moment of F at the point A

Assumption: None

Solution: Using vector product



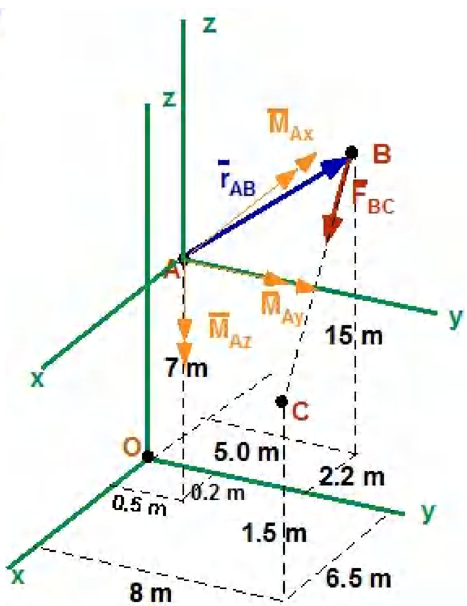


Figure 5.2.2 The vectors and components

Setting up elements of Eqn. (5.2) and calculating the vector product

$$M_A = \bar{r}_{AB} \times \bar{F}_{BC}$$

$$\bar{r}_{AB} = \hat{i}(-2.2-0.2) + \hat{j}(5-0.5) + \hat{k}(15-7)$$

$$= -2.4\hat{i} + 4.5\hat{j} + 8\hat{k}\,[m[$$

$$\bar{F}_{BC} = 350\frac{\hat{i}(6.5-(-2.2)) + \hat{j}(8-5) + \hat{k}(1.5-15)}{\sqrt{8.7^2+3^2+13.5^2}} = 186.372\hat{i} + 64.266\hat{j} - 289.197\hat{k}\,[N[$$

$$M_A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2.4 & 4.5 & 8 \\ 186.37 & 64.27 & -289.2 \end{vmatrix}$$

$$= \hat{i}\left[(-4.5\times 289.2) - (8\times 64.27)\right]$$

$$+ \hat{j}\left[(8\times 186.37) - (2.4\times 289.2)\right]$$

$$+ \hat{k}\left[(-2.4\times 64.27) - (4.5\times 186.37)\right]$$

$$= -1815.5\hat{i} + 796.9\hat{j} - 992.9\hat{k}\,[Nm[$$

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.2.1- Example 5.3
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 5.3\n')
fprintf('----------------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data (in meters)
%% Data
O = [0 0 0]; A = [0.2 0.5 7]; B = [-2.2 5 15]; C = [6.5 8 1.5];
Fv = 350;
%% calculations
rAB = B-A;
rBC = (C - B);
eBC = rBC/norm(rBC);
FBC = Fv*eBC;

MA = cross(rAB,FBC);
MAv = norm(MA);
```

```
%% Printing
fprintf('Point O [m]   : '),disp(O)
fprintf('Point A [m]   : '),disp(A)
fprintf('Point B [m]   : '),disp(B)
fprintf('Point C [m]   : '),disp(C)
fprintf('Force BC [N] : '),disp(Fv)

fprintf('\nPosition vector AB [m]    = '),disp(rAB)
fprintf('Force vector FBC [N]      = '),disp(FBC)
fprintf('Moment of F about A [Nm] = '),disp(MA)
fprintf('Magnitude of the Moment [Nm] = '), disp(MAv)
```

In Command Window

```
----------------------------------------
Example 5.3
-----------------------------------------
Point O [m]  :      0      0      0
Point A [m]  :           0.2          0.5            7
Point B [m]  :          -2.2            5           15
Point C [m]  :           6.5            8          1.5
Force BC [N] :    350

Position vector AB [m]    =         -2.4          4.5            8
Force vector FBC [N]      =       186.37       64.266       -289.2
Moment of F about A [Nm] =       -1815.5        796.9      -992.91
Magnitude of the Moment [Nm] =        2217.4
```

An interesting inclusion in the problem statement was "*after having notched the branch at the point A with a saw*." This is something you may practically do but how does it help. The notch creates a region for stress concentration - more stress than expected. Since this is bending it will be normal stress. This encourages the branch to fail at the point by handling stress beyond the yield point.

5.2.2 Scalar multiplications to calculate the Moment

The vector multiplication in Example 5.3 is natural and easily applied but requires vector algebra. You can also calculate the moment by multiplying the the force times the distance and keeping track of the direction using the right hand rule. This is a more complicated way to calculate the moment in Example 5.3 and is illustrated in Figure 5.2.3

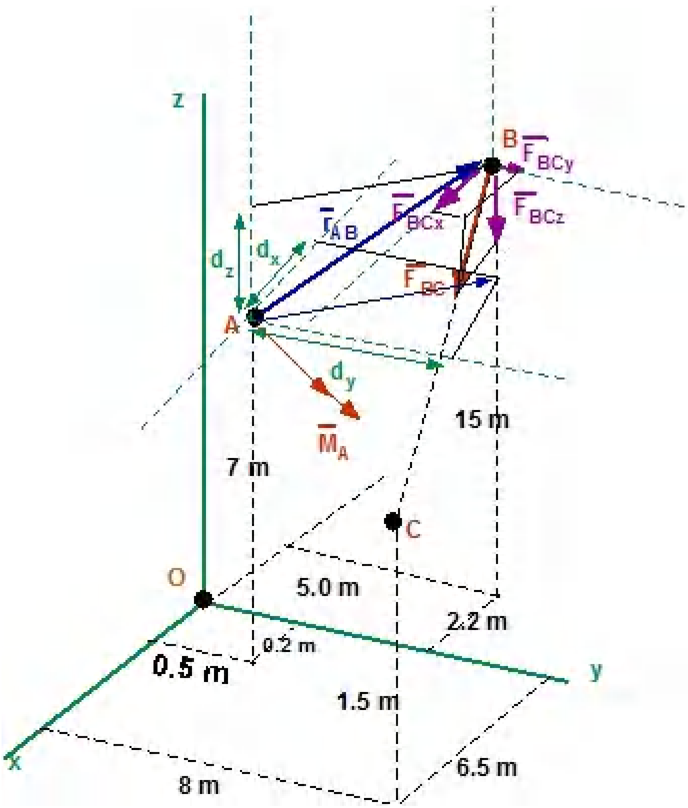


Figure 5.2.3 Scalar multiplication in Example 5.3

$$\bar{M}_A = \hat{i}\left[-\left(F_{BCz}\right)\left(d_y\right)-\left(F_{BCy}\right)\left(d_z\right)\right]+\hat{j}\left[-\left(F_{BCz}\right)\left(d_x\right)+\left(F_{BCx}\right)\left(d_z\right)\right]$$
$$+\hat{k}\left[-\left(F_{BCy}\right)\left(d_x\right)-\left(F_{BCx}\right)\left(d_y\right)\right]$$
$$\bar{M}_A = \hat{i}\left[\left(-4.5\times 289.2\right)-\left(8\times 64.27\right)\right]+\hat{j}\left[-\left(2.4\times 289.2\right)+\left(8\times 186.37\right)\right]$$
$$+\hat{k}\left[\left(-2.4\times 64.27\right)-\left(4.5\times 186.37\right)\right]$$
$$\bar{M}_A = \left(-1815.56\right)\hat{i}+\left(796.88\right)\hat{j}+\left(-992.13\right)\hat{k} \quad [Nm]$$

5.2.3 Example 5.4

The sign in front of the establishment is being battered by the wind. You are concerned that it will be uprooted. The effect of the wind can be reproduced by a concentrated force F, located at A, of magnitude 1500 N and at an angle of α of 55 degrees to the plane of the sign which is in the yz plane. The stiffened sign weighs 980 N and the center of mass of the sign is at B. The lengths L_1, L_2, and L_3 are 6.5 m, 0.6m, and 1.2 m respectively. What is the net moment (resultant moment) at the point O?

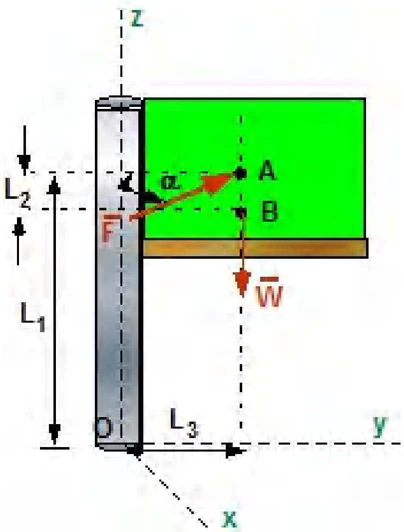


Figure 5.2.4 Example 5.4

Data: F = 1500 [N]; W = 980 [N], α= 55 [deg]; L_1=6.5 [m]; L_2= 0.6 [m]; L_3=1.2 [m]

Find: Net Moment at the point O

Assumption: None

Solution: Moments can be added since they are vectors. We will use the cross product to compute the moments of the forces about O and add them. Draw the displacement vectors to assist in setting up the problem

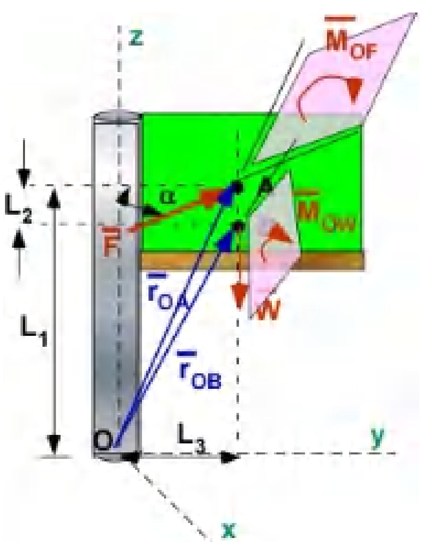


Figure 5.2.5 Plane of the position vector and force

$$\bar{M}_O = \bar{M}_{FO} + \bar{M}_{WO} = \left(\bar{r}_{OA} \times \bar{F}\right) + \left(\bar{r}_{OB} \times \bar{W}\right)$$

$$\bar{r}_{OA} = 1.2\hat{j} + 6.5\hat{k}\,[m[; \quad \bar{r}_{OB} = 1.2\hat{j} + 5.9\hat{k}\,[m[$$

$$\bar{F} = 1500\left(-sin55\hat{i} + cos\,55\,\hat{j}\right) = -1.2287\times10^3\hat{i} + 0.8604\times10^3\,\hat{j}\,[N[$$

$$\bar{W} = -980\hat{k}\,[N[$$

$$\bar{M}_O = \bar{M}_{FO} + \bar{M}_{WO} = \left(\bar{r}_{OA} \times \bar{F}\right) + \left(\bar{r}_{OB} \times \bar{W}\right)$$

$$= \left(1.2\hat{j} + 6.5\hat{k}\right)\times\left(-1.2287\times10^3\hat{i} + 0.8604\times10^3\,\hat{j}\right) + \left(1.2\hat{j} + 5.9\hat{k}\right)\times\left(-980\hat{k}\right)$$

$$\bar{M}_O = (1.2)\left(-1.2287\times10^3\right)\left(\hat{j}\times\hat{i}\right) + (1.2)\left(0.8604\times10^3\right)\cancel{\left(\hat{j}\times\hat{j}\right)} +$$

$$(6.5)\left(-1.2287\times10^3\right)\left(\hat{k}\times\hat{i}\right) + (6.5)\left(0.8604\times10^3\right)\left(\hat{k}\times\hat{j}\right) +$$

$$(1.2)(-980)\left(\hat{j}\times\hat{k}\right) + (5.9)(-980)\cancel{\left(\hat{k}\times\hat{k}\right)}$$

$$\bar{M}_O = (1.2)\left(-1.2287\times10^3\right)\left(\hat{j}\times\hat{i}\right) + (6.5)\left(-1.2287\times10^3\right)\left(\hat{k}\times\hat{i}\right) +$$

$$(6.5)\left(0.8604\times10^3\right)\left(\hat{k}\times\hat{j}\right) + (1.2)(-980)\left(\hat{j}\times\hat{k}\right)$$

$$\hat{j}\times\hat{i} = -\hat{k}; \quad \hat{k}\times\hat{i} = \hat{j}; \quad \hat{k}\times\hat{j} = -\hat{i}; \quad \hat{j}\times\hat{k} = \hat{i}$$

$$\bar{M}_O = -6.7684\times10^3\hat{i} - 7.9867\times10^3\,\hat{j} + 1.4745\times10^3\hat{k}\,[Nm[$$

The moments in the x and y axis will try and bend the pole. The moment along the z axis will cause the pole to twist.
In this example we calculated the moment at a point O at the base of the sign pole due to more than one force. In your experience you may have realized that is easy to break some structure by trying to bend it. That is because moments are able to cause large stresses. In this particular case we can examine the failure due to bending (bending moment) and torsion (torsional moment). While they can influence each other we will regard them as separately and independently applied.

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.2.4- Example 5.4
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-------------------------------------------\n')
fprintf('Example 5.4\n')
fprintf('--------------------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%% Data (in meters)
%% Data
F = 1500;   W = 980;  % the forces
L1 = 6.5;    L2 = 0.6;   L3 = 1.2;  % the distances
alpha = 55;   alphar = alpha*pi/180;  % alpha in radians
%% calculations
O = [0 0 0]; % point O
A = [0 L3 L1];  % point A
B = [0 L3 (L1 - L2)]; % point B

eF = [-sin(alphar) cos(alphar) 0];  % unit vector in F direction
Fw = F*eF ; % vector F
Wv = W*[0 0 -1] ; % vector W

rOB = B - O ; % displacemnet vector from O to B
rOA = A - O  ;% displacement vector from O to A

MF = cross(rOA,Fw); % Moment due to F
MW = cross(rOB,Wv); % Moment due to W
MTot = MF + MW   ;  %Total moment at O
MagMTot = norm(MTot);   % Mangnitude of the total Moment

MTot1 = cross(rOA,(Fw+Wv));
%% Printing
fprintf('Point O [m]  : '),disp(O)
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)

fprintf('\nWeight  [N]  : '),disp(W)
fprintf('Force F [N]  : '),disp(F)
fprintf('alpha [deg]  : '),disp(alpha)

fprintf('\nPosition vector rOB [m]   = '),disp(rOB)
fprintf('Position vector rOA [m]   = '),disp(rOA)
fprintf('Force vector Fw [N]       = '),disp(Fw)
fprintf('Moment due to F [Nm]      = '),disp(MF)
fprintf('Moment due to W [Nm]      = '),disp(MF)
fprintf('Moment of Forces about O [Nm] = '),disp(MTot)
fprintf('Magnitude of the Moment [Nm] = '), disp(MagMTot)
```

In the Command Window

```
----------------------------------------
Example 5.4
-----------------------------------------
Point O [m]  :      0      0      0
Point A [m]  :              0           1.2          6.5
Point B [m]  :              0           1.2          5.9

Weight  [N]  :    980
Force F [N]  :         1500
alpha [deg]  :     55

Position vector rOB [m]   =            0          1.2          5.9
Position vector rOA [m]   =            0          1.2          6.5
Force vector Fw [N]       =      -1228.7       860.36            0
```

```
Moment due to F [Nm]       =       -5592.4      -7986.7       1474.5
Moment due to W [Nm]       =        -1176          0             0
Moment of Forces about O [Nm] =       -6768.4      -7986.7       1474.5
Magnitude of the Moment [Nm] =        10572
```

5.2.4 Moment about a Line

In the previous examples we have calculated moment about a point. On many occasions you will be interested in calculating a moment about an axis - or a line. Consider what happens as you open the door by pulling on the handle. You are applying a force but the door turns around hinges that are located along a line. Mathematically this involves finding the components of the moment vector along the line - or the product of the moment vector with the unit vector defining the line. It is also calculated easily as a triple product. Let us modify Example 5.4 to calculate the moment about a line.

Example 5.5

The sign in front of the establishment is being battered by the wind. You are concerned that it will be uprooted. The effect of the wind can be reproduced by a concentrated force F, located at A, of magnitude 1500 N and at an angle of α of 55 degrees to the plane of the sign which is in the yz plane. The stiffened sign weighs 980 N and the weight passes through the point A. The unusually wet weather has corroded the pole and you notice a line of rust and material weakness at C along the line CD. The various distances shown on the figure are in meters. You are concerned that the pole will break at the point C and would like to know the value of the moment along the direction CD at the point C.

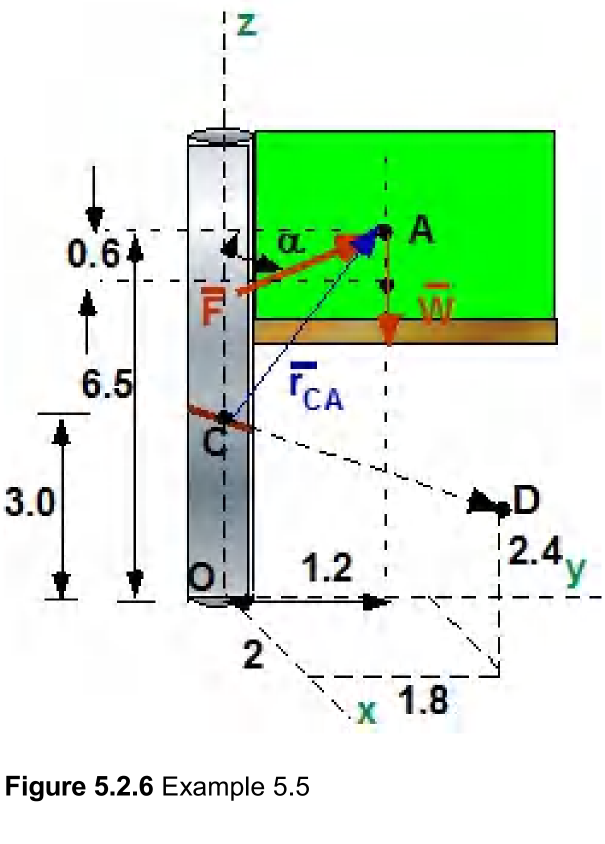


Figure 5.2.6 Example 5.5

Data: F = 1500 [N]; W = 980 [N], α= 55 [deg]; L_1=6.5 [m]; L_2= 0.6 [m]; L_3=1.2 [m]; Point C : [0,0,3]; Point D:[2,1.8,2.4]

Find: Net Moment about line CD

Assumption: None

Solution: To calculate the moment about a line (line CD) we first need a point on the line (point C) and the unit vector along the line (e_{CD}). Another useful feature we can exploit is to notice that the line of action of the weight W, and the force F due the wind can be combined at A as the line of action of both forces pass through A. We can consider $\mathbf{F_A}$ is the resultant of the force system. The solution is the moment due to this resultant force about C rather than the sum of the individual moments.

$$M_{CD} = \hat{e}_{CD} \bullet \bar{M}_C \qquad = \hat{e}_{CD} \bullet \left(\bar{r}_{CA} \times \bar{F}_A \right)$$

$$\bar{F}_A = -1.2287 \times 10^3\, \hat{i} + 0.8604 \times 10^3\, \hat{j} - 980\, \hat{k} \quad [N[$$

$$\hat{e}_{CD} = \frac{\overline{CD}}{\left|\overline{CD}\right|} = 0.7255\hat{i} + 0.6529\,\hat{j} - 0.2176\hat{k}$$

$$\bar{r}_{CA} = 0\hat{i} + 1.2\,\hat{j} + 3.5\,\hat{k} \quad [m[$$

$$\bar{M}_C = \bar{r}_{CA} \times \bar{F}_A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1.2 & 3.5 \\ -1.2287 \times 10^3 & 0.8604 \times 10^3 & -980 \end{vmatrix}$$

$$= -4.1873 \times 10^3\, \hat{i} - 4.3005 \times 10^3\, \hat{j} + 1.4745 \times 10^3\, \hat{k} \quad [Nm[$$

$$M_{CD} = \hat{e}_{CD} \bullet \bar{M}_C = (0.7255)\left(-4.1873 \times 10^3\right) + (0.6529)\left(-4.3005 \times 10^3\right) + (-0.2176)\left(1.4745 \times 10^3\right)$$

$$= -6.1666 \times 10^3\, [Nm[$$

Scalar Triple Product: There is no reason to calculate $\mathbf{M_C}$ as we can use the scalar triple product shown below.

$$M_{CD} = \hat{e}_{CD} \bullet \left(\bar{r}_{CA} \times \bar{F}_A \right)$$

$$M_{CD} = \begin{vmatrix} e_{CDx} & e_{CDy} & e_{CDz} \\ r_{CAx} & r_{CAy} & r_{CAz} \\ F_{Ax} & F_{Ay} & F_{Az} \end{vmatrix} = e_{CDx} \begin{vmatrix} r_{CAy} & r_{CAz} \\ F_{Ay} & F_{Az} \end{vmatrix} - e_{CDy} \begin{vmatrix} r_{CAx} & r_{CAz} \\ F_{Ax} & F_{Az} \end{vmatrix} + e_{CDz} \begin{vmatrix} r_{CAx} & r_{CAy} \\ F_{Ax} & F_{Ay} \end{vmatrix}$$

$$M_{CD} = \begin{vmatrix} 0.7255 & 0.6529 & -0.2176 \\ 0 & 1.2 & 3.5 \\ -1.2287 \times 10^3 & 0.8604 \times 10^3 & -980 \end{vmatrix} = -6.1666 \times 10^3 \,[Nm]$$

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.2.4- Example 5.5
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 5.5\n')
fprintf('-----------------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data (in meters)
%% Data
F = 1500;   W = 980;  % the forces
L1 = 6.5;    L2 = 0.6;   L3 = 1.2;  % the distances
alpha = 55;   alphar = alpha*pi/180;  % alpha in radians
%% calculations
O = [0 0 0]; % point O
A = [0 L3 L1];  % point A
B = [0 L3 (L1 - L2)]; % point B
C =[0,0,3];  D = [2,1.8,2.4];

rCD = D-C;  eCD = rCD/norm(rCD);
eF = [-sin(alphar) cos(alphar) 0];  % unit vector in F direction
Fw = F*eF ; % vector F
Wv = W*[0 0 -1] ; % vector W
FA = Fw + Wv;

rCA = A -C;
MC = cross(rCA,FA); % Moment due to FA at C
MCD = dot(eCD,cross(rCA,FA))  % Moment along the line CD

%% Printing
fprintf('Point O [m]  : '),disp(O)
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point C [m]  : '),disp(C)
fprintf('Point D [m]  : '),disp(D)

fprintf('\nWeight  [N]  : '),disp(W)
fprintf('Force F [N]    : '),disp(F)
fprintf('alpha [deg]    : '),disp(alpha)

fprintf('\nPosition vector rCD [m]   = '),disp(rCD)
fprintf('Net Force vector FA [N]     = '),disp(FA)
fprintf('Moment due to FA at C [Nm]  = '),disp(MC)
fprintf('Moment along CD [Nm]        = '),disp(MCD)
```

In Command Window

```
----------------------------------------
Example 5.5
-----------------------------------------
MCD =
      -6166.6
Point O [m]  :      0     0     0
Point A [m]  :             0          1.2          6.5
Point B [m]  :             0          1.2          5.9
Point C [m]  :      0     0     3
Point D [m]  :             2          1.8          2.4
```

```
Weight  [N]  :    980
Force F [N]    :        1500
alpha [deg]    :     55

Position vector rCD [m]   =            2          1.8         -0.6
Net Force vector FA [N]     =       -1228.7       860.36         -980
Moment due to FA at C [Nm]  =       -4187.3      -4300.5       1474.5
Moment along CD [Nm]        =       -6166.6
```

Execution in Octave

The code is the same as in MATLAB

In Octave Command Window

```
----------------------------------------
Example 5.5
-----------------------------------------
Point O [m]  :   0  0  0
Point A [m]  :   0  1.2  6.5
Point B [m]  :   0  1.2  5.9
Point C [m]  :   0  0  3
Point D [m]  :   2  1.8  2.4

Weight  [N]  : 980
Force F [N]    : 1500
alpha [deg]    : 55

Position vector rCD [m]   =   2  1.8  -0.6
Net Force vector FA [N]     =   -1228.7  860.36  -980
Moment due to FA at C [Nm]  =   -4187.3  -4300.5  1474.5
Moment along CD [Nm]        = -6166.6
```

The solutions are the same as in MATLAB.

5.2.5 Couple and Couple Moment ***(This material appeared earlier in Section 2.12)***

A couple is a pair of equal and opposite forces separated by a distance. The couple has no net force on the object but they will generate a moment - couple moment. The couple moment is normal to the plane formed by the forces. It can be moved parallel to itself on the object without changing the effect on the object and hence is called a free vector. The force in the couple can be in any direction but the plane of the forces will always be two dimensional. The couple moment is a free vector normal to the plane. The magnitude of the moment is the force multiplied by the shortest distance between the forces. It can be formally calculated as the moment by both forces at any point on or off the rigid body.

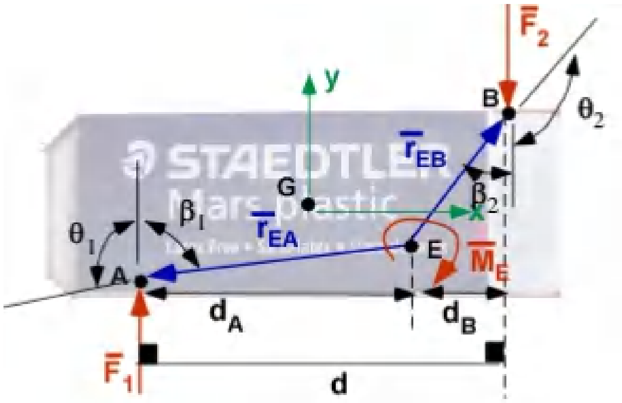


Figure 5.2.7. Couple Moment

The couple is created by the pair of forces **F₁** and **F₂**. They are equal and opposite but numbered differently for illustrating the calculation. The couple moment is calculated by summing the moment due to both forces at a arbitrary point E.

$$\bar{M}_E = \left(\bar{r}_{EA} \times \bar{F}_1\right) + \left(\bar{r}_{EB} \times \bar{F}_2\right) = F_1 r_{EA} \sin\theta_1\left(-\hat{k}\right) + F_2 r_{EB} \sin\theta_2\left(-\hat{k}\right)$$

$$\theta_1 = \pi - \beta_1; \quad \theta_2 = \pi - \beta_2; \quad \sin\left(\pi - \beta_1\right) = \sin\beta_1; \quad \sin\left(\pi - \beta_2\right) = \sin\beta_2$$

$$r_{EA} \sin\beta_1 = d_A; \quad r_{EB} \sin\beta_2 = d_B$$

$$\bar{M}_E = \left(-\hat{k}\right)\left(F_1 d_A + F_2 d_B\right) = \left(-\hat{k}\right) F_1 d \quad = \quad \left(-\hat{k}\right) F_2 d$$

The couple moment is *F*d* and the direction is normal to the plane in the sense of the couple.

Example 5.6

Three couples are shown acting on the prismatic body shown. The magnitude of forces F_1, F_2, F_3, acting in x, y, and z direction respectively are 10, 12, and 15 [N]. (a) Calculate the moment due to the three couples; (b) Identify the resultant couple.

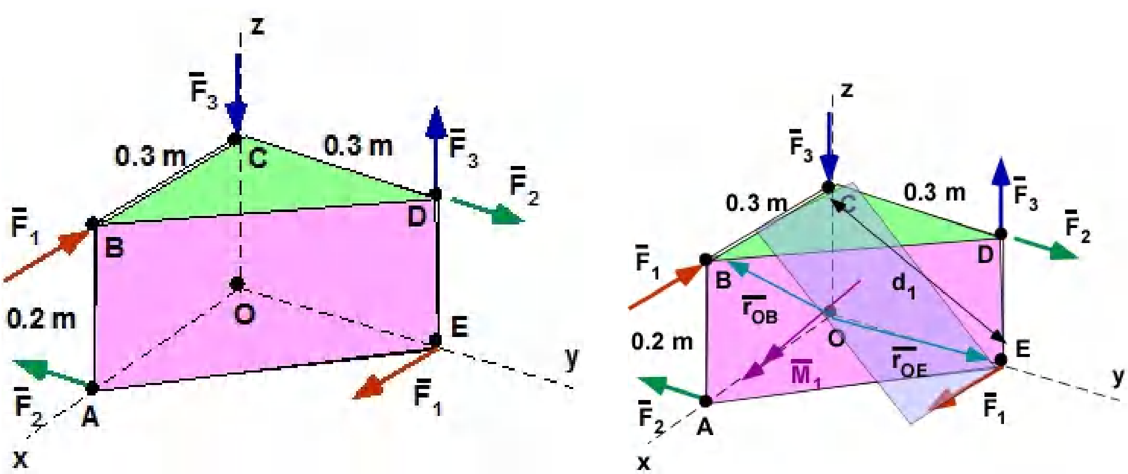


Figure 5.2.8 Example 5.6

Data: F1 = 10 [N], F_2 = 12 [N], F_3 = 15 [N]. Location shown on figure. [note that they are couples since the forces (i) have the same magnitude; (ii) have opposite direction; and (iii) are separated].

Find: (a) the individual couple moments (**M_1, M_2, M_3**), and (b) sum of the couples

Assumption: none

Solution:

$$\left|\bar{M}_1\right| = M_1 = F_1 d_1$$

$$\bar{M}_1 = (\bar{r}_{OB}) \times \left(-F_1\,\hat{i}\right) + (\bar{r}_{OE}) \times \left(F_1\,\hat{i}\right)$$

$$\bar{M}_1 = \left(0.3\hat{i} + 0.2\hat{j}\right) \times \left(-10\hat{i}\right) + \left(0.3\hat{j}\right) \times \left(10\hat{i}\right) = -2\hat{j} - 3\hat{k}$$

$$\left|\bar{M}_1\right| = 3.6056\left[Nm\right]$$

$$\left|\bar{M}_2\right| = M_2 = F_2 d_2$$

$$\bar{M}_2 = \left(0.3\hat{i}\right) \times \left(-12\hat{j}\right) + \left(0.3\hat{j} + 0.2\hat{k}\right) \times \left(12\hat{j}\right) = -3.6\hat{k} - 2.4\hat{i}; M_2 = 4.3267\left[Nm\right]$$

$$\bar{M}_3 = (15)(0.3)\hat{i} = 4.5\hat{i}\ \left[Nm\right]$$

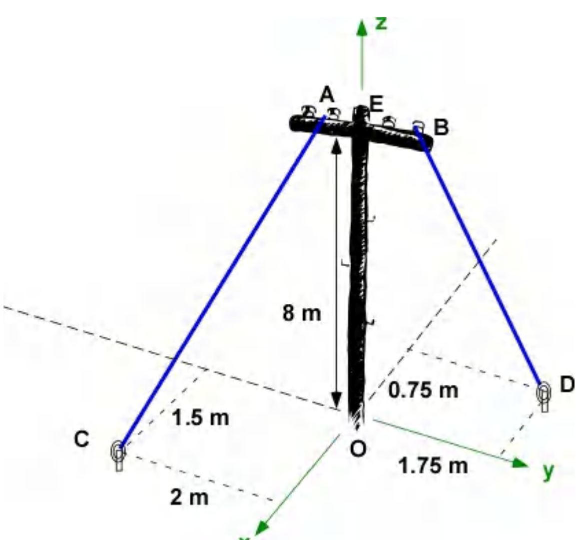
$$\sum \bar{M} = \left[-2\hat{j} - 3\hat{k}\right] + \left[-3.6\hat{k} - 2.4\hat{i}\right] + \left[4.5\hat{i}\right] = 2.1\hat{i} - 2\hat{j} - 6.6\hat{k}\left[Nm\right]$$

5.2.6 Additional Problems

Solve the following problems on paper and using MATLAB/Octave. For each problem you must draw the position vector and force vector along with the coordinate system if they are not available. Choose your own coordinate system if they are not given.

Problem 5.2.1

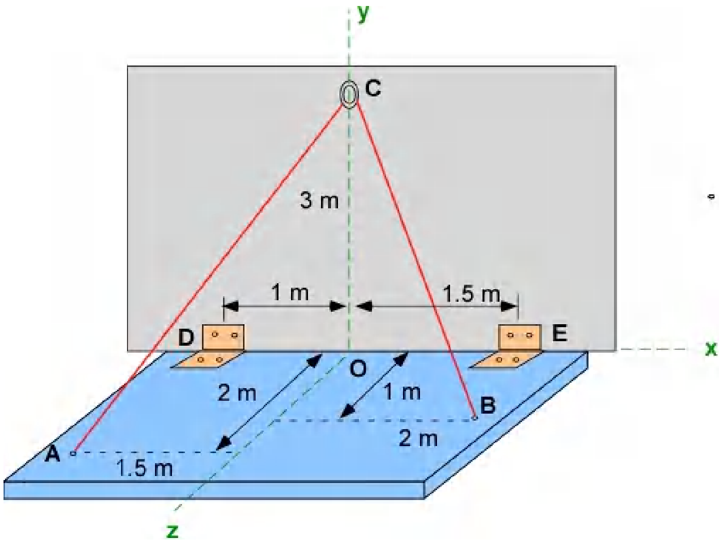
Both cables are in tension. The force in cable AC is 500 N. The force in cable BD is 350 N. (a) Calculate the moment of the force in cable AC at O. (b) (a) Calculate the moment of the force in cable BD at O. (c) Express the net moment at O due to items (a) and (b).



Problem 5.2.1

Problem 5.2.2

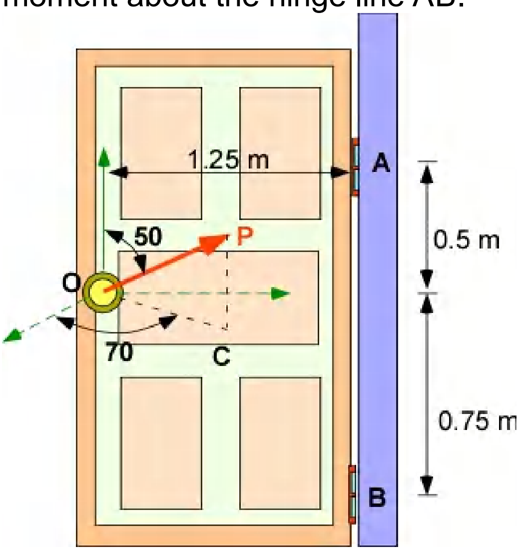
The single cable passes around the hook at C. The tension in the cable is 250 N. (a) Calculate the resultant force due to the cable at C. (b) Calculate the moment due to the force in (a) about the point D. (c) Calculate the moment about the hinge line DE.



Problem 5.2.2

Problem 5.2.3

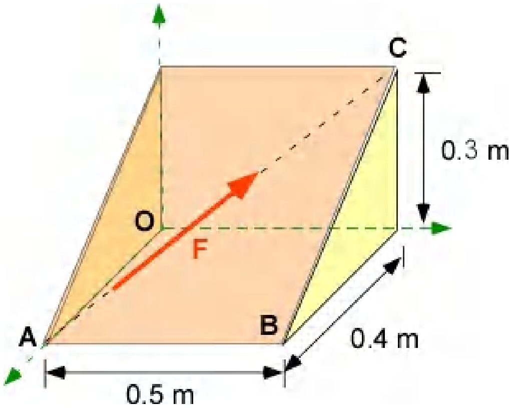
The force P on the door handle is 15 N in the direction shown. (a) Express the force P as a vector. (b) Calculate the moment of the force about the point A where the hinge is located. (c) Calculate the moment about the hinge line AB.



Problem 5.2.3

Problem 5.2.4

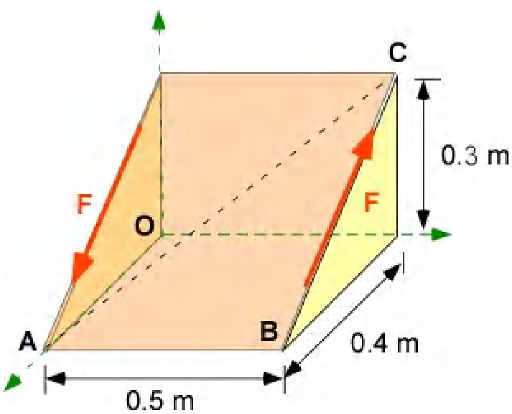
The force F has a magnitude of 25 N and is directed from A to C. (a) Calculate the moment of this force at O. (b) Calculate the moment of this force at B.



Problem 5.2.4

Problem 5.2.5

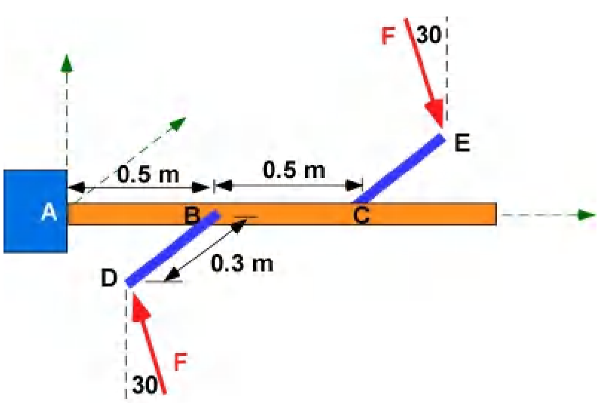
The pair of forces of magnitude 25 N generates a couple moment M. (a) Express the couple moment as a vector. (b) What is the magnitude of the couple moment?



Problem 5.2.5

Problem 5.2.6

The pair of forces in the vertical plane create a couple moment M. The magnitude of the force F is 55 N. (a) Calculate the magnitude of the couple moment. (b) Express the couple moment as a vector.



Problem 5.2.6

5.3 RIGID BODY EQUILIBRIUM IN TWO DIMENSIONS

Now that we can compute the moment we can move on to study rigid bodies under **equilibrium**. This is the main principle in **Statics**. The rigid body should not move under the application of forces, moments, and couples. All analysis require establishing a free body diagram (FBD). This means that on a FBD the sum of the forces must be zero and the sum of the moments at any /every point "O" on the rigid body must be zero. They translate into the following vector set of equations:

$$\sum \bar{F} = 0;$$
$$\sum \bar{M}_O = 0; \qquad (5.6)$$

In two dimensions (x - y coordinates) this provides us with the following scalar equations:

$$\sum F_x = 0;$$
$$\sum F_y = 0; \qquad (5.7)$$
$$\sum M_{Oz} = 0;$$

The problem is essentially three dimensional since the moment is in the z-direction. However it is the plane of the x-y coordinates. This provides three equations which can be used to solve a problem with three unknowns.

There are *alternate equilibrium equations* to the ones in Eqn. (5.7).

In some problems we can replace one of the force equilibrium equations with a moment equation about another point. If we consider A and B as two points on the rigid body. If we were to use the equilibrium equation for the force in the x-direction then the line AB must not be perpendicular to the x-direction. We can write the equilibrium equations:

$$\sum F_x = 0;$$
$$\sum M_{Az} = 0; \qquad (5.8)$$
$$\sum M_{Bz} = 0;$$

In some problems we can apply the moment equilibrium about three points A, B, C that are not collinear as

$$\sum M_{Az} = 0;$$

$$\sum M_{Bz} = 0; \qquad (5.9)$$

$$\sum M_{Cz} = 0;$$

Most textbooks will include special sections where the system of forces may have a particular geometry that can be exploited to find the solution easily. For example the forces can be:

i. Collinear - The applied and reactive forces along a line.
ii. Concurrent - The applied and reactive forces pass through a single point
iii. Parallel: The applied and reactive forces are parallel
iv. General: The applied and reactive forces have no geometric relations.

In all of the above special cases the equilibrium equations (Eqn. 5.7) must be satisfied. It is just easier in some cases. In this book we will uniformly apply Eqn. (5.7) to all problems without seeking special advantage. This way there is no need to remember special cases. We will develop our topic using the example that appeared earlier.

5.3.1 Rectilinear Motion Only

Example 5.7

The person is trying to move a heavy chest of drawers. Since this is a rigid body we need more information about the size and shape. We will probably need to know or be able to compute the center of mass and the moment of inertia. We need to know the location of the loads precisely. External loads can be forces or moments/couples. In this will example The person is applying a horizontal force **F**. We wish to know the largest force he can apply before the chest will move. Note that the chest can move to the right, or tilt in place, or a combination of the two. The mass of the chest is 250 kg. The coefficient of friction is 0.12. While the problem is naturally three dimensional we will stipulate that there is no loss of information due to the two dimensional representation. The problem is essentially being solved in the plane that contains the center of mass and we are able to apply point forces. In addition we also assume that the center of mass coincides with the geometric center. This is a lot of additional assumptions that we did not make in solving equilibrium problems with particles.

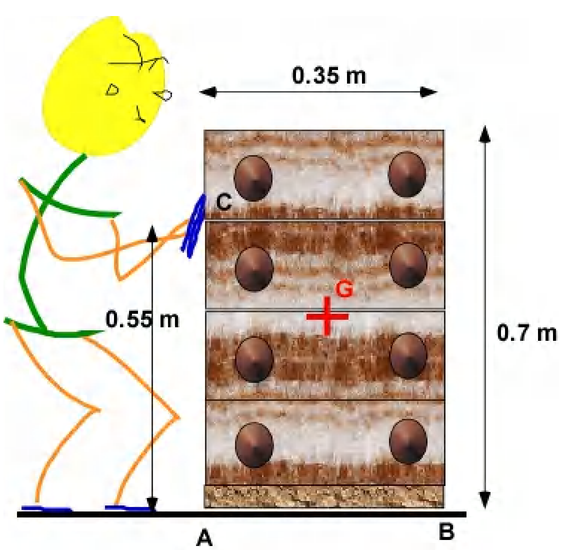


Figure 5.3.1 Problem description

As we develop the model and the FBD of the rigid body we have to consider the following important issues/assumptions:

1. We will first solve this problem by requiring that the chest is just about to move to the right because the force applied is the maximum that can be applied under the model for dry friction. If the applied force is less than this threshold then the chest will not move.

2. For the model to be in equilibrium with the applied force there must be friction ($\mathbf{F_r}$) as otherwise the applied force cannot be neutralized. This reaction must be produced by the ground.

3. As we set up the FBD we cannot have the line of action of the normal reaction from the ground (**N**) pass through the mass center. If it did then the the FBD is not in equilibrium as the applied force and the frictional force will cause the chest to rotate. The location of the normal force must be closer to edge B for the couple created by the weight and the normal force to balance the couple from the applied force and friction.

The FBD in Figure 5.3.2 captures this information

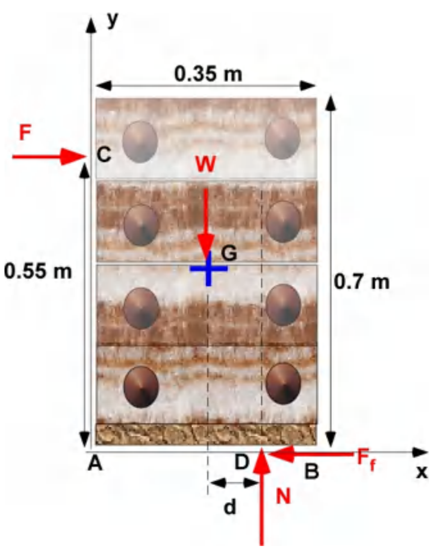


Figure 5.3.2 FBD of chest for translation

Data: m = 250 kg; W = mg; μ = 0.12

Find: Maximum F before it starts moving

Assumptions: Chest is in equilibrium - no translation or rotation. Mass center is at the centroid based on symmetry.

Solution: The location of the Normal force N is also an unknown. For a rigid body we do have an extra equation of equilibrium indicating that the moment of the body is zero. We enforce that by setting up the moment about any point on the body. Due to the available definition of the distance d, we take the moment about the point D.

$$\sum F_x = F - F_f = 0;$$
$$\sum F_y = N - W = 0;$$
$$\sum M_{Dz} = Wd - 0.55F = 0$$

We will now solve for the three unknowns:

$$N = W = 2452.5[N]$$
$$F_f = 0.12 * N = 294.3[N]$$
$$F = 294.3[N]$$
$$d = 0.55 * 294.3 / 2452.5 = 0.066[m]$$

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.3.1- Example 5.7
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 5.7\n')
fprintf('-----------------------------------------\n')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Data
%% Data
m = 250;   W = m*9.81;  % the forces
w = 0.35;    h = 0.7;   hf = 0.55;  % the distances
mu = 0.12;
%% calculations
syms F d N
A = [0,0,0]; B = [w,0,0];  C = [0,hf,0];  G = [w/2,h/2,0];
D = [0.5*w + d,0,0]; % points

eF = [1,0,0];  % unit vector in F direction
eW  = [0,-1,0];  eN = [0,1,0];  eFf = [-1,0,0];

FF = F*eF;  WW = W*eW; NN = N*eN;
Ff = mu*N;  FFf = Ff*eFf;

%% force equilibrium
SumF = FF + WW + NN + FFf

%% Moment equilibium
%  Moment about a point is calculated using r X F  (cross product)
rDC = C-D;    rDG = G - D;
MomD = cross(rDC,FF) + cross(rDG,WW)

sol = solve(SumF(1),SumF(2),MomD(3));
F = double(sol.F);
```

```
N = double(sol.N);
d = double(sol.d);

%% Printing
fprintf('mass (kg)       : '),disp(m)
fprintf('width -w [m]    : '),disp(w)
fprintf('height - h[m]  : '),disp(h)
fprintf('force height [m]   : '),disp(hf)
fprintf('coeff. friction    : '),disp(mu)

fprintf('\nPosition vector rDC [m]    = '),disp(rDC)
fprintf('Position vector rDG [m]    = '),disp(rDG)

fprintf('\nSumF : '),disp(SumF)
fprintf('SumM : '),disp(MomD)
fprintf('\nSolution: Applied Force F[N]      = '),disp(F)
fprintf('Solution: Normal Force  N[N]      = '),disp(N)
fprintf('Solution: distance d [m]          = '), disp(d)
```

In the Command Window

```
----------------------------------------
Example 5.7
-----------------------------------------
mass (kg)      :    250
width -w [m]   :          0.35
height - h[m] :           0.7
force height [m]   :         0.55
coeff. friction    :         0.12

Position vector rDC [m]   = [ - d - 7/40, 11/20, 0]
Position vector rDG [m]   = [ -d, 7/20, 0]

SumF : [ F - (3*N)/25, N - 4905/2, 0]
SumM : [ 0, 0, (4905*d)/2 - (11*F)/20]

Solution: Applied Force F[N]     =         294.3
Solution: Normal Force  N[N]     =        2452.5
Solution: distance d [m]         =         0.066
```

5.3.2 Rotation Only

Now we consider that the force F will cause the chest to rotate. We expect it to rotate about the point B. We will be solving just as the rotation is going to start, because once it starts then it will continue and equilibrium is lost.

Example 5.8

The FBD is exaggerated to illustrate the rotation, but the forces F and W, and their location, will have the same geometry/orientation as in the previous case. The normal N and the friction F_f must move to the point B as there is only a single point of contact.

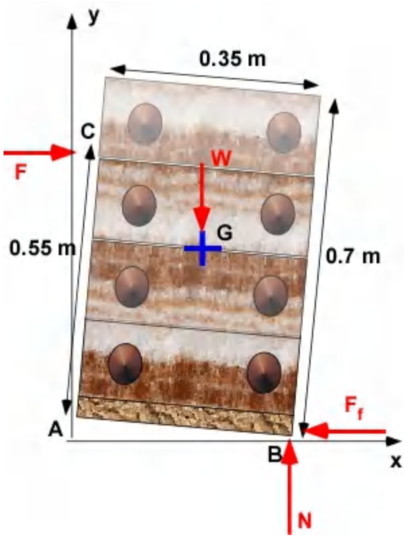


Figure 5.3.3 FBD for Example 5.8

Data: m = 250 kg;   W = mg;   m = 0.12

Find: Maximum F before it starts rotating

Assumptions: Chest is in equilibrium - no translation or rotation. Mass center is at the centroid based on symmetry. *Problem is solved for the moment it is about to rotate.*

Solution: *The location of the Normal force N is at B*

$$\sum F_x = F - F_f = 0;$$
$$\sum F_y = N - W = 0;$$
$$\sum M_B = W(0.35/2) - 0.55F = 0$$

The solution is

$$N = W = 2452.5[N]$$
$$F_f = F$$

$$F = 2452.5 * 0.35/2/0.55 = 780.34[N]$$

For equilibrium, F_f is equal to **F**. It is no longer determined through the dry friction relation. Note that for the dimensions of this chest and the location of the applied force the required force is over 2.5 times the previous case.

Also note that the lower the force is applied on the chest the greater is the required force. You will have learned this from direct experience as you try to move heavy things in real life.

For this problem, the chest is more likely to move with the application of force greater than 294 [N] since it needs a lot more force to cause it to rotate about a point. Once the chest starts moving, our model changes and we must switch to the application of knowledge from dynamics.

5.3.3 Support Reactions

In many structural design problems which experience loads the important consideration is how to support the structure. Usually the main focus on these kind of equilibrium problem is to determine the support reactions. These will depend on the nature of this support. While supports can be complicated in the following discussions they are pinned or rigid supports. You can have flexible supports but these must include an appropriate model for determining the reactions associated with them. These reactions are usually the *unknowns* in the problem. There are lists of these supports available in most text books. The supports are explained when they initial appear. You can also get a fairly intuitive sense of these reactions through practice. In general we use the equilibrium equations to solve for the unknowns.

Statically determinate problem: When the number of reactions are equal to the number of independent equilibrium equations (from statics) available to solve them. For a two dimensional problem this will be three and for a three-dimensional problem it will be six. These are set up by force and moment equilibrium equations in the coordinate directions (Eqn. 5.7)

Statically indeterminate problem: Here the number of unknowns exceed the number of independent equilibrium equations (from statics) available to solve them. This will allows you to implicitly determine some of the unknowns in terms of some of the other unknowns. For explicit values of the reactions you have to include additional equations. Frequently this is developed by using constraints on the problem, particularly displacement of certain points of the structure. This will require the knowledge of the relations between stress and displacement (strength of materials). Requiring the structure to be elastic allows us to uses the Hooke's Law. This allows you to create additional equations. Exploiting displacement constraints is illustrated in Example 3.4.

One of the strong reasons to combine the presentation of statics and strength of materials is to solve the statically independent problem which are usually found in real structures. The primary reason for strength of materials is to be able to design a structure to avoid failure.

Example 5.9

In this design we have a horizontal force creating a force on a rigid inclined plate. This is like transferring a force in magnitude and direction. Imagine pressing the brakes in the car. That causes the car to slow down - because the inclined plate is connected to the wheels in some way (not shown). You can Imagine even more actions. the inclined plate is hinged at the left end and is connected to a spring. This brings in multiple rigid bodies. We are mostly going to be looking at a single rigid body. The rigid body is the angular device ABC. We require the reactions on the device. The applied force F is 300 N. The geometry and dimensions are shown in the problem (Figure 5.3.4)

The reactions, on the rigid body ABC, are provided by the pin at B and the contact at C. We can assume smooth contact. We can also solve for friction at C using the standard dry friction model.

The pin at B will only provide a force reaction. It can have a magnitude and direction or two unknown force components for this two dimensional example.

The contact at C will have a force reaction normal to the plane of contact. If this plane is defined then only the magnitude of the force is the unknown.

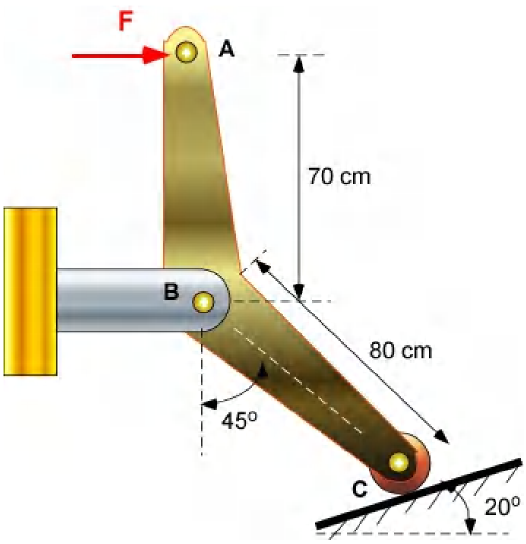


Figure 5.3.4 Example 5.9 **-** Problem Definition

Data: F = 300 [N];   Point B : [0,0,0]; Point A:[0,0.7,0]; Point C:[0.8*sin45), -0.8*cos(45),0]; The coordinates are at B.

Find: The reactions at B (B_x, B_y) and F_c

Assumption: No friction at C and B. Ignore weight of the rigid body

Solution: To calculate the reaction we need the FBD of the rigid body.

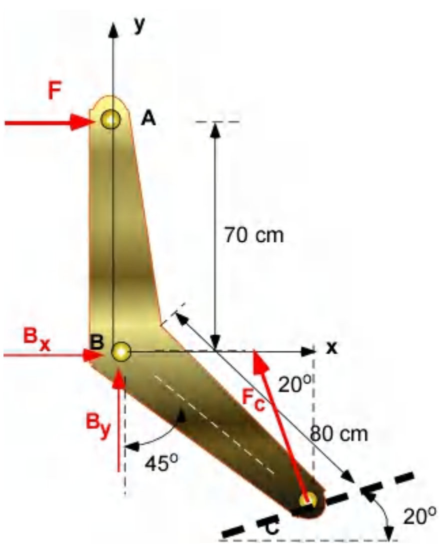


Figure 5.3.5 FBD of ABC

There are 3 unknowns and we have three equilibrium equations

$$\sum F_x = F + B_x - F_c \sin(20) = 0$$

$$\sum F_y = B_y + F_c \cos(20) = 0$$

$$\sum M_B = \bar{r}_{BA} \times \bar{F} + \bar{r}_{BC} \times \bar{F}_c = 0$$

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.3.3- Example 5.9
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-----------------------------------------\n')
fprintf('Example 5.9\n')
fprintf('------------------------------------------\n')
%% Data
F = 300;     % the forces
L1 = 0.7;     L2 = 0.8;
alpha1 = 45;   alpha1r = alpha1*pi/180;  % alpha in radians
alpha2 = 20;   alpha2r = alpha2*pi/180;  % alpha in radians
%% calculations
syms Bx By Fc
B = [0 0 0]; % point B
A = [0 L1 0];  % point A
C = [L2*sin(alpha1r),-L2*cos(alpha1r),0]; % point C
rBA = A - B; % vector from B to A
rBC = C - B; % veB to Cctor from

FC = [-Fc*sin(alpha2r),Fc*cos(alpha2r),0];
FF = [F,0,0];
BB = [Bx,By,0];

%% equilibrim
SumF = FF + BB + FC;    % Sum of the forces
SumM = cross(rBA,FF) + cross(rBC,FC);    % Sum of moments at B

sol = solve(SumF(1),SumF(2),SumM(3)); % solution is a structure

%% Printing
fprintf('F [ N]   : '),disp(F)
fprintf('L1  [m]   : '),disp(L1)
fprintf('L2  [m]   : '),disp(L2)

fprintf('\nPosition vector rBA [m]    = '),disp(rBA)
fprintf('Position vector rBC [m]    = '),disp(rBC)

% displaying symbolic values to three decimals
fprintf('\nSumF : '),disp(vpa(SumF,3))
fprintf('SumM : '),disp(vpa(SumM,3))

fprintf('\nBx [N]      = '),disp(double(sol.Bx))
fprintf('By [N]      = '),disp(double(sol.By))
fprintf('Fc [N]      = '),disp(double(sol.Fc))
```

In the Command Window

```
-----------------------------------------
Example 5.7
```

```
----------------------------------------
F [ N]   :     300
L1 [m]   :           0.7
L2 [m]   :           0.8

Position vector rBA [m]   =          0        0.7          0
Position vector rBC [m]   =    0.56569   -0.56569          0

SumF : [ Bx - 0.342*Fc + 300.0, By + 0.94*Fc, 0]
SumM : [ 0, 0, 0.338*Fc - 210.0]

Bx [N]      =       -87.562
By [N]      =       -583.67
Fc [N]      =        621.13
```

5.3.4 Cable-Pulley Support

Example 5.10

This is another 2D example with cable-pulley support. This includes a frictionless pulley and inextensive cable. The bar AD must be horizontal after the application of the 500 N load. You need to ensure the cable will not break. For this you will need to determine the force in the cable. This will also give you the reactions at the pin support at D. We will ignore the size of the pulley.

The reaction due to cable is the force along the cable.
The reaction at the pin at D is a vector force reaction.

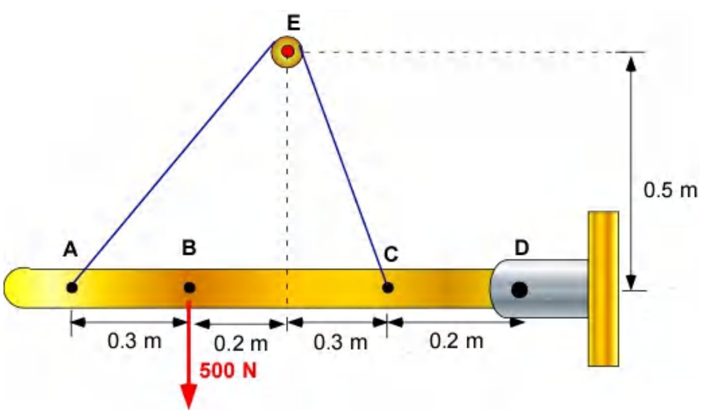


Figure 5.3.6 Example 5.10 - Problem Definition

To solve the problem we first draw the FBD of the bar in Figure 5.3.7. At first glance there are there are 4 unknowns suggesting a possible indeterminate problem. We have three equilibrium equations (a 2D problem will have a maximum of three equations). We also have a constraint that the tension in the cable must be same on either side of the frictionless pulley (F_{AE} = F_{CE})The inextensible cable means there will be change in the length of cable and the geometry will bot change. Therefore this problem is **statically determinate**.

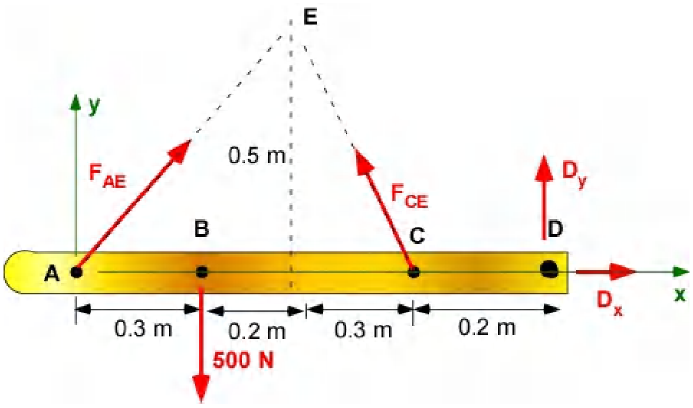


Figure 5.3.7 FBD of Example 5.10

Data: Applied force F = 500 [N]; Point A:[0,0,0]; Point B : [0.3,0,0]; ; Point C:[0.8,0,0]; Point D: [1,0,0]; Point E: [0.5,0.5,0];

Find: The reactions at D (D_x, D_y) and F_{AE} (= F_{CE})

Assumption: Ignore weight of the rigid body AD

Solution: Using the FBD of the rigid body. The formal equations can be written as

$$\sum F_x = F_{AEx} - F_{CEx} + D_x = 0$$
$$\sum F_y = F_{AEy} + F_{CEy} + D_y - 500 = 0$$
$$\sum M_{Dz} = \left(\bar{r}_{DA} \times \bar{F}_{AE}\right) + \left(\bar{r}_{DB} \times \left(-500\hat{j}\right)\right) + \left(\bar{r}_{DC} \times \bar{F}_{CE}\right) = 0$$
$$\left|F_{AE}\right| = \left|F_{CE}\right|$$

From Figure 5.3.7 you can verify that these equations can also be written as:

$$F_{AE}\cos(45) - F_{AE}\cos(59) + D_x = 0$$
$$F_{AE}\sin(45) + F_{AE}\sin(59) + D_y = 500$$
$$-F_{AE}\sin(45)(1) - F_{AE}\sin(59)(0.2) + (500)(0.7) = 0$$

You can then set this up as a matrix equation:

$$\begin{bmatrix} \cos(45) - \cos(59) & 1 & 0 \\ \sin(45) + \sin(59) & 0 & 1 \\ -\sin(45) - (0.2)\sin(59) & 0 & 0 \end{bmatrix} \begin{bmatrix} F_{AE} \\ D_x \\ D_y \end{bmatrix} = \begin{bmatrix} 0 \\ 500 \\ -500(0.7) \end{bmatrix}$$

You can verify if you get the same solution using MATLAB

Solution Using MATLAB

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.3.4- Example 5.10
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 5.10\n')
fprintf('-----------------------------------------\n')

%% Data
F = 500;     % the applied force
A = [0 0 0]; % point A
B = [0.3 0 0];  % point B in m
C = [0.8,0,0];
D = [1,0,0];
E = [0.5,0.5,0];

%% calculations
syms Dx Dy Fae

rAE = E - A;  eAE = rAE/norm(rAE);
rCE = E - C;  eCE = rCE/norm(rCE);
rDA = A - D;
rDB = B - D;
rDC = C - D;

FAE = Fae*eAE;
FCE = Fae*eCE;
FB = F*[0,-1,0];
FD = [Dx,Dy,0];

%% equilibrim
SumF = FAE + FCE + FB + FD ;  % Sum of the forces
SumM = cross(rDA,FAE) + cross(rDB,FB)+ cross(rDC,FCE) ;
% Sum of moments at D

sol = solve(SumF(1),SumF(2),SumM(3));
% solution is a structure

%% Printing
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point C [m]  : '),disp(C)
fprintf('Point D [m]  : '),disp(D)
fprintf('Point E [m]  : '),disp(E)

fprintf('\nPosition vector rAE [m]    = '),disp(rAE)
fprintf('Position vector rCE[m]    = '),disp(rCE)
fprintf('Position vector rDA [m]    = '),disp(rDA)
fprintf('Position vector rDB [m]    = '),disp(rDB)
fprintf('Position vector rDC [m]    = '),disp(rDC)

fprintf('\nF [ N]   : '),disp(F)
```

```
% displaying symbolic values to three decimals
fprintf('\nSumF : '),disp(vpa(SumF,3))
fprintf('SumM : '),disp(vpa(SumM,3))

fprintf('\nResults:\n')
fprintf('-----------\n')
fprintf('Dx  [N]     = '),disp(double(sol.Dx))
fprintf('Dy  [N]     = '),disp(double(sol.Dy))
fprintf('Fae[N]      = '),disp(double(sol.Fae))
```

In the Command Window

```
----------------------------------------
Example 5.10
-----------------------------------------
Point A [m]  :      0     0     0
Point B [m]  :          0.3           0            0
Point C [m]  :          0.8           0            0
Point D [m]  :      1     0     0
Point E [m]  :          0.5         0.5            0

Position vector rAE [m]   =          0.5          0.5            0
Position vector rCE[m]   =          -0.3          0.5            0
Position vector rDA [m]   =      -1       0      0
Position vector rDB [m]   =          -0.7            0            0
Position vector rDC [m]   =          -0.2            0            0

F [ N]  :    500

SumF : [ Dx + 0.193*Fae, Dy + 1.56*Fae - 500.0, 0]
SumM : [ 0, 0, 350.0 - 0.879*Fae]

Results:
-----------
Dx  [N]     =       -76.728
Dy  [N]     =       -123.27
Fae[N]      =        398.36
```

Execution in Octave

The code is the same as in MATLAB except for the additional statements below . The changes are highlighted. You must include the symbolic package and if you do not wish to see warnings you include the command warning off as shown

```
clc, clear, format compact, close all, format short G, warning off

pkg load symbolic
sympref display flat
%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

In Octave Command Window

```
----------------------------------------
Example 5.10
```

```
-----------------------------------------
Point A [m]  :   0  0  0
Point B [m]  :   0.3  0  0
Point C [m]  :   0.8  0  0
Point D [m]  :   1  0  0
Point E [m]  :   0.5  0.5  0

Position vector rAE [m]   =   0.5  0.5  0
Position vector rCE[m]   =   -0.3  0.5  0
Position vector rDA [m]   =   -1  0  0
Position vector rDB [m]   =   -0.7  0  0
Position vector rDC [m]   =   -0.2  0  0

F [ N]  : 500

SumF :   [Dx + 0.192*Fae  Dy + 1.56*Fae - 500.0  0]
SumM :   [0  0  -0.878*Fae + 350.0]

Results:
-----------
Dx [N]     = -76.728
Dy [N]     = -123.27
Fae[N]     = 398.36
```

The results are the same as in MATLAB.

5.3.5 Statically Indeterminate Problem

Example 5.11

Let us make minor modification to the support D and make it a rigid one as shown in Figure 5.3.8. We will keep the same dimensions and applied force.

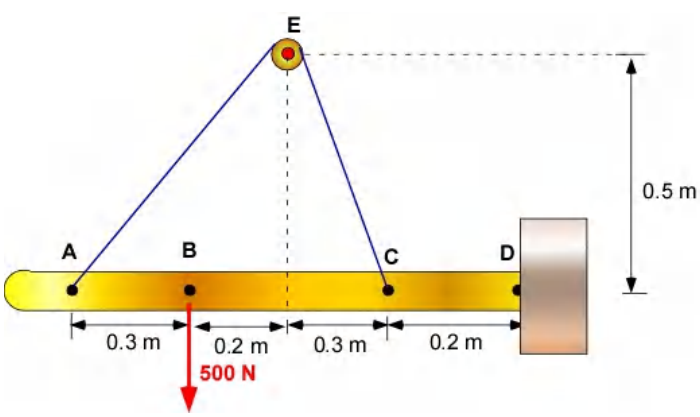


Figure 5.3.8 Example 5.11 (Example 5.10 modified)

The *fixed* or *rigid* or *cantilevered* support will have two reactions for two dimensional problems:

A force vector: The force will have two unknowns, either magnitude and direction or two force components.

A moment vector: This is a single unknown establishing the magnitude of the moment since for 2D problems as it is directed out of the plane of the problem.

Therefore in addition to D_x and D_y there will be another unknown, M_D, in the problem and this is shown in the FBD in Figure 5.9.4. The number of equations are still the same but we have one additional unknown. This problem is **statically indeterminate**. It does not mean it *cannot be solved*. It just means that you cannot solve it by the equations of statics alone. We will need to bring in additional considerations to solve this problem. This will become a design problem. We can limit the force in the cable so that it will not fail. This means the material of the cable and the area of cross-section of the cable will need to be specified. On the other hand we can use the resourcefulness of MATLAB and obtain the solution to the reactions as force in the cable is varied between prescribed limits.

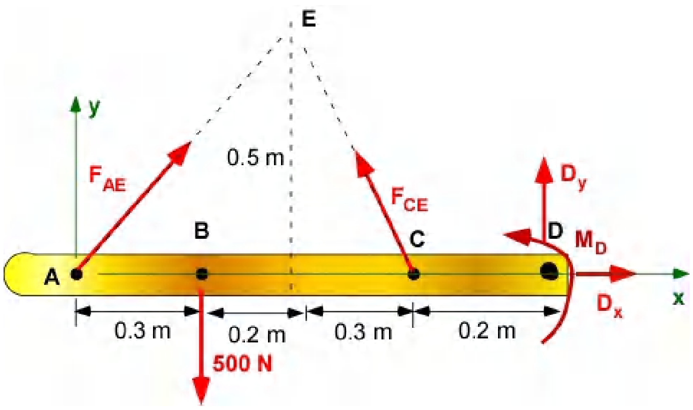


Figure 5.3.9. FBD of Example 5.11

Data: Applied force F = 500 [N]; Point A:[0,0,0]; Point B : [0.3,0,0]; ; Point C:[0.8,0,0]; Point D: [1,0,0]; Point E: [0.5,0.5,0];

Find: The reactions at D (D_x, D_y) ;$F_{AE;}$ and $M_{D.}$ Plot D_x, D_y, M_D as a function of F_{AE}.

Assumption: Ignore weight of the rigid body AD

Solution: Using the FBD of the rigid body

$$\sum F_x = F_{AEx} - F_{CEx} + D_x = 0$$
$$\sum F_y = F_{AEy} + F_{CEy} + D_y - 500 = 0$$
$$\sum M_D = \left(\bar{r}_{DA} \times \bar{F}_{AE}\right) + \left(\bar{r}_{DB} \times \left(-500\hat{j}\right)\right) + \left(\bar{r}_{DC} \times \bar{F}_{CE}\right) + \bar{M}_D = 0$$
$$\left|F_{AE}\right| = \left|F_{CE}\right|$$

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.3.5- Example 5.11
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-----------------------------------------\n')
fprintf('Example 5.11\n')
fprintf('------------------------------------------\n')
```

```
%% Data
F = 500;     % the applied force
A = [0 0 0]; % point A
B = [0.3 0 0];  % point B in m
C = [0.8,0,0];
D = [1,0,0];
E = [0.5,0.5,0];

%% calculations
syms Dx Dy Fae Md

rAE = E - A;   eAE = rAE/norm(rAE);
rCE = E - C;   eCE = rCE/norm(rCE);
rDA = A - D;
rDB = B - D;
rDC = C - D;

FAE = Fae*eAE;
FCE = Fae*eCE;
FB = F*[0,-1,0];
FD = [Dx,Dy,0];
MD = Md*[0,0,1];

%% equilibrim
SumF = FAE + FCE + FB + FD ;  % Sum of the forces
SumM = cross(rDA,FAE) + cross(rDB,FB)+ cross(rDC,FCE)+ MD ;
% Sum of moments at D

sol = solve(SumF(1),SumF(2),SumM(3),Dx,Dy,Md) ;
% Solve for Dx, Dy, and Md

%% Printing
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point C [m]  : '),disp(C)
fprintf('Point D [m]  : '),disp(D)
fprintf('Point E [m]  : '),disp(E)

fprintf('\nPosition vector rAE [m]    = '),disp(rAE)
fprintf('Position vector rCE[m]    = '),disp(rCE)
fprintf('Position vector rDA [m]    = '),disp(rDA)
fprintf('Position vector rDB [m]    = '),disp(rDB)
fprintf('Position vector rDC [m]    = '),disp(rDC)

fprintf('\nF [ N]  : '),disp(F)

% displaying symbolic values to three decimals
fprintf('\nSumF : '),disp(vpa(SumF,3))
fprintf('SumM : '),disp(vpa(SumM,3))

fprintf('\nResults:\n')
fprintf('-----------\n')
fprintf('Dx [N]     = '),disp(sol.Dx)
fprintf('Dy [N]     = '),disp(sol.Dy)
fprintf('Md [N]     = '),disp(sol.Md)
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  Plot results of Dx, Dy, Md as a function of Fae
% Design of the structure
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
set(gcf,'Position',[25,50,400,350])
hp0 = ezplot(sol.Dx,[200,1000]);
set(hp0,'Color','b','LineWidth',2)
hold on
hp1 = ezplot(sol.Dy,[200,1000]);
set(hp1,'Color','r','LineWidth',2)
hp2 = ezplot(sol.Md,[200,1000]);
set(hp2,'Color','k','LineWidth',2)
grid on
xlabel('Fae')
```

In the Command Window

```
----------------------------------------
Example 5.11
-----------------------------------------
Point A [m]  :      0     0     0
Point B [m]  :           0.3            0            0
Point C [m]  :           0.8            0            0
Point D [m]  :      1     0     0
Point E [m]  :           0.5          0.5            0

Position vector rAE [m]   =          0.5          0.5             0
Position vector rCE[m]   =         -0.3          0.5             0
Position vector rDA [m]   =     -1     0     0
Position vector rDB [m]   =         -0.7            0             0
Position vector rDC [m]   =         -0.2            0             0

F [ N]  :    500

SumF : [ Dx + 0.193*Fae, Dy + 1.56*Fae - 500.0, 0]
SumM : [ 0, 0, Md - 0.879*Fae + 350.0]

Results:
-----------
Dx [N]     = (3*34^(1/2)*Fae)/34 - (2^(1/2)*Fae)/2
Dy [N]     = 500 - (5*34^(1/2)*Fae)/34 - (2^(1/2)*Fae)/2
Md [N]     = (2^(1/2)*Fae)/2 + (34^(1/2)*Fae)/34 - 350
```

In the Figure Window

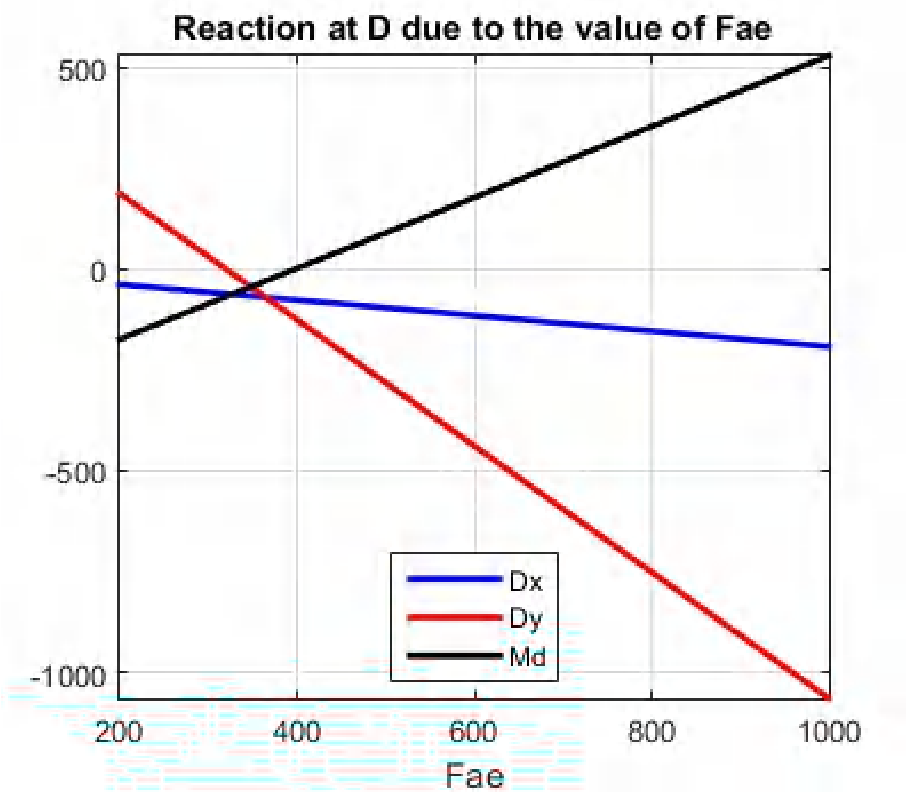


Figure 5.3.10 Example 5.11- Reactions as function of cable force

Notice that Figure 5.3.10 includes the solution to the original Example 5.10. We could have also calculated the reactions as a function of stress for a given diameter of the cable. We could have fixed the force in the cable and calculate the reactions with changes in the diameter of the cable. These kind of problems take us into design of the structure.

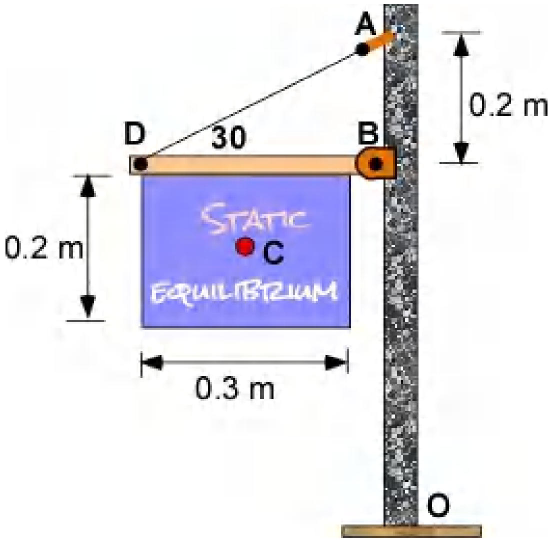
> **More unknowns than reactions is not really a disadvantage. We can turn the problem from a dull routine problem to an exciting design problem.**

5.3.6 Additional Problems

Solve the following problems on paper and using MATLAB/Octave. For each problem you must first draw the FBD. You must set up a coordinate system if it is not given. You should draw the force/moment vector on the FBD. Draw the position vector for calculating moment about the point you decide to apply the moment equation. Please record the assumptions for each problem. Please continue to see each can be a design problem. The problems in Section 5.1 can also be studies as problems in equilibrium. Note that all problems in this section are idealized as 2D even though they are actually 3D.

Problem 5.3.1

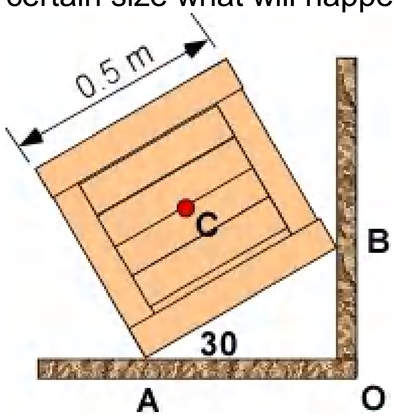
The heavy sign has a mass of 200 kg and it is uniformly distributed over the rectangular region. It is supported by a cable and a pin support at B. (a) Calculate the force in the cable. (b) Calculate the force at the pin? (c) Extend this to a design problem in selecting the cable. Use your own assumptions. (d) Extend this as a design problem in determining the dimensions of the pin in double shear at B.



Problem 5.3.1

Problem 5.3.2

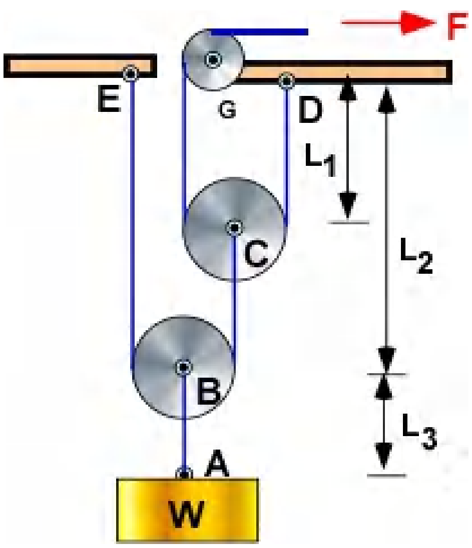
The heavy crate weighs 700 N. The geometry is square and the weight is uniformly distributed. It rests against the two plane surfaces as shown. (a) Calculate the reactions at A and B. (b) Solve the problem by considering moment equations about A and B. (c) Consider the crate is a rectangular instead and consider crates with increasing size in the direction normal to the line AB. Beyond a certain size what will happen to the crate?



Problem 5.3.2

Problem 5.3.3

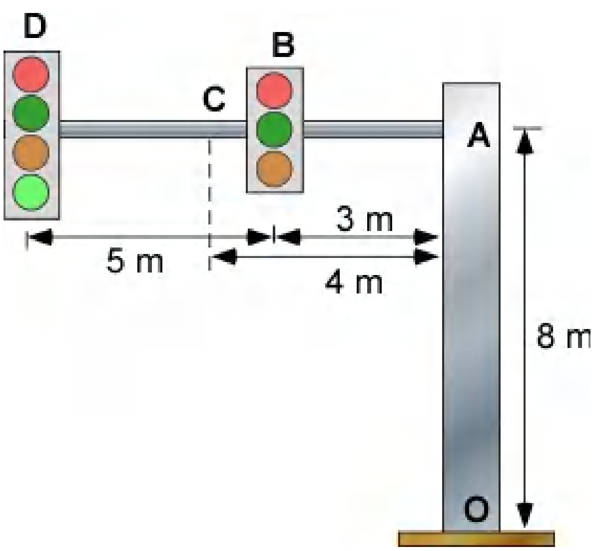
In this problem you can see the mechanical advantage of the pulley system. Without the pulley system you have to support the weight **W.** The system is in static equilibrium. The force applied **F** keeps the weight **W** in place. Consider a weight **W** of 1000 N. The diameters of pulley B and C are 0.2 m. The diameter of pulley G is 0.1 m. The cables are inextensible and the weight of the pulleys can be ignored. The pulleys are frictionless. The lengths L_1, L_2, and L_3 are 0.4m, 0.8 m, and 0.2 m respectively. Consider the FBD of pulley B and pulley C to solve the problem. (a) Calculate the force **F** for static equilibrium. (b) What is the force in the pin supporting pulley G? (c) Do you need the diameters of the pulleys to solve the problem? (d) Do you need the lengths of the cables to solve the problem.



Problem 5.3.3

Problem 5.3.4

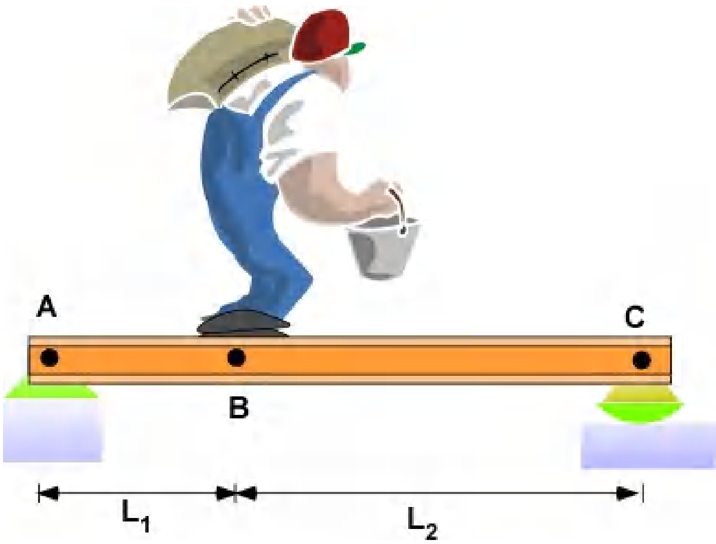
The two traffic signals are mounted on a horizontal rod of weight 450 N. The rod is attached at A to a solid column AO. The end at O is fixed. Thw weight of the signal at D is 250 N. The weight of the signal at B is 200 N. (a) Calculate the reactions at A. (b) Calculate the reactions at O.(c) What would be the design considerations at point A?



Problem 5.3.4

Problem 5.3.5

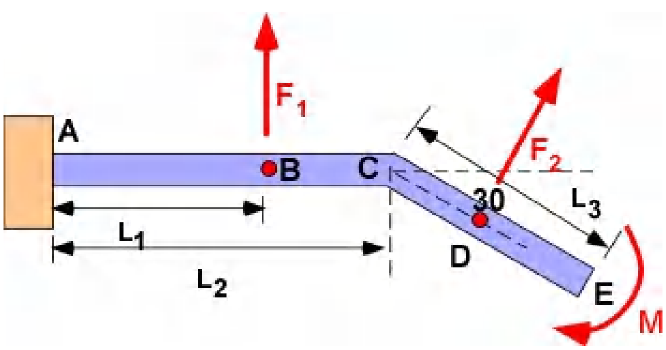
The heavy set man (mass = 150 kg) carrying a sack (mass 30 kg), and a full bucket of water (mass 15 kg) is standing on simple plank across the stream at B. The plank is pin supported at A and has a roller support at C. The length L_1 is 3 m while L_2 is 6 m. (a) Calculate the reaction at A. (b) Calculate the reaction at C. (c) Is the man exerting a moment at B ? (d) Solve the problem if the point C was also a pin. You can use your own assumptions.



Problem 5.3.5

Problem 5.3.6

The bent bar is subject to the loading shown. $\mathbf{F_1}$ and $\mathbf{F_2}$ are 250 N while **M** is 250 Nm. L_1 and L_3 are 0.3 m. L_2 is 0.5 m. (a) Calculate the reactions at the end A. (b) Move the force $\mathbf{F_1}$ to end A. (c) Move the force F_2 to end A. (d) Move the moment **M** to end A. (e) What is the difference between the result in (a) and the sum of the results in (b), (c) and (d)?



Problem 5.3.6

5.4 RIGID BODY EQUILIBRIUM - THREE DIMENSIONS

We first revisit Example 5.4 and explore it as a rigid body equilibrium problem and rename it as Example 5.12 below. You should notice that most of the computations are the same. The equilibrium vector equations are :

$$\sum \bar{F} = 0;$$
$$\sum \bar{M}_O = 0; \qquad (5.6)$$

This translates to six scalar equations with three force equations and three moment equations about any point O:

$$\sum F_x = 0;$$
$$\sum F_y = 0;$$
$$\sum F_z = 0;$$
$$\sum M_{Ox} = 0; \qquad (5.10)$$
$$\sum M_{Oy} = 0;$$
$$\sum M_{Oz} = 0;$$

In theory, Eqn. (5.10) is a set of six equations that can be used to solve six unknowns for a statically determinate problem. These are typically linear equations. Solving six equations in six unknowns is exhausting using elimination and this is where matrix handling and MATLAB are most useful.

5.4.1 Example 5.12

Problem Description

The sign in front of the establishment is being battered by the wind. You are concerned that it will be uprooted. The effect of the wind can be reproduced by a concentrated force F, located at A, of magnitude 1500 N and at an angle of α of 55 degrees to the plane of the sign which is in the yz plane. The stiffened sign weighs 980 N and the center of mass of the sign is at B. The lengths L_1, L_2, and L_3 are 6.5 m, 0.6m, and 1.2 m respectively. Find the reactions at the base of the pole.

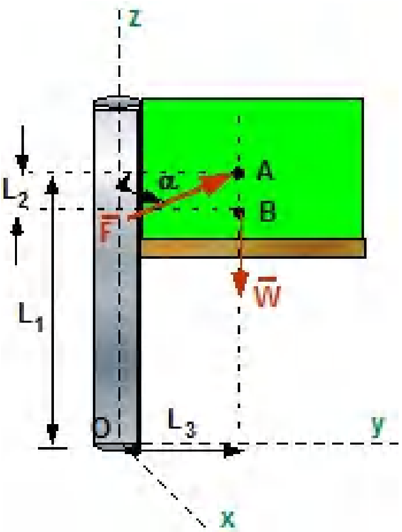


Figure 5.4.1 Problem description

Reaction

We have used the word reaction in the previous section. What does the word **reaction** mean? A simple way to understand it is to cut the structure at some point and consider the FBD of the cut portion structure subject to external loads. If the original structure was in equilibrium, then the cut portion is also in equilibrium. The cut section requires additional forces and moments at the cut to maintain equilibrium of the structure subject to external loads. These are known as **reactions**.

Another explanation for reactions is the loads required to support the structure to keep it operational. In this example, the ground must support the structure from moving and turning under the application of the weight and the force due to the wind. The ground provides a rigid support in this example. A rigid structure will require a force and a moment at the cut. A pinned structure will only require a force at the structure. Usually we are trying to solve for these reactions in structural problems.

On the other hand, a design problem is one in which the limit on the reactions are known and therefor one must find the largest force that can be applied to the structure to avoid failure.

Between rigid and pinned there are many other types of reactions possible but that must be identified before the problem can be solved. It is there for important to assess the type of reactions in the problem and displaying this on a FBD. If we are dealing with a two dimensional structure then the force is two dimensional and we have a single component for the moment or the couple for a rigid structure. This provides a total of three unknowns that must be obtained applying the equations of equilibrium. For a three dimensional rigid structure there will be a three component force and a three components of moment to be determined. A hinge or a pin requires no moment at the point or along the axis of the hinge.

Data: F = 1500 [N]; W = 980 [N], α= 55 [deg]; L_1=6.5 [m]; L_2= 0.6 [m]; L_3=1.2 [m];
Point C : [0,0,3]; Point D:[2,1.8,2.4]

Find: Reactions at the point O

Assumption: None

Solution: To calculate the reaction we need the FBD of the structure isolated from the ground. The ground will be represented by the reactions. There is force in an unknown direction represented by three components (O_x, O_y, and O_z). There is a moment whose direction is unknown and is represented by three components at O (M_x, M_y, M_z).

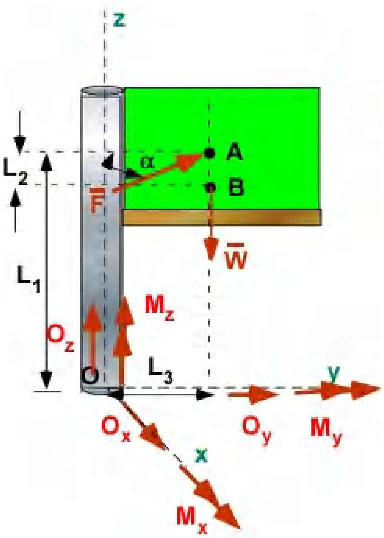


Figure 5.4.2 FBD of the structure with reactions

The equilibrium equations (Eqn. 5.6):

$$\sum \bar{F} = \bar{F} + \bar{W} + \bar{O} = 0$$

$$\sum \bar{M}_o = \bar{M} + \left(\bar{r}_{OA} \times \bar{F}\right) + \left(\bar{r}_{OB} \times \bar{W}\right) = 0$$

We have 6 unknowns and 6 equations to solve for the unknowns.

$$\bar{F} = 1500\left(-\sin 55\hat{i} + \cos 55\hat{j}\right) = -1.2287 \times 10^3\hat{i} + 0.8604 \times 10^3\hat{j}$$

$$\bar{W} = -980\hat{k}$$

$$\bar{O} = O_x\hat{i} + O_y\hat{j} + O_z\hat{k}$$

$$\left(\bar{r}_{OA} \times \bar{F}\right) = \left(1.2\hat{j} + 6.5\hat{k}\right) \times \left(-1.2287 \times 10^3\hat{i} + 0.8604 \times 10^3\hat{j}\right)$$

$$\left(\bar{r}_{OB} \times \bar{W}\right) = \left(1.2\hat{j} + 5.9\hat{k}\right) \times \left(-980\hat{k}\right)$$

$$\bar{M} = M_x\hat{i} + M_y\hat{j} + M_z\hat{k}$$

Substituting in (Eqn. 5.6)

$$\sum \bar{F} = 0 = \left(-1.2287\times10^3\,\hat{i} + 0.8604\times10^3\,\hat{j}\right) + \left(-980\hat{k}\right) +$$
$$\left(O_x\,\hat{i} + O_y\,\hat{j} + O_z\hat{k}\right)$$
$$Along\, i : -1.2287\times10^3 + O_x = 0; \quad O_x = 1.23\,kN$$
$$Along\, j : 0.8604\times10^3 + O_y = 0; \quad O_y = -0.86\,kN$$
$$Along\, k : -980 + O_z = 0; \quad O_z = 0.98\,kN$$
$$\sum \bar{M}_O = 0 = \left(M_x\hat{i} + M_y\hat{j} + M_z\hat{k}\right) + \left(1.2\,\hat{j} + 6.5\hat{k}\right)\times\left(-1.2287\times10^3\,\hat{i} + 0.8604\times10^3\,\hat{j}\right) +$$
$$\left(1.2\,\hat{j} + 5.9\hat{k}\right)\times\left(-980\,\hat{k}\right)$$
$$Along\, i : M_x - 6768.4.0 = 0; \quad M_x = 6768.4\,Nm$$
$$Along\, j : M_y - 7986.7 = 0; \quad M_y = 7986.7\,Nm$$
$$Along\, k : M_z + 1474.5 = 0; \quad M_z = -1474.5\,Nm$$

The equilibrium equations will give us six equations in six variables. They can be solved directly in this particular example as the equations are simple and each equations give a direct answer. In the following we use MATLAB.

Solution Using MATLAB

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.4.1- Example 5.12
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('-----------------------------------------\n')
fprintf('Example 5.12\n')
fprintf('------------------------------------------\n')

%% Data
F = 1500;    W = 980;  % the forces
L1 = 6.5;     L2 = 0.6;    L3 = 1.2;  % the distances
alpha = 55;    alphar = alpha*pi/180;  % alpha in radians
O = [0 0 0]; % point O
A = [0 L3 L1];   % point A
B = [0 L3 (L1 - L2)]; % point B
%% calculations
syms Ox Oy Oz Mx My Mz

FO = [Ox, Oy, Oz];
MO = [Mx, My, Mz];
eF = [-sin(alphar) cos(alphar) 0];  % unit vector in F direction
```

```
Fw = F*eF ; % vector F
Wv = W*[0 0 -1] ; % vector W

rOB = B - O ; % displacemnet vector from O to B
rOA = A - O  ;% displacement vector from O to A

MF = cross(rOA,Fw); % Moment due to F
MW = cross(rOB,Wv); % Moment due to W

%% equilibrim
SumF = FO + Fw + Wv ; % Sum of the forces
SumM = MO + MF + MW ;  % Sum of moments at O

sol = solve([SumF,SumM]);  % solution is a structure

%% Printing
fprintf('Point O [m]  : '),disp(O)
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
% fprintf('Point D [m]  : '),disp(D)
% fprintf('Point E [m]  : '),disp(E)

fprintf('\nPosition vector rOB [m]   = '),disp(rOB)
fprintf('Position vector rOA[m]   = '),disp(rOA)
% fprintf('Position vector rDA [m]   = '),disp(rDA)
% fprintf('Position vector rDB [m]   = '),disp(rDB)
% fprintf('Position vector rDC [m]   = '),disp(rDC)

fprintf('\nW [ N]    : '),disp(Wv)
fprintf('MF [ Nm]  : '),disp(MF)
fprintf('MW [ Nm]  : '),disp(MW)

% displaying symbolic values to three decimals
fprintf('\nSumF : '),disp(vpa(SumF,3))
fprintf('SumM : '),disp(vpa(SumM,3))

fprintf('\nResults:\n')
fprintf('-----------\n')
fprintf('Ox [N]      = '),disp(double(sol.Ox))
fprintf('Oy [N]      = '),disp(double(sol.Oy))
fprintf('Oz [N]      = '),disp(double(sol.Oz))
fprintf('Mx [Nm]     = '),disp(double(sol.Mx))
fprintf('My [Nm]     = '),disp(double(sol.My))
fprintf('Mz [Nm]     = '),disp(double(sol.Mz))
```

In the Command Window

```
----------------------------------------
Example 5.12
-----------------------------------------
Point O [m]  :      0      0      0
Point A [m]  :          0          1.2          6.5
Point B [m]  :          0          1.2          5.9

Position vector rOB [m]   =            0          1.2           5.9
```

```
Position vector rOA[m]    =              0          1.2           6.5

W [ N]     :       0      0   -980
MF [ Nm]   :        -5592.4       -7986.7       1474.5
MW [ Nm]   :         -1176            0            0

SumF : [ Ox - 1233.0, Oy + 860.0, Oz - 980.0]
SumM : [ Mx - 6777.0, My - 7999.0, Mz + 1477.0]

Results:
-----------
Ox [N]       =        1228.7
Oy [N]       =       -860.36
Oz [N]       =    980
Mx [Nm]      =        6768.4
My [Nm]      =        7986.7
Mz [Nm]      =       -1474.5
```

5.4.2 Some Support Reactions

You can find a table of support and appropriate reactions in any text book on Statics. Figure 5.4.3 is a collection of a few support types and the reactions.

Generally

- If the support does not allow change of linear displacement in any direction there will be a reaction force in that direction.
- If the support does not permit angular displacement along an axis there will be a reaction moment along that axis.

Displacement in any direction translates to components along the coordinate directions. Rotation along any axis translates to rotations along coordinate directions.

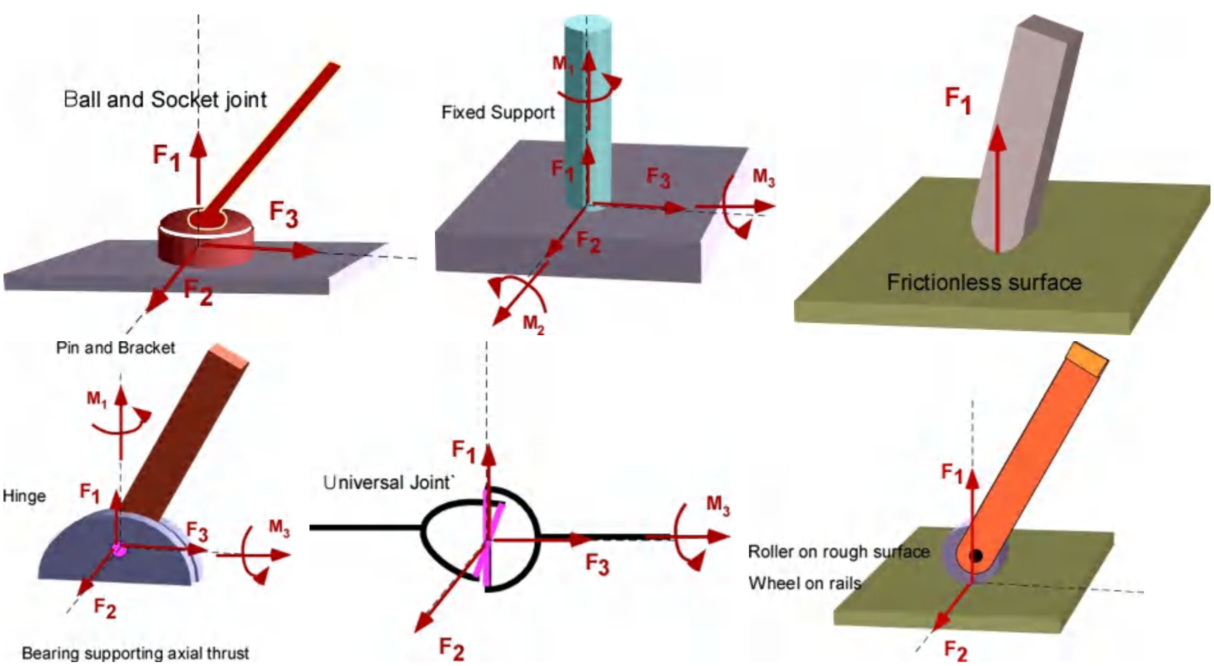


Figure 5.4.3 Some useful supports and corresponding reactions

5.4.3 Example 5.13

This example features a ball and socket joint. This can be considered a three-dimensional pin since it only provides force reaction and cannot support a moment along any axis.In this example we will hang a heavy sign, 500 kg, outside the House of Statics. The sign will be fixed to metal post whose weight is negligible compared to the weight of the sign. Since there are strong winds in the area, we will use a ball and socket support to attach the post to the wall. To ensure the sign stays in place we will attach three cables from the walls to two locations on the post. What are the tensions in the cable to keep the post horizontal. The design is illustrated in Figure 5.4.4 with all the dimensions. Cable supported problems are common for 3D equilibrium as they provided a single unknown in the magnitude of the force.

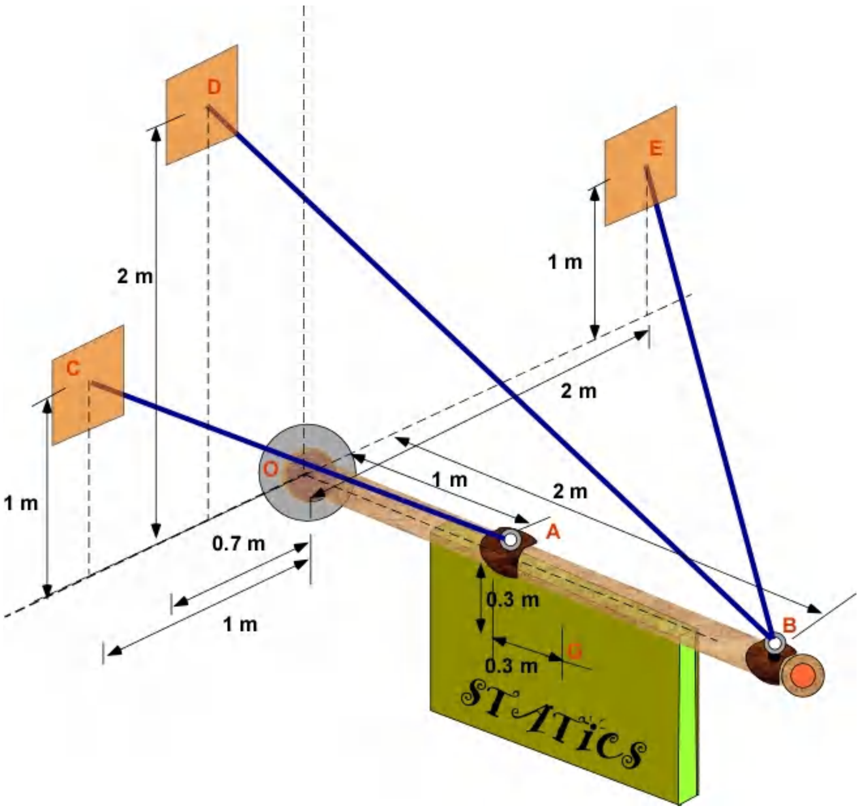


Figure 5.4.4 Problem description

Reactions: There are three reactions at the ball and socket joint which we do not know. There are three additional unknowns due to the force in each of the cable. We have a total of six unknowns and six equations for their solution. This is a statically determinate problem.

Note: However, this does not guarantee that the design will work. We need to make sure that the design can withstand forces other than the weight. If there are other forces on the sign in other directions it appears that the cables will be able to hold it in place. Could we have designed the system with one less cable? Remove cable AC. With strong winds from the right will the sign move all the way to left? The ball and socket joint will not resist a moment. If you have less unknowns than equations, the problem is considered ***under constrained***. If you have more unknowns than equations you have a ***statically indeterminate*** system. For this problem as originally specified we have a ***well constrained*** problem. It really does not answer the question is the design is satisfactory. For this you need ***knowledge*** and ***experience***. You need to view the problem as more than a ***mathematical exercise***. Let us go ahead and solve this problem

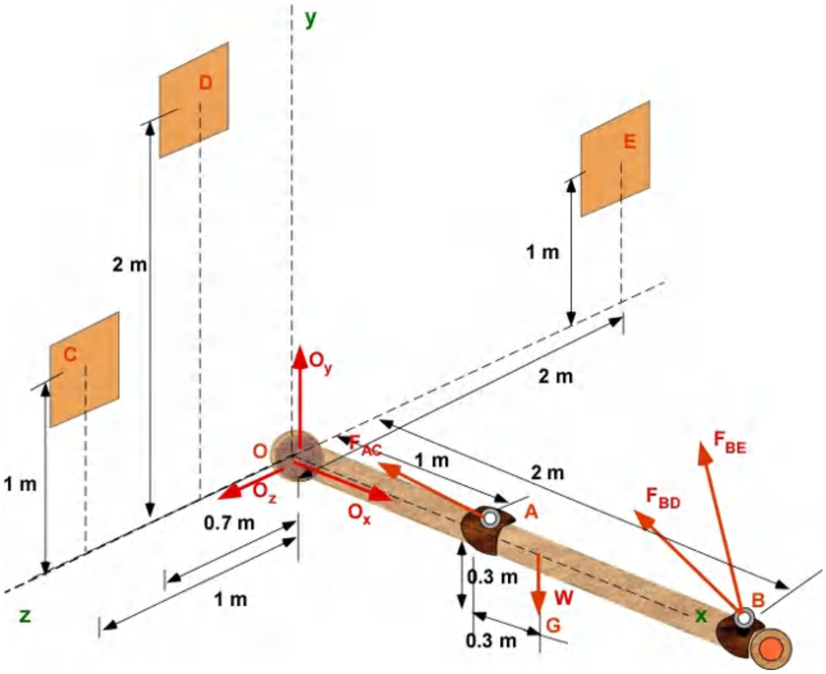


Figure 5.4.5 FBD of Example 5.13

Data: Applied force W = 500*9.81 [N]; Point O:[0,0,0]; Point A : [1,0,0]; ; Point B:[2,0,0]; Point G: [1.3,-0.3,0]; Point C: [0,1,1]; Point D: [0,2,0.7]; Point E: [0,1,-2]

Find: The reactions at O (O_x, O_y, O_z) and magnitude of forces $F_{AC,}$ F_{BD}, F_{BE}

Assumption: Ignore weight of the rigid body OB

Solution: Using the FBD of the rigid body

$$\sum \bar{F} = \bar{O} + \bar{W} + \bar{F}_{AC} + \bar{F}_{BD} + \bar{F}_{BE} = 0 \qquad (i)$$

$$\sum \bar{M}_O = \left(\bar{r}_{OA} \times \bar{F}_{AC}\right) + \left(\bar{r}_{OG} \times \bar{W}\right) + \left(\bar{r}_{OB} \times \bar{F}_{BD}\right) + \left(\bar{r}_{OB} \times \bar{F}_{BE}\right) = 0 \qquad (ii)$$

Solution Process:

For force equilibrium - (i)

- Find unit vector AC and define force F_{AC} with unknown magnitude
- Find unit vector BD and define force F_{BD} with unknown magnitude
- Find unit vector BE and define force F_{BE} with unknown magnitude
- Apply equilibrium and separate the terms in the coordinate directions

For moment equilibrium (ii)

- Find position vector r_{OA} and calculate moment of force F_{AC} about point O
- Find position vector r_{OG} and calculate moment of force W about point O
- Find position vector r_{OB} and calculate moment of force F_{BD} about point O
- Calculate moment of force F_{BE} about point O
- Apply equilibrium and separate the terms in the coordinate directions

Technically this should give **six** equations for **six** unknowns. *Does it?*
All forces intersect the x-axis. There can be no moment about the x-axis due to any of the forces since there is no distance from it. This will give you only *5 - equations to solve six unknowns.*

Please work out the details and see if the set of equations matches that identified by MATLAB.

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.4.3- Example 5.13
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('---------------------------------------\n')
fprintf('Example 5.13\n')
fprintf('----------------------------------------\n')

%%% Data
w = 500*9.81;     % the applied force
O = [0 0 0]; A = [1,0,0];  B= [2,0,0]; G = [1.3,-0.3,0];
C = [0,1,1];  D = [0,2,0.7];  E = [0,1,-2];

%% calculations
syms Ox Oy Oz Fac Fbd Fbe real

rAC = C - A;  eAC = rAC/norm(rAC);
rBD = D - B;  eBD = rBD/norm(rBD);
rBE = E - B;  eBE = rBE/norm(rBE);
rOA = A - O;
rOG = G - O;
rOB = B - O;

FAC = Fac*eAC;  FBD = Fbd*eBD;  FBE = Fbe*eBE;
W = w*[0,-1,0];
FO = [Ox, Oy, Oz];

%% equilibrim
SumF = FO + W + + FAC + FBD + FBE ;  % Sum of the forces
SumMO = cross(rOA,FAC) + cross(rOG,W)+ ...
    cross(rOB,FBD)+ cross(rOB,FBE) ;  % Sum of moments at O

sol = solve([SumF,SumMO]); % solution is a structure
%% Printing
fprintf('Point O [m]  : '),disp(O)
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point G [m]  : '),disp(G)
fprintf('Point C [m]  : '),disp(C)
fprintf('Point D [m]  : '),disp(D)
fprintf('Point E [m]  : '),disp(E)

fprintf('\nPosition vector rOA [m]   = '),disp(rOA)
fprintf('Position vector rOG[m]   = '),disp(rOG)
fprintf('Position vector rOB [m]   = '),disp(rOB)
fprintf('Unit vector eAC [m]       = '),disp(eAC)
fprintf('Unit vector eBD [m]       = '),disp(eBD)
```

```
fprintf('Unit vector eBE [m]        = '),disp(eBE)

fprintf('\nW [ N]     : '),disp(W)

% displaying symbolic values to three decimals
fprintf('\nSumF : \n'),disp(vpa(SumF',4))
fprintf('SumMO : \n'),disp(vpa(SumMO',4))

fprintf('\nResults:\n')
fprintf('------------\n')
fprintf('Ox [N]      = '),disp(double(sol.Ox))
fprintf('Oy [N]      = '),disp(double(sol.Oy))
fprintf('Oz [N]      = '),disp(double(sol.Oz))
fprintf('Fac [N]     = '),disp(double(sol.Fac))
fprintf('Fbd [N]     = '),disp(double(sol.Fbd))
fprintf('Fbe [N]     = '),disp(double(sol.Fbe))
```

In the Command Window

```
----------------------------------------
Example 5.13
-----------------------------------------
Point O [m]  :      0     0     0
Point A [m]  :      1     0     0
Point B [m]  :      2     0     0
Point G [m]  :          1.3          -0.3            0
Point C [m]  :      0     1     1
Point D [m]  :            0             2          0.7
Point E [m]  :      0     1    -2

Position vector rOA [m]   =      1     0     0
Position vector rOG[m]   =          1.3          -0.3            0
Position vector rOB [m]   =      2     0     0
Unit vector eAC [m]       =     -0.57735      0.57735      0.57735
Unit vector eBD [m]       =      -0.6864       0.6864      0.24024
Unit vector eBE [m]       =     -0.66667      0.33333     -0.66667

W [ N]    :            0       -4905           0

SumF :
          Ox - 0.6864*Fbd - 0.6667*Fbe - 0.5774*Fac
 0.5774*Fac + 0.6864*Fbd + 0.3333*Fbe + Oy - 4905.0
          0.5774*Fac + 0.2402*Fbd - 0.6667*Fbe + Oz
SumMO :
                                            0
          1.333*Fbe - 0.4805*Fbd - 0.5774*Fac
 0.5774*Fac + 1.373*Fbd + 0.6667*Fbe - 6376.0

Results:
------------
Ox [N]      =        3663.1
Oy [N]      =        1716.8
Oz [N]      =      0
Fac [N]     =      0
Fbd [N]     =        3953.1
Fbe [N]     =        1424.5
```

Revisiting Equilibrium Equations

We will write the equilibrium equations in matrix form as it may be easier to interpret them. The equations are numbered for discussion

$$\begin{bmatrix} 1 & 0 & 0 & -0.6864 & -0.6667 & -0.5774 \\ 0 & 1 & 0 & 0.6864 & 0.3333 & 0.5774 \\ 0 & 0 & 1 & 0.2402 & -0.6667 & 0.5774 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4805 & 1.333 & -0.5774 \\ 0 & 0 & 0 & 1.373 & 0.6667 & 0.5774 \end{bmatrix} \begin{bmatrix} O_x \\ O_y \\ O_z \\ F_{bd} \\ F_{be} \\ F_{ac} \end{bmatrix} = \begin{bmatrix} 0 \\ 4905 \\ 0 \\ 0 \\ 0 \\ 6376 \end{bmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$$

Adding Eq.(3) + 0.5*Eq.(5) will result in Oy +0.5*0.5774*Fac = 0. Therefore Oy = —0.2887 Fac.

A simple solution is Oy = Fac = 0 which is what MATLAB provides. It is difficult to establish the solution analytically.

Effect on Design - an Exploration:

Do we really need both cables AC and BD? Eliminating one we can solve this problem and makes a simple design since both attempt to do the same in the problem. If we remove cable AC we have the following results.
Setting Fac = 0 in MATLAB code yields the following results:

In the Command Window

```
Results:
------------
Ox [N]      =        3663.1
Oy [N]      =        1716.8
Oz [N]      =      0
Fbd [N]     =        3953.1
Fbe [N]     =        1424.5
```

Another Exploration: Another Solution by Taking Moment about point E

First the moment equilibrium equation can be applied about any point in the problem. You should get the same solution. If this is correct then the choice of the moment equation is a convenient issue. Let us consider an inconvenient point E:

- This should determine the same solution
- This will verify that we can apply the moment about any point
- This may include a complete set of six equitations in six unknowns

The equilibrium equations:

$$\sum \bar{F} = \bar{O} + \bar{W} + \bar{F}_{AC} + \bar{F}_{BD} + \bar{F}_{BE} = 0 \quad (i)$$

$$\sum \bar{M}_E = \left(\bar{r}_{EA} \times \bar{F}_{AC}\right) + \left(\bar{r}_{EG} \times \bar{W}\right) + \left(\bar{r}_{EB} \times \bar{F}_{BD}\right) + \left(\bar{r}_{EB} \times \bar{F}_{BE}\right) + \left(\bar{r}_{EO} \times \bar{F}_O\right) = 0 \, (ii)$$

This involves simple change to the previous MATLAB Code. The results are included below.

In the Command Window

```
-----------------------------------------
Example 5.13- ver 2
------------------------------------------
Point 0 [m]  :      0     0     0
Point A [m]  :      1     0     0
Point B [m]  :      2     0     0
Point G [m]  :          1.3         -0.3            0
Point C [m]  :      0     1     1
Point D [m]  :            0             2          0.7
Point E [m]  :      0     1    -2

Position vector rEA [m]   =      1    -1     2
Position vector rEG[m]   =          1.3          -1.3            2
Position vector rEB [m]   =      2    -1     2
Unit vector eAC [m]       =      -0.57735      0.57735      0.57735
Unit vector eBD [m]       =       -0.6864       0.6864      0.24024
Unit vector eBE [m]       =      -0.66667      0.33333     -0.66667

W [ N]    :           0       -4905           0

SumF :
          Ox - 0.6864*Fbd - 0.6667*Fbe - 0.5774*Fac
 0.5774*Fac + 0.6864*Fbd + 0.3333*Fbe + Oy - 4905.0
          0.5774*Fac + 0.2402*Fbd - 0.6667*Fbe + Oz
SumME :
 9810.0 - 1.613*Fbd - 2.0*Oy - 1.0*Oz - 1.732*Fac
                   2.0*Ox - 1.853*Fbd - 1.732*Fac
                         0.6864*Fbd + Ox - 6376.0

Results:
------------
Ox [N]     =        3663.1
Oy [N]     =        1716.8
Oz [N]     =      0
Fac [N]    =      0
Fbd [N]    =        3953.1
Fbe [N]    =        1424.5
```

We get the same solution. We also have a six equations for equilibrium. The solutions are obtained explicitly.

5.4.4 Example 5.14 : Statically Indeterminate Problem

Example 5.13 Modified with Fixed Support

Let us consider Example 5.13 with a fixed support at O instead of a ball and socket joint at O and two cables BD and BE. In this new design we really do not need any cables. However, because the sign is heavy this will require a large post, whose weight might be consequential. In the interest of public safety it might be required to have the cables BD and BE to lighten the loads at the support O, as well as providing providing some backup in case the metal post starts to wear down or corrode. This is now a design problem. We will start with the FBD.

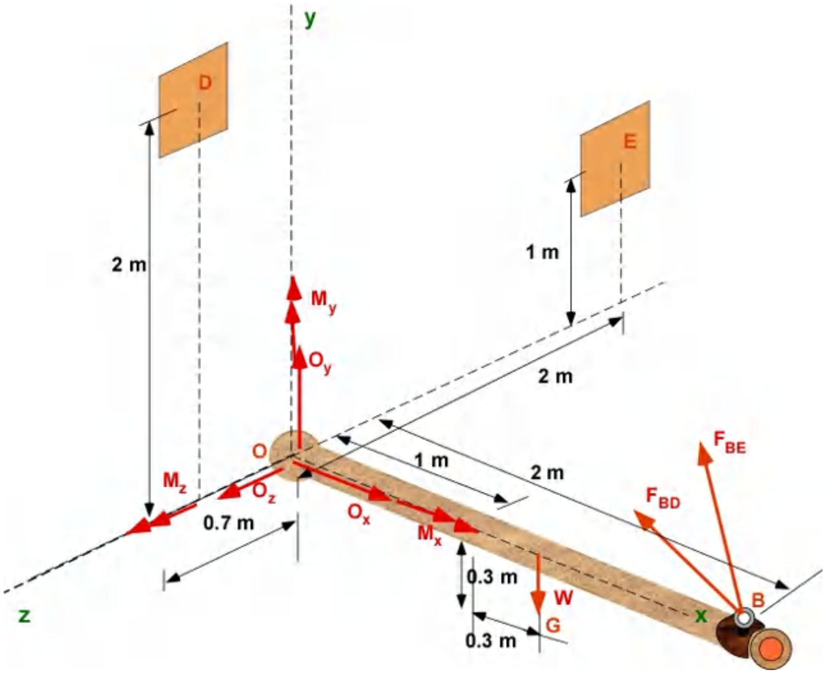


Figure 5.4.6 FBD with fixed support

As you can see we now have 6 unknown reactions at O plus to forces in the cables as unknowns. This is statically indeterminate to degree two as there are only 6 equations of equilibrium. Once again as a design exercise we can solve the reactions in terms of the two cable forces $\mathbf{F_{bd}}$ and $\mathbf{F_{be}}$ and plot the results easily. Note that the cables themselves can be designed to have minimum diameter.

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.4.4- Example 5.14
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 5.14 \n')
fprintf('-----------------------------------------\n')

%% Data
w = 500*9.81;     % the applied force
O = [0 0 0]; A = [1,0,0];  B= [2,0,0]; G = [1.3,-0.3,0];
C = [0,1,1];  D = [0,2,0.7];  E = [0,1,-2];

%% calculations
syms Ox Oy Oz Mx My Mz Fbd Fbe real

% rAC = C - A;  eAC = rAC/norm(rAC);
rBD = D - B;  eBD = rBD/norm(rBD);
```

```
rBE = E - B;  eBE = rBE/norm(rBE);
rEA = A - E;
rEG = G - E;
rEB = B - E;
rEO = O - E;

FBD = Fbd*eBD;  FBE = Fbe*eBE;
W = w*[0,-1,0];
FO = [Ox, Oy, Oz];
MO = [Mx, My, Mz];

%% equilibrim
SumF = FO + W  + FBD + FBE;  % Sum of the forces
SumME = MO + cross(rEG,W)+ ...
    cross(rEB,FBD)+ cross(rEB,FBE) + cross(rEO,FO) ;
% Sum of moments at O

sol = solve([SumF,SumME],Ox, Oy, Oz, Mx, My, Mz);
% solution is a structure

% Printing
fprintf('Point O [m]  : '),disp(O)
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point G [m]  : '),disp(G)
% fprintf('Point C [m]  : '),disp(C)
fprintf('Point D [m]  : '),disp(D)
fprintf('Point E [m]  : '),disp(E)

fprintf('\nPosition vector rEA [m]   = '),disp(rEA)
fprintf('Position vector rEG[m]    = '),disp(rEG)
fprintf('Position vector rEB [m]   = '),disp(rEB)
fprintf('Position vector rEO [m]   = '),disp(rEO)
fprintf('Unit vector eBD [m]       = '),disp(eBD)
fprintf('Unit vector eBE [m]       = '),disp(eBE)

fprintf('\nW [ N]    : '),disp(W)
%
% displaying symbolic values to three decimals
fprintf('\nSumF : \n'),disp(vpa(SumF',4))
fprintf('SumME : \n'),disp(vpa(SumME',4))

fprintf('\nResults:\n')
fprintf('------------\n')
fprintf('Ox [N]     = '),disp(vpa(sol.Ox,4))
fprintf('Oy [N]     = '),disp(vpa(sol.Oy,4))
fprintf('Oz [N]     = '),disp(vpa(sol.Oz,4))
fprintf('Mx [Nm]    = '),disp(vpa(sol.Mx,4))
fprintf('My [Nm]    = '),disp(vpa(sol.My,4))
fprintf('Mz [Nm]    = '),disp(vpa(sol.Mz,4)
```

In the Command Window

```
----------------------------------------
Example 5.14
```

```
-----------------------------------------
Point 0 [m]  :      0     0     0
Point A [m]  :      1     0     0
Point B [m]  :      2     0     0
Point G [m]  :           1.3         -0.3            0
Point D [m]  :             0            2          0.7
Point E [m]  :      0     1    -2

Position vector rEA [m]    =      1    -1     2
Position vector rEG[m]     =           1.3          -1.3             2
Position vector rEB [m]    =      2    -1     2
Position vector rEO [m]    =      0    -1     2
Unit vector eBD [m]        =       -0.6864       0.6864       0.24024
Unit vector eBE [m]        =      -0.66667      0.33333      -0.66667

W [ N]    :            0       -4905           0

SumF :
          Ox - 0.6667*Fbe - 0.6864*Fbd
 0.6864*Fbd + 0.3333*Fbe + Oy - 4905.0
          0.2402*Fbd - 0.6667*Fbe + Oz
SumME :
 Mx - 1.613*Fbd - 2.0*Oy - 1.0*Oz + 9810.0
                   My - 1.853*Fbd + 2.0*Ox
             0.6864*Fbd + Mz + Ox - 6376.0

Results:
------------
Ox [N]      = 0.6864*Fbd + 0.6667*Fbe
Oy [N]      = 4905.0 - 0.3333*Fbe - 0.6864*Fbd
Oz [N]      = 0.6667*Fbe - 0.2402*Fbd
Mx [Nm]     = 0.0
My [Nm]     = 0.4805*Fbd - 1.333*Fbe
Mz [Nm]     = 6376.0 - 0.6667*Fbe - 1.373*Fbd
```

The reactions are all *linear functions* of two unknown - the cable forces $\mathbf{F_{bd}}$ and $\mathbf{F_{be}}$. This can generate a 3D plot. Unlike 2D plots it is difficult to interpret multiple 3D plots on the same figure. It would be a good idea to draw a single plot for each reaction. Since there is no moment resisted along the x-axis, and from the view point of failure, the moments are more important, let us plot the reactions M_y and M_z to different values of $\mathbf{F_{be}}$ and $\mathbf{F_{bd}}$

In the Editor (continued)

```
%%%%%%%%%%%%%%%%%%%%%%
%% Plotting
%%%%%%%%%%%%%%%%%%%%%%
set(gcf,'Position',[25,50,350,300])
ezcontour(sol.My,[0,1000,0,1000]);
xlabel('Fbd')
ylabel('Fbe')
title('Reaction Moment My at O [Nm]')
colorbar
grid on
```

```
figure
set(gcf,'Position',[25,50,350,300])
ezcontour(sol.Mz,[0,1000,0,1000]);
xlabel('Fbd')
ylabel('Fbe')
title('Reaction Moment Mz at O [Nm]')
colorbar
grid on
```

In the Figure Window

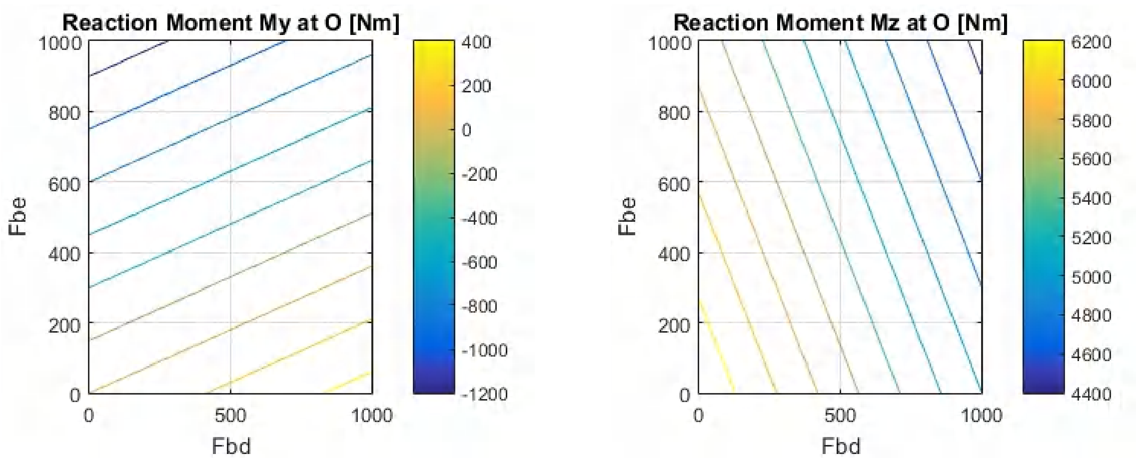


Figure 5.4.7 Reaction Moment M_y and M_z at O - Example 5.14

As you expected, increase in F_{be} and F_{bd} decreases the moment reactions. Whil we can design the cables, the bending moments introduces stresses on the rigid body. We cannot complete our failure analysis as we have not yet learned how to calculate stresses associated with forces moments on a rigid body. That should soon appear in the following chapters. We have just learned that problem definition is very important.

You can now see that you ***cannot have more than one fixed support*** in a structural problem because of ***indeterminacy.*** There are only six equations of equilibrium for a rigid body. It will be difficult to keep track of so many unknown reactions. While software can do it possibly, we still need additional relations to identify the reactions usefully. These come from constraints on deflections and stresses - particularly to avoid failure. For example, in the original problem 5.13 of this section, the moment equation could have solved for forces in the cable directly, since we can easily determine if the cable's will fail because of the one-dimensional stress calculations associated with the cable.

5.4.5 Example 5.15

Here in another example with indeterminacy. Notice we continue to use cables as they just require one unknown. A single unknown requires a single equation and this can be established by taking a moment about an axis rather than about a point. In this example, we are interested in the force in the cable to select the diameter of the cable. Figure 5.4.8 is the problem definition. There are two ball and socket connections in the problem at A and D. There are three reactions at each of the ball and socket support at A and D. The force in the cable is also an unknown but F_{bf} and F_{be} are the same since the same cable passes through a ring at B. We have 7 unknowns so that the problem is statically indeterminate. We can determine the cable force by taking a moment along the line AD. This will completely eliminate the reactions at A and D.

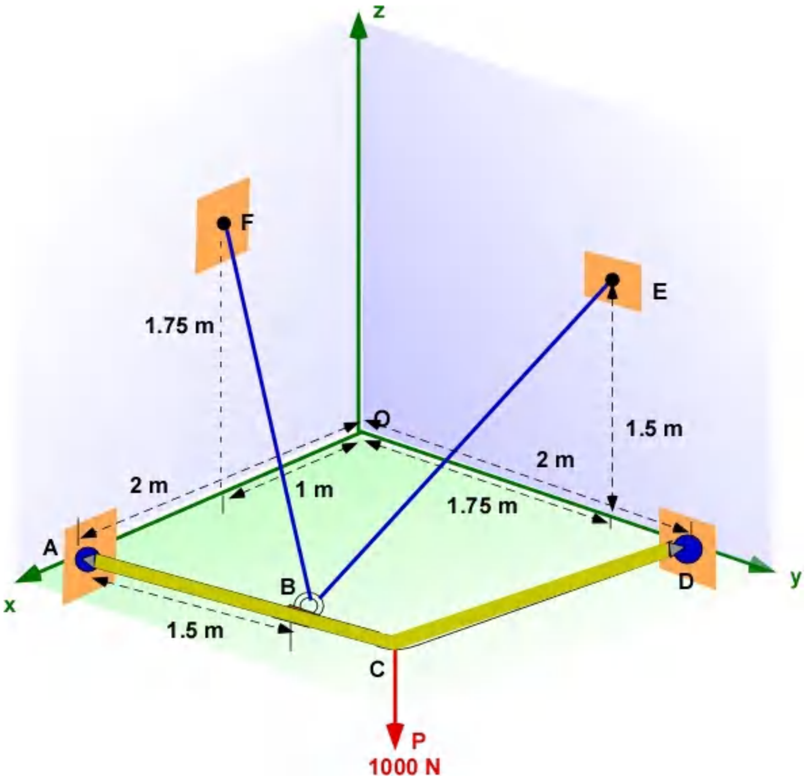


Figure 5.4.8 Problem definition - Example 5.15

Data: Applied force P = 1000 [N]; Point O:[0, 0, 0]; Point A : [2, 0, 0]; ; Point B:[2,1.5,0]; Point C: [2, 2,0]; Point D: [0, 2 0]; Point E: [0, 1.75, 1.5]; Point F: [1, 0, 1.75]

Find: Force in the cable F_{BF} (or F_{BF})

Assumption: Ignore weight of the rigid body ACD

Solution: Consider the FBD of rigid body ACD in Figure 5.4.9. We can take moments first at D and then use the dot product with the unit vector from A to D to calculate the moment along the line AD. The only contribution to moment along AD will be from F_{BF}, F_{BF} and the applied force P. Note there are 7 unknowns in this problem.

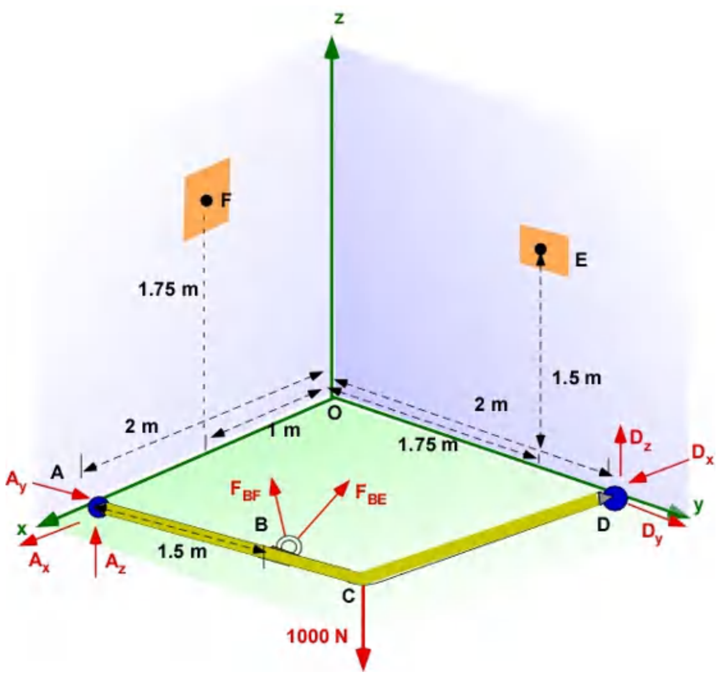


Figure 5.4.9 FBD of Example 5.14

Equilibrium Equations

$$\sum \bar{F} = \bar{A} + \bar{D} + \bar{P} + \bar{F}_{BE} + \bar{F}_{BF} = 0 \qquad (i)$$

$$\sum \bar{M}_{AD} = \hat{e}_{AD} \times \left(\sum \bar{M}_D\right) \quad = 0 \qquad (ii)$$

with

$$\sum \bar{M}_D = \left(\bar{r}_{DA} \times \bar{A}\right) + \left(\bar{r}_{DB} \times \bar{F}_{BE}\right) + \left(\bar{r}_{DB} \times \bar{F}_{BF}\right) + \left(\bar{r}_{DC} \times \bar{P}\right)$$

Please solve the problem and compare your solution at various steps with MATLAB below.

Solution Using MATLAB
In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 5.4.5- Example 5.15
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('----------------------------------------\n')
fprintf('Example 5.15 \n')
fprintf('-----------------------------------------\n')

%% Data
p = 1000;     % the applied force
O = [0 0 0]; A = [2,0,0];  B= [2,1.5,0]; C = [2,2,0]; D = [0,2,0];
E = [0,1.75,1.5];  F = [1,0,1.75];

%% calculations
syms Ax Ay Az Dx Dy Dz Fbf  real % Fde = Fbf in magnitude
```

```
rDA = A - D;  rAD = D - A;  eAD = rAD/norm(rAD);
rDB = B - D;
rDC = C - D;
rBF = F - B;  eBF = rBF/norm(rBF);
rBE = E - B;  eBE = rBE/norm(rBE);

FBF = Fbf*eBF;  FBE = Fbf*eBE;
P = p*[0,0,-1];
A = [Ax Ay, Az];
D = [Dx, Dy, Dz];
%% Moment about line AD
SumMD = cross(rDA,A)+ cross(rDB,FBF)+ cross(rDB,FBE) + cross(rDC,P);
% Sum of moments at D

SumM_AD = dot(SumMD,eAD);
sol =solve(SumM_AD);
%%%%%%%%%%%%%%%%%%%%%%
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('Point 0 [m]  : '),disp(O)
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point C [m]  : '),disp(C)
fprintf('Point D [m]  : '),disp(D)
fprintf('Point E [m]  : '),disp(E)
fprintf('Point F [m]  : '),disp(F)

fprintf('\nPosition vector rDA [m]   = '),disp(rDA)
fprintf('Position vector rDB[m]    = '),disp(rDB)
fprintf('Position vector rDC [m]   = '),disp(rDC)
% fprintf('Position vector rEO [m]   = '),disp(rEO)
fprintf('Unit vector eBF [m]       = '),disp(eBF)
fprintf('Unit vector eBE [m]       = '),disp(eBE)

fprintf('\nP [ N]     : '),disp(P)

% displaying symbolic values to four decimals
fprintf('\nSumMD : \n'),disp(vpa(SumMD',4))
fprintf('\nSumM_AD : \n'),disp(vpa(SumM_AD',4))

fprintf('\nResults:\n')
fprintf('------------\n')
fprintf('Fbf [N]      = '),disp(double(sol))
```

In the Command Window

```
----------------------------------------
Example 5.15
-----------------------------------------
Point 0 [m]  :      0      0      0
Point A [m]  : [ Ax, Ay, Az]
Point B [m]  :            2          1.5            0
Point C [m]  :      2      2      0
Point D [m]  : [ Dx, Dy, Dz]
Point E [m]  :            0         1.75          1.5
```

```
Point F [m]   :              1            0         1.75

Position vector rDA [m]    =      2    -2     0
Position vector rDB[m]     =             2          -0.5            0
Position vector rDC [m]    =      2     0     0
Unit vector eBF [m]        =      -0.39801     -0.59702      0.69653
Unit vector eBE [m]        =      -0.79603     0.099504      0.59702

P [ N]    :            0           0       -1000

SumMD :
       - 2.0*Az - 0.6468*Fbf
 2000.0 - 2.587*Fbf - 2.0*Az
 2.0*Ax + 2.0*Ay - 1.592*Fbf

SumM_AD :
1414.0 - 1.372*Fbf

Results:
------------
Fbf [N]     =        1030.8
```

Execution in Octave

The code is the same as in MATLAB except for the additional statements below . The changes are highlighted. You must include the symbolic package and if you do not wish to see warnings you include the command warning off as shown

```
clc, clear, format compact, close all, format short G, warning off

pkg load symbolic
sympref display flat
%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

In Octave Command Window

```
----------------------------------------
Example 5.15
-----------------------------------------
Point 0 [m]   :   0  0  0
Point A [m]   :   Matrix([[Ax, Ay, Az]])
Point B [m]   :   2  1.5  0
Point C [m]   :   2  2  0
Point D [m]   :   Matrix([[Dx, Dy, Dz]])
Point E [m]   :   0  1.75  1.5
Point F [m]   :   1  0  1.75

Position vector rDA [m]    =   2  -2  0
Position vector rDB[m]     =   2  -0.5  0
Position vector rDC [m]    =   2  0  0
Unit vector eBF [m]        =   -0.39801  -0.59702  0.69653
Unit vector eBE [m]        =   -0.79603  0.099504  0.59702

P [ N]    :   0  0  -1000

SumMD :
```

```
  Matrix([[-2.0*Az - 0.6468*Fbf], [-2.0*Az - 2.587*Fbf + 2000.0], [2.0*Ax
+ 2.0*Ay - 1.592*Fbf]])

SumM_AD :
  -1.372*Fbf + 1414.0

Results:
------------
Fbf [N]     = 1030.8
```

The results are the same as in MATLAB. However the fprintf command did not express SumMD in column form. This needs investigation. Nevertheless the information is correct.

5.4.6 Example 5.16

This appeared as a problem in understanding three dimensional vectors. This is a good example for an equilibrium problem even if it appears simple. We will use it to understand some essential features of applying equilibrium equations.

Problem: A heavy structure of W = 2000 N is displayed by suspending it through three cables from three towers which are 30 m tall. The tower base is on a circle of radius 10 m and the angular offset is shown on the figure. Can D also be at the same height as the towers?. Should it sag? Let us solve the problem for point D to be at 27 m above O.

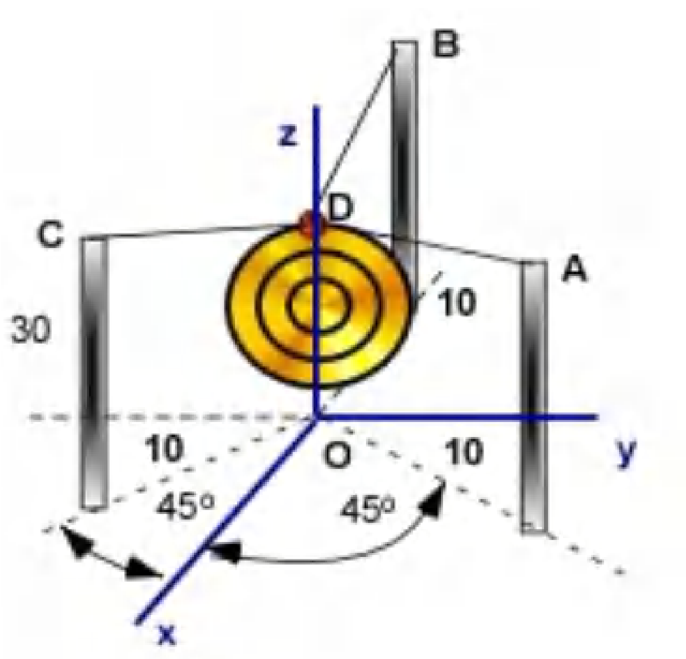


Figure 5.4.10 Example 5.16

The FBD of the problem is the FBD of the p oint D since all the force intersect There.

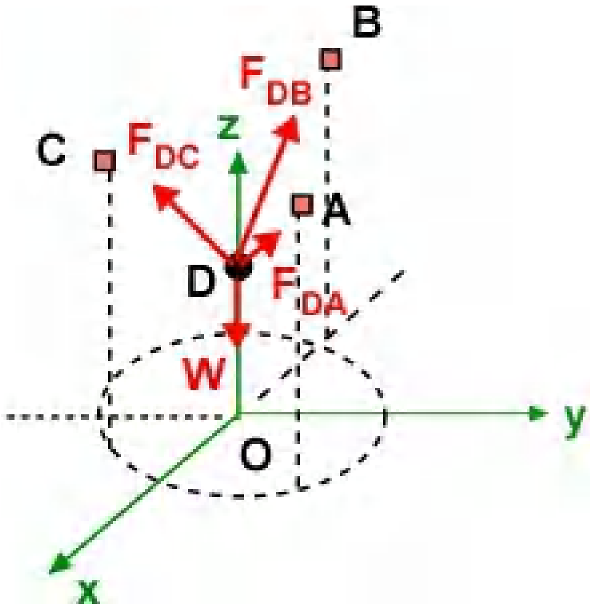


Figure 5.4.11 FBD of Example 5.16

Data: W = 2000 [N]; Point O:[0, 0, 0]; Point A : [10*cos45, 10*sin45, 30]; ; Point B:[-10,0,30]; Point C: [10*cos45,-10*sin45,30]; Point D: [0, 0, 27];

Find: Force in the cable F_{AD}, F_{BD}, F_{DC}

Assumption: This is a concurrent force system.

Solution 1: Consider the FBD of rigid body at D in Figure 5.4.11.

Prior to working out the solution it appears that there are three unknowns and the concurrent force suggests that force equilibrium should be sufficient to solve this problem. The location of the towers suggest that the problem is symmetric about the x-axis. Therefore you can expect the solution for F_{da} and F_{dc} will be the same.

Force Equilibrium:

$$\sum \bar{F} = 0 = \bar{F}_{DA} + \bar{F}_{DB} + \bar{F}_{DC} + \bar{W}$$

You can verify that:

$$\bar{F}_{DA} = F_{da}\left[0.68\hat{i} + 0.68\hat{j} + 0.29\hat{k}\right]$$

$$\bar{F}_{DB} = F_{db}\left[-0.96\hat{i} + 0\hat{j} + 0.29\hat{k}\right]$$

$$\bar{F}_{DC} = F_{dc}\left[0.68\hat{i} - 0.68\hat{j} + 0.29\hat{k}\right]$$

The equilibrium equations along the coordinate directions are:

$$\sum F_x = 0 = 0.68F_{da} - 0.96F_{db} + 0.68F_{dc}$$

$$\sum F_y = 0 = 0.68F_{da} - 0.68F_{dc}$$

$$\sum F_z = 0 = 0.29F_{da} + 0.29F_{db} + 0.29F_{dc} - 2000$$

The solution is:

$$F_{da} = 2038.6[N]$$

$$F_{db} = 2883[N]$$

$$F_{dc} = 2038.6[N]$$

The solution is symmetric as expected.

Design Discussion:
We accept the the cables must be stretched. Let us use cables of the same material. Consider the following questions:

Q1. Are the stretched lengths the same for all three cables. They should be. The base is located along a circle. The point D is above O. You can check it out.
Q2. Can we assume that the unstretched lengths for the cables are the same? - For example the value of the radius.
Q3. If the final lengths are same and the initial lengths are the same then is the strain is the same in each cable?
Q4.Are the stress in each cable the same?
Q5 What is the design issue involved.

Solution 2 (Using Moment Equilibrium)
We have three unknowns and need three equations. Can you solve the problem by applying moment equilibrium about any point?
Point D is not useful as the moment equation is identically zero?
Let us try and apply it about point O. Do you think this will be useful?

$$\sum \bar{M}_O = 0 = \left(\bar{r}_{OD} \times \bar{F}_{DA}\right) + \left(\bar{r}_{OD} \times \bar{F}_{DB}\right) + \left(\bar{r}_{OD} \times \bar{F}_{DC}\right) + \left(\bar{r}_{OD} \times \bar{W}\right)$$

Applying the moment equations we obtain the following equations for moment equilibrium about point O

$$\sum M_{Ox} = 18.29F_{dc} - 18.29F_{da} = 0$$

$$\sum M_{Oy} = 18.29F_{da} - 25.86F_{db} + 18.29F_{dc} = 0$$

$$\sum M_{Oz} = 0$$

There are only two equations to solve for three unknowns.
The first of the above equations sets up $F_{dc} = F_{da}$ which is expected.
The second equation sets up $F_{da} = 0.707\ F_{db}$. This appears correct from Solution 1.
However we cannot explicitly solve for the force. The weight is not involved in the set of equilibrium

equations.

If you include the equation force equilibrium in the z-direction to complete the set of equations then you get the same result as in Solution 1.

The thumb rule is that *force equilibrium enjoys priority in equilibrium problems.*

Solution 3. (Moment equilibrium about B)
What if you applied the moment equilibrium about point B?

$$\sum \bar{M}_B = 0 = \left(\bar{r}_{BD} \times \bar{F}_{DA}\right) + \left(\bar{r}_{BD} \times \bar{F}_{DB}\right) + \left(\bar{r}_{BD} \times \bar{F}_{DC}\right) + \left(\bar{r}_{BD} \times \bar{W}\right)$$

Applying the above equation:

$$\sum M_{Ox} = 2.032F_{da} - 2.032F_{dc} = 0$$
$$\sum M_{Oy} = 20000 - 4.905F_{dc} - 4.905F_{da} = 0$$
$$\sum M_{Oz} = 6.773F_{da} - 6.773F_{dc} = 0$$

You will see three equations in two unknowns F_{da} and F_{dc}.
The first and the third equations sets up $F_{dc} = F_{da}$ which is expected.
The second solves for $F_{da} = F_{dc} = 2038.6$ N

But you cannot solve for F_{db}.

5.4.7 Example 5.17

Once again we will explore a simple problem to understand some additional design issues. In this example we have a three legged bar stool and the weight is offset from the center. The problem is defined in Figure 5.4.12

Reactions: The reactions of the legs with the floor is a force normal to the floor at A, B, and C. The stool is not restrained against horizontal movement or rotations. There are three unknowns.

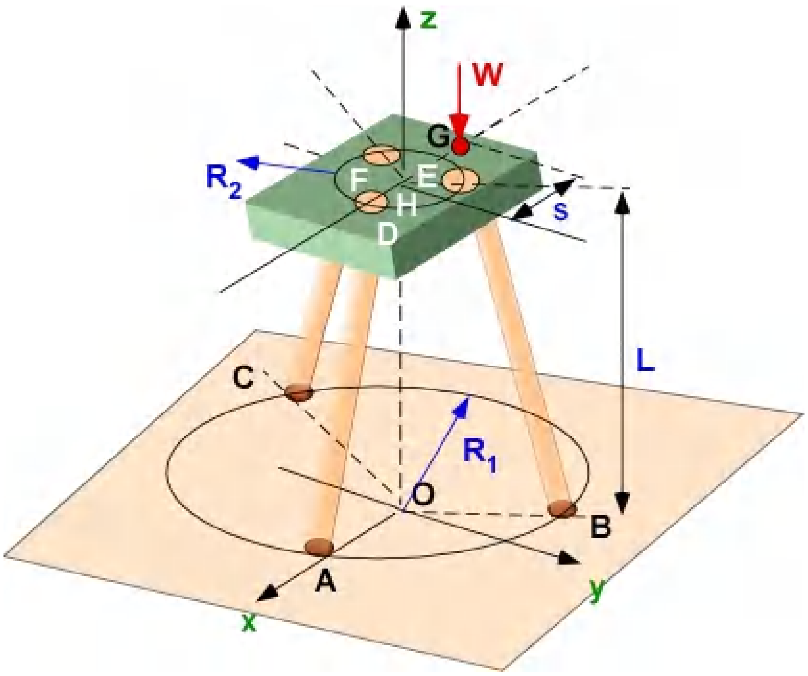


Figure 5.4.12 Example 5.17 Problem Definition

Problem. The three legs of the stool can be defined in two planes. One at the floor level (points A, B, and C) and the other at the seat level (points D,E, and F). The points are located on a circle of radius R_1 = 0.2 m at the floor and at a radius of R_2 = 0.12 m at the seat level. The points A, B, C and D, E, F subtend an angle of 120 degrees about the center of the respective circles. In fact the legs (AD, BE, and CF) are in planes that are located 120 degrees between each other. The height of the stool is 0.75 m. A person of mass 90 kg sits on the stool eccentrically with his weight W idealized at the point s = 0.15 m. Neglect the weight of the stool itself.   What the reactions at A, B, and C?

FBD: The FBD is in Figure 5.4.13

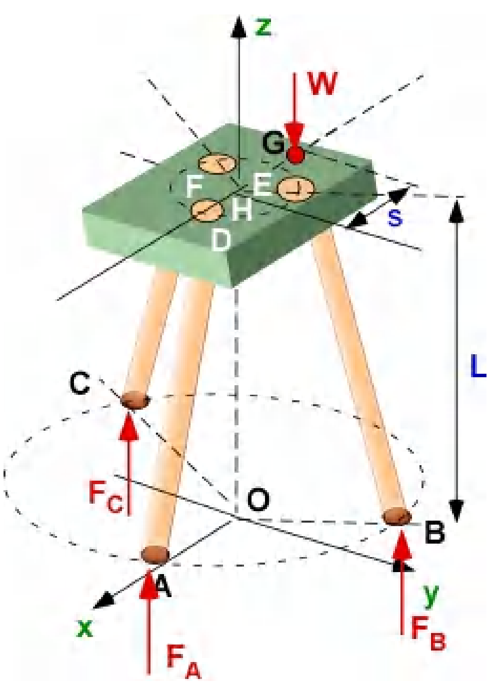


Figure 5.4.13 FBD of Example 5.17

Data: W = 882.9 [N];   Points: O:[0, 0, 0];   H : [0,0,0.75] ;   A : [0.2, 0, 0];   B : [-0.1,0.173,0];

C : [-0.1,-0.173,0]; D : [0.12, 0, 0.75]; E : [-0.06, 0.173, 0.75]; F : [-0.06, -0.173, 0.75];
H : [0, 0, 0.75]; G : [-0.15, 0, 0.75]; R_1 = 0.2 m; R_2 = 0.12 m; L = 0.75 m; s = 0.15 m

Find: F_A, F_B, F_C

Assumption: This is a parallel force system force system.

Solution 1: Consider the FBD of the stool in in Figure 5.4.13.

There are three unknown forces and the weight in the z-direction. The force equilibrium will yield only one equation. We have to also use moment equilibrium to solve the problem.
The forces are all in the z-direction so they will not be able to generate any moment along the z-axis.
The force and moment equilibrium should provide three equations to solve the problem

Equilibrium:

$$\sum \bar{F} = 0 \Rightarrow \sum F_z = 0 = F_A + F_B + F_C$$

$$\sum \bar{M}_O = 0 = \left(\bar{r}_{OA} \times \bar{F}_A\right) + \left(\bar{r}_{OB} \times \bar{F}_B\right) + \left(\bar{r}_{OC} \times \bar{F}_C\right) + \left(\bar{r}_{OG} \times \bar{W}\right)$$

The Equations
You can verify the equations as:

$$F_a + F_b + F_c - 882.9 = 0$$

$$0.1732F_b - 0.1732F_c = 0$$

$$0.1F_b - 0.2F_a + 0.1F_c - 132.4 = 0$$

The solution:
(using lower case for the magnitudes)

$$F_a = -147.15[N]$$

$$F_b = 515.02[N]$$

$$F_c = 515.02[N]$$

About this solution:

- Is the solution acceptable?
- What is the implication of negative sign of F_A?
- This means that the floor is trying to pull down the stool at the point A?
- Should we pin the stool at point A since it it is trying to leave contact with the floor?
- If the stool has no contact with floor at A then should F_A even exist?
- If the stool has no contact with A that means it is in rotating - and this implies that it is not in moment equilibrium. It is moving and no longer static! This also suggests that the design is unstable (this topic is usually discussed in dynamics).

In this arrangement the patron will fall if he tries to sit on this stool at G. This is a bad outcome or bad design.

Design Changes

- Change the offset s = 0 then each leg sees the same force.
- Change the offset so that it is along the positive x - axis
- What happens if the offset along the positive x- axis exceeds the point A? The stool will not be in equilibrium again?
- What if the seat of the stool is small so that the weight is located at H or close to it? It may not be very comfortable for some persons but it is stable?
- Will a four leg stool be better?

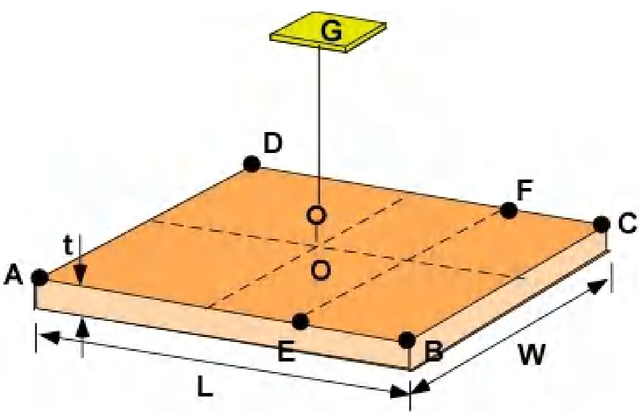
Use MATLAB to explore the solution as s is changed to get a better understanding of the solution.

5.4.8 Additional Problems

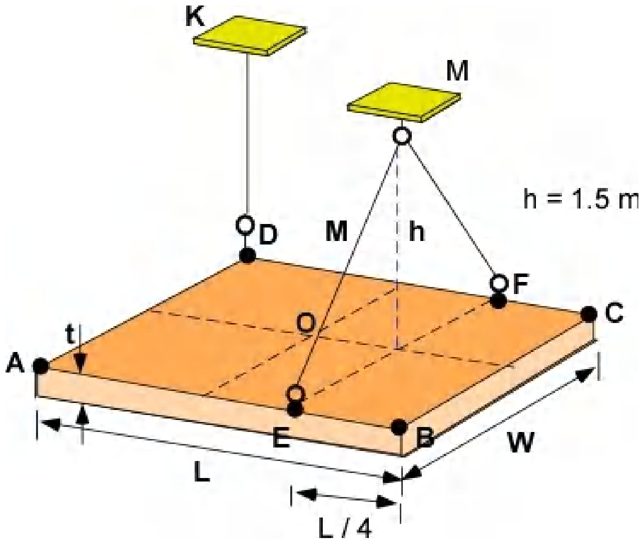
Solve the following problems on paper and using MATLAB/Octave. For each problem you must draw the FBD and work with a coordinate system. Use your own if one is not prescribed. Clearly identify support reactions.

Problem 5.4.1

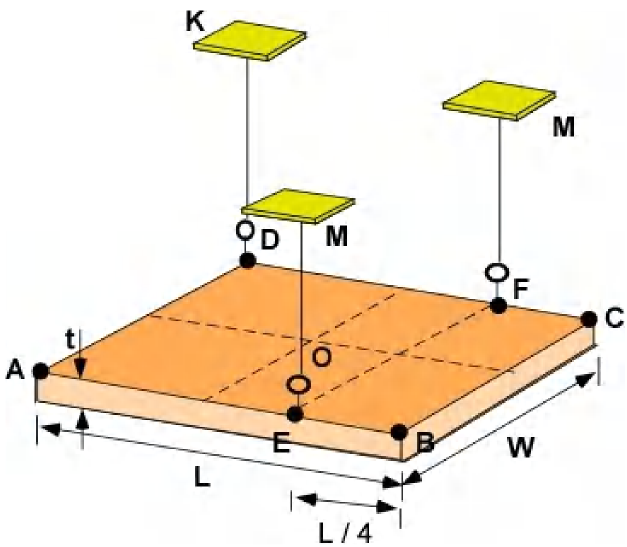
The sheet of steel of L = 1 m and W = 0.7 m and thickness t = 5 cm is required to be hung horizontally as part of an art exhibit. You are exploring the three options listed in the figures below. The first is a single cable suspension. The second is a two-cable suspension. The third is a three cable suspension. For each case determine the size of the nylon cable used for suspension. Use cable of a single size and use a factor of safety of 10. Identify the advantages and disadvantages of the three options. Use your own data for the problem. List your assumptions. Point O is the center of the sheet. Points E and F are located at 1/4 L.



Problem 5.4.1 (a)



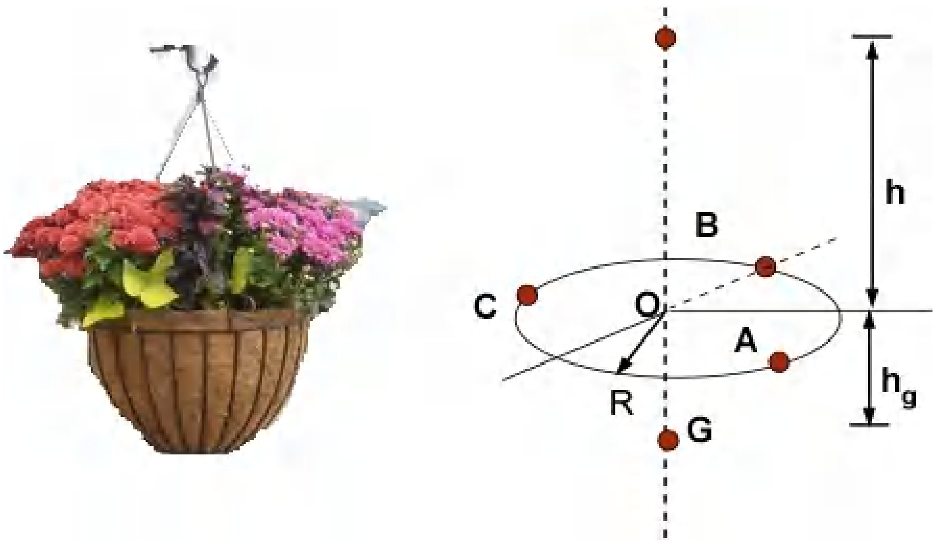
Problem 5.4.1 (b)



Problem 5.4.1 (c)

Problem 5.4.2

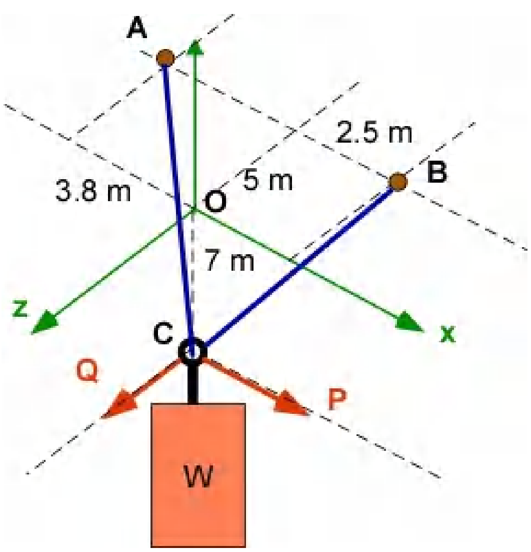
The simple flower basket weighs 10 kg. It is suspended using plastic cable at the points A, B, C which are on a radius of 15 cm sub tending 120 degree angle about the center O. The suspension height is h = 0.55 m above O and the weight can be considered to be located at h_g = 0.15 m. Calculate the reactions in cable. Solve the problem effortlessly.



Problem 5.4.2. Problem and template for FBD

Problem 5.4.3

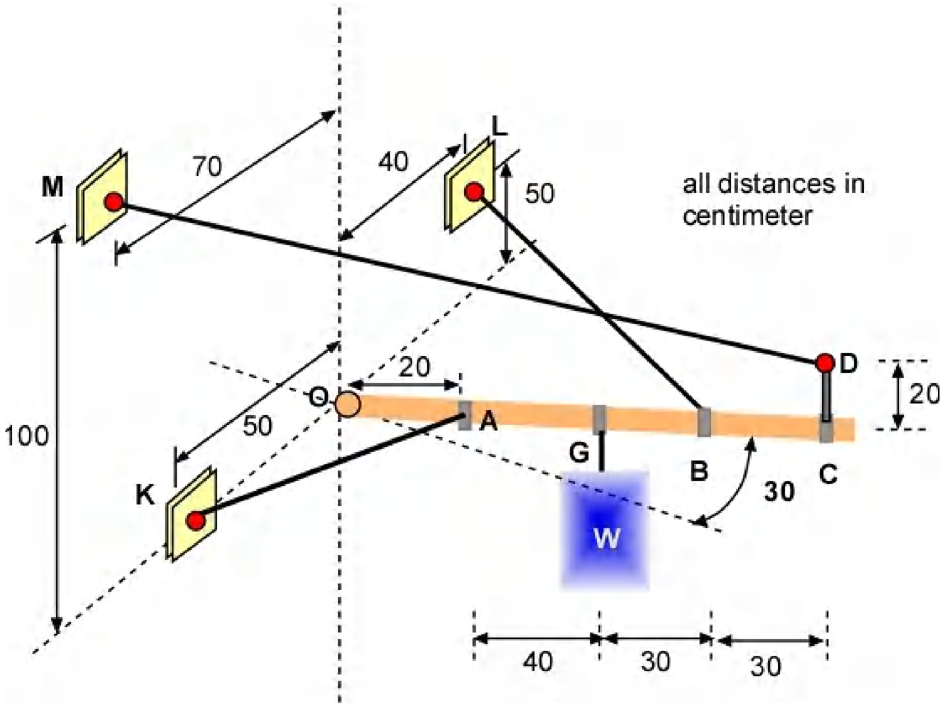
The weight W is suspended at C, directly below O at a distance of 7 m. It is suspended by two steel cables AC and BC and A and B are in the same plane as O. The forces P and Q are along the coordinate directions and have values of 500 N and 325 N respectively. What are the forces in the cable and the value of the weight?



Problem 5.4.3

Problem 5.4.4

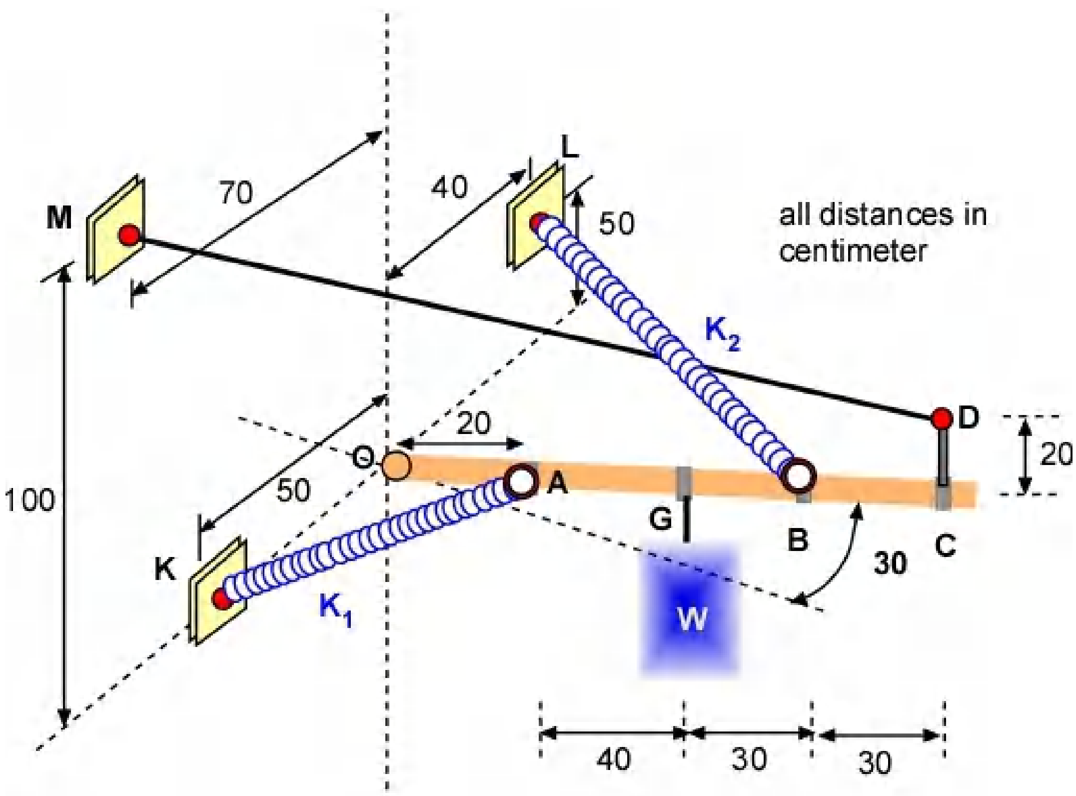
The complicated rigging system has three cables holding the bar with a heavy weight W = 1500 kg in the configuration shown. Find the forces in the cable and the reaction at O. The bar is held by a ball and socket joint at O.



Problem 5.4.4

Problem 5.4.5

The complicated rigging system has one cable and two linear springs of different spring constants holding the bar with a heavy weight W = 1500 kg in the configuration shown. The springs are unstretched when the bar is aligned along the axis (angle is zero degrees). Find the force in the cable, the spring constants, and the reaction at O. The bar is held by a ball and socket joint at O.



Problem 5.4.5

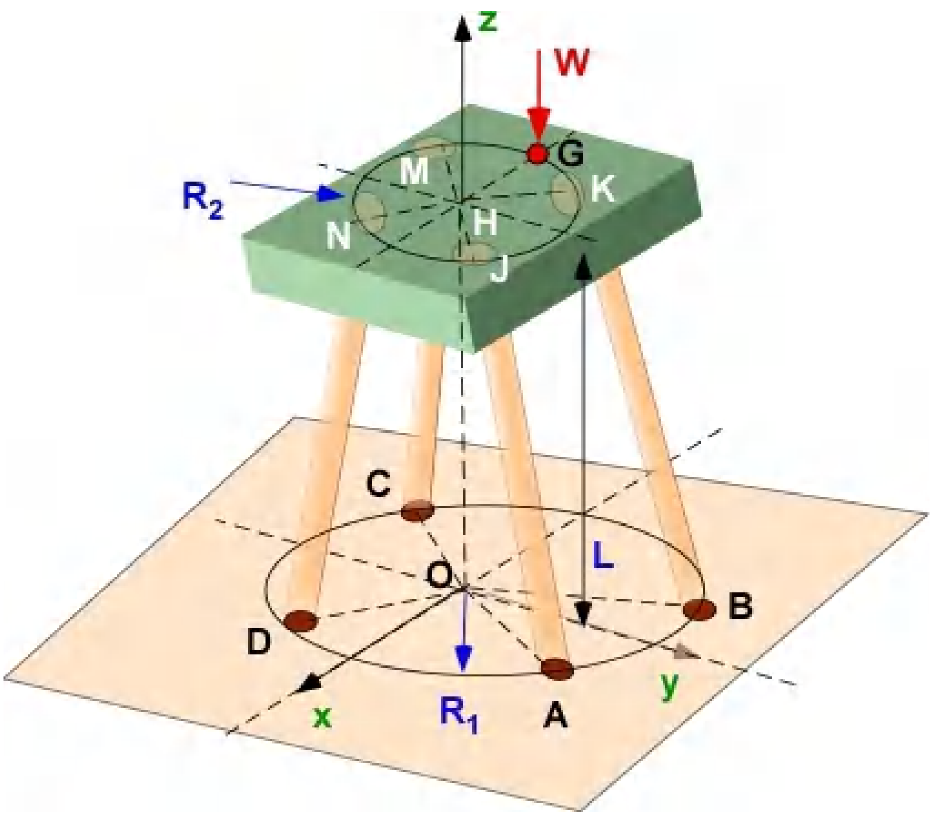
Problem 5.4.6

We looked at the three legged stool in Example 5.17 and found it unsteady and quick to rotate with an eccentric weight applied. We think we can prevent the rotation with four legs as shown.

The legs of the stool can be defined in two planes. One at the floor level (points A, B, C, and D) and the other at the seat level (points J, K, M, and N). The points are located on a circle of radius R_1 = 0.2 m at the floor and at a radius of R_2 = 0.12 m at the seat level. The points A, B, C, D and J, K, M, N subtend an angle of 90 degrees about the center of the respective circles. The height of the stool is 0.75 m. A person of mass 90 kg sits on the stool eccentrically with his weight W idealized at the point on the circle R_2 as shown. . Neglect the weight of the stool itself. What the reactions at A, B, C and D?

Some questions before you begin:

- Is the problem symmetric?
- Is it statically determinate?
- Developed any additional insights using design constraints if necessary.



Problem 5.4.6

6. TRUSS

Truss is a special structure that can support large loads. Usually, it is fairly light compared to the loads it can support. You can see it in bridges, roofs, tall buildings and in many other designs. It is usually constructed by long members bolted at the ends. Figure 6.1 includes some examples of truss.

Figure 6.1 Examples of truss structures (Images from Wikimedia Commons)

You can see that there is a lot of empty spaces between straight elements, which we call members. In our analysis of such structures we assume these members are connected at junctions by **pins**. You can also see that a lot of these members lie in a plane. Figure 6.2 is an example of such an image.

Figure 6.2 Member connections (from Wikimedia Commons)

Figure 6.2 stretches the truth regarding the **pins** at the junctions as such connections are really *fixed* and seriously *constrained*. They cannot move or rotate relatively. Replacing these connections by reactions that include both forces and moments will cause severe indeterminacy and render it practically insolvable through simple analysis. It has been determined that assuming pin connections in the members can determine the force reactions reasonably well. This is usually sufficient for design. Therefore, we will assume *simple single pin* connections for the members that are joined at the ends. Before we start to analyze these structures we examine these pins connections in the next section.

Before we move on to analyzing trusses we will spend a moment about the stresses in the pins and the stress they cause at the connections. They are primarily bearing stress and shearing stresses introduced in the next section.

6.1 STRESSES AT THE PIN

Bolts are used to transmit forces. The same is true of pins and rivets. Their analysis are all similar. In places where bolts are used, like in truss structures, the failure of the bolt is more likely the culprit for the failure of the structure.Consider an example in Figure 6.1.1. It shows two bolts being used to transmit the force F. Each bolt is considered to be in *double* **shear.** One can use a single bolt and a simpler arrangement as shown in Figure 6.1.2. which is in *single* **shear**. It is expected that the bolt will fail due to the shear stress it carries.

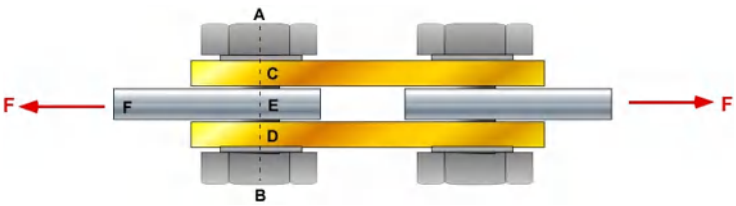


Figure 6.1.1 Bolt in double shear

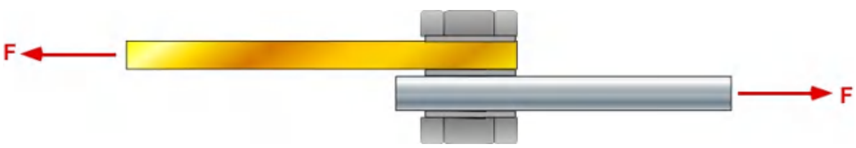


Figure 6.1.2 Bolt in single shear

6.1.1 Shearing Stress on Bolt

The stress in the bolt can be calculated using the FBD of the bolt. Consider the equilibrium of the bolt shown in Figure 6.1.1. Here symmetry is assumed so that force equilibrium will split force F in half in each segment. This is insight based on our knowledge from previous sections. You can actually prove it to be true by applying the equilibrium equations. Symmetry is a very powerful argument for arriving at common sense solution.

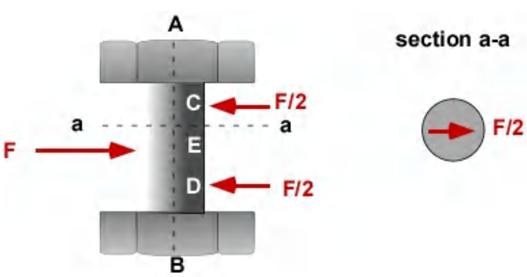


Figure 6.1.3 Bolt shear

The shear stress in the bolt is the force divided by the cross sectional area which is easily calculated using the diameter of the bolt.

$$\tau = \frac{F/2}{A_c} \qquad (6.1)$$

For the bolt in single shear in Figure 6.1.2 it is simple to show that the shear stress is

$$\tau = \frac{F}{A_s} \qquad (6.2)$$

This means that the bolt in Figure 6.1.2 is more likely to fail because it is resisting twice the shear stress of the first bolt. This can drive the selection of the bolt material and its dimension. These are calculations based on static loading. One can generally use a higher factor of safety for dynamic loading. This should be accompanied by inspections for wear.

6.1.2 Bearing Stress on Plate

The plate carrying the load F also interacts with the bolt. The interaction is shown in the top, side, and an exploded view in Figure 6.1.3. It represents the gray plate held in either single or double shear in Figure 6.1.1 or 6.1.2. Unlike the stress in the bolt, the stress in the plate a normal stress. It is called a **bearing stress** since the bolt is pressing down on the plate within the hole that accommodates the bolt.

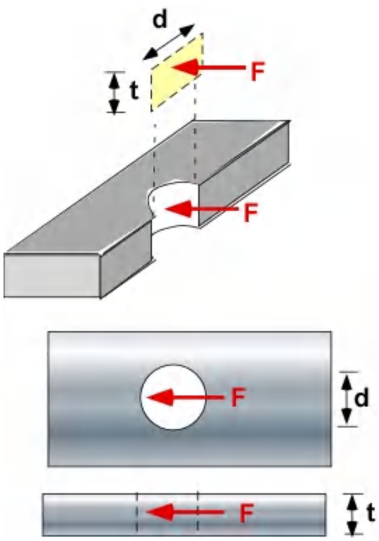


Figure 6.1.4 Illustration of bearing stress (exposed, top and side view)

The bearing stress is calculated as:

$$\sigma_b = \frac{F}{td} \qquad (6.3)$$

Figure 6.1.4 is not in equilibrium. It is drawn to indicate the bearing stress. The rest of the material in the cross section of the plate, where the bearing stress is calculated, maybe carry a lower stress depending on the cross-section of the plate minus the projected bolt area on the plate. If you move away from the bolt, between the bolt and the end where the force F is shown in Figure 6.1.1, the stress will get distributed over the entire cross-section. This value should be lower than the bearing stress. The bearing stress maybe a important consideration for determining failure. To complete the discussion, the FBD of the plate in the section shown in Figure 6.1.4

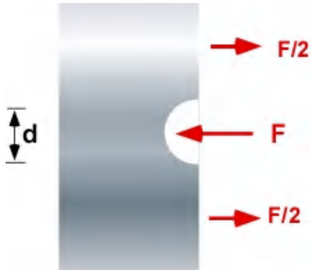


Figure 6.1.5 Bearing stress in equilibrium.

Note that these figures were missing a coordinate system. They were omitted for clarity of discussion.

6.1.3 Example 6.1

The ideas from shearing stress and bearing stress are useful in completing detail design of supports which are largely overlooked when trying to design larger members of the structure. Consider a simple example of using statics to convert a vertical force to a horizontal force as shown in Figure 6.1.6. The vertical force of 10 kN produces a horizontal force P in a cable at B. Note that the arrangement also scales the applied force. The structure is pin supported at A. In addition to standard processing of such problems through equilibrium we need to design the cable at B as well as the pin and bracket at A. The additional design information/constraint are as follows. The cable is steel and has an ultimate normal stress of 450 MPa. Use a FOS of 10. The pin at C will have an ultimate shearing stress of stress of 250 MPa and an FOS of 6. The bracket will also be made of steel with an ultimate normal stress of 400 MPa with a FOS of 6.

Design Details:

- The diameter of the cable at B (normal stress)
- The diameter of the pin at A (shear stress)
- The thickness of the bracket (bearing stress)

In the simplest case the material is known and design involves identifying the corresponding dimensions. This may include design constraints like FOS. A more involved design may require exploring different materials and accompanying dimensions

Steps:

(a) Apply equilibrium and determine the forces at the pin A and in the cable (P)
(b) Determine the stress based on support design.

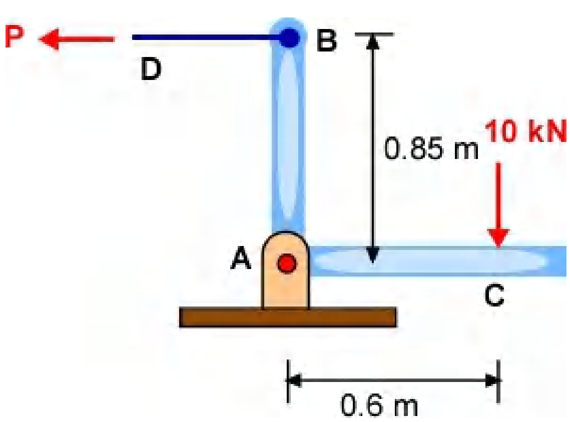


Figure 6.1.6 Example 6.1

Data: F = 10 [kN]; Locations of forces are known. For cable σ_u is 450 MPa and FOS of 10. For pin

τ_u is 250 MPa with a FOS of 6. For bracket the ultimate bearing stress σ_u is 400 MPa with a FOS of 6.

Find: (a) Force P; (b) Force at support A (A_x, A_y); (c) diameter of cable at B; (d) diameter of pin at A; (e) thickness of bracket at A.

Assumption: Pinned support at A.

Solution: Draw FBD and apply equilibrium

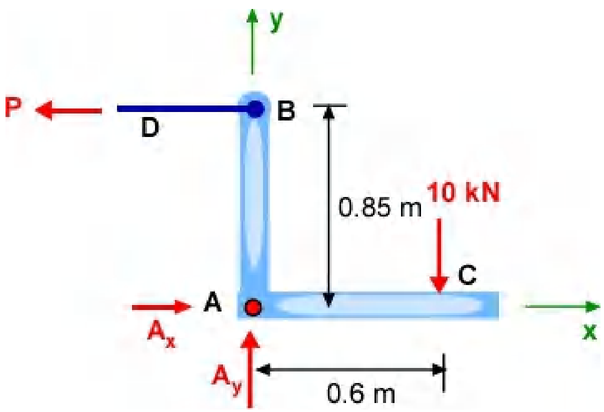


Figure 6.1.7 Example 1 - FBD

Equilibrium (2D):

$$\sum F_x = A_x - P = 0$$

$$\sum F_y = A_y - 10000 = 0$$

$$\sum M_A = 0.85P - 10000 \times 0.6 = 0$$

The solution is :

$$A_y = 10000[N]$$

$$P = 10000\frac{0.6}{0.85} = 7058.8[N]$$

$$A_x = 7058.8[N]; \quad A = \sqrt{10000^2 + 7058.8^2} = 12240[N]$$

Detail Design: For detail design we look at the design of bracket and the FBD of the pin in Figure 6.1.8.

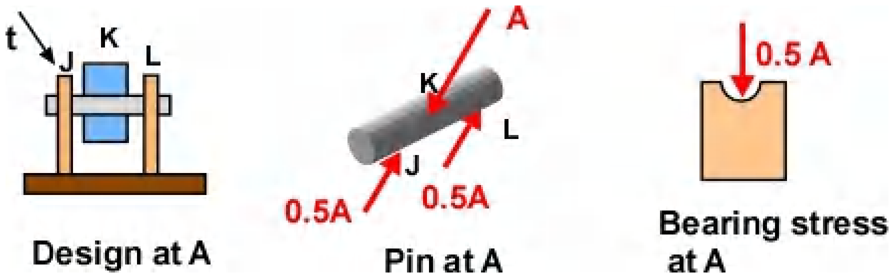


Figure 6.1.8 Details of bracket

The cable: Normal stress

$$\frac{P}{\frac{\pi d_B^2}{4}} = \frac{\sigma_u}{10}; \quad d_B = \sqrt{\frac{4P\times 10}{\pi\sigma_u}} = 0.0141[m]$$

Pin: The pin is in double shear. The shear force on the pin is 0.5A.

$$\frac{0.5A}{\frac{\pi d_A^2}{4}} = \frac{\tau_u}{FOS} = \frac{250\times 10^6}{6}; \quad d_A = \sqrt{\frac{2A\times 6}{\pi\tau_u}} = 0.0137[m]$$

Bracket: bearing stress

$$\frac{0.5A}{td_A} = \frac{\sigma_u}{FOS}; \quad t = \frac{0.5AFOS}{\sigma_u d_A} = 0.0067[m]$$

6.1.4 Additional Problems

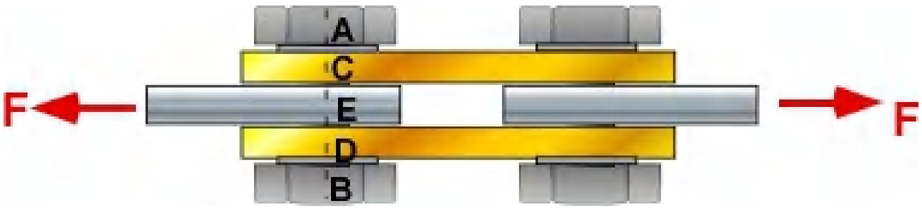
Please use Table 6.1 for your calculations.

Table 6.1

Material	Aluminum	Brass	Steel	Wood
E [GPa]	70	105	200	13
Ultimate Stress (tension) [MPa]	300	500	450	100
Ultimate stress (shear) [MPa]	70	200	250	10

Problem 6.1.1

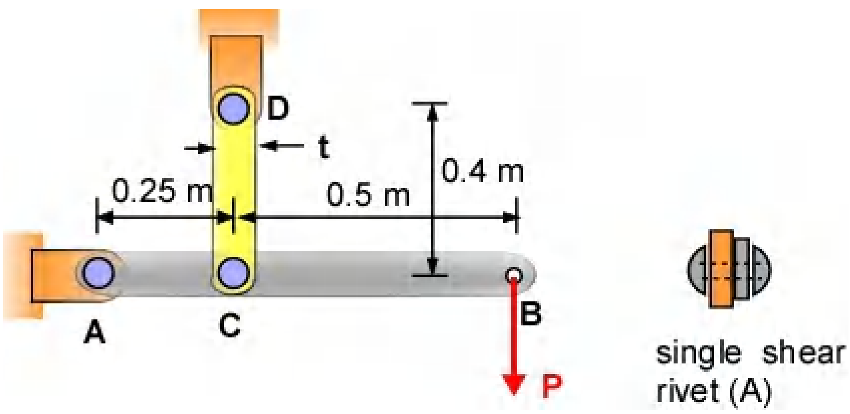
The brass plates (C, D) are bolted along with the pair of aluminum plate (E) to handle/transfer the force F = 10 kN. The bolts are made of steel. All of the structure will have a factor of safety of at least 4.5. Design the mechanism. The plates can have different thickness but they all have the same diameter and width.



Problem 6.1.1

Problem 6.1.2

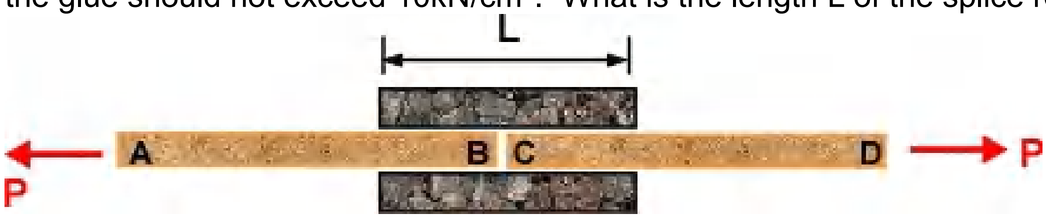
The force P is 8000 N. The connections riveted and in single shear as shown in the figure. The rivet is made of steel. The structure must have a factor of safety of 3.5. AB is aluminum and CD is brass.. The width of the bars do not have to be the same. Design the system



Problem 6.1.2

Problem 6.1.3

The wooden beams carrying the load P = 6 kN is spliced with two wooden blocks using glue applied uniformly. The width of the beam and splice are the same at 10 cm. The average shearing stress in the glue should not exceed 10kN/cm^2. What is the length L of the splice required.



Problem 6.1.3

6.2 PLANE TRUSS

Trusses make it easy to carry heavy loads. A bridge over the river, roof over your house, the giant crane used in construction or unloading heavy cargo from a ship are some of common examples. Trusses are light too. It is just a framework. You use them to support the forces so that the structure will not come tumbling down. Let us start with a figure of a simple truss in Figure 6.2.1 (from Wikipedia) which will handle the load generated by automobiles/trucks on the bridge. On the right I have idealized the bridge in terms of simple elements in the front plane: (i) the straight members; (ii) the pins; and (iii) the support for the straight elements at the bottom. The bridge is three-dimensional, and I have shown the information for the front plane (that will be two-dimensional). Plane truss implies two-dimensional truss. The pins are deliberate to avoid static indeterminacy.

Example 6.2.1 A simple bridge truss -and simplified.

The solution for truss problem is based on rigid body equilibrium? A simple question is whether a truss is a rigid body?

Looking at Figure 6.2.1 it is clear that the truss has a lot of empty spaces. It is nevertheless designed to function almost as a rigid body since you want the structure to carry the vehicles that move across it without significantly changing its shape. In effect this appears to be a more efficient way to handle large forces without using a lot of material.

The truss appears to be pieces of rigid body (members) connected together at the ends (pins/joints). The truss supports external loads which are applied at some of the pins and stays in place through supports (support reactions). The members are long and the axis of the members are along the length. The cross-section is ignored for initial calculations. The members can only carry forces along the axis and this can either be tension (positive) and compression (negative).

For the purpose of calculations
The entire truss is in static rigid body equilibrium (FBD of the entire truss). Force and moment equilibrium (at some point O) :

$$\sum \bar{F} = 0 \qquad (6.4)$$

$$\sum \bar{M}_O = 0 \qquad (6.5)$$

This is also enforced at each pin which must also being in static equilibrium (FBD of the pin).

$$\sum \bar{F} = 0; \qquad (6.6)$$

This allows us to solve for:

- Support Reactions
- Member forces

The truss is not a rigid body. It is permitted to deflect through the deflection of the pins. However the deflections are small and are elastic. The deflections are along the axis of the members. This is a one-dimensional elastic deformation which we have used earlier. This allows the force and the deflection of each member to be calculated through:

$$\delta_{AB} = \frac{F_{AB} L_{AB}}{E_{AB} A_{AB}} \qquad (6.7)$$

$$\delta_{AB} = change\ in\ length\ of\ member\ AB$$

$$F_{AB} = force\ in\ member\ AB$$

$$L_{AB} = length\ of\ member\ AB$$

$$E_{AB} = modulus\ of\ Elasticity\ of\ member\ AB$$

$$A_{AB} = cross-section\ of\ member\ AB$$

6.2.1 Simple Two-Dimensional Truss

We will use an example to define the various features of solving a plane truss problem. We will break out the analysis in various subsections.

Example 6.2: In the interest of keeping the calculations modest I have shortened the truss to a simpler geometry and loading as shown in Figure 6.2.2. This is an idealized description of the truss.

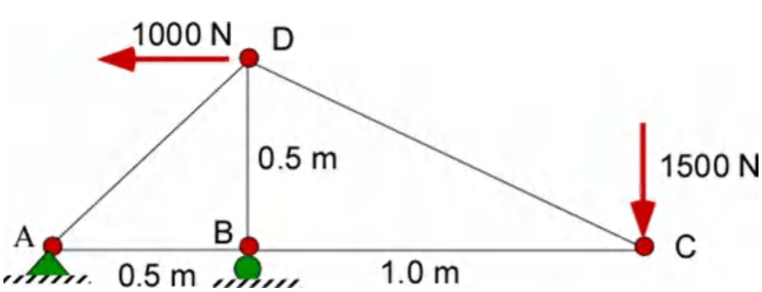


Figure 6.2.2 A simple 2D truss

There are four joints/pins (***n***), which we have identified as A, B, C, and D. There are five members (***m***) in the truss, and we refer to them through the pair of letters AD, AB, BD, BC, and CD. There are two supports for the truss at A and B. The support at A is a pin, and is not allowed to displace. The support at B is termed as a roller support and can move horizontally, but cannot move in the vertical direction. The truss carries two forces/loads in the direction shown. Points C and D are not restricted from moving. You should decide easily that this is a two-dimensional truss.

Stability
The stability of the truss is given by a simple relation for a two dimensional truss:

K = 2*n - 3 (6.8)

If K = m, the truss is structurally stable

We seek two types of solution to the problem expressed in Figure 6.2.2.

- First, we would like to know how much load each of the members will carry. This will help us to design the truss so that it will not fail under loading. This is considered as the *Member Forces*
- Second, we would also like to compute the final geometry of the truss with the loads as some pins will move under the applied loads. We can use this information to ensure that the point C, for example, will not hit against another machine (not shown here). This is the solution to the *truss deformation.*

There are two methods for solving the member forces. The first is the **Method of Joints** which will determine all member forces. Alternately, if we are certain that a certain member is critical, then we can seek the force only in that member to design it to avoid failure. This is called the **Method of Sections**. This could require less calculations for force computation. However, for calculating the displacements all the forces are necessary.

For either method we start the calculations using a (FBD) free body diagram of the entire truss and apply the equation of statics. We will assume that the truss is *statically determinate*. The solution process consist of the following steps:

- Find the support reactions. It is possible to find these reactions explicitly.
- Calculate the force in the various members to help design the members.
- Calculate the displacements of the pins that can move.

6.2.2 Support Reactions - Example 6.2

The FBD of the truss is shown in Figure 6.2.3. Here we are replacing the support reactions on the truss and the applied loads.

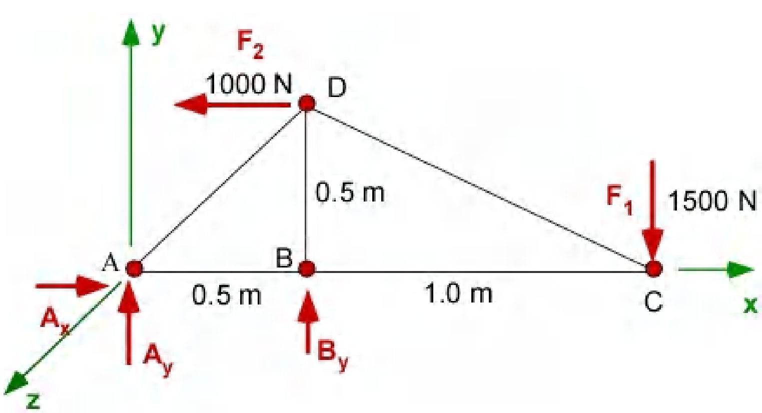


Figure 6.2.3 FBD of Example 6.2

Data: A(0, 0, 0); B(0, 0.5, 0); C(1.5, 0, 0); D(0.5; 0.5, 0);
F1 = 1500 [N]; F2 = = 1000 [N] (direction on Figure);
All members are made up of the same material with the modulus of elasticity value of 11.4×10^6 [Pa].

All of the members have the same cross-sectional area of 50×10^{-4} [m^2].

Find: (a) A_x, A_y, B_y; (b) F_{AB}, F_{BC}, F_{BD}, F_{CD}; (c) dB_x, dC_x, dC_y, dD_x, dD_y

Assumption: Pinned support at A, roller support at B

Solution: Apply Equilibrium

The first thing we observe is that there are **three** unknown reactions. This works exceedingly well since we have **three** equations of equilibrium to solve them. This is a **statically determinate** problem. If the roller support at B was changed to a pin support we would have a **statically indeterminate** problem and would have trouble solving them from the equations of statics alone. We will have to solve for the displacements to use the new extra displacement information at B (the horizontal displacement is now zero). We will consider *statically indeterminate* example later on. Our first step is to determine the support reactions in Figure 6.2.3.

Support Reactions:
The equilibrium equations:

$$\sum F_x = 0 = A_x - 1000 \qquad (i)$$

$$\sum F_y = 0 = A_y + B_y - 1500 \qquad (ii)$$

$$\sum M_A = 0 = \left(\bar{r}_{AD} \times -1000\,\hat{i}\right) + \left(\bar{r}_{AB} \times \bar{B}_y\right) + \left(\bar{r}_{AC} \times -1500\,\hat{j}\right) \qquad (iii)$$

$$0 = 0.5 \times B_y + (0.5 \times 1000) - (1.5 \times 1500)$$

The solution can be obtained as:

$$A_x = 1000 \quad [N]$$
$$B_y = 3500 \quad [N]$$
$$A_y = -2000 \quad [N]$$

Solution Using MATLAB
The MATLAB Code is broken into segments to calculate items of interest in the current section. - The next piece of code continues after the current piece and will depend on information generated by the previous code segment (You should collect the code in a single file to solve the complete problem).

In the Editor

```
% Essential Mechanics
% P. Venkataraman
% Section 6.2.2- Example 6.1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, close all, format short G
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('------------------------------------------\n')
fprintf('Example 6.1 \n')
fprintf('-------------------------------------------\n')
```

```
%% Data
F1 = 1500; F2 = 1000; % the applied force
A = [0 0 0]; B = [0.5,0,0];  C = [1.5,0,0]; D = [0.5,0.5,0];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Support reaction calculations
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
syms Ax Ay By real % Unknowns

rAB = B - A;  rAD = D - A;  rAC = C - A;
FA = [Ax, Ay, 0];  FB = [0,By,0];  FC = [0,-F1,0];  FD = [-F2,0,0];

%% Equilibrium
SumF = FA + FB + FC + FD ; % Sum of Forces
SumMA = cross(rAD,FD)+ cross(rAB,FB)+ cross(rAC,FC);
% Sum of moments at A

sol = solve(SumF(1),SumF(2), SumMA(3));
Ax = double(sol.Ax);
Ay = double(sol.Ay);
By = double(sol.By);

%%%%%%%%%%%%%%%%%%%%%%
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point C [m]  : '),disp(C)
fprintf('Point D [m]  : '),disp(D)

fprintf('Position vector rAB[m]    = '),disp(rAB)
fprintf('Position vector rAC[m]    = '),disp(rAC)
fprintf('Position vector rAD[m]   = '),disp(rAD)

fprintf('\nF1 [ N]    : '),disp(F1)
fprintf('F2 [ N]    : '),disp(F2)

% displaying symbolic values to four decimals
fprintf('\nSumF : \n'),disp(vpa(SumF',4))
fprintf('\nSumMA : \n'),disp(vpa(SumMA',4))

fprintf('\nReactions:\n')
fprintf('------------\n')
fprintf('Ax [N]     = '),disp(Ax)
fprintf('Ay [N]     = '),disp(Ay)
fprintf('By [N]     = '),disp(By)
```

In the Command Window

```
-------------------------------------
Example 6.1
--------------------------------------
Point A [m]  :     0     0     0
Point B [m]  :          0.5          0          0
Point C [m]  :          1.5          0          0
```

```
Point D [m]   :            0.5           0.5            0
Position vector rAB[m]     =           0.5            0             0
Position vector rAC[m]     =           1.5            0             0
Position vector rAD[m]    =          0.5           0.5             0

F1  [ N]     :         1500
F2  [ N]     :         1000

SumF :
      Ax - 1000.0
 Ay + By - 1500.0
                0

SumMA :
               0
               0
 0.5*By - 1750.0

Reactions:
------------
Ax  [N]      =          1000
Ay  [N]      =         -2000
By  [N]      =          3500
```

Execution in Octave

The code is the same as in MATLAB except for the additional statements below . The changes are highlighted. You must include the symbolic package and if you do not wish to see warnings you include the command warning off as shown

clc, clear, format compact, close all, format short G, warning off
pkg load symbolic;

In Octave Command Window

```
-----------------------------------------
Example 6.1
------------------------------------------
Point A [m]  :   0  0  0
Point B [m]  :   0.5  0  0
Point C [m]  :   1.5  0  0
Point D [m]  :   0.5  0.5  0
Position vector rAB[m]    =   0.5  0  0
Position vector rAC[m]    =   1.5  0  0
Position vector rAD[m]   =   0.5  0.5  0

F1  [ N]    : 1500
F2  [ N]    : 1000

SumF :
  Matrix([[Ax - 1000.0], [Ay + By - 1500.0], [0]])

SumMA :
  Matrix([[0], [0], [0.5*By - 1750.0]])

Reactions:
------------
Ax  [N]     = 1000
```

```
Ay [N]      = -2000
By [N]      = 3500
```

Note: The elements of SumF and SumMA are reported differently in Octave.

6.2.3 Member Forces - Example 6.2

Method of Joints

There are two important assumptions in solving for the member forces

- The force is directed along the member. The member can therefore be in one of two states - **tension or compression.** The **unknowns** are just the **magnitude** of the force in the members. The orientation of the member **is** obtained from problem geometry. This is called a **two-force** member.
- The load will not change the original geometry of the truss (even if the pins deflect). Deflections are considered to be very small.

All members forces are unknowns. We have five member forces and we will declare their magnitudes as F_{AB}, F_{AD}, F_{BC}, F_{BD}, F_{CD}. The member forces are obtained by considering the equilibrium of **each pin** or **joint**. This is the **Method of Joints**. Since we will be developing equilibrium equations for each pin, or a point - there are only two force equilibrium equations available at each pin for a 2D problem. To solve for the forces explicitly we cannot have more than two unknowns. It is useful to cycle through the pins so that the forces are explicitly solved with a numerical value at each pin (true only for **statically determinate** problems). It is therefore important to start at a pin with not more than two unknown member forces. We have to draw the FBD of the pins. Figure 6.2.4 illustrates the FBD of pins A, B, C. Member forces drawn **heading away** from the pin are in **tension.** From Figure 6.2.4 we should not start our calculation cycle at pin B (***Why?***).

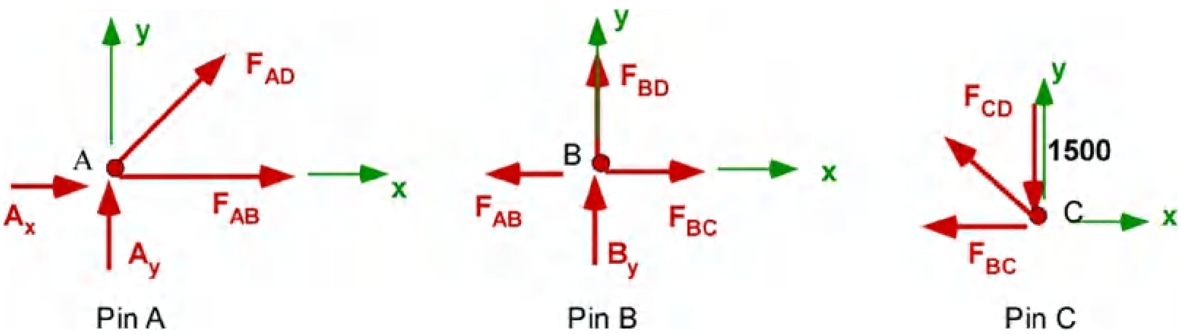


Figure 6.2.4 FBD of pins

Solving for the member forces is simple but tiresome. The current set of equations are solved using previous known solutions for the reactions. The components of forces can be easily computed through the unit vectors based on the location of the pins. They can be reviewed in the MATLAB code and are not explicitly registered here. For each pin we are applying the equilibrium of the forces, which is shown along the coordinate directions below. A lot of statements in the code are used for printing. They are similar to the ones encountered earlier in this book.

The analysis is easy if you can recognize that F_{AD} is inclined 45 degrees to the x-axis. And F_{CD} is inclined26.57 degrees to negative x-axis.

Pin A:

$$\sum \bar{F} = 0 = \bar{F}_A + \bar{F}_{AB} + \bar{F}_{AD}$$

$$\sum F_x = A_x + F_{AB} + F_{AD}(\cos 45) = 0$$

$$\sum F_y = A_y + F_{AD}(\sin 45) = 0$$

$$F_{AD} = -\frac{-2000}{.707} = 2828.4 \quad [N]$$

$$F_{AB} = -1000 - 2828.4 \cos 45 = -3000 \quad [N]$$

Pin B: (setting up co-ordinate equations directly)

$$\sum F_x = -F_{AB} + F_{BC} = 0$$

$$\sum F_y = B_y + F_{BD} = 0$$

$$F_{BC} = F_{AB} = -3000 \quad [N]$$

$$F_{BD} = -B_y = -3500 \quad [N]$$

Pin C: (setting up coordinate equations directly)

$$\sum F_x = -F_{BC} - F_{CD} \cos 26.57 = 0$$

$$\sum F_y = F_{CD} \sin 26.57 - 1500 = 0$$

$$F_{CD} = \frac{1500}{\sin 26.57} = 3353.5 [N]$$

We solve these equations using MATLAB. The **positive values** for the member forces indicate **tension**. The **negative** values indicate that the member is in **compression**.

Solving for member forces using MATLAB: (Remember this piece of code continues the previous code)

In the Editor

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Member force calculations
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% substitute for known values
FA = subs(FA); FB = subs(FB);

%% Pin A
fprintf('\nPin A:\n')
fprintf('------------\n')
syms Fab Fad real
rAB = B-A;   eAB = rAB/norm(rAB);
rAD = D-A;   eAD = rAD/norm(rAD);
```

```
SumA = FA + Fab*eAB + Fad*eAD;

sola = solve(SumA(1),SumA(2));
Fab = double(sola.Fab);
Fad = double(sola.Fad);

fprintf('SumA : \n'),disp(vpa(SumA',4))
fprintf('Fab [N]      = '),disp(Fab)
fprintf('Fad [N]      = '),disp(Fad)

%% Pin B
fprintf('\nPin B:\n')
fprintf('------------\n')
syms Fbc Fbd real
rBC = C - B;  eBC = rBC/norm(rBC);
rBD = D - B;  eBD = rBD/norm(rBD);
SumB = -Fab*eAB + FB + Fbc*eBC + Fbd*eBD;
solb = solve(SumB(1),SumB(2));
Fbc = double(solb.Fbc);
Fbd = double(solb.Fbd);

fprintf('SumB : \n'),disp(vpa(SumB',4))
fprintf('Fbc [N]      = '),disp(Fbc)
fprintf('Fbd [N]      = '),disp(Fbd)

%% Pin C
fprintf('\nPin C:\n')
fprintf('------------\n')
syms Fcd real
rCD = D - C;  eCD = rCD/norm(rCD);
% only one unknown - need only one equation to solve for it
SumC = -Fbc*eBC + FC + Fcd*eCD ;
solc = solve(SumC(1));
Fcd = double(solc);

fprintf('SumC : \n'),disp(vpa(SumC',4))
fprintf('Fcd [N]      = '),disp(Fcd)
```

In The Command Window

```
Pin A:
------------
SumA :
 Fab + 0.7071*Fad + 1000.0
       0.7071*Fad - 2000.0
                         0
Fab [N]     =        -3000
Fad [N]     =        2828.4

Pin B:
------------
SumB :
 Fbc + 3000.0
 Fbd + 3500.0
            0
```

```
Fbc [N]      =        -3000
Fbd [N]      =        -3500

Pin C:
------------
SumC :
 3000.0 - 0.8944*Fcd
 0.4472*Fcd - 1500.0
                  0
Fcd [N]      =        3354.1
```

The values by hand and MATLAB match well.

Execution in Octave

The code is the same as in MATLAB except for the additional statements below . The changes are highlighted. Remember this code is appended to the previous code and saved in a single file. The symbolic substitution in Octave needs atention and work around.

In Octave Editor

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Member force calculations
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% substitute for known values

##FA = subs(FA); FB = subs(FB);
##This did not work in OCTAVE with the following error
##error: subs: we do not support single-input w/ substitution from
workspace
##error: called from
##     subs at line 151 column 5

% this provides work around
FA = subs(FA,Ax);
FA = subs(FA,Ay);

FB = subs(FB,By)

fprintf('\nThe reaction vectors :\n')  % printing to see if it worked
fprintf('------------\n')
fprintf('FA [N]      = '),disp(FA)
fprintf('FB [N]      = '),disp(FB)


##fprintf('SumA : \n'),disp(vpa(SumA',4))
## The column printing seens to create an extra line feed
fprintf('SumA : \n'),disp(vpa(SumA,4))


##fprintf('SumB : \n'),disp(vpa(SumB',4))
fprintf('SumB : \n'),disp(vpa(SumB,4))


##fprintf('SumC : \n'),disp(vpa(SumC',4))
fprintf('SumC : \n'),disp(vpa(SumC,4))
```

In Octave Command Window

```
The reaction vectors :
------------
FA [N]      =    [1000  -2000  0]
FB [N]      =    [0  3500  0]

Pin A:
------------
SumA :
  [Fab + 0.7071*Fad + 1000.0  0.7071*Fad - 2000.0  0]
Fab [N]     = -3000
Fad [N]     = 2828.4

Pin B:
------------
SumB :
  [Fbc + 3000.0  Fbd + 3500.0  0]
Fbc [N]     = -3000
Fbd [N]     = -3500

Pin C:
------------
SumC :
  [-0.8944*Fcd + 3000.0  0.4472*Fcd - 1500.0  0]
Fcd [N]     = 3354.1
```

Note: There are some changes to the MATLAB Code. I have tried to highlight them. The vector equations are written in row form for appearance. There are no changes in the values. Just to emphasize the Octave code is the same MATLAB m-file with the highlighted changes. *At this time I am not programming in Octave. I am only adjusting the MATLAB code.*

6.2.4 Displacement of the Truss - Example 6.2

This is a missing topic in standard engineering curriculum. Calculating the displacement of the truss is not taught in the basic Statics course because it uses the stress and strain of the members to set up the problem. It is not covered in the basic Strength of Materials courses because Trusses are not revisited there. If you enroll in a Finite Element course you are likely to solve for truss displacements numerically. You might solve for truss displacements using energy methods in an advanced course on Strength of Materials. These problems are usually solved using software today.

We will solve for the displacements in this section using simple geometry and the relation between the displacement and the area of cross-section of the member, the length of the member, and the modulus of elasticity (the material property). We need additional information for the problem - the cross-sectional dimensions and the material for the member. This is based on stress, strain, and Hooke's law. It does get cumbersome with increase in the number of members. Before we venture into the solution let us consider the displacement of the Example truss in Figure 6.2.5.

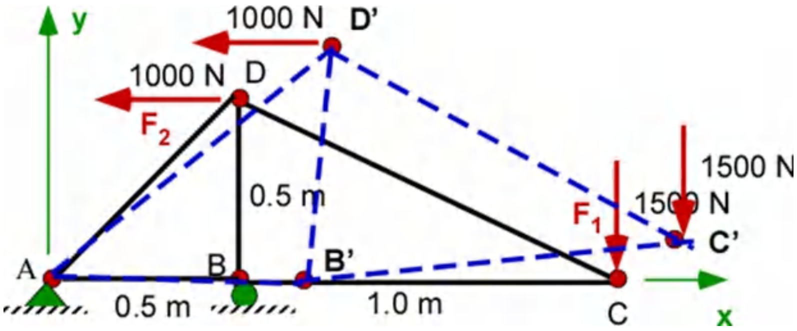


Figure 6.2.5 Original and displaced (exaggerated) truss

The pins of the truss are free to move if they are not constrained by supports. Pin B can move horizontally and hence its unknown displacement is $\mathbf{dB_x}$. Pins C and D can move in any direction and therefore we can describe their displacement of C defined by $\mathbf{dC_x}$ and $\mathbf{dC_y}$. The displacement of D is similarly described by $\mathbf{dD_x}$ and $\mathbf{dD_y}$. We have shown the displacement of C and D to be positive, a standard procedure for deriving the formula in all of science and engineering. In 2D space we can resolve the general displacement of the pin into **two displacement components** in the coordinate directions - components in x and y directions. This is illustrated in Figure 6.2.6 for the member CD - both as part of the truss and separately as a FBD. **Note:** the original geometry does not change because the displacement are considered small.

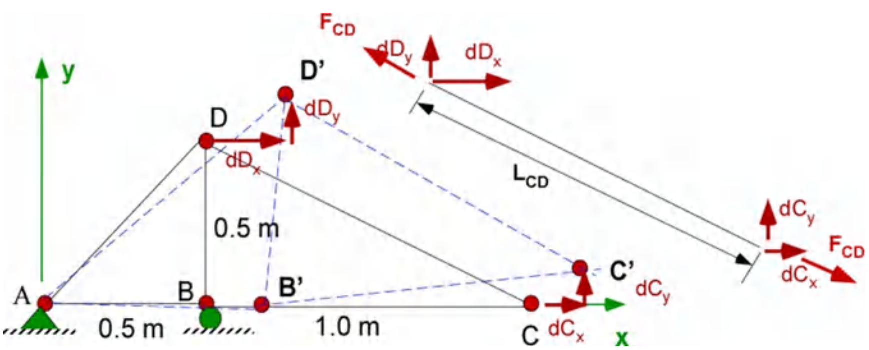


Figure 6.2.6 Displacement of member CD

To solve for the displacements of member CD through Figure 6.2.7a below:

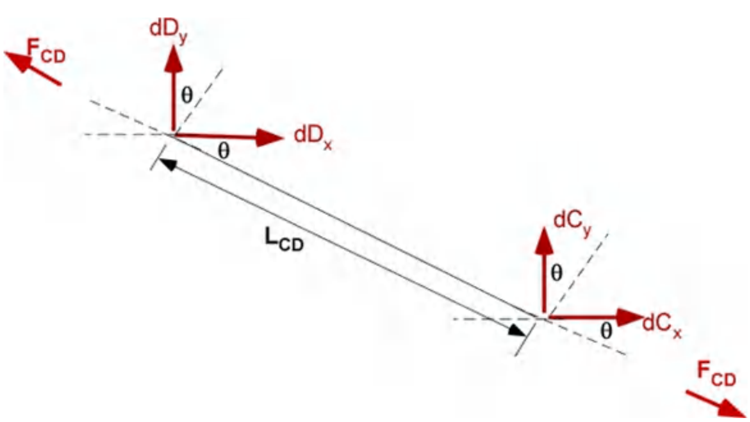


Figure 6.2.7a Force-displacement of member CD

The total change in length of the member CD (δ_{CD}) **along the member CD** (ignoring the small changes and based on undeformed original geometry) is the **relative difference** between the **displacement of the end C** and the **displacement of the end D** ***along the direction of member***

CD. We will assume that **end C displaces more than end D** to indicate tension in the member. This is not a limiting assumption. You could have assumed that end D moves more than end C to get the same results. It is advisable to be **consistent** in the development of the displacement relations. In Figure 6.2.7 this is computed as as follows with ɵ = 26.57 degrees.

$$\delta_{CD} = \left(\delta_C\right)_{along\ CD} - \left(\delta_D\right)_{along\ CD}$$

$$\delta_{CD} = \left(dC_x \cos\theta - dC_y \sin\theta\right) - \left(dD_x \cos\theta - dD_y \sin\theta\right)$$

$$\varepsilon_{CD} = \frac{\delta_{CD}}{L_{CD}}$$

$$\sigma_{CD} = \frac{F_{CD}}{A_{CD}} = E\,\varepsilon_{CD} = E\,\frac{\delta_{CD}}{L_{CD}} = E\,\frac{\left(dC_x \cos\theta - dC_y \sin\theta\right) - \left(dD_x \cos\theta - dD_y \sin\theta\right)}{L_{CD}}$$

$$F_{CD} = \frac{EA_{CD}}{L_{CD}}\left(dC_x \cos\theta - dC_y \sin\theta\right) - \left(dD_x \cos\theta - dD_y \sin\theta\right)$$

$$\delta_{CD} = \left(dC_x \cos\theta - dC_y \sin\theta\right) - \left(dD_x \cos\theta - dC_y \sin\theta\right) = \frac{F_{CD}L_{CD}}{A_{CD}E_{CD}} \qquad (6.9)$$

We made use of the definitions of **strain** and **stress** in the member CD. We use the **Hooke's law** to relate the stress and strain in the member to arrive at a relation between the **force** $\mathbf{F_{CD}}$ in the member and the **unknown displacements** of the ends of the member CD - $\mathbf{dC_x, dC_y, dD_x, dD_y}$. Equation (6.9) is valid only for the elastic region. In addition we are limited to small displacements. That is the displacements of the truss will not change the geometry of the truss significantly. This will allow us to identify the angles θ based on the pre-deformed geometry.

If the displacements are defined as vectors, then the total change of length along the member is the **dot product** of the unit vector along the member times the displacement vector of the ends. This is easy to incorporate in MATLAB/Octave.

$$d\bar{C} = \left[dC_x; dC_y; 0\right]; \quad d\bar{D} = \left[dD_x; dD_y; 0\right]$$

$$\delta_{CD} = \hat{e}_{CD} \cdot d\bar{C} - \hat{e}_{CD} \cdot d\bar{D} = \frac{F_{CD}L_{CD}}{E_{CD}A_{CD}} \qquad (6.10)$$

In Example 6.2, we have to develop one equation relating the force in the member to the displacements of the ends of the member, for each of the members in the example. Each member will have a FBD like Figure 6.2.7a to help develop the relations. This is shown in Figure 6.2.7b.

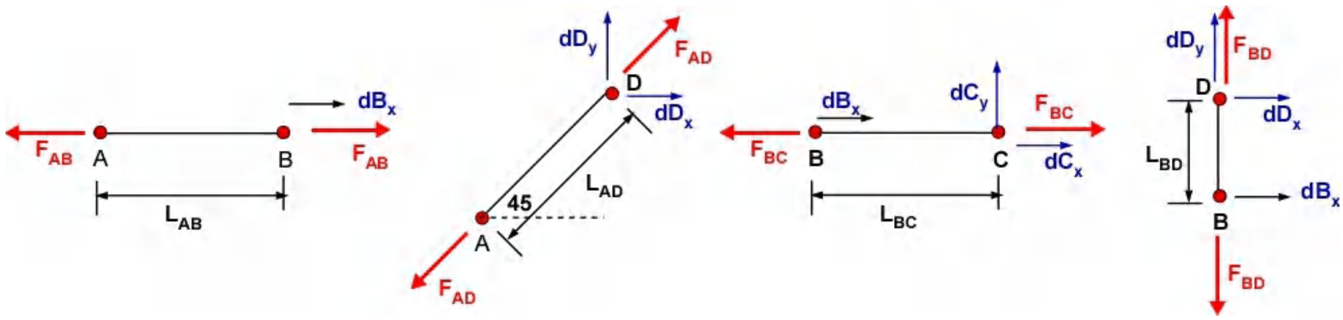


Figure 6.2.7b. Force - displacement diagram for other members

We will make additional choices to obtain numerical results for this example. All members are made up of the same material with the modulus of elasticity value of 11.4x10^6 [Pa]. All of the members have the same cross-sectional area of 50x10^{-4} [m^2].

Note: The FBD of each member is drawn in tension. The actual sign of the magnitude should be used in the calculations to yield the right results. The equations are aligned along the member axis so the coordinate system is not necessary.

Member AB:

$$d\bar{B} = [dB_x, 0, 0]; \quad d\bar{A} = [0, 0, 0]$$

$$\delta_{AB} = \hat{e}_{AB} \cdot d\bar{B} - \hat{e}_{AB} \cdot d\bar{A} = \frac{F_{AB} L_{AB}}{E_{AB} A_{AB}}$$

$$dB_x = \frac{F_{AB} L_{AB}}{E_{AB} A_{AB}} = -0.0263$$

Member AD:

$$d\bar{D} = [dD_x, dD_y, 0]; \quad d\bar{A} = [0, 0, 0];$$

$$\delta_{AD} = \hat{e}_{AD} \cdot d\bar{D} - \hat{e}_{AD} \cdot d\bar{A} = \frac{F_{AD} L_{AD}}{E_{AD} A_{AD}}$$

$$0.707\, dD_x + 0.707\, dD_y = \frac{F_{AD} L_{AD}}{E_{AD} A_{AD}} = 0.035$$

Member BC:

$$d\bar{C} = [dC_x, dC_y, 0]; \quad d\bar{B} = [dB_x, 0, 0]$$

$$\delta_{BC} = \hat{e}_{BC} \cdot d\bar{C} - \hat{e}_{BC} \cdot d\bar{B} = \frac{F_{BC} L_{BC}}{E_{BC} A_{BC}}$$

$$dC_x - dB_x = \frac{F_{BC} L_{BC}}{E_{BC} A_{BC}} = -0.053$$

Member BD:

$$d\bar{D} = \left[dD_x; dD_y; 0\right]; \quad d\bar{B} = \left[dB_x; 0; 0\right];$$

$$\delta_{BD} = \hat{e}_{BD} \cdot d\bar{D} - \hat{e}_{BD} \cdot d\bar{B} = \frac{F_{BD}L_{BD}}{E_{BD}A_{BD}}$$

$$dD_y = \frac{F_{BD}L_{BD}}{E_{BD}A_{BD}} = -0.031$$

Member DC:

$$d\bar{C} = \left[dC_x; dC_y; 0\right]; \quad d\bar{D} = \left[dD_x; dD_y; 0\right]$$

$$\delta_{CD} = \hat{e}_{CD} \cdot d\bar{C} - \hat{e}_{CD} \cdot d\bar{D} = \frac{F_{CD}L_{CD}}{E_{CD}A_{CD}}$$

$$0.8944dC_x - 0.4472dC_y - 0.8944dD_x + 0.4472dD_y = \frac{F_{CD}L_{CD}}{E_{CD}A_{CD}} = 0.0294$$

We can solve these 5 equations for the unknowns dB_x, dC_x, dC_y, dD_x, dD_y.

Solving for the displacements using MATLAB: (Remember this piece of code continues the previous code)

In the Editor

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%  Displacements
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Calculating displacements
syms dBx dCx dCy dDx dDy  % the unknowns

% Properties
Elas = 11.4e06;   Area = 50e-04;
% displacement vectors
dA = [0,0,0];  dB = [dBx, 0,0];  dC = [dCx,dCy,0];
dD = [dDx, dDy,0];

% length of members and unit vectors  (Copy and Paste)
rAB = B - A;  LAB = norm(rAB);  eAB = rAB/LAB;
rAD = D - A;  LAD = norm(rAD);  eAD = rAD/LAD;
rBC = C - B;  LBC = norm(rBC);  eBC = rBC/LBC;
rBD = D - B;  LBD = norm(rBD);  eBD = rBD/LBD;
rDC = C - D;  LDC = norm(rDC);  eDC = rDC/LDC;

% set up the equations  with zero on right hand side (Copy and Paste)
Eq(1) = dot(eAB,dB)-dot(eAB,dA) - (Fab*LAB/(Area*Elas)); % Member AB
Eq(2) = dot(eAD,dD)-dot(eAD,dA) - (Fad*LAD/(Area*Elas)); % Member AD
Eq(3) = dot(eBC,dC)-dot(eBC,dB) - (Fbc*LBC/(Area*Elas)); % Member BC
Eq(4) = dot(eBD,dD)-dot(eBD,dB) - (Fbd*LBD/(Area*Elas)); % Member BD
Eq(5) = dot(eDC,dC)-dot(eDC,dD) - (Fcd*LAB/(Area*Elas)); % Member DC

soldis = solve(Eq);

fprintf('\nThe Equations\n')
fprintf('------------------\n')
```

```
fprintf('Eq(1)  :  '),disp(vpa(Eq(1),4))
fprintf('Eq(2)  :  '),disp(vpa(Eq(2),4))
fprintf('Eq(3)  :  '),disp(vpa(Eq(3),4))
fprintf('Eq(4)  :  '),disp(vpa(Eq(4),4))
fprintf('Eq(5)  :  '),disp(vpa(Eq(5),4))

fprintf('\nDisplacements\n')
fprintf('-----------------\n')
fprintf('dBx [m] : '),disp(double(soldis.dBx))
fprintf('dCx [m] : '),disp(double(soldis.dCx))
fprintf('dCy [m] : '),disp(double(soldis.dCy))
fprintf('dDx [m] : '),disp(double(soldis.dDx))
fprintf('dDy [m] : '),disp(double(soldis.dDy))
```

In the Command Window

```
The Equations
-----------------
Eq(1) : dBx + 0.02632
Eq(2) : 0.7071*dDx + 0.7071*dDy - 0.03509
Eq(3) : dCx - 1.0*dBx + 0.05263
Eq(4) : dDy + 0.0307
Eq(5) : 0.8944*dCx - 0.4472*dCy - 0.8944*dDx + 0.4472*dDy - 0.02942

Displacements
-----------------
dBx [m] :     -0.026316
dCx [m] :     -0.078947
dCy [m] :      -0.41503
dDx [m] :      0.080323
dDy [m] :     -0.030702
```

Execution in Octave

The code is the same as in MATLAB. Remember this code is appended to the previous code and saved in a single file.

In Octave Editor

The code is same as in the MATLAB above - no changes

In Octave Command Window

```
The Equations
-----------------
Eq(1) :   dBx + 0.02632
Eq(2) :   0.7071*dDx + 0.7071*dDy - 0.03509
Eq(3) :   -dBx + dCx + 0.05263
Eq(4) :   dDy + 0.0307
Eq(5) :   0.8944*dCx - 0.4472*dCy - 0.8944*dDx + 0.4472*dDy - 0.02942

Displacements
-----------------
dBx [m] : -0.026316
dCx [m] : -0.078947
dCy [m] : -0.41503
```

```
dDx [m] : 0.080323
dDy [m] : -0.030702
```

We have now obtained (1) the **support reaction**, (ii) **member forces**, and (iii) the **displacements** of the truss. We have also performed the same calculations in MATLAB and in Octave. This is a complete design solution to the truss problem.

Observation: Now let us look closely at the solution values.

- The displacements are large. The highest is 40 cm. This is clearly unacceptable.
- If we change the cross-section area to $50x10^{-3}$ [m^2] the displacements reduce by an order of magnitude with a maximum displacement of 4 cm.
- If we increase the value of E = 70 MPa with original cross-section then the maximum is about 7 cm.

These were simple changes to property values in MATLAB. This is design exploration and it is easy if your problem can be solved by code. You should be able to include FOS in the analysis. What we have done is essentially a *what if?* analysis.

6.2.5 Method of Sections - Example 6.2

Method of Sections

The method of sections is a quick way to determine the forces in selected members. It works best if the support reactions are known. To expose the force in a member (or members) you will have to cut/section a member (or members) typically using a line across the member (or members) and draw the FBD of the cut section exposing the forces. The exact location of the cut is not material.

Lets start with Figure 6.2.3 and have solved for the unknowns A_x, A_y, and B_y. We now seek the value for the force in the member CD. We will cut or section the truss to expose the member forces of interest. Consider a line sectioning the members BC and DC at the tail of the forces in Figure 6.2.8. We apply the equation of statics to the sectioned FBD shown in Figure 6.2.8.

Note: You cannot just section only member DC. You have to include member BC too. You have to section the truss and not an individual member.

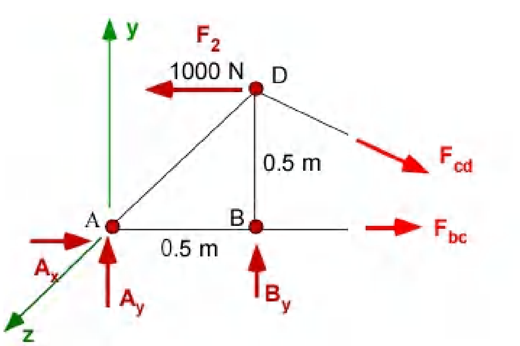


Figure 6.2.8 FBD - Method of Sections

We know apply the equilibrium equations to the sectioned truss

$$\sum \bar{F} = 0 = \bar{F}_A + \bar{F}_B + \bar{F}_{CD} + \bar{F}_{BC} + \bar{F}_D$$

$$\sum \bar{M}_A = 0 = \left(\bar{r}_{AD} \times \bar{F}_D\right) + \left(\bar{r}_{AD} \times \bar{F}_{CD}\right) + \left(\bar{r}_{AB} \times \bar{F}_B\right)$$

F_{bc} will not provide a moment about A. Since there are only two unknown we can solve for the forces from the force equation itself

Solving for the member forces using MATLAB: (Remember this piece of code continues the previous code)
Method of Sections

In the Editor

```
%% Method of sections  - we will use previous code for calculation of
reactions

syms Fcb Fdc real  % we will use a different variable for forces
clear SumF SumMA

rBC = C - B;  eBC = rBC/norm(rBC);
rDC = C - D;  eDC = rDC/norm(rDC);

FCB = Fcb*eBC;   FDC = Fdc*eDC;

SumF = FA + FB + FD + FCB + FDC;
solMS = solve(SumF(1),SumF(2));

Fcb = double(solMS.Fcb);
Fdc = double(solMS.Fdc);

% displaying symbolic values to four decimals
fprintf('\nSumF : \n'),disp(vpa(SumF',4))


fprintf('\nMethod of Sections\n')
fprintf('----------------------\n')
fprintf('Fcb [N]      = '),disp(Fcb)
fprintf('Fad [N]      = '),disp(Fdc)
```

In the Command Window

```
SumF :
    Fcb + 0.8944*Fdc
 1500.0 - 0.4472*Fdc
                   0

Method of Sections
----------------------
Fcb [N]      =        -3000
Fad [N]      =        3354.1
```

Execution in Octave
The code is the same as in MATLAB except for the additional statements below . The changes are highlighted. Remember this code is appended to the previous code and saved in a single file.

In Octave Editor
The code is same as in the MATLAB above - with the following changes

```
##fprintf('\nSumF : \n'),disp(vpa(SumF',4))
fprintf('\nSumF : \n'),disp(vpa(SumF,4))
```

In Octave Command Window
The output values is the same as in MATLAB

```
SumF :
  [Fcb + 0.8944*Fdc  -0.4472*Fdc + 1500.0  0]

Method of Sections
----------------------
Fcb [N]     = -3000
Fad [N]     = 3354.1
```

These are the same values obtained previously through method of joints.

6.2.6 Example 6.3

Example 6.3 in Figure 6.2.9 is another 2D truss with more pins and more members. Force $\mathbf{F_1}$ is 2 kN while $\mathbf{F_2}$ is 1.5 kN. The geometry of the truss is given in the figure. A and H are pinned supports. The area of cross-section of all members are the same and is $8x10^{-3}$ [m^2]. The material is steel with E = 200 [MPa]. Find (a) the reactions at the support; (b) the force in the members; and (c) the displacement of the truss.

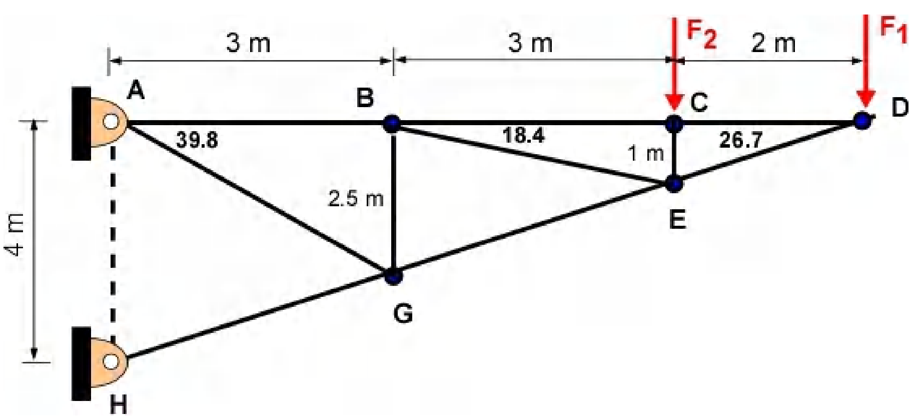


Figure 6.2.9 Example 6.3

The initial appearance of the truss suggests that this truss is likely to be statically indeterminate due to the two pinned support. That means that we cannot explicitly apply our step-by-step procedure of reactions- member forces - deflection in that order. If you start the analysis at the pin D then we can see that it is possible to obtain member forces easily by the method of joints without knowing the reactions. The member forces will allow us to determine the reactions at pins A and H eventually. The FBD of the truss is in Figure 6.2.10. In this Example we will adapt the solution process as:

- Determine member forces
- Determine reactions

- Determine displacement of the truss

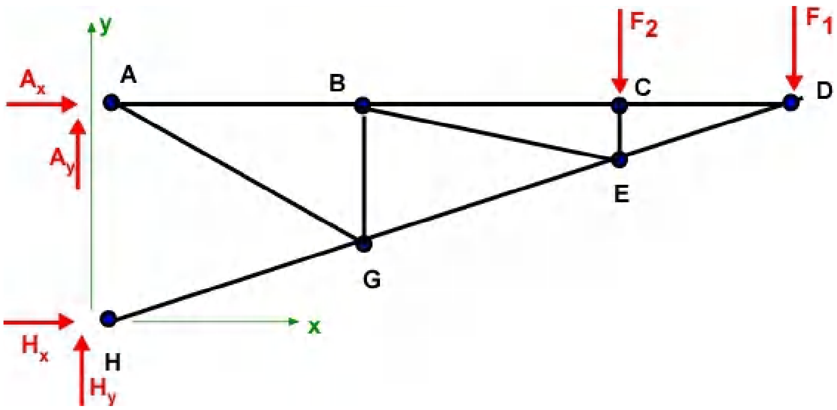


Figure 6.2.10a Example 6.3 - FBD of truss

Data: F_1= 12 [kN]; F_2 = 1.5 kN;
Locations of pins: A(0,4,0); B(3,4,0); C(6,4,0); D(8,4,0); H(0,0,0); G(3,1.5,0); E(6,3,0);
Pinned support at A and H
E = 200 [MPa]; Area of cross-section = 8x10^{-3} [m^2] - for all members

Find: (a) Member forces; (b) Reactions at A and H; (c) Displacement of the truss

Assumption: Material is elastic

Solution:
(a) Member forces
Draw FBD of pins and apply equilibrium. Start with pin D, then C, E, B, G.

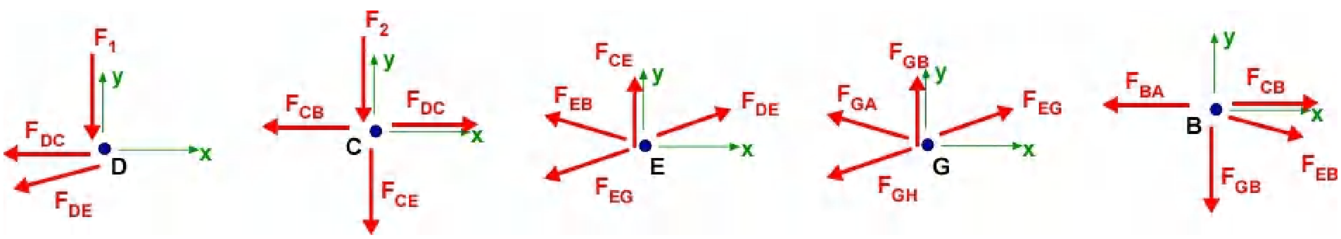


Figure 6.2.10b Example 6.3 - FBD of pins

Pin D:

$$\sum F_x = -F_{DC} - F_{DE}\cos(26.56) = 0$$
$$\sum F_y = -F_1 - F_{DE}\sin(26.56) = 0$$
$$F_{DC} = 4000[N]; \quad F_{DE} = -4472.1[N]$$

Pin C:

$$\sum F_x = F_{DC} - F_{CB} = 0$$
$$\sum F_y = -F_2 - F_{CE} = 0$$
$$F_{CE} = -1500[N]; \quad F_{CB} = 4000[N]$$

Pin E:

$$\sum F_x = F_{DE}\cos(26.56) - F_{EG}\cos(26.56) - F_{EB}\cos(18.4) = 0$$
$$\sum F_y = F_{DE}\sin(26.56) - F_{EG}\sin(26.56) + F_{EB}\sin(18.4) + F_{CE} = 0$$
$$F_{EB} = 1897.4[N]; \quad F_{EG} = -6484.6[N]$$

Pin B:

$$\sum F_x = F_{CB} - F_{BA} + F_{EB}\cos(18.4) = 0$$
$$\sum F_y = -F_{EB}\sin(18.4) - F_{GB} = 0$$
$$F_{BA} = 5840[N]; \quad F_{GB} = -600[N]$$

Pin G:

$$\sum F_x = F_{EG}\cos(26.56) - F_{GA}\cos(39.8) - F_{GH}\cos(26.56) = 0$$
$$\sum F_y = F_{EG}\sin(26.56) + F_{GA}\sin(39.8) - F_{GH}\sin(26.56) + F_{GB} = 0$$
$$F_{GA} = 585.8[N]; \quad F_{GH} = -6987.7[N]$$

(b) Reactions at the PIns

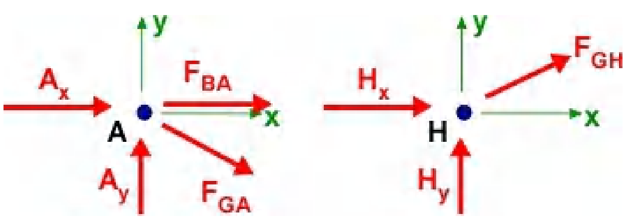


Figure 6.2.10c Example 6.3 - FBD of pins A and H

Pin A:

$$\sum F_x = F_{BA} + F_{GA}\cos(39.8) + A_x = 0$$
$$\sum F_y = A_y - F_{GA}\sin(39.8) = 0$$
$$A_x = -6250[N]; \quad A_y = 375[N]$$

Pin H:

$$\sum F_x = H_x + F_{GH}\cos(26.56) = 0$$

$$\sum F_y = H_y + F_{GH}\sin(26.56) = 0$$

$$H_x = 6250[N]; \quad H_y = 3125[N]$$

Note the reactions are not necessary to calculate the displacements. They are useful for designing supports.

(c) Displacement of Truss Joints

This is fairly long because there are 10 members with 10 unknown displacements. We have to generate 10 equations and they are all linear and some of them can be solved directly. It will probably be smarter to make MATLAB to do the grunt work.

Unknowns: dB_x, dB_y, dC_x, dC_y, dD_x, dD_y, dE_x, dE_y, dG_x, dG_y

Known: For all members: Area of cross-section is $8\text{x}10^{-3}$ [m^2]; E = 200 [MPa].

Note: The FBD of each member is drawn in tension. The actual sign of the magnitude should be used in the calculations to yield the right results. The equations are aligned along the member axis so the coordinate system is not necessary.

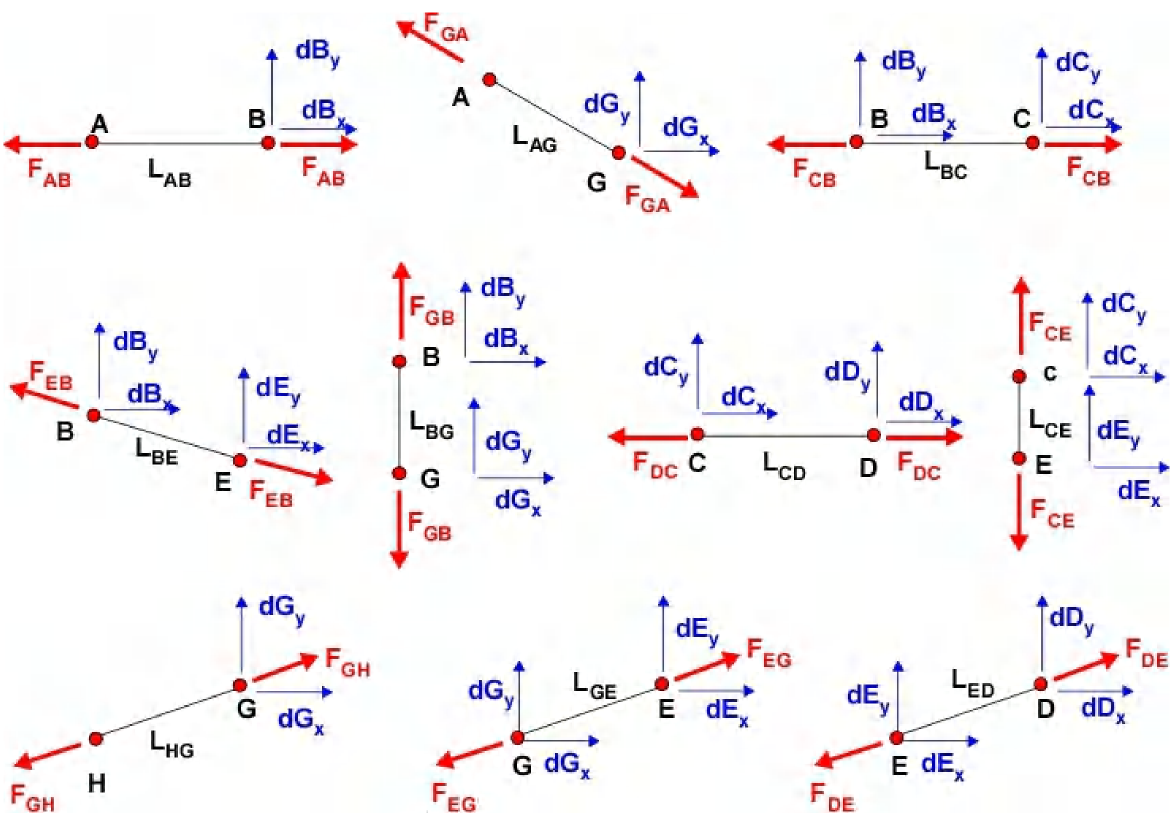


Figure 6.2.10d Example 6.3 - FBD of the members

Member AB:

$$d\bar{B} = \left[dB_x, dB_y, 0\right]; \quad d\bar{A} = \left[0, 0, 0\right]$$

$$\delta_{AB} = \hat{e}_{AB} \cdot d\bar{B} - \hat{e}_{AB} \cdot d\bar{A} = \frac{F_{AB} L_{AB}}{E_{AB} A_{AB}}$$

$$dB_x = \frac{F_{AB} L_{AB}}{E_{AB} A_{AB}} = 0.0109$$

Member AG:

$$d\bar{A} = \left[0, 0, 0\right]; \quad d\bar{G} = \left[dG_x, dG_y, 0\right];$$

$$\delta_{AG} = \hat{e}_{AG} \cdot d\bar{G} - \hat{e}_{AG} \cdot d\bar{A} = \frac{F_{AG} L_{AG}}{E_{AG} A_{AG}}$$

$$0.768\, dG_x - 0.64\, dG_y = 0.00143$$

Member BC:

$$d\bar{B} = \left[dB_x, dB_y, 0\right]; \quad d\bar{C} = \left[dC_x, dC_y, 0\right];$$

$$\delta_{BC} = \hat{e}_{BC} \cdot d\bar{C} - \hat{e}_{BC} \cdot d\bar{B} = \frac{F_{CB} L_{BC}}{E_{BC} A_{BC}}$$

$$dC_x - dB_x = 0.0075$$

Member BE:

$$d\bar{B} = \left[dB_x, dB_y, 0\right]; \quad d\bar{E} = \left[dE_x, dE_y, 0\right];$$

$$\delta_{BE} = \hat{e}_{BE} \cdot d\bar{E} - \hat{e}_{BE} \cdot d\bar{B} = \frac{F_{EB} L_{BE}}{E_{BE} A_{BE}}$$

$$0.316\, dB_y - 0.95\, dB_x + 0.95\, E_x - 0.316\, E_y = 0.00375$$

Member BG:

$$d\bar{B} = \left[dB_x, dB_y, 0\right]; \quad d\bar{G} = \left[dG_x, dG_y, 0\right];$$

$$\delta_{BG} = \hat{e}_{BG} \cdot d\bar{G} - \hat{e}_{BG} \cdot d\bar{B} = \frac{F_{GB} L_{BG}}{E_{BG} A_{BG}}$$

$$dB_y - dG_y = -0.0009375$$

Member CD:

$$d\bar{C} = \left[dC_x; dC_y; 0\right]; \quad d\bar{D} = \left[dD_x; dD_y; 0\right];$$

$$\delta_{CD} = \hat{e}_{CD} \cdot d\bar{D} - \hat{e}_{CD} \cdot d\bar{D} = \frac{F_{DC} L_{CD}}{E_{CD} A_{CD}}$$

$$dD_x - dC_x = 0.005$$

Member CE:

$$d\bar{C} = \left[dC_x; dC_y; 0\right]; \quad d\bar{E} = \left[dE_x; dE_y; 0\right];$$

$$\delta_{CE} = \hat{e}_{CE} \cdot d\bar{E} - \hat{e}_{CE} \cdot d\bar{C} = \frac{F_{CE} L_{CE}}{E_{CE} A_{CE}}$$

$$dC_y - dE_y = -0.0009375$$

Member HG:

$$d\bar{H} = [0; 0; 0]; \quad d\bar{G} = \left[dG_x; dG_y; 0\right];$$

$$\delta_{HG} = \hat{e}_{HG} \cdot d\bar{G} - \hat{e}_{HG} \cdot d\bar{H} = \frac{F_{GH} L_{HG}}{E_{HG} A_{HG}}$$

$$0.894\, dG_x + 0.447\, dG_y = -0.0146$$

Member GE:

$$d\bar{G} = \left[dG_x; dG_y; 0\right]; \quad d\bar{E} = \left[dE_x; dE_y; 0\right];$$

$$\delta_{GE} = \hat{e}_{GE} \cdot d\bar{E} - \hat{e}_{GE} \cdot d\bar{G} = \frac{F_{EG} L_{GE}}{E_{GE} A_{GE}}$$

$$0.894\, dE_x + 0.447\, dE_y - 0.894\, dG_x - 0.447\, dG_y = -0.0136$$

Member ED:

$$d\bar{E} = \left[dE_x; dE_y; 0\right]; \quad d\bar{D} = \left[dD_x; dD_y; 0\right];$$

$$\delta_{ED} = \hat{e}_{ED} \cdot d\bar{D} - \hat{e}_{ED} \cdot d\bar{E} = \frac{F_{DE} L_{ED}}{E_{ED} A_{ED}}$$

$$0.894\, dD_x + 0.447\, dD_y - 0.894\, dE_x - 0.447\, dE_y = -0.0063$$

Substituting know values and solving for the unknowns through the equations above we should calculate the displacements as:

$$dB_x = 0.01087[m]; \quad dB_y = -0.01462[m];$$
$$dC_x = 0.01837[m]; \quad dC_y = -0.06247[m];$$
$$dD_x = 0.02337[m]; \quad dD_y = -0.12388[m];$$
$$dE_x = -0.00081[m]; \quad dE_y = -0.06153[m];$$
$$dG_x = -0.00953[m]; \quad dG_y = -0.01367[m];$$

Design and Discussion:
1. The maximum deflection of over 12 cm is a cause for concern. Maybe the members carrying large loads should have an increased cross-sectional area.

2. The value of K is 2*7-3 = 11. There are only 10 members. Is this truss design acceptable. Note that this is a statically indeterminate problem. We can consider a phantom member between A and G and now it is satisfies the condition. We may assume that the condition is valid for statically determinate truss structure.

6.2.7 Example 6.4
Example 6.4 is the statically determinate version of Example 6.3 and shown in Figure 6.2.11. The support at H is roller support.

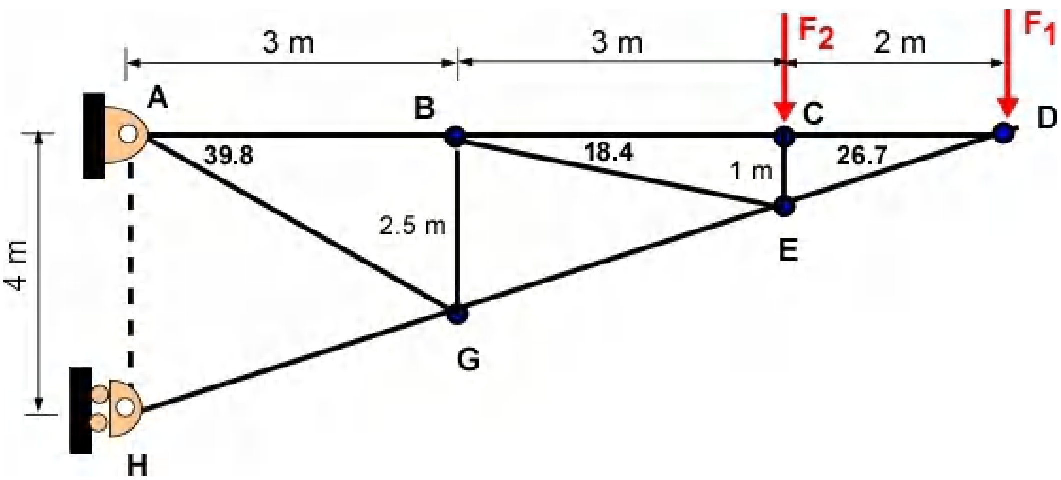


Figure 6.2.11 Example 6.4 - initial

Data: F_1= 12 [kN]; F_2 = 1.5 kN;
Locations of pins: A(0,4,0); B(3,4,0); C(6,4,0); D(8,4,0); H(0,0,0); G(3,1.5,0); E(6,3,0);
Pinned support at A and roller support at H
E = 200 [MPa]; Area of cross-section = 8x10^{-3} [m^2] - for all members

Find: (a) Member forces; (b) Reactions at A and H; (c) Displacement of the truss

Assumption: Material is elastic

You would really not expect this example to be a serious challenge after having solved Example 6.3 in detail. You could still start at the left end and work your way to the right.

Roller Support at H
Instead let us focus at the roller support H.
(1) Since H is not constrained in the vertical direction (free to move) - where is the final position of H? Is this truss stable? Will it hold together?

(2) Consider the FBD of the pin at H:

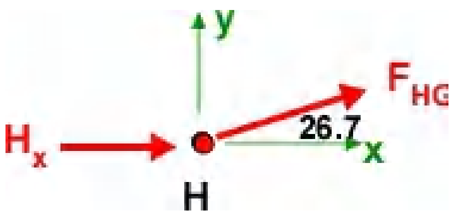


Figure 6.2.12 FBD of Pin H

$$\sum F_x = H_x + F_{HG}\cos 26.7 = 0;$$
$$\sum F_x = F_{HG}\sin 26.7 = 0;$$
$$F_{HG} = 0; \quad H_x = 0;$$

This implies that the member HG is not useful. In the previous example this member carried the most load.

(3) The value of K is 2*7-3 = 11. There are only 10 members for this determinate problem. The truss is missing a member to make it stable.

(4) The new member should be between A and H. Notice this makes the member HG more useful and it will no longer be a zero force member. The revised example is displayed in Figure 6.2.13

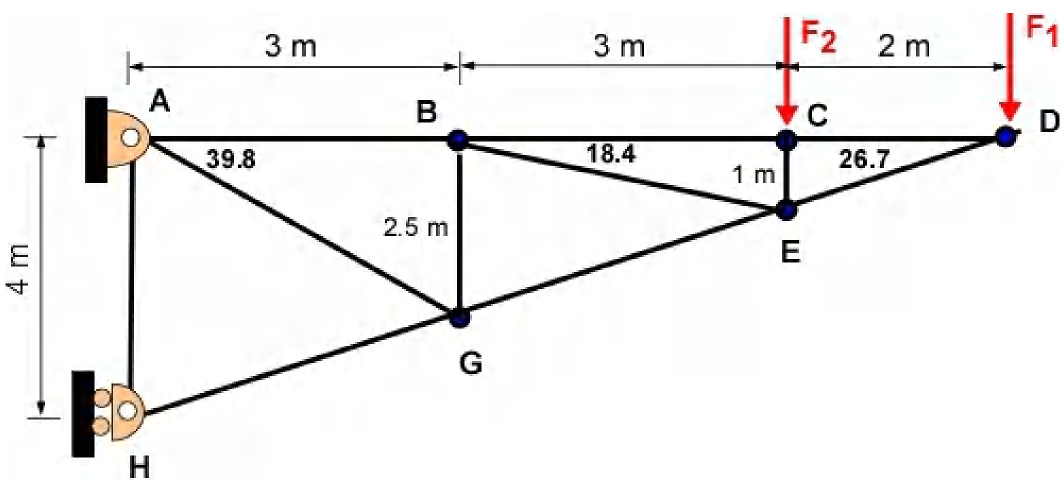


Figure 6.2.13 Example 6.4 - New

The information for Example 6.4 is the same as Example 6.3. Force $\mathbf{F_1}$ is 2 kN while $\mathbf{F_2}$ is 1.5 kN. The geometry of the truss is given in the figure. A is a pinned support while H is a roller support. The area of cross-section of all members are the same and is 8x10^{-3} [m^2]. The material is steel with E = 200 [MPa]. Find (a) the reactions at the support; (b) the force in the members; and (c) the displacement of the truss.

Exercise: Solve this example and compare it to the solution of Example 6.3. It is a good idea to solve the example in MATLAB/Octave to create a template code.

6.2.8 Additional Problems

Solve the following problems by hand on paper and using MATLAB/Octave. For each problem you must draw the FBD and work with a coordinate system. Use your own if one is not prescribed. For all problems obtain support reactions, member forces , and displacement of the truss. Use a factor of safety of 6. Choose your material and area of cross-section and ensure you have an acceptable design.

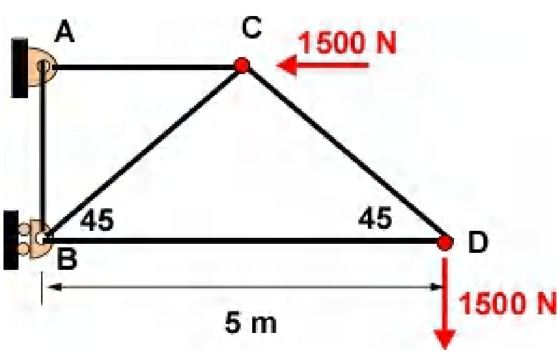
Please use Table 6.1 for your calculations.

Table 6.1

Material	**Aluminum**	**Brass**	**Steel**	**Wood**
E [GPa]	70	105	200	13
Ultimate Stress (tension) [MPa]	300	500	450	100
Ultimate stress (shear) [MPa]	70	200	250	10

Problem 6.2.1

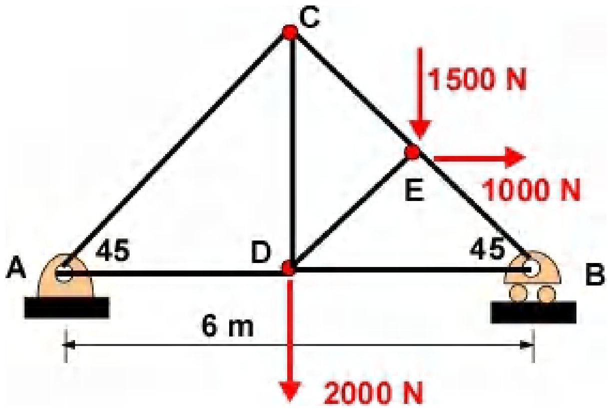
The truss is shown below



Problem 6.2.1

Problem 6.2.2

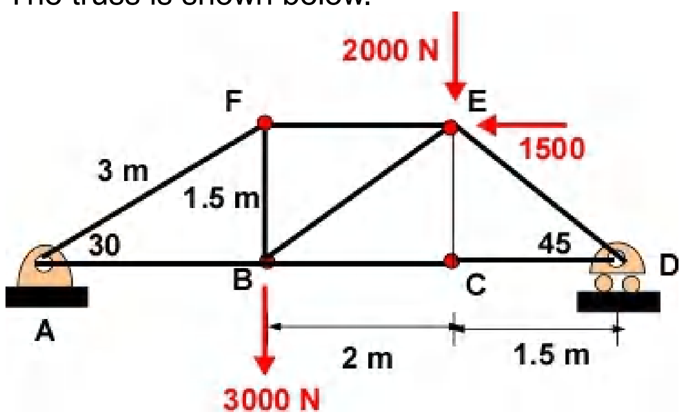
The truss is shown below



Problem 6.2.2

Problem 6.2.3

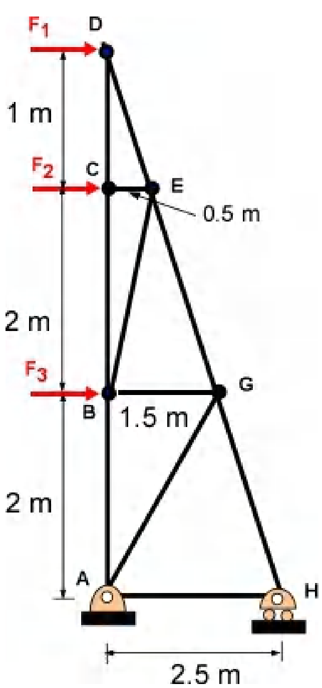
The truss is shown below.



Problem 6.2.3

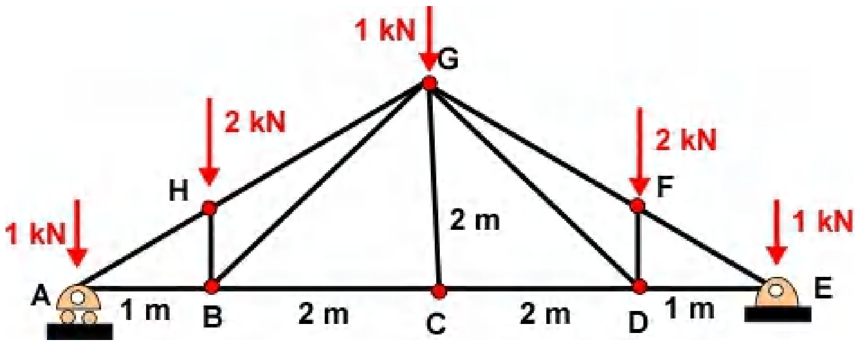
Problem 6.2.4

The truss is shown below. This is a vertical version of Example 6.4. Take F_1, F_2, and F_3 as 1000 N, 1500 N, and 2000 N respectively.



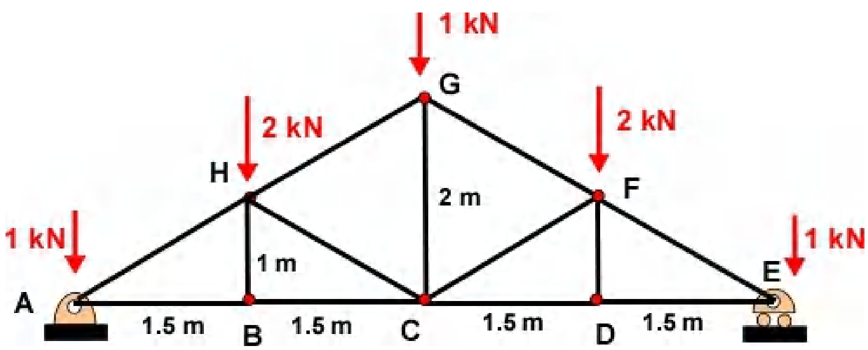
Problem 6.2.4

Problem 6.2.5
This truss is called Pratt roof truss. Since loads can be placed only on the pins/joints the snow load on the truss is translated to the load at the joints.



Problem 6.2.5

Problem 6.2.6
This truss is called Howe roof truss. Note the truss in Problems 6.2.5 and 6.2.6 carry similar loads and they span the same distance. You can therefore have a preference for one kind of truss over another.
Before solving the problem take a look at the truss and see if you can identify truss members that have zero forces by just inspection.
Consider member BH. Can it be a zero force member? If so. Why?
Are there additional zero-force members?
Can we just remove member BH if it is a zero-force member?



Problem 6.2.6

6.3 PLANE TRUSS -STATICALLY INDETERMINATE

In statically indeterminate problems we need additional relations to solve the problem. Frequently this is generated from constraints. One source of constraint is displacement at the support of the structure. Calculating displacements typically include assumptions of elastic behavior and requires additional properties of the structure like material and cross-sectional dimensions.

We will consider a simple recasting of Example 6.1 to Example 6.5 and follow the same analysis. The loads and the geometry are the same. A small change is made to the support. The roller is made into another pin support at B. This makes the problem statically indeterminate. It will be instructive to see the change in the values of the forces etc. What should be apparent is that the method of the solution and implementation is the same as in Example 6.1. The values of the forces may not be explicit along the way. In this example we need *one more equation* because of the indeterminacy and that is directly provided by the requirement of the *x-displacement of B must be zero.* In Example 6.1 it was free and had to be evaluated. Figure 6.3.1 describes the problem and the FBD.

Symbolic algebra can be a useful tool for solving this problem. We can collect all the equations and solve them at the end instead of incrementally as we did in the previous example. The incremental solutions were useful to keep the hand calculations smaller and sustainable.

Example 6.5

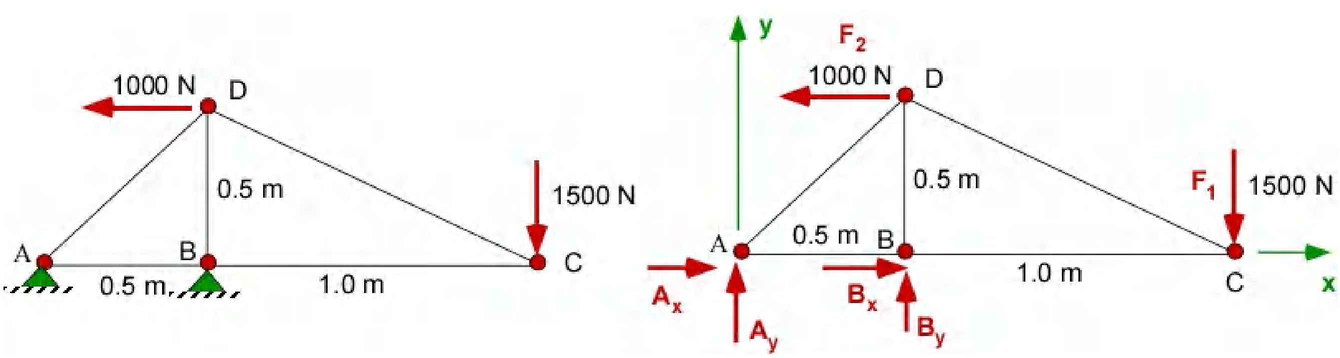


Figure 6.3.1 Example 6.2 and FBD

Data: A(0, 0, 0); B(0, 0.5, 0); C(1.5, 0, 0); D(0.5; 0.5, 0);
F1 = 1500 [N]; F2 = = 1000 [N] (direction on Figure);
All members are made up of the same material with the modulus of elasticity value of $11.4\text{x}10^6$ [Pa].
All of the members have the same cross-sectional area of $50\text{x}10^{-4}$ [m^2].

Find: (a) A_x, A_y, B_x, B_y; (b) F_{AB}, F_{BC}, F_{BD}, F_{CD}; (c) dB_x, dC_x, dC_y, dD_x, dD_y

Assumption: Pinned support at A, roller support at B

Solution: Apply Equilibrium
(These above steps can be folded into discrete steps to solve truss problems).

In the following we will assign equation numbers for reference later. We can solve for the support reactions through equilibrium:

We have four unknowns: A_x, A_y, B_x, B_y. Total unknowns so far = 4.

$$\sum F_x = 0 = A_x + B_x - 1000 \qquad (1)$$

$$\sum F_y = 0 = A_y + B_y - 1500 \qquad (2)$$

$$\sum M_A = 0 = \left(\bar{r}_{AD} \times -1000\,\hat{i}\right) + \left(\bar{r}_{AB} \times \bar{F}_B\right) + \left(\bar{r}_{AC} \times -1500\,\hat{j}\right)$$

$$0 = 0.5 \times B_y + (0.5 \times 1000) - (1.5 \times 1500) \qquad (3)$$

Equation (3) will give us B_y and Equation (2) will yield the value of A_y. B_x and A_x cannot be explicitly determined.

6.3.1 Member Forces - Method of Joints

The member forces are obtained by considering the equilibrium of each pin or joint. Since we will be developing equilibrium equations for each pin, or a point - there are only two force equilibrium equations available at each pin for a 2D problem. We will set up the equations for pins A, B and C as was done previouly in Example 6.1. The FBD of the pins are shown in Figure 6.3.2.

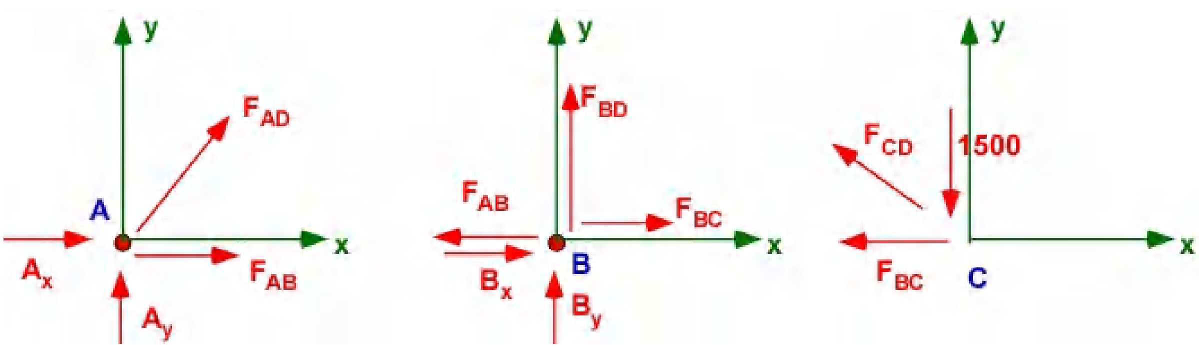


Figure 6.3.2 FBD of pins/joints

Applying force equilibrium at the pins:

Pin A: (Two additional unknowns : F_{AB}, F_{AD}; Total unknowns: 6)

$$\sum \bar{F} = 0 = \bar{F}_A + \bar{F}_{AB} + \bar{F}_{AD}$$

$$\sum F_x = A_x + F_{AB} + F_{ADx} = 0 \qquad (4)$$

$$\sum F_y = A_y + F_{ADy} = 0 \qquad (5)$$

Pin B: (Two additional unknowns : F_{BC}, F_{BD}; Total unknowns: 8)

$$\sum F_x = B_x - F_{AB} + F_{BC} = 0 \qquad (6)$$

$$\sum F_y = B_y + F_{BD} = 0 \qquad (7)$$

Pin C: (One additional unknowns : F_{CD}; Total unknowns: 9)

$$\sum F_x = -F_{BC} - F_{CDx} = 0$$
$$\sum F_y = F_{CDy} - 1500 = 0 \qquad (8)$$

At this stage we have 9 unknowns and 8 equations.

6.3.2 Truss Displacement

We will develop the relation between member forces and member displacement using the stress, strain, and Hooke's law, and the original geometry of the member. These equations are identical to the ones in Example 6.1. The exception is that $dB_x = 0$ and is not an unknown. This provides an extra equation to solve for support reactions.
New unknowns : dC_x, dC_y, dD_x, dD_y. Total number of unknowns = 13
Number of new equations = 5. Total number of equations = 13

This will provide enough equations to solve for the unknowns.

Member AB:

$$d\bar{B} = [0, 0, 0]; \quad d\bar{A} = [0, 0, 0]$$
$$\delta_{AB} = \hat{e}_{AB} \cdot d\bar{B} - \hat{e}_{AB} \cdot d\bar{A} = \frac{F_{AB} L_{AB}}{E_{AB} A_{AB}} \qquad (9)$$

Member AD

$$d\bar{D} = [dD_x, dD_y, 0]; \quad d\bar{A} = [0, 0, 0];$$
$$\delta_{AD} = \hat{e}_{AD} \cdot d\bar{D} - \hat{e}_{AD} \cdot d\bar{A} = \frac{F_{AD} L_{AD}}{E_{AD} A_{AD}} \qquad (10)$$

Member BC

$$d\bar{C} = [dC_x, dC_y, 0]; \quad d\bar{B} = [0, 0, 0]$$
$$\delta_{BC} = \hat{e}_{BC} \cdot d\bar{C} - \hat{e}_{BC} \cdot d\bar{B} = \frac{F_{BC} L_{BC}}{E_{BC} A_{BC}} \qquad (11)$$

Member BD

$$d\bar{D} = [dD_x, dD_y, 0]; \quad d\bar{B} = [0, 0, 0];$$
$$\delta_{BD} = \hat{e}_{BD} \cdot d\bar{D} - \hat{e}_{BD} \cdot d\bar{B} = \frac{F_{BD} L_{BD}}{E_{BD} A_{BD}} \qquad (12)$$

Member CD:

$$d\bar{C} = [dC_x, dC_y, 0]; \quad d\bar{D} = [dD_x, dD_y, 0]$$
$$\delta_{CD} = \hat{e}_{CD} \cdot d\bar{C} - \hat{e}_{CD} \cdot d\bar{D} = \frac{F_{CD} L_{CD}}{E_{CD} A_{CD}} \qquad (13)$$

We will construct these 13 equations and solve them using MATLAB. The code is exactly the same as in Example 6.1 but is consolidated. The solution is also deferred to the end.

Solution Using MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, Jan 2015
%  Example 6.5 Section: 6.3.1
% Same as Example 6.1 but statically indeterminate
% Truss 2D
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(3)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 6.5\n')
fprintf('------------\n')
%% Data
F1 = 1500; F2 = 1000; % the applied force
A = [0 0 0]; B = [0.5,0,0];  C = [1.5,0,0]; D = [0.5,0.5,0];

%% Support reaction calculations
syms Ax Ay Bx By real % Unknowns

rAB = B - A;  rAD = D - A;  rAC = C - A;
FA = [Ax, Ay, 0];  FB = [Bx,By,0];  FC = [0,-F1,0];  FD = [-F2,0,0];

%% Equilibrium
SumF = FA + FB + FC + FD;  % Sum of Forces
SumMA = cross(rAD,FD)+ cross(rAB,FB)+ cross(rAC,FC);
% Sum of moments at A

Eq(1) = SumF(1);
Eq(2) = SumF(2);
Eq(3) = SumMA(3);

%%%%%%%%%%%%%%%%%%%%%%
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point C [m]  : '),disp(C)
fprintf('Point D [m]  : '),disp(D)

fprintf('Position vector rAB[m]    = '),disp(rAB)
fprintf('Position vector rAC[m]    = '),disp(rAC)
fprintf('Position vector rAD[m]   = '),disp(rAD)

fprintf('\nF1 [ N]     : '),disp(F1)
fprintf('F2 [ N]     : '),disp(F2)

fprintf('\nEquilibrium\n')
fprintf('----------------------')
fprintf('\nSumF : \n'),disp(vpa(SumF',4))
fprintf('\nSumMA : \n'),disp(vpa(SumMA',4))
```

```
%% Pin A
fprintf('\nPin A:\n')
fprintf('------------\n')
syms Fab Fad
rAB = B-A;  eAB = rAB/norm(rAB);
rAD = D-A;  eAD = rAD/norm(rAD);
SumA = FA + Fab*eAB + Fad*eAD

Eq(4) = SumA(1);
Eq(5) = SumA(2);

%% Pin B
fprintf('\nPin B:\n')
fprintf('------------\n')
syms Fbc Fbd
rBC = C - B;  eBC = rBC/norm(rBC);
rBD = D - B;  eBD = rBD/norm(rBD);
SumB = -Fab*eAB + FB + Fbc*eBC + Fbd*eBD

Eq(6) = SumB(1);
Eq(7) = SumB(2);

%% Pin c
fprintf('\nPin C:\n')
fprintf('------------\n')
syms Fcd
rCD = D - C;  eCD = rCD/norm(rCD);
% only one unknown - need only one equation to solve for it
SumC = -Fbc*eBC + FC + Fcd*eCD

Eq(8)= SumC(1);

%% Calculating displacements
syms  dCx dCy dDx dDy  % the unknowns

% Properties
Elas = 11.4e06;   Area = 50e-04;
% displacement vectors
dA = [0,0,0];  dB = [0,0,0];  dC = [dCx,dCy,0];
dD = [dDx, dDy,0];

% length of members and unit vectors  (Copy and Paste)
rAB = B - A;  LAB = norm(rAB);  eAB = rAB/LAB;
rAD = D - A;  LAD = norm(rAD);  eAD = rAD/LAD;
rBC = C - B;  LBC = norm(rBC);  eBC = rBC/LBC;
rBD = D - B;  LBD = norm(rBD);  eBD = rBD/LBD;
rDC = C - D;  LDC = norm(rDC);  eDC = rDC/LDC;

% set up the equations
fprintf('\nDisplacement Equations:\n')
fprintf('------------------------------\n')
Eq(9) = dot(eAB,dB)-dot(eAB,dA) - (Fab*LAB/(Area*Elas)); % Member AB
fprintf('Eq(9) = '),disp(Eq(9))
Eq(10) = dot(eAD,dD)-dot(eAD,dA) - (Fad*LAD/(Area*Elas)); % Member AD
fprintf('Eq(10) = '),disp(Eq(10))
```

```
Eq(11) = dot(eBC,dC)-dot(eBC,dB) - (Fbc*LBC/(Area*Elas)); % Member BC
fprintf('Eq(11) = '),disp(Eq(11))
Eq(12) = dot(eBD,dD)-dot(eBD,dB) - (Fbd*LBD/(Area*Elas)); % Member BD
fprintf('Eq(12) = '),disp(Eq(12))
Eq(13) = dot(eDC,dC)-dot(eDC,dD) - (Fcd*LDC/(Area*Elas)); % Member DC
fprintf('Eq(13) = '),disp(Eq(13))

%% Solution
sol = solve(Eq);

%% Print solution
fprintf('\nReactions:\n')
fprintf('------------\n')
fprintf('Ax [N]     = '),disp(double(sol.Ax))
fprintf('Ay [N]     = '),disp(double(sol.Ay))
fprintf('Bx [N]     = '),disp(double(sol.Bx))
fprintf('By [N]     = '),disp(double(sol.By))

fprintf('\nMember Forces:\n')
fprintf('----------------\n')
fprintf('Fab [N]     = '),disp(double(sol.Fab))
fprintf('Fad [N]     = '),disp(double(sol.Fad))
fprintf('Fbc [N]     = '),disp(double(sol.Fbc))
fprintf('Fbd [N]     = '),disp(double(sol.Fbd))
fprintf('Fcd [N]     = '),disp(double(sol.Fcd))

fprintf('\nJoint Displacements:\n')
fprintf('----------------------\n')
fprintf('dCx [m] : '),disp(double(sol.dCx))
fprintf('dCy [m] : '),disp(double(sol.dCy))
fprintf('dDx [m] : '),disp(double(sol.dDx))
fprintf('dDy [m] : '),disp(double(sol.dDy))
```

In the Command Window

```
Example 6.5
------------
Point A [m]  :      0      0      0
Point B [m]  :           0.5            0             0
Point C [m]  :           1.5            0             0
Point D [m]  :           0.5          0.5             0
Position vector rAB[m]    =           0.5             0             0
Position vector rAC[m]    =           1.5             0             0
Position vector rAD[m]   =           0.5           0.5             0

F1 [ N]    :         1500
F2 [ N]    :         1000

Equilibrium
----------------------
SumF :
 Ax + Bx - 1000.0
 Ay + By - 1500.0
               0
```

```
SumMA :
                0
                0
 0.5*By - 1750.0

Pin A:
------------
SumA =
[ Ax + Fab + (2^(1/2)*Fad)/2, Ay + (2^(1/2)*Fad)/2, 0]

Pin B:
------------
SumB =
[ Bx - Fab + Fbc, By + Fbd, 0]

Pin C:
------------
SumC =
[ - Fbc - (2*5^(1/2)*Fcd)/5, (5^(1/2)*Fcd)/5 - 1500, 0]

Displacement Equations:
------------------------------
Eq(9) = -Fab/114000
Eq(10) = (2^(1/2)*dDx)/2 - (2^(1/2)*Fad)/114000 + (2^(1/2)*dDy)/2
Eq(11) = dCx - Fbc/57000
Eq(12) = dDy - Fbd/114000
Eq(13) = (2*5^(1/2)*dCx)/5 - (5^(1/2)*Fcd)/114000 - (5^(1/2)*dCy)/5 -
(2*5^(1/2)*dDx)/5 + (5^(1/2)*dDy)/5

Reactions:
------------
Ax [N]     =        -2000
Ay [N]     =        -2000
Bx [N]     =         3000
By [N]     =         3500

Member Forces:
----------------
Fab [N]     =      0
Fad [N]     =         2833
Fbc [N]     =        -3000
Fbd [N]     =        -3500
Fcd [N]     =         3355

Joint Displacements:
----------------------
dCx [m] :       -0.0526
dCy [m] :        -0.444
dDx [m] :        0.0803
dDy [m] :       -0.0307
```

Note that this is a consolidated MATLAB Code. We defer all printing till the end. You can do the same for Example 6.1 and any other problem you are going to solve.

Execution in Octave

The code is the same as in MATLAB except for the additional statements and/or highlighted changes

A lot of these changes relate to print statements to obtain better information in the command window. Code relating to calculations are mostly the same.
Flattening the symbolic display made the expressions very long. Hence the symbolic numbers are printed in rational form.

In Octave Editor

```
clc, clear, format compact, format shortg, close all
warning off
pkg load symbolic;


SumA = FA + Fab*eAB + Fad*eAD;
fprintf('\nSumA : \n'),disp(vpa(SumA,4))

SumB = -Fab*eAB + FB + Fbc*eBC + Fbd*eBD ;
fprintf('\nSumB : \n'),disp(vpa(SumB,4))

SumC = -Fbc*eBC + FC + Fcd*eCD ;
fprintf('\nSumC : \n'),disp(vpa(SumC,4))

Eq(9) = dot(eAB,dB)-dot(eAB,dA) - (Fab*LAB/(Area*Elas)); % Member AB
fprintf('Eq(9) = \n'),disp(Eq(9))
Eq(10) = dot(eAD,dD)-dot(eAD,dA) - (Fad*LAD/(Area*Elas)); % Member AD
fprintf('Eq(10) = \n'),disp(Eq(10))
Eq(11) = dot(eBC,dC)-dot(eBC,dB) - (Fbc*LBC/(Area*Elas)); % Member BC
fprintf('Eq(11) = \n'),disp(Eq(11))
Eq(12) = dot(eBD,dD)-dot(eBD,dB) - (Fbd*LBD/(Area*Elas)); % Member BD
fprintf('Eq(12) = \n'),disp(Eq(12))
Eq(13) = dot(eDC,dC)-dot(eDC,dD) - (Fcd*LDC/(Area*Elas)); % Member DC
fprintf('Eq(13) =\n '),disp(Eq(13))
```

In Octave Command Window
The output values is the same as in MATLAB

```
Example 6.5
------------
Point A [m]  :   0   0   0
Point B [m]  :   0.5   0   0
Point C [m]  :   1.5   0   0
Point D [m]  :   0.5   0.5   0
Position vector rAB[m]    =   0.5   0   0
Position vector rAC[m]    =   1.5   0   0
Position vector rAD[m]   =   0.5   0.5   0

F1 [ N]    : 1500
F2 [ N]    : 1000

Equilibrium
----------------------
SumF :
  [Ax + Bx - 1000.0  Ay + By - 1500.0  0]

SumMA :
```

```
  [0  0  0.5*By - 1750.0]

Pin A:
------------

SumA :
  [Ax + Fab + 0.7071*Fad  Ay + 0.7071*Fad  0]

Pin B:
------------

SumB :
  [Bx - Fab + Fbc  By + Fbd  0]

Pin C:
------------

SumC :
  [-Fbc - 0.8944*Fcd  0.4472*Fcd - 1500.0  0]

Displacement Equations:
-----------------------------
Eq(9) =
  -Fab
  ------
  114000
Eq(10) =
    569*pi*Fad   569*pi*dDx   569*pi*dDy
  - ---------- + ---------- + ----------
    144096000       2528         2528
Eq(11) =
     Fbc
  - ----- + dCx
    57000
Eq(12) =
     Fbd
  - ------ + dDy
    114000
Eq(13) =
     963*Fcd    2584*dCx   1292*dCy   2584*dDx   1292*dDy
  - -------- + -------- - -------- - -------- + --------
    49096000     2889       2889       2889       2889

Reactions:
------------
Ax [N]     = -2000
Ay [N]     = -2000
Bx [N]     = 3000
By [N]     = 3500

Member Forces:
----------------
Fab [N]     = 0
Fad [N]     = 2828.4
Fbc [N]     = -3000
Fbd [N]     = -3500
```

```
Fcd [N]     = 3354.1

Joint Displacements:
----------------------
dCx [m] : -0.052632
dCy [m] : -0.44372
dDx [m] : 0.080323
dDy [m] : -0.030702
```

6.3.3 Compare Example 6.2 and Example 6.5

We have solved the same truss example with two different constraints. Let us compare the values for the various quantities and see if there is some information to be gleamed.

Table 6.2 2 Comparison of Solutions

Quantity	Example 6.1	Example 6.5
Ax [N]	1000	-2000
Ay [N]	-2000	-2000
Bx [N]	0	3000
By [N]	3500	3500
Fab [N]	-3000	0
Fad [N]	2833	2833
Fbc [N]	-3000	-3000
Fbd [N]	-3500	-3500
Fcd [N]	3355	3355
dCx [m]	-0.0789	-0.0526
dCy [m]	-0.415	-0.444
dDx [m]	0.0804	0.0803
dDy [m]	-0.0307	-0.0307
dBx [m]	-0.0263	0

Observations:

- Member force in AB is different because of the constraint but other member forces are the same.
- The force in AB in Example 6.5 is zero. Is member AB really necessary? Can it be a phantom member.
- The pin at B in Example 6.5 handles the force in the member AB in Example 6.1
- The x-displacement at B is absorbed by the x-displacement at C
- The y-reaction forces are unaffected.
- The displacements are large and hence changes in material or area of cross-section or both must be made for a safer truss structure.

6.3 4 Truss Example 6.6

We have covered most of the important issues with respect to truss problems. We have not used ***Method of Sections*** much since solving for displacements required all member forces. Nevertheless applying the method of joints consistently is sufficient for solving truss problems. We have seen that MATLAB can be an effective tool for complete truss analysis and that is encouraged in this book. Let us consider one more example of a statically indeterminate truss and complete it by graphically comparing the displacements before and after the load is applied.

The use of software like MATLAB/Octave takes care of a lot of drudge work in truss problems. Note it is also important to use it in a consistent way. The successful solution depends on your use and its application. The following are some of your responsibilities:

- Recognizing the type of problem.
- Setting up the problem in the right way.
- Organizing the solutions as a sequence of smaller problems.
- Representing the small and the large problems using a correct FBD.
- Applying the analysis consistently.
- Reporting the results and checking them for error.
- Using a calculator to verify some in-between results.
- Developing an instinctive feel for the problem.
- Using parameters in the code that will allow you to vary the solution for acceptable design based on FOS and material strength.

Example 6.6

A 2D truss is shown in Figure 6.6.1 It is a roof truss and is withstanding a blizzard. The wind is from the left and the snow load is on the right. Remember in a truss the loads are only applied at the pins. In this instance the actual loads on the roof is transfered to the joints. How do you apply the vertical load on pin D which is internal to the roof? This is a text book problem and we can take liberties. You can recognize that it is statically indeterminate. Is it stable? The number of joints (n) is 7. The number of members (m) is 11. It is stable. You should verify it. We plan to solve for the displacement of the truss under the loads. We will dispense with the initial identification of Data, Find, Assumptions in favor of discrete steps in truss problems.

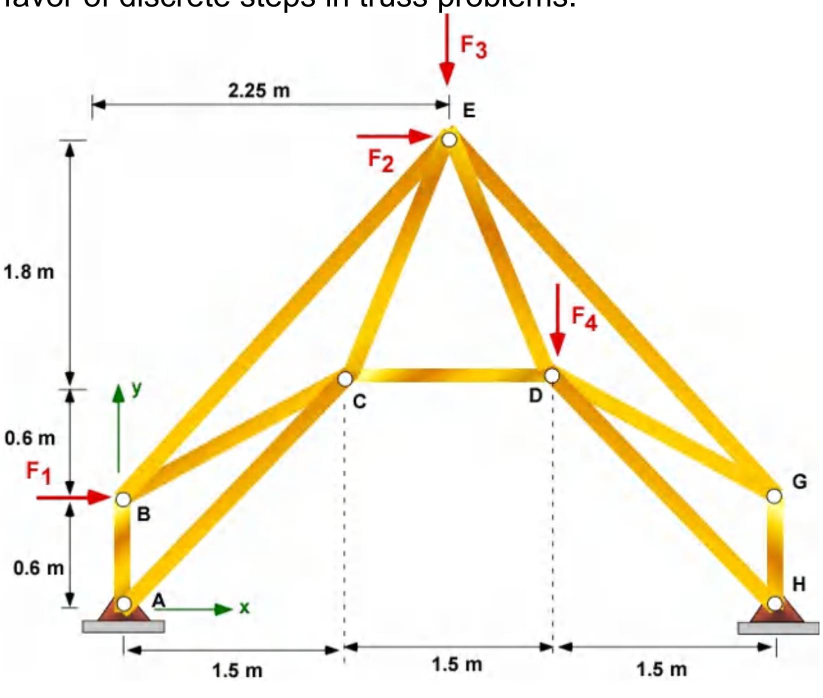


Figure 6.3.3 Example 6.6

There are a lot of joints and members. It is useful to make further observations about the truss to see if we can reduce the calculations. The truss structure itself is symmetrical. If the loading was also symmetrical, then one needs to solve for one-half the truss. Unfortunately the loading is is not symmetrical and therefore we have to expect that the loads in the symmetrically located members will be different.

Another question we can ask:

Is it less cumbersome to apply the **method of sections**?

Let us look at the truss again from the end E, away from the support where we know the problem is indeterminate. We cannot section the truss to solve for only two unknowns. If the applied force F_2 was not present, then we can use symmetry and section the truss through members BE, CE, DE, GE and solve for the forces in the members assuming force in member BE was the same as the force in GE, and the force in CE is the same as in DE. Then we can move to joint B and solve for force in members AB and BC. We can go from joint to joint until all the forces are resolved, including the reactions. Given the current loading we have to apply the **method of joints**. We will use F_1 as 500 N, F_2 as 800 N, F_3 as 750 N and F_4 as 1000 N. All members are made up of the same material (wood) with the modulus of elasticity value of 11.4×10^6 [Pa]. All of the members have the same cross-sectional area of (2 in x 6 in) or 0.008 [m^2].

Support Reactions

We start with FBD of the truss and write the equilibrium equations taking the moment about A. The pin locations are known.

There are 4 unknowns at this stage A_x, A_y, H_x, H_y with 3 equations of statics.

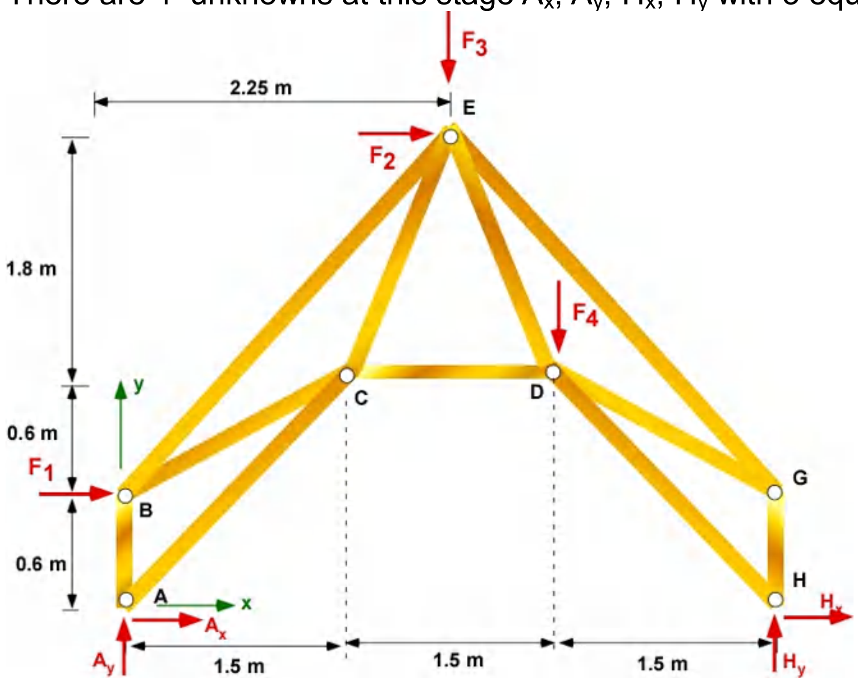


Figure 6.3.4 FBD of Example 6.6

We will once again locally number equations for reference. The three equations of equilibrium are generated from:

$$\sum \bar{F} = 0 = \bar{F}_A + \bar{F}_H + \bar{F}_B + \bar{F}_E + \bar{F}_D \qquad (1)$$

$$\sum \bar{M}_A = 0 = \left(\bar{r}_{AB} \times \bar{F}_B\right) + \left(\bar{r}_{AE} \times \bar{F}_E\right) + \left(\bar{r}_{AD} \times \bar{F}_D\right) + \left(\bar{r}_{AH} \times \bar{F}_H\right) \qquad (2)$$

Member Forces

There are 11 member force unknowns as we have 11 members with equations applied at 6 joints. These will require FBD of the Joints. We will draw FBD of joints A, B, C, E, D, and H.

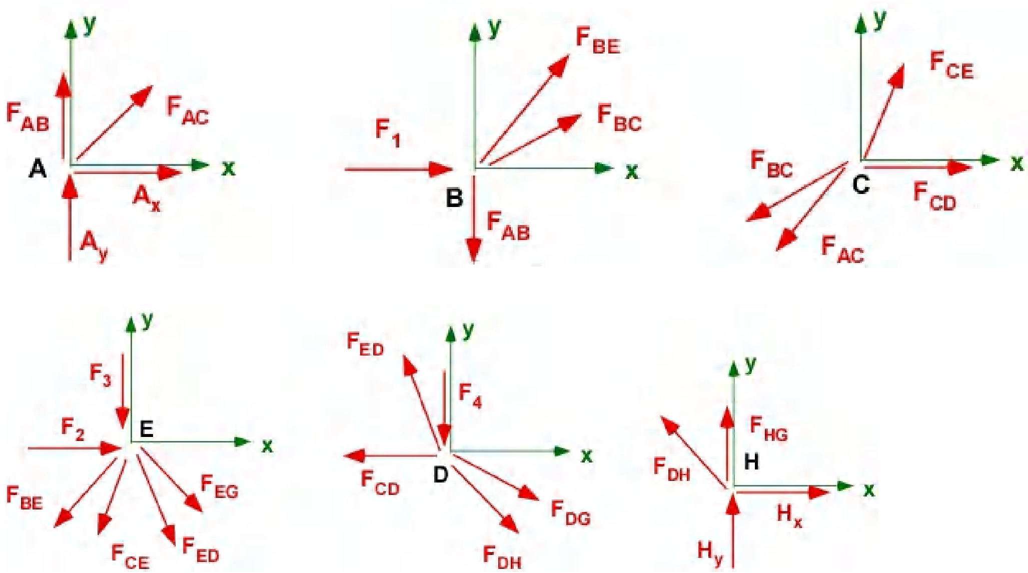


Figure 6.3.5 FBD of the joints

The vector equilibrium equations for the pins/joints are:

Joint A: $$\sum \bar{F} = 0 = \bar{F}_{AB} + \bar{F}_{AC} + \bar{F}_{A} = 0 \qquad (3)$$

Joint B: $$\sum \bar{F} = 0 = \bar{F}_{BA} + \bar{F}_{BE} + \bar{F}_{BC} + \bar{F}_{B} = 0 \qquad (4)$$

Joint C: $$\sum \bar{F} = 0 = \bar{F}_{CB} + \bar{F}_{CA} + \bar{F}_{CE} + \bar{F}_{CD} = 0 \qquad (5)$$

Joint E: $$\sum \bar{F} = 0 = \bar{F}_{EB} + \bar{F}_{EC} + \bar{F}_{EG} + \bar{F}_{ED} + \bar{F}_{E} = 0 \qquad (6)$$

Joint D: $$\sum \bar{F} = 0 = \bar{F}_{DC} + \bar{F}_{DE} + \bar{F}_{DG} + \bar{F}_{DH} + \bar{F}_{D} = 0 \qquad (7)$$

Joint H: $$\sum \bar{F} = 0 = \bar{F}_{HD} + \bar{F}_{HG} + \bar{F}_{H} = 0 \qquad (8)$$

There are now 15 unknowns and 14 useful equations. The joints yield 2 equations each except the joint H which contributes only one equation. The equation numbers are not consistent with the number of unknowns as the above are vector equations.

Joint Displacements

These introduce another 10 unknowns but there are 11 equations - one for each member. Altogether we have 25 unknowns and 25 equations and a solution should be possible.

We use a representative member BC to develop the relation between the joint displacement and member forces in Figure 6.3.6

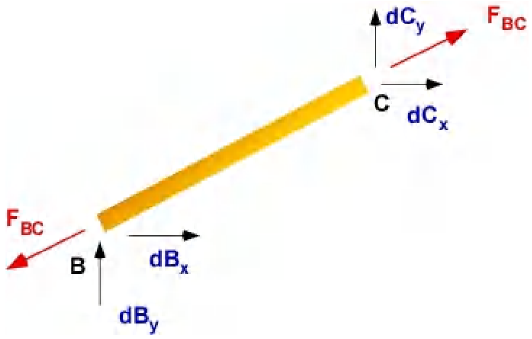


Figure 6.3.6 FBD of Member BC

Member BC:

$$d\bar{C} = \left[dC_x, dC_y, 0\right]; \quad d\bar{B} = \left[dB_x, dB_y, 0\right]$$

$$\delta_{BC} = \hat{e}_{BC} \cdot d\bar{C} - \hat{e}_{BC} \cdot d\bar{B} = \frac{F_{BC} L_{BC}}{E_{BC} A_{BC}} \qquad (9)$$

Member AB:

$$d\bar{B} = \left[dB_x, dB_y, 0\right]; \quad d\bar{A} = [0, 0, 0]$$

$$\delta_{AB} = \hat{e}_{AB} \cdot d\bar{B} - \hat{e}_{AB} \cdot d\bar{A} = \frac{F_{AB} L_{AB}}{E_{AB} A_{AB}} \qquad (10)$$

Member AC:

$$d\bar{C} = \left[dC_x, dC_y, 0\right]; \quad d\bar{A} = [0, 0, 0]$$

$$\delta_{AC} = \hat{e}_{AC} \cdot d\bar{C} - \hat{e}_{AC} \cdot d\bar{A} = \frac{F_{AC} L_{AC}}{E_{AC} A_{AC}} \qquad (11)$$

Member BE:

$$d\bar{E} = \left[dE_x, dE_y, 0\right]; \quad d\bar{B} = \left[dB_x, dB_y, 0\right]$$

$$\delta_{BE} = \hat{e}_{BE} \cdot d\bar{E} - \hat{e}_{BE} \cdot d\bar{B} = \frac{F_{BE} L_{BE}}{E_{BE} A_{BE}} \qquad (12)$$

Member CD:

$$d\bar{D} = \left[dD_x, dD_y, 0\right]; \quad d\bar{C} = \left[dC_x, dC_y, 0\right]$$

$$\delta_{CD} = \hat{e}_{CD} \cdot d\bar{D} - \hat{e}_{CD} \cdot d\bar{C} = \frac{F_{CD} L_{CD}}{E_{CD} A_{CD}} \qquad (13)$$

Member CE:

$$d\bar{E} = \left[dE_x, dE_y, 0\right]; \quad d\bar{C} = \left[dC_x, dC_y, 0\right]$$

$$\delta_{CE} = \hat{e}_{CE} \cdot d\bar{E} - \hat{e}_{CE} \cdot d\bar{C} = \frac{F_{CE} L_{CE}}{E_{CE} A_{CE}} \qquad (14)$$

Member ED:

$$d\bar{D} = \left[dD_x; dD_y; 0\right]; \quad d\bar{E} = \left[dE_x; dE_y; 0\right]$$

$$\delta_{ED} = \hat{e}_{ED} \cdot d\bar{D} - \hat{e}_{ED} \cdot d\bar{E} = \frac{F_{ED}L_{ED}}{E_{ED}A_{ED}} \qquad (15)$$

Member EG:

$$d\bar{G} = \left[dG_x; dG_y; 0\right]; \quad d\bar{E} = \left[dE_x; dE_y; 0\right]$$

$$\delta_{EG} = \hat{e}_{EG} \cdot d\bar{G} - \hat{e}_{EG} \cdot d\bar{E} = \frac{F_{EG}L_{EG}}{E_{EG}A_{EG}} \qquad (16)$$

Member DG:

$$d\bar{G} = \left[dG_x; dG_y; 0\right]; \quad d\bar{D} = \left[dD_x; dD_y; 0\right]$$

$$\delta_{DG} = \hat{e}_{DG} \cdot d\bar{G} - \hat{e}_{DG} \cdot d\bar{D} = \frac{F_{DG}L_{DG}}{E_{DG}A_{DG}} \qquad (17)$$

Member DH:

$$d\bar{H} = \left[0; 0; 0\right]; \quad d\bar{D} = \left[dD_x; dD_y; 0\right]$$

$$\delta_{DH} = \hat{e}_{DH} \cdot d\bar{H} - \hat{e}_{DH} \cdot d\bar{D} = \frac{F_{DH}L_{DH}}{E_{DH}A_{DH}} \qquad (18)$$

Member HG:

$$d\bar{G} = \left[dG_x; dG_y; 0\right]; \quad d\bar{H} = \left[0; 0; 0\right]$$

$$\delta_{HG} = \hat{e}_{HG} \cdot d\bar{G} - \hat{e}_{DH} \cdot d\bar{H} = \frac{F_{HG}L_{HG}}{E_{HG}A_{HG}} \qquad (19)$$

6.3.5 Solution of Example 6.6 Using MATLAB

The complete solution is obtained using MATLAB

In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, Jan 2015
%  Example 6.6 - Statically indeterminate - Section 6.3.5
% 25 equations and 25 unknowns - symbolic setup and solution
% Truss 2D
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(3)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 6.6\n')
fprintf('------------\n')
%% Data
F1 = 500; F2 = 800; F3 = 750;  F4 = 1000; % the applied force
A = [0 0 0]; B = [0,0.6,0];  C = [1.5,1.2,0]; D = [3.0,1.2,0];
```

```
E = [2.25,3.0,0];  G = [4.5,0.6,0];  H = [4.5,0,0];

%% Support reaction calculations
syms Ax Ay Hx Hy real % Unknowns

rAB = B - A;  rAD = D - A;  rAE = E - A; rAH = H - A;
FA = [Ax, Ay, 0];  FB = [F1,0,0];  FE = [F2,-F3,0];  FD = [0,-F4,0];
FH = [Hx, Hy,0];

%% Equilibrium

SumF = FA + FB + FE + FD +FH;  % Sum of Forces
SumMA = cross(rAB,FB)+ cross(rAE,FE)+ cross(rAD,FD) + cross(rAH,FH);
% Sum of moments at A

Eq(1) = SumF(1);
Eq(2) = SumF(2);
Eq(3) = SumMA(3);


%%%%%%%%%%%%%%%%%%%%%%
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('Point A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point C [m]  : '),disp(C)
fprintf('Point D [m]  : '),disp(D)
fprintf('Point E [m]  : '),disp(E)
fprintf('Point G [m]  : '),disp(G)
fprintf('Point H [m]  : '),disp(H)

fprintf('Position vector rAB[m]    = '),disp(rAB)
fprintf('Position vector rAD[m]    = '),disp(rAD)
fprintf('Position vector rAE[m]   = '),disp(rAE)
fprintf('Position vector rAH[m]   = '),disp(rAH)

fprintf('\nF1 [ N]     : '),disp(F1)
fprintf('F2 [ N]     : '),disp(F2)
fprintf('F3 [ N]     : '),disp(F3)
fprintf('F4 [ N]     : '),disp(F4)

fprintf('\nEquilibrium\n')
fprintf('----------------------\n')
fprintf('Eq(1) = '),disp(Eq(1))
fprintf('Eq(2) = '),disp(Eq(2))
fprintf('Eq(3) = '),disp(Eq(3))

% fprintf('\nSumF : \n'),disp(vpa(SumF',4))
% fprintf('\nSumMA : \n'),disp(vpa(SumMA',4))

%% Pin A
fprintf('\nPin A:\n')
fprintf('------------\n')
syms Fab Fac real
rAB = B-A;  eAB = rAB/norm(rAB);
rAC = C-A;  eAC = rAC/norm(rAC);
```

```
SumA = FA + Fab*eAB + Fac*eAC;

Eq(4) = SumA(1);
Eq(5) = SumA(2);
fprintf('Eq(4) = '),disp(Eq(4))
fprintf('Eq(5) = '),disp(Eq(5))

%% Pin B
fprintf('\nPin B:\n')
fprintf('------------\n')
syms Fbc Fbe real
rBC = C - B;  eBC = rBC/norm(rBC);
rBE = E - B;  eBE = rBE/norm(rBE);
SumB = -Fab*eAB + FB + Fbc*eBC + Fbe*eBE;

Eq(6) = SumB(1);
Eq(7) = SumB(2);
fprintf('Eq(6) = '),disp(Eq(6))
fprintf('Eq(7) = '),disp(Eq(7))

%% Pin C
fprintf('\nPin C:\n')
fprintf('------------\n')
syms Fcd Fce real
rCD = D - C;  eCD = rCD/norm(rCD);
rCE = E - C;  eCE = rCE/norm(rCE);
SumC = -Fbc*eBC -Fac*eAC + Fcd*eCD + Fce*eCE;

Eq(8)= SumC(1);
Eq(9)= SumC(2);
fprintf('Eq(8) = '),disp(Eq(8))
fprintf('Eq(9) = '),disp(Eq(9))

%% Pin E
fprintf('\nPin E:\n')
fprintf('------------\n')
syms Fed Feg real
rED = D - E;  eED = rED/norm(rED);
rEG = G - E;  eEG = rEG/norm(rEG);
SumE = -Fbe*eBE -Fce*eCE + Fed*eED + Feg*eEG + FE;

Eq(10)= SumE(1);
Eq(11)= SumE(2);
fprintf('Eq(10) = '),disp(Eq(10))
fprintf('Eq(11) = '),disp(Eq(11))

%% Pin D
fprintf('\nPin D:\n')
fprintf('------------\n')
syms Fdh Fdg real
rDH = H - D;  eDH = rDH/norm(rDH);
rDG = G - D;  eDG = rDG/norm(rDG);
SumD = -Fed*eED -Fcd*eCD + Fdh*eDH + Fdg*eDG + FD;

Eq(12)= SumD(1);
Eq(13)= SumD(2);
```

```
fprintf('Eq(12) = '),disp(Eq(12))
fprintf('Eq(13) = '),disp(Eq(13))

%% Pin H
fprintf('\nPin H:\n')
fprintf('------------\n')
syms Fhg real
rHG = G - H;  eHG = rHG/norm(rHG);

SumH = -Fdh*eDH + Fhg*eHG  + FH;

Eq(14)= SumH(2);
fprintf('Eq(14) = '),disp(Eq(14))

%% Calculating displacements
syms  dBx dBy dCx dCy dDx dDy dEx dEy dGx dGy real % the unknowns

% Properties
Elas = 11.4e06;   Area = 0.008;

% displacement vectors
dA = [0,0,0];  dH = [0,0,0];  dC = [dCx,dCy,0];
dB = [dBx, dBy,0];  dD = [dDx, dDy,0]; dE = [dEx, dEy,0];
dG = [dGx, dGy,0];

% length of members and unit vectors  (Copy and Paste)
rAB = B - A;  LAB = norm(rAB);  eAB = rAB/LAB;
rAC = C - A;  LAC = norm(rAC);  eAC = rAC/LAC;
rBC = C - B;  LBC = norm(rBC);  eBC = rBC/LBC;
rBE = E - B;  LBE = norm(rBE);  eBE = rBE/LBE;
rCD = D - C;  LCD = norm(rCD);  eCD = rCD/LCD;
rCE = E - C;  LCE = norm(rCE);  eCE = rCE/LCE;
rED = D - E;  LED = norm(rED);  eED = rED/LED;
rEG = G - E;  LEG = norm(rEG);  eEG = rEG/LEG;
rDG = G - D;  LDG = norm(rDG);  eDG = rDG/LDG;
rDH = H - D;  LDH = norm(rDH);  eDH = rDH/LDH;
rHG = G - H;  LHG = norm(rHG);  eHG = rHG/LHG;

fprintf('\nDisplacement Equations:\n')
fprintf('-----------------------------\n')
Eq(15) = dot(eAB,dB)-dot(eAB,dA) - (Fab*LAB/(Area*Elas)); % Member AB
fprintf('Eq(15) = '),disp(Eq(15))

Eq(16) = dot(eAC,dC)-dot(eAC,dA) - (Fac*LAC/(Area*Elas)); % Member AC
fprintf('Eq(16) = '),disp(Eq(16))

Eq(17) = dot(eBC,dC)-dot(eBC,dB) - (Fbc*LBC/(Area*Elas)); % Member BC
fprintf('Eq(17) = '),disp(Eq(17))

Eq(18) = dot(eBE,dE)-dot(eBE,dB) - (Fbe*LBE/(Area*Elas)); % Member BE
fprintf('Eq(18) = '),disp(Eq(18))

Eq(19) = dot(eCD,dD)-dot(eCD,dC) - (Fcd*LCD/(Area*Elas)); % Member CD
fprintf('Eq(19) = '),disp(Eq(19))

Eq(20) = dot(eCE,dE)-dot(eCE,dC) - (Fce*LCE/(Area*Elas)); % Member CE
```

```
fprintf('Eq(20) = '),disp(Eq(20))

Eq(21) = dot(eED,dD)-dot(eED,dE) - (Fed*LED/(Area*Elas)); % Member ED
fprintf('Eq(21) = '),disp(Eq(21))

Eq(22) = dot(eEG,dG)-dot(eEG,dE) - (Feg*LEG/(Area*Elas)); % Member EG
fprintf('Eq(22) = '),disp(Eq(22))

Eq(23) = dot(eDG,dG)-dot(eDG,dD) - (Fdg*LDG/(Area*Elas)); % Member DG
fprintf('Eq(23) = '),disp(Eq(23))

Eq(24) = dot(eDH,dH)-dot(eDH,dD) - (Fdh*LDH/(Area*Elas)); % Member DH
fprintf('Eq(24) = '),disp(Eq(24))

Eq(25) = dot(eHG,dG)-dot(eHG,dH) - (Fhg*LHG/(Area*Elas)); % Member HG
fprintf('Eq(25) = '),disp(Eq(25))

%% Solution
sol = solve(Eq);

%% Print solution
fprintf('\nReactions:\n')
fprintf('------------\n')
fprintf('Ax [N]     = '),disp(double(sol.Ax))
fprintf('Ay [N]     = '),disp(double(sol.Ay))
fprintf('Hx [N]     = '),disp(double(sol.Hx))
fprintf('Hy [N]     = '),disp(double(sol.Hy))
%
fprintf('\nMember Forces:\n')
fprintf('----------------\n')
fprintf('Fab [N]     = '),disp(double(sol.Fab))
fprintf('Fac [N]     = '),disp(double(sol.Fac))
fprintf('Fbc [N]     = '),disp(double(sol.Fbc))
fprintf('Fbe [N]     = '),disp(double(sol.Fbe))
fprintf('Fcd [N]     = '),disp(double(sol.Fcd))
fprintf('Fce [N]     = '),disp(double(sol.Fce))
fprintf('Fed [N]     = '),disp(double(sol.Fed))
fprintf('Feg [N]     = '),disp(double(sol.Feg))
fprintf('Fdh [N]     = '),disp(double(sol.Fdh))
fprintf('Fdg [N]     = '),disp(double(sol.Fdg))
fprintf('Fhg [N]     = '),disp(double(sol.Fhg))
%
fprintf('\nJoint Displacements:\n')
fprintf('----------------------\n')
fprintf('dBx [m] : '),disp(double(sol.dBx))
fprintf('dBy [m] : '),disp(double(sol.dBy))
fprintf('dCx [m] : '),disp(double(sol.dCx))
fprintf('dCy [m] : '),disp(double(sol.dCy))
fprintf('dDx [m] : '),disp(double(sol.dDx))
fprintf('dDy [m] : '),disp(double(sol.dDy))
fprintf('dEx [m] : '),disp(double(sol.dEx))
fprintf('dEy [m] : '),disp(double(sol.dEy))
fprintf('dGx [m] : '),disp(double(sol.dGx))
fprintf('dGy [m] : '),disp(double(sol.dGy))
```

The program is long but most of it is copy, paste, and edit. It follows the set up we have above. To make it work it was important that the ***y-equation*** at the pin **H** needs to be chosen for ***Eq(14)*** as it contains the unknown $\mathbf{F_{hg}}$. Selecting the x-equation will not produce a solution.

In the Command Window

```
Example 6.6
------------
Point A [m]  :      0     0     0
Point B [m]  :            0          0.6            0
Point C [m]  :          1.5          1.2            0
Point D [m]  :            3          1.2            0
Point E [m]  :         2.25            3            0
Point G [m]  :          4.5          0.6            0
Point H [m]  :          4.5            0            0
Position vector rAB[m]    =            0          0.6            0
Position vector rAD[m]    =            3          1.2            0
Position vector rAE[m]   =          2.25            3            0
Position vector rAH[m]   =           4.5            0            0

F1 [ N]    :    500
F2 [ N]    :    800
F3 [ N]    :    750
F4 [ N]    :         1000

Equilibrium
----------------------
Eq(1) = Ax + Hx + 1300
Eq(2) = Ay + Hy - 1750
Eq(3) = (9*Hy)/2 - 14775/2

Pin A:
------------
Eq(4) = Ax + (5*41^(1/2)*Fac)/41
Eq(5) = Ay + Fab + (4*41^(1/2)*Fac)/41

Pin B:
------------
Eq(6) = (5*29^(1/2)*Fbc)/29 + (15*481^(1/2)*Fbe)/481 + 500
Eq(7) = (2*29^(1/2)*Fbc)/29 - Fab + (16*481^(1/2)*Fbe)/481

Pin C:
------------
Eq(8) = Fcd + (5*Fce)/13 - (5*41^(1/2)*Fac)/41 - (5*29^(1/2)*Fbc)/29
Eq(9) = (12*Fce)/13 - (4*41^(1/2)*Fac)/41 - (2*29^(1/2)*Fbc)/29

Pin E:
------------
Eq(10) = (5*Fed)/13 - (5*Fce)/13 - (15*481^(1/2)*Fbe)/481 +
(15*481^(1/2)*Feg)/481 + 800
Eq(11) = - (12*Fce)/13 - (12*Fed)/13 - (16*481^(1/2)*Fbe)/481 -
(16*481^(1/2)*Feg)/481 - 750

Pin D:
------------
Eq(12) = (5*29^(1/2)*Fdg)/29 - (5*Fed)/13 - Fcd + (5*41^(1/2)*Fdh)/41
Eq(13) = (12*Fed)/13 - (2*29^(1/2)*Fdg)/29 - (4*41^(1/2)*Fdh)/41 - 1000
```

```
Pin H:
------------
Eq(14) = Fhg + Hy + (4*41^(1/2)*Fdh)/41

Displacement Equations:
-----------------------------
Eq(15) = dBy - Fab/152000
Eq(16) = (5*41^(1/2)*dCx)/41 - (41^(1/2)*Fac)/304000 + (4*41^(1/2)*dCy)/41
Eq(17) = (5*29^(1/2)*dCx)/29 - (5*29^(1/2)*dBx)/29 - (2*29^(1/2)*dBy)/29 -
(29^(1/2)*Fbc)/304000 + (2*29^(1/2)*dCy)/29
Eq(18) = (15*481^(1/2)*dEx)/481 - (15*481^(1/2)*dBx)/481 -
(16*481^(1/2)*dBy)/481 - (481^(1/2)*Fbe)/608000 + (16*481^(1/2)*dEy)/481
Eq(19) = dDx - dCx - Fcd/60800
Eq(20) = (5*dEx)/13 - (5*dCx)/13 - (12*dCy)/13 - (13*Fce)/608000 +
(12*dEy)/13
Eq(21) = (5*dDx)/13 - (13*Fed)/608000 - (12*dDy)/13 - (5*dEx)/13 +
(12*dEy)/13
Eq(22) = (16*481^(1/2)*dEy)/481 - (15*481^(1/2)*dEx)/481 -
(481^(1/2)*Feg)/608000 + (15*481^(1/2)*dGx)/481 - (16*481^(1/2)*dGy)/481
Eq(23) = (2*29^(1/2)*dDy)/29 - (5*29^(1/2)*dDx)/29 - (29^(1/2)*Fdg)/304000
+ (5*29^(1/2)*dGx)/29 - (2*29^(1/2)*dGy)/29
Eq(24) = (4*41^(1/2)*dDy)/41 - (5*41^(1/2)*dDx)/41 - (41^(1/2)*Fdh)/304000
Eq(25) = dGy - Fhg/152000

Reactions:
------------
Ax [N]     =          48.7
Ay [N]     =    108
Hx [N]     =        -1355
Hy [N]     =         1644

Member Forces:
----------------
Fab [N]     =         -69.4
Fac [N]     =         -62.4
Fbc [N]     =   -750
Fbe [N]     =    287
Fcd [N]     =   -612
Fce [N]     =   -344
Fed [N]     =    280
Feg [N]     =        -1233
Fdh [N]     =        -1733
Fdg [N]     =    909
Fhg [N]     =   -563

Joint Displacements:
----------------------
dBx [m] :        0.0125
dBy [m] :     -0.000456
dCx [m] :      -0.00225
dCy [m] :      0.000708
dDx [m] :       -0.0123
dDy [m] :       -0.0736
dEx [m] :        0.0646
dEy [m] :       -0.0351
```

```
dGx [m] :          0.033
dGy [m] :        -0.0037
```

This is impressive. We have an exact solution to the problem with 25 variables.
We must investigate if it makes sense. A simple check on the reactions is a start. It matches except for a round off error.
The largest displacement is about 7.3 cm. Is this a concern?

It will be curious to see the displacement of the truss visually. We can also do that in MATLAB.

Execution in Octave

The code is same as in the MATLAB above - with the following changes. You can also flatten the symbolic display

In Octave Editor

```
clc, clear, format compact, format shortg, close all
warning off
pkg load symbolic;

fprintf('Eq(1)  = \n'),disp(Eq(1))
fprintf('Eq(2)  = \n'),disp(Eq(2))
fprintf('Eq(3)  = \n'),disp(Eq(3))

fprintf('Eq(4)  = \n'),disp(Eq(4))
fprintf('Eq(5)  = \n'),disp(Eq(5))

fprintf('Eq(6)  = \n'),disp(Eq(6))
fprintf('Eq(7)  = \n'),disp(Eq(7))

fprintf('Eq(8)  = \n'),disp(Eq(8))
fprintf('Eq(9)  = \n'),disp(Eq(9))

fprintf('Eq(10) =\n '),disp(Eq(10))
fprintf('Eq(11) = \n'),disp(Eq(11))

fprintf('Eq(12) = \n'),disp(Eq(12))
fprintf('Eq(13) = \n'),disp(Eq(13))

fprintf('Eq(15) = \n'),disp(Eq(15))

fprintf('Eq(16) = \n'),disp(Eq(16))

fprintf('Eq(17) = \n'),disp(Eq(17))

fprintf('Eq(18) = \n'),disp(Eq(18))

fprintf('Eq(19) = \n'),disp(Eq(19))

fprintf('Eq(20) = \n'),disp(Eq(20))

fprintf('Eq(21) = \n'),disp(Eq(21))

fprintf('Eq(22) = \n'),disp(Eq(22))
```

```
fprintf('Eq(23) = \n'),disp(Eq(23))

fprintf('Eq(24) = \n'),disp(Eq(24))

fprintf('Eq(25) = \n'),disp(Eq(25))
```

In Octave Command Window

The output values is the same as in MATLAB

```
Example 6.6
------------
Symbolic pkg v2.7.1: Python communication link active, SymPy v1.3.
Point A [m]  :   0  0  0
Point B [m]  :   0  0.6  0
Point C [m]  :   1.5  1.2  0
Point D [m]  :   3  1.2  0
Point E [m]  :   2.25  3  0
Point G [m]  :   4.5  0.6  0
Point H [m]  :   4.5  0  0
Position vector rAB[m]    =   0  0.6  0
Position vector rAD[m]    =   3  1.2  0
Position vector rAE[m]   =   2.25  3  0
Position vector rAH[m]   =   4.5  0  0

F1 [ N]    : 500
F2 [ N]    : 800
F3 [ N]    : 750
F4 [ N]    : 1000

Equilibrium
----------------------
Eq(1) =
  Ax + Hx + 1300
Eq(2) =
  Ay + Hy - 1750
Eq(3) =
  9*Hy   14775
  ---- - -----
   2       2

Pin A:
------------
Eq(4) =
       431*pi*Fac
  Ax + ----------
          1734
Eq(5) =
             862*pi*Fac
  Ay + Fab + ----------
                4335

Pin B:
------------
Eq(6) =
  701*Fbc   1069*Fbe
  ------- + -------- + 500
```

```
    755       1563
Eq(7) =
         878*pi*Fbc   205*Fbe
  -Fab + ---------- + -------
            7427        281

Pin C:
------------
Eq(8) =
    431*pi*Fac   701*Fbc         5*Fce
  - ---------- - ------- + Fcd + -----
       1734        755             13
Eq(9) =
    862*pi*Fac   878*pi*Fbc   12*Fce
  - ---------- - ---------- + ------
       4335         7427        13

Pin E:
------------
Eq(10) =
     1069*Fbe   5*Fce   5*Fed   1069*Feg
  - -------- - ----- + ----- + -------- + 800
      1563       13      13      1563
Eq(11) =
    205*Fbe   12*Fce   12*Fed   205*Feg
  - ------- - ------ - ------ - ------- - 750
      281       13       13       281

Pin D:
------------
Eq(12) =
         701*Fdg   431*pi*Fdh   5*Fed
  -Fcd + ------- + ---------- - -----
           755        1734        13
Eq(13) =
    878*pi*Fdg   862*pi*Fdh   12*Fed
  - ---------- - ---------- + ------ - 1000
       7427         4335        13

Pin H:
------------
Eq(14) =
  862*pi*Fdh
  ---------- + Fhg + Hy
     4335

Displacement Equations:
-----------------------------
Eq(15) =
     Fab
  - ------ + dBy
    152000
Eq(16) =
     41*Fac   431*pi*dCx   862*pi*dCy
  - ------- + ---------- + ----------
    1946550      1734         4335
```

```
Eq(17) =
    19*pi*Fbc   701*dBx   878*pi*dBy   701*dCx   878*pi*dCy
  - --------- - ------- - ---------- + ------- + ----------
     3369600      755        7427        755        7427
Eq(18) =
    131*pi*Fbe   1069*dBx   205*dBy   1069*dEx   205*dEy
  - ---------- - -------- - ------- + -------- + -------
     11409120      1563       281       1563       281
Eq(19) =
     Fcd
  - ----- - dCx + dDx
    60800
Eq(20) =
    13*Fce   5*dCx   12*dCy   5*dEx   12*dEy
  - ------ - ----- - ------ + ----- + ------
    608000     13      13       13      13
Eq(21) =
    13*Fed   5*dDx   12*dDy   5*dEx   12*dEy
  - ------ + ----- - ------ - ----- + ------
    608000     13      13       13      13
Eq(22) =
    131*pi*Feg   1069*dEx   205*dEy   1069*dGx   205*dGy
  - ---------- - -------- + ------- + -------- - -------
     11409120      1563       281       1563       281
Eq(23) =
    19*pi*Fdg   701*dDx   878*pi*dDy   701*dGx   878*pi*dGy
  - --------- - ------- + ---------- + ------- - ----------
     3369600      755        7427        755        7427
Eq(24) =
     41*Fdh   431*pi*dDx   862*pi*dDy
  - ------- - ---------- + ----------
    1946550      1734         4335
Eq(25) =
     Fhg
  - ------ + dGy
    152000

Reactions:
------------
Ax [N]     = 48.723
Ay [N]     = 108.33
Hx [N]     = -1348.7
Hy [N]     = 1641.7

Member Forces:
----------------
Fab [N]     = -69.355
Fac [N]     = -62.395
Fbc [N]     = -749.58
Fbe [N]     = 286.53
Fcd [N]     = -612.45
Fce [N]     = -343.81
Fed [N]     = 280.19
Feg [N]     = -1234.1
Fdh [N]     = -1727.2
```

```
Fdg [N]      = 909.05
Fhg [N]      = -562.69

Joint Displacements:
----------------------
dBx [m] : 0.012517
dBy [m] : -0.00045628
dCx [m] : -0.0022496
dCy [m] : 0.00070821
dDx [m] : -0.012323
dDy [m] : -0.07364
dEx [m] : 0.064587
dEy [m] : -0.035104
dGx [m] : 0.032996
dGy [m] : -0.0037019
```

The values re the same. The solution did take significantly longer in Octave. Please be patient.

6.3.6 Picturing the Truss Displacement in Example 6.6

We will use MATLAB to draw the original truss and the displaced truss. MATLAB can actually draw a scaled drawing. This will help you understand the solution and develop instinct about truss and loading.
The MATLAB code is appended to the previous code so that the previous results are available.

In the Editor

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Graphical description of Solution
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
set(gcf,'Position',[10,20,600,600],'Color','w')
axis([-0.1,5,-0.1,4])
axis equal

%% draw the original truss members
line([A(1),B(1)],[A(2),B(2)],'LineWidth',2,'Color','k')
line([A(1),C(1)],[A(2),C(2)],'LineWidth',2,'Color','k')
line([B(1),C(1)],[B(2),C(2)],'LineWidth',2,'Color','k')
line([B(1),E(1)],[B(2),E(2)],'LineWidth',2,'Color','k')
line([C(1),D(1)],[C(2),D(2)],'LineWidth',2,'Color','k')
line([C(1),E(1)],[C(2),E(2)],'LineWidth',2,'Color','k')
line([D(1),E(1)],[D(2),E(2)],'LineWidth',2,'Color','k')
line([D(1),G(1)],[D(2),G(2)],'LineWidth',2,'Color','k')
line([D(1),H(1)],[D(2),H(2)],'LineWidth',2,'Color','k')
line([H(1),G(1)],[H(2),G(2)],'LineWidth',2,'Color','k')
line([E(1),G(1)],[E(2),G(2)],'LineWidth',2,'Color','k')

%% set the text
text(A(1)-0.1,A(2),'\bfA','Color','b')
text(B(1)-0.1,B(2),'\bfB','Color','b')
text(C(1),C(2)-0.1,'\bfC','Color','b')
text(D(1)+0.1,D(2)+0.1,'\bfD','Color','b')
text(E(1),E(2)+0.1,'\bfE','Color','b')
text(G(1)+0.1,G(2),'\bfG','Color','b')
```

```
text(H(1)+0.1,H(2),'\bfH','Color','b')

%% Calculate the change in locations in decimals
dBx = double(sol.dBx);
dBy = double(sol.dBy);
dCx = double(sol.dCx);
dCy = double(sol.dCy);
dDx = double(sol.dDx);
dDy = double(sol.dDy);
dEx = double(sol.dEx);
dEy = double(sol.dEy);
dGx = double(sol.dGx);
dGy = double(sol.dGy);

%% vector displacements
dC = subs(dC); dB = subs(dB);  dD = subs(dD); dE = subs(dE);
dG = subs(dG);

%% displacement of pins
AA = A + dA;  BB = B + dB;  CC = C + dC;  DD = D + dD;
EE = E + dE;  GG = G + dG; HH = H + dH;

%% draw lines for displaced truss
line([AA(1),BB(1)],
[AA(2),BB(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([AA(1),CC(1)],
[AA(2),CC(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([BB(1),CC(1)],
[BB(2),CC(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([BB(1),EE(1)],
[BB(2),EE(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([CC(1),DD(1)],
[CC(2),DD(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([CC(1),EE(1)],
[CC(2),EE(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([DD(1),EE(1)],
[DD(2),EE(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([DD(1),GG(1)],
[DD(2),GG(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([DD(1),HH(1)],
[DD(2),HH(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([HH(1),GG(1)],
[HH(2),GG(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([EE(1),GG(1)],
[EE(2),GG(2)],'LineWidth',2,'Color','r','LineStyle','--')

title('\bfOriginal and Displaced Truss - Example 6.6')
```

In the Figure Window

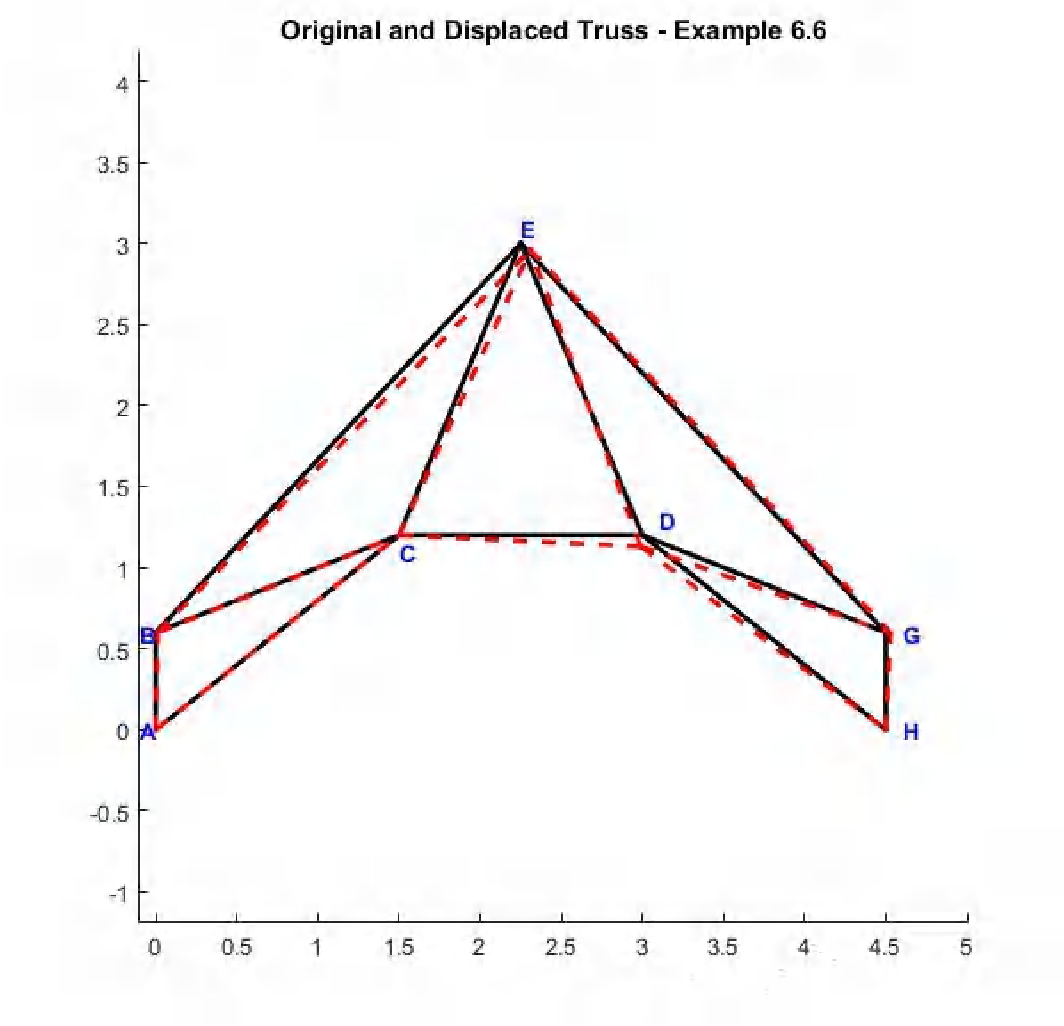


Figure 6.3.7a Original and displaced truss- Example 6.6

You can also add the applied loads/reactions graphically on the truss to make it more informative. This is am impressive application of MATLAB to solve two-dimensional truss problems. It is a complete solution.

The code is about 300 lines long. You will see a majority of the code is for printing information and values to the command window and graphically illustrating the truss. A lot of the code is generated by copying, pasting , and editing. Being methodical, patient, and understanding what each line of code is doing is the best attribute to solve these problems correctly using MATLAB or Octave.

Execution in Octave

The code is the same as in MATLAB. Since the symbolic solution takes a long time the displacement values are input to the program.

In the Octave Editor:

```
% Essential Foundations in Mechanics
% P. Venkataraman, Jan 2015
%  Example 6.6 - Plotting the deflected truss
% Truss 2D
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 6.6\n')
fprintf('------------\n')
%% Data
F1 = 500; F2 = 800; F3 = 750;  F4 = 1000; % the applied force
A = [0 0 0]; B = [0,0.6,0];  C = [1.5,1.2,0]; D = [3.0,1.2,0];
E = [2.25,3.0,0];  G = [4.5,0.6,0];  H = [4.5,0,0];

dBx = 0.012517;
dBy = -0.00045628;
dCx = -0.0022496;
dCy = 0.00070821;
dDx = -0.012323;
dDy = -0.07364;
dEx = 0.064587;
dEy = -0.035104;
dGx = 0.032996;
dGy = -0.0037019;

% displacement vectors
dA = [0,0,0];  dH = [0,0,0];  dC = [dCx,dCy,0];
dB = [dBx, dBy,0];  dD = [dDx, dDy,0]; dE = [dEx, dEy,0];
dG = [dGx, dGy,0];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Graphical description of Solution
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
set(gcf,'Position',[10,20,600,600],'Color','w')
axis([-0.1,5,-0.1,4])
axis equal

%% draw the original truss members
line([A(1),B(1)],[A(2),B(2)],'LineWidth',2,'Color','k')
line([A(1),C(1)],[A(2),C(2)],'LineWidth',2,'Color','k')
line([B(1),C(1)],[B(2),C(2)],'LineWidth',2,'Color','k')
line([B(1),E(1)],[B(2),E(2)],'LineWidth',2,'Color','k')
line([C(1),D(1)],[C(2),D(2)],'LineWidth',2,'Color','k')
line([C(1),E(1)],[C(2),E(2)],'LineWidth',2,'Color','k')
line([D(1),E(1)],[D(2),E(2)],'LineWidth',2,'Color','k')
line([D(1),G(1)],[D(2),G(2)],'LineWidth',2,'Color','k')
line([D(1),H(1)],[D(2),H(2)],'LineWidth',2,'Color','k')
line([H(1),G(1)],[H(2),G(2)],'LineWidth',2,'Color','k')
line([E(1),G(1)],[E(2),G(2)],'LineWidth',2,'Color','k')

%% set the text
text(A(1)-0.1,A(2),'\bfA','Color','b')
text(B(1)-0.1,B(2),'\bfB','Color','b')
text(C(1),C(2)-0.1,'\bfC','Color','b')
text(D(1)+0.1,D(2)+0.1,'\bfD','Color','b')
text(E(1),E(2)+0.1,'\bfE','Color','b')
text(G(1)+0.1,G(2),'\bfG','Color','b')
text(H(1)+0.1,H(2),'\bfH','Color','b')

%% vector displacements
##dC = subs(dC); dB = subs(dB);  dD = subs(dD); dE = subs(dE);
##dG = subs(dG);
```

```
%% displacement of pins
AA = A + dA;  BB = B + dB;  CC = C + dC;  DD = D + dD;
EE = E + dE;  GG = G + dG; HH = H + dH;

%% draw lines for displaced truss
line([AA(1),BB(1)],
[AA(2),BB(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([AA(1),CC(1)],
[AA(2),CC(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([BB(1),CC(1)],
[BB(2),CC(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([BB(1),EE(1)],
[BB(2),EE(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([CC(1),DD(1)],
[CC(2),DD(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([CC(1),EE(1)],
[CC(2),EE(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([DD(1),EE(1)],
[DD(2),EE(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([DD(1),GG(1)],
[DD(2),GG(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([DD(1),HH(1)],
[DD(2),HH(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([HH(1),GG(1)],
[HH(2),GG(2)],'LineWidth',2,'Color','r','LineStyle','--')
line([EE(1),GG(1)],
[EE(2),GG(2)],'LineWidth',2,'Color','r','LineStyle','--')

title('\bfOriginal and Displaced Truss - Example 6.6')
```

In the Octave Figure Window

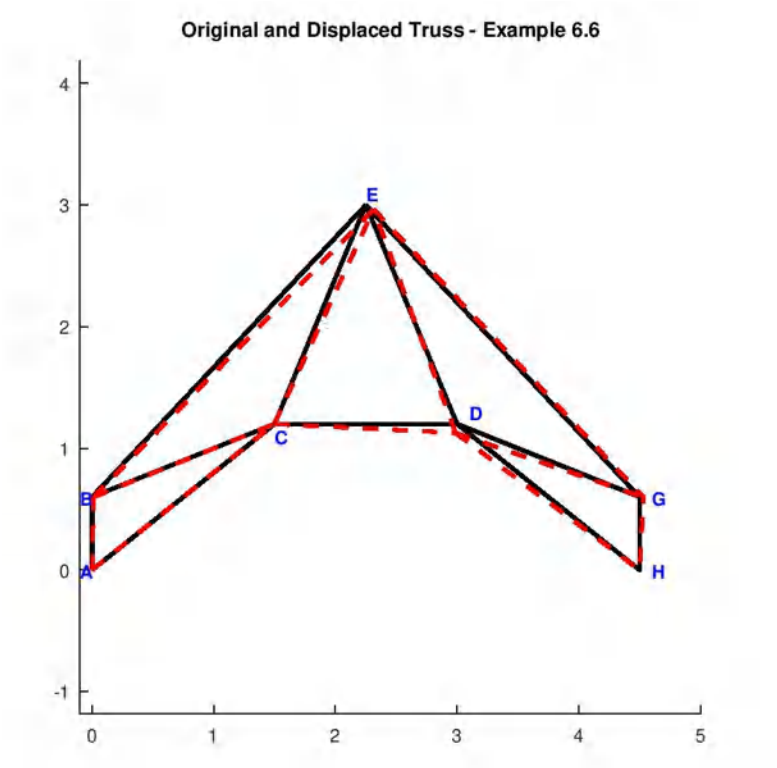


Figure 6.3.7b Original and displaced truss- Example 6.6 - Octave

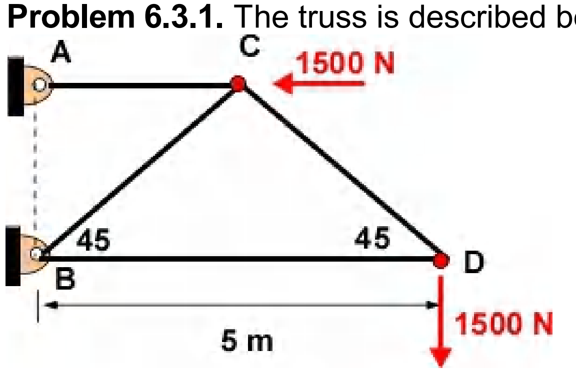
6.3.7 Additional Problems

Set up the following problems by hand on paper and solve them using MATLAB/Octave and confirm some of your calculations using a calculator. For each problem you must draw the FBD and work with a coordinate system. For all problems obtain support reactions, member forces , and displacement of the truss. Use a factor of safety of 6. Choose your material and area of cross-section and ensure you have an acceptable design. Identify zero-force members before you start your solution and confirm it through calculations.

The first 4 problems are similar to problems from Section 6.2. Compare the solutions for insight.

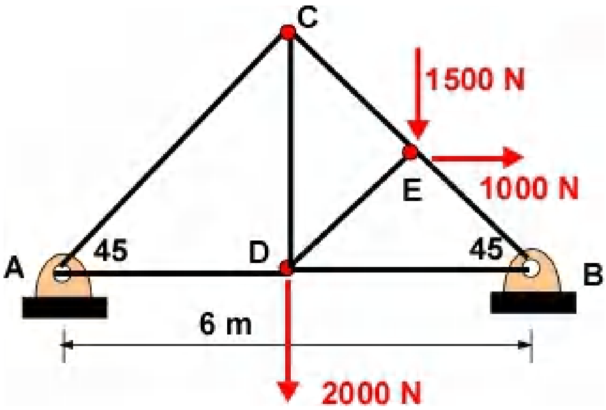
Please use Table 6.1 for your calculations.

Problem 6.3.1. The truss is described below.



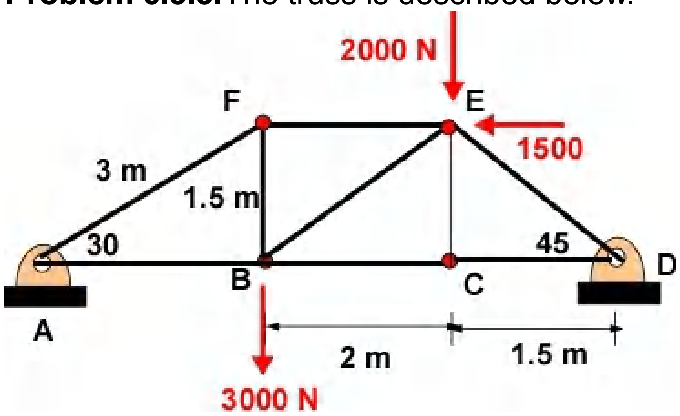
Problem 6.3.1

Problem 6.3.2. The truss is described below.



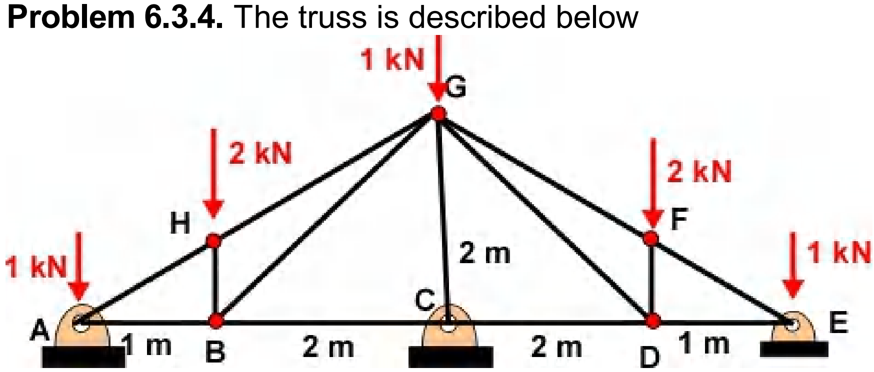
Problem 6.3.2

Problem 6.3.3. The truss is described below.



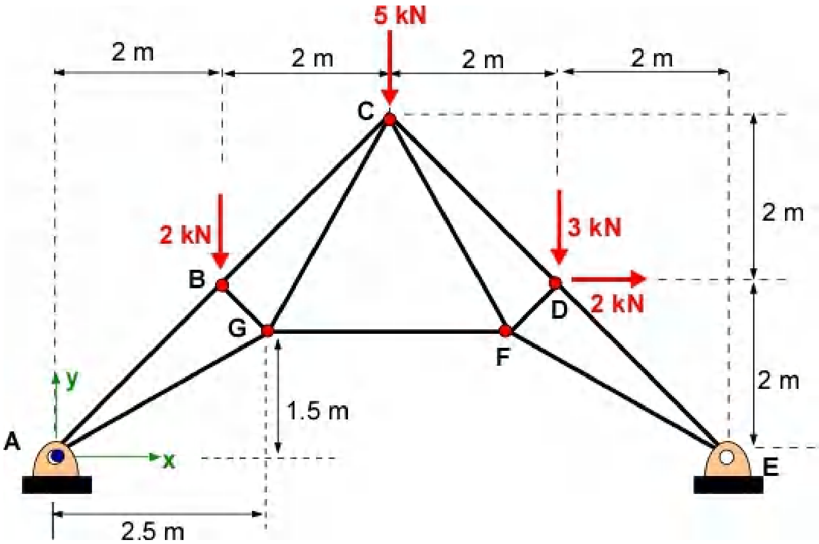
Problem 6.3.3

Problem 6.3.4. The truss is described below



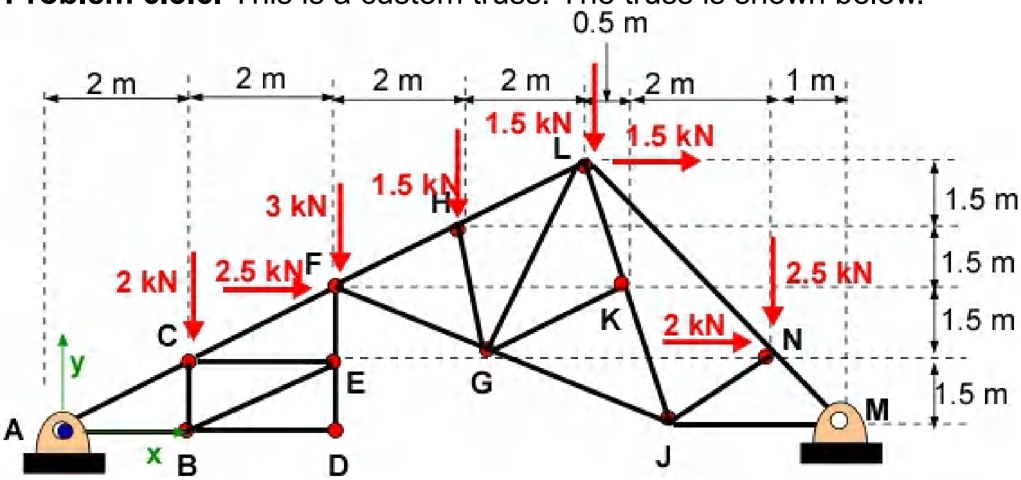
Problem 6.3.4

Problem 6.3.5. This is a geometric symmetric Fink truss. The truss is described below.



Problem 6.3.5

Problem 6.3.6. This is a custom truss. The truss is shown below.



Problem 6.3.6

6.4 SPACE TRUSS

Our first illustration of a simple wooden bridge truss was actually a 3D truss. These are also termed as space truss or space frames. Large roof structures enclosing large sport venues have very visible 3D truss structures both as a structural component and for visual attraction as well. Figure 6.4.1 illustrates some examples of 3D truss.

Figure 6.4.1 Examples of 3D truss
(images from Wikipedia)

Once again our solution to these examples is to help us avoid failure, last forever, and look aesthetic. Coming to this section with awareness of the calculations for a 2D truss it should be evident we will be handling more unknowns, more equations of equilibrium, and more calculations. In practice these problems are usually solved through computer software that includes many engineers working as a team. Our interest in the subsequent development is to look at text book examples where the analysis is modest and the calculations are reasonable. It will provide an exposure to the nature of analysis involved in 3D structures. We are going to approach it as an extension of the calculations in the 2D truss examples of the previous sections.

The following paragraphs were used in defining the 2D truss in Section 6.2. Notice that it applies to 3D truss examples too without any change!

The truss appears to be pieces of rigid body (members) connected together at the ends (pins/joints). The truss supports external loads which are applied at some of the pins and stays in place through supports (support reactions). The members are long and the axis of the members are along the length. The cross-section is ignored for initial calculations. The members can only carry forces along the axis and this can either be tension (positive) and compression (negative).

For the purpose of calculations
The entire truss is in static rigid body equilibrium (FBD of the entire truss). Force and moment equilibrium (at some point O)

$$\sum \bar{F} = 0 \qquad (6.4)$$

$$\sum \bar{M}_O = 0 \qquad (6.5)$$

This is also enforced at each pin which must also being in static equilibrium (FBD of the pin).

$$\sum \bar{F} = 0; \qquad (6.6)$$

This allows us to solve for:

- Support Reactions
- Member forces

The truss is not a rigid body. It is permitted to deflect through the deflection of the pins. However the deflections are small and are elastic. The deflections are along the axis of the members. This allows the force and the deflection of each member to be calculated through:

$$\delta_{AB} = \frac{F_{AB} L_{AB}}{E_{AB} A_{AB}} \qquad (6.7)$$

$$\delta_{AB} = change\ in\ length\ of\ member\ AB$$

$$F_{AB} = force\ in\ member\ AB$$

$$L_{AB} = length\ of\ member\ AB$$

$$E_{AB} = modulus\ of\ Elasticity\ of\ member\ AB$$

$$A_{AB} = cross-section\ of\ member\ AB$$

You should agree that the equations and procedure for solving the truss problems are the same for two and three dimensional. We will only have to incorporate 3D geometry.

6.4.1 Example 6.7

Example 6.7 is a simple 3D truss that may be part of a larger truss system. In Figure 6.4.2 the truss is carrying three loads at the end E while being supported at the pins/joints A, B, C, and D, which in this example are in a plane - vertical plane in the back. Let us examine some choices for the support. Examine the truss again and conclude that we must have support at all of these four locations

locations to avoid excessive deformation of the truss. The solution is broken into steps with discussion so we will dispense with the formal setting up Data, Assumption, and Solution steps.

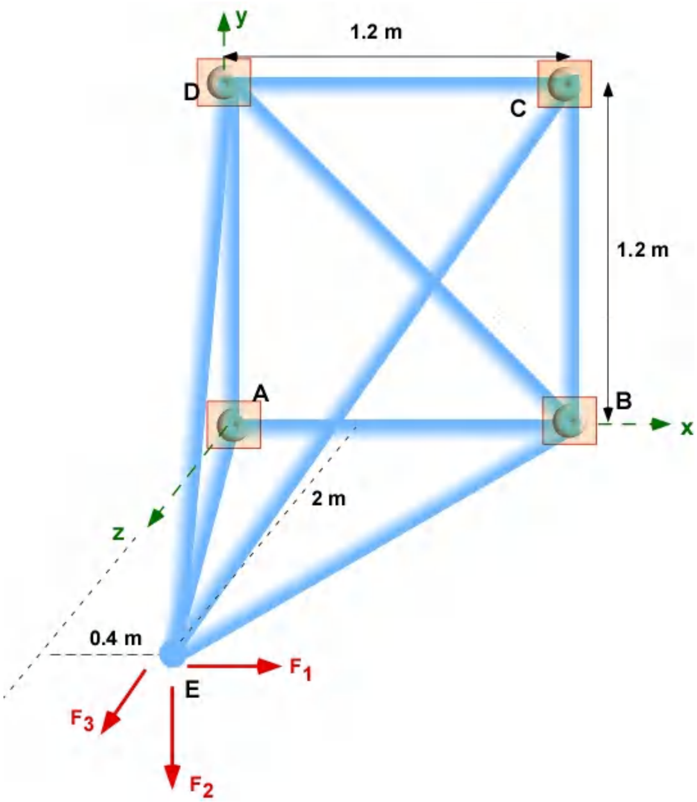


Figure 6.4.2 Example 6.7

This is a 3D problem and we only have **6** equations for the static equilibrium of the entire truss.

- If we **clamp** the truss at all the four points then we have **6** support reactions (three forces and three moments) at each point for a total **24** unknowns making it a statically indeterminate problem of high degree.
- It is possible to have a **ball and socket** joint at each support location with three force unknowns at each ball and socket. This is the same as a **3D pin**. That will lead to **12** unknown support reactions, making the problem difficult to solve for the reactions from statics alone.
- We have to introduce something like a **roller support** found in Example 6.2 for this problem. This is delivered through **short links**, that will prevent displacement in the direction of the links However the pin can be displaced in a plane normal to the link. If we have **1 ball and socket joint** at A, and **1 short link** at B, C, and D then we will have exactly **6** unknown support reactions.

This kind of discussion is unnecessary if we are solving the problem using finite element software where the focus is on solving displacements and therefore indeterminacy is automatically handled. Figure 6.4.3 is the statically determinate version of Example 6.7 that we will address here. Reactions at C and D is in the z- direction only. At B the reaction is in the y direction. There are no moment reactions at A, B, C or D. To complete the problem definition we will assume $\mathbf{F_1}$, $\mathbf{F_2}$, and $\mathbf{F_3}$ to be 500 N, 2500 N, and 200 N respectively.

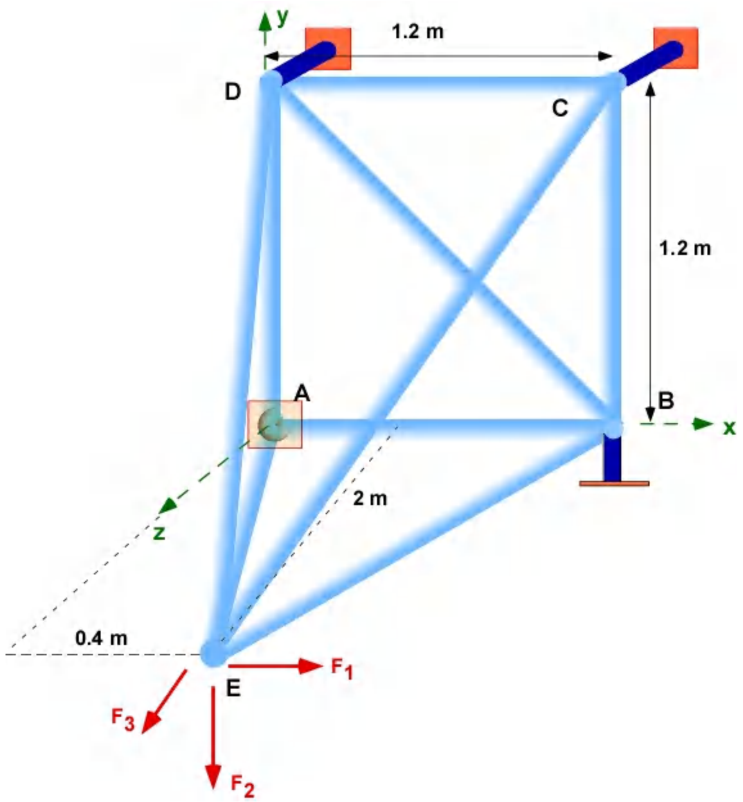


Figure 6.4.3 Statically determinate Example 6.7

There are **five** joints/pins (***n***), which we have identified as A, B, C, D, and E. There are **nine** members (***m***) in the truss, and we refer to them through the pair of letters AD, AB, BC, BD, CD, AE, BE, CE, and DE. There are four supports for the truss. A is a ball and socket joint with three unknown forces. B, C, and D have short links with one force unknown in the directions of the links. The pins at B, C, and D can only displace in the directions of the short links.

The stability of the truss is given by a simple relation for a three dimensional truss:

$K = 3*n - 6$ (6.11)

If K = m, the truss is structurally stable
Example 6.7 is a stable truss.

6.4.2 Support Reactions

We will use the equations of static equilibrium to solve for the support reactions. We start with the FBD of the truss. Only the external loads and the reactions are present on the FBD.

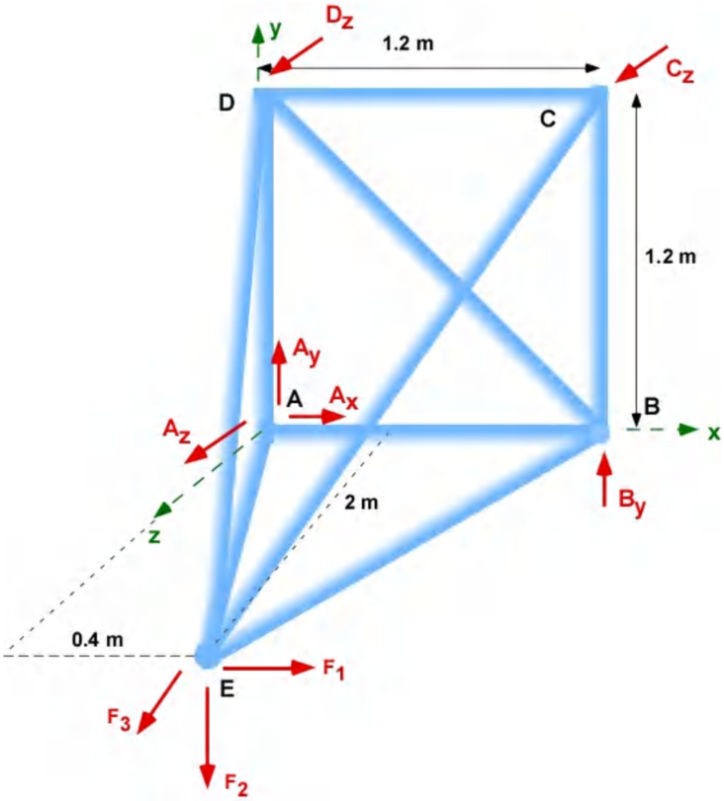


Figure 6.4.4 FBD of Example 6.7

The 6 equations of motion are generated by three force components and three moment component equations. We will take the moment about the origin A. We will represent the force at E by **F =** F_1 i+ F_2 j + F_3 k, and the reactions at the other pins by $\mathbf{F_A}$, $\mathbf{F_B}$, $\mathbf{F_C}$, and $\mathbf{F_D}$ respectively. We should be able to solve for the unknown reactions A_x, A_y, A_z, B_y, C_z, D_z.

In the following we represent the vectors within square paranthesis representing the ***i***, ***j***, and the ***k*** components respectively.

Force Equilibrium

$$\sum \bar{F} = \bar{F} + \bar{F}_A + \bar{F}_B + \bar{F}_C + \bar{F}_D = 0$$

$$[500, -2500, 200] + [A_x, A_y, A_z] + [0, B_y, 0] + [0, 0, C_z] + [0, 0, D_z] = 0$$

$$A_x + 500 = 0;$$

$$A_y + B_y - 2500 = 0$$

$$A_z + C_z + D_z = 0$$

Moment equilibrium about pin A

$$\sum \bar{M}_A = \left(\bar{r}_{AE} \times \bar{F}\right) + \left(\bar{r}_{AB} \times \bar{F}_B\right) + \left(\bar{r}_{AC} \times \bar{F}_C\right) + \left(\bar{r}_{AD} \times \bar{F}_D\right)$$

$$[0.4, 0, 2] \times [500, -2500, 200] + [1.2, 0, 0] \times \left[0, B_y, 0\right] +$$

$$\left[1.2, 1.2, 0\right] \times \left[0, 0, C_z\right] + \left[0, 1.2, 0\right] \times \left[0, 0, D_z\right]$$

$$1.2C_z + 1.2D_z + 5000 = 0$$

$$920 - 1.2C_z = 0$$

$$1.2B_y - 1000 = 0$$

The reactions

$$A_x = -500[N]$$

$$A_y = 1677[N]$$

$$A_z = 3977[N]$$

$$B_y = 833[N]$$

$$C_z = 767[N]$$

$$D_z = -4933[N]$$

We will implement these calculations in MATLAB/Octave. Like in previous sections we will separate the code into segments that carry out particular calculations.

Solution Using MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, Jan 2015
%  Example 6.7 - 3D Truss - Section 6.4.2
%
% Truss 3D
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(3)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 6.7\n')
fprintf('------------\n')
%% Data
F1 = 500; F2 = 2500; F3 = 200; % the applied force
A = [0 0 0]; B = [1.2,0,0];  C = [1.2,1.2,0]; D = [0,1.2,0];
E = [0.4,0,2];

%% Support reaction calculations
syms Ax Ay Az By Cz Dz real % Unknowns

FA = [Ax, Ay, Az];  FB = [0,By,0];  FC = [0,0,Cz];
FD = [0,0,Dz];  F = [F1,-F2,F3];
```

```
rAE = E - A;  rAB = B - A;  rAC = C - A; rAD = D - A;

%% Equilibrium
SumF = FA + FB + FC + FD +F; % Sum of Forces
SumMA = cross(rAE,F)+ cross(rAB,FB)+ cross(rAC,FC) + cross(rAD,FD);
  % Sum of moments at A

sol = solve([SumF,SumMA]);
Ax = double(sol.Ax);
Ay = double(sol.Ay);
Az = double(sol.Az);
By = double(sol.By);
Cz = double(sol.Cz);
Dz = double(sol.Dz);

%%%%%%%%%%%%%%%%%%%%%%
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('Point A [m]   : '),disp(A)
fprintf('Point B [m]   : '),disp(B)
fprintf('Point C [m]   : '),disp(C)
fprintf('Point D [m]   : '),disp(D)
fprintf('Point E [m]   : '),disp(E)

fprintf('\nPosition vector rAE[m]    = '),disp(rAE)
fprintf('Position vector rAB[m]     = '),disp(rAB)
fprintf('Position vector rAC[m]    = '),disp(rAC)
fprintf('Position vector rAD[m]    = '),disp(rAD)

fprintf('\nF1 [ N]     : '),disp(F1)
fprintf('F2 [ N]     : '),disp(F2)
fprintf('F3 [ N]     : '),disp(F3)


fprintf('\nEquilibrium\n')
fprintf('----------------------\n')
% fprintf('Eq(1) = '),disp(Eq(1))
% fprintf('Eq(2) = '),disp(Eq(2))
% fprintf('Eq(3) = '),disp(Eq(3))

fprintf('\nSumF : \n'),disp(vpa(SumF',4))
fprintf('\nSumMA : \n'),disp(vpa(SumMA',4))

fprintf('\nReactions:\n')
fprintf('------------\n')
fprintf('Ax [N]      = '),disp(Ax)
fprintf('Ay [N]      = '),disp(Ay)
fprintf('Az [N]      = '),disp(Az)
fprintf('By [N]      = '),disp(By)
fprintf('Cz [N]      = '),disp(Cz)
fprintf('Dz [N]      = '),disp(Dz)

%% substitute for the unknowns
FA = subs(FA);  FB = subs(FB);  FC = subs(FC);  FD = subs(FD);
```

In the Command Window

```
Example 6.7
------------
Point A [m]  :      0     0     0
Point B [m]  :          1.2          0          0
Point C [m]  :          1.2        1.2          0
Point D [m]  :            0        1.2          0
Point E [m]  :          0.4          0          2

Position vector rAE[m]   =         0.4          0            2
Position vector rAB[m]    =         1.2          0            0
Position vector rAC[m]   =         1.2        1.2            0
Position vector rAD[m]   =           0        1.2            0

F1 [ N]    :    500
F2 [ N]    :         2500
F3 [ N]    :    200

Equilibrium
----------------------

SumF :
           Ax + 500.0
     Ay + By - 2500.0
 Az + Cz + Dz + 200.0

SumMA :
 1.2*Cz + 1.2*Dz + 5000.0
           920.0 - 1.2*Cz
          1.2*By - 1000.0

Reactions:
------------
Ax [N]     =   -500
Ay [N]     =         1677
Az [N]     =         3977
By [N]     =    833
Cz [N]     =    767
Dz [N]     =        -4933
```

Execution in Octave

The code is same as in the MATLAB above - with the following changes

In Octave Editor

```
clc, clear, format compact, format shortg, close all
warning off
pkg load symbolic;

fprintf('\nSumF : '),disp(vpa(SumF,4))
fprintf('\nSumMA : '),disp(vpa(SumMA,4))

% FA = subs(FA);
FA = subs(FA,Ax);
FA = subs(FA,Ay);
```

```
FA = subs(FA,Az);
% FB = subs(FB);
FB = subs(FB,By);
% FC = subs(FC);
FC = subs(FC,Cz);
% FD = subs(FD);
FD = subs(FD,Dz);
```

In Octave Command Window
The output values is the same as in MATLAB

```
Example 6.7
------------
Point A [m]  :   0  0  0
Point B [m]  :   1.2  0  0
Point C [m]  :   1.2  1.2  0
Point D [m]  :   0  1.2  0
Point E [m]  :   0.4  0  2

Position vector rAE[m]   =   0.4  0  2
Position vector rAB[m]    =   1.2  0  0
Position vector rAC[m]   =   1.2  1.2  0
Position vector rAD[m]   =   0  1.2  0

F1 [ N]    : 500
F2 [ N]    : 2500
F3 [ N]    : 200

Equilibrium
----------------------

SumF :   [Ax + 500.0  Ay + By - 2500.0  Az + Cz + Dz + 200.0]

SumMA :   [1.2*Cz + 1.2*Dz + 5000.0  -1.2*Cz + 920.0  1.2*By - 1000.0]

Reactions:
------------
Ax [N]     = -500
Ay [N]     = 1666.7
Az [N]     = 3966.7
By [N]     = 833.33
Cz [N]     = 766.67
Dz [N]     = -4933.3
```

6.4.3 Member Forces - Method of Joints

To solve for the member forces we enforce the equilibrium of the pins/joints. The joints are only in force equilibrium and hence at any joint you can have a maximum of three unknowns. We start with a joint with no more than three unknowns and proceed to the next joint (with three or less unknowns) until all the member forces have been obtained, or all joints have been examined. Figure 6.4.5 is the FBD of the joints in order of examination in this section. Some of the member forces are three dimensional.

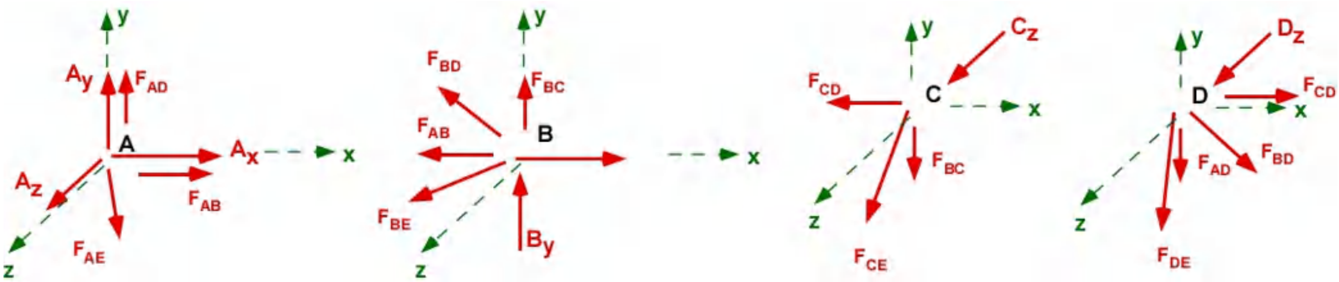


Figure 6.4.5 FBD of the pins/joints of Example 6.7

The equilibrium equations at the pins:

Pin A:

$$\bar{F}_A + \bar{F}_{AB} + \bar{F}_{AD} + \bar{F}_{AE} = 0$$

$$[-500, 1677, 3977] + F_{ab}[1, 0, 0] + F_{ad}[0, 1, 0] + F_{ae}[0.196, 0, 0.98] = 0$$

$$F_{ab} + 0.196F_{ae} - 500 = 0$$

$$F_{ad} + 1677 = 0$$

$$0.98F_{ae} + 3977 = 0$$

$$F_{ab} = 1300[N]; \quad F_{ad} = -1688[N]; \quad F_{ae} = -4066[N]$$

Pin B:

$$\bar{F}_B + \bar{F}_{BA} + \bar{F}_{BC} + \bar{F}_{BD} + \bar{F}_{BE} = 0$$

$$[0, 833, 0] + 1300[-1, 0, 0] + F_{bc}[0, 1, 0] + F_{bd}[-0.707, 0.707, 0] + F_{be}[-0.37, 0, 0.93] = 0$$

$$-0.707F_{bd} - 0.37F_{be} - 1300 = 0$$

$$F_{bc} + 0.707F_{bd} + 833 = 0$$

$$0.93F_{be} = 0$$

$$F_{be} = 0[N]; \quad F_{bd} = -1844[N]; \quad F_{bc} = 467[N]$$

Pin C:

$$\bar{F}_C + \bar{F}_{CB} + \bar{F}_{CD} + \bar{F}_{CE} = 0$$

$$[0, 0, 767] + 467[0, -1, 0] + F_{cd}[-1, 0, 0] + F_{ce}[-0.32, -0.49, 0.81] = 0$$

$$-F_{cd} - 0.32F_{ce} = 0$$

$$-0.49F_{ce} - 467 = 0$$

$$0.81F_{ce} + 767 = 0$$

$$F_{cd} = 311[N]; \quad F_{ce} = -960[N]$$

Pin D:

$$\bar{F}_D + \bar{F}_{DA} + \bar{F}_{DB} + \bar{F}_{DC} + \bar{F}_{DE} = 0$$

$$[0, 0, -4933] + (-1688)[0, -1, 0] + (-1844)[0.707, -0.707, 0] +$$
$$(311)[1, 0, 0] + F_{de}[0.17, -0.51, 0.84] = 0$$

$$0.17F_{de} - 993 = 0$$

$$2992 - 0.51F_{de} = 0$$

$$0.84F_{de} - 4933 = 0$$

$$F_{de} = 5844[N]$$

The following MATLAB code is used to obtain member forces. It is appended to previous code.

Solution Using MATLAB
In the Editor

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Member forces - Method of Joints
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Joint A
fprintf('-------------\n')
fprintf('**************\n')
fprintf('-------------\n')
fprintf('Pin A:\n')
fprintf('-----------------\n')
syms Fad Fae Fab  real % Unknown member forces
rAD = D - A;   eAD = rAD/norm(rAD);
rAB = B - A;   eAB = rAB/norm(rAB);
rAE = E - A;   eAE = rAE/norm(rAE);

fprintf('Unit vector eAD[m]    = '),disp(eAD)
fprintf('Unit vector eAB[m]    = '),disp(eAB)
fprintf('Unit vector eAE[m]    = '),disp(eAE)
% equilibrium of pin
sumA = FA + Fab*eAB + Fae*eAE + Fad*eAD;
solA = solve(sumA);
fprintf('\nSum of Forces at A : \n'),disp(vpa(sumA',4))

Fab = double(solA.Fab);
Fad = double(solA.Fad);
Fae = double(solA.Fae);


fprintf('-------------\n')
fprintf('Member Forces:\n')
fprintf('-----------------\n')
fprintf('Fab [N]      = '),disp(Fab)
fprintf('Fad [N]      = '),disp(Fad)
fprintf('Fae [N]      = '),disp(Fae)

%% Joint B
fprintf('-------------\n')
```

```
fprintf('*************\n')
fprintf('------------\n')
fprintf('Pin B:\n')
fprintf('----------------\n')
syms Fbc Fbd Fbe real % unknown member forces
rBC = C - B;  eBC = rBC/norm(rBC);
rBD = D - B;  eBD = rBD/norm(rBD);
rBE = E - B;  eBE = rBE/norm(rBE);
rBA = A - B;  eBA = rBA/norm(rBA);

fprintf('Unit vector eBC[m]   = '),disp(eBC)
fprintf('Unit vector eBD[m]   = '),disp(eBD)
fprintf('Unit vector eBE[m]   = '),disp(eBE)
fprintf('Unit vector eBA[m]   = '),disp(eBA)

% equilibrium of pin
sumB = FB + Fab*eBA + Fbc*eBC + Fbd*eBD + Fbe*eBE;
solB = solve(sumB);
fprintf('\nSum of Forces at B : \n'),disp(vpa(sumB',5))

Fbc = double(solB.Fbc);
Fbd = double(solB.Fbd);
Fbe = double(solB.Fbe);


fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fbc [N]      = '),disp(Fbc)
fprintf('Fbd [N]      = '),disp(Fbd)
fprintf('Fbe [N]      = '),disp(Fbe)

%% Joint C
fprintf('------------\n')
fprintf('*************\n')
fprintf('------------\n')
fprintf('Pin C:\n')
fprintf('----------------\n')

syms Fcd Fce real  % only two unknowns
rCD = D - C;  eCD = rCD/norm(rCD);
rCE = E - C;  eCE = rCE/norm(rCE);
rCB = B - C;  eCB = rCB/norm(rCB);

fprintf('Unit vector eCD[m]   = '),disp(eCD)
fprintf('Unit vector eCE[m]   = '),disp(eCE)
fprintf('Unit vector eCB[m]   = '),disp(eCB)

% equilibrium of pin
sumC = FC + Fbc*eCB + Fcd*eCD + Fce*eCE;
solC = solve(sumC(1),sumC(2));
fprintf('\nSum of Forces at C : \n'),disp(vpa(sumC',5))

Fcd = double(solC.Fcd);
Fce = double(solC.Fce);
```

```
fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fcd [N]      = '),disp(Fcd)
fprintf('Fce [N]      = '),disp(Fce)

%% Joint D
fprintf('------------\n')
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin D:\n')
fprintf('----------------\n')

syms Fde real  % only one unknown
rDE = E - D;  eDE = rDE/norm(rDE);
rDA = A - D;  eDA = rDA/norm(rDA);
rDB = B - D;  eDB = rDB/norm(rDB);
rDC = C - D;  eDC = rDC/norm(rDC);

fprintf('Unit vector eDE[m]   = '),disp(eDE)
fprintf('Unit vector eDA[m]   = '),disp(eDA)
fprintf('Unit vector eDB[m]   = '),disp(eDB)
fprintf('Unit vector eDC[m]   = '),disp(eDC)

% equilibrium of pin
sumD = FD + Fde*eDE + Fad*eDA + Fbd*eDB + Fcd*eDC;
fprintf('\nSum of Forces at D : \n'),disp(vpa(sumD',5))
fde = solve(sumD(3));
Fde = double(fde);

fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fde [N]      = '),disp(Fde)
```

In the Command Window

```
------------
**************
------------
Pin A:
----------------
Unit vector eAD[m]   =      0     1     0
Unit vector eAB[m]   =      1     0     0
Unit vector eAE[m]   =       0.19612            0       0.98058

Sum of Forces at A :
 Fab + 0.1961*Fae - 500.0
             Fad + 1677.0
      0.9806*Fae + 3977.0
------------
Member Forces:
----------------
Fab [N]     =         1300
Fad [N]     =        -1688
Fae [N]     =        -4066
```

```
------------
**************
------------
Pin B:
----------------
Unit vector eBC[m]   =      0     1     0
Unit vector eBD[m]   =      -0.70711      0.70711            0
Unit vector eBE[m]   =      -0.37139            0      0.92848
Unit vector eBA[m]   =     -1     0     0

Sum of Forces at B :
 - 0.70711*Fbd - 0.37139*Fbe - 1300.0
            Fbc + 0.70711*Fbd + 833.0
                          0.92848*Fbe
------------
Member Forces:
----------------
Fbc [N]     =    467
Fbd [N]     =        -1844
Fbe [N]     =      0
------------
**************
------------
Pin C:
----------------
Unit vector eCD[m]   =     -1     0     0
Unit vector eCE[m]   =      -0.32444     -0.48666      0.81111
Unit vector eCB[m]   =      0    -1     0

Sum of Forces at C :
 - 1.0*Fcd - 0.32444*Fce
   - 0.48666*Fce - 467.0
     0.81111*Fce + 767.0
------------
Member Forces:
----------------
Fcd [N]     =    311
Fce [N]     =   -960
------------
**************
------------
Pin D:
----------------
Unit vector eDE[m]   =       0.16903     -0.50709      0.84515
Unit vector eDA[m]   =      0    -1     0
Unit vector eDB[m]   =       0.70711     -0.70711            0
Unit vector eDC[m]   =      1     0     0

Sum of Forces at D :
  0.16903*Fde - 992.9
 2991.9 - 0.50709*Fde
 0.84515*Fde - 4933.0
------------
Member Forces:
----------------
Fde [N]     =         5844
```

Execution in Octave
The code is same as in the MATLAB above - with the following changes

In Octave Editor
```
fprintf('\nSum of Forces at A : \n'),disp(vpa(sumA,5))

fprintf('\nSum of Forces at B : \n'),disp(vpa(sumB,5))

fprintf('\nSum of Forces at C : \n'),disp(vpa(sumC,5))

fprintf('\nSum of Forces at D : \n'),disp(vpa(sumD,5))
```

In Octave Command Window
The output values are similar to MATLAB. There appears to be a perceptible small difference. I cannot discern at this time if it is due to round off.

```
------------
**************
------------
Pin A:
----------------
Unit vector eAD[m]    =   0  1  0
Unit vector eAB[m]    =   1  0  0
Unit vector eAE[m]    =   0.19612  0  0.98058

Sum of Forces at A :
  [Fab + 0.19612*Fae - 500.0  Fad + 1666.7  0.98058*Fae + 3966.7]
------------
Member Forces:
----------------
Fab [N]     = 1293.3
Fad [N]     = -1666.7
Fae [N]     = -4045.2
------------
**************
------------
Pin B:
----------------
Unit vector eBC[m]    =   0  1  0
Unit vector eBD[m]    =   -0.70711  0.70711  0
Unit vector eBE[m]    =   -0.37139  0  0.92848
Unit vector eBA[m]    =   -1  0  0

Sum of Forces at B :
  [-0.70711*Fbd - 0.37139*Fbe - 1293.3  Fbc + 0.70711*Fbd + 833.33
0.92848*Fbe]
------------
Member Forces:
----------------
Fbc [N]     = 460
Fbd [N]     = -1829
Fbe [N]     = 0
------------
**************
```

```
------------
Pin C:
----------------
Unit vector eCD[m]    =    -1  0  0
Unit vector eCE[m]    =    -0.32444  -0.48666  0.81111
Unit vector eCB[m]    =    0  -1  0

Sum of Forces at C :
  [-Fcd - 0.32444*Fce  -0.48666*Fce - 460.0  0.81111*Fce + 766.67]
------------
Member Forces:
----------------
Fcd [N]     = 306.67
Fce [N]     = -945.21
------------
**************
------------
Pin D:
----------------
Unit vector eDE[m]    =    0.16903  -0.50709  0.84515
Unit vector eDA[m]    =    0  -1  0
Unit vector eDB[m]    =    0.70711  -0.70711  0
Unit vector eDC[m]    =    1  0  0

Sum of Forces at D :
  [0.16903*Fde - 986.67  -0.50709*Fde + 2960.0  0.84515*Fde - 4933.3]
------------
Member Forces:
----------------
Fde [N]     = 5837.2
```

6.4.4 Joint Displacements

We have to start with the unknown displacements. Joint A will not move. Joint E is free to displace in space. Joints B, C, and D have two unknowns each. We show these on Figure 6.4.6.

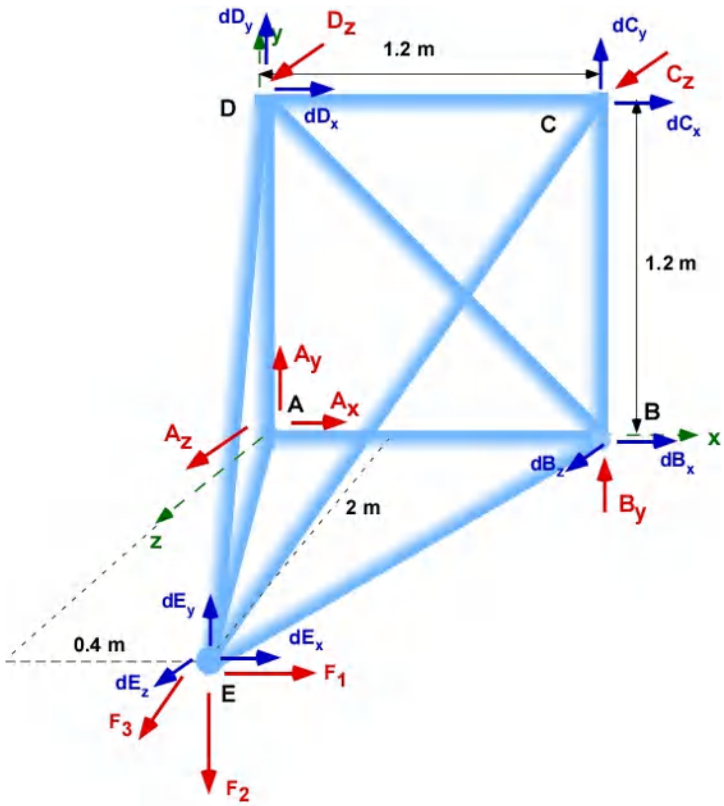


Figure 6.4.6 Unknown displacements.

There are 9 unknowns and it is not a coincidence that there are 9 members. These unknowns are solved by setting up the force displacement relations for each member. Like the two dimensional truss this will require the stress strain relations using Hooke's law as in Eqn (6.7). Once again we will develop the relations for one member CE and then directly extend the analysis to the remaining members. Figure 6.4.7 forms the basis for the development. Unlike 2D truss it is probably more convenient to work with direction cosines in generating the angle information to resolve the displacements along the member. It is also convenient to use the the dot product to resolve vector displacements along member axis. We are considering both pins C and D have general displacement. Specific constraints are directly incorporated into the displacement vector definition.

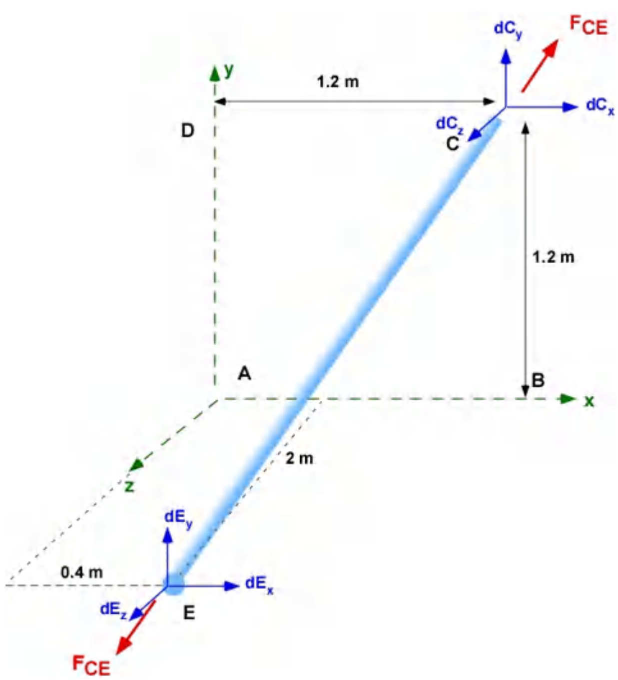


Figure 6.4.7 Member CE

Figure 6.4.7 Member CE

$$\delta_{CE} = \left(\delta_C\right)_{along\,CE} - \left(\delta_E\right)_{along\,CE}$$

$$d\bar{C} = \left[dC_x ; dC_y ; dC_z\right]; \quad d\bar{E} = \left[dE_x ; dE_y ; dE_z\right];$$

$$\delta_{CE} = \hat{e}_{CE} \cdot d\bar{C} - \hat{e}_{CE} \cdot d\bar{E} = \frac{F_{CE} L_{CE}}{E_{CE} A_{CE}}$$

An important observation is the similarity of the equations with the 2D truss example. That is because we have been working with 3D vectors all the time. We have covered the detailed implementation of Eqn. (6.7) in Examples 6.2 and 6.3. They are quite time consuming and long. We will implement and solve for the displacements using MATLAB. Once again, for convenience, we will assume that the material and cross-sectional area is the same for all members. The modulus of elasticity is 200 GPa and the area is 5×10^{-5} [m^2]. The displacement equations for each member is set up as:

$$\delta_{AB} = \hat{e}_{AB} \cdot d\bar{B} - \hat{e}_{AB} \cdot d\bar{A} = \frac{F_{AB} L_{AB}}{EA} \qquad (1)$$

$$\delta_{AD} = \hat{e}_{AD} \cdot d\bar{D} - \hat{e}_{AD} \cdot d\bar{A} = \frac{F_{AD} L_{AD}}{EA} \qquad (2)$$

$$\delta_{AB} = \hat{e}_{AE} \cdot d\bar{E} - \hat{e}_{AE} \cdot d\bar{A} = \frac{F_{AE} L_{AE}}{EA} \qquad (3)$$

$$\delta_{BC} = \hat{e}_{BC} \cdot d\bar{C} - \hat{e}_{BC} \cdot d\bar{B} = \frac{F_{BC} L_{BC}}{EA} \qquad (4)$$

$$\delta_{BD} = \hat{e}_{BD} \cdot d\bar{D} - \hat{e}_{BD} \cdot d\bar{B} = \frac{F_{BD} L_{BD}}{EA} \qquad (5)$$

$$\delta_{BE} = \hat{e}_{BE} \cdot d\bar{E} - \hat{e}_{BE} \cdot d\bar{B} = \frac{F_{BE} L_{BE}}{EA} \qquad (6)$$

$$\delta_{CD} = \hat{e}_{CD} \cdot d\bar{D} - \hat{e}_{CD} \cdot d\bar{C} = \frac{F_{CD} L_{CD}}{EA} \qquad (7)$$

$$\delta_{CE} = \hat{e}_{CE} \cdot d\bar{E} - \hat{e}_{CE} \cdot d\bar{C} = \frac{F_{CE} L_{CE}}{EA} \qquad (8)$$

$$\delta_{DE} = \hat{e}_{DE} \cdot d\bar{E} - \hat{e}_{DE} \cdot d\bar{D} = \frac{F_{DE} L_{DE}}{EA} \qquad (9)$$

The nine equations and nine unknowns are solved using MATLAB code below: These equations are also printed to the command window so that you can check on some of them. This code is appended after the code for the earlier sections.

Solution Using MATLAB

In the Editor:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Calculating displacements
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
syms dBx dBz dCx dCy dDx dDy dEx dEy dEz real  % the unknowns

%% Properties
Elas = 200e09;   Area = 50e-06;
fprintf('\n---------------\n')
fprintf('Properties\n')
fprintf('---------------\n')
fprintf('Modulus of Elasticity [Pa]: '),disp(Elas)
fprintf('Crosssectional Area   [m2]: '),disp(Area)
%% displacement vectors
dA = [0,0,0];  dB = [dBx,0,dBz]; dC = [dCx,dCy,0];
dD = [dDx,dDy,0];  dE = [dEx,dEy,dEz];

% length of members and unit vectors  (Copy and Paste)
rAB = B - A;  LAB = norm(rAB);  eAB = rAB/LAB;
rAD = D - A;  LAD = norm(rAD);  eAD = rAD/LAD;
rAE = E - A;  LAE = norm(rAE);  eAE = rAE/LAE;
rBC = C - B;  LBC = norm(rBC);  eBC = rBC/LBC;
rBD = D - B;  LBD = norm(rBD);  eBD = rBD/LBD;
rBE = E - B;  LBE = norm(rBE);  eBE = rBE/LBE;
rCD = D - C;  LCD = norm(rCD);  eCD = rCD/LCD;
rCE = E - C;  LCE = norm(rCE);  eCE = rCE/LCE;
rDE = E - D;  LDE = norm(rDE);  eDE = rDE/LDE;

%% set up the displacement equations   (Copy and Paste)
% Member Forces  Fad Fae Fab Fbc Fbd Fbe Fcd Fce Fde

fprintf('\nDisplacement Equations:\n')
fprintf('-----------------------------\n')
Eq(1) = dot(eAB,dB)-dot(eAB,dA) - (Fab*LAB/(Area*Elas)); % Member AB
fprintf('Eq(1) = '),disp(vpa(Eq(1),5))

Eq(2) = dot(eAD,dD)-dot(eAD,dA) - (Fad*LAD/(Area*Elas)); % Member AD
fprintf('Eq(2) = '),disp(vpa(Eq(2),5))

Eq(3) = dot(eAE,dE)-dot(eAE,dA) - (Fae*LAE/(Area*Elas)); % Member AE
fprintf('Eq(3) = '),disp(vpa(Eq(3),5))

Eq(4) = dot(eBC,dC)-dot(eBC,dB) - (Fbc*LBC/(Area*Elas)); % Member BC
fprintf('Eq(4) = '),disp(vpa(Eq(4),5))

Eq(5) = dot(eBD,dD)-dot(eBD,dB) - (Fbd*LBD/(Area*Elas)); % Member BD
fprintf('Eq(5) = '),disp(vpa(Eq(5),5))

Eq(6) = dot(eBE,dE)-dot(eBE,dB) - (Fbe*LBE/(Area*Elas)); % Member BE
fprintf('Eq(6) = '),disp(vpa(Eq(6),5))

Eq(7) = dot(eCD,dD)-dot(eCD,dC) - (Fcd*LCD/(Area*Elas)); % Member CD
fprintf('Eq(7) = '),disp(vpa(Eq(7),5))

Eq(8) = dot(eCE,dE)-dot(eCE,dC) - (Fce*LCE/(Area*Elas)); % Member CE
```

```
fprintf('Eq(8)  =  '),disp(vpa(Eq(8),5))

Eq(9) = dot(eDE,dE)-dot(eDE,dD) - (Fde*LDE/(Area*Elas)); % Member DE
fprintf('Eq(9)  =  '),disp(vpa(Eq(9),5))


%% Solution
soldis = solve(Eq);

fprintf('\nJoint Displacements:\n')
fprintf('----------------------\n')

fprintf('dBx [m] : '),disp(double(soldis.dBx))
fprintf('dBz [m] : '),disp(double(soldis.dBz))
fprintf('dCx [m] : '),disp(double(soldis.dCx))
fprintf('dCy [m] : '),disp(double(soldis.dCy))
fprintf('dDx [m] : '),disp(double(soldis.dDx))
fprintf('dDy [m] : '),disp(double(soldis.dDy))
fprintf('dEx [m] : '),disp(double(soldis.dEx))
fprintf('dEy [m] : '),disp(double(soldis.dEy))
fprintf('dEz [m] : '),disp(double(soldis.dEz))
```

In the Command Window

```
--------------
Properties
--------------
Modulus of Elasticity [Pa]:         2e+11
Crosssectional Area    [m2]:         5e-05

Displacement Equations:
-----------------------------
Eq(1) = dBx - 0.000156
Eq(2) = dDy + 0.00020256
Eq(3) = 0.19612*dEx + 0.98058*dEz + 0.0008293
Eq(4) = dCy - 0.00005604
Eq(5) = 0.70711*dBx - 0.70711*dDx + 0.70711*dDy + 0.00031294
Eq(6) = 0.37139*dBx - 0.92848*dBz - 0.37139*dEx + 0.92848*dEz
Eq(7) = dCx - 1.0*dDx - 0.00003732
Eq(8) = 0.32444*dCx + 0.48666*dCy - 0.32444*dEx - 0.48666*dEy +
0.81111*dEz + 0.00023671
Eq(9) = 0.50709*dDy - 0.16903*dDx + 0.16903*dEx - 0.50709*dEy +
0.84515*dEz - 0.0013829

Joint Displacements:
----------------------
dBx [m] :      0.000156
dBz [m] :      -0.00312
dCx [m] :      0.000433
dCy [m] :       5.6e-05
dDx [m] :      0.000396
dDy [m] :     -0.000203
dEx [m] :       0.00389
dEy [m] :      -0.00447
dEz [m] :      -0.00162
```

Execution in Octave
The code is same as in the MATLAB above.

In Octave Command Window
The output values are similar to MATLAB. There appears to be a very small difference.

```
--------------
Properties
--------------
Modulus of Elasticity [Pa]: 2e+11
Crosssectional Area   [m2]: 5e-05

Displacement Equations:
------------------------------
Eq(1) =   dBx - 0.0001552
Eq(2) =   dDy + 0.0002
Eq(3) =   0.19612*dEx + 0.98058*dEz + 0.00082507
Eq(4) =   dCy - 5.52e-5
Eq(5) =   0.70711*dBx - 0.70711*dDx + 0.70711*dDy + 0.0003104
Eq(6) =   0.37139*dBx - 0.92848*dBz - 0.37139*dEx + 0.92848*dEz
Eq(7) =   dCx - dDx - 3.68e-5
Eq(8) =   0.32444*dCx + 0.48666*dCy - 0.32444*dEx - 0.48666*dEy +
0.81111*dEz + 0.000233
  07
Eq(9) =   -0.16903*dDx + 0.50709*dDy + 0.16903*dEx - 0.50709*dEy +
0.84515*dEz - 0.00138
  13

Joint Displacements:
----------------------
dBx [m] : 0.0001552
dBz [m] : -0.0031054
dCx [m] : 0.00043097
dCy [m] : 5.52e-05
dDx [m] : 0.00039417
dDy [m] : -0.0002
dEx [m] : 0.0038768
dEy [m] : -0.0044578
dEz [m] : -0.0016168
```

6.4.5 Graphical Description of Deflected Truss

It is possible to complete the problem by drawing the deflected truss in MATLAB. The deflections for this example are small and it is likely that the deflected position will not be different from the original truss. We will scale the deflections by 10 to be viewable. The code is same as in Section 6.3 except the line command also requires the z-component of the point.

Solution Using MATLAB
In the Editor:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Graphical description of Solution
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
set(gcf,'Position',[10,20,600,600],'Color','w')
axis([-0.1,1.5,-0.1,1.5,-0.1,2.1])
axis equal


%% draw the original truss members
line([A(1),B(1)],[A(2),B(2)],[A(3),B(3)], ...
    'LineWidth',2,'Color','k')  % line AB

line([A(1),D(1)],[A(2),D(2)],[A(3),D(3)], ...
    'LineWidth',2,'Color','k')  % line AD

line([D(1),C(1)],[D(2),C(2)],[D(3),C(3)], ...
    'LineWidth',2,'Color','k')  % line DC

line([D(1),B(1)],[D(2),B(2)],[D(3),B(3)], ...
    'LineWidth',2,'Color','k')  % line DB

line([B(1),C(1)],[B(2),C(2)],[B(3),C(3)], ...
    'LineWidth',2,'Color','k')  % line BC

line([D(1),E(1)],[D(2),E(2)],[D(3),E(3)], ...
    'LineWidth',2,'Color','k')  % line DE

line([B(1),E(1)],[B(2),E(2)],[B(3),E(3)], ...
    'LineWidth',2,'Color','k')  % line BE

line([C(1),E(1)],[C(2),E(2)],[C(3),E(3)], ...
    'LineWidth',2,'Color','k')  % line CE

line([A(1),E(1)],[A(2),E(2)],[A(3),E(3)], ...
    'LineWidth',2,'Color','k')  % line AE

%% set the text
text(A(1)-0.1,A(2),A(3),'\bfA','Color','b')
text(B(1)+0.1,B(2),B(3),'\bfB','Color','b')
text(C(1),C(2)+0.1,C(3),'\bfC','Color','b')
text(D(1)-0.1,D(2)+0.1,D(3),'\bfD','Color','b')
text(E(1),E(2)+0.1,E(3),'\bfE','Color','b')

%% Calculate the change in locations in decimals
sf = 10;  % scale factor
dBx = sf*double(soldis.dBx);
dBz = sf*double(soldis.dBz);
dCx = sf*double(soldis.dCx);
dCy = sf*double(soldis.dCy);
dDx = sf*double(soldis.dDx);
dDy = sf*double(soldis.dDy);
dEx = sf*double(soldis.dEx);
dEy = sf*double(soldis.dEy);
dEz = sf*double(soldis.dEz);

%% vector displacements
dC = subs(dC); dB = subs(dB);  dD = subs(dD); dE = subs(dE);

%% displacement of pins
```

```
AA = A;  BB = B + dB;  CC = C + dC;  DD = D + dD;
EE = E + dE;

%% draw lines for displaced truss

line([AA(1),BB(1)],[AA(2),BB(2)],[AA(3),BB(3)], ...
    'LineWidth',2,'Color','r','LineStyle','--') % line AB

line([AA(1),DD(1)],[AA(2),DD(2)],[AA(3),DD(3)], ...
    'LineWidth',2,'Color','r','LineStyle','--') % line AD

line([DD(1),CC(1)],[DD(2),CC(2)],[DD(3),CC(3)], ...
    'LineWidth',2,'Color','r','LineStyle','--')  % line DC

line([DD(1),BB(1)],[DD(2),BB(2)],[DD(3),BB(3)], ...
    'LineWidth',2,'Color','r','LineStyle','--')  % line DB

line([BB(1),CC(1)],[BB(2),CC(2)],[BB(3),CC(3)], ...
    'LineWidth',2,'Color','r','LineStyle','--')  % line BC

line([DD(1),EE(1)],[DD(2),EE(2)],[DD(3),EE(3)], ...
    'LineWidth',2,'Color','r','LineStyle','--')  % line DE

line([BB(1),EE(1)],[BB(2),EE(2)],[BB(3),EE(3)], ...
    'LineWidth',2,'Color','r','LineStyle','--') % line BE

line([CC(1),EE(1)],[CC(2),EE(2)],[CC(3),EE(3)], ...
    'LineWidth',2,'Color','r','LineStyle','--')  % line CE

line([AA(1),EE(1)],[AA(2),EE(2)],[AA(3),EE(3)], ...
    'LineWidth',2,'Color','r','LineStyle','--')  % line AE

title('\bfOriginal and Displaced Truss - Example 6.6')

xlabel('x')
ylabel('y')
zlabel('z')
camup([0,1,0])% supposed to make y axis vertical - did not work
view(3)
grid
```

In the Figure Window

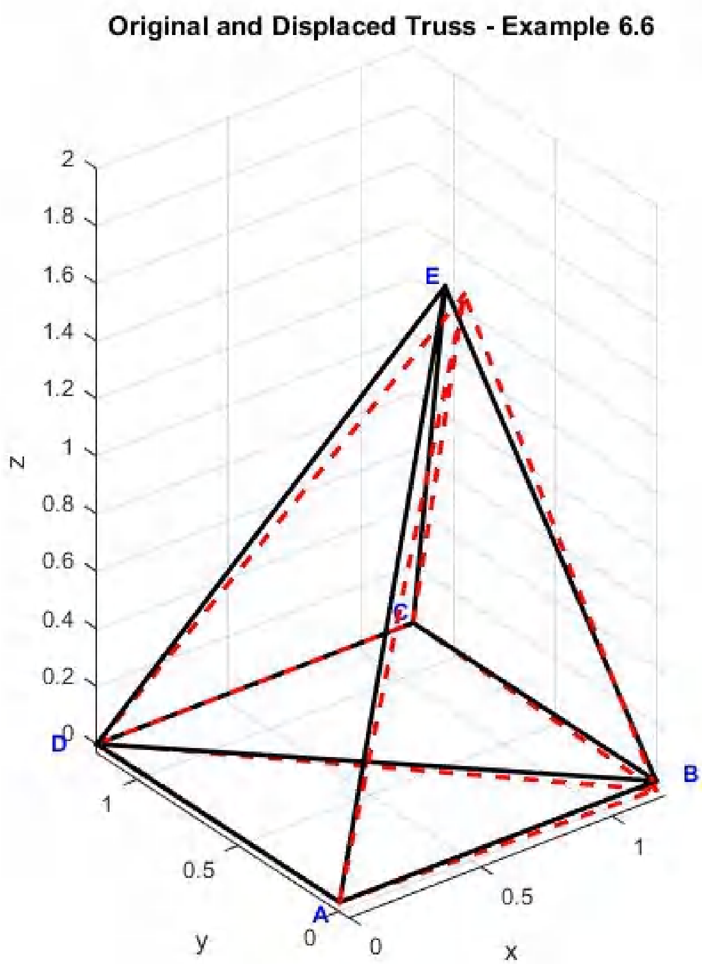


Figure 6.4.8a Deflected truss - Example 6.7

Execution in Octave

The code is same as in the MATLAB above except for the changes highlighted below

In Octave Editor

```
% dC = subs(dC);
dC = subs(dC,dCx);
dC = double(subs(dC,dCy));

% dB = subs(dB);
dB = subs(dB,dBx)
dB = double(subs(dB,dBz))

% dD = subs(dD);
dD = subs(dD,dDx);
dD = double(subs(dD,dDy));

% dE = subs(dE);
dE = subs(dE,dEx);
dE = subs(dE,dEy);
dE = double(subs(dE,dEz));
```

In Octave Figure Window

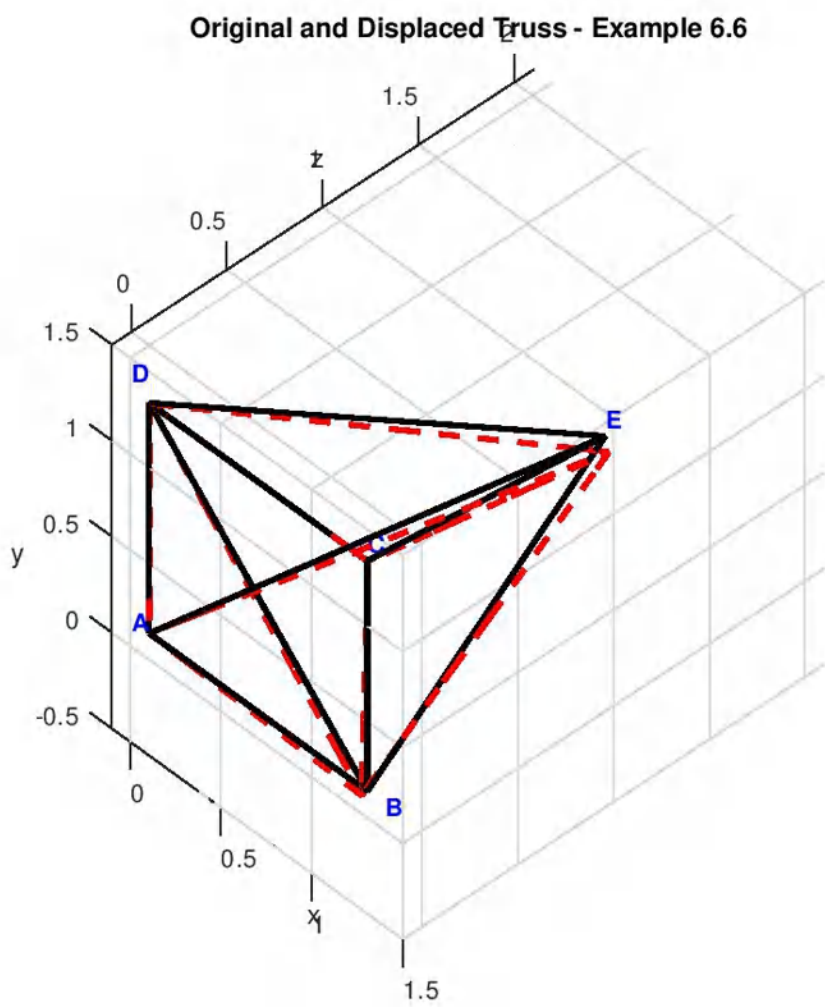


Figure 6.4.8b Deflected truss - Example 6.7

At this time this is the figure produced my MATLAB code. I am sure that the axis can be swapped in Octave. I will leave it to you to investigate this.

6.4.6 3D Truss - Example 6.8

We will re-examine Example 6.7 with a small change. We will remove the member BD. We now have only n = 5 joints and m = 8 members. If you have not already noted , we have violated the formula for stability since K is not equal to m. We will start with Figure 6.4.9 that defines the problem

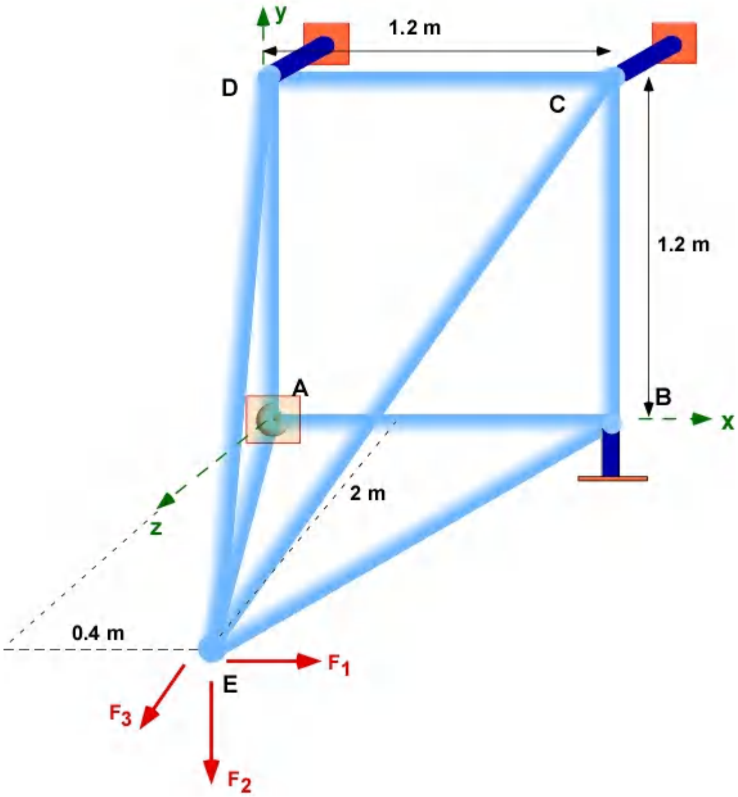


Figure 6.4.9 Example 6.8

We will apply the method of joints. The calculations will remain the same. To keep the programming modest we will keep the same dimensions, the same material, and the same area of cross-section for the members. Before starting our calculation an observation from Figure 6.4.9 suggests that joints B and D can seriously displace towards each other. D can displace in x-y plane and joint B can displace in the x-z plane. If this displacement is large, particularly in the x direction the truss may collapse. The member BD acts as a *brace*. If we look out to the solution of the displacement we will have 9 unknowns and only 8 member equations. This is an under-determined system. From a design point of view we may be able to introduce an extra constraint at B or D to make the problem determinate. If professionally you dealt with truss all the time, this kind of deduction will be intuitive. We will use MATLAB and borrow from previous code to solve the problem.

Equilibrium and Support Reactions

We draw the FBD of the truss with $\mathbf{F_1}$, $\mathbf{F_2}$, and $\mathbf{F_3}$ to be 500 N, 2500 N, and 200 N respectively as in Example 6.7. Note you can re-solve the truss with different values of the applied load without re-writing any code.

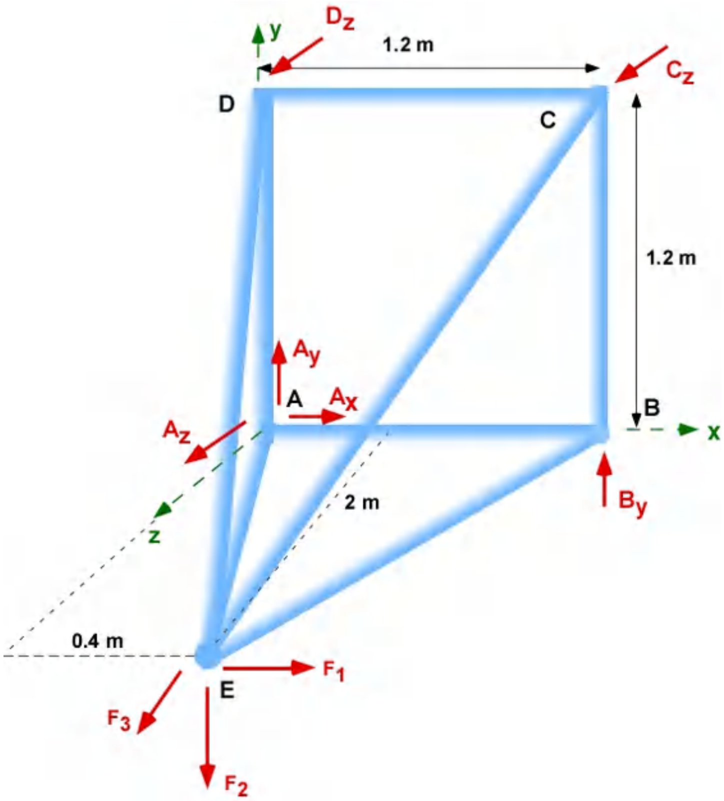


Figure 6.4.10 FBD of Truss

This is a statically determinate problem. We should be able to solve for the unknown reactions $\mathbf{A_x}$, $\mathbf{A_y}$, $\mathbf{A_z}$, $\mathbf{B_y}$, $\mathbf{C_z}$, $\mathbf{D_z}$.

$$\sum \bar{F} = 0 = \bar{F} + \bar{F}_A + \bar{F}_B + \bar{F}_C + \bar{F}_D$$

$$\sum \bar{M}_A = \left(\bar{r}_{AE} \times \bar{F}\right) + \left(\bar{r}_{AB} \times \bar{F}_B\right) + \left(\bar{r}_{AC} \times \bar{F}_C\right) + \left(\bar{r}_{AD} \times \bar{F}_D\right)$$

Solution Using MATLAB

The code is same as in Example 6.7 with same results (not shown)

In the Command Window

```
Example 6.8
------------
Point A [m]  :      0     0     0
Point B [m]  :           1.2          0          0
Point C [m]  :           1.2        1.2          0
Point D [m]  :             0        1.2          0
Point E [m]  :           0.4          0          2

Position vector rAE[m]   =          0.4           0            2
Position vector rAB[m]    =          1.2           0            0
Position vector rAC[m]   =          1.2         1.2            0
Position vector rAD[m]   =            0         1.2            0

F1 [ N]    :    500
F2 [ N]    :           2500
F3 [ N]    :    200
```

```
Equilibrium
----------------------

SumF :
           Ax + 500.0
     Ay + By - 2500.0
 Az + Cz + Dz + 200.0

SumMA :
 1.2*Cz + 1.2*Dz + 5000.0
           920.0 - 1.2*Cz
          1.2*By - 1000.0
------------
Reactions:
------------
Ax [N]      =   -500
Ay [N]      =         1677
Az [N]      =         3977
By [N]      =    833
Cz [N]      =    767
Dz [N]      =        -4933
```

Member Forces - Method of Joints

Since member DB does not exist the member forces must be solved with the new pin connections. To solve for the member forces we enforce the equilibrium of the pins/joints. The FBD of the joints are similar to Example 6.7 except that there is no member BD.

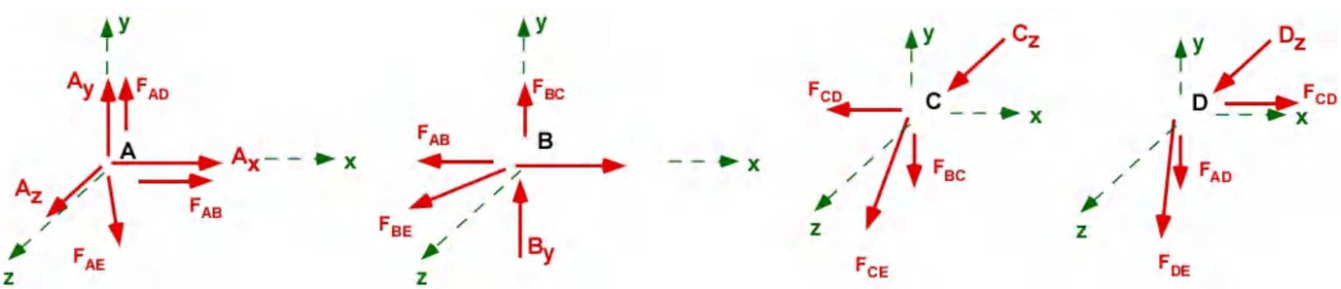


Figure 6.4.11 FBD of the joints

Pin A: $\bar{F}_A + \bar{F}_{AB} + \bar{F}_{AD} + \bar{F}_{AE} = 0$

Pin B: $\bar{F}_B + \bar{F}_{BA} + \bar{F}_{BC} + \bar{F}_{BE} = 0$

Pin C: $\bar{F}_C + \bar{F}_{CB} + \bar{F}_{CD} + \bar{F}_{CE} = 0$

Pin D: $\bar{F}_D + \bar{F}_{DA} + \bar{F}_{DC} + \bar{F}_{DE} = 0$

We will use MATLAB to obtain the member forces. The code is the same except at pin B there are only two unknowns. The code is the same and only information about member BD is edited out. You will expect some member forces to change to distribute the force on member BD to other members.

Solution Using MATLAB
In the Editor

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Member forces - Method of Joints
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Joint A
fprintf('------------\n')
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin A:\n')
fprintf('----------------\n')
syms Fad Fae Fab  real % Unknown member forces
rAD = D - A;  eAD = rAD/norm(rAD);
rAB = B - A;  eAB = rAB/norm(rAB);
rAE = E - A;  eAE = rAE/norm(rAE);

fprintf('Unit vector eAD[m]    = '),disp(eAD)
fprintf('Unit vector eAB[m]    = '),disp(eAB)
fprintf('Unit vector eAE[m]    = '),disp(eAE)
% equilibrium of pin
sumA = FA + Fab*eAB + Fae*eAE + Fad*eAD;
solA = solve(sumA);
fprintf('\nSum of Forces at A : \n'),disp(vpa(sumA',4))

Fab = double(solA.Fab);
Fad = double(solA.Fad);
Fae = double(solA.Fae);


fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fab [N]      = '),disp(Fab)
fprintf('Fad [N]      = '),disp(Fad)
fprintf('Fae [N]      = '),disp(Fae)

%% Joint B
fprintf('------------\n')
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin B:\n')
fprintf('----------------\n')
syms Fbc Fbe real % unknown member forces
rBC = C - B;  eBC = rBC/norm(rBC);
rBE = E - B;  eBE = rBE/norm(rBE);
rBA = A - B;  eBA = rBA/norm(rBA);

fprintf('Unit vector eBC[m]    = '),disp(eBC)
fprintf('Unit vector eBE[m]    = '),disp(eBE)
fprintf('Unit vector eBA[m]    = '),disp(eBA)

% equilibrium of pin
sumB = FB + Fab*eBA + Fbc*eBC + Fbe*eBE;
solB = solve(sumB(1),sumB(2)); % only two unknowns
fprintf('\nSum of Forces at B : \n'),disp(vpa(sumB',5))
% note the last equation makes Fbe = 0 and the first equation
```

```
% solves Fbe is not zero.  This is inconsistent and third equation is
% ignored

Fbc = double(solB.Fbc);
Fbe = double(solB.Fbe);

fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fbc [N]      = '),disp(Fbc)
fprintf('Fbe [N]      = '),disp(Fbe)

%% Joint C
fprintf('------------\n')
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin C:\n')
fprintf('----------------\n')

syms Fcd Fce real  % only two unknowns
rCD = D - C;  eCD = rCD/norm(rCD);
rCE = E - C;  eCE = rCE/norm(rCE);
rCB = B - C;  eCB = rCB/norm(rCB);

fprintf('Unit vector eCD[m]   = '),disp(eCD)
fprintf('Unit vector eCE[m]   = '),disp(eCE)
fprintf('Unit vector eCB[m]   = '),disp(eCB)

% equilibrium of pin
sumC = FC + Fbc*eCB + Fcd*eCD + Fce*eCE;
solC = solve(sumC(1),sumC(2));
fprintf('\nSum of Forces at C : \n'),disp(vpa(sumC',5))

Fcd = double(solC.Fcd);
Fce = double(solC.Fce);

fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fcd [N]      = '),disp(Fcd)
fprintf('Fce [N]      = '),disp(Fce)
%% Joint D
fprintf('------------\n')
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin D:\n')
fprintf('----------------\n')

syms Fde real  % only one unknown
rDE = E - D;  eDE = rDE/norm(rDE);
rDA = A - D;  eDA = rDA/norm(rDA);
rDC = C - D;  eDC = rDC/norm(rDC);

fprintf('Unit vector eDE[m]   = '),disp(eDE)
fprintf('Unit vector eDA[m]   = '),disp(eDA)
fprintf('Unit vector eDC[m]   = '),disp(eDC)
```

```
% equilibrium of pin
sumD = FD + Fde*eDE + Fad*eDA +  Fcd*eDC;
fprintf('\nSum of Forces at D : \n'),disp(vpa(sumD',5))
fde = solve(sumD(3));
Fde = double(fde);

fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fde [N]      = '),disp(Fde)
```

In the Command Window

```
------------
**************
------------
Pin A:
----------------
Unit vector eAD[m]   =      0     1     0
Unit vector eAB[m]   =      1     0     0
Unit vector eAE[m]   =       0.19612          0      0.98058

Sum of Forces at A :
 Fab + 0.1961*Fae - 500.0
             Fad + 1677.0
      0.9806*Fae + 3977.0
------------
Member Forces:
----------------
Fab [N]      =         1300
Fad [N]      =        -1688
Fae [N]      =        -4066
------------
**************
------------
Pin B:
----------------
Unit vector eBC[m]   =      0     1     0
Unit vector eBE[m]   =      -0.37139          0      0.92848
Unit vector eBA[m]   =     -1     0     0

Sum of Forces at B :
 - 0.37139*Fbe - 1300.0
            Fbc + 833.0
            0.92848*Fbe
------------
Member Forces:
----------------
Fbc [N]      =   -833
Fbe [N]      =        -3500
------------
**************
------------
Pin C:
----------------
Unit vector eCD[m]   =     -1     0     0
```

```
Unit vector eCE[m]   =      -0.32444     -0.48666      0.81111
Unit vector eCB[m]   =      0    -1      0

Sum of Forces at C :
 - 1.0*Fcd - 0.32444*Fce
     833.0 - 0.48666*Fce
     0.81111*Fce + 767.0
------------
Member Forces:
----------------
Fcd [N]     =   -555
Fce [N]     =         1711
------------
**************
------------
Pin D:
----------------
Unit vector eDE[m]   =       0.16903     -0.50709      0.84515
Unit vector eDA[m]   =      0    -1      0
Unit vector eDC[m]   =      1     0      0

Sum of Forces at D :
  0.16903*Fde - 555.0
 1688.0 - 0.50709*Fde
 0.84515*Fde - 4933.0
------------
Member Forces:
----------------
Fde [N]     =         5844
```

Member Stress:
One useful piece of information that we can generate before trying to solve for the displacements is to calculate the stress in the members. We have the forces. We have the material properties and cross-sectional area. Stress is just force/area.

Solution Using MATLAB
In the Editor

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Calculating Stress in Members
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Stress in the Members
Elas = 200e09;   Area = 50e-06;
fprintf('\n Properties\n')
fprintf('--------------\n')
fprintf('Modulus of Elasticity [Pa]: '),disp(Elas)
fprintf('Crosssectional Area    [m2]: '),disp(Area)
fprintf('\n Member Stress\n')
fprintf('----------------\n')
fprintf('Sigma_ab [MPa]     = '),disp(Fab/Area/1e06)
fprintf('Sigma_ad [MPa]     = '),disp(Fad/Area/1e06)
fprintf('Sigma_ae [MPa]     = '),disp(Fae/Area/1e06)
fprintf('Sigma_bc [MPa]     = '),disp(Fbc/Area/1e06)
fprintf('Sigma_be [MPa]     = '),disp(Fbe/Area/1e06)
fprintf('Sigma_cd [MPa]     = '),disp(Fcd/Area/1e06)
```

```
fprintf('Sigma_ce [MPa]      = '),disp(Fce/Area/1e06)
fprintf('Sigma_de [MPa]      = '),disp(Fde/Area/1e06)
```

We are working with steel and the yield is about 200 MPa.

In The Command Window

```
Properties
--------------
Modulus of Elasticity [Pa]:         2e+11
Crosssectional Area   [m2]:         5e-05

 Member Stress
----------------
Sigma_ab [MPa]     =     26
Sigma_ad [MPa]     =        -33.76
Sigma_ae [MPa]     =        -81.32
Sigma_bc [MPa]     =        -16.66
Sigma_be [MPa]     =    -70
Sigma_cd [MPa]     =         -11.1
Sigma_ce [MPa]     =         34.22
Sigma_de [MPa]     =        116.88
```

The stresses are quite high and the design will not meet a factor of safety of 2. The area of cross section should be increased. This information could have been obtained for all the previous examples too.

Joint Displacements

A problem can be anticipated here since there are 8 members and 9 displacement unknowns. There are insufficient number of equations to solve for the displacements explicitly.
It is possible to obtain an implicit solution in terms of one of the displacements. We can therefore solve for the displacements in terms of the z-displacement **dE_z**. MATLAB will express the solution in terms of the last symbolic variable on its internal list of variables. This is usually alphabetic.

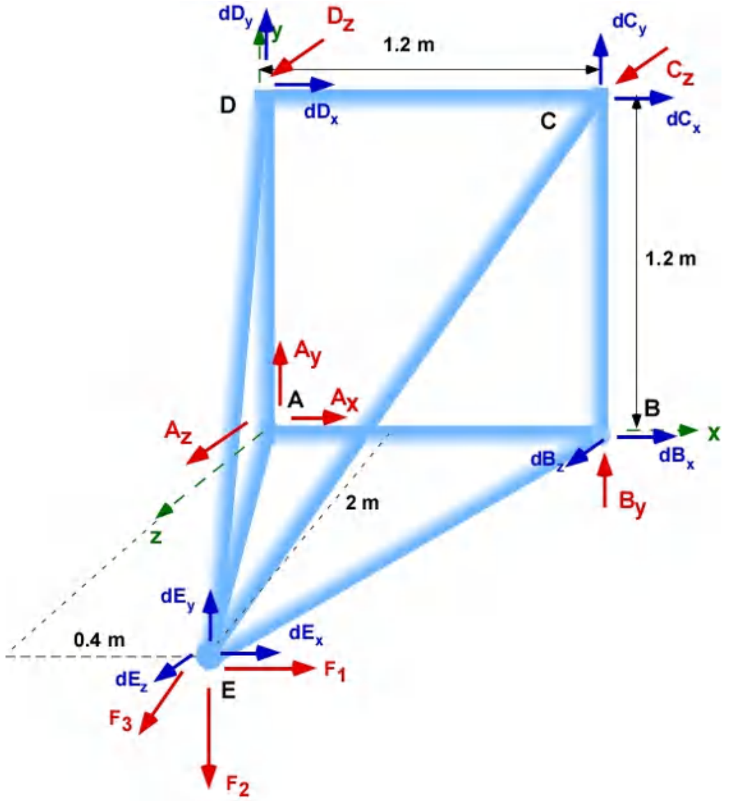


Figure 6.4.12 Joint displacements

The development of the displacement and force relations for each members is the same as in Example 6.7 We have no new information or process to add. We will just include the development of equations along member CE for the sake of completeness. Figure 6.4.13 provides the figure for the development.

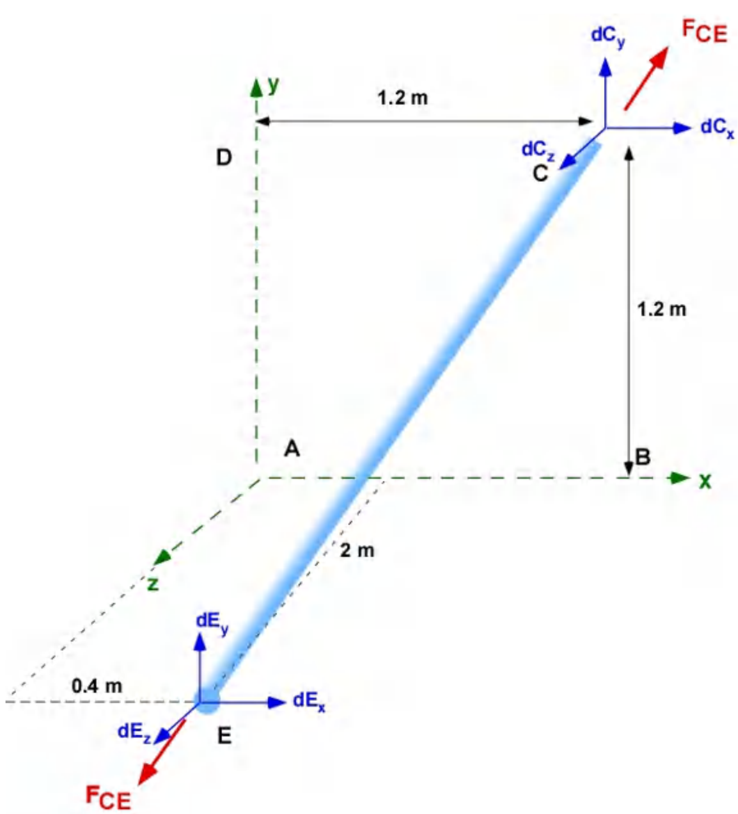


Figure 6.4.13 Member CE

$$\delta_{CE} = \left(\delta_C\right)_{along\,CE} - \left(\delta_E\right)_{along\,CE}$$

$$d\bar{C} = \left[dC_x, dC_y, dC_z\right]; \quad d\bar{E} = \left[dE_x, dE_y, dE_z\right];$$

$$\delta_{CE} = \hat{e}_{CE} \cdot d\bar{C} - \hat{e}_{CE} \cdot d\bar{E} = \frac{F_{CE} L_{CE}}{E_{CE} A_{CE}}$$

The remaining members have the same form as in Example 6.7 and Section 6.4.4 and is not repeated here. Member BD is not included is not in the calculations.

Solution Using MATLAB
In the Editor

```
%% Calculating displacements
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
syms dBx dBz dCx dCy dDx dDy dEx dEy dEz real  % the unknowns
%% displacement vectors
dA = [0,0,0];  dB = [dBx,0,dBz]; dC = [dCx,dCy,0];
dD = [dDx,dDy,0];  dE = [dEx,dEy,dEz];

% length of members and unit vectors  (Copy and Paste)
rAB = B - A;  LAB = norm(rAB);  eAB = rAB/LAB;
rAD = D - A;  LAD = norm(rAD);  eAD = rAD/LAD;
rAE = E - A;  LAE = norm(rAE);  eAE = rAE/LAE;
rBC = C - B;  LBC = norm(rBC);  eBC = rBC/LBC;
% rBD = D - B;  LBD = norm(rBD);  eBD = rBD/LBD;
rBE = E - B;  LBE = norm(rBE);  eBE = rBE/LBE;
rCD = D - C;  LCD = norm(rCD);  eCD = rCD/LCD;
rCE = E - C;  LCE = norm(rCE);  eCE = rCE/LCE;
rDE = E - D;  LDE = norm(rDE);  eDE = rDE/LDE;

%% set up the displacement equations   (Copy and Paste)
% Member Forces  Fad Fae Fab Fbc Fbd Fbe Fcd Fce Fde

fprintf('\nDisplacement Equations:\n')
fprintf('-----------------------------\n')
Eq(1) = dot(eAB,dB)-dot(eAB,dA) - (Fab*LAB/(Area*Elas)); % Member AB
fprintf('Eq(1) = '),disp(vpa(Eq(1),5))

Eq(2) = dot(eAD,dD)-dot(eAD,dA) - (Fad*LAD/(Area*Elas)); % Member AD
fprintf('Eq(2) = '),disp(vpa(Eq(2),5))

Eq(3) = dot(eAE,dE)-dot(eAE,dA) - (Fae*LAE/(Area*Elas)); % Member AE
fprintf('Eq(3) = '),disp(vpa(Eq(3),5))

Eq(4) = dot(eBC,dC)-dot(eBC,dB) - (Fbc*LBC/(Area*Elas)); % Member BC
fprintf('Eq(4) = '),disp(vpa(Eq(4),5))

% Eq(5) = dot(eBD,dD)-dot(eBD,dB) - (Fbd*LBD/(Area*Elas)); % Member BD

Eq(5) = dot(eBE,dE)-dot(eBE,dB) - (Fbe*LBE/(Area*Elas)); % Member BE
fprintf('Eq(5) = '),disp(vpa(Eq(5),5))
```

```
Eq(6) = dot(eCD,dD)-dot(eCD,dC) - (Fcd*LCD/(Area*Elas)); % Member CD
fprintf('Eq(6) = '),disp(vpa(Eq(6),5))

Eq(7) = dot(eCE,dE)-dot(eCE,dC) - (Fce*LCE/(Area*Elas)); % Member CE
fprintf('Eq(7) = '),disp(vpa(Eq(7),5))

Eq(8) = dot(eDE,dE)-dot(eDE,dD) - (Fde*LDE/(Area*Elas)); % Member DE
fprintf('Eq(8) = '),disp(vpa(Eq(8),5))

%% Solution
soldis = solve(Eq);

fprintf('\nJoint Displacements:\n')
fprintf('----------------------\n')

fprintf('dBx [m] : '),disp(vpa(soldis.dBx))
fprintf('dBz [m] : '),disp(vpa(soldis.dBz))
fprintf('dCx [m] : '),disp(vpa(soldis.dCx))
fprintf('dCy [m] : '),disp(vpa(soldis.dCy))
fprintf('dDx [m] : '),disp(vpa(soldis.dDx))
fprintf('dDy [m] : '),disp(vpa(soldis.dDy))
fprintf('dEx [m] : '),disp(vpa(soldis.dEx))
fprintf('dEy [m] : '),disp(vpa(soldis.dEy))
% fprintf('dEz [m] : '),disp(vpa(soldis.dEz))
```

In the Command Window

```
Joint Displacements:
----------------------
dBx [m] : 1.56e-4
dBz [m] : 3.0*dEz + 0.00257
dCx [m] : - 5.0*dEz - 0.00621
dCy [m] : -1.0e-4
dDx [m] : - 5.0*dEz - 0.00615
dDy [m] : -2.03e-4
dEx [m] : - 5.0*dEz - 0.00423
dEy [m] : 1.67*dEz - 0.00229
```

The displacements dB_x, dC_y, and dD_y are constant and the remaining are linear function of dE_z. The displacement for dE_z in Example 6.7 was -0.00162. For this value the displacements for Example 6.8 are

dBx [m] : 0.000156

dBz [m] : -0.00229

dCx [m] : 0.00189

dCy [m] : -0.0001

dDx [m] : 0.00195

dDy [m] : -0.000203

dEx [m] : 0.00387

dEy [m] : -0.00499

You can see that dD_x in Example 6.8 is much higher than in Example 6.7. We can also compare the member forces which should change because the load carried by member BD in Example 6.7.

We have explored the space truss (three dimensional truss) problem in its entirety. We have developed a complete set of MATLAB code to solve the problem and to explore its design. You can gain insight by using different material for members, different areas of cross-sections for the members, different geometry of truss itself, and different end support schemes. You can accomplish this by copying the original code and then editing it to reflect the new problem. Understanding the changes, being patient and diligent in editing the code is necessary for a successful resolutions to other three dimensional truss problems.

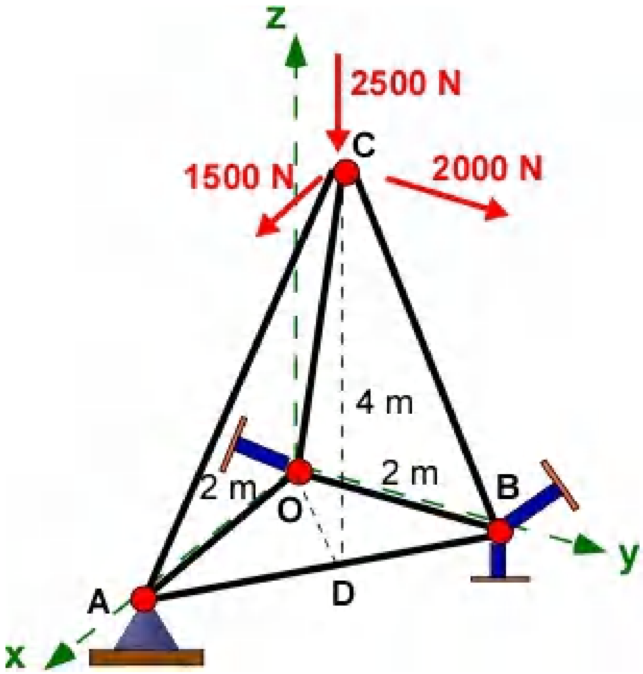
Execution in Octave
The code works in Octave without any changes. You will have adjust the code for better printing. You are recommended to run this example in Octave.

6.4.7 Additional Problems

Set up the following problems by hand on paper and solve them using MATLAB and confirm some of your calculations using a calculator. For each problem you must draw the FBD and work with a coordinate system. For all problems obtain support reactions, member forces , stresses, and displacement of the truss. Use a factor of safety of 6. Choose your material and area of cross-section and ensure you have an acceptable design. Identify zero-force members before you start your solution and confirm it through calculations.

The short link supports are aligned along the axis of the coordinate system in all problems
Please use Table 6.1 for your calculations.

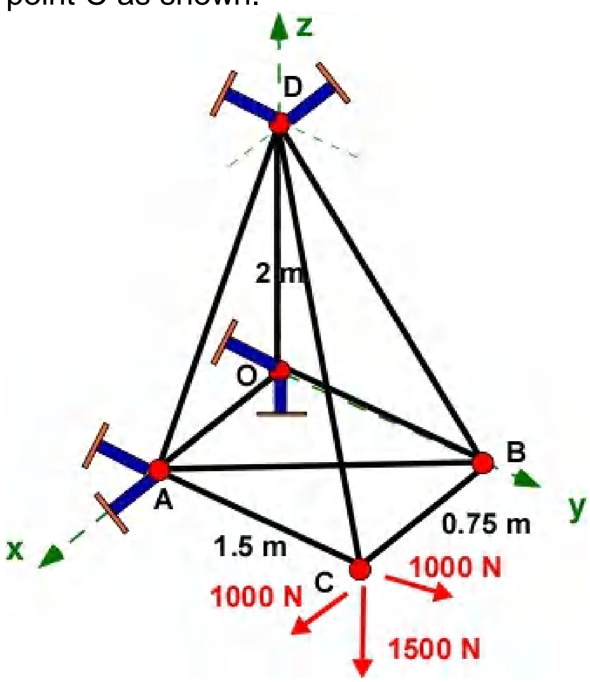
Problem 6.4.1
In this space truss a ball and socket joint provides support at A. Two short links generate reactions at B and a single link at O. Three forces are applied along the axes at C.



Problem 6.4.1

Problem 6.4.2

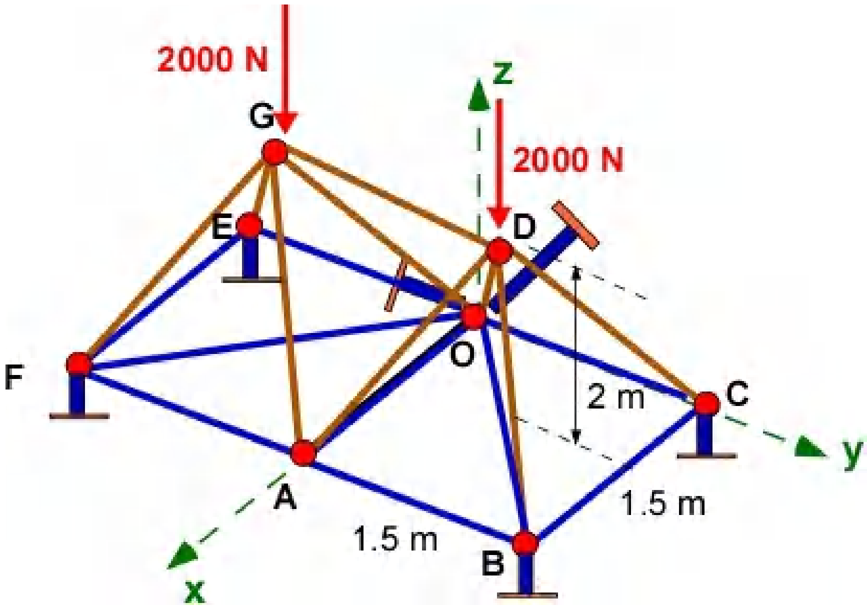
In this space truss two short links provide support at pins O, A and D. Three forces are applied at point C as shown.



Problem 6.4.2

Problem 6.4.3

This space truss is made of a pair of pyramidal trusses. The base of the pyramid is square of side 1.5 m and the vertical height of 2 m over the center of the square area. Member AO is shared between the trusses. Short links provide support in the z-direction at pins B, C, E and F. At O two short links provide support along the x and y directions. The truss is loaded symmetrically at D and G with a force of 2000 N. For better visualization the blue members are all in the same plane. The yellow members are above this plane. The argument of symmetry will be necessary for this problem.



Problem 6.4.3

6.5 SPACE TRUSS - STATICALLY INDETERMINATE

This section considers a severely constrained three dimensional truss. The problem will be statically indeterminate and therefore there must be sufficient constraints to finally solve the problem using deflections. The approach is the same as in the case of plane truss in Example 6.3. The increase in analysis make it more useful to deploy MATLAB/Octave to solve such problems right from the beginning. MATLAB can easily handle the large system of dependent linear equations in as many unknown. Octave can too while it takes longer. This has been demonstrated through several examples and possibly through the additional problems at the end of the sections.

6.5.1 Example 6.9

The space truss in Example 6.9 is shown in Figure 6.5.1. The truss is rigidly supported at pins A, D , J and G. There are three force and three moment unknown reactions at each of these pins . This is a total of 24 unknowns. That requires solving for 24 reactions. There are only 6 equations of statics. We have not employed rigid support before as they also require that the pins not move and also should not rotate. This rotation is difficult to implement in pin connections since pins are free to rotate. There are n = 10 joints and m = 18 members. The simple equation establishing stability does not apply well. We can see that the structure is stable, since it is rigidly held and cannot collapse on itself even though the equation does not say so. Rigid member supported truss are very rare in text books.

There are many additional questions that arise with fixed supports. The first is that if all members are two force members how is the moment at the pin support traverse through the truss? It is unclear what a moment along the member would mean?
The second question are the joints/pins away from the support. Do these function act as ball and socket joints?

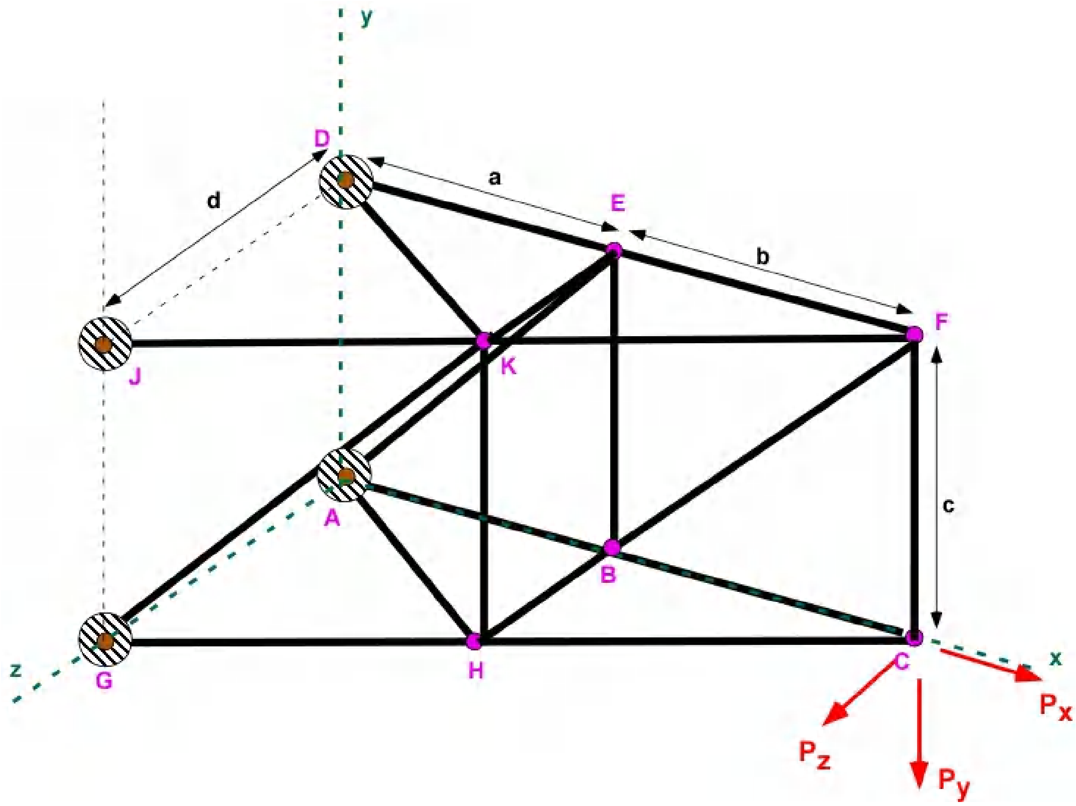


Figure 6.5.1 Example 6.6 - 3D Truss

Solution Strategy: As in Example 6.3 this is phony indeterminate problem. This particular problem can be solved creatively. The important consideration here is to solve the problem without solving for the reactions. Starting at Joint C we note that we can obtain the member forces at C. Moving to joint F we can determine the member forces connected at F. We can move from one member to another until all the member forces are established. We can decide if the structure will fail or what is the factor of safety. There are 6 joints that can displace. This creates 18 unknown displacements. We have 18 members and therefore we should be able to solve for the displacements too. We can also establish the force reactions at the fixed joints too. The problem can also have additional external forces at pins E, K, H, B, and F and that will not alter our strategy. This way we avoid solving for the reactions completely. In effect this is now a determinate problem like Example 6.3.

The example is solved using MATLAB.. Applying equilibrium at the pin involves only force equilibrium and we can avoid the FBD diagrams of the pins by carefully working with Figure 6.5.1 You just need member forces directed along the members.

In this example the locations of the pin are also [parameterized. We will establish the pin locations as:
A(0,0,0); B(a,0,0); C(a+b,0,0); D(0,c,0); E(a,c,0); F(a+b,c,0);
G(0,0,d); H(a,0,ek); J(0,c,d); K(a,c,ek);

For this example a = 0.5 m; b = 0.5 m, c= 0.6 m, and d = 0.8 m. ek = d*b/(a+b);

The material is steel E = 200 GPa. The area of cross section of all pins is the same and is 0.002 m^2. The applied loads are Px = 300 N; Py = 2000 N, and Pz = 500 N

Solution Using MATLAB
In the Editor

Data:

```
% Essential Foundations in Mechanics
% P. Venkataraman, Jan 2015
%  Example 6.9 - 3D Truss - Section 6.5.1
%
% Truss 3D
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(3)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 6.9\n')
fprintf('------------\n')
%% Data
a = 0.5; b = 0.5;  c = 0.6; d = 0.8;  ek = d*b/(a+b);
Px = 300;  Py = 2000;  Pz = 500;
E = 200e09;  Area = 0.002;

% parameterized data
A = [0 0 0]; B = [a,0,0];  C = [a+b,0,0]; D = [0,c,0];
E = [a,c,0];
F =[a+b,c,0];  G = [0,0,d];  H = [a,0,ek];  J = [0,c,d];
K = [a,c,ek];
FP = [Px,-Py,Pz];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% No Equilibrium investigation for reactions
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('Parameters')
fprintf('\n-----------------------\n')
fprintf('a [m] = '),disp(a)
fprintf('b [m] = '),disp(b)
fprintf('c [m] = '),disp(c)
fprintf('d [m] = '),disp(d)
fprintf('ek [m] = '),disp(ek)

fprintf('\nPoint A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point C [m]  : '),disp(C)
fprintf('Point D [m]  : '),disp(D)
fprintf('Point E [m]  : '),disp(E)
fprintf('Point F [m]  : '),disp(F)
fprintf('Point G [m]  : '),disp(G)
fprintf('Point H [m]  : '),disp(H)

fprintf('\nPx [N] : '),disp(Px)
fprintf('Py [N] : '),disp(Py)
fprintf('Pz [N]  : '),disp(Pz)
```

Member Forces:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Member forces - Method of Joints
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%% Joint A
fprintf('------------\n')
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin C:\n')
fprintf('----------------\n')
syms Fch Fcf Fcb real % Unknown member forces
rCH = H - C;  eCH = rCH/norm(rCH);
rCB = B - C;  eCB = rCB/norm(rCB);
rCF = F - C;  eCF = rCF/norm(rCF);

fprintf('Unit vector eCH[m]   = '),disp(eCH)
fprintf('Unit vector eCB[m]   = '),disp(eCB)
fprintf('Unit vector eCF[m]   = '),disp(eCF)
% equilibrium of pin
sumC = FP + Fch*eCH + Fcf*eCF + Fcb*eCB;
solC = solve(sumC);
fprintf('\nSum of Forces at C : \n'),disp(vpa(sumC',4))

Fch = double(solC.Fch);
Fcf = double(solC.Fcf);
Fcb = double(solC.Fcb);

fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fch [N]     = '),disp(Fch)
fprintf('Fcf [N]     = '),disp(Fcf)
fprintf('Fcb [N]     = '),disp(Fcb)

%% Joint F
fprintf('------------\n')
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin F:\n')
fprintf('----------------\n')

syms Ffe Ffk Ffb real % Unknown member forces
rFE = E - F;  eFE = rFE/norm(rFE);
rFK = K - F;  eFK = rFK/norm(rFK);
rFB = B - F;  eFB = rFB/norm(rFB);
rFC = C - F;  eFC = rFC/norm(rFC);
fprintf('Unit vector eFE[m]   = '),disp(eFE)
fprintf('Unit vector eFK[m]   = '),disp(eFK)
fprintf('Unit vector eFB[m]   = '),disp(eFB)
fprintf('Unit vector eFC[m]   = '),disp(eFC)

sumF = Ffe*eFE + Ffk*eFK + Ffb*eFB + Fcf*eFC;
solF = solve(sumF);
fprintf('\nSum of Forces at F : \n'),disp(vpa(sumF',5))

Ffe = double(solF.Ffe);
Ffk = double(solF.Ffk);
Ffb = double(solF.Ffb);

fprintf('------------\n')
```

```
fprintf('Member Forces:\n')
fprintf('-----------------\n')
fprintf('Ffe [N]      = '),disp(Ffe)
fprintf('Ffk [N]      = '),disp(Ffk)
fprintf('Ffb [N]      = '),disp(Ffb)

%% Joint B
fprintf('------------\n')
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin B:\n')
fprintf('-----------------\n')
syms Fba Fbe Fbh  real % unknown member forces
rBH = H - B;  eBH = rBH/norm(rBH);
rBE = E - B;  eBE = rBE/norm(rBE);
rBA = A - B;  eBA = rBA/norm(rBA);
rBF = F - B;  eBF = rBF/norm(rBF);
rBC = C - B;  eBC = rBC/norm(rBC);


fprintf('Unit vector eBH[m]    = '),disp(eBH)
fprintf('Unit vector eBE[m]    = '),disp(eBE)
fprintf('Unit vector eBA[m]    = '),disp(eBA)
fprintf('Unit vector eBF[m]    = '),disp(eBF)
fprintf('Unit vector eBC[m]    = '),disp(eBC)

% equilibrium of pin
sumB = Fba*eBA + Fbe*eBE + Fbh*eBH + Ffb*eBF + Fcb*eBC;
solB = solve(sumB);
fprintf('\nSum of Forces at B : \n'),disp(vpa(sumB',5))

Fba = double(solB.Fba);
Fbe = double(solB.Fbe);
Fbh = double(solB.Fbh);

fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('-----------------\n')
fprintf('Fba [N]      = '),disp(Fba)
fprintf('Fbe [N]      = '),disp(Fbe)
fprintf('Fbh [N]      = '),disp(Fbh)


%% Joint H
fprintf('------------\n')
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin H:\n')
fprintf('-----------------\n')

syms Fhg Fhk Fha  real % unknown member forces
rHG = G - H;  eHG = rHG/norm(rHG);
rHK = K - H;  eHK = rHK/norm(rHK);
rHA = A - H;  eHA = rHA/norm(rHA);
rHC = C - H;  eHC = rHC/norm(rHC);
rHB = B - H;  eHB = rHB/norm(rHB);
```

```
fprintf('Unit vector eHG[m]    = '),disp(eHG)
fprintf('Unit vector eHK[m]    = '),disp(eHK)
fprintf('Unit vector eHA[m]    = '),disp(eHA)
fprintf('Unit vector eHC[m]    = '),disp(eHC)
fprintf('Unit vector eHB[m]    = '),disp(eHB)

sumH = Fhg*eHG + Fhk*eHK + Fha*eHA + Fch*eHC + Fbh*eHB;
solH = solve(sumH);
fprintf('\nSum of Forces at H : \n'),disp(vpa(sumH',5))
Fhg = double(solH.Fhg);
Fhk = double(solH.Fhk);
Fha = double(solH.Fha);

fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fhg [N]      = '),disp(Fhg)
fprintf('Fhk [N]      = '),disp(Fhk)
fprintf('Fha [N]      = '),disp(Fha)

%% Joint E
fprintf('------------\n')
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin E:\n')
fprintf('----------------\n')

syms Fek Fea Fed real  % unknown member forces
rEK = K - E;  eEK = rEK/norm(rEK);
rEA = A - E;  eEA = rEA/norm(rEA);
rED = D - E;  eED = rED/norm(rED);
rEF = F - E;  eEF = rEF/norm(rEF);
rEB = B - E;  eEB = rEB/norm(rEB);
fprintf('Unit vector eEK[m]    = '),disp(eEK)
fprintf('Unit vector eEA[m]    = '),disp(eEA)
fprintf('Unit vector eED[m]    = '),disp(eED)
fprintf('Unit vector eEF[m]    = '),disp(eEF)
fprintf('Unit vector eEB[m]    = '),disp(eEB)

sumE = Fek*eEK + Fea*eEA + Fed*eED + Ffe*eEF + Fbe*eEB;
solE = solve(sumE);
fprintf('\nSum of Forces at E : \n'),disp(vpa(sumE',5))

Fek = double(solE.Fek);
Fea = double(solE.Fea);
Fed = double(solE.Fed);

fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fek [N]      = '),disp(Fek)
fprintf('Fea [N]      = '),disp(Fea)
fprintf('Fed [N]      = '),disp(Fed)

%% Joint K
fprintf('------------\n')
```

```
fprintf('**************\n')
fprintf('------------\n')
fprintf('Pin K:\n')
fprintf('----------------\n')

syms Fkj Fkd Fkg real  % unknown member forces
rKJ = J - K;  eKJ = rKJ/norm(rKJ);
rKD = D - K;  eKD = rKD/norm(rKD);
rKG = G - K;  eKG = rKG/norm(rKG);
rKE = E - K;  eKE = rKE/norm(rKE);
rKH = H - K;  eKH = rKH/norm(rKH);
fprintf('Unit vector eKJ[m]   = '),disp(eKJ)
fprintf('Unit vector eKD[m]   = '),disp(eKD)
fprintf('Unit vector eKG[m]   = '),disp(eKG)
fprintf('Unit vector eKE[m]   = '),disp(eKE)
fprintf('Unit vector eKH[m]   = '),disp(eKH)

sumK = Fkj*eKJ + Fkd*eKD + Fkg*eKG + Fek*eKE + Fhk*eKH
solK = solve(sumK);
fprintf('\nSum of Forces at K : \n'),disp(vpa(sumK',5))

Fkj = double(solK.Fkj);
Fkd = double(solK.Fkd);
Fkg = double(solK.Fkg);

fprintf('------------\n')
fprintf('Member Forces:\n')
fprintf('----------------\n')
fprintf('Fkj [N]      = '),disp(Fkj)
fprintf('Fkd [N]      = '),disp(Fkd)
fprintf('Fkg [N]      = '),disp(Fkg)
```

Stress in Members:

```
%% Stress in the Members
% prints greek letter sigma to the command window

Elas = 200e09;   Area = 0.002;
fprintf('\n Properties\n')
fprintf('--------------\n')
fprintf('Modulus of Elasticity [Pa]: '),disp(Elas)
fprintf('Crosssectional Area   [m2]: '),disp(Area)
fprintf('\n Member Stress (non zero)\n')
fprintf('----------------------------\n')
fprintf('\x03c3(ch) [MPa]  = '),disp(Fch/Area/1e06)
fprintf('\x03c3(cf) [MPa]  = '),disp(Fcf/Area/1e06)
fprintf('\x03c3(cb) [MPa]  = '),disp(Fcb/Area/1e06)
fprintf('\x03c3(fe) [MPa]  = '),disp(Ffe/Area/1e06)
fprintf('\x03c3(fb) [MPa]  = '),disp(Ffb/Area/1e06)
fprintf('\x03c3(ba) [MPa]  = '),disp(Fba/Area/1e06)
fprintf('\x03c3(be) [MPa]  = '),disp(Fbe/Area/1e06)
fprintf('\x03c3(hg) [MPa]  = '),disp(Fhg/Area/1e06)
fprintf('\x03c3(ea) [MPa]  = '),disp(Fea/Area/1e06)
fprintf('\x03c3(ed) [MPa]  = '),disp(Fed/Area/1e06)
```

Displacement:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Calculating displacements
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
syms dCx dCy dCz dFx dFy dFz dHx dHy dHz dBx dBy dBz real
syms dKx dKy dKz dEx dEy dEz real
%% displacement vectors
dA = [0,0,0];  dD = [0,0,0];  dJ = [0,0,0];dG = [0,0,0];
dC = [dCx,dCy,dCz];  dF = [dFx,dFy,dFz];
dH = [dHx,dHy,dHz];  dK = [dKx,dKy,dKz];
dB = [dBx,dBy,dBz];  dE = [dEx,dEy,dEz];

% set up the equations
% Member Forces
%   Fch Fcf Fcb Ffe Ffk Ffb Fba Fbe Fbh Fhg Fhk Fha Fek Fea
% Fed Fkj Fkd Fkg

rCF = F - C;  LCF = norm(rCF);  eCF = rCF/LCF;
rCH = H - C;  LCH = norm(rCH);  eCH = rCH/LCH;
rCB = B - C;  LCB = norm(rCB);  eCB = rCB/LCB;
rFE = E - F;  LFE = norm(rFE);  eFE = rFE/LFE;
rFK = K - F;  LFK = norm(rFK);  eFK = rFK/LFK;
rFB = B - F;  LFB = norm(rFB);  eFB = rFB/LFB;
rBA = A - B;  LBA = norm(rBA);  eBA = rBA/LBA;
rBE = E - B;  LBE = norm(rBE);  eBE = rBE/LBE;
rBH = H - B;  LBH = norm(rBH);  eBH = rBH/LBH;
rHG = G - H;  LHG = norm(rHG);  eHG = rHG/LHG;
rHK = K - H;  LHK = norm(rHK);  eHK = rHK/LHK;
rHA = A - H;  LHA = norm(rHA);  eHA = rHA/LHA;
rEK = K - E;  LEK = norm(rEK);  eEK = rEK/LEK;
rEA = A - E;  LEA = norm(rEA);  eEA = rEA/LEA;
rED = D - E;  LED = norm(rED);  eED = rED/LED;
rKJ = J - K;  LKJ = norm(rKJ);  eKJ = rKJ/LKJ;
rKD = D - K;  LKD = norm(rKD);  eKD = rKD/LKD;
rKG = G - K;  LKG = norm(rKG);  eKG = rKG/LKG;


fprintf('\nDisplacement Equations:\n')
fprintf('-----------------------------\n')
Eq(1) = dot(eCH,dH)-dot(eCH,dC) - (Fch*LCH/(Area*Elas)); % Member CH
Eq(2) = dot(eCF,dF)-dot(eCF,dC) - (Fcf*LCF/(Area*Elas)); % Member CF
Eq(3) = dot(eCB,dB)-dot(eCB,dC) - (Fcb*LCB/(Area*Elas)); % Member CB
Eq(4) = dot(eFE,dE)-dot(eFE,dF) - (Ffe*LFE/(Area*Elas)); % Member FE
Eq(5) = dot(eFK,dK)-dot(eFK,dF) - (Ffk*LFK/(Area*Elas)); % Member FK
Eq(6) = dot(eFB,dB)-dot(eFB,dF) - (Ffb*LFB/(Area*Elas)); % Member FB
Eq(7) = dot(eBA,dA)-dot(eBA,dB) - (Fba*LBA/(Area*Elas)); % Member BA
Eq(8) = dot(eBE,dE)-dot(eBE,dB) - (Fbe*LBE/(Area*Elas)); % Member BE
Eq(9) = dot(eBH,dH)-dot(eBH,dB) - (Fbh*LBH/(Area*Elas)); % Member BH
Eq(10) = dot(eHG,dG)-dot(eHG,dH) - (Fhg*LHG/(Area*Elas)); % Member HG
Eq(11) = dot(eHK,dK)-dot(eHK,dH) - (Fhk*LHK/(Area*Elas)); % Member HK
Eq(12) = dot(eHA,dA)-dot(eHA,dH) - (Fha*LHA/(Area*Elas)); % Member HA
Eq(13) = dot(eEK,dK)-dot(eEK,dE) - (Fek*LEK/(Area*Elas)); % Member EK
Eq(14) = dot(eEA,dA)-dot(eEA,dE) - (Fea*LEA/(Area*Elas)); % Member EA
Eq(15) = dot(eED,dD)-dot(eED,dE) - (Fed*LED/(Area*Elas)); % Member ED
Eq(16) = dot(eKJ,dJ)-dot(eKJ,dK) - (Fkj*LKJ/(Area*Elas)); % Member KJ
Eq(17) = dot(eKD,dD)-dot(eKD,dK) - (Fkd*LKD/(Area*Elas)); % Member KD
```

```
Eq(18) = dot(eKG,dG)-dot(eKG,dK) - (Fkg*LKG/(Area*Elas)); % Member KG


for i = 1:18
    fprintf('Eq(%d) = ',i),disp(vpa(Eq(i),5))
end
%% Solution
soldis = solve(Eq);

fprintf('\nJoint Displacements:\n')
fprintf('----------------------\n')

fprintf('dCx [m] : '),disp(double((soldis.dCx)))
fprintf('dCy [m] : '),disp(double((soldis.dCy)))
fprintf('dCz [m] : '),disp(double((soldis.dCz)))
fprintf('dFx [m] : '),disp(double((soldis.dFx)))
fprintf('dFy [m] : '),disp(double((soldis.dFy)))
fprintf('dFz [m] : '),disp(double((soldis.dFz)))
fprintf('dHx [m] : '),disp(double((soldis.dHx)))
fprintf('dHy [m] : '),disp(double((soldis.dHy)))
fprintf('dHz [m] : '),disp(double((soldis.dHz)))
fprintf('dBx [m] : '),disp(double((soldis.dBx)))
fprintf('dBy [m] : '),disp(double((soldis.dBy)))
fprintf('dBz [m] : '),disp(double((soldis.dBz)))
fprintf('dKx [m] : '),disp(double((soldis.dKx)))
fprintf('dKy [m] : '),disp(double((soldis.dKy)))
fprintf('dKz [m] : '),disp(double((soldis.dKz)))
fprintf('dEx [m] : '),disp(double((soldis.dEx)))
fprintf('dEy [m] : '),disp(double((soldis.dEy)))
fprintf('dEz [m] : '),disp(double((soldis.dEz)))
```

In the Command Window

```
Example 6.9
------------
Parameters
----------------------
a [m] =          0.5
b [m] =          0.5
c [m] =          0.6
d [m] =          0.8
ek [m] =          0.4

Point A [m]  :     0     0     0
Point B [m]  :          0.5            0            0
Point C [m]  :     1     0     0
Point D [m]  :            0          0.6            0
Point E [m]  :          0.5          0.6            0
Point F [m]  :            1          0.6            0
Point G [m]  :            0            0          0.8
Point H [m]  :          0.5            0          0.4
Point J [m]  :            0          0.6          0.8
Point K [m]  :          0.5          0.6          0.4

Px [N] :     300
Py [N] :          2000
```

```
Pz [N]  :    500
------------
**************
------------
Pin C:
----------------
Unit vector eCH[m]   =      -0.78087             0       0.6247
Unit vector eCB[m]   =     -1     0     0
Unit vector eCF[m]   =      0     1     0

Sum of Forces at C :
 300.0 - 0.7809*Fch - 1.0*Fcb
                 Fcf - 2000.0
           0.6247*Fch + 500.0
------------
Member Forces:
----------------
Fch [N]     =   -800
Fcf [N]     =         2000
Fcb [N]     =    925
------------
**************
------------
Pin F:
----------------
Unit vector eFE[m]   =     -1     0     0
Unit vector eFK[m]   =      -0.78087             0       0.6247
Unit vector eFB[m]   =      -0.64018     -0.76822            0
Unit vector eFC[m]   =      0    -1     0

Sum of Forces at F :
 - 0.64018*Ffb - 1.0*Ffe - 0.78087*Ffk
                - 0.76822*Ffb - 2000.0
                            0.6247*Ffk
------------
Member Forces:
----------------
Ffe [N]     =         1677
Ffk [N]     =      0
Ffb [N]     =        -2600
------------
**************
------------
Pin B:
----------------
Unit vector eBH[m]   =      0     0     1
Unit vector eBE[m]   =      0     1     0
Unit vector eBA[m]   =     -1     0     0
Unit vector eBF[m]   =       0.64018      0.76822            0
Unit vector eBC[m]   =      1     0     0

Sum of Forces at B :
 - 1.0*Fba - 739.48
       Fbe - 1997.4
                Fbh
------------
```

```
Member Forces:
----------------
Fba [N]     =   -739
Fbe [N]     =         2000
Fbh [N]     =      0
------------
*************
------------
Pin H:
----------------
Unit vector eHG[m]   =      -0.78087            0       0.6247
Unit vector eHK[m]   =      0     1     0
Unit vector eHA[m]   =      -0.78087            0      -0.6247
Unit vector eHC[m]   =       0.78087            0      -0.6247
Unit vector eHB[m]   =      0     0    -1

Sum of Forces at H :
 - 0.78087*Fha - 0.78087*Fhg - 624.7
                              Fhk
    0.6247*Fhg - 0.6247*Fha + 499.76
------------
Member Forces:
----------------
Fhg [N]     =   -800
Fhk [N]     =      0
Fha [N]     =      1.82e-14
------------
*************
------------
Pin E:
----------------
Unit vector eEK[m]   =      0     0     1
Unit vector eEA[m]   =      -0.64018     -0.76822            0
Unit vector eED[m]   =     -1     0     0
Unit vector eEF[m]   =      1     0     0
Unit vector eEB[m]   =      0    -1     0

Sum of Forces at E :
 1677.0 - 1.0*Fed - 0.64018*Fea
        - 0.76822*Fea - 2000.0
                           Fek
------------
Member Forces:
----------------
Fek [N]     =      0
Fea [N]     =        -2600
Fed [N]     =         3344
------------
*************
------------
Pin K:
----------------
Unit vector eKJ[m]   =      -0.78087            0       0.6247
Unit vector eKD[m]   =      -0.78087            0      -0.6247
Unit vector eKG[m]   =       -0.5698     -0.68376      0.45584
Unit vector eKE[m]   =      0     0    -1
```

```
Unit vector eKH[m]   =      0    -1     0

Sum of Forces at K :
 - 0.78087*Fkd - 0.5698*Fkg - 0.78087*Fkj
                             -0.68376*Fkg
    0.45584*Fkg - 0.6247*Fkd + 0.6247*Fkj
------------
Member Forces:
----------------
Fkj [N]     =      0
Fkd [N]     =      0
Fkg [N]     =      0

 Properties
--------------
Modulus of Elasticity [Pa]:          2e+11
Crosssectional Area   [m2]:          0.002

 Member Stress (non zero)
---------------------------
σ(ch) [MPa]  =          -0.4
σ(cf) [MPa]  =      1
σ(cb) [MPa]  =        0.4625
σ(fe) [MPa]  =        0.8385
σ(fb) [MPa]  =          -1.3
σ(ba) [MPa]  =       -0.3695
σ(be) [MPa]  =      1
σ(hg) [MPa]  =          -0.4
σ(ea) [MPa]  =          -1.3
σ(ed) [MPa]  =         1.672

Displacement Equations:
-----------------------------
Eq(1) = 0.78087*dCx - 0.6247*dCz - 0.78087*dHx + 0.6247*dHz + 1.2806e-6
Eq(2) = dFy - 1.0*dCy - 3.0e-6
Eq(3) = dCx - 1.0*dBx - 1.1563e-6
Eq(4) = dFx - 1.0*dEx - 2.0962e-6
Eq(5) = 0.78087*dFx - 0.6247*dFz - 0.78087*dKx + 0.6247*dKz
Eq(6) = 0.64018*dFx - 0.76822*dBy - 0.64018*dBx + 0.76822*dFy + 5.0767e-6
Eq(7) = dBx + 9.2375e-7
Eq(8) = dEy - 1.0*dBy - 3.0e-6
Eq(9) = dHz - 1.0*dBz
Eq(10) = 0.78087*dHx - 0.6247*dHz + 1.2806e-6
Eq(11) = dKy - 1.0*dHy
Eq(12) = 0.78087*dHx + 0.6247*dHz - 2.9134e-23
Eq(13) = dKz - 1.0*dEz
Eq(14) = 0.64018*dEx + 0.76822*dEy + 5.0767e-6
Eq(15) = dEx - 4.18e-6
Eq(16) = 0.78087*dKx - 0.6247*dKz
Eq(17) = 0.78087*dKx + 0.6247*dKz
Eq(18) = 0.5698*dKx + 0.68376*dKy - 0.45584*dKz

Joint Displacements:
----------------------

Displacements
```

```
-----------------
dCx [m] :       2.32e-07
dCy [m] :      -2.87e-05
dCz [m] :       4.39e-06
dFx [m] :       6.28e-06
dFy [m] :      -2.57e-05
dFz [m] :       7.85e-06
dHx [m] :       -8.2e-07
dHy [m] :       0
dHz [m] :       1.02e-06
dBx [m] :      -9.24e-07
dBy [m] :      -1.31e-05
dBz [m] :       1.02e-06
dKx [m] :       0
dKy [m] :       0
dKz [m] :       0
dEx [m] :       4.18e-06
dEy [m] :      -1.01e-05
dEz [m] :       0
```

Execution in Octave

The code works in Octave as is. You may need to address some formatting issues for better printing in the Command Window.

Once again the displacements are too small to visualize on a plot. Plotting of displacements is left to the student. Evaluating the reaction forces at the pins A, H, G, and J is also an exercise to be completed by the student.

Discussion

- This problem was initially posed as a severe statically indeterminate problem using fixed constraints at A, D, G and J.
- It was creatively solved for member forces, member stress and member displacements.
- The reactions were not necessary.
- It is possible to solve for force reactions at A, D, G and J (see exercise above).
- The displacements at A, D, G, and J were assumed to be zero - linearly fixed.
- This is equivalent to using a 3D pin- or a ball and socket joint - at these locations .
- The initial statement refers to fixed constraint at A, D, G , and J. This implies that the members are rotationally fixed too.
- There is no way to solve for the moment reactions.
- The FBD of any member that is attached to one of these supports cannot transfer a moment through the truss as they are two force members.
- It is difficult to evaluate true fixed supports for pinned truss. We can only handle pinned truss.

6.5.2 Additional Problems

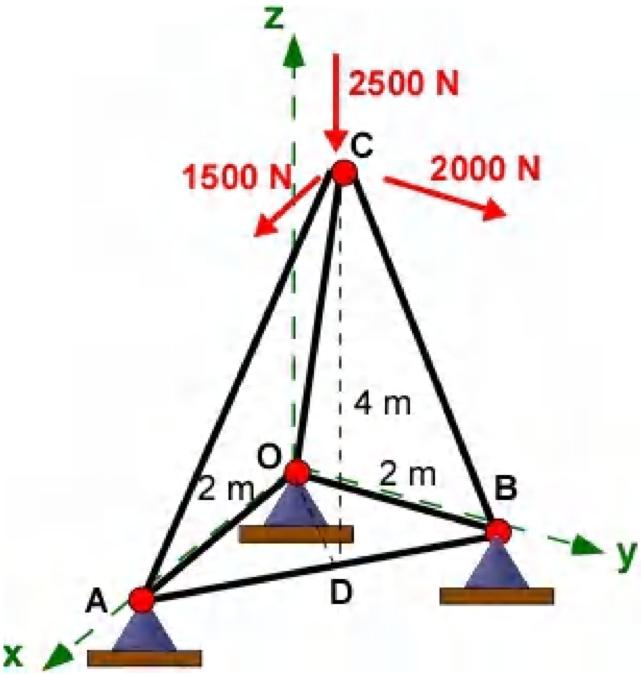
Set up the following problems by hand on paper and solve them using MATLAB/Octave and confirm some of your calculations using a calculator. For each problem you must draw the FBD and work with a coordinate system. For all problems obtain support reactions, member forces , stresses, and displacement of the truss. Use a factor of safety of 6. Choose your material and area of cross-section and ensure you have an acceptable design. Identify zero-force members before you start your solution and confirm it through calculations.

The short link supports are aligned along the axis of the coordinate system
Please use Table 6.1 for your calculations.

The problems in this section have appeared before in Section 6.4 and the Example 6.9 above. Please compare the solutions for insight.

Problem 6.5.1

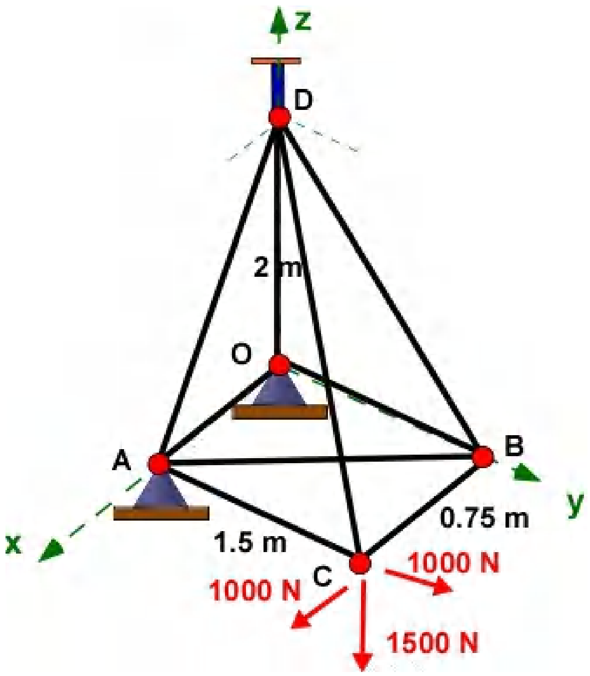
In this space truss the ball and socket joint provides support at O, A, and B. Three forces are applied along the axes at C.



Problem 6.5.1

Problem 6.5.2

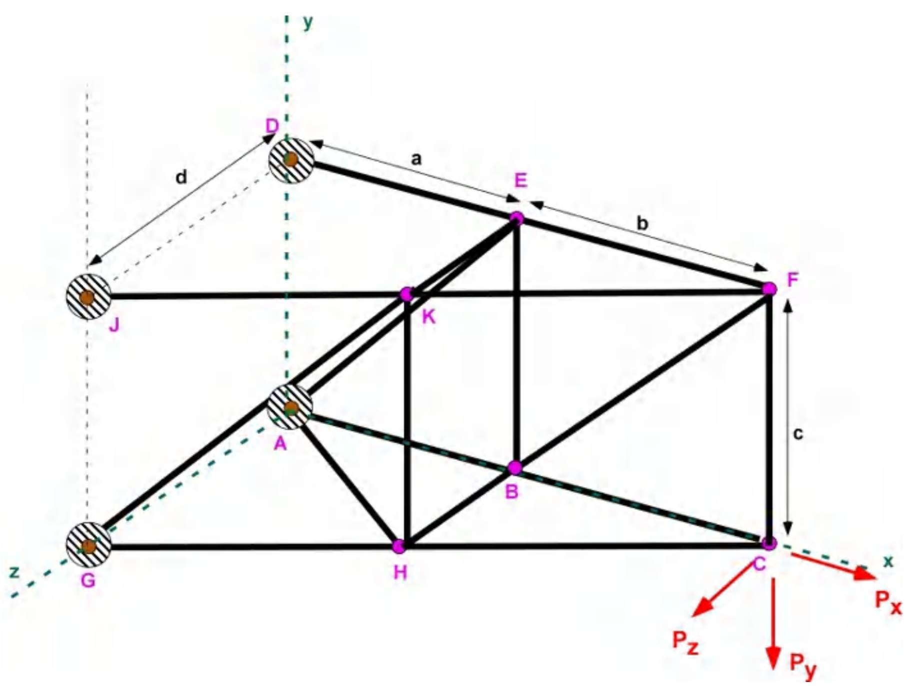
In this space truss pins O and A are supported by a ball and socket joint. At pin D there is a short link along the z-direction. Three forces are applied at point C as shown.



Problem 6.5.2

Problem 6.5.3

In Example 6.9 there were lots of zero-force members. One of them was member KD. Solve for the truss with member KD removed. Use the same values for the geometry and materials as in Example 6.9.



Problem 6.5.3

6.6 FRAMES

A frame is very similar to truss except that external loads loads can be applied along the length of the member rather than only at the pins. Let us look at a simple example of the gymnast on parallel bars. He is balancing himself by his grip at point D, on a single bar. This point is not a pin. Therefore it is not a truss problem by definition even though the structure appears as a truss. It should be apparent to hold this pose (statics) he has to exert a force and a moment at the point D. In this example the moment must be three dimensional and could resisted at the end supports. In actual situation the bar will also bend through its length resulting in a bending displacement. The idealization of the bar is cast as a two-dimensional problem in Figure 6.6.1. The FBD is simple and we avoided the statical indeterminacy by ignoring the horizontal reactions at the pins. We have two unknowns and we should be able to determine them with ease. This is the type of problems in Chapter 5. In general we could have many other types of loading along the member. The approach is still the same. Draw the FBD diagram of the structure as a whole and the FBD of individual members and apply repeatedly the equations of equilibrium until all unknowns are established. This allows discussion of problems in Chapter 5 with connected parts. This could that could include trusses connected to other structures..

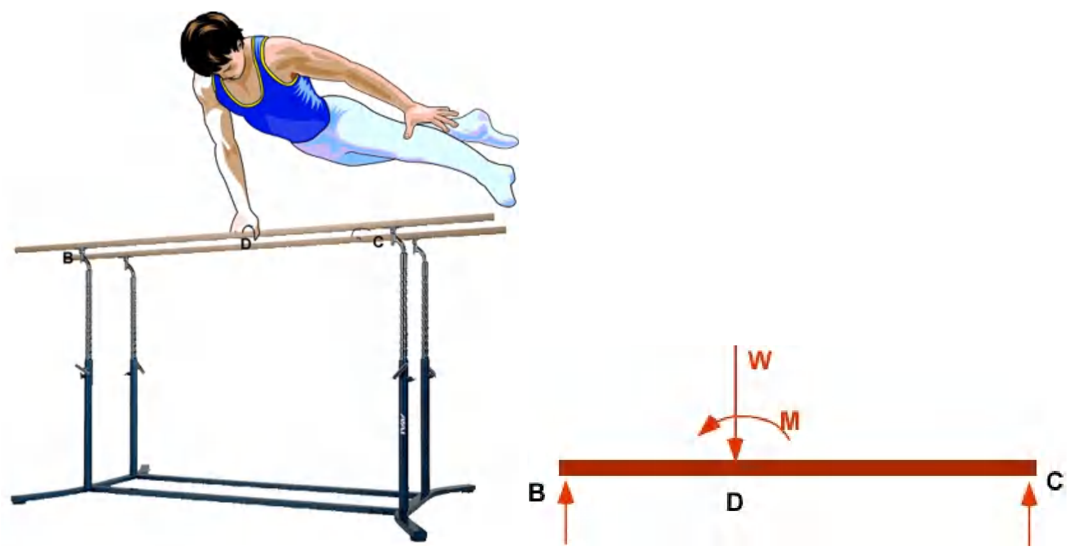


Figure 6.6.1 Example of a Frame and FBD

6.6.1 Example 6.10

We will look a at an example of a structure (two-dimensional) with two members/links that are connected using a pin and a slot. There are two loads applied on the structure - a force F on one member and a moment M on the other. The point B will influence both links. We expect that the moment will cause the yellow link to apply a force at the point B that will attempt to bend the rose colored link down. The force on the same link will increase this bending. The link is fixed at D and is capable of resisting both force and moments at the fixed end D.

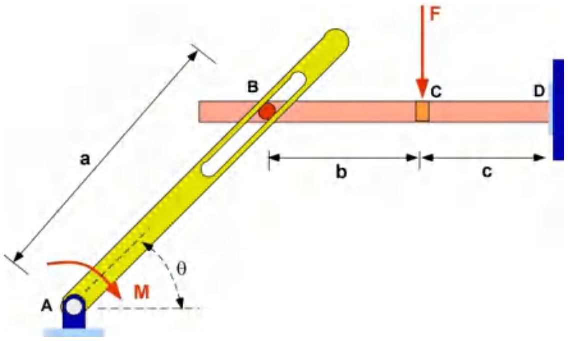


Figure 6.6.2 Example 6.10

We start with the FBD of the overall structure in Figure 6.6.3 to compute support reactions. There are five unknowns A_x, A_y, D_x D_y and M_D. This problem is statically indeterminate with a degree of two. According to previous sections we use displacements to provide additional relations to solve the statically indeterminate problems. Calculating the displacement of the link BD will appear in the next chapter and currently beyond the scope. This implies we must rely on equilibrium equations to solve such problems. The connection of the two members should provide additional relations. This connection is shown in Figure 6.6.4.

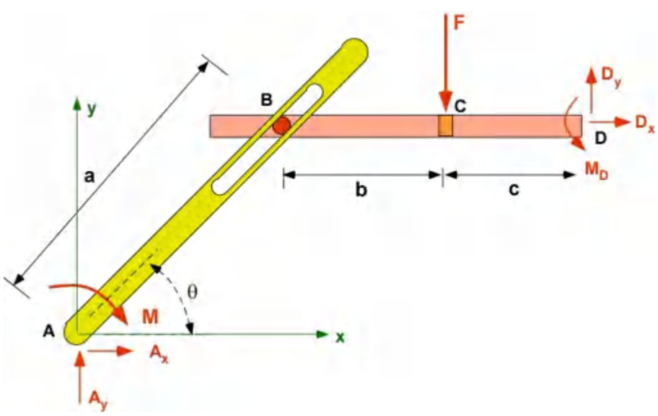


Figure 6.6.3 FBD of the structure

Since there are two members we can explore the FBD of each member thereby doubling the number of available equations of equilibrium. This will introduce additional **internal reaction forces** on the members at their connection/interaction points. These forces do not appear in the overall FBD of the structure since they cancel each other out as an **action and reaction** pair. In Figure 6.6.4 we have the internal force of contact R. Between the two structure we have six equations and six unknowns and it is now statically determinate.

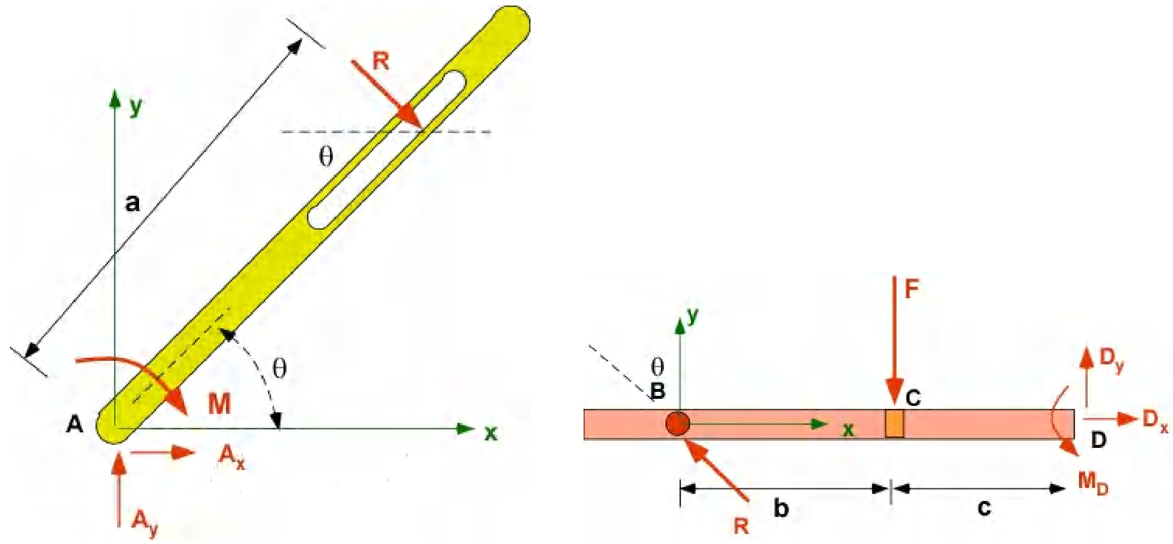


Figure 6.6.4 FBD of the two members/links

This problem is now reduced to applying the equations of equilibrium to two rigid bodies. These are coupled equations.

Data: A(0, 0, 0); B(acosθ, asinθ , 0); C(acosθ +b, asinθ, 0); D(acosθ +b + c; asinθ, 0);
Given M [Nm]; F [N]; (direction on Figure)

Find: (a) A_x, A_y, D_x, D_y, M_D

Assumption: Pinned support at A, Fixed support at D

Solution: Apply Equilibrium equations to the two links in Figure 6.6.4

Equations of Equilibrium - Example 6.10
The solution can be expressed in terms of the parameters a, b, c, θ , M, F.
We will explore if we can parameterize the problem in MATLAB/OCTAVE.

Slotted Member:

$$\sum F_x = A_x + R\sin\theta = 0 \qquad (1)$$

$$\sum F_y = A_y - R\cos\theta = 0 \qquad (2)$$

$$\sum M_A = -M - Ra = 0 \qquad (3)$$

$$R = -\frac{M}{a}; \quad A_x = -\left(-\frac{M}{a}\right)\sin\theta; \quad A_y = \left(-\frac{M}{a}\right)\cos\theta$$

Straight Member:

$$\sum F_x = D_x - R\sin\theta = 0 \qquad (4)$$

$$\sum F_y = D_y + R\cos\theta - F = 0 \qquad (5)$$

$$\sum M_D = M_D + Fc - R\cos\theta(b+c) = 0 \qquad (6)$$

$$D_x = \left(-\frac{M}{a}\right)\sin\theta; \quad D_y = F - \left(-\frac{M}{a}\right)\cos\theta;$$

$$M_D = -Fc + \left(-\frac{M}{a}\right)\cos\theta(b+c)$$

Note that we can express numerical solution for any values of a, b, c, θ , M, F. By substituting in the symbolic solution

Solution Using MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, Mar 2015
%  Example 6.10 - Frame (2D)
%  Section 6.6.1
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(3)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 6.10\n')
fprintf('------------\n')
%% Data
syms Ax Ay Dx Dy Md F M R real
syms a b c tht real

% parameterized data
% slotted arm
A = [0,0,0];  B = [a*cos(tht),a*sin(tht),0];
rAB = B-A;
FA = [Ax,Ay,0];  MA = [0,0,-M];
FR = [R*sin(tht),-R*cos(tht),0];


% Horizontal Member - origin at B
B1 = [0,0,0];  C = [b,0,0];  D = [b+c,0,0];
rDC = C - D;  rDB = D - B1;
FB = -FR;  FC = [0,-F,0];  FD = [Dx,Dy,0];
MD = [0,0,Md];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Equilibrium of connected system
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
sumF = FA + FR;
sumMA = MA + cross(rAB,FR);
sumF1 = FB + FC + FD;
sumMD = cross(rDC,FC) + cross(rDB,FB) + MD;
```

```
sol=solve([sumF(1),sumF(2),sumMA(3),sumF1(1),sumF1(2),sumMD(3)], ...
    [Ax,Ay,Dx,Dy,R,Md]);


% sol = solve([SumF,SumMA]);
Ax = (sol.Ax);
Ay = (sol.Ay);
Dx = (sol.Dx);
Dy = (sol.Dy);
R =  (sol.R);
Md = (sol.Md);
%
%%%%%%%%%%%%%%%%%%%%%%
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('In term of Parameters')
fprintf('\n----------------------')
fprintf('\nPoint A [m]   : '),disp(A)
fprintf('Point B [m]   : '),disp(B)
fprintf('Point B1 [m]   : '),disp(B1)
fprintf('Point C [m]   : '),disp(C)
fprintf('Point D [m]   : '),disp(D)

fprintf('\nEquilibrium - Link AB\n')
fprintf('---------------------------')
fprintf('\nSumF : \n'),disp(vpa(sumF',4))
fprintf('SumMA : \n'),disp(vpa(sumMA',4))


fprintf('\nEquilibrium - Link BD\n')
fprintf('---------------------------')
fprintf('\nSumF1 : \n'),disp(vpa(sumF1',4))
fprintf('SumMD : \n'),disp(vpa(sumMD',4))
%
fprintf('------------\n')
fprintf('Reactions:\n')
fprintf('------------\n')
fprintf('Ax [N]      = '),disp(Ax)
fprintf('Ay [N]      = '),disp(Ay)
fprintf('Dx [N]      = '),disp(Dx)
fprintf('Dy [N]      = '),disp(Dy)
fprintf('R [N]       = '),disp(R)
fprintf('MD [N]      = '),disp(Md)
```

In the Command Window

```
Example 6.10
------------
In term of Parameters
----------------------
Point A [m]   :      0      0      0
Point B [m]   : [ a*cos(tht), a*sin(tht), 0]
Point B1 [m]   :      0      0      0
Point C [m]   : [ b, 0, 0]
```

```
Point D [m]   : [ b + c, 0, 0]

Equilibrium - Link AB
---------------------------
SumF :
     Ax + R*sin(tht)
 Ay - 1.0*R*cos(tht)
                   0
SumMA :
                                                 0
                                                 0
 - 1.0*M - 1.0*R*a*cos(tht)^2 - 1.0*R*a*sin(tht)^2

Equilibrium - Link BD
---------------------------
SumF1 :
     Dx - 1.0*R*sin(tht)
 Dy - 1.0*F + R*cos(tht)
                       0
SumMD :
                             0
                             0
 Md + F*c + R*cos(tht)*(b + c)
------------
Reactions:
------------
Ax [N]      = (M*sin(tht))/(a*cos(tht)^2 + a*sin(tht)^2)
Ay [N]      = -(M*cos(tht))/(a*cos(tht)^2 + a*sin(tht)^2)
Dx [N]      = -(M*sin(tht))/(a*cos(tht)^2 + a*sin(tht)^2)
Dy [N]      = (F*a*cos(tht)^2 + M*cos(tht) + F*a*sin(tht)^2)/(a*cos(tht)^2
+ a*sin(tht)^2)
R [N]       = -M/(a*cos(tht)^2 + a*sin(tht)^2)
MD [N]      = (M*b*cos(tht) + M*c*cos(tht) - F*a*c*cos(tht)^2 -
F*a*c*sin(tht)^2)/(a*cos(tht)^2 + a*sin(tht)^2)
```

Please compare the two solutions. Please remember

$$a\cos^2\theta + a\sin^2\theta = a$$

Execution in Octave

The code is same as in the MATLAB above - with the following changes for formatting purposes

In Octave Editor

```
clc, clear, format compact, format shortg, close all,
warning off
pkg load symbolic;

fprintf('\nSumF : \n'),disp(vpa(sumF,4))
fprintf('SumMA : \n'),disp(vpa(sumMA,4))

fprintf('\nSumF1 : \n'),disp(vpa(sumF1,4))
fprintf('SumMD : \n'),disp(vpa(sumMD,4))
```

```
fprintf('Ax [N]      =  \n'),disp(Ax)
fprintf('\nAy [N]      = \n'),disp(Ay)
fprintf('\nDx [N]      = \n'),disp(Dx)
fprintf('\nDy [N]      = \n'),disp(Dy)
fprintf('\nR [N]       = \n'),disp(R)
fprintf('\nMD [N]      = \n'),disp(Md)
```

In Octave Command Window

```
Example 6.10
------------
In term of Parameters
----------------------
Point A [m]  :   0   0   0
Point B [m]  :   [a*cos(tht)   a*sin(tht)   0]
Point B1 [m]  :   0   0   0
Point C [m]  :   [b   0   0]
Point D [m]  :   [b + c   0   0]

Equilibrium - Link AB
---------------------------
SumF :
  [Ax + R*sin(tht)   Ay - R*cos(tht)   0]
SumMA :
  [                   2                2     ]
  [0   0   -M - R*a*sin (tht) - R*a*cos (tht)]

Equilibrium - Link BD
---------------------------
SumF1 :
  [Dx - R*sin(tht)   Dy - F + R*cos(tht)   0]
SumMD :
  [0   0   F*c + Md + R*(b + c)*cos(tht)]
------------
Reactions:
------------
Ax [N]      =
  M*sin(tht)
  ----------
      a

Ay [N]      =
  -M*cos(tht)
  ------------
       a

Dx [N]      =
  -M*sin(tht)
  ------------
       a

Dy [N]      =
      M*cos(tht)
  F + ----------
```

```
            a

R [N]       =
  -M
  ---
   a

MD [N]      =
  -F*a*c + M*(b + c)*cos(tht)
  ---------------------------
                 a
```

It appears that Octave cleans up the final expression better than MATLAB. Probably there is a better way to use disp in Octave.

Exercises
To complete the problem finish the following exercises.

1. Choose value for the parameters and compute the corresponding numerical values for the reactions.
2. Use the values selected for a, b, c, θ, F in exercise 1 above and plot the reactions as a function of M.

6.6.2 Example 6.11

In this frame example we will throw in a support structure, a frictionless pulley, and an inextensible cord/cable. As we walk through the analysis please take note that there are no new ideas required for solving the problem. The problem is described in Figure 6.6.5. The mass of W is 500 kg. Note the main structure looks like a truss but is not one since forces are present at points within the member. In a truss the forces are applied at pins at the ends of the members.

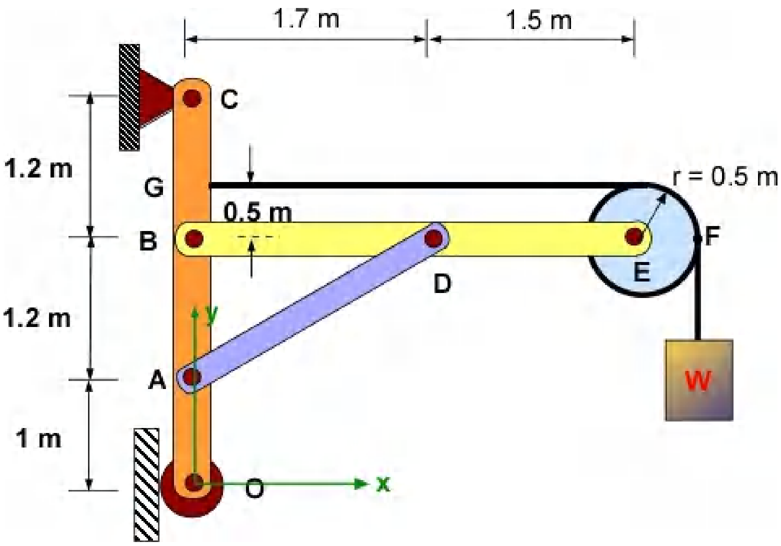


Figure 6.6.5 Example 6.11

Data: O(0, 0, 0); A(0, 1, 0); B(0, 2.2, 0), C(0, 3.4, 0); D(1.7, 2.2, 0); E(3.2, 2.2, 0)
W = 500x9.81 [N];

Pin support at C; Roller support at O

Find: (a) Reaction at the supports; (b) Forces at the pins

Assumption: Statically determinate, frictionless pulley; inextensible cord. Ignore the weight of the members.

Solution: Apply Equilibrium equations to entire structure and individual members

(a) Reactions - see Figure 6.6.6

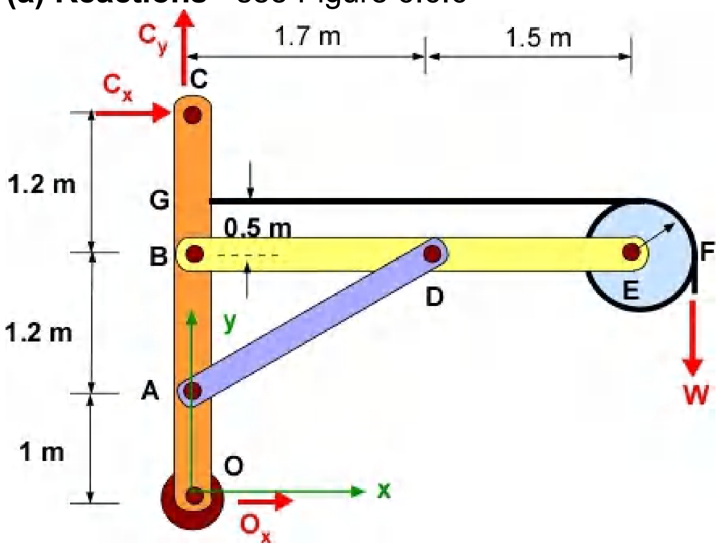


Figure 6.6.6 FBD of entire structure in Example 6.11

$$\sum F_x = C_x + O_x = 0;$$
$$\sum F_y = C_y - 4905 = 0; \quad C_y = 4905[N];$$
$$\sum M_C = 3.4O_x - 3.7 \times 4905 = 0; \quad O_x = 5337.6[N]$$

(b) Forces at the Pins - see Figure 6.6.7

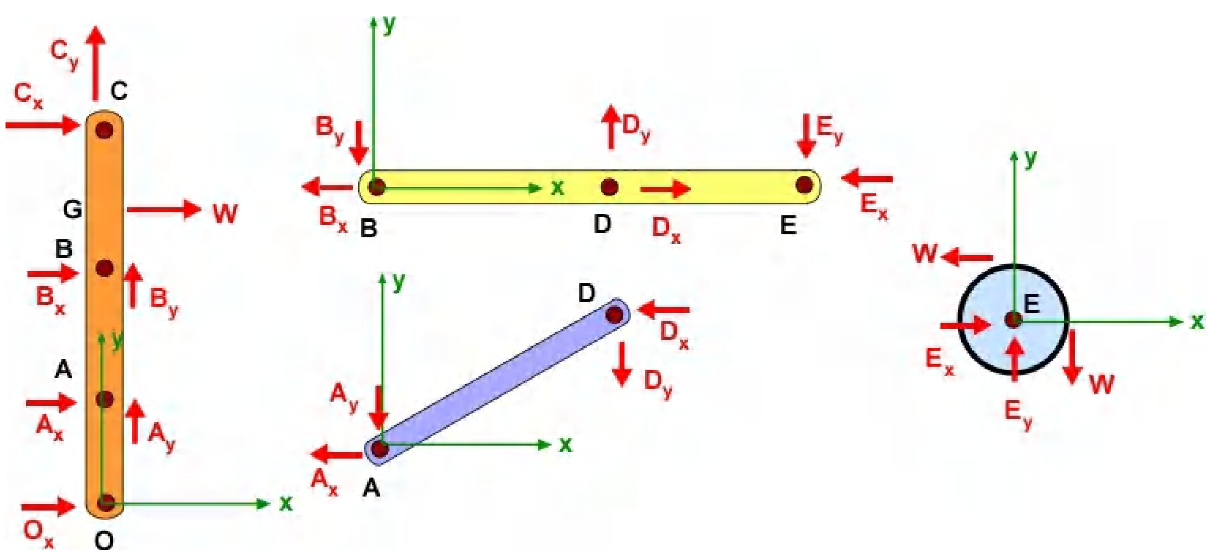


Figure 6.6.7 FBD of components in Example 6.11

The unknowns are A_x, A_y, B_x, B_y, D_x, D_y, E_x, E_y. Some of them can be related easily through the component FBD. Note the reactions on the components are equal and opposite at the same pin.

Consider FBD of pulley at E:

$$E_x = W = 4905[N];$$
$$E_y = W = 4905[N];$$

Consider FBD of member AD:

$$A_x + D_x = 0; \quad D_x = -A_x;$$
$$A_y + D_y = 0; \quad D_y = -A_y;$$

There is also another consideration. AB is a two force member. If the force must be along the member then we have a single unknown F_{AD}.

$$A_x = F_{AD}\frac{1.7}{\sqrt{1.7^2+1.2^2}} = 0.817F_{AD};$$
$$A_y = F_{AD}\frac{1.2}{\sqrt{1.7^2+1.2^2}} = 0.577F_{AD};$$

Consider FBD of member OC:

$$\sum F_x = C_x + B_x + 0.817F_{AD} + O_x + W = 0$$
$$= -5337.6 + B_x + 0.817F_{AD} + 5337.6 + 4905 = 0$$
$$\sum F_y = C_y + B_y + 0.577F_{AD} = 0$$
$$= 4905 + B_y + 0.577F_{AD} = 0;$$
$$\sum M_B = 2.2O_x + 1.2A_x - 0.5W - 1.2C_x = 0$$
$$= 2.2\times 5337.6 + 1.2\times 0.817F_{AD} - 0.5\times 4905 - 1.2\times(-5337.6) = 0$$
$$F_{AD} = -16010[N]; \quad B_x = 8175[N]; \quad B_y = 4328[N]$$

Using the value of FAD the forces A_x, A_y, B_x, B_y, D_x, D_y can be determined.

Design Consideration

The frame problems are solved through equilibrium of connected/coupled rigid bodies. At this stage it it is difficult to calculate deformation of these bodies. Applying loads in between members will cause bending and that is addressed in the next chapter. In addition problems are mostly pin supported to avoid transferring moments through the structure.

In this specific example It is possible to design for the cable/cord, the pins, and the member AD. Members OC and BE are beyond the scope at this time.

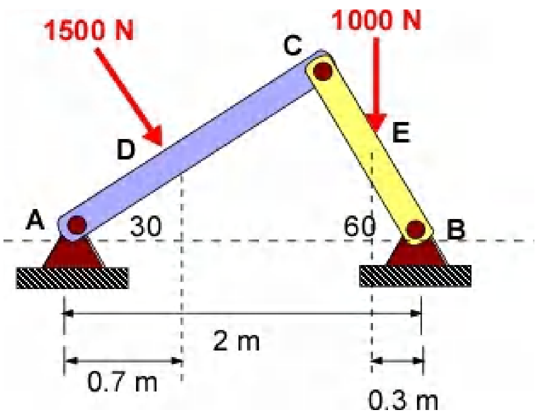
6.6.3 Additional Problems

Set up the following problems by hand on paper and solve them using MATLAB/Octave and confirm some of your calculations using a calculator. For each problem you must draw the FBD and work with a coordinate system. For all problems obtain support reactions and other required forces. Calculate stresses if information is available. If exploring design use a factor of safety of 6. Choose your material and area of cross-section and ensure you have an acceptable design.

Please use Table 6.1 for your calculations if necessary

Problem 6.6.1

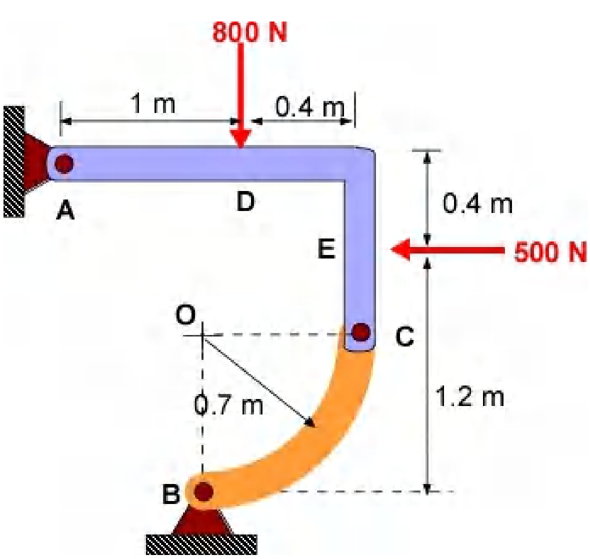
The problem appears statically indeterminate. However there are six equations of equilibrium from the FBD of two links



Problem 6.6.1

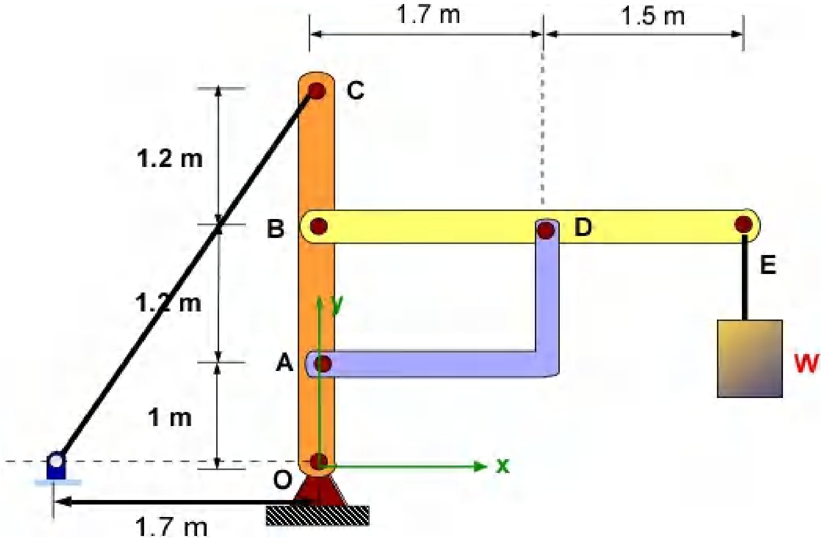
Problem 6.6.2

The link BC is a quarter arc. The problem appears statically indeterminate. However there are six equations of equilibrium from the FBD of two links.
Will the problem change if the the circular link was replaced by a straight link?



Problem 6.6.2

Problem 6.6.3
This problem has some similarity to Example 6.11.
Will the problem change if link AD was straight?



Problem 6.6.3

6.7 MACHINES

In most texts the study of frames ends with the study of machines. Machines are a very broad category and encompass many things. Here, machines refer to mechanisms - a device that allow us to transfer, amplify, create, and deliver forces or moments to solve a particular design problem. The analysis will involve the same concepts that we saw in the previous examples on frames. Machines were intrinsic to human development and we often delight at the way our ancestors solved one problem after another to make our life less of a struggle everyday by employing the wheel, the fulcrum, the screw, and the balance. Figure 1 (Wikipedia) is a table of simple mechanisms, from Chamber's Cyclopedia in 1728. Simple machines provide a "vocabulary" for understanding more complex machines.

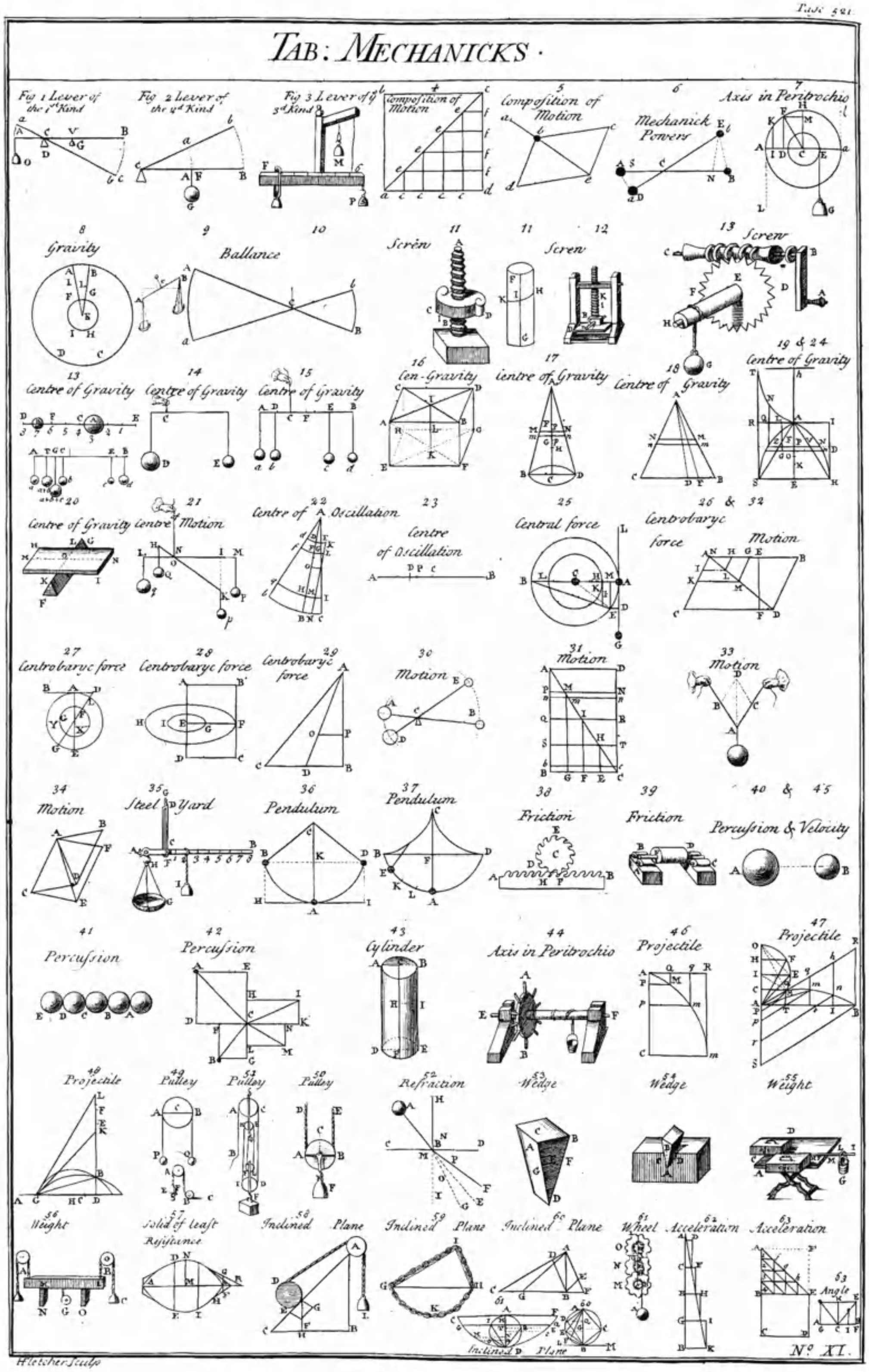


Figure 6.7.1 Collection of simple mechanisms

6.7.1 Example 6.12

One of the machines you see everyday moving massive amounts of dirt is a backhoe. Figure 6.9.2 is an image from the Caterpillar back hoe (from Caterpillar Inc.) and a simple representation of this mechanism with a loaded bucket in the horizontal position is shown in Figure 6.7.3.

Figure 6.7.2 Backhoe (Courtesy Caterpillar)

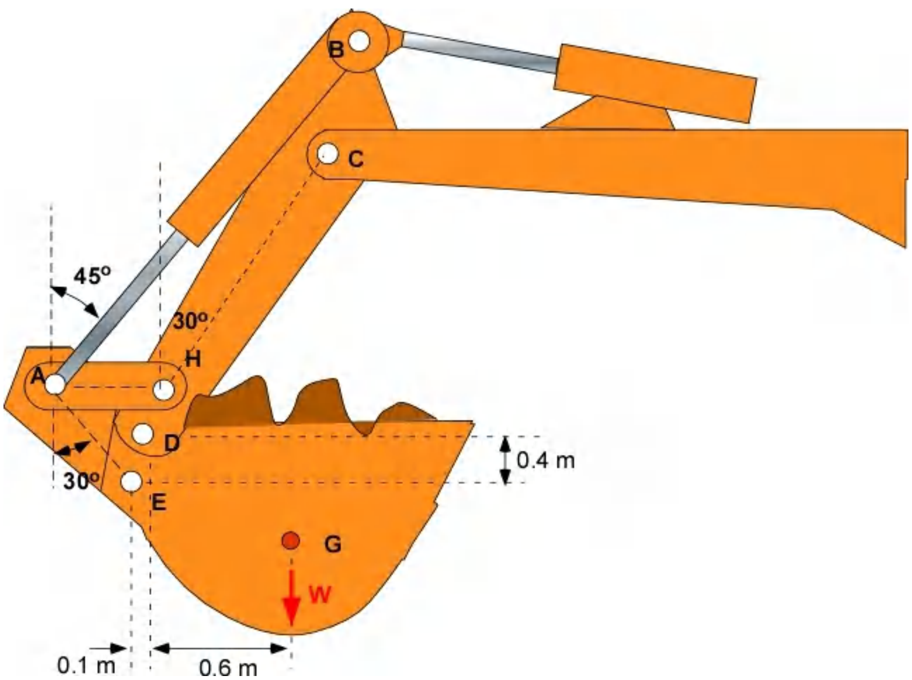


Figure 6.7.3 Bucket in horizontal Position

Problem: Find the force in the links AH, AE and the force in AB when the bucket is horizontal position as shown with a mass of 500 kg.

Assumptions: The elements involved in this structure are massive and quite distinct from the slender members that we used during previous truss problems. Nevertheless we can approach the problem in the same manner. They are pinned members. We assume the force in the links are directed along the axis of the member - **that is the line joining the pins.** This was evident in the problems in the previous section on frames. We can use the weights of the members if given to be located at the center of mass of the member - if known. We are introducing *no new information* or *procedure* to this class of problems - except multiple connected segments. We should be able to the method of joints and equilibrium to solve the problem. Pulleys and springs can also be used with these problems.

Solution: From the information given we are likely to use the equilibrium of the bucket and the equilibrium of the pin A to solve the problem. We will generalize the information so that the solution is available in parametric form. The FBD of the pin at A and the bucket are shown.

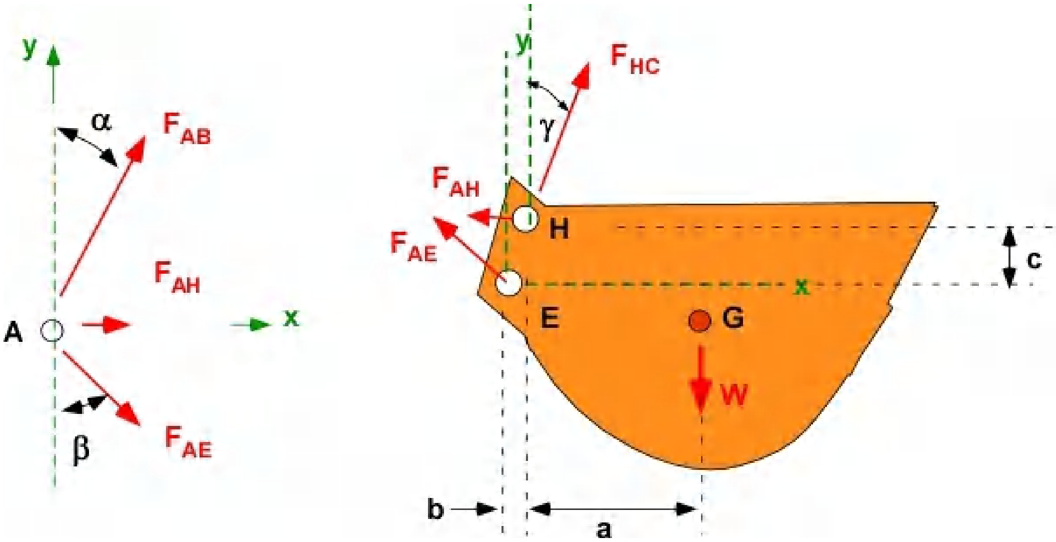


Figure 6.7.4 FBD of the pin A and the bucket.

Pin A:

$$\sum F_x = 0 = F_{AH} + F_{AB}\sin\alpha + F_{AE}\sin\beta$$
$$\sum F_y = 0 = F_{AB}\cos\alpha - F_{AE}\cos\beta$$

Bucket:

$$\sigma_{max}(P) = \frac{P}{A}\left[1 + \frac{ec}{k^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right];$$

We have 5 equations for four variables $\mathbf{F_{AB}}$, $\mathbf{F_{AH}}$, $\mathbf{F_{AE}}$ and $\mathbf{F_{HC}}$. We need only four of the equations to solve the problem.

The problem looks complicated but actually turns out quite simple. This problem is very dynamic as this orientation is one of the many the backhoe will experience in its range of motion.

Discussion
Knowing the pins in the members we can design the cross section and the size of the pins by requiring the pins and the members to avoid failure.

The actual design itself may be designed by dynamic loading. For example the sharp lip of the bucket is hit hard against the ground to break up the earth before picking it up. That requires an **impact force** that may be many times the **dead loading** the bucket carries like in this problem. Dynamic analysis is beyond the scope of the book, but it can be reduced it to a case of static loading which can be handled by the methods we know.

6.7.2 Example 6.13

This section looks at a simple machine - the bicycle foot pump. Figure 6.7.5 shows the pump in two positions. The left position is the storage position and is held down using a simple hook to overcome the torsional moment produced by the coiled spring located under the dial and influencing the link with the foot rest. The hook can be seen on the far link. This spring provides a resisting moment. The right position is the open position from which the operation of the pump begins by using the foot to press the black foot rest. This is a dynamic machine and only works by the repeated pushing on the downward motion which pumps air through the attached hose. When this force is removed the pump restores the open position to begin the next pumping action. The rubber feet prevents the motion of the pump during action.

A simple two-dimensional essential static model (not to scale) is shown in Figure 6.7.6 for some instant during the pumping action. The rubber foot is modeled as a pin with reaction along the direction DC. The left support is modeled as a free pin. The applied force is 100 N inclined 20 degrees as shown. The resisting moment on the pump bar OAB is is 1.5 Nm for this configuration.

An important consideration is to decide if the resisting moment is an *external* moment or an *internal* moment. More specifically is the resisting moment applied on the pump through the bar. or only on the bar. If the moment M_{OA} is only applied on the bar OAB then the moment is *internal* and can be ignored for the FBD of the pump to calculate the reactions. This moment will be applied on the FBD of the link OAB only. In the analysis below this resisting moment/torque develops because of the application of P. It is considered an external moment and therefore it will be included in both calculations.

Latched with hook

Open

Figure 6.7.5 Simple machine - the foot pump.

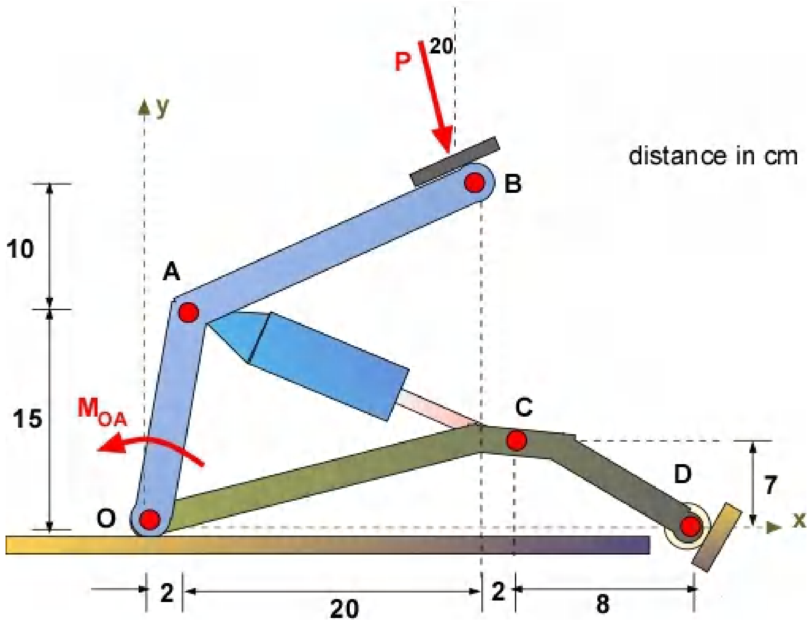


Figure 6.7.6 Two-dimensional model for the foot pump.

Data: [cm] :O(0, 0, 0); A(2, 15, 0); B(22, 25, 0), C(24, 7, 0); D(32, 0, 0);
P = 100 [N]; M_{OA} = 1.5 [Nm];
Pin support at O; Roller support at D

Find: (a) Force in the piston: F_{AC};

Assumption: Statically determinate, Connected members.
Force in link OAB is along the direction OB.
Force in link OCD at O is along the direction OC
Reaction at D is along DC

Solution: Apply Equilibrium equations to entire structure (reactions) and then link OAB to calculate the force in the piston

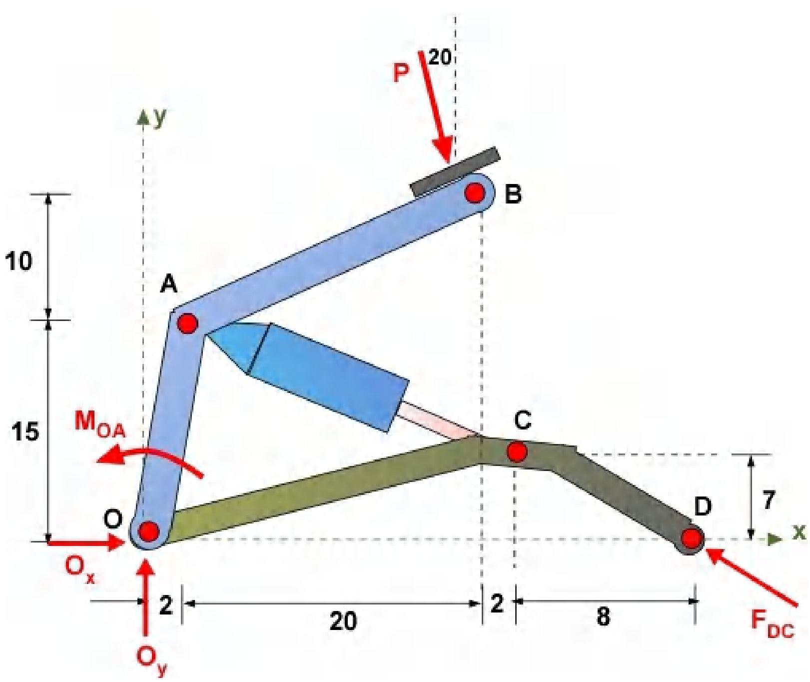


Figure 6.7.7 FBD of the foot pump.

Equilibrium of Pump:

$$\sum F_x = O_x + 100\sin 20 - F_{DC}\frac{8}{\sqrt{8^2+7^2}} = 0$$

$$\sum F_y = O_y - 100\cos 20 + F_{DC}\frac{7}{\sqrt{8^2+7^2}} = 0$$

$$\sum M_O = 1.5 - (100\sin 20)25 - (100\cos 20)22 + F_{DC}\frac{7}{\sqrt{8^2+7^2}}32 = 0$$

$$F_{DC} = 132[N]; \quad O_x = 64.8[N]; \quad O_y = 7.33[N];$$

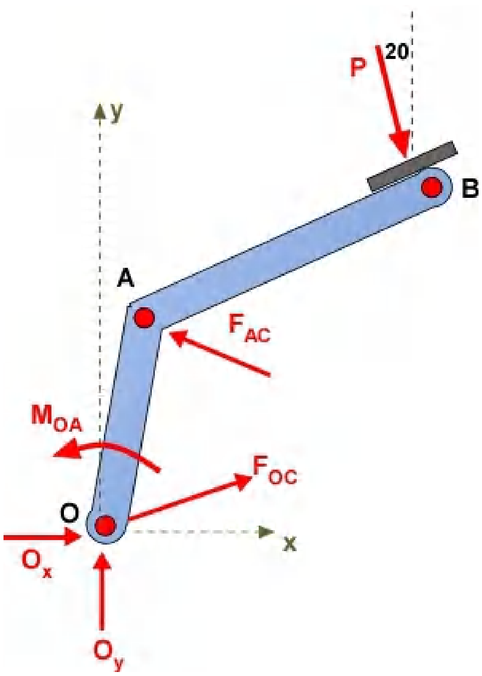


Figure 6.7.8 FBD of link OAB

Equilibrium of link OAB

$$\bar{r}_{OA} = 0.02i + 0.15j + 0k$$

$$\bar{r}_{OB} = 0.22i + 0.25j + 0k$$

$$\hat{e}_{OC} = 0.96i + 0.28j + 0k$$

$$\hat{e}_{CA} = -0.94i + 0.34j + 0k$$

$$\sum F_x = 64.8 + 34.2 + 0.96F_{OC} - 0.94F_{AC} = 0$$

$$\sum F_y = 7.33 - 93.97 + 0.28F_{OC} + 0.34F_{AC} = 0$$

$$\sum M_O = 1.5 - 34.2 \times 0.25 - 93.97 \times 0.22 + 0.94F_{AC} \times 0.15 + 0.34F_{AC} \times 0.02 = 0$$

$$F_{AC} = 188[N]; \quad F_{OC} = 80.5[N]$$

Only two of the three equations are necessary to solve for the forces.

6.7.3 Additional Problems

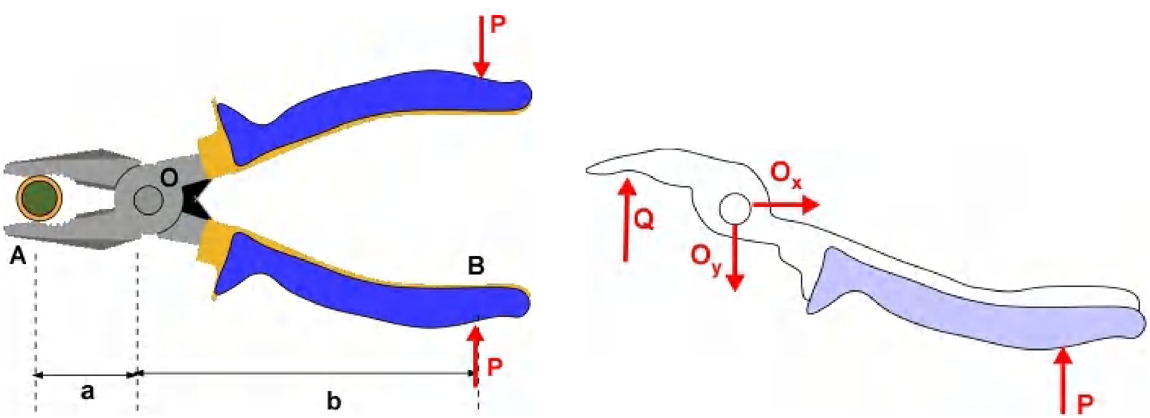
Set up the following problems by hand on paper and solve them using MATLAB/Octave and confirm some of your calculations using a calculator. For each problem you must draw the FBD and work with a coordinate system. Calculate stresses if information is available. If exploring design use a factor of safety of 6. Choose your material and area of cross-section and ensure you have an acceptable design. The problems in this section may appear complex but they are simple to solve with proper assumtions.

Please use Table 6.1 for your calculations if necessary

Problem 6.7.1

Consider the pliers as a simple machine. You apply equal and opposite force P to squeeze the bottle cap. What is the force on the bottle cap?

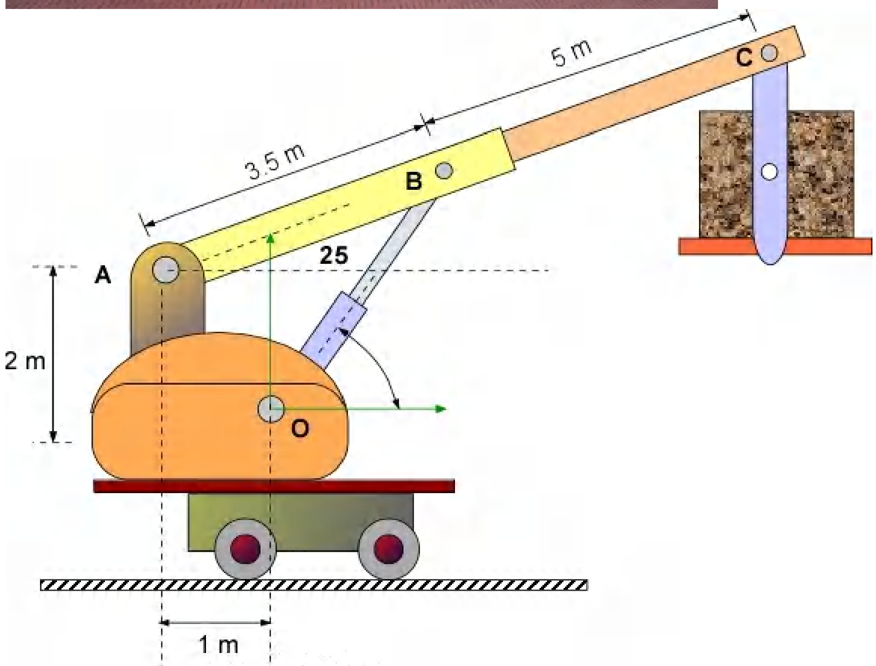
The pliers is not a single rigid body. It is made up of at least two parts that act as one - or connected rigid bodies. The FBD of one part of the pliers is also shown in the figure



Problem 6.7.1

Problem 6.7.2

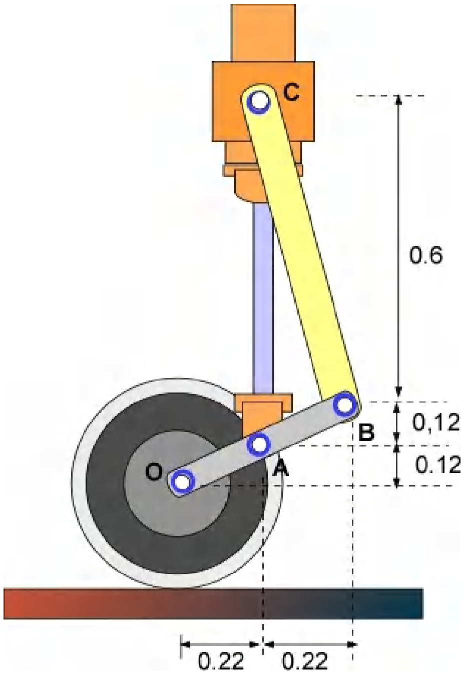
This problem looks at a cheery picker type of machine which is used to deliver heavy materials to different elevations. Below is an actual machine and also a simple two-dimensional model of it. Note that the machine must have dead weight to prevent from toppling. Take the weight being lifted as 5000 N. The center of mass is directly below C. (a) Calculate the force in the hydraulic piston. (b) Can you identify a counterweight and location so that both wheels are in contact with the ground.



Problem 6.7 2

Problem 6.7.3

The machine in this problem is a simple two-dimensional model of an aircraft landing gear. The aircraft is taxing on the runway. The wheel is providing a force of 25 kN at the pin O. Make your assumptions and calculate the force in the hydraulic piston at A.



Problem 6.7.3

7. BEAMS AND BENDING

We will start with our illustration of the gymnast on the parallel bars (seen in Chapter 6) and the corresponding FBD in Figure 7.1. If you have watched the event it is clear that the representation does not capture the true picture. There is a lot of give to the flexible bars and that can be seen by the bar actually deforming. It takes a bent shape and we classify this as bending shown on the right in the same figure. The same thing happens when a line of cars are parked on the bridge or the beam carrying the load of your entire house located in the basement. In these cases you cannot observe the bending because they are kept small by design. It is also clear that you do not want the wooden beam to fail with the gymnast holding his stance. He will injure himself and the manufacturer could be liable.

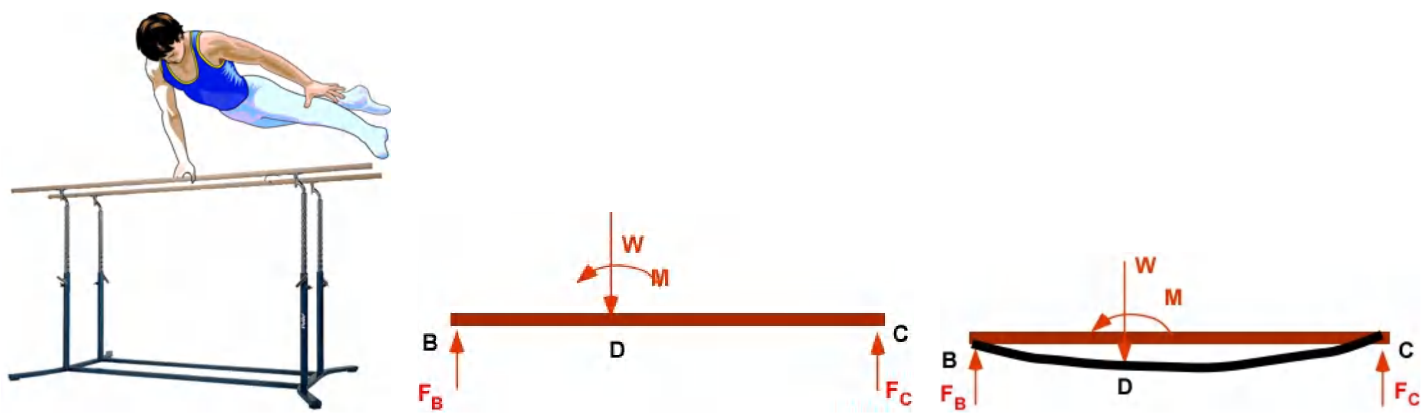


Figure 7.1 Bending

The standard design practice is to ensure the bar does not fail. This means that the stresses in the bars do not exceed the failure criteria. In addition the displacements must not exceed the elastic limit since then the bar will not return to its horizontal position when the gymnast dismounts making the next gymnast ill at ease.

The stresses and displacements in the bar are generated by the forces and moments present in the material throughout the bar. It is likely that these vary through the length of the bar. Note in the FBD above these are ***not*** shown. For example in the previous section the equal and opposite forces at the pin connections were not used to calculate the support reaction. It was mainly the external forces on the entire structure was use in the the first part of the analysis. The forces now are distributed along the bar rather than at a point. One important assumption used in these analysis is that these forces will be ***continuous***, except where the point loads are present where it is possible to observe jumps in forces and moments.

Example 7.1. Let us explore the gymnast on the bar as Example 7.1

Internal Forces and Moments

To understand and solve for these stresses and displacement we need to introduce a new concept - **internal** forces and moments. These will generate internal stresses and displacements. In the FBD in Figure 7.1 we see only concentrated forces and a moment at points B, D and C. These are considered external forces and point loads. **External force** is a force that acts on the surface of the beam.The reactions at the ends are obtained through the static equilibrium of the entire bar/beam are external forces too. We will use the term **beam** for these problems. We will regard the **beam** as a long structural member (of uniform cross section for the time being) carrying a variety of loads across its length. The question then is what happens between B and D or between D and C.

Consider a point E between B ad D and let us look at the segment of the beam BE. This is shown in Figure 7.2. If the entire beam is in equilibrium then every segment of the beam must be in equilibrium. Therefore BE must be in equilibrium. The end B carries a force $\mathbf{F_B}$. This force must be balanced for equilibrium by an equal and opposite force.

Where can it be located?

It cannot be along BE since if it existed it would appear on the FBD similar to the force F_B. The only possibility is that it must be at the end E - that is **in the cross-section of the beam** at E. This is why we call an **internal force**. Placing this force **V** at E with the same magnitude as F_B will balance the forces. However this creates a couple and therefore a moment, **M** must also be present to keep the segment BE in equilibrium and prevent rotation of the segment. If we looked at the segment CE instead we would require a force and a moment too. Both are shown Figure 7.2.

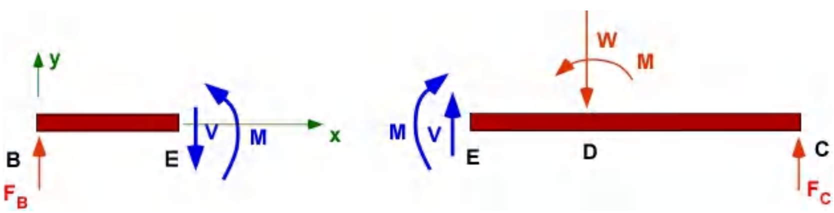


Figure 7.2 Internal force and moment

In Figure 7.2, **V** is the **internal force** and **M** is the **internal moment**. **V** is definitely a **shear force** as it is applied over the cross-sectional area. If the beam BC is made whole again then V and M will vanish since they will cancel each other. They only exist when the beam is cut an any point along the length to expose them . This is another reason they are called **internal loads**. The internal moment **M** is also called the **bending moment** since it will attempt to bend the beam. Note that this moment is directed along the z- axis.

The **direction** of this **internal** shear force and bending moment (also referred to as internal loads) as shown in Figure 7.2 is considered **positive** in further discussions. *We will take this for granted*. This is the sign convention. *This does not depend on the coordinate system*. We illustrate this with a succinct figure - the positive direction of the internal loads shown on the right or the left of the *cut* or ***sectioned*** beam. We should *section the beam on only one side* during any calculation.

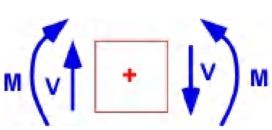


Figure 7.3 Sign Convention:Positive internal loads

Reactions:

Let us look at the problem with numerical values. The overall bar is 3.5 m long and let us say the supports are 3 m apart. The gymnast is balancing 1.4 m from the end B. The gymnast weighs 165 kg and the moment he applies is 500 Nm at the point D. This is a statically determinate problem and we can calculate the forces F_B as 1029.95 N which we will round to 1030 N. The force F_C is 588.7 or 589

N. The weight is rounded to 1619 N.

Internal Shear Force and Bending Moment Distribution

The internal loads that we defined before are rarely constant across the beam and depend on the external loading. They can vary in strength and direction along the beam and can be solved by applying the equations of equilibrium by sectioning the beam at different points along the length - location ***x***. For the example in Figure 7.2 the point E was arbitrary and is representative of any point between B and D. Let us also consider point G between the point D and C which is representative of all points between C and D. These points are located at the generic point ***x*** - measured from the left end where the origin of the coordinate system is located. Let us section the beam at E and G and consider the equilibrium of the sectioned beam

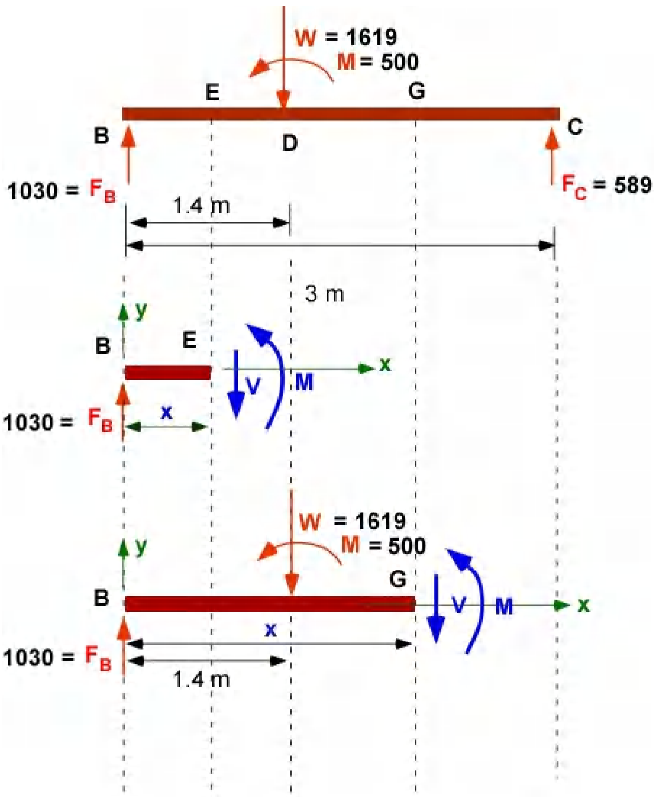


Figure 7.4 FBD for developing the shear and bending relations

$$\text{Section } BE: \quad 0 \le x < 1.4$$
$$\sum F_y = 0 = 1030 - V(x); \quad V(x) = 1030 [N]$$
$$\sum M_B = 0 = M(x) - Vx; \quad M(x) = 1030\,x [Nm]$$

$$\text{Section } BG: \quad 1.4 < x < 3$$
$$\sum F_y = 0 = 1030 - 1619 - V(x); \quad V(x) = -589 [N]$$
$$\sum M_B = 0 = 500 + M(x) - 1619 \times 1.4 - Vx;$$
$$M(x) = 2266.6 - 500 + Vx = 1766.6 - 589\,x \quad [Nm]$$

We have the internal shear force V(x) and the internal bending moment M(x) calculated over the beam. The shear force appears to be constant in each section. The bending moment is linear and is a function of the distance from the end B. The next step is to plot them to provide a better picture. An important information is the location of the maximum values of the shear force and bending moment. The beam is more likely to fail at that point. Since we know about failure and they depend on principal stress we should consider both the shear force and bending moment simultaneously to identify failure. We will deal with this later as we learn how to translate the shear force and bending moment to stresses.

Why did we only calculate for *two* segments above. This should be obvious from the model of the problem. The shear force and bending moment are expected to be ***continuous*** functions unless it comes across point loads where these continuous functions are expected to change. We must section the beam over regions where the loadings are different and develop the shear force and bending moment relations for each of them.

Shear Force and Bending Moment Diagram

We will use the calculations in the previous section to draw the shear force and bending moment diagram along the beam.

Shear Force Diagram

Between BD the shear force has a constant value of 1030 N. Since the left end at B has a force directed up (+y), according to our sign convention this will be positive. In the section DC the shear force has a negative value and we will draw a constant negative value. We can see that there is a jump at D of exactly 1619 - which is the value of W. Note that is is in the same direction of W. We look for these things to build intuition and very often we can avoid using algebra like we did in the previous section to establish the shear force diagram.

Bending Moment Diagram

There is no bending moment at B and C. If there is it would appear on the beam FBD. In section BE the moment increase linearly until D. In the section DC the bending moment is again linear but has a slope of -589. There is a jump in the bending moment at the point D of value 500 - which is the value of the external moment at D. The numerical values suggests that the moment is zero at C. Unlike the shear force there is negative jump for a positive (coordinate wise) lumped external moment at D. Remember to build this in as part of your intuition.

These diagrams are shown in Figure 7.5

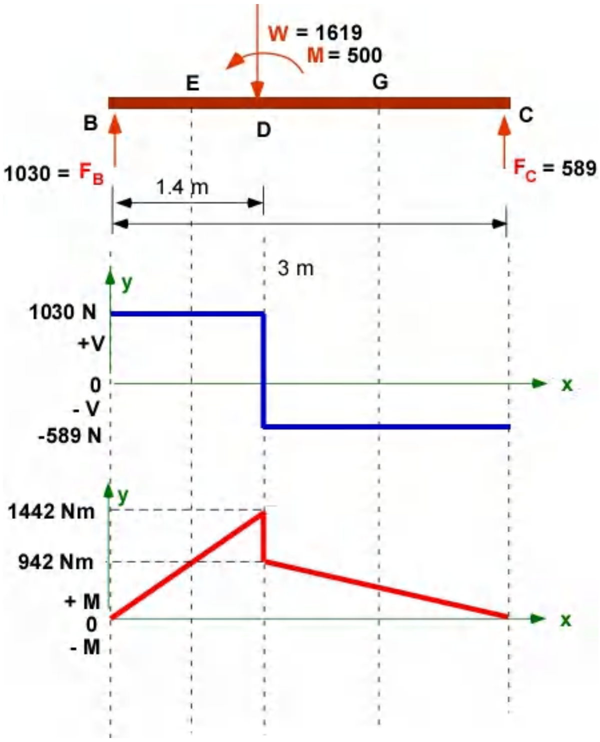


Figure 7.5 Shear force and bending moment diagram

Relations between Distributed force, Shear Force and Bending Moment through Calculus

We see that we can represent the shear force and bending moment diagrams as piecewise continuous functions with jumps along the length of the beam. The derivation through equilibrium also suggest that these functions are related to each other. Calculus provides an efficient procedure to establish the shear force and bending moment diagram. We will explore it in the next section with a new example.

7.1 LOAD, SHEAR FORCE, AND BENDING MOMENT

We will use an example to help develop the analysis of bending and its associated topics.

Example 7.2: Example 7.2 is used to anchor the analysis. It is shown in Figure 7.1.1. It is a challenging example to develop the analysis for the first time but it has many of the features in beam bending problems. Example 7.1 was a very simple example and provides expectation of what is involved in these problems. Example 7.2 is used to explore many ways of solving beam bending problems. The example is based on a cantilevered beam with a concentrated load of 500 N applied off the beam through a bracket at H. There are two distributed loads. One is constant at 300 N/m. The other is a triangular distribution with a peak of 500 N at the right end of the beam. This is a *statically determinate* problem since there are only three unknown reactions at the left end A.

The first simplification is to bring all the loads to bear on the beam itself by eliminating the bracket. This transfers as a concentrated load and a concentrated moment at the point **D** of the beam (transferring a force parallel to itself must include the force itself and an additional moment -Chapter 4). This is shown in the adjacent illustration in Figure 7.1.1. This is the final problem to be analyzed.

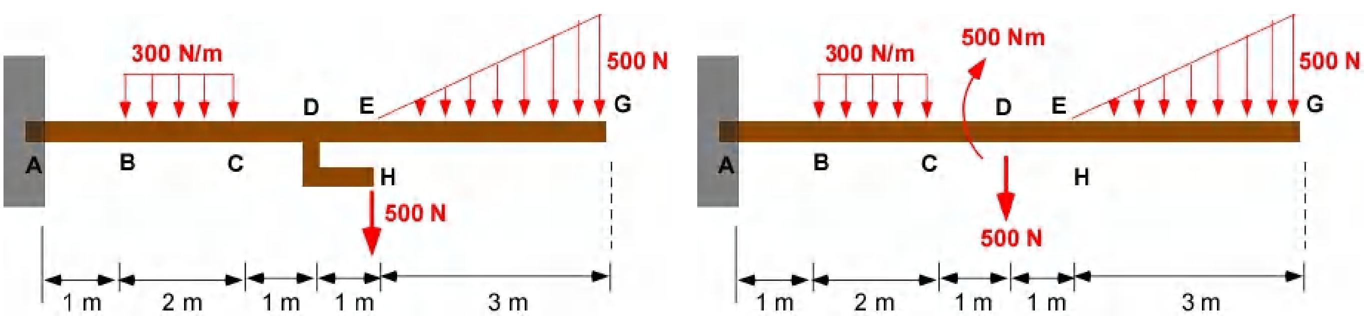


Figure 7.1.1 Example 7.2

7.1.1 Support Reactions

This example is solved in distinct steps for this class of problems - beam bending. Like in the previous sections the support reactions are first determined.
We will start with the FBD of the beam. This requires us to remove the cantilevered support at the end A and replace it by reactions. For the two dimensional problem this will involve two forces and a moment at the end A.

To draw the FBD we can replace the **distributed forces** by **concentrated forces**. The magnitude of the corresponding concentrated force is the area under the load distribution. The location of this concentrated force is at the **centroid** of this distribution. The centroid of the *constant* distributed load is at the halfway point. The centroid of the *triangular* distribution 2/3 of the base from the point E. This is *only used* for *the calculation of the reactions*. The calculation of shear force and moment distribution will require the actual distribution. The concentrated load representation of the FBD is shown in Figure 7.1.2.

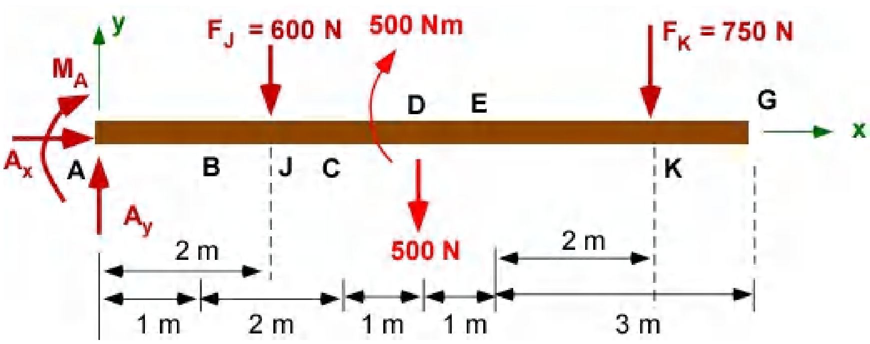


Figure 7.1.2 FBD of the beam

To calculate the reactions A_x, A_y and M_A we have three equations.

$$\Sigma F_x = A_x = 0;$$

$$\Sigma F_y = A_y - 600 - 500 - 750 = 0; \quad A_y = 1850;$$

$$\Sigma M_A = -M_A - (2\times 600) - 500 - (4\times 500) - (7\times 750) = 0; \quad M_A = -8950;$$

We could have also worked with the distributions directly. This will require applying simple calculus to set up the problem. It will help us in later steps. To start, we recognize that the coordinate ***x*** is measured from the end A. We then determine the two load distributions as $\mathbf{w_1(x)}$ = -300 and $\mathbf{w_2(x)}$ = 2500/3 - (500*x)/3. For each distribution we consider an infinitesimal distance ***dx*** at some point ***x*** within the distributions. The corresponding concentrated load at **x** is **w(x)dx**. The corresponding moment about A of this load is **x*w(x)dx**. Here **w(x)** is a *placeholder* for $w_1(x)$ or $w_2(x)$. Then

The total load produced by each distributed load is just the integral of the **w(x)dx** between the limits.

The moment produced by each distributed load at A is the integral of the **x*w(x)dx** between the limits.

This is illustrated in Figure 7.1.3.

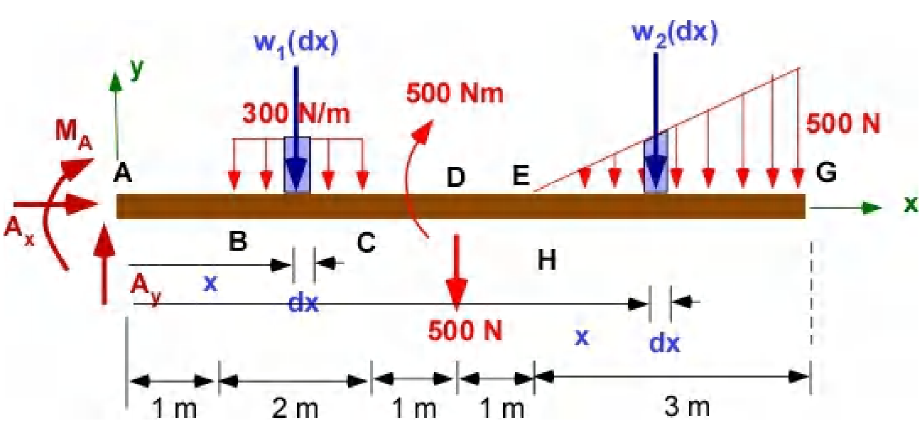


Figure 7.1.3 Calculating Force and Moments due to distributions

The equations of equilibrium are:

$$\sum F_x = 0 = A_x;$$

$$\sum F_y = 0 = A_y + \int_1^3 w_1(x)\,dx - 500 + \int_5^8 w_2(x)\,dx;$$

$$\sum M_A = -M_A + \int_1^3 w_1(x)\,x\,dx - (4\times 500) - 500 + \int_5^8 w_2(x)\,x\,dx;$$

Solution Using MATLAB
In the Editor

```
% Essential Mechanics Dec 2015
%  Example 7.2 - Beam (2D)
%  Section 7.1.1
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(5)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 7.2\n')
fprintf('-------------\n')
%% Data
%% Data
% parameterizing location and load
a = 0; b = 1;  c = 3; d = 4; e = 5; g = 8; % parameterizing the data
wd = -500;  md = -500;
w1 = -300;  % constant load value
w2min = 0; w2max = -500;  x21 = e;  x22 = g;  % linear load with limits

% coordinates
A = [a,0,0];  B = [b,0,0];  C = [c,0,0];  D = [d,0,0]; E = [e,0,0];
G = [g,0,0];

% calculate load distribution
syms x a2 b2
w1x = w1;  % constant load
% linear load distribution  w2(x) = a2 + b2*x
w2x = a2 + b2*x;
Eq(1) = a2 + b2*x21-w2min;
Eq(2) = a2 +b2*x22 - w2max;
[a2, b2]  = solve(Eq);
w2x = subs(w2x);  % substiture for known values

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Reactions _Equilibrium
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
syms Ax Ay Ma real
FA = [Ax,Ay,0];  MA = [0,0,-Ma];  FD = [0,wd,0];  MD = [0,0,md];
AD = D-A;
% force equilibrium
SumF = FA + FD +[0,int(w1x,x,b,c),0] + [0,int(w2x,x,e,g),0];

% moment equilibrium
x1 = [x,0,0];  x2 = [x,0,0];  w1v = [0,w1x,0];  w2v = [0,w2x,0];
```

```
SumMA = MA + MD + ...
    int(cross(x1,w1v),x,b,c) + ...
    int(cross(x2,w2v),x,e,g) + ...
    cross(AD,FD);

sol = solve(SumF(1),SumF(2),SumMA(3));

Ax = (sol.Ax);
Ay = (sol.Ay);
Ma = (sol.Ma);

%%%%%%%%%%%%%%%%%%%%%%
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('Coordinates')
fprintf('\n-----------------------')
fprintf('\nPoint A [m]   : '),disp(A)
fprintf('Point B [m]   : '),disp(B)
fprintf('Point C [m]   : '),disp(C)
fprintf('Point D [m]   : '),disp(D)
fprintf('Point G [m]   : '),disp(G)


fprintf('\nLoads')
fprintf('\n-----------------------')
fprintf('\nLoad at D [N]      = '),disp(wd)
fprintf('Moment at D[N/m]    = '),disp(md)
fprintf('w1(x)[N/m]          = '),disp(w1x)
fprintf('w2(x)[N/m]          = '),disp(w2x)

fprintf('\nEquilibrium - Beamn\n')
fprintf('---------------------------')
fprintf('\nSumF : \n'),disp(vpa(SumF',4))
fprintf('SumMA: \n'),disp(vpa(SumMA',4))

fprintf('------------\n')
fprintf('Reactions:\n')
fprintf('------------\n')
fprintf('Ax [N] = '),disp(double(Ax))
fprintf('Ay [N] = '),disp(double(Ay))
fprintf('Ma [Nm]= '),disp(double(Ma))
```

In Command Window

```
Example 7.2
------------
Coordinates
----------------------
Point A [m]   :      0      0      0
Point B [m]   :      1      0      0
Point C [m]   :      3      0      0
Point D [m]   :      4      0      0
Point G [m]   :      8      0      0

Loads
```

```
----------------------
Load at D [N]      =   -500
Moment at D[N/m]    =   -500
w1(x)[N/m]          =   -300
w2(x)[N/m]          = 2500/3 - (500*x)/3

Equilibrium - Beamn
---------------------------
SumF :
          Ax
 Ay - 1850.0
           0
SumMA:
                 0
                 0
 - 1.0*Ma - 8950.0
------------
Reactions: is
------------
Ax [N] =      0
Ay [N] =         1850
Ma [Nm]=        -8950
```

The values are the same obtained through calculation.

Execution in Octave

The code is same as in the MATLAB above - with the highlighted changes

In Octave Editor

```
clc, clear, format compact, format shortg, close all,
warning off
pkg load symbolic;

%w2x = subs(w2x);  % substitution for known values
a2 = double(a2);
b2 = double(b2);
w2x=subs(w2x,'a2',a2);
w2x=subs(w2x,'b2',b2);

fprintf('\nSumF : \n'),disp(vpa(SumF,4))
fprintf('SumMA: \n'),disp(vpa(SumMA,4))
```

In Octave Command Window

The output values are the same to MATLAB.

```
Example 7.2
------------
Symbolic pkg v2.7.1: Python communication link active, SymPy v1.3.
Coordinates
----------------------
Point A [m]  :   0  0  0
Point B [m]  :   1  0  0
Point C [m]  :   3  0  0
```

```
Point D [m]  :   4  0  0
Point G [m]  :   8  0  0

Loads
----------------------
Load at D [N]      = -500
Moment at D[N/m]    = -500
w1(x)[N/m]          = -300
w2(x)[N/m]          =
    500*x   2500
  - ----- + ----
      3      3

Equilibrium - Beam
---------------------------
SumF :
  [Ax  Ay - 1850.0  0]
SumMA:
  [0  0  -Ma - 8950.0]
------------
Reactions:
------------
Ax [N] = 0
Ay [N] = 1850
Ma [Nm]= -8950
```

7.1.2 Relating Load, Shear Force, and Bending Moment

The assumption in this derivation are:

- The load, the shear force, and the bending moment are varying with the distance **x** along the beam - they are all functions of ***x***.
- They are piecewise continuous functions of ***x,*** where ***x*** is measured from one end of the beam - preferably the left end of the beam.
- The beam is of uniform cross section
- The **y** coordinate is positive upwards.
- The origin is at the centroid of the cross-section (derived later)
- It is standard in most derivation to assume that the shear force and moment increase in the x - direction.
- There is a sign convention involved in setting the positive directions of the shear and bending moments is the cross-section (Figure 7.3)

We consider an infinitesimal distance **dx** along the beam and draw the FBD of this portion with the load distribution considered constant (the variation over dx will not matter as we will be using order of magnitude analysis). The increase in shear force and the bending moment are differential (infinitesimal) too. This is shown in Figure 7.1.4 adjacent to the FBD.

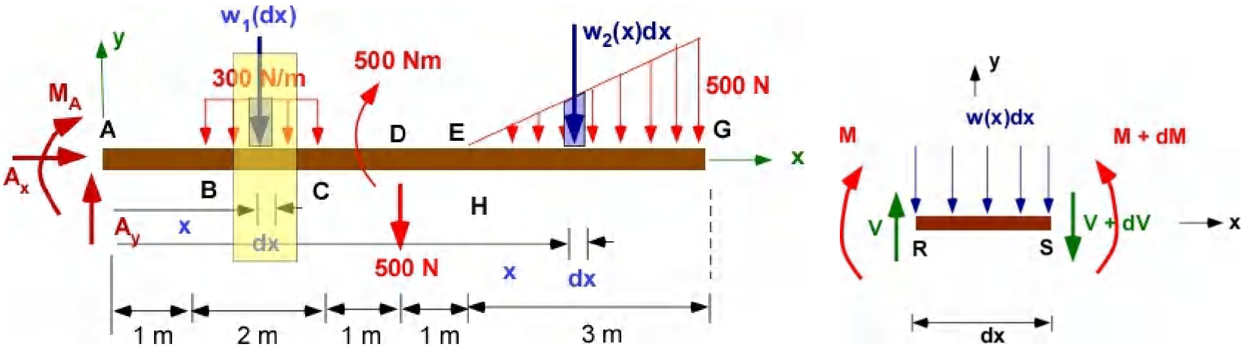


Figure 7.1.4 The FBD of a small beam element (right)

The equilibrium of this beam element using order of magnitude argument

$$\sum F_y = V-(V+dV)-w(x)dx = 0 = -dV - w(x)dx;$$

$$\frac{dV}{dx} = -w(x) \quad \left[w(x)\ \textit{is biased to be negative}\right]$$

$$\sum M_R = 0 = -M + (M+dM) - w(x)dx\left(\frac{dx}{2}\right) - (V+dV)\,dx$$

$$Vdx \gg w(x)\frac{dx^2}{2} \gg dV\,dx$$

$$\frac{dM}{dx} = V = V(x)$$

The negative sign in the first differential relation is due to the fact that w(x) is drawn downward - in the negative 'y' direction. The differential relations have corresponding integral components which are used to quickly sketch the shear force and bending moment diagrams between limits x_1 and x_2 where the function is continuous. These relations are:

$$\frac{dV}{dx} = -w(x); \quad V_2 - V_1 = \int_{x_1}^{x_2} -w(x)\,dx \qquad (7.1)$$

$$\frac{dM}{dx} = V(x); \quad M_2 - M_1 = \int_{x_1}^{x_2} V(x)\,dx \qquad (7.2)$$

7.1.3 Calculation of the Shear Force and Bending Moment - Example 7.2

There is a sign convention associated with further discussion. This is **independent of the coordinate system**. In Figure 7.1.5 (also Figure 7.3) the positive directions for the shear and bending moment on either side of the beam is shown along with the coordinate system. The moment at the left end is positive even though it is clockwise - with respect to the coordinate system. Similarly the shear on the right side is positive even if it is shown in the negative y direction.

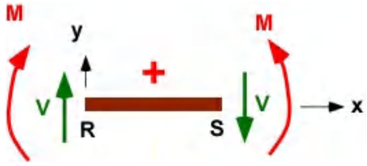


Figure 7.1.5 Positive Sign Convention

We start with the FBD of the beam with the given loading and the reactions in Figure 7.1.6. We then solve for the shear force and bending moments in the different segments of the beam using Eqn. 7.1 and 7.2 for each segment. These **segments are visually determined**. They are identified when there is a change in distributed loading or concentrated forces or concentrated moments. For Example 7.2 from Figure 7.1.6 these segments are **AB**, **BC**, **CD**, **DE**, and **EG**. You must identify these segments before proceeding. Note in Figure 7.1.6 the load distribution is also included.

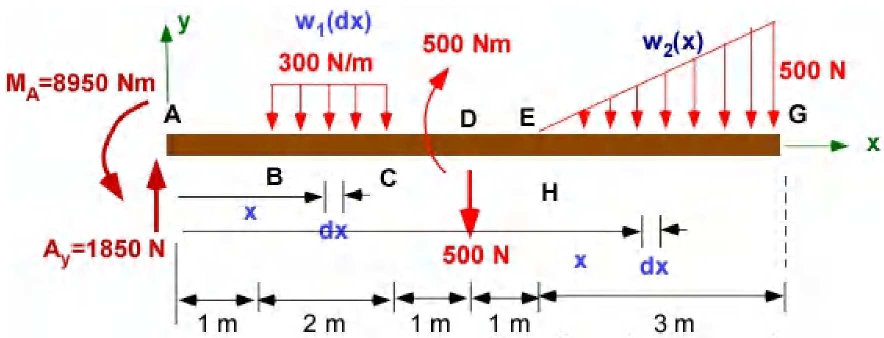


Figure 7.1.6 The Beam with the loading

The functions describing ***w(x)***, ***V(x)***, and ***M(x)*** in various segments are derived below. The positive convention in Figure 7.1.5 applies. In the analysis below Eqn. 7.1 and 7.2 are applied both in *integral* form and *differential* form. In integral form they establish the loads at the limits. In differential form they establish the nature of the distribution in between. Starting with w(x) in the segment the shear force and bending moment are calculated.

Section AB:

$$AB: \quad 0 \le x \le 1$$

$$w(x) = 0 = \frac{dV}{dx}$$

$$V_B - V_A = \int_{x_A}^{x_B} (0)\, dx; \quad V_B = V_A = 1850$$

$$M_B - M_A = \int_{x_A}^{x_B} V(x)\, dx; M_B = -8950 + 1850 \times 1 = -7100$$

$$\frac{dV}{dx} = 0; \quad V(x) = C = 1850;$$

$$\frac{dM}{dx} = V = 1850; \quad M(x) = 1850x + C; \quad M(0) = -8950 = C; \quad M(x) = 1850x - 89$$

$$check: \quad M_B = M(1) = 1850 - 8950 = -7100$$

Section BC:

$$BC: \quad 1 \le x \le 3$$

$$w(x) = -300 = \frac{dV}{dx}$$

$$V_C = V_B + \int_1^3 -300\,dx = 1850 - 300(3-1) = 1250;$$

$$\frac{dV}{dx} = -300; \quad V(x) = -300x + C; \quad C = V(1) + 300 = 2150; \quad V(x) = 2150 - 300x; \quad V_C = 2150 - 6$$

$$M_C = M_B + \int_1^3 (2150 - 300x)\,dx = -7100 + 2150(3-1) - \frac{300}{2}\left(3^2 - 1^2\right) = -4000;$$

$$\frac{dM}{dx} = 2150 - 300x; \quad M(x) = 2150x - 150x^2 + C; \quad C = -7100 - 2150 + 150 = -9100;$$

$$M(x) = -9100 + 2150x - 150x^2; \quad check: M_C = -9100 + 2150 \times 3 - 150 \times 3^2 = -4000;$$

Section CD:

$$CD: \quad 3 \le x < 4\left(just\ before\ D\right)$$

$$w(x) = 0 = \frac{dV}{dx};$$

$$V_{D-} = V_C = 1250; \quad V(x) = C = 1250;$$

$$M_{D-} = M_C + \int_3^4 1250\,dx = M_C + 1250(4-3) = -4000 + 1250 = -2750;$$

$$\frac{dM}{dx} = 1250; \quad M(x) = 1250x + C; \quad C = -4000 - 1250 \times 3 = -7750;$$

$$M(x) = -7750 + 1250x; \quad check: \quad M_{D-} = -7750 + 1250 \times 4 = -2750;$$

$$V_{D+} = 1250 - 500 = 750;$$

$$M_{D+} = -2750 + 500 = -2250;$$

Section DE:

$$DE: \quad 4 \le x < 5$$

$$w(x) = 0 = \frac{dV}{dx};$$

$$V_E = V_{D+} = 750; \quad V(x) = 750;$$

$$M_E = M_{D+} + \int_4^5 750\,dx = -2250 + 750(5-4) = -1500;$$

$$\frac{dM}{dx} = 750; \quad M(x) = 750x + C; \quad C = -2250 - 750 \times 4 = -5250;$$

$$M(x) = -5250 + 750x; \quad check: \quad M_E = -5250 + 750 \times 5 = -1500;$$

Section EG: (the original figure is changed to fit the page)

$$EG: \quad 5 \le x \le 8$$

$$w(x) = \left(\frac{2500}{3}\right) - \left(\frac{500}{3}\right)x = \frac{dV}{dx};$$

$$V_G = V_E + \int_5^8 \left[\left(\frac{2500}{3}\right) - \left(\frac{500}{3}\right)x\right]dx = 750 + \left(\frac{2500}{3}\right)(8-5) - \left(\frac{500}{6}\right)(8^2 - 5^2) = 0;$$

$$\frac{dV}{dx} = \left(\frac{2500}{3}\right) - \left(\frac{500}{3}\right)x; \quad V(x) = \left(\frac{2500}{3}\right)x - \left(\frac{500}{6}\right)x^2 + C; \quad C = 750 - \left(\frac{2500}{3}\right)\times 5 + \left(\frac{500}{6}\right)\times 5^2 = -\frac{4000}{3};$$

$$V(x) = \left(\frac{2500}{3}\right)x - \left(\frac{500}{6}\right)x^2 - \frac{4000}{3};$$

$$M_G = M_E + \int_5^8 \left[\left(\frac{2500}{3}\right)x - \left(\frac{500}{6}\right)x^2 - \frac{4000}{3}\right]dx = -1500 + \left(\frac{2500}{6}\right)(8^2 - 5^2) - \left(\frac{500}{18}\right)(8^3 - 5^3) - \left(\frac{4000}{3}\right)(8-5) = 0;$$

$$\frac{dM}{dx} = \left(\frac{2500}{3}\right)x - \left(\frac{500}{6}\right)x^2 - \frac{4000}{3}; \quad M(x) = \left(\frac{2500}{6}\right)x^2 - \left(\frac{500}{18}\right)x^3 - \frac{4000}{3}x + C; \quad C = -1500 + \frac{4000}{3}5 - \left(\frac{2500}{6}\right)5^2 + \left(\frac{500}{18}\right)5^3 = -17777.77$$

$$M(x) = \left(\frac{2500}{6}\right)x^2 - \left(\frac{500}{18}\right)x^3 - \frac{4000}{3}x - 1777.77;$$

$$check: \quad M_G = \left(\frac{2500}{6}\right)8^2 - \left(\frac{500}{18}\right)8^3 - \frac{4000}{3}8 - 1777.77 = 0;$$

These calculation, specially the values at the end points of the segments should help us sketch the shear force and bending moment over the beam. Remember to use the slope information to improve the sketch. Let us summarize the set of hoary calculations above for statically determinate problems. Keep in mind that you are establishing functions that are only **piecewise continuous**.

Step 1. Establish the reactions for the overall beam.

Step 2. Break the beam up into segments where you will establish the functions. This is done visually.

Step 3. Start with the left end (x = 0) and for each segment ***a*** <= ***x*** < **b**

- **Step i.** Identify w(x) - It is negative if it is downward
- **Step ii**. Solve for V(x) using indefinite integration and using the left end for establishing integration constants
- **Step iii.** Solve for V_b from definite integration and V_a introducing any jumps if necessary at b
- **Step iv**. Solve for M(x) using indefinite integration and using the left end for establishing integration constants
- **Step v.** Solve for M_b from definite integration and M_b introducing any jumps if necessary at b

Step 4. Sketch the shear force and bending moment diagram using end point value at the segments (next section).

7.1.4 The Load, Shear, and Bending Moment Diagrams

To draw the diagrams we will use the functions identified in the above section along with the values of shear force and bending moments at the end points. We also use nature of the function and the slope of the function to sketch of the distribution. It is important that these functions are piecewise continuous with jumps only at the end points of the sections of the beam.

There are three distributions drawn below each other with guidance provided by the FBD of the entire beam with reactions on the top. These stacked figures have the same transition locations. The first distribution is the distributed load w(x). This is under the FBD. The shear force distribution V(x) is below w(x). The lowest is M(x) distribution. Each distribution depends on the one immediately above it.

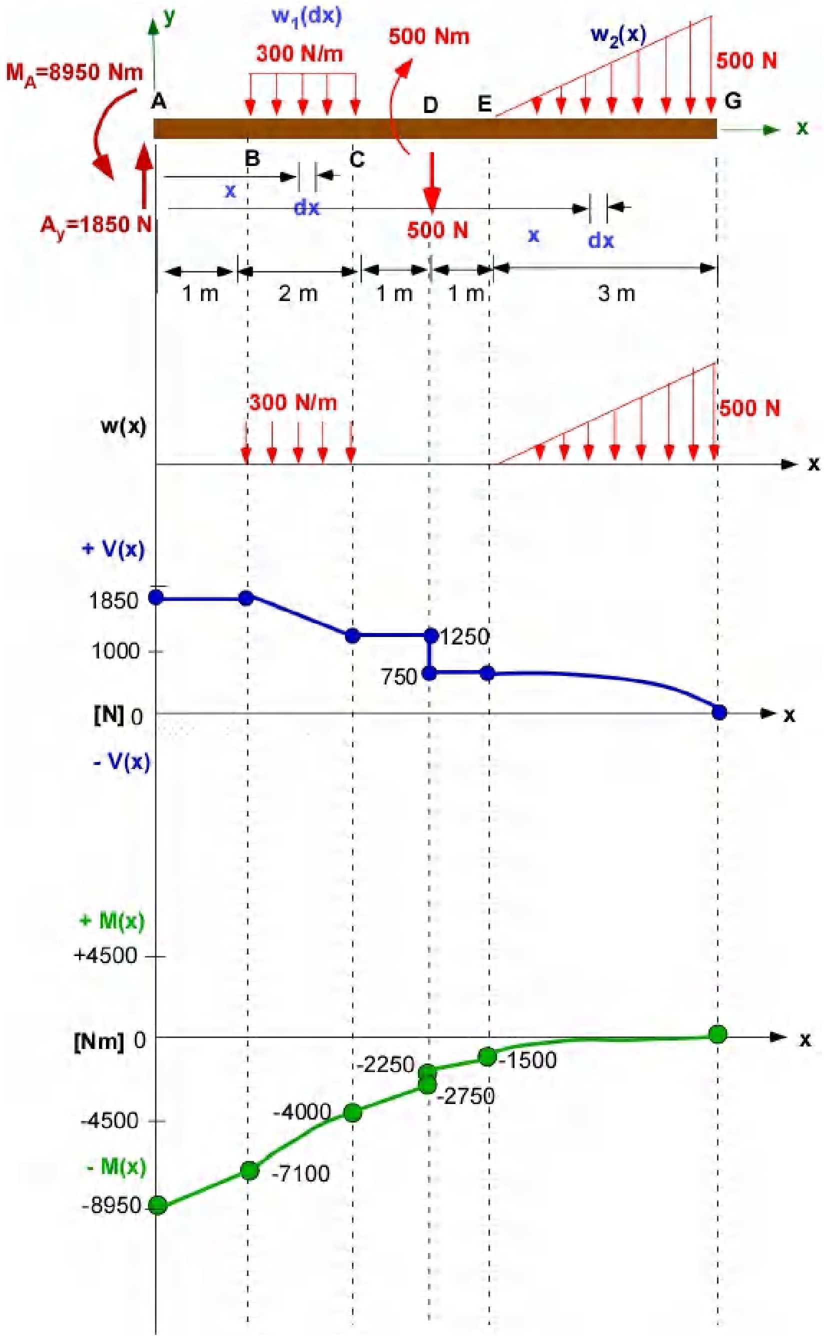


Figure 7.1.7 Load, shear, and bending diagram

7.1.5 The Intuitive Approach to Shear, and Bending Moment Diagrams

With practice it is possible, for simple distributions, to draw these diagrams intuitively. The point load only occur at the ends of the segments. Within a segment the integral of a continuous function is replaced by the area under the curve. The point loads cause appropriate jumps in the forces and moments at the locations they occur. More specifically:

- The change in the shear force between the end of segments is the area under the w(x) curve
- The change in the bending moments between the ends of the segments is the area under the V(x) curve
- The jump in the shear force is in the same direction as the point force
- The jump is the bending moment is positive for a clockwise bending moment.

To do this for Example 7.2 let us start from the left end with Figure 7.1.8. The segments are still the same - **AB**, **BC**, **CD**, **DE**, and **EG.**

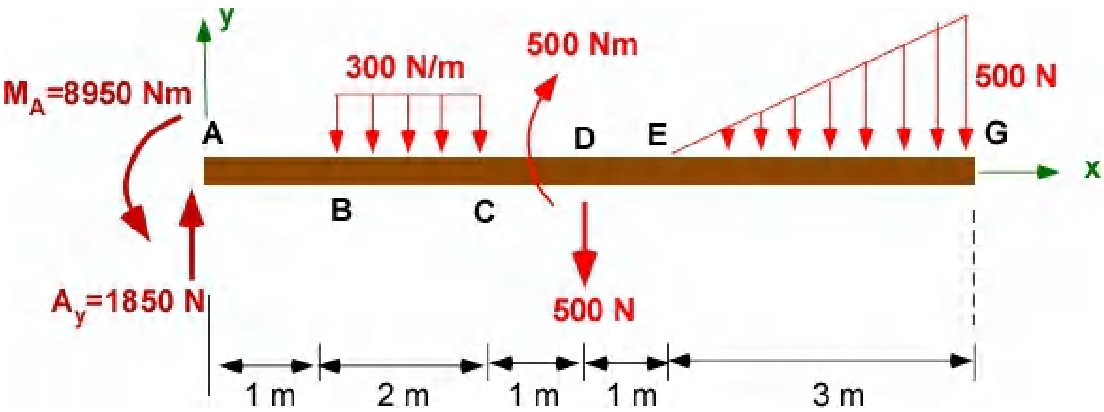


Figure 7.1.8 Distributed, shear, and bending loads and reactions

Shear Diagram
We will first draw the *shear force* diagram V(x). This depends on w(x) and point forces.

Section AB:

Figure 7.1.9a V(x) Section AB

- On the left end A_y is directed up. It is positive.
- w(x) is 0.   Shear at B is same as shear at A. V_B = 1850.

Section BC:

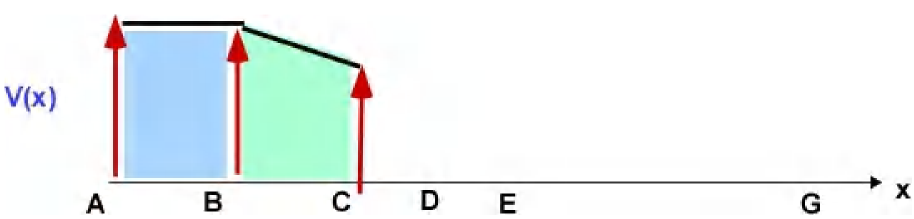


Figure 7.1.9b V(x) Section BC

- w(x) - 300;
- integral w(x) between B and C = -600.
- V_C = 1250
- Connect V_B and V_C by a straight line

Section CD:

Figure 7.1.9c V(x) Section CD

- w(x) is 0.   Shear at D is same as shear at C. V_D = 1250.
- At D there is a jump of - 500. Final V_D = 750

Section DE:

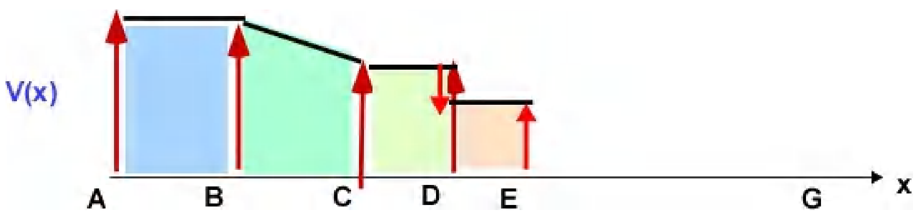


Figure 7.1.9d V(x) Section DE

- w(x) is 0.   Shear at E is same as shear at D. V_E = 750.

Section EG:

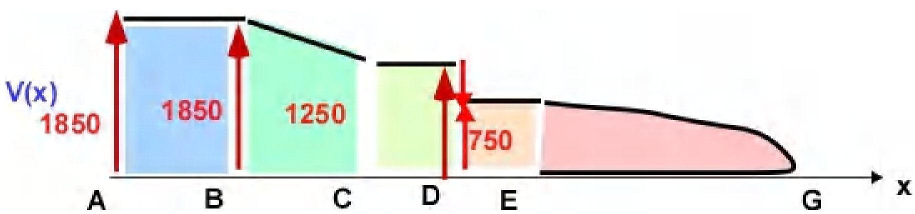


Figure 7.1.9e V(x) Section EG

- w(x) - linear;
- integral w(x) between E and C=G = -3*300/2=-750
- V_G= 750 - 750 = 0
- The curve between E and G is quadratic

Bending Moment Diagram
Now we will sketch the bending moment diagram M(x). This depends on V(x) diagram we just sketched and and point loads (moments).
We will start with the original loading Figure 7.1.8 and the V(x) diagram in Figure 7.1.9e included below for easy reference

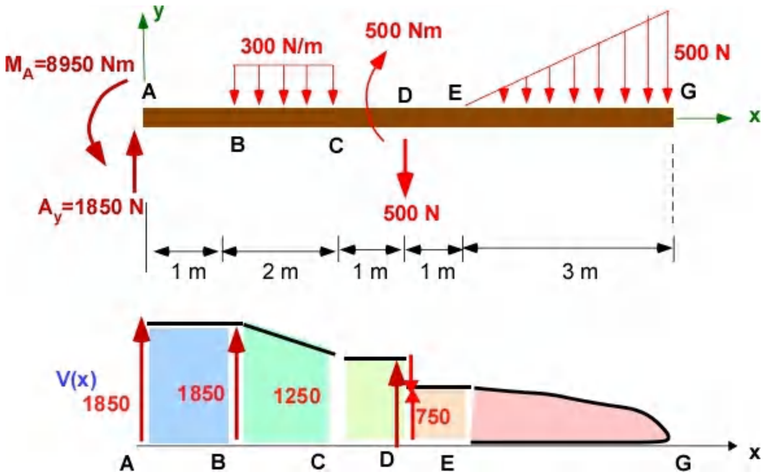


Figure 7.1.10 Figures for drawing Bending Moment Diagram

Section AB:

Figure 7.1.11a M(x) in Section AB

- On the left end M_A is clockwise and therefor negative. M_A = -8950.
- V(x) = constant = 1850. .   Area under the curve for V(x) between A and B is 1850*1 = 1850.
- M_B = -8950 + 1850 = -7100
- M(x) is linear between A and B

Section BC:

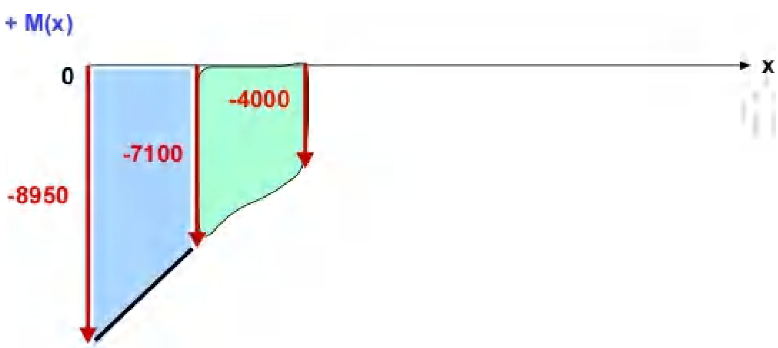


Figure 7.1.11b M(x) in Section BC

- V(x) is linear and area under V(x) curve between B and C is 0.5*(1850+1250)*2 = 3100
- M_C = -7100 + 3100 = -4000
- M(x) between B and C is quadratic with decreasing positive slope

Section CD :

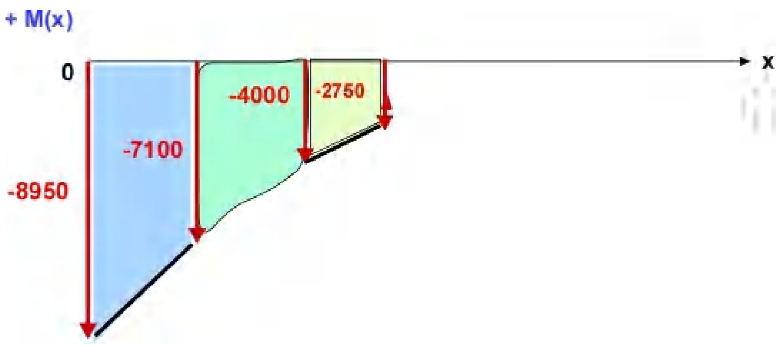


Figure 7.1.11c M(x) in Section CD

- V(x) = constant = 1250 . Area under the curve for V(x) between C and D is 1250*1 = 1250.
- M_D = -4000 + 1250 = -2750
- M(x) is linear between C and D
- There is a jump in moment at D of + 500. Final M_D = -2750 + 500 = -2250.

Section DE :

Figure 7.1.11d M(x) in Section DE

- V(x) = constant = 750 . Area under the curve for V(x) between D and E is 750*1 = 750.
- M_E = -2250 + 1750 = -1500
- M(x) is linear between D and E

Section EG :

Figure 7.1.11e M(x) in Section EG

- V(x) is quadratic between E and G. It is not trivial to evaluate the area under the V(x) curve between E and G (done above in Section 7.1.3)

- But $M_G = 0$ (from FBD)
- M(x) is cubic between E and G. An approximate sketch is shown using just the broad features.

Design Discussion:
For beam bending problems the shear force and bending moment diagrams are very important for design, particularly for constant cross-sectional area beams.

- The maximum shear force location identifies the location and maximum shear stress - shear failure
- The maximum bending moment identifies the location and the maximum normal stress - normal stress failure
- To use failure criteria you must calculate the location of the maximum principal stress - which will belong to a set of the above locations.
- The bending moment distribution is related to the beam displacement (will appear later in the book).

7.1.6 The Engineering Approach to Shear, and Bending Moment Diagram

The previous sections have demonstrated the formal approach to obtaining the shear force and bending moment distributions for beam bending problems. They were based on the loads on a small infinitesimal piece of the beam. This model provided simple ordinary differential equations/relations for piecewise continuous shear force and bending moments on the beam. The same results can be developed using an traditional engineering analysis involving FBD of portions on the beam as illustrated in Example 7.1 in the introduction to this Chapter. This engineering approach is based on the static equilibrium of portions of the beam and discovers the explicit relations for the shear force and bending moment distributions. We start with Figure 7.1.8 and show the corresponding FBD for each section in Figure 7.1.12. The beam is cut an at arbitrary point K for each section whose location is the value x measured from the left end A. The subsequent analysis is based on this figure. Please understand the figure before analysis. You should see a simple pattern.

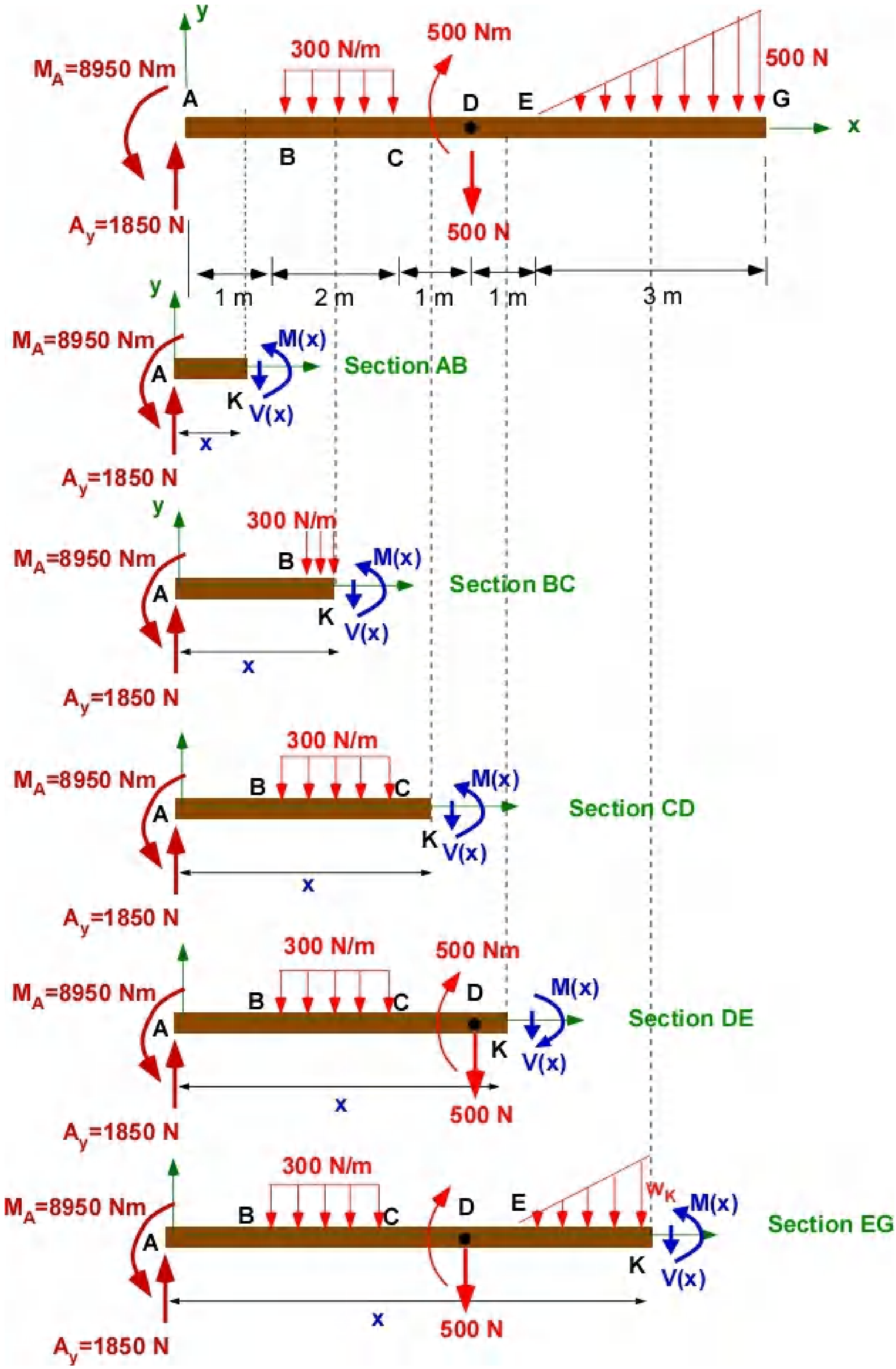


Figure 7.1.12 Distributed, shear, and bending loads and reactions

For each section we will write the two equilibrium equations to solve for V(x) and M(x). The FBD for each section is available in and identified in Figure 7.1.12. The solution for comparison is available in Section 7.1.3

Section AB:

$$\sum F_y = 0 = 1850 - V(x); \quad V(x) = 1850;$$

$$\sum M_A = 0 = 8950 + M(x) - xV(x); \quad M(x) = -8950 + 1850x;$$

Section BC:

$$\sum F_y = 0 = 1850 - 300(x-1) - V(x);$$

$$V(x) = 2150 - 300x;$$

$$\sum M_A = 0 = 8950 + M(x) - xV(x) - 300(x-1)\left(x - \frac{[x-1]}{2}\right);$$

$$M(x) = -9100 + 2150x - 150x^2;$$

Section CD:

$$\sum F_y = 0 = 1850 - 300 \times 2 - V(x);$$

$$V(x) = 1250;$$

$$\sum M_A = 0 = 8950 + M(x) - xV(x) - 300 \times 2 \times 2;$$

$$M(x) = -7750 + 1250x;$$

Section DE:

$$\sum F_y = 0 = 1850 - 300 \times 2 - 500 - V(x);$$

$$V(x) = 750;$$

$$\sum M_A = 0 = 8950 + M(x) - xV(x) - 500 \times 4 - 300 \times 2 \times 2;$$

$$M(x) = -5250 + 750x;$$

Section EG:

Prior to applying the equations we need to model the triangular load distribution on the FBD. as a point force and calculate its moment about A. Figure 7.1.13 is useful for this determination:

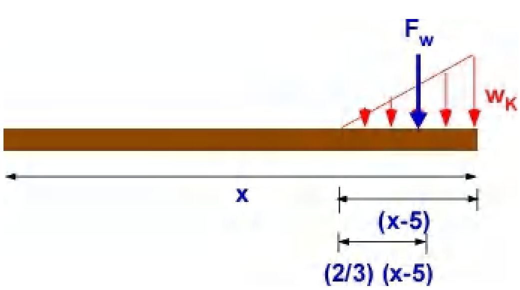


Figure 7.1.13 The triangular load distribution

$$w_K = \left(\frac{500}{3}\right)(x-5) = \left(\frac{500}{3}\right)x - \frac{2500}{3}$$

$$F_W = \frac{1}{2}(x-5)w_K = \frac{250}{3}(x-5)^2$$

$$\textit{Moment of } F_W \textit{ about } A = F_W\left(5+\frac{2}{3}[x-5]\right) = \frac{250}{3}(x-5)^2\left(5+\frac{2}{3}[x-5]\right)$$

$$= \frac{500}{9}(x-5)^3 + \frac{1250}{3}(x-5)^2$$

Equilibrium of the beam segment for section EG:

$$\Sigma F = 0 = 1850 - 300\times 2 - 500 - V(x) - \frac{250}{3}(x-5)^2;$$

$$V(x) = 750 - \frac{250}{3}(x-5)^2;$$

$$\Sigma M_A = 0 = 8950 + M(x) - xV(x) - 500\times 4 - 300\times 2\times 2 - \left\{\frac{500}{9}(x-5)^3 + \frac{1250}{3}(x-5)^2\right\};$$

$$M(x) = -5250 + \left\{750 - \frac{250}{3}(x-5)^2\right\}x + \left\{\frac{500}{9}(x-5)^3 + \frac{1250}{3}(x-5)^2\right\};$$

This form is more convenient then the results in Section 7.1.3. Please verify they are the same.

Design Discussion:

This section explored an example that was more than simple to outline several approaches to solve the beam bending problems. The example had point loads, two types of distributed loads, and a load applied on an appendage to the main beam. If you have understood this problem and the analysis you should be fairly comfortable with this type of problems. These are very important problems in structural design and architecture. Large civil structures, buildings, and even the single family home have beams to support the various loads like beds, furniture etc.

This section presented three ways to obtain the solution to beam bending problems.

- The first used calculus and was based on the simple differential equations for the beam element.
- The second was the simplest. It was an intuitive approach that relied on basic ideas from calculus.
- The third was based on our solid understanding of static equilibrium and consistent with the methods we have employed throughout the book so far. It is very consistent with they way we have analyzed problems until now. It is straight forward to apply and does not need any information from calculus, though it discovers the same solution.

The engineering approach is strongly recommended. Here is another idea for contemplation! What about using MATLAB/Octave to completely solve these problems as formulated in this section? It is clear that would involve much work, patience, and careful implementation. Probably it is better to do it on paper by the methods above rather than writing code to solve it since it depends so much on the layout of each section.

7.1.7 Example 7.3

This example is simpler than the previous one. It provides one more opportunity to use the engineering approach and explore how to code the the various segments in the analysis. The example is introduced in Figure 7.1.14.

Example 7.3

The beam AD is simply supported at end A and has a roller support at end D. A point load of 3000 N is applied as shown at B. A uniform distributed load of 500 N/m is applied between C and D. The length dimensions are available on the figure.

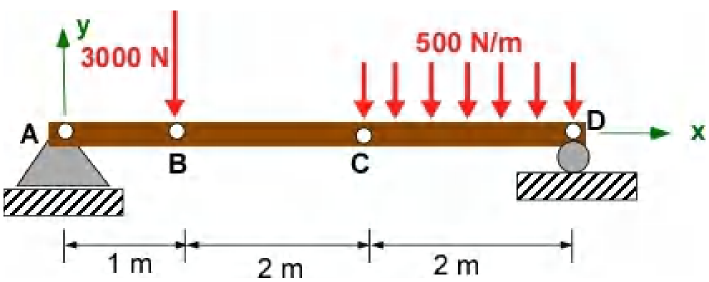


Figure 7.1.14a Example 7.3

Data: Beam, loading and locations are shown in the figure. A = [0,0,0]; B = [1,0,0]; C = [3,0,0]; D = [5,0,0];
F_B = -3000 N; w = -500 N/m.
Pinned at A; Roller at D - Statically Determinate.

Find: (a) Obtain reactions; (b) Obtain shear force and bending moment distributions; (c) Draw the load, shear force and bending moment diagrams.

Assumption: None

Solution: Use engineering approach to solve for the distributions
(a) Draw FBD of the complete beam and solve for the reactions

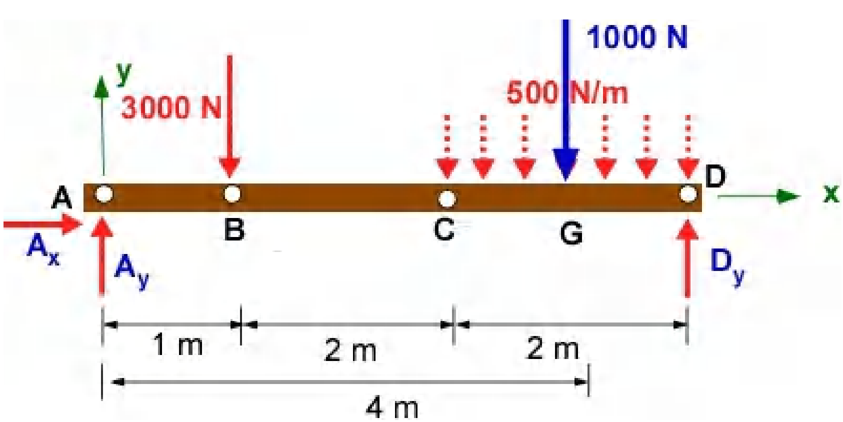


Figure 7.1.14b FBD of Example 7.3

$$\sum F_x = A_x = 0;$$

$$\sum F_y = A_y + D_y - 3000 - 1000 = 0;$$

$$\sum M_A = 0 = -(3000 \times 1) - (1000 \times 4) + 5D_y;$$

$$A_x = 0; \quad D_y = 1400[N]; \quad A_y = 2600[N];$$

Solution Using MATLAB

The MATLAB code is broken into specific analysis and should be assembled together for solving the entire problem.

In the Editor:

```
% Essential Mechanics Dec 2015
%  Example 7.3 - Beam (2D)
%  Section 7.1.7
%  P. Venkataraman
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(5)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 7.3\n')
fprintf('------------\n')
%% Data
w = -500;  dw = 2;
fb = -3000;  fg = w*dw;

% coordinates
A = [0,0,0];  B = [1,0,0];  C = [3,0,0];  D = [5,0,0];
G = [4,0,0];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Reactions _Equilibrium
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
syms Ax Ay Dy real
FA = [Ax,Ay,0];  FB = [0,fb,0];  FD = [0,Dy,0];  FG = [0,fg,0];

% force equilibrium
SumF = FA + FB + FG + FD;

% moment equilibrium
rAB = B - A;  rAG = G - A;  rAD = D - A;

SumMA = cross(rAB,FB) + cross(rAG,FG) + cross(rAD, FD);

sol = solve(SumF(1),SumF(2),SumMA(3));

Ax = (sol.Ax);
Ay = (sol.Ay);
Dy = (sol.Dy);

%%%%%%%%%%%%%%%%%%%%%%
```

```
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('Coordinates')
fprintf('\n-----------------------')
fprintf('\nPoint A [m]   : '),disp(A)
fprintf('Point B [m]   : '),disp(B)
fprintf('Point C [m]   : '),disp(C)
fprintf('Point D [m]   : '),disp(D)
fprintf('Point G [m]   : '),disp(G)


fprintf('\nLoads')
fprintf('\n-----------------------')
fprintf('\nLoad at B [N]       = '),disp(fb)
fprintf('w(x)[N/m]           = '),disp(w)

fprintf('\nEquilibrium - Beamn\n')
fprintf('----------------------------')
fprintf('\nSumF : \n'),disp(vpa(SumF',5))
fprintf('SumMA: \n'),disp(vpa(SumMA',5))

fprintf('------------\n')
fprintf('Reactions:\n')
fprintf('------------\n')
fprintf('Ax [N] = '),disp(double(Ax))
fprintf('Ay [N] = '),disp(double(Ay))
fprintf('Dy [Nm]= '),disp(double(Dy))
```

In Command Window:

```
Example 7.3
------------
Coordinates
-----------------------
Point A [m]   :      0     0     0
Point B [m]   :      1     0     0
Point C [m]   :      3     0     0
Point D [m]   :      5     0     0
Point G [m]   :      4     0     0

Loads
-----------------------
Load at B [N]       =        -3000
w(x)[N/m]           =    -500

Equilibrium - Beam
----------------------------
SumF :
               Ax
 Ay + Dy - 4000.0
                0
SumMA:
               0
               0
 5.0*Dy - 7000.0
------------
```

```
Reactions:
------------
Ax [N] =      0
Ay [N] =          2600
Dy [Nm]=          1400
```

(b) Shear Force and Bending Moment Distributions

There are three sections for determining the functions corresponding to the shear force and bending moment. They are based on individual FBD diagrams consolidated in Figure 7.1.15

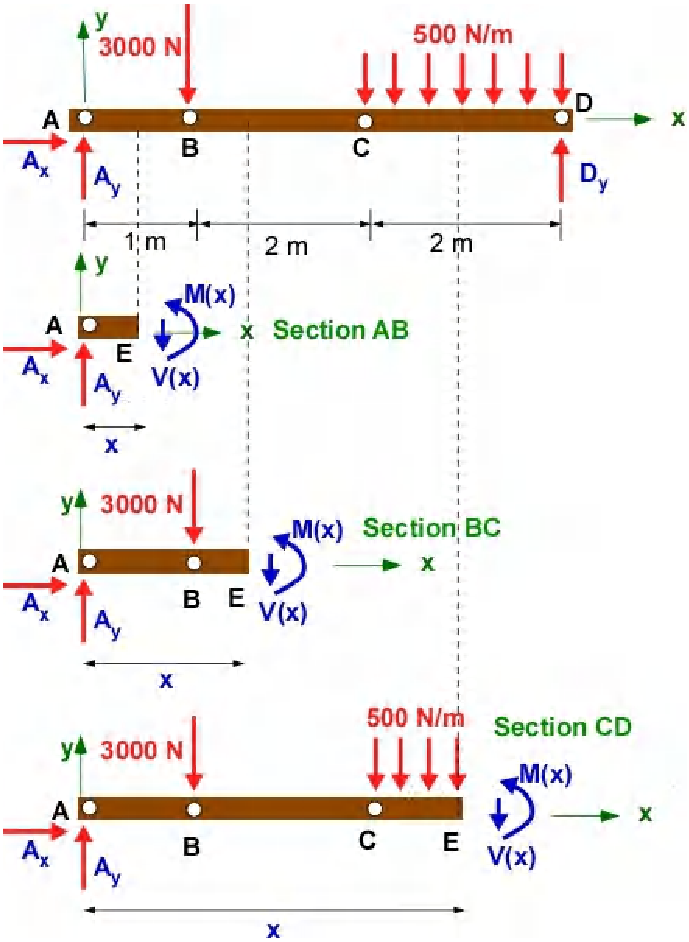


Figure 7.1.15 FBD for various sections

Section AB:

$$\sum F_x = 2600 - V(x) = 0; \quad V(x) = 2600;$$
$$\sum M_A = M(x) - xV(x) = 0; \quad M(x) = 2600x$$

Section BC:

$$\sum F_x = 2600 - 3000 - V(x) = 0; \quad V(x) = -400;$$
$$\sum M_A = M(x) - xV(x) - 3000 \times 1 = 0; \quad M(x) = 3000 - 400x$$

Section CD:

$$\sum F_y = 2600 - 3000 - 500(x-3) - V(x) = 0; \quad V(x) = 1100 - 500x;$$

$$\sum M_A = M(x) - xV(x) - 3000 \times 1 - 500(x-3)\left[3 + \frac{x-3}{2}\right] = 0;$$

$$M(x) = 3000 + x(1100 - 500x) + \frac{500}{2}(x^2 - 9)$$

$$M(x) = 750 + 1100x - 250x^2$$

MATLAB Code

In the Editor : (appended to previous code) - The *x*-equilibrium equation is not necessary

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% (b)  V(x) and M(x)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Section AB
FA = subs(FA);
FD = subs(FD);

syms x V M real
rAE = [x,0,0];
VV = [0,-V,0];  MM = [0,0,M];
SumF1 = FA + VV;
SumMA1 = MM +cross(rAE,VV);

sol1 = solve([SumF1(2),SumMA1(3)],[V,M]);
Vab = sol1.V;
Mab = sol1.M;

% Section BC
SumF2 = FA + +FB + VV;
SumMA2 = MM +cross(rAE,VV) + cross(rAB,FB);

sol2 = solve([SumF2(2),SumMA2(3)],[V,M]);
Vbc = sol2.V;
Mbc = sol2.M;

% Section CD
FW = [0,w*(x-3),0];
dw = 3 + (x-3)/2;  rAW = [dw,0,0];
SumF3 = FA + +FB + VV + FW ;
SumMA3 = MM +cross(rAE,VV) + cross(rAB,FB) + cross(rAW,FW);

sol3 = solve([SumF3(2),SumMA3(3)],[V,M]);
Vcd = sol3.V;
Mcd = sol3.M;

fprintf('-----------------------------------------\n')
fprintf('Shear and Moment Distribution in Sections\n')
fprintf('-------------------------------------------\n')
```

```
fprintf('Equilibrium - Section AB\n')
fprintf('---------------------------')
fprintf('\nSumF1 : \n'),disp(vpa(SumF1',5))
fprintf('SumMA1: \n'),disp(vpa(SumMA1',5))
fprintf('\nSolution - Section AB\n')
fprintf('---------------------------\n')
fprintf('Vab(x)  [N] = '),disp(vpa(Vab,3))
fprintf('Mab(x) [Nm] = '),disp(vpa(Mab,3))

fprintf('\n-------------------------\n')
fprintf('Equilibrium - Section BC\n')
fprintf('---------------------------')
fprintf('\nSumF2 : \n'),disp(vpa(SumF2',5))
fprintf('SumMA2: \n'),disp(vpa(SumMA2',5))
fprintf('\nSolution - Section BC\n')
fprintf('---------------------------\n')
fprintf('Vbc(x)  [N] = '),disp(vpa(Vbc,3))
fprintf('Mbc(x) [Nm] = '),disp(vpa(Mbc,3))

fprintf('\n-------------------------\n')
fprintf('Equilibrium - Section CD\n')
fprintf('---------------------------')
fprintf('\nSumF3 : \n'),disp(vpa(SumF3',5))
fprintf('SumMA3: \n'),disp(vpa(SumMA3',5))
fprintf('\nSolution - Section CD\n')
fprintf('---------------------------\n')
fprintf('Vcd(x)  [N] = '),disp(vpa(Vcd,3))
fprintf('Mcd(x) [Nm] = '),disp(vpa(Mcd,3))
```

In Command Window:

```
Solution - Section AB
---------------------------
Vab(x)  [N] = 2600.0
Mab(x) [Nm] = 2600.0*x

-------------------------
Equilibrium - Section BC
---------------------------
SumF2 :
               0
 - 1.0*V - 400.0
               0
SumMA2:
                    0
                    0
 M - 1.0*V*x - 3000.0

Solution - Section BC
---------------------------
Vbc(x)  [N] = -400.0
Mbc(x) [Nm] = 3000.0 - 400.0*x

-------------------------
Equilibrium - Section CD
```

```
---------------------------
SumF3 :
                        0
 1100.0 - 500.0*x - 1.0*V
                        0
SumMA3:
                                                                   0
                                                                   0
 M - 1.0*V*x - 1.0*(500.0*x - 1500.0)*(0.5*x + 1.5) - 3000.0

Solution - Section CD
---------------------------
Vcd(x) [N] = 1100.0 - 500.0*x
Mcd(x)[Nm] = - 250.0*x^2 + 1100.0*x + 750.0
```

(c) Load, Shear Force, and Bending Moment Diagram
(Drawn Intuitively - not to scale)

To draw the diagram intuitively with the positive sign convention in Figure 7.1.5 :

- First draw the w(x) diagram
- If w(x) is zero, V(x) is constant, M(x) is linear
- If w(x) is constant, V(x) is linear, M(x) is constant
- If w(x) is linear, V(x) is quadratic, M(x) is cubic
- Point loads are applied at the end of the segments

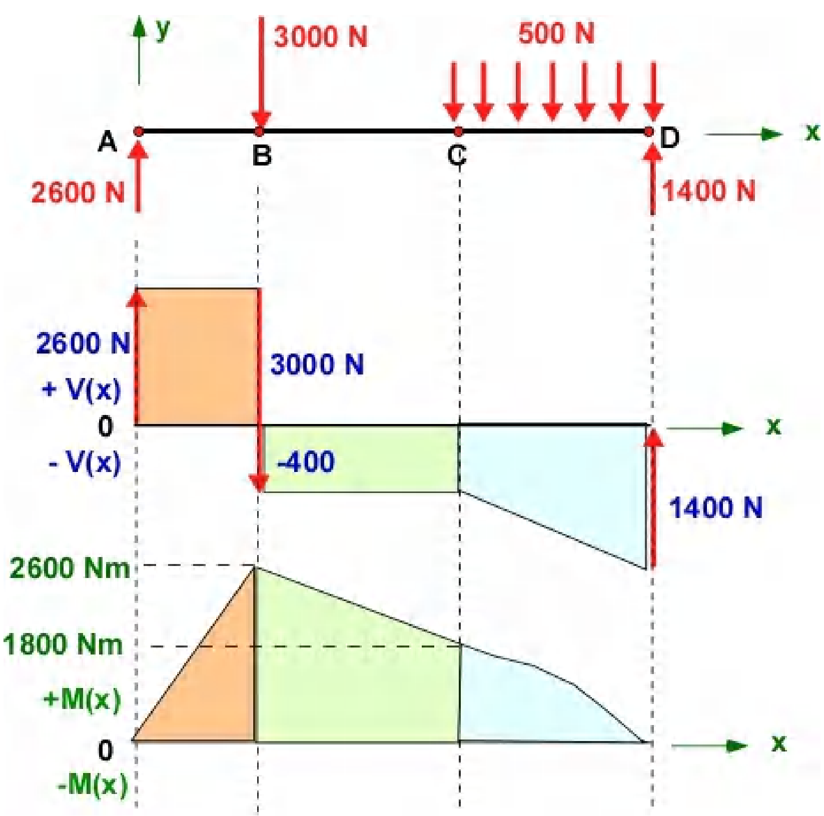


Figure 7.1.16 load,shear, moment diagram - intuitive

MATLAB Code
In the Editor : (appended to previous code)

```
%%%%%%%%%%%%%%%%%%%%%%
%% Graphics
%%%%%%%%%%%%%%%%%%%%%%

%% locate the figure
set(gcf,'Position',[50,50,400,350])
hold on

%% Draw w(x)  diagram betwwen
syms wx
wx = -500;
hp0 = ezplot(0*x/x,[0,3]);
set(hp0,'Color','r','LineWidth',2);

hp1 = ezplot(wx*x/x,[3,5]);
set(hp1,'Color','r','LineWidth',2);

%% draw V(x) diagram
hp2 = ezplot(Vab,[0,1]);
set(hp2,'Color','b','LineWidth',2);
hp3 = ezplot(Vbc,[1,3]);
set(hp3,'Color','b','LineWidth',2);
hp4 = ezplot(Vcd,[3,5]);
set(hp4,'Color','b','LineWidth',2);

%% draw M(x) diagram
hp5= ezplot(Mab,[0,1]);
set(hp5,'Color','g','LineWidth',2);
hp6 = ezplot(Mbc,[1,3]);
set(hp6,'Color','g','LineWidth',2);
hp7 = ezplot(Mcd,[3,5]);
set(hp7,'Color','g','LineWidth',2);

grid
axis([0,5,-2000,3000])
title('Example 7.3 - Load, Shear, and Bending Moment')
text(4,2500,'Load   ','Color','r','FontWeight','b')
text(4,2200,'Shear  ','Color','b','FontWeight','b')
text(4,1900,'Moment ','Color','g','FontWeight','b')

% Filling patches for readability
% only for shear diagram
Vab0 = double(subs(Vab,x,0));
Vab1 = double(subs(Vab,x,1));
Vbc1 = double(subs(Vbc,x,1));
Vbc3 = double(subs(Vbc,x,3));
Vcd3 = double(subs(Vcd,x,3));
Vcd5 = double(subs(Vcd,x,5));

patch([0,1,1,0],[0,0,Vab1,Vab0],[0,0.5,0],'FaceAlpha',0.3)
patch([1,3,3,1],[0,0,Vbc3,Vbc1],[0,0.75,0.75],'FaceAlpha',0.3)
patch([3,3,5,5,3],[0,Vcd3,Vcd5,0,0],[0.75,0.75,0],'FaceAlpha',0.3);
```

In Figure Window

Since all three diagrams are on the same figure only the shear diagram is filled to avoid clutter. The color is filled with transparency to show the other plots. The plot is also to scale unlike the intuitive sketch.

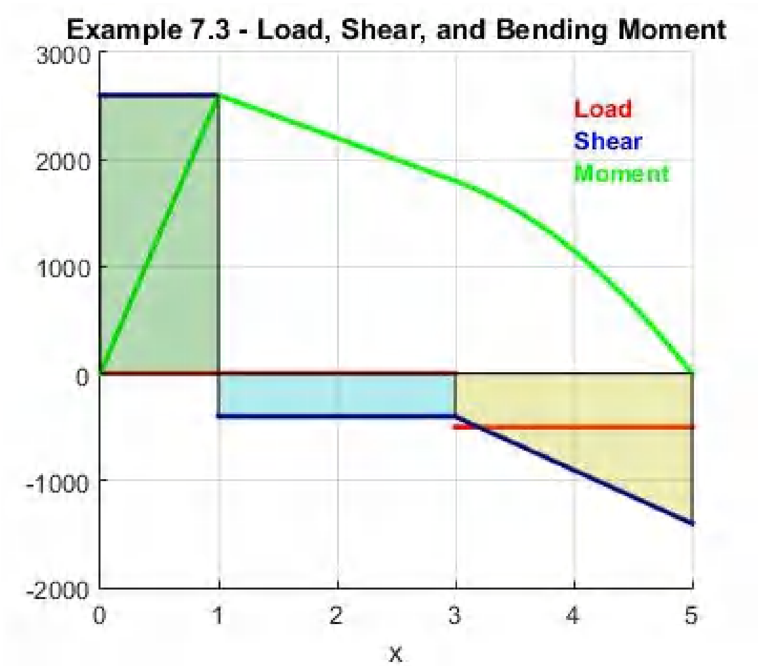


Figure 7.1.17a Load, Shear, Moment diagram - MATLAB

Execution in Octave

The code is same as in the MATLAB above. We can expect to make changes for formatted printing and vector substitution. The MATLAB code is assembled together and only the changes are indicated below

In Octave Editor

```
clc, clear, format compact, format shortg, close all,
warning off
pkg load symbolic;

%FA = subs(FA);
FA = subs(FA,Ax);
FA = subs(FA,Ay);
FD = subs(FD,Dy);

fprintf('\nSumF1 : \n'),disp(vpa(SumF1,5))
fprintf('SumMA1: \n'),disp(vpa(SumMA1,5))

fprintf('\nSumF2 : \n'),disp(vpa(SumF2,5))
fprintf('SumMA2: \n'),disp(vpa(SumMA2,5))

fprintf('\nSumF3 : \n'),disp(vpa(SumF3,5))
fprintf('SumMA3: \n'),disp(vpa(SumMA3,5))
```

In Octave Command Window

```
Example 7.3
------------
Coordinates
----------------------
Point A [m]  :   0  0  0
Point B [m]  :   1  0  0
Point C [m]  :   3  0  0
```

```
Point D [m]  :   5  0  0
Point G [m]  :   4  0  0

Loads
----------------------
Load at B [N]      = -3000
w(x)[N/m]          = -500

Equilibrium - Beamn
---------------------------
SumF :
  [Ax  Ay + Dy - 4000.0  0]
SumMA:
  [0  0  5.0*Dy - 7000.0]
------------
Reactions:
------------
Ax [N] = 0
Ay [N] = 2600
Dy [Nm]= 1400
------------------------------------------
Shear and Moment Distribution in Sections
--------------------------------------------
Equilibrium - Section AB
---------------------------
SumF1 :
  [0  -V + 2600.0  0]
SumMA1:
  [0  0  M - V*x]

Solution - Section AB
---------------------------
Vab(x) [N] =   2.60e+3
Mab(x)[Nm] =   2.6e+3*x

-------------------------
Equilibrium - Section BC
---------------------------
SumF2 :
  [0  -V - 400.0  0]
SumMA2:
  [0  0  M - V*x - 3000.0]

Solution - Section BC
---------------------------
Vbc(x) [N] =   -400.
Mbc(x)[Nm] =   -400.0*x + 3.0e+3

-------------------------
Equilibrium - Section CD
---------------------------
SumF3 :
  [0  -V - 500.0*x + 1100.0  0]
SumMA3:
  [0  0  M - V*x + (-500.0*x + 1500.0)*(0.5*x + 1.5) - 3000.0]
```

```
Solution - Section CD
---------------------------
Vcd(x) [N] =   -500.0*x + 1.1e+3
Mcd(x)[Nm] =             2
  - 250.0*x  + 1.1e+3*x + 750.0
```

In Octave Figure Window

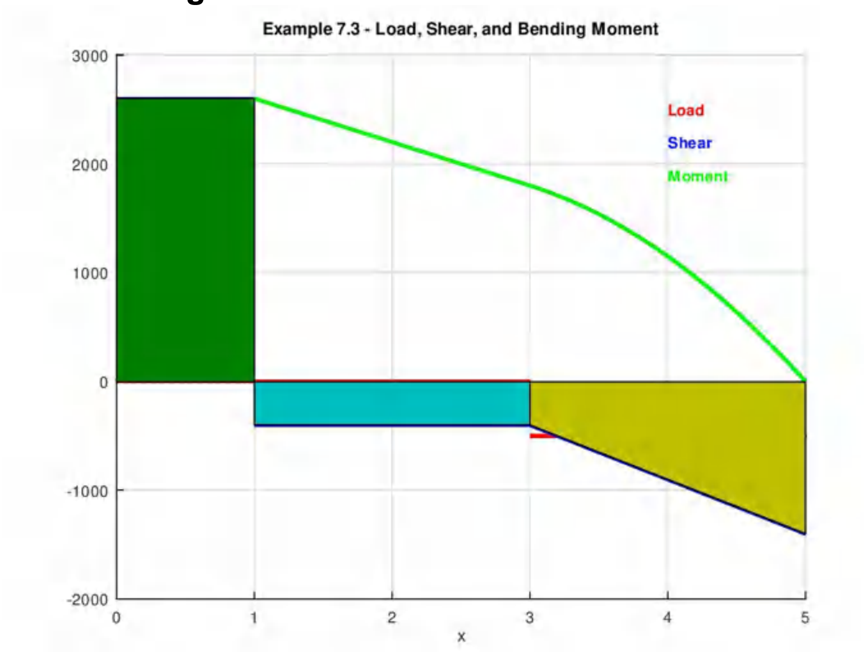


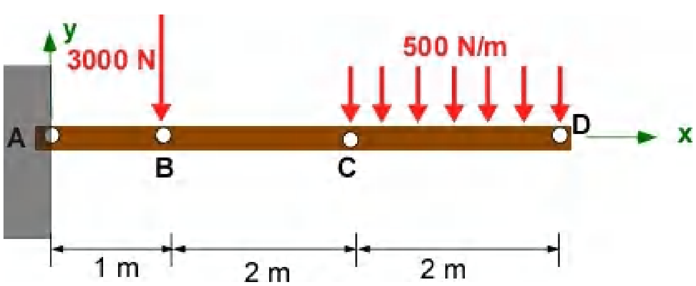
Figure 7.1.17b Load, Shear, Moment diagram - Octave

The figure actually appears the same as that in MATLAB in Octave figure window. Saving it as .jpg or .png (to print it here) appears to loose the alpha (transparency) information.

7.1.8 Additional Problems

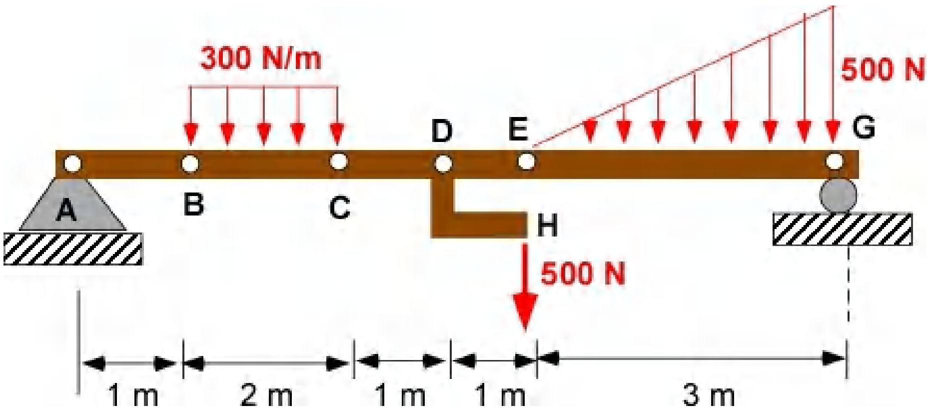
Set up the following problems by hand on paper and solve them on paper and using MATLAB or Octave. For each problem you must draw the FBD and work with a coordinate system. For all problems check for static indeterminacy and obtain support reactions. Obtain and plot the distributed load - w(x), shear force - V(x), and the bending moment diagrams M(x) using the engineering approach.

Problem 7.1.1 (you should recognize this problem)



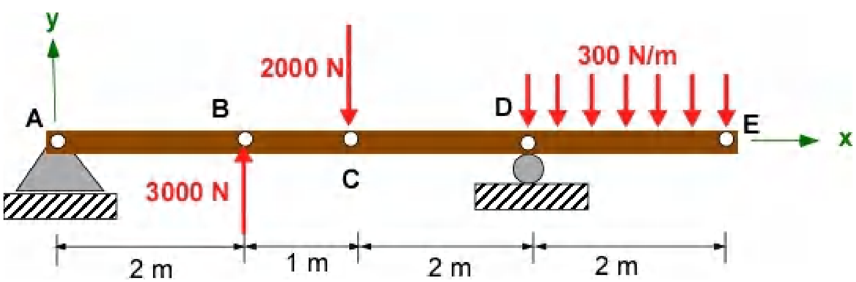
Problem 7.1.1

Problem 7.1.2 (you should recognize this problem)



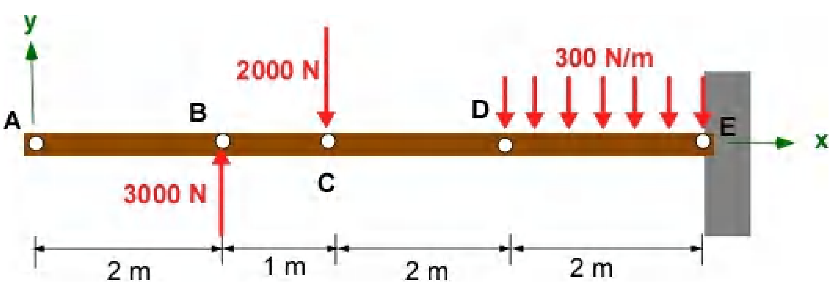
Problem 7.1.2

Problem 7.1.3 -A beam with overhang load and an upward force.



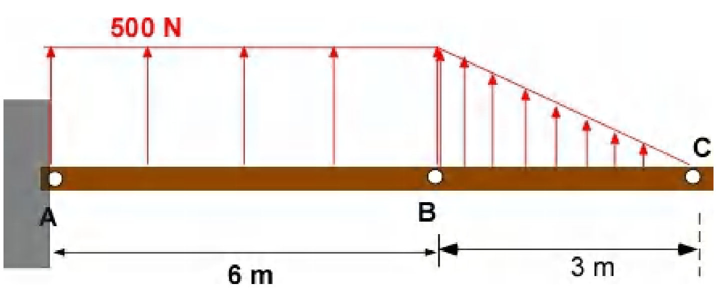
Problem 7.1.3

Problem 7.1.4 - Boundary conditions on the right



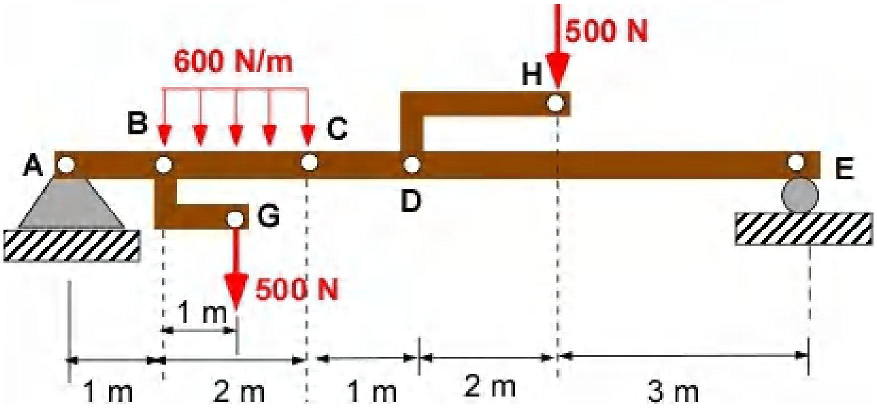
Problem 7.1.4

Problem 7.1.5 - Upward forces - like aerodynamics on an aircraft wing.



Problem 7.1.5

Problem 7.1.6 - Transfer forces to central beam



Problem 7.1.6

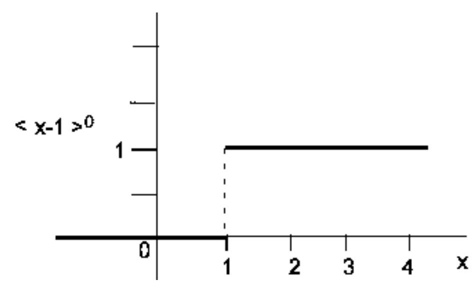
7.2 SINGULARITY FUNCTION

The calculation of the shear force and bending moment distribution required serious effort in the previous section. The nature of these distributions are piecewise continuous with possible concentrated loadings at the junctions. The piecewise nature of these distributions provide a simpler way to calculate the shear and bending moment using special mathematical functions known as ***singularity functions.*** These functions are used to model step functions that are characterized by bimodal behavior at a certain point in the domain. Prior to this point they have one kind of property and after the point they have a different representation. For the functions used here during the first part they do not exist (have the value zero). In the second part they have a specific behavior - like the load distribution in the problems of the previous section. They are a class of discontinuous functions and have a special notation. They have a vast literature but here they are introduced and used only in the context of beam bending problems we have explored in Section 7.1 For an integer **n** the bimodal/step functions are defined using angular parenthesis as:

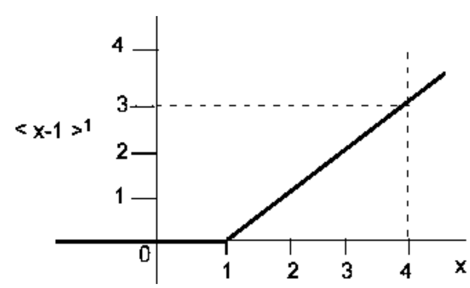
$$\langle x-a\rangle^n = \begin{cases} (x-a)^n, & x \geq a \\ 0, & x < a \end{cases} \qquad (7.3)$$

In Eqn 7.3, the function has zero value until x = a and then after it is an n^{th} order polynomial. The function is therefore discontinuous at the point **a**. This is illustrated in Figure 7.2.1 for a = 1 and n = 0, 1, and 2.

$$\langle x-1\rangle^0 = \begin{cases} (x-1)^0 = 1, & x \geq 1 \\ 0, & x < 1 \end{cases}$$



$$\langle x-1\rangle^1 = \begin{cases} (x-1)^1, & x \geq 1 \\ 0, & x < 1 \end{cases}$$



$$\langle x-1\rangle^2 = \begin{cases} (x-1)^2, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

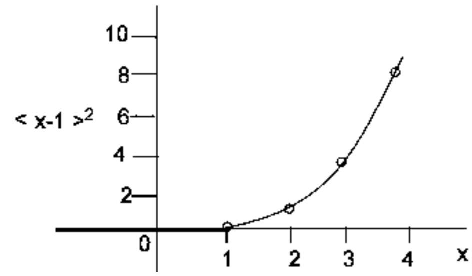


Figure 7.2.1 Illustration of singularity functions

For n = 0 it is a step function at x = 1. For n = 1 it is a ramp at x = 1 with a slope of 1 - the coefficient of the function. For n = 2 it is a quadratic ramp at x = 1 with a coefficient of 1 These are similar to the functions used in defining the distributed loads in the previous example. In addition the relations from calculus, particularly the derivative and integral of these functions are very useful in developing the shear and bending moment diagram.

$$\frac{d\langle x-a\rangle^n}{dx} = n\frac{d\langle x-a\rangle^{n-1}}{dx}; n > 1$$

$$\int \langle x-a\rangle^n \, dx = \begin{cases} (x-a)^{n+1}, n \leq 0 \\ \dfrac{(x-a)^{n+1}}{n+1}, \quad n \geq 0 \end{cases} \qquad (7.4)$$

Some texts like to use negative integers but in this book we will not be using them in the chapter. We will use the functions as they appear.

One important note about these functions is that they can be ***defined at any point in the beam*** , but they will ***continue to exist all the way to the end of the beam.*** Hence to define these functions over a finite length of the beam, you will have to be creative with the definition. Let us illustrate this feature with the uniform loading from one of the previous examples.

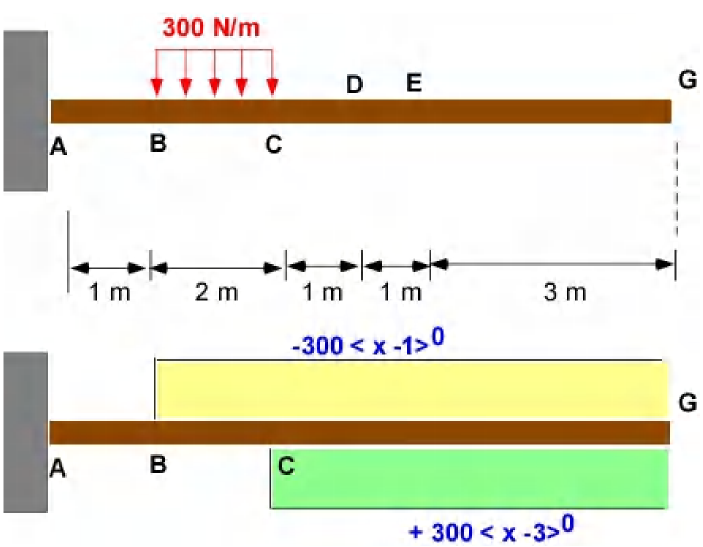


Figure 7.2.2 replacing distributed loading using singularity functions

The step function starting at B (x = 1) will be present all the way to the end of the beam. To cancel the loads after point C (x = 3) a step function of same value and opposite in sign is introduced at C which will stay all the way to the end of the beam. Doing so for linear functions will require greater creativity but with the same idea.

7.2.1 Solving Example 7.2 using Singularity Functions

The use of singularity function with beam bending is illustrated using example 7.2 from the previous section. It starts with the FBD of the beam after the reactions have been determined. These functions are set up in order of *w(x)*, *V(x)* and *M(x)*. *V(x)* is determined by integrating *w(x)* and including concentrated forces as step functions. *M(x)* follows by integrating *V(x)* and including concentrated moments as step functions. This is the procedure. The functions include the stepping point. An exponent of zero is a constant step function. An exponent of one is a linear ramp. We start with the FBD borrowed from the previous section (Figure 7.1.6).

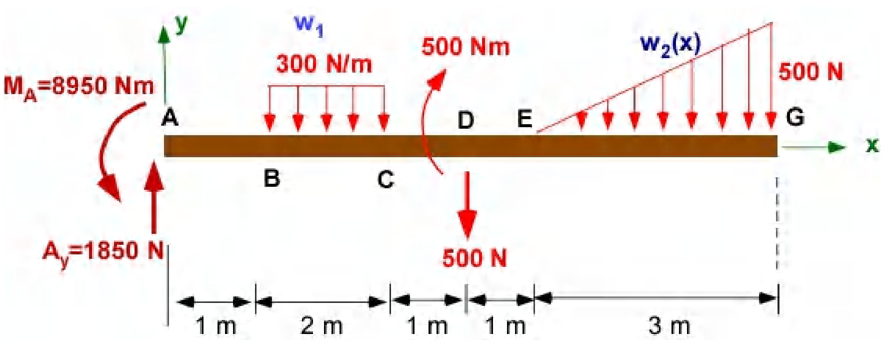


Figure 7.2.3 Example 7.2 - FBD

In the function below we have used the integration define in Eqn. (7.4)

$$w(x) = -300\langle x-1\rangle^0 + 300\langle x-3\rangle^0 - \left(\frac{500}{3}\right)\langle x-5\rangle^1$$

$$V(x) = -300\langle x-1\rangle^1 + 300\langle x-3\rangle^1 - \left(\frac{500}{3}\right)\frac{\langle x-5\rangle^2}{2} +$$
$$1850\langle x-0\rangle^0 - 500\langle x-4\rangle^0$$

$$M(x) = -300\frac{\langle x-1\rangle^2}{2} + 300\frac{\langle x-3\rangle^2}{2} - \left(\frac{500}{3}\right)\frac{\langle x-5\rangle^3}{3\times 2}$$
$$+1850\langle x-0\rangle^1 - 500\langle x-4\rangle^1 - 8950\langle x-0\rangle^0 + 500\langle x-4\rangle^0$$

The above relations are written just by observing the FBD. There are a couple of notes to keep in mind.

- The negative sign in w(x) is incorporated in the definition for w(x).
- The linear distribution for w(x) at the end of the beam requires the value of slope as the coefficient
- The reaction at A has a positive coefficient using the sign convention
- The moment reaction at A has a negative coefficient using the sign convention
- The jump in the concentrated shear at D is negative and based on its direction on the FBD (or considered left of the point D for applying the sign convention)

- The jump in the concentrated moment at D is positive sign convention (considering the moment is to the left of point D)

Let us compute the values for shear force and bending moment at D+ (just past the point D)

$$w\left(4^{+}\right)=-300\langle 3\rangle^{0}+300\langle 1\rangle^{0}=0$$

$$V\left(4^{+}\right)=-300\times 3+300\times 1+1850-500=750$$

$$M\left(4^{+}\right)=-300\times\frac{9}{2}+300\times\frac{1}{2}+1850\times 4-500\times 0-8950+500=-2250$$

These are the same values obtained previously. These functions can be evaluated at any point on the beam. We can use those values to plot them two. MATLAB or OCTAVE can calculate and plot these functions easily.

7.2.2 Example 7.2 using Singularity functions in MATLAB

We use the previous FBD (for completeness) and the functions established earlier (reproduced below again)

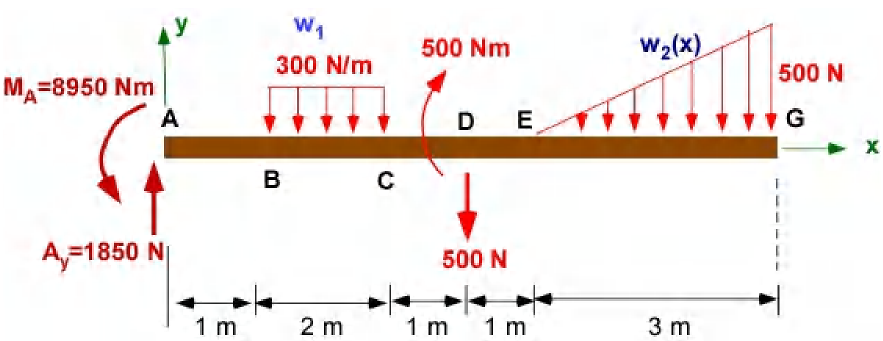


Figure 7.2.3 Example 7.2 - FBD

$$w(x)=-300\langle x-1\rangle^{0}+300\langle x-3\rangle^{0}-\left(\frac{500}{3}\right)\langle x-5\rangle^{1}$$

$$V(x)=-300\langle x-1\rangle^{1}+300\langle x-3\rangle^{1}-\left(\frac{500}{3}\right)\frac{\langle x-5\rangle^{2}}{2}+$$
$$1850\langle x-0\rangle^{0}-500\langle x-4\rangle^{0}$$

$$M(x)=-300\frac{\langle x-1\rangle^{2}}{2}+300\frac{\langle x-3\rangle^{2}}{2}-\left(\frac{500}{3}\right)\frac{\langle x-5\rangle^{3}}{3\times 2}$$
$$+1850\langle x-0\rangle^{1}-500\langle x-4\rangle^{1}-8950\langle x-0\rangle^{0}+500\langle x-4\rangle^{0}$$

The singularity function are used numerically. We use a logical check to define the function to be different from zero. We also implement the singularity function as an ***anonymous*** function.
The jumps in the functions as defined below do not happen at the same value of x. They take place over a finite (though small) distance. By taking a large number of values for x we can make these

jumps appear to take place at the same point.

Solution Using MATLAB
In the Editor

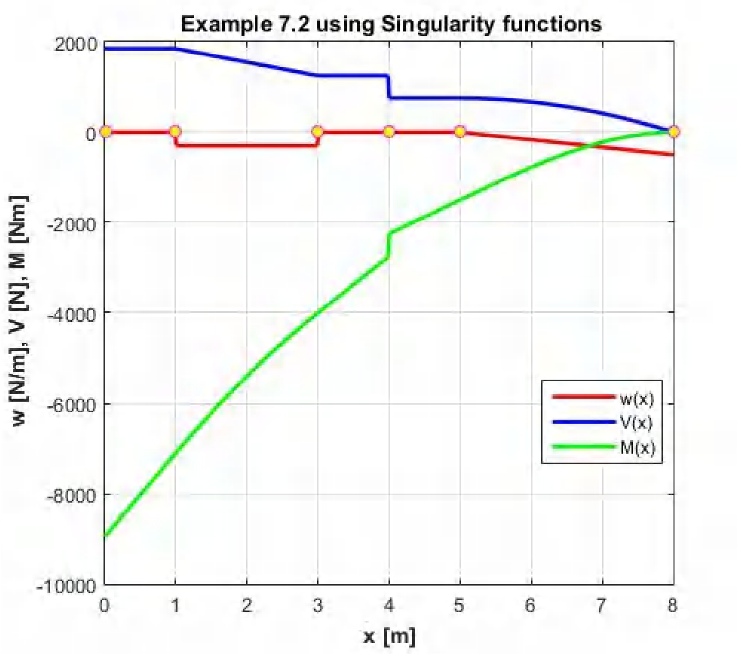
```
% Essential Mechanics Dec 2015
%  Example 7.2 - Beam (2D)
%  Section 7.2.3 - Singularity function
%  P. Venkataraman
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(5)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 7.2 - Singularity Function\n')
fprintf('--------------------------------------\n')

%% Data
% The constants - from FBD
w1 = 300;
w2 = -500/3;
Ay = 1850;  Dy = -500;
Ma = -8950;  Md = +500;
xpoints = [0,1,3,4,5,8];  % Points for discontinuities in all functions
ypoints = [0,0,0,0,0,0];

%% Define an anonymous function (used to be called inline function)
% the approach is numerical
% n is expected to be greater than or equal to zero
% the function takes in the value of x, a, and n and returns the
% singulatity function for the entire x value
singfun = @(x,a,n) (x-a).^n.*(x >= a);

%% assemble the functions
% this is not symbolic implementation
% use very close points for x so it appears as a step at same
% point instead of incline
x = linspace(0,8,501);
w = -w1*singfun(x,1,0) + w1*singfun(x,3,0) + w2*singfun(x,5,1);

% for V(x) the w(x) is integrated - increase exponent by 1 and divide by
% exponent
V = -w1*singfun(x,1,1)/1 + w1*singfun(x,3,1)/1 + w2*singfun(x,5,2)/2 ...
    + Ay*singfun(x,0,0)+ Dy*singfun(x,4,0);

% for M(x) the V(x) is integrated - increase exponent by 1 and divide by
% new exponent
M = -w1*singfun(x,1,2)/1/2 +w1*singfun(x,3,2)/1/2 + w2*singfun(x,5,3)/2/3
...
    +Ay*singfun(x,0,1)/1+Dy*singfun(x,4,1)/1 ...
        + Ma*singfun(x,0,0) + Md*singfun(x,4,0);

%% plot the functions
figure
set(gcf,'Position',[25,50,500,450], ...
    'Color','w');
plot(x,w,'r-','LineWidth',2)
hold on
```

```
plot(x,V,'b-','LineWidth',2)
plot(x,M,'g-','LineWidth',2)
xlabel('\bfx [m]')
ylabel('\bf w [N/m], V [N], M [Nm]')
grid
legend('w(x)','V(x)','M(x)','Location','Best')
title('\bf Example 7.2 using Singularity functions')
 % points on the beam where functoions change
plot(xpoints,ypoints,'mo','MarkerFaceColor','y')
hold off

%%%%%%%%%%%%%%%%%%%%%%
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('Constants')
fprintf('\n----------------------')
fprintf('\nw1 [N/m]  : '),disp(w1)
fprintf('Slope of w2(x) [N/m]  : '),disp(w2)
fprintf('Ay [N]  : '),disp(Ay)
fprintf('Dy [N]  : '),disp(Dy)
fprintf('Ma [Nm] : '),disp(Ma)
fprintf('Md [Nm] : '),disp(Md)
```

In the Command Window

```
Example 7.2 - Singularity Function
--------------------------------------
Constants
----------------------
w1 [N/m]  :    300
slope of w2(x) [N/m]  :       -166.67
Ay [N]  :          1850
Dy [N]  :   -500
Ma [Nm] :         -8950
Md [Nm] :    500
```

In the Figure Window



Example 7.2.4 Example 7.2 shear and bending moment diagram

Figure 7.2.4 compares well with Figure 7.1.7. Figure 7.2.4 is scaled. The same scale applies to all the three entities. It is difficult to make out the linear and nonlinear behavior of the M(x) function.

Notice it is easy to obtain and plot the load, shear, and bending moment diagrams using the singularity function compared to the analysis in the previous section.

7.2.3 Example 7.3 using Singularity functions

This example is detailed in Section 7.1.7 and we pick up the FBD in Figure 7.1.14. In the example $A_x = 0$; $A_y = 2600$ N ; and $D_y = 1400$ N.

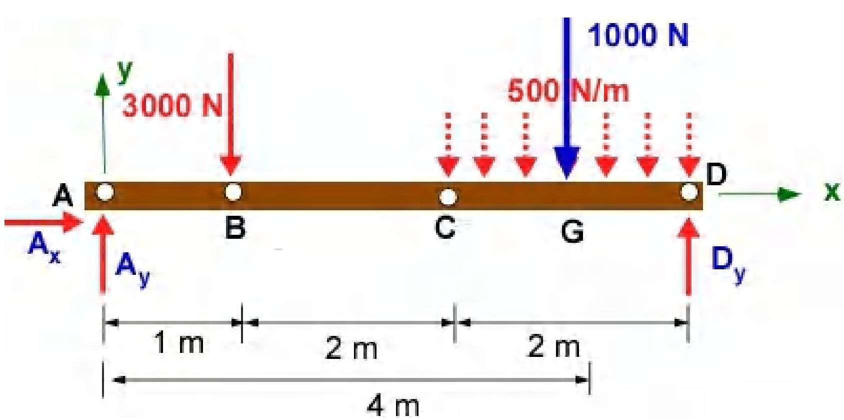


Figure 7.1.14 FBD - Example 7.3

The singularity functions for Example 7.3 can be expressed as:

$$w(x) = -500(x-3)^0$$
$$V(x) = -500(x-3)^1/1 + 2600(x-0)^0 - 3000(x-1)^0$$
$$M(x) = -500(x-3)^2/1/2 + 2600(x-0)^1/1 - 3000(x-1)^1/1$$

These are the load, shear force and bending moment distributions for the example. We will plot these functions using MATLAB and compare to the results in Figure 7.1.17.

Solution Using MATLAB
In the Editor

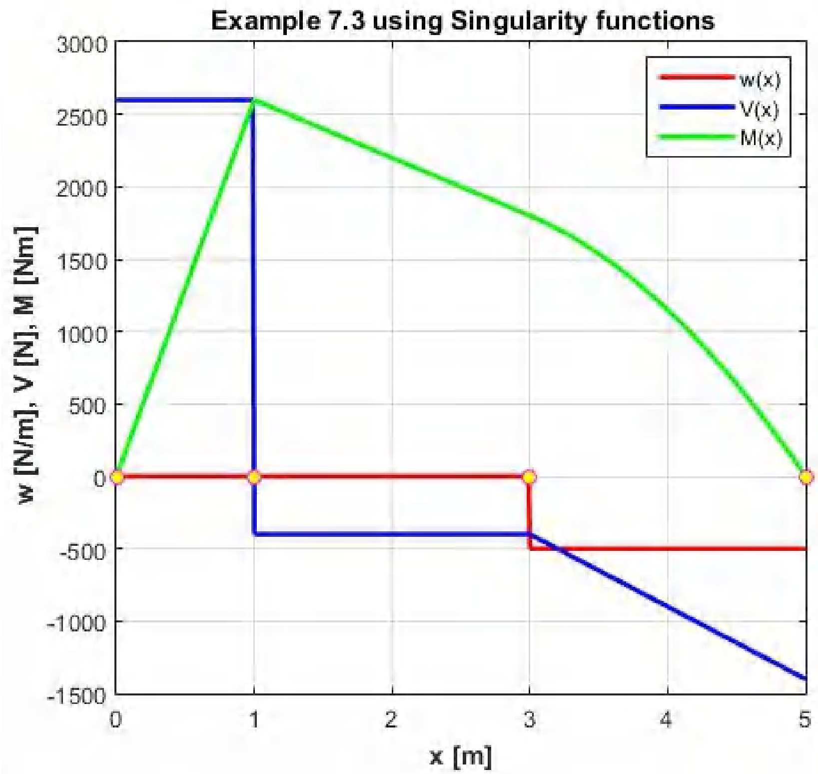
```
% Essential Mechanics Dec 2015
%  Example 7.3 - Beam (2D)
%  Section 7.2.4 - Singularity function
%  P. Venkataraman
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(5)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 7.3 - Singularity Function\n')
fprintf('--------------------------------------\n')

%% Data
% The constants - from FBD
w1 = -500;
Ay = 2600; By = -3000;  Dy = 1400;

xpoints = [0,1,3,5];  % Points for discontinuities in all functions
ypoints = [0,0,0,0];

%% Define an anonymous function (used to be called inline function)
% the approach is numerical
% n is expected to be greater than or equal to zero
% the function takes in the value of x, a, and n and returns the
% singulatity function for the entire x value
singfun = @(x,a,n) (x-a).^n.*(x >= a);

%% assemble the functions
% this is not symbolic implementation
% use very close points for x so it appears as a step at same point
x = linspace(0,5,501);
w = w1*singfun(x,3,0) ;

% for V(x) the w(x) is integrated - increase exponent by 1 and divide by
% exponent
V = w1*singfun(x,3,1)/1 + ...
    + Ay*singfun(x,0,0) + By*singfun(x,1,0);

% for M(x) the V(x) is integrated - increase exponent by 1 and divide by
% new exponent
M = w1*singfun(x,3,2)/1/2 ...
    + Ay*singfun(x,0,1)/1+ By*singfun(x,1,1)/1 ;

%% plot the functions
```

```
figure
set(gcf,'Position',[25,50,500,450], ...
    'Color','w');
plot(x,w,'r-','LineWidth',2)
hold on
plot(x,V,'b-','LineWidth',2)
plot(x,M,'g-','LineWidth',2)
xlabel('\bfx [m]')
ylabel('\bf w [N/m], V [N], M [Nm]')
grid
legend('w(x)','V(x)','M(x)','Location','Best')
title('\bf Example 7.3 using Singularity functions')
 % points on the beam where functoions change
plot(xpoints,ypoints,'mo','MarkerFaceColor','y')
hold off

%%%%%%%%%%%%%%%%%%%%%%
%% Printing
%%%%%%%%%%%%%%%%%%%%%%
fprintf('Constants')
fprintf('\n-----------------------')
fprintf('\nw1 [N/m]   : '),disp(w1)

fprintf('Ay [N]   : '),disp(Ay)
fprintf('By [N]   : '),disp(By)
fprintf('Dy [N]   : '),disp(Dy)
```

In the Command Window

```
Example 7.3 - Singularity Function
---------------------------------------
Constants
----------------------
w1 [N/m]   :   -500
Ay [N]   :         2600
By [N]   :        -3000
Dy [N]   :         1400
```

In the Figure Window



Example 7.2.5a Example 7.3 - Load, shear and bending moment diagram

The figure compares well to the one in Figure 7.1.17. You can see that the singularity functions are very helpful in solving beam bending problems once you have the reactions and the FBD of the beam.

Execution in OCTAVE

In Octave Editor

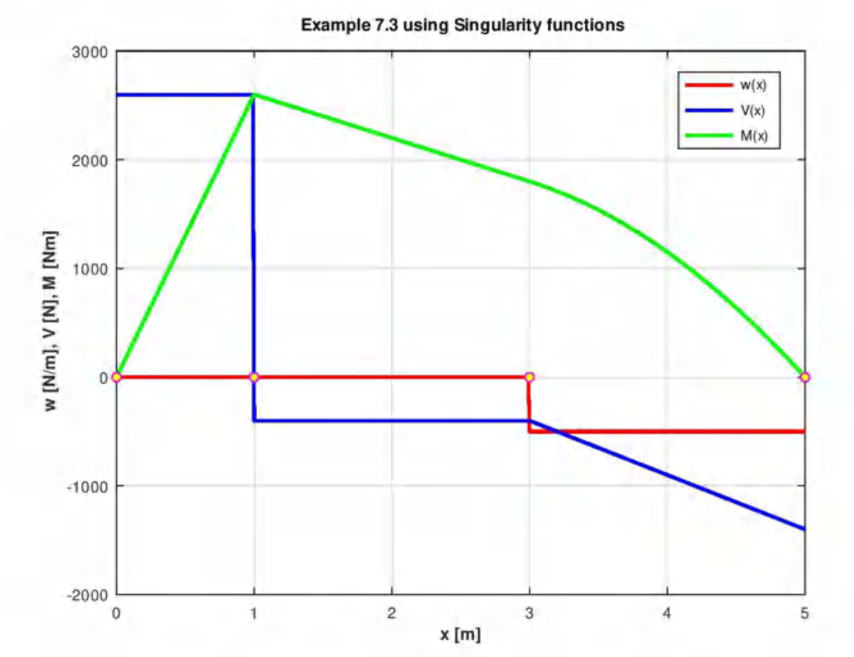
The code is same as in the MATLAB above. Changes are highlighted. Does not like digits unless Python is invoked through the symbolic package

```
clc, clear, format compact, format shortg, close all,
```

In the Octave Command Window

```
Example 7.3 - Singularity Function
--------------------------------------
Constants
----------------------
w1 [N/m]  : -500
Ay [N]  : 2600
By [N]  : -3000
Dy [N]  : 1400
```

In Octave Figure Window

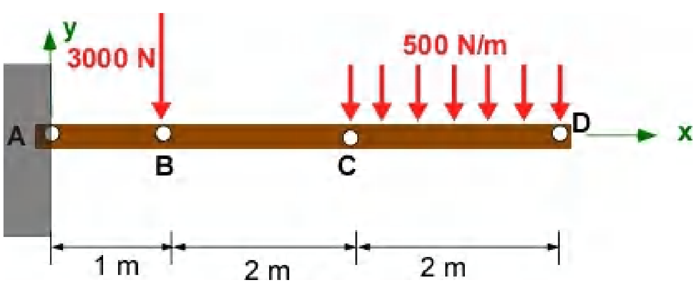


Example 7.2.5b Example 7.3 - Load, shear and bending moment diagram - OCTAVE

7.2.4 Additional Problems

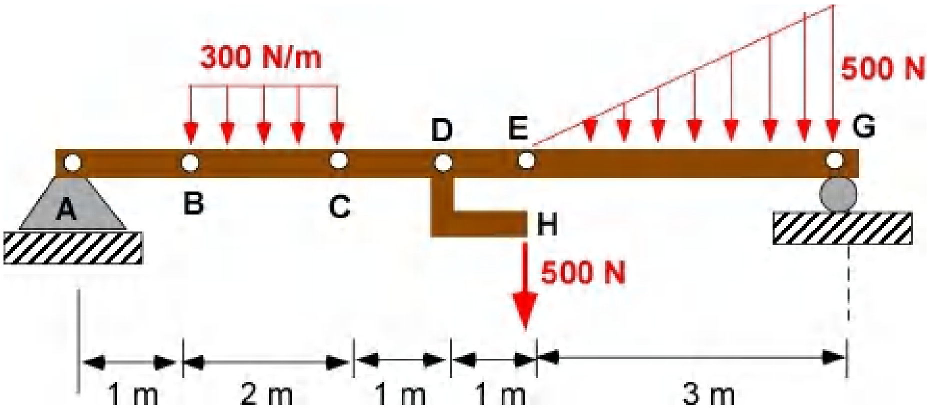
These problems are the same as in Section 7.1 Plot the load, shear and bending moment diagrams using singularity function and compare them to those obtained in the previous section. You can borrow the FBD diagram if you are confident that it is correct. Solve by hand and using MATLAB/Octave.

Problem 7.2.1



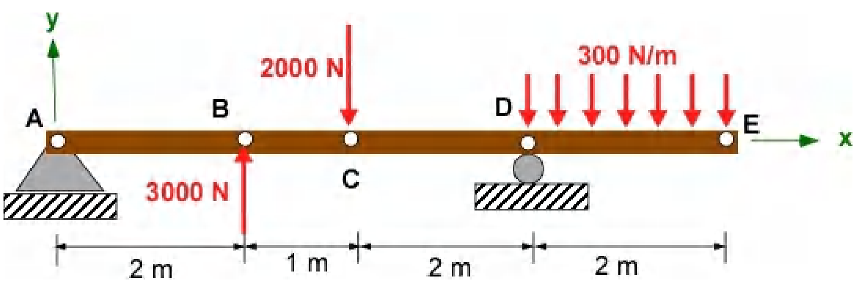
Problem 7.2.1

Problem 7.2.2 (you should recognize this problem)



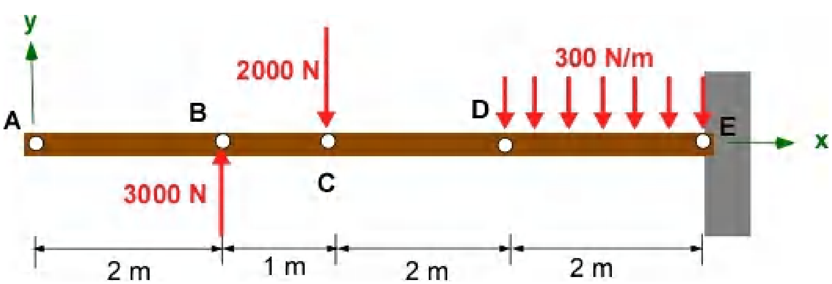
Problem 7.2.2

Problem 7.2.3 -A beam with overhang load and an upward force.



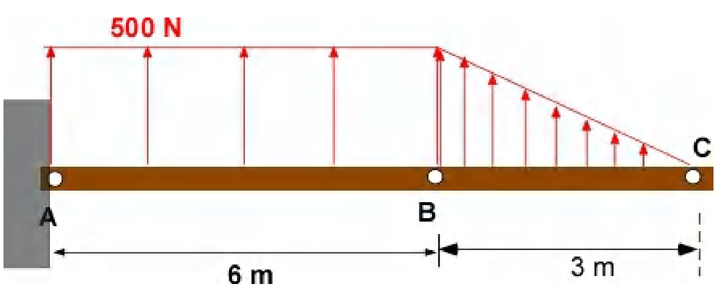
Problem 7.2.3

Problem 7.2.4 - Boundary conditions on the right



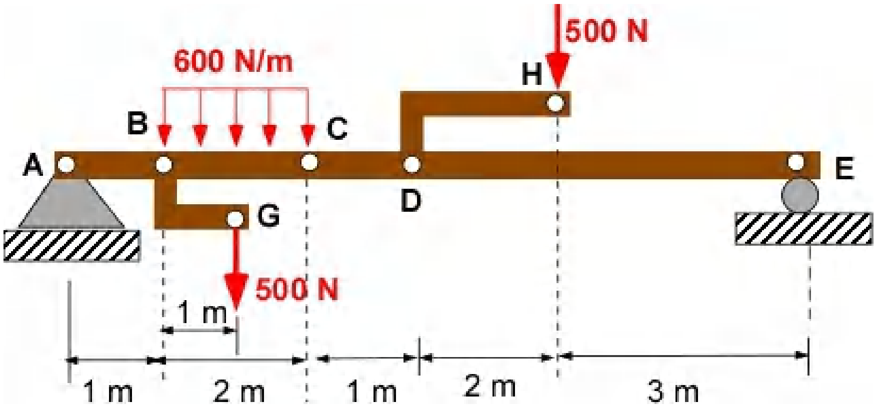
Problem 7.2.4

Problem 7.2.5 - Upward forces - like aerodynamics on an aircraft wing.



Problem 7.2.5

Problem 7.2.6 - Transfer forces to central beam



Problem 7.2.6

7.3 BEAM IN PURE BENDING

We now can calculate the shear force and the bending moment along the beam for any type of loads placed on it. This is not the end of the **design** problem. We can feel that the beam must **deflect** with all these loads it carries over the length, like the diving board bend when the diver is at the edge of the board over the pool. Or for that matter, you can break a branch of the tree to fashion a walking stick by bending it around the knee. If you can break the branch that means the branch has failed. We have learned that this happens because you exerted a stress that was greater than the rupture stress (remember you break things by stress and not force). Bending, or the application of a bending moment can create large stresses. As a consequence this creates deflections since there will be strains and strains cause displacement. For example the loaded beam of Example 7.2 can change shape under the applied load and is indicated as ***y(x)*** (exaggerated of course) in Figure 7.3.1. Such deformation is obtained under the assumption of small deformations so that linearity can be assumed.

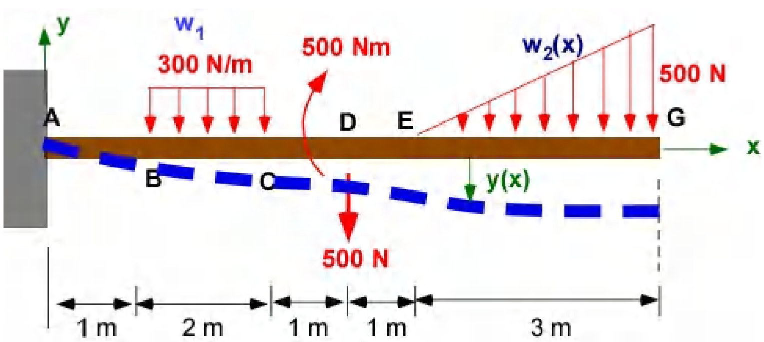


Figure 7.3.1 Beam Deflection

The design problem with beams requires that we have to be sure that the beam will not break and its deflection is elastic so that it recovers the original shape when the loads are removed. In other words the beam shall not fail under loading - that is the Von Mises stresses are satisfied. Off course we will need a factor of safety. We can assume that the beam can fail at a particular point along its length where it is likely to see the maximum stress. The next question is will it be *normal stress* or *shear stress*. It turns out that the beam is subjected to both kinds of stresses when loaded. We have to connect the shear force and bending moment to these stresses. To relate stress to displacement we need the property of the material of the beam. A wooden beam may deflect more than an Aluminum beam which will deflect more than a steel beam. We can also expect a thin beam to deflect more than a thick beam. In Figure 7.3.1 we can expect the beam to deflect a lot at the right end and therefore we probably should use an additional support at that end. That will change the design since it will change the shear force and bending moment diagram.

Let us summarize our current thoughts regarding beam design:

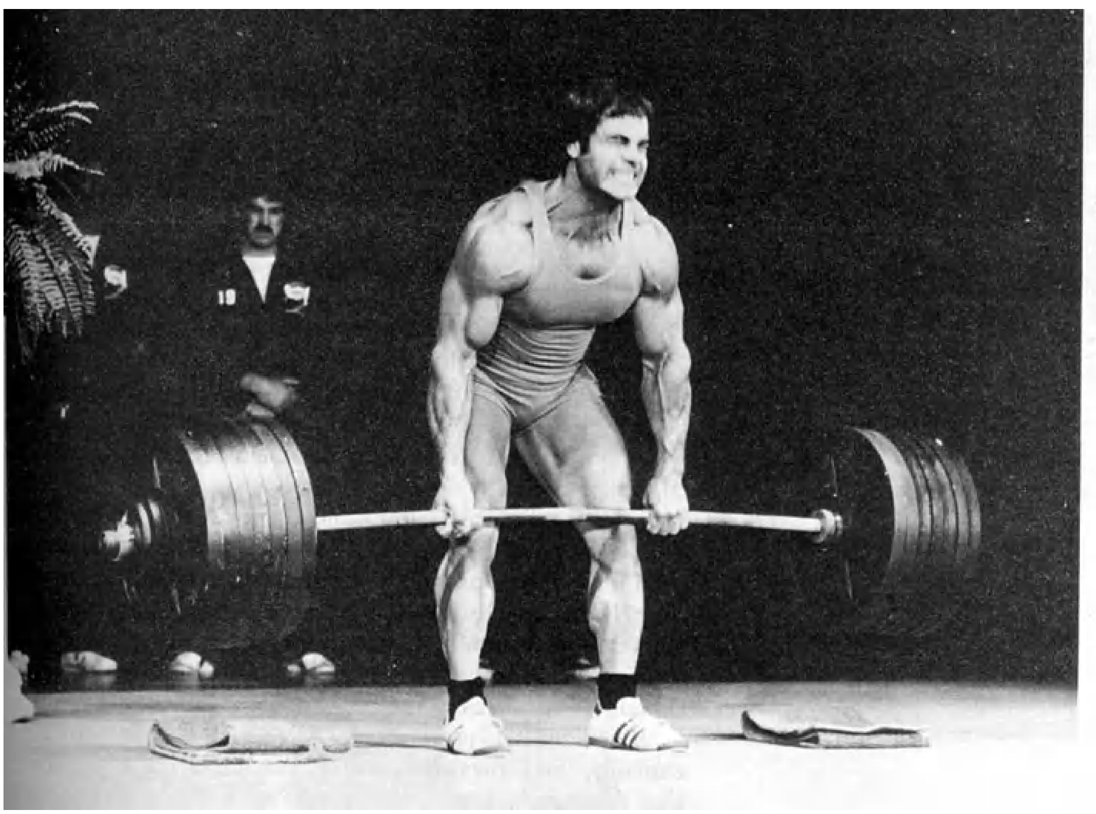
- Beams will deflect under loading
- Beam deflection will depend on the shear force and bending moment distribution along the beam
- These distributions can be changed by supporting the beams differently along the length or at the ends
- The shear force and bending moment on the beam will give arise to shear and normal stresses

in the beam
- Beam deflection will depend on the material of the beam
- Beam will most likely fail at a point along the length where the stresses are at a maximum and when they exceed allowable stress
- Beam deflection will depend on the size of the beam (particularly the cross section)
- Beam design must be subject to the failure criteria
- Beam must be subject to less than design loads to avoid failure
- Beam should deflect only elastically under the maximum applied loading (most circumstances) so that it is load and geometry is not permanently altered

There is a lot to discuss in the design of beams and we will do it through several sub-sections below. In this section we will relate the bending moment on the beam to normal stresses and deflection.

7.3.1 Pure Bending - Deformation

Let us start with Figure 7.3.2 which illustrates bar bending - a term that describes the ability to bend the barbell by lifting significant amount of weights. The simple FBD alongside the photograph suggests that the bar between CD is in pure bending - only a bending moment is carried/resisted throughout the beam - that is in every cross- section. There is no shear force in CD. Also note that it appears to bend in an arc of a circle (larger the radius less the bending?). As we continue with the analysis we will incorporate several idealizations.



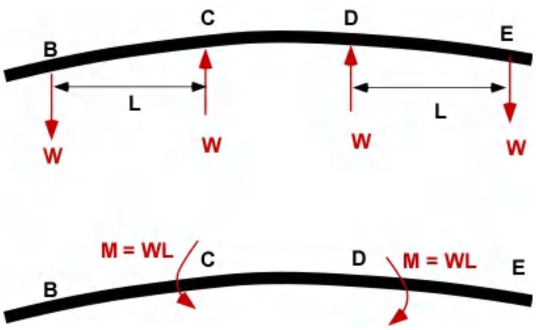
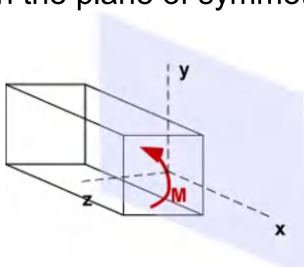


Figure 7.3.2 Pure Bending (Wikimedia Commons)

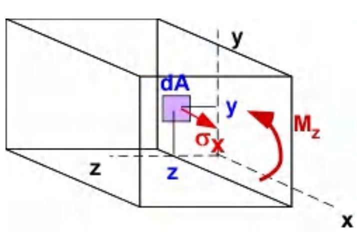
Idealizations

1. If we revisit the shear force and bending moment distributions in the previous section we recognize that we had idealized the problem as two-dimensional - the beam was only loaded along a single line. It's depth was ignored in the discussion. In this illustration the barbell is round but the FBD alongside ignores the three-dimensionality of the barbell. Where is the moment at C and D? For the sake of simplicity and illustration we will assume the bar is of square cross-section. It is possibly that the moment is in the mid plane - the vertical plane through the center of the barbell. In the development of the mathematical relations we will ignore the three dimensionality of the problem. This is helpful (initially) if the vertical plane through the center of the beam is also a plane of symmetry.

2. We like our positive x-axis to flow right and the y-axis to go up. The z-axis is out of the screen/page. A positive bending moment will be opposite to the illustration in Figure 7.3.2. It is shown in the plane of symmetry and is directed along the positive z-axis.



3. Now what kind of stresses will be created by this moment in the cross-section, or equivalently, what kind of stress distribution in the cross-section will result in a moment ? Let us consider a small area **dA** in the cross-section in the positive quadrant. It is located at a distance y and z (the cross-section is the *yz* plane). The stresses on this **dA** will determine the elemental force **dF**. This force can create elemental moments about the axis. These elemental values can be integrated over the cross-section **A** to provide the bending moment



Since there only a bending moment in the cross-section there must be a distribution of normal stress σ_x to create the bending moment $\mathbf{M_z}$. We can set

$$M_z = \int_A -y\,\sigma_x\, dA \qquad (7.5)$$

The negative sign relates the direction of the bending moment and the moment produced by the normal stress to be the same. In order for the moment produced by the normal stress not to cancel over the different areas in the cross-section, the sign of σ_x must change above and below the z-axis. There is also a problem with assuming the normal stress having a constant value. This causes the normal stress to be discontinuous at y = 0. At this time we would like to work with a continuous distribution.

We know that integral of σ_x over the cross-section must equal a normal force in the x-direction which must be **zero** since there is no normal force present in the cross-section. There is only $\mathbf{M_z}$ in the cross-section. This also suggest that the normal stress must change sign over the cross-section to cancel . We can set this up as a constraint

$$F_x = \int_A \sigma_x\, dA = 0 \qquad (7.6)$$

The same σ_x can also cause $\mathbf{M_y}$ which does not exist in this example. This provides another constraint which can be set up as:

$$M_y = \int_A z\,\sigma_x\, dA = 0 \qquad (7.7)$$

This does not require that the normal stress change sign in the z-direction.The normal stress distribution in the cross-section must satisfy these three equations/constraints. This is an overdetermined system. Our unknown is σ_x. While Eqn. (7.5) connects the M_z and σ_x there is no way to explicitly determine σ_x .

If we had applied shear load in the cross-section that would lead to shear stresses. For this example we have none.

4. We have arrived at a statically indeterminate system. We cannot solve for the stress unless we include additional information. This can come from understanding the deflection or the deformation of the beam under bending.

5. Since every cross-section in the beam sees the same bending moment, every cross-section will bend the same way.

Bending Deformation (Pure Bending)

Pure bending will deform the beam in an arc of a circle. In Figure 7.3.3 we see that the straight bar subject to bending can bend in at least two ways.

- In the first case (Case 1) the top surface AB and the bottom surface CD can bend with the same radius, have the same length after bending but with the center of the circle at different points. This suggests that the strains might be the same everywhere - and therefore the stresses. That is not what we were expecting in our previous idealizations of bending expressed earlier.
- In the second case (Case 2) the center of the circle is the same for both surfaces, but AB will shorten and CD will elongate compared to EF. Therefore, Case 2 requires that there will be strains and deformation in bending will vary in the cross-section. This agrees with our

expectation in item 3 above.

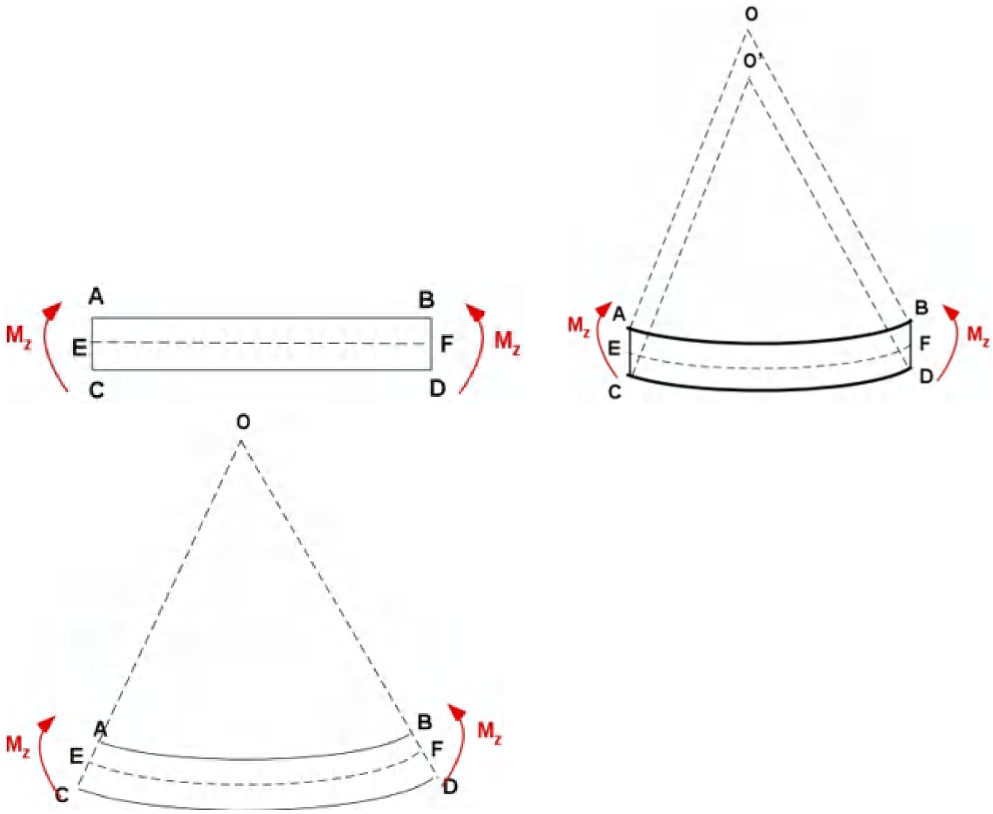


Figure 7.3.3 Original beam, Bending - Case 1, Bending - Case 2

Bending Case 2: This describes the deformation in pure bending. This leads to another idealization which suggests that the points in the cross-sections will be in the same plane before and after bending moment is applied - *plane sections remain plane*. It is a cleaner description of bending than allowing multiple centers for bending deformation in Case 1. In Figure 7.3.4 we look at Case 2 in more detail to derive the relations between strain, stress, and bending moment.

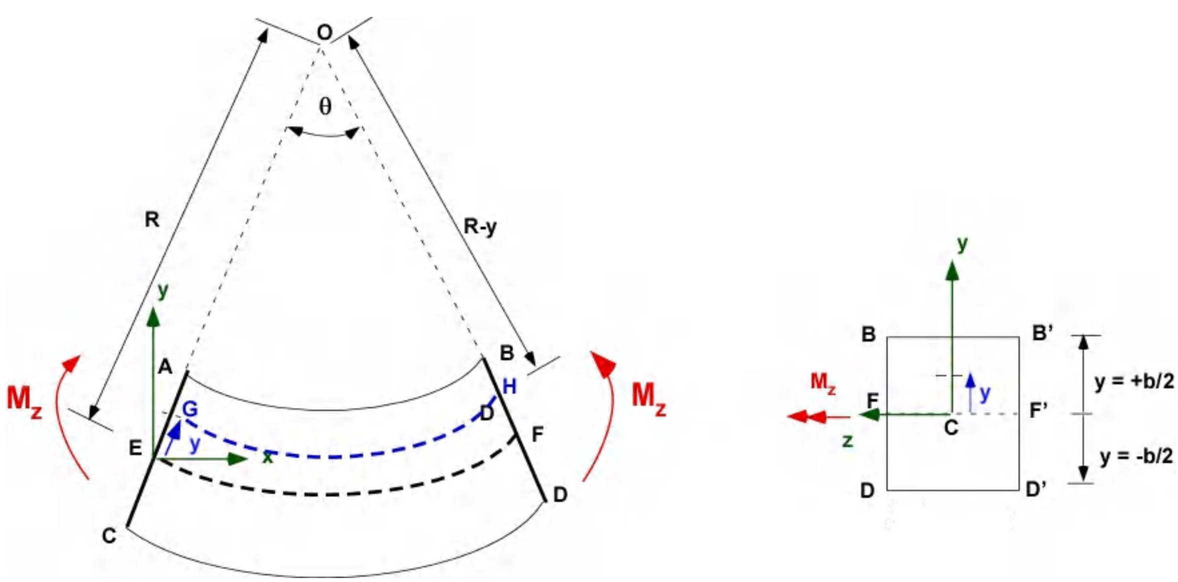


Figure 7.3.4 Pure Bending and deformation

Figure 7.3.4 needs to be understood well to follow bending calculations.

- The pure bending moment $\mathbf{M_z}$ bends an original **straight bar** ABCD into an **arc of a circle of**

radius R. O is the center of curvature and the radius is measured from O to a special line/surface EF.

- The deformation is drawn showing *plane sections remain plane*.
- The cross-section is rectangular and is also shown.
- Prior to bending, for the straight bar the lengths AB, CD, and EF all have the same length (L).
- Once the bending is applied AB will **shorten** (compression) and CD will **expand** (elongation/tension) since plane sections remain plane. These are **normal strains** as they are normal to the cross-section.
- EF has the same length before and after bending. EF has a special name and is called the **neutral axis (NA)**. EF suffers no strain and therefore no stress.
- Since the cross-section is doubly symmetric it is easy to locate the neutral axis passing through a special point - the **centroid**. Later on we will show that this is also the case for non-symmetric sections too.
- The angle subtended by the bar at the center is θ. The radius of curvature is **R** and is drawn from O to the **neutral axis** EF as shown.
- The line GH (or the surface GH) is at a distance +**y** measured from the **NA**.

One of the assumptions we make is that the angle θ is small. Since the bending of the bar is represented by an arc of a circle we can calculate

$EF = L = R\,\theta$
$GH = (R - y)\,\theta$
$AB = (R-b/2)\,\theta$; $\qquad$ $CD = (R + b/2)\,\theta$;

Change in length at the distance y from the neutral axis and the subsequent strain is calculated as

$$\delta = GH - EF = (R - y)\theta - R\theta = -y\theta$$
$$\varepsilon(y) = \frac{\delta}{L} = \frac{-y\theta}{R\theta} = \frac{-y}{R} \qquad (7.8)$$
$$\varepsilon_B = \frac{-b}{2R}; \quad \varepsilon_D = \frac{+b}{2R}$$

The following are some of the important issues based on this derivation:

- The negative sign indicates shortening above the neutral axis and also indicates lengthening below the neutral axis for negative values of y.
- Another thing to note is that the strain is linear and is zero at the neutral axis.
- A very important fact in this relation is that the linear strain relation is independent of the type of the cross-section as long as the distance is measured from the NA.
- It therefore follows the maximum strains are felt by the fibers on the top or the bottom.
- In our illustration they are the same but of opposite sign because the cross-section is symmetric.
- The maximum strain depends on the distance furtherest from the NA and must include the sign.
- Finally, **every cross-section** of the bar is subject to the same strain distribution.

We have related the bending deformation to normal strains. This is due to the resistance of the beam to bending alone. The shear force distribution that always accompanies the bending moment distribution in real problems will create shear stresses on the beam and that is discussed separately.

7.3.2 Normal Bending Stresses

Pure bending introduces a linear normal stress distribution in the cross-section of the beam. Both tensile and compressive stresses are created in the same cross-section. The maximum stresses are on the outer fibers of the beam. Hence failure is likely to occur either at the top or the bottom of the beam, whichever is higher in magnitude. Since our design will usually be limited to the elastic range, the stress distribution can be related to the strain in pure bending using the modulus of elasticity of the material and Eq. (7.8). Using this stress distribution we can then relate the stress and the applied bending moment through the integral established in Eqn. (7.5). Note the bending radius R is the same for the entire beam.

$$\sigma_x(y) = E\varepsilon_x(y) = -E\frac{y}{R} = -\frac{E}{R}y \qquad (7.9)$$

$$M_z = \int_A -y\sigma_x(y)\,dA = \int_A -y\left(-\frac{E}{R}y\right)dA = \frac{E}{R}\int_A y^2\,dA = \frac{E}{R}I_{zz} \qquad (7.10)$$

$$\frac{E}{R} = \frac{M_z}{I_{zz}}$$
$$\sigma_x(y) = -\frac{M_z}{I_{zz}}y \qquad (7.11)$$

Since y is measured with respect the NA, $\boldsymbol{I_{zz}}$ is the second moment of area (or the moment of inertia - MOI) about the NA. What about the NA itself? Let us look at Eqn. (7.6) and use Eqn. (7.11)

$$F_x = 0 = \int_A \sigma_x(y)\,dA = \int_A\left(-\frac{M_z}{I_{zz}}y\right)dA = -\frac{M_z}{I_{zz}}\int_A y\,dA$$
$$\int_A y\,dA = 0 \qquad (7.12)$$

Eqn. (7.12) is satisfied if the NA passes through the centroid for the area of cross-section. Or, the z-centroidal axis of the cross-section is the NA.

You will notice that we used a rectangular cross-section for our illustration of bending though the nature of the cross-section did not feature in our discussion. We discovered the NA is very important for bending and bending deformation (also strain and stress) are measured from this axis with the deformation being zero on the NA. In Eqn. (7.12) we now assert that the NA must be located at the centroid of the cross-section. As a consequence the stress in Eqn. (7.11) is based on the MOI about the centroidal axis.

Thumb Rule (design): We can see that to keep the bending stress small our beam must have a large $\boldsymbol{I_{zz}}$ in the cross-section. This means we must have most of the area as far from the NA as possible. To handle large bending moment you can see the following beam cross-sections used in the building or bridge construction or aircraft wings.

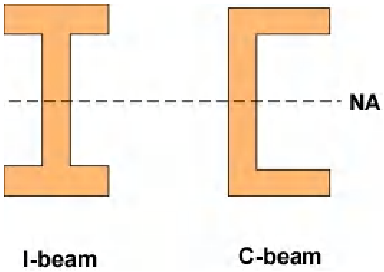


Figure 7.3.5 Favorite cross-section for beam bending

The third integral from Section 7.3.1 can also be addressed with this new information. Substituting the stress relation in the integral we get:

$$M_y = 0 = \int_A z\,\sigma_x\,dA = \int_A z\left(-\frac{E}{R}y\right)dA = -\frac{E}{R}\int_A yz\,dA = -\frac{E}{R}I_{yz} \qquad (7.13)$$

$$I_{yz} = 0$$

This implies that for **pure bending** about the z-axis the cross-section must have an axis of symmetry about the centroid for the product of inertia to be zero.

7.3.3 Bending Deflection

We will go back to the beginning of the section and examine Figure 7.3.1 again. We really would like to know how is displacement of the beam ***y(x)*** connected to the bending moment moment $\mathbf{M_z}$. For pure bending this is connected to the radius of curvature **R** that appears in the equations above. From calculus the radius **R** and **y(x)** are related through the expression from calculus:

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{d^2y}{dx^2} \quad \left(\text{for small } \frac{dy}{dx}\right) \qquad (7.14)$$

We know that the slope of the deflections will be small. From the previous relation we can then write:

$$\frac{1}{R} = \frac{d^2y}{dx^2} = \frac{M_z}{EI_{zz}} \qquad (7.15)$$

The latter defines a second order *differential equation* for **y(x)**. We know that the bending moment $\mathbf{M_z}$ could be a function of length of the beam, **x,** and $\mathbf{I_{zz}}$ could also change with **x** if the cross-section varies. The differential equation can be integrated twice with appropriate boundary conditions to establish **y(x)** . The simplest way of solving for the deflection of the beam is to *directly extend the use of the singularity functions beyond the calculation of $M_z(x)$*.

Bending Moment Variation Along The Beam:
We have obtained all of the equations above for pure bending - this means a constant bending moment on the beam. From sections 7.1 and 7.2 we know that for many real problems we will rarely have beams subject to pure bending. Bending will depend on the load in the cross-section in most cases vary along the cross-section. This introduces shear load variation in the cross-section too. Do we need to develop additional equations fro these problems. The answer is No! We can use the relations above to define instantaneous relations at any position along the beam, **x** due to the value of $\mathbf{M_z(x)}$ at that location. In real design we prefer to use the same cross-section of the beam. This also implies that the instantaneous radius of curvature will change with location. *We will avoid pure bending problems* since the analysis is *no different from problems where bending moment varies.*

7.3.4 Example 7.4 - Calculating Bending Deflection

This example is very similar to Example 7.3. The beam AB is pin supported at end A and has roller support at end B. It has a point load and a constant distributed load. We have examined these problems in previous section. The cross-section of the beam is a ‘T’ section with dimensions shown in Figure 7.3.6. This is new information.

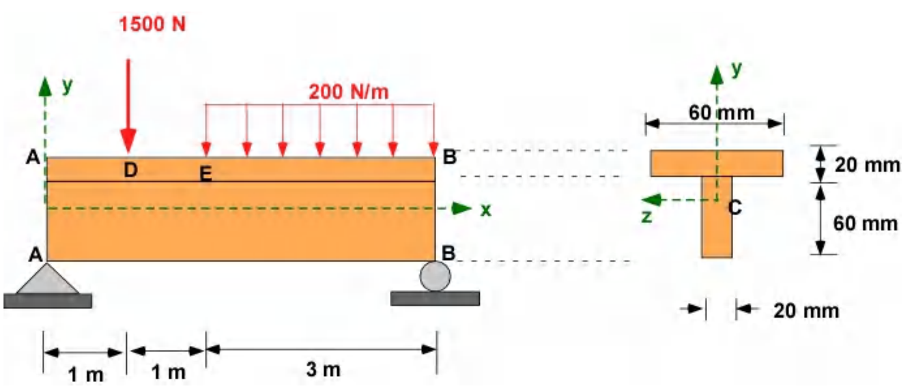


Figure 7.3.6 Example 7.4

Let us make note of the following features of the problem:

1. The bean subjected to a concentrated load and a distributed load.
2. It is supported by a pin on the left end and a roller on the right making the problem statically determinate.
3. The beam has uniform cross-section along the length.
4. The cross-section of the beam is a ‘T’ section. This has a single axis of symmetry and therefore the product of inertia I_{yz} is zero.
5. The cross-section dimensions and the length of the beam have different scales.
6. The equations require that the coordinate system be located at the centroid, which will have to be calculated.
7. The maximum stress, both in compression and tension, will be either on the top or the bottom of the beam. They will not be the same as the centroid is closer to the top of the beam.
8. We expect to have a shear force and bending moment distributed over the beam
9. Maximum bending stresses on the beam will be located at the location of maximum bending moment.

Let us assume that the beam is made of standard construction steel A36. This has a density of 7,800 kg/m^3 (0.28 lb/cu in), Young's modulus for is 200 GPa (29,000,000 psi), a Poisson's ratio of 0.26, and a shear modulus of 75 GPa (10,900,000 psi) . It has a minimum yield strength of 250 MPa (36,000 psi) and ultimate tensile strength of 400 - 550 MPa (58,000–80,000 psi) ***- Wikipedia***.

For this example we will step through the following different (but extensive) calculations in sequence:

i. The reactions at the support.
ii. The shear force and bending moment diagram.
iii. The location of the centroid.
iv. The calculation of the moment of inertia.
v. The maximum bending stress and the corresponding factor of safety in bending for the given load.
vi. The deflection of the beam

These calculations are based on equations developed above. Chapter 2 provides information on centroid and MOI. We will use singularity functions for developing the shear and bending moment diagrams. We will use MATLAB for all our calculations and verify we have the same results.

i. Reactions

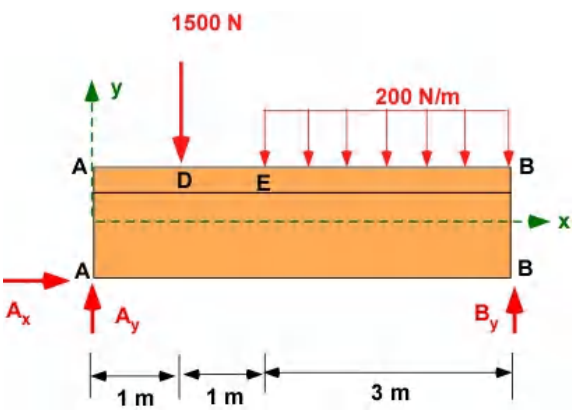


Figure 7.3.7 Example 7.4 -Reactions

$$\sum F_x = A_x = 0;$$
$$\sum F_y = A_y + B_y - 1500 - 200 \times 3 = 0; \quad A_y + B_y = 2100;$$
$$\sum M_A = -(1500 \times 1) - (600 \times 3.5) + B_y \times 5 = 0;$$
$$B_y = 720\,N; \quad A_y = 1380\,N; \quad A_x = 0;$$

ii. Shear Force and Bending Moment Diagram (using singularity functions)

$$w(x) = -200\langle x-2\rangle^0$$
$$V(x) = -200\langle x-2\rangle^1 + 1380\langle x-0\rangle^0 - 1500\langle x-1\rangle^0$$
$$M(x) = -200\langle x-2\rangle^2/2 + 1380\langle x-0\rangle^1 - 1500\langle x-1\rangle^1$$

We draw the load, shear and moment diagrams by intuition

The w(x) is the given distributed load between x = 2 and x = 5.

The V(x) diagram starts at x = 0 with + 1380. It stays flat till x = 1. Then it drops by 1500 at x = 1. It stays flat till x = 2. It is then linear with a value of -720 at x = 5.

The M(x) diagram starts at 0 at x = 0. It increases linearly to 1380 at x = 1 . This is the area under the V(x) curve till that point. It decrease linearly till x = 2 to a value of 1260. This is the negative area in the V(x) diagram. It decreases to zero at x = 5 quadratically.

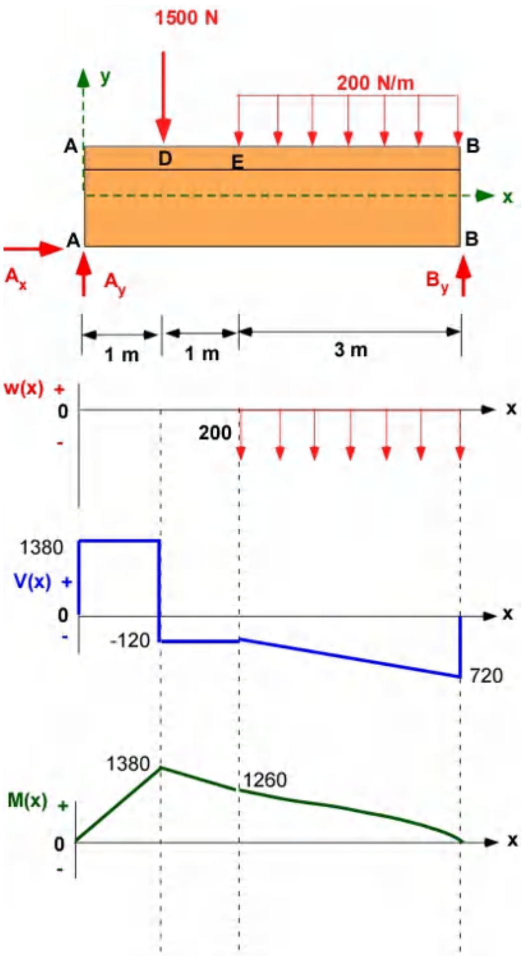


Figure 7.3.8 The load, shear, and moment diagrams for Example 7.4

iii. Location of Centroid

Although the actual coordinates in the cross-section are y and z located at the centroid, for convenience we will use x-y coordinates to locate the centroid to help us recollect our earlier experience for such calculations.. Since there is an axis of symmetry the centroid must lie on the axis of symmetry - which will be place the y-axis through the center of the cross-section. As we have not calculated the centroid in a while we will identify the centroid location formally.

For our calculation we are going to use composite areas. We recognize that we can break up the cross-section into two areas identified in the figure as area 1 and area 2. There areas are A_1 and A_2. We locate the origin of the transient coordinates at the left bottom corner. The composite areas have their own centroid located at (x_1, y_1) and (x_2, y_2) - Figure 7.3.9. We saw this very early in Chapter 2.

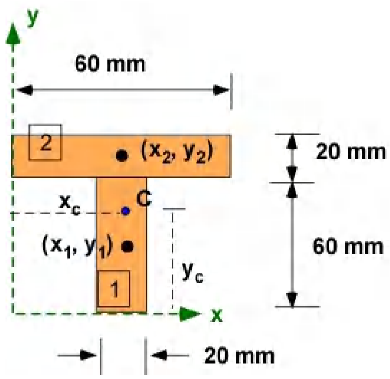


Figure 7.3.9 Layout for centroid calculations

$$x_c = \frac{x_1 A_1 + x_2 A_2}{A_1 + A_2} = \frac{30\times(60\times20)+30\times(60\times20)}{(60\times20)+(60\times20)} = 30\ mm$$

$$y_c = \frac{y_1 A_1 + y_2 A_2}{A_1 + A_2} = \frac{30\times(60\times20)+70\times(60\times20)}{(60\times20)+(60\times20)} = 50\ mm$$

We have formally established the y-axis I through center of the cross-section. This is also obtained by arguing symmetry.

iv. Moment of Inertia (I_{zz})

We have bending moment only along the x-axis - M_z in the problem. We need only only the MOI about the z-axis through the centroid C - I_{zz} . We will use the parallel axis theorem to calculate the MOI about the centroid.

Let w1 and w2 be the width and h1 and h2 be the height of the two rectangles respectively.

$$\begin{aligned} I_z &= \frac{w_1 h_1^3}{12} + w_1 h_1 (y_1 - y_c)^2 + \frac{w_2 h_2^3}{12} + w_2 h_2 (y_2 - y_c)^2 \\ &= \frac{20\times60^3}{12} + 20\times60\times(30-50)^2 + \frac{60\times20^3}{12} + 20\times60\times(70-50)^2 \\ &= 1360000\ mm^4 = 1.36\times10^{-6}\ m^4 \end{aligned}$$

v. Maximum Bending Stress

From Figure 7.3.8 the maximum bending moment occurs at x = 1 m. This is the location on the beam where the maximum bending stress will be present. The bending moment is positive with a value of M_z = 1380 Nm. The normal stress in this cross-section is linear. The bending moment at the top fiber in the cross-section will be compressive (-x direction) while the bending moment at the bottom fiber in the cross-section will be in tension (+x direction). The maximum stress in tension will be greater than the maximum stress in compression because its distance from the NA (centroid) is greater. This is shown in the figure below.

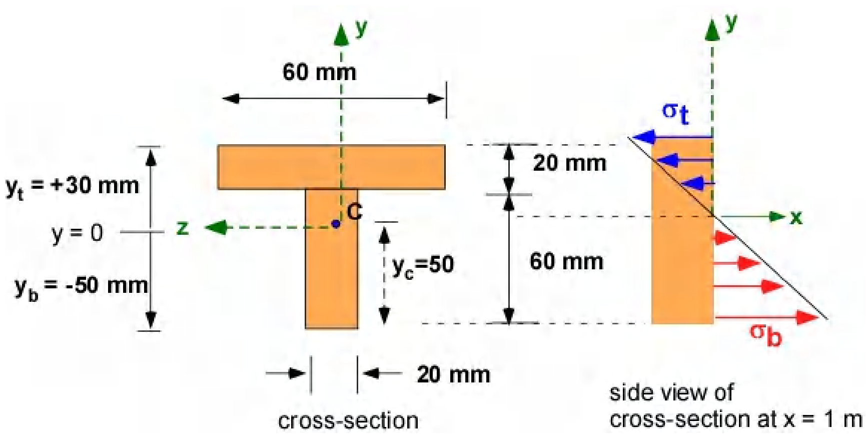


Figure 7.3.10 Stress in cross-section

We calculate the stresses at x = 1 m using Eq. (7.11) as:

$$\sigma_x(y_t) = -\frac{M_z y_t}{I_{zz}} = -\frac{1380\times(30/1000)}{1.36\times10^{-6}} = -30.44\,MPa$$

$$\sigma_x(y_b) = -\frac{M_z y_t}{I_{zz}} = -\frac{1380\times(-50/1000)}{1.36\times10^{-6}} = +50.73\,MPa$$

Note these stresses are independent of the material the beam for a given loading. For a given loading they are based on the geometry of the cross-section. If we believe that the beam responds the same in tension and compression we use the maximum of the stresses to calculate the factor of safety (FOS). We define the FOS as the ratio of the maximum strength of the material to the maximum stress in the beam.

FOS for steel A36 (using yield stress) = 250/50.73 = 4.92
FOS for steel A36 (using ultimate stress) = 400/50.73 = 7.88

The current design (the ***combination of load, cross-section and material***) is more than adequate and we may be able to save material by reducing the dimensions. However, if that requires custom beam fabrication that may not be a good idea because of increased costs.

Let us say we decided to use Ash Wood for the beam of the same cross-section dimensions.

Table 7.1 Mechanical Strength for Ash Wood (Black)

Nature of Wood	Modulus of Elasticity (MPa)	Compression: parallel to the grain (kPa)	Compression: perpendicular to grain (kPa)	Shear : parallel to grain (kPa)	Tension: perpendicular to grain (kPa)
Green	7200	15,900	2400	5900	3400
12 % moisture	11,000	41,200	5200	10800	4800

(Mechanical Properties of Wood David W. Green, Jerrold E. Winandy, and David E. Kretschmann)

Let us assume that the grains are perpendicular to the beam cross-section and we choose the reduced moisture wood. We then will calculate

FOS(tension) = 4.8/50.73 = 0.095 (inadequate)

FOS (compression) = 41.2/30.44 = 1.35 (barely allowable)

We either have to increase the cross-section dimensions if we are going to use Ash wood for the design. *Notice now we are also talking about the design of the beam instead of just analysis of the beam*.

vi Beam Deflection

We will calculate the beam deflection using Eq. (7.13) and singularity functions. The boundary conditions for the beam support requires the deflection to be zero at x = 0 and x = 5. We will use the singularity finction for $M_z(x)$ to continue. We will use E = 200 GPa (different material since wood was inadequate).

$$EI_{zz}\frac{d^2y}{dx^2} = -200\langle x-2\rangle^2/2 + 1380\langle x-0\rangle^1 - 1500\langle x-1\rangle^1$$

$$EI_{zz}\frac{dy}{dx} = -200\langle x-2\rangle^3/2/3 + 1380\langle x-0\rangle^2/2 - 1500\langle x-1\rangle^2/2 + C_1$$

$$EI_{zz}y(x) = -200\langle x-2\rangle^4/2/3/4 + 1380\langle x-0\rangle^3/2/3 - 1500\langle x-1\rangle^3/2/3 + C_1x + C_2$$

$$y(0) = 0 = C_2$$

$$y(5) = 0 = -200\times 3^4/24 + 1380\times 5^3/6 - 1500\times 4^4/6 + C_1\times 5;\ C_1 = -2415$$

$$y(x) = \left[\frac{1}{\left(200\times 10^9\right)\left(1.36\times 10^{-6}\right)}\right]\left\{-200\langle x-2\rangle^4/2/3/4 + 1380\langle x-0\rangle^3/2/3 - 1500\langle x-1\rangle^3/2/3 - 2415x\right\}$$

You can estimate and plot the displacement as a function of x. This is done easier with MATLAB in the next sub-section.

To summarize we have the complete analysis for Example 7.4 through a sequence of calculations. These calculations extend the earlier calculations on equilibrium, load and moment distribution, to obtain stress and deformation. This is significant for beam design. The same sequence will be used for all beam bending and design problems, particularly if they are statically determinate.. The sequence is repeated below for completeness and emphasis. The sequence of calculations are:

i. The reactions at the support.
ii. The shear force and bending moment diagram.
iii. The location of the centroid.
iv. The calculation of the moment of inertia.
v. The maximum bending stress and the corresponding factor of safety in bending for the given load.
vi. The deflection of the beam

7.3.5 Example 7.4 using MATLAB

Please review the code . All of the sections are clearly identified. The calculations are the same as in the section above. Make sure you verify them. Please note the **symbolic x** is replaced by the **numerical x** after singularity functions are used.

In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, June 2016
%  Example 7-4
% Beams 2D - Reactions, Shear and bending diagram (singularity)
% centroid , moment of inertia
% stress, factor of safety
% deflection
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(5)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 7.4 - Beam Bending - Complete Analysis\n')
fprintf('----------------------------------------------------\n')

%% Data
%% The constants
w1 = -200;
Fd = -1500;
xd = 1; xe = 2; xb = 5; xf = (xe+xb)/2; % parameterizing location
% beam location
A= [0,0,0]; D = [xd,0,0]; E = [xe,0,0];  B = [xb,0,0];
F = [xf,0,0];  % point load location for the distributed load

fprintf('Coordinates')
fprintf('\n-----------------------')
fprintf('\nPoint A [m]   : '),disp(A)
fprintf('Point D [m]   : '),disp(D)
fprintf('Point E [m]   : '),disp(E)
fprintf('Point B [m]   : '),disp(B)
fprintf('Point F [m]   : '),disp(F)

fprintf('\nLoads')
fprintf('\n-----------------------')
fprintf('\nLoad at D [N]      = '),disp(Fd)
fprintf('w(x)[N/m]          = '),disp(w1)
%% (i) reactions
syms Ax Ay By x real
FA = [Ax,Ay,0];  FD = [0,Fd,0];  FB = [0,By,0];
Fed = int(w1,x,E(1),B(1));
FED = [0,Fed,0];
SumF = FA+FB+FD+FED  % sum of forces

rAD = D-A;  rAF = F -A;  rAB = B - A;
SumMA = cross(rAD,FD) + cross(rAF,FED) + cross(rAB,FB);
% sum of moments about A

fprintf('\nEquilibrium - Beam\n')
fprintf('----------------------------')
fprintf('\nSumF : \n'),disp(vpa(SumF',5))
fprintf('SumMA: \n'),disp(vpa(SumMA',5))

sol1 = solve(SumF(1),SumF(2),SumMA(3));
Ax = double(sol1.Ax);
Ay = double(sol1.Ay);
By = double(sol1.By);
```

```
FA = subs(FA);  FB = subs(FB);

fprintf('\nAx [N] = '),disp(Ax)
fprintf('Ay [N] = '),disp(Ay)
fprintf('By [N] = '),disp(By)
fprintf('----------------------------------------------------\n')

%% (ii) Shear force and bending moment diagram
xpoints = [A(1),D(1),E(1),B(1)];  % Points for discontinuities in all
functions
ypoints = [A(2),D(2),E(2),B(2)];

% Define an an0nymous function (used to be called inline function)
% n is expected to be greater than or equal to zero
singfun = @(x,a,n) (x-a).^n.*(x >= a);
%
% assemble the functions
% this is not symbolic implementation
% use very close points for x so you cannot see the jump
x = linspace(0,xb,501);  % x is no longer symbolic
w = w1*singfun(x,xe,0);
V = w1*singfun(x,xe,1)/1 + Ay*singfun(x,0,0)+ Fd*singfun(x,xd,0);
M = w1*singfun(x,xe,2)/1/2  +Ay*singfun(x,0,1)/1 + Fd*singfun(x,xd,1)/1;

% plot the functions
figure
set(gcf,'Position',[25,50,500,450], ...
    'Color','w');
plot(x,w,'r-','LineWidth',2)
hold on
plot(x,V,'b-','LineWidth',2)
plot(x,M,'g-','LineWidth',2)
xlabel('\bfx [m]')
ylabel('\bf w [N/m], V [N], M [Nm]')
grid
legend('w(x)','V(x)','M(x)','Location','Best')
title('\bf Example 7.4 using Singularity functions')
 % points on the beam where functoions change
plot(xpoints,ypoints,'mo','MarkerFaceColor','y')
hold off

%% (iii) Centroid Calculations
% area 1 is the tall rectangle; area 2 is the wide rectangle

Ar= [60*20,60*20];   % area of the rectangles
xcen =[30,30];  ycen= [30,70];  % centroid of the respective rectangles

fprintf('Centroid Cross-section\n')
fprintf('---------------------------\n')
% centrod
xc = sum(xcen.*Ar)/sum(Ar);
yc = sum(ycen.*Ar)/sum(Ar);
fprintf('x - location of centroid [mm] : '),disp(xc)
fprintf('y - location of centroid [mm] : '),disp(yc)

% formal POSITION of recatngle in MATLAB
```

```
% [left bottom corner (x,y),width,height]
rect1 = [20,0,20,60];
rect2 = [0,60,60,20];

% plotting the rectangle and centroid
figure
set(gcf,'Position',[25,50,500,450], ...
    'Color','w');
hr1 = rectangle('Position',rect1);
set(hr1,'FaceColor','g');

hr2 = rectangle('Position',rect2);
set(hr2,'FaceColor','y');
hold on

% draw axis and coordinates
line([xc,xc-25],[yc,yc],'Color','r','LineWidth',2)
plot(xc-25,yc,'<')
text(xc-25,yc-3,'z_c')

line([xc,xc],[yc,yc+35],'Color','r','LineWidth',2)
plot(xc,yc+35,'^')
text(xc+3,yc+35,'y_c')

text(xc+2,yc,'C')
axis square
title('Centroid of cross-section - Example 7.4')
hold off

fprintf('-------------------------------------------------\n')
fprintf('MOI Cross-section\n')
fprintf('---------------------------\n')
wid = [20,60];  hgt = [60,20];  % height and width of beam

% implementing parallel axis theorem
Iz1 = (wid.*hgt.^3./12) + (Ar.*(ycen-yc).^2);
Izz = sum(Iz1);  Izz = Izz/1000/1000/1000/1000;
fprintf('MOI - Izz [m^4] = '),disp(Izz)
fprintf('-------------------------------------------------\n')

%% (v) Maximum bending stress and FOS
fprintf('--------------------------------------------------\n')
fprintf('Maximum bending stress and FOS\n')
fprintf('---------------------------\n')
youter = [80,0]; %top/bottom of beam original reference
ymax = (youter-yc)/1000;  % top and bottom wrt centroid
[Mmax,xmax] = max(M);  % maximum bending moment and location
% yt = 30/1000;   yb = -50/1000;
sigm = -Mmax*ymax/Izz;

if sigm(1) > 0
    sigt = sigm(1);
    sigc = sigm(2);
else
    sigt = sigm(2);
    sigc = sigm(1);
```

```
end

fprintf('Maximum Tensile Stress [MPa]      = '),disp(sigt/1000000);
fprintf('Maximum Compressive Stress [MPa] = '),disp(sigc/1000000);


SigY = 250e06;  SigF = 400e06;
FOSy = SigY/max(abs([sigt,sigc]));
FOSf = SigF/max(abs([sigt,sigc]));
fprintf('Maximum Yield Stress [MPa]     = '),disp(SigY/1000000);
fprintf('Maximum Fracture Stress [MPa] = '),disp(SigF/1000000);
fprintf('Maximum FOS (yield)     ='),disp(FOSy)
fprintf('Maximum FOS (fracture) ='),disp(FOSf)
fprintf('------------------------------------------------------\n')

%% (vi) Beam deflection
fprintf('------------------------------------------------------\n')
fprintf('Beam Deflection\n')
fprintf('----------------------------\n')

EL = 200e09;
fprintf('Modulus of Elasticity [GPa] =     = '),disp(EL/1e9);

syms C1 C2 xx
dydx = w1*singfun(x,xe,3)/1/2/3  +Ay*singfun(x,0,2)/1/2 + ...
    Fd*singfun(x,xd,2)/1/2 + C1;
yx = w1*singfun(x,xe,4)/1/2/3/4  +Ay*singfun(x,0,3)/1/2/3 + ...
    Fd*singfun(x,xd,3)/1/2/3 + C1*x + C2;
eq1 = yx(1);
eq2 = yx(end);

fprintf('\nBoundary Conditions - Beam\n')
fprintf('----------------------------')
fprintf('\neq1: '),disp(vpa(eq1,5))
fprintf('eq2: '),disp(vpa(eq2,5))

sol = solve(eq1,eq2);
C1 = sol.C1;  C2 = sol.C2;
yx = double(subs(yx));
yx = yx/EL/Izz;  % actual displacement
[maxdef,xdef] = min(yx);
fprintf('C1 = '),disp(C1)
fprintf('C2 = '),disp(C2)

fprintf('\nMaximum deflection [m] = '),disp(maxdef)
fprintf('at location  [m] = '),disp(xdef*xb/501)

fprintf('------------------------------------------------\n')

% draw the beam deflection curve
figure
set(gcf,'Position',[25,50,500,450], ...
    'Color','w');

plot(x,yx,'b-','LineWidth',2)
hold on
```

```
xlabel('\bfx [m]')
ylabel('\bf y [m]')
grid
title('\bf Example 7.4; y(x) using Singularity functions')
 % points on the beam where functoions change
plot(xpoints,ypoints,'mo','MarkerFaceColor','y')
hold off
```

In the Command Window

```
Example 7.4 - Beam Bending - Complete Analysis
---------------------------------------------------
Coordinates
----------------------
Point A [m]  :      0      0      0
Point D [m]  :      1      0      0
Point E [m]  :      2      0      0
Point B [m]  :      5      0      0
Point F [m]  :           3.5            0            0

Loads
----------------------
Load at D [N]     =        -1500
w(x)[N/m]         =   -200
SumF =
[ Ax, Ay + By - 2100, 0]
SumMA =
[ 0, 0, 5*By - 3600]

Equilibrium - Beam
---------------------------
SumF :
               Ax
 Ay + By - 2100.0
                0
SumMA:
               0
               0
 5.0*By - 3600.0

Ax [N] =      0
Ay [N] =         1380
By [N] =    720
--------------------------------------------------------------
Centroid Cross-section
---------------------------
x - location of centroid [mm] :     30
y - location of centroid [mm] :     50
-----------------------------------------------------------
MOI Cross-section
---------------------------
MOI - Izz [m^4] =      1.36e-06
-----------------------------------------------------------
-----------------------------------------------------------
Maximum bending stress and FOS
---------------------------
```

```
Maximum Tensile Stress [MPa]      =        50.735
Maximum Compressive Stress [MPa] =        -30.441
Maximum Yield Stress [MPa]     =    250
Maximum Fracture Stress [MPa] =    400
Maximum FOS (yield)     =       4.9275
Maximum FOS (fracture) =       7.8841
----------------------------------------------------------------
-----------------------------------------------------------
Beam Deflection
---------------------------
Modulus of Elasticity [GPa] =     =    200

Boundary Conditions - Beam
---------------------------
eq1: C2
eq2: 5.0*C1 + C2 + 12075.0
C1 = -2415
C2 = 0

Maximum deflection [m] =     -0.012154
at location  [m] =       2.3353
----------------------------------------------------------------
```

In Figure Window(s)

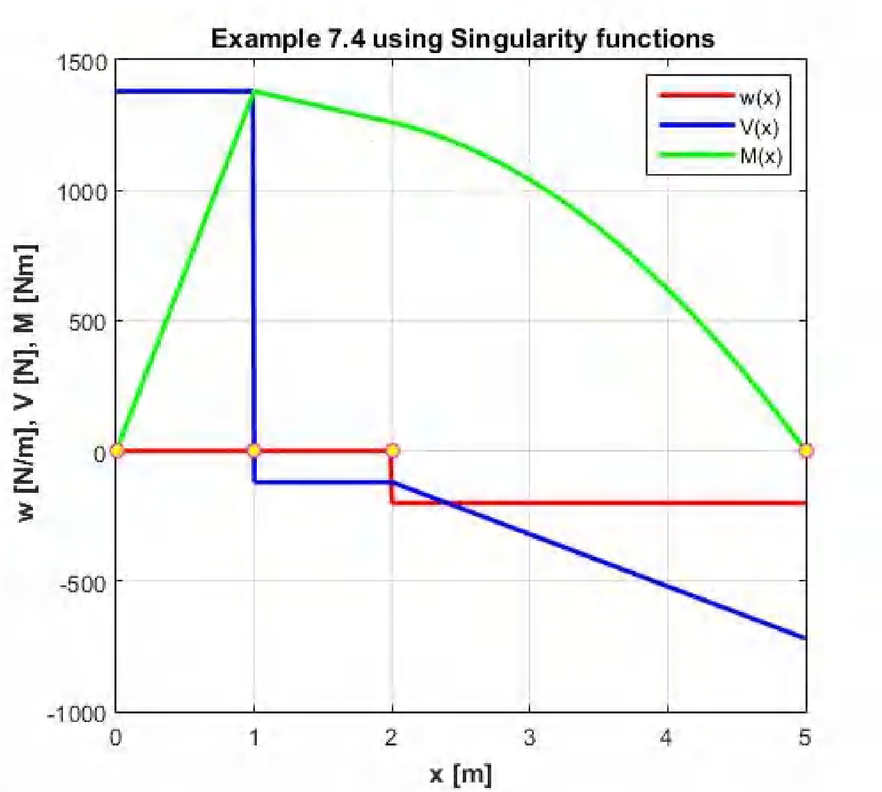


Figure 7.3.11a Load,shear,and bending moment

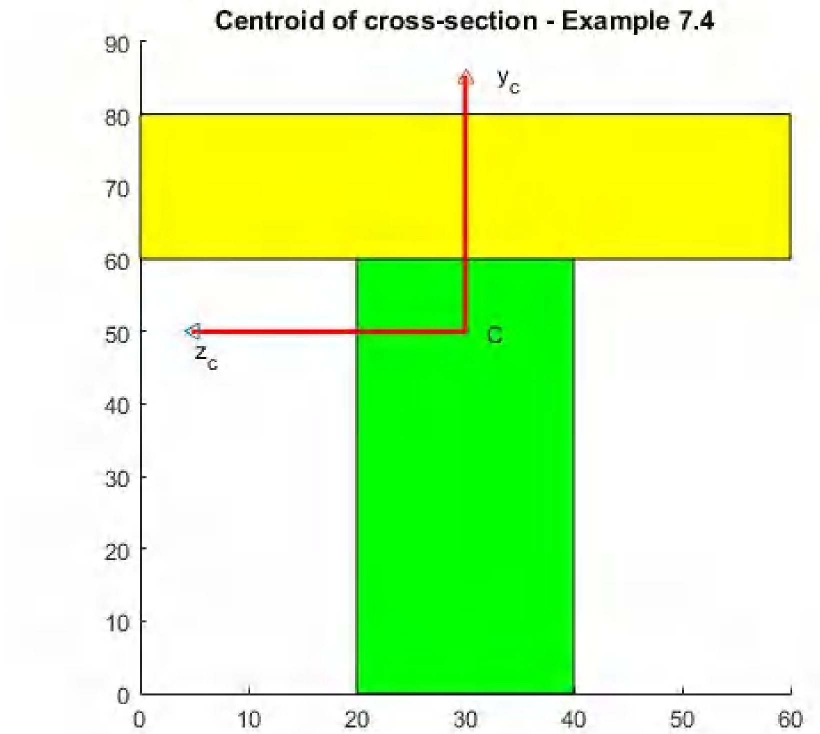


Figure 7.3.11b Centroid determination

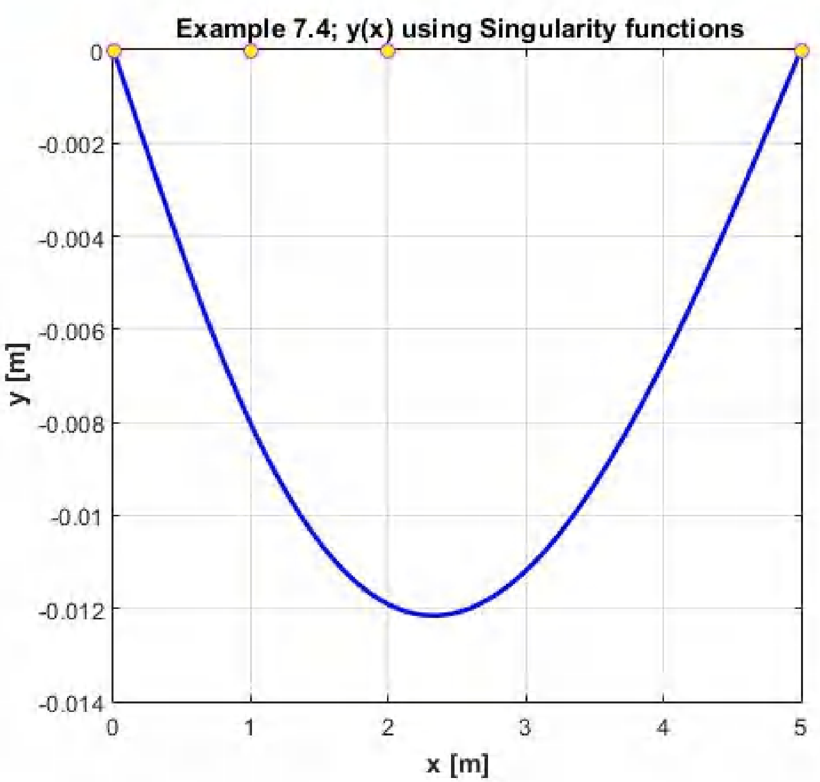


Figure 7.3.11c. Beam deflection

Summary

There was a lot of material developed in this section. The first question in this section is how does the beam respond to the application of pure bending. The beam cross-section is the same along the length (uniform).

1. Pure bending means that the the beam carries a constant bending moment. If this moment was not constant then we can expect the beam to also have a shear force and then it would not be pure bending.

2. Pure bending causes the beam to deflect in an arc of a circle - the radius is usually very large because the deflections are usually small.

3. This deflection causes a linear normal stress distribution about the cross-section about the centroid. At the centroid itself the stress is zero. Therefore maximum stress are at the outer fibers of the beam (top and bottom). The top and the bottom are subject to oppositely directed normal stresses.

4. The beam deflection - or the inverse of the radius of curvature of the beam is directly related to the bending moment and inversely related to the modulus of elasticity of the material of the beam and also inversely related to the MOI about the centroid parallel to the axis of the applied bending moment.

5. You will notice that we did not introduce an example in pure bending but went on to work through Example 7.4 (similar to Example 7.3) that was not pure bending. We used all the relations that we developed in this section and they are not restricted to pure bending. We just assumed that there was instantaneous pure bending at every point on the beam.

6. A pure bending problem is very easy to analyze compared to Example 7.4 since every cross-section of the beam will have the same stress distribution as the bending moment is constant along the beam. It is rare to find pure bending in real design situations. For pure bending you would follow the same analysis as in this example at any cross-section instead of focusing on location of the maximum bending moment in the beam.

7. You can see that MATLAB is very useful to solve these kinds of problems. In this chapter we will use MATLAB extensively. We will off course explicitly calculate the expressions when they appear for the first time.

Execution in OCTAVE
In Octave Editor

The code is same as in the MATLAB except for the highlighted changes.

```
clc, clear, format compact, format shortg, close all, digits(5)
warning off
pkg load symbolic;

fprintf('\nSumF : \n'),disp(vpa(SumF,5))
fprintf('SumMA: \n'),disp(vpa(SumMA,5))

%FA = subs(FA);  FB = subs(FB);
FA = subs(FA,Ax)
FA = subs(FA,Ay)
FB = subs(FB,By);

%yx = double(subs(yx));
yx = subs(yx,C1);
yx = subs(yx,C2);
```

```
yx = double(yx);
```

In Octave Command Window
The results are the same as in MATLAB Command window (except for some formatting changes) and are not included here.

In Octave Figure Window
There are three figures which are the same as those included in the MATLAB Figure Window and only one is included below. The default font appears to be smaller in Octave.

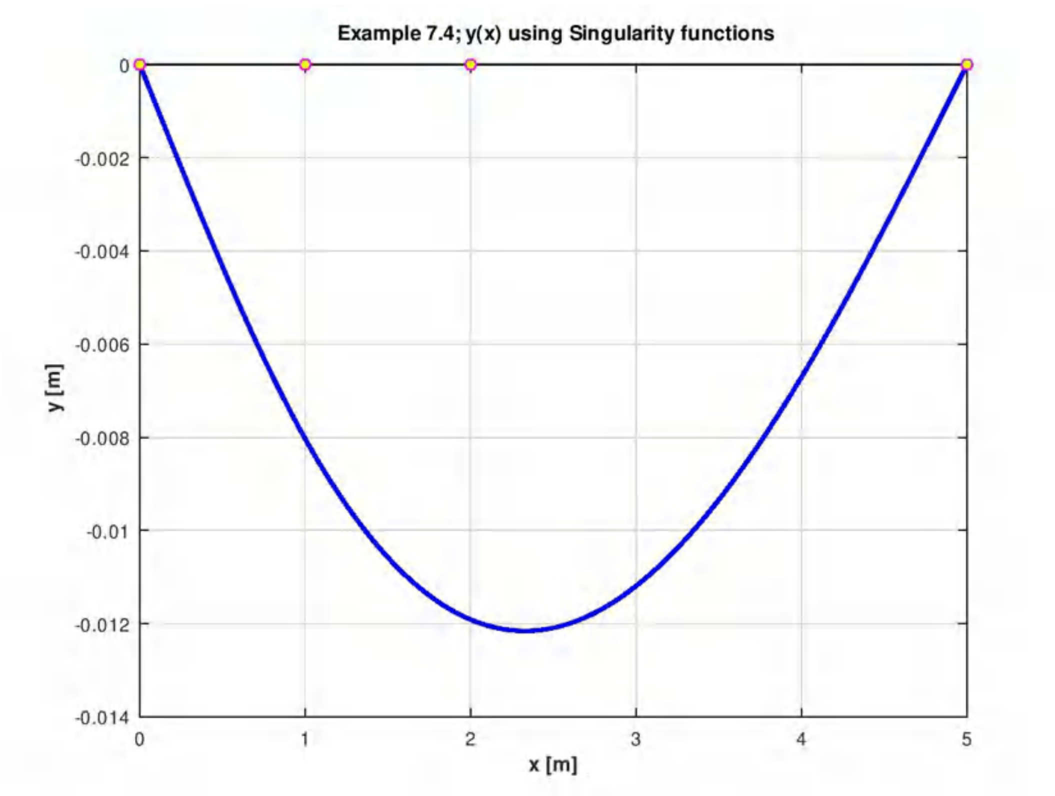


Figure 7.3.11d. Beam deflection - OCTAVE

7.3.6 Example 7.5

Example 7.5 is defined in Figure 7.3.12. The cross-section of the beam is symmetric and shown in the adjacent figure. This is a π shaped beam. This beam is cantilevered at the left end with a concentrated force and a triangular distribution that is reversed from that of Example 7.2. This simple change in distribution brings so much complexity to the analysis. It also allows us to be creative in application. We will perform a complete analysis of beam bending for this example using the engineering approach. Let us assume that the beam is made of Aluminum Alloy. It has a density of 2700 kg/m^3 , Young's modulus of 70 GPa , a Poisson's ratio of 0.32, and a shear modulus of 26 GPa. It has an yield strength of 95 MPa and ultimate tensile strength of 110 MPa in tension. You are then challenged to apply the same analysis using MATLAB.

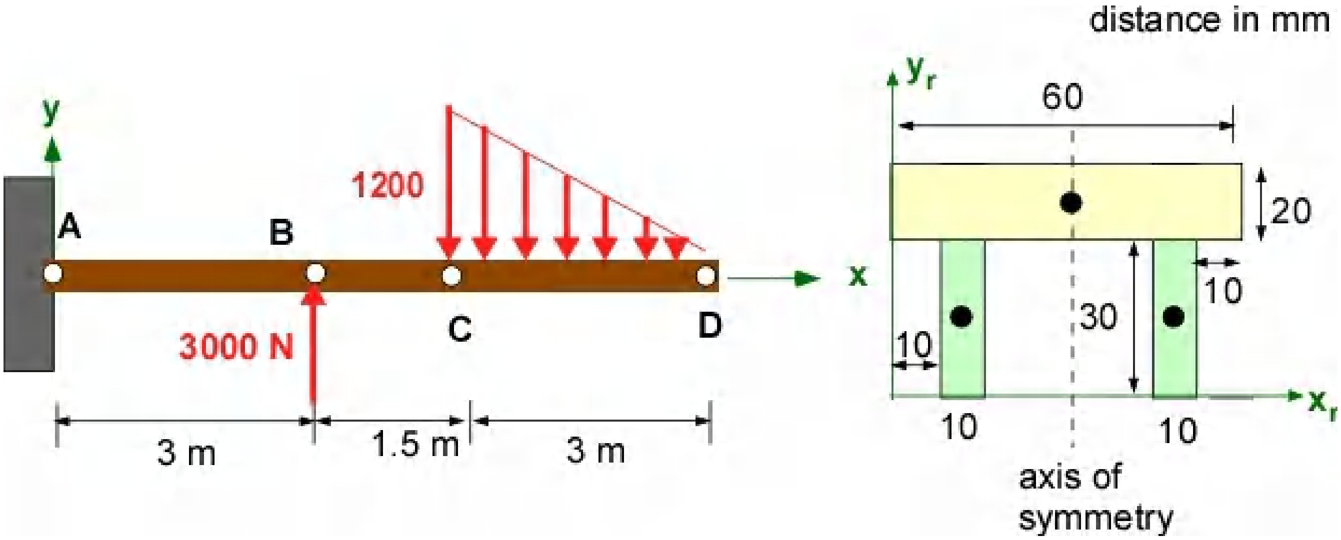


Figure 7.3.12. Example 7.5

Data: Beam, loading and locations are shown in the figure. A = [0,0,0]; B = [1,0,0]; C = [3,0,0]; D = [5,0,0];
F_B = -3000 N; w = -500 N/m.
Cantilevered at A -Statically Determinate.
σ_y = 95 MPa; σ_U = 110 MPa

Find: Use engineering approach to solve for:

i. The reactions at the support.
ii. The shear force and bending moment diagram.
iii. The location of the centroid.
iv. The calculation of the moment of inertia.
v. The maximum bending stress and the corresponding factor of safety in bending for the given load.
vi. The deflection of the beam

Assumption: Cross-section has an axis of symmetry.

Solution:

i. The reactions at the support.

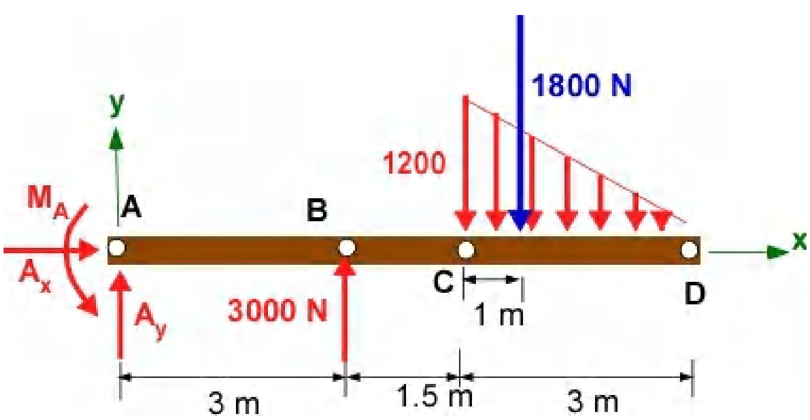


Figure 7.3.13. Example 7.5 - FBD

Using the FBD above:

$$\sum F_x = 0 = A_x;$$

$$\sum F_y = 0 = A_y + 3000 + \frac{1}{2}(-1200)\times 3; \quad A_y = -1200[N];$$

$$\sum M_A = 0 = M_A + (3000\times 3) - (1800\times 5.5); \quad M_A = 900[Nm];$$

ii. The shear force and bending moment diagram.

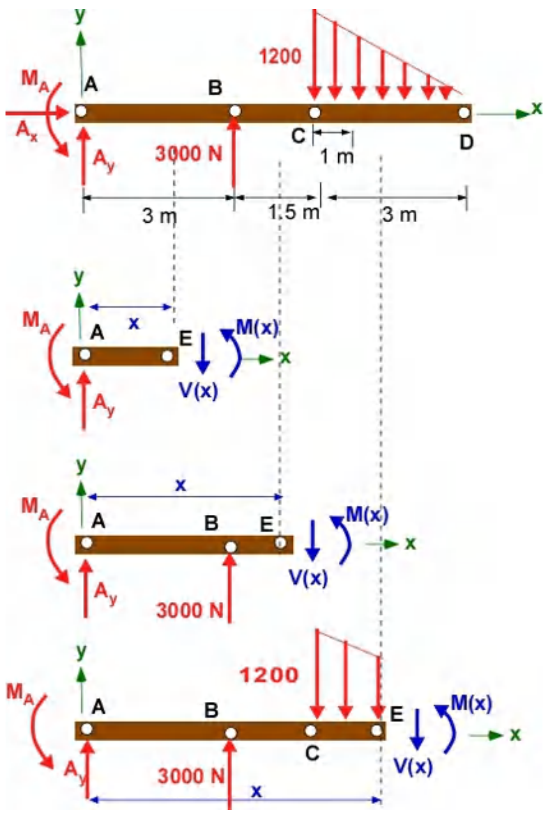


Figure 7.3.14. FBD for obtaining shear and moment distributions

Section AB:

$$\sum F_y = 0 = -1200 - V(x); \quad V(x) = -1200[N];$$

$$\sum M_A = 0 = 900 + M(x) - xV(x); \quad M(x) = -900 - 1200x[Nm];$$

Section BC:

$$\sum F_y = 0 = -1200 + 3000 - V(x); \quad V(x) = 1800[N];$$

$$\sum M_A = 0 = 900 + M(x) + 3000\times 3 - xV(x); \quad M(x) = -900 - 9000 + 1800x[Nm];$$

Section CD:

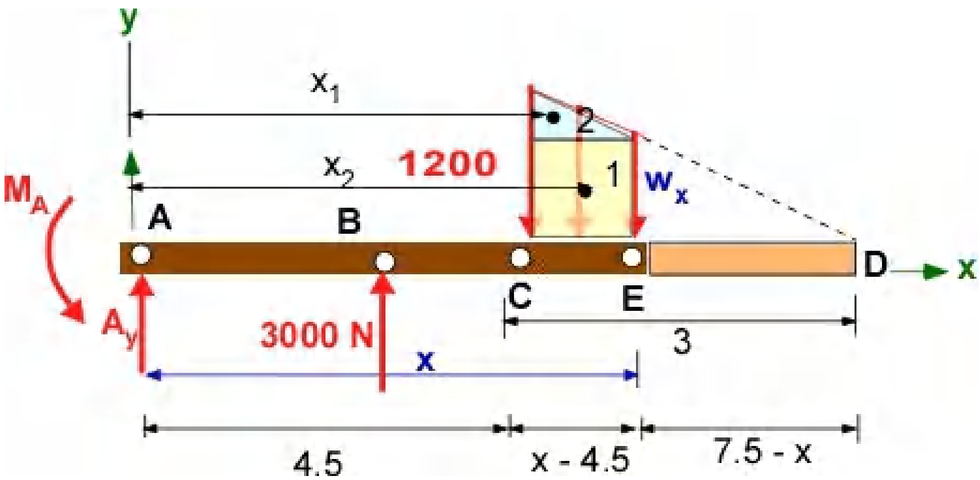


Figure 7.3.15. FBD for obtaining shear and moment distributions in section CD

The load between CE can be assembled as a rectangular load $\mathbf{F_1}$ with centroid at $\mathbf{x_1}$ and a triangular load $\mathbf{F_2}$ with the centroid at $\mathbf{x_2}$.

The values are obtained through geometry - similar triangles) as:

$$w_x = \frac{-1200}{3}(7.5-x) = -400(7.5-x) = 400x - 3000;$$

$$F_1 = w_x(x-4.5) = (400x-3000)(x-4.5);$$

$$x_1 = 4.5 + \frac{1}{2}(x-4.5) = 0.5x + 2.25;$$

$$F_2 = (-1200 - w_x)\frac{1}{2}(x-4.5) = -(200x-900)(x-4.5);$$

$$x_2 = 4.5 + \frac{1}{3}(x-4.5) = 0.3333x + 3;$$

$$\sum F_y = 0 = -1200 + 3000 + F_1 + F_2 - V(x);$$

$$V(x) = 1800 + (400x-3000)(x-4.5) - (200x-900)(x-4.5)[N];$$

$$V(x) = 200x^2 - 3000x + 11250[N];$$

$$\sum M_A = 0 = 9900 + M(x) + x_1F_1 + x_2F_2 - xV(x);$$

$$M(x) = 66.667x^3 - 1500x^2 + 11250x - 28125[Nm];$$

iii. The location of the centroid.

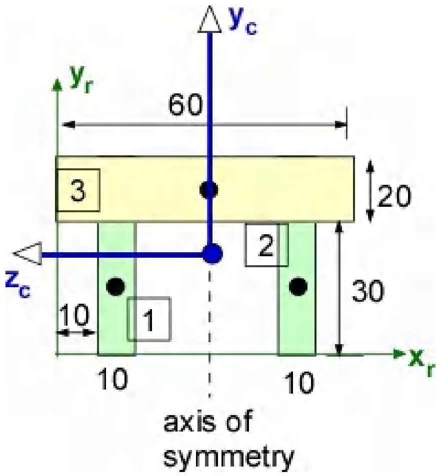


Figure 7.3.16. Beam cross-section for centroid calculation

The cross-section has a single axis of symmetry and therefore the centroid is located on it. We need to calculate only the y-centroid for this cross-section. There are three rectangles identified on the figure using integers. The reference coordinates are indicated on the figure. The caluclations are in mm.

$$A_1 = A_2 = 10 \times 30 = 300; \quad A_3 = 60 \times 20 = 1200;$$

$$x_1 = 15; \quad x_2 = 45; \quad x_3 = 30;$$

$$y_1 = 15, \quad y_2 = 15; \quad y_3 = 40;$$

$$x_c = \frac{15 \times 300 + 45 \times 300 + 30 \times 1200}{300 + 300 + 1200} = 30 [mm]$$

$$y_c = \frac{15 \times 300 + 15 \times 300 + 40 \times 1200}{300 + 300 + 1200} = 31.67 [mm]$$

iv. The calculation of the moment of inertia.

Only the *z*-MOI about the centroid is calculated. The parallel axis theorem is used for all three triangles.

$$w_1 = w_2 = 10; \quad w_3 = 60;$$

$$h_1 = h_2 = 30; \quad h_3 = 20;$$

$$I_{\zeta\zeta} = \frac{1}{12}10\times 30^3 + 300\times(31.67-15)^2 +$$

$$\frac{1}{12}10\times 30^3 + 300\times(31.67-15)^2 +$$

$$\frac{1}{12}60\times 20^3 + 1200\times(31.67-40)^2 = 335000\left[mm^4\right]$$

$$= 3.35\times 10^{-7}\left[m^4\right]$$

v. The maximum bending stress and the corresponding factor of safety in bending for the given load.

The maximum bending stress depends on the maximum bending moment on the beam. The shear and bending moment diagrams can be sketched intuitively or formally obtained by plotting the distributions in Step (ii). The plot of the symbolic functions is shown below. The maximum bending moment is -4500 Nm and its location is at 3 m.

The negative bending moment implies that the top of the cross-section is in tension and the bottom is in compression.

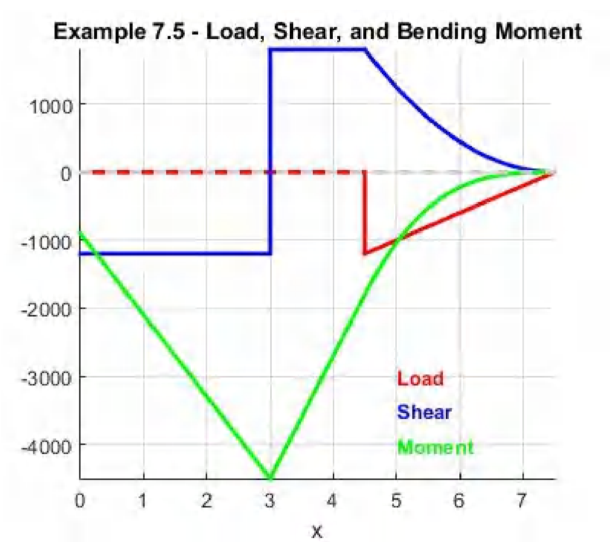


Figure 7.3.17. Load, shear and bending distributions

With σ_y = 95 MPa; σ_U = 110 MPa

$$\sigma_t(\max) = 4500(50-31.67)/\left(3.35\times10^{-7}\times1000\right) = 246.3[MPa];$$

$$\sigma_c(\max) = 4500(31.67-0)/\left(3.35\times10^{-7}\times1000\right) = -425.4[MPa];$$

$$FOS_y = \frac{95}{425.4} = 0.22;$$

$$FOS_U = \frac{110}{425.4} = 0.26;$$

These FOS numbers are terrible. This design will most likely fail.

Design Discussion:
This design is not acceptable as we are targeting a FOS of 6. To improve the FOS for a beam of uniform cross-section we have three options:

- **Decrease the load:** This is unlikely because the beam is being designed to handle given loads. It is assumed that you have no control over the placement or the magnitude of the loads or their direction.
- **Change the material:** This is not helpful for this particular problem. Changing to steel will double the FOS but it will be still below 1. It does not prevent the beam from failing.
- **Change the MOI:** This is usually where the design takes place. To decrease the stresses we must increase the MOI. Since we must work with the same shape of cross-section, the MOI well increase if we can locate more area away from the centroid. We can increase area above and below the current centroid by the same ratio so that the centroid location stays the same but the inertia will increase. This will also increase the stresses but less significantly.

vi. The deflection of the beam

The deflection is easily obtained using singularity functions. Using calculus is a little involved as you have to deal with piecewise continuous functions and integration constants. For a cantilever beam the boundary conditions are that the deflection and the slope of the deflection are zero at the cantilevered end. In this case **x = 0, y = 0**; and **x = 0, dy/dx = 0.**

We only need the moment distribution M(x) for calculating the deformation of the beam. E = 70 GPa; and $I_{zz} = 3.35\text{x}10^{-7}$ m^4.

Important consideration: The beam deflection must be continuous. A non continuous beam under loading is a beam that has failed. This continuity is imposed in the derivation below.

$$EI_{zz}\frac{d^2y}{dx^2} = M(x) = -900-1200x; \qquad 0 < x < 3$$

$$EI_{zz}\frac{d^2y}{dx^2} = M(x) = -9900+1800x; \qquad 3 < x < 4.5$$

$$EI_{zz}\frac{d^2y}{dx^2} = M(x) = 66.667x^3 - 1500x^2 + 11250x - 28125; \qquad 4.5 < x < 7.5$$

Calculating the slope:

$$EI_{zz}\frac{dy}{dx}=-900x-1200\frac{x^2}{2}+C; \qquad 0<x<3$$

$$x=0;\quad \frac{dy}{dx}=0;\quad C=0;\quad EI_{zz}\frac{dy}{dx}(3)=-8100;$$

$$EI_{zz}\frac{dy}{dx}=-9900x+1800\frac{x^2}{2}+C; \qquad 3<x<4.5$$

$$x=3;\quad EI_{zz}\frac{dy}{dx}(3)=-9900\times3+1800\frac{9}{2}+C=-8100;\quad C=13500;$$

$$EI_{zz}\frac{dy}{dx}=-9900x+1800\frac{x^2}{2}+13500;\quad EI_{zz}\frac{dy}{dx}(4.5)=-12825;$$

$$EI_{zz}\frac{dy}{dx}=\frac{66.667x^4}{4}-\frac{1500x^3}{3}+\frac{11250x^2}{2}-28125x+C; \qquad 4.5<x<7.5$$

$$x=4.5;\quad EI_{zz}\frac{dy}{dx}(4.5)=\frac{66.667x^4}{4}-\frac{1500x^3}{3}+\frac{11250x^2}{2}-28125x+C=-12825;\quad C=308475/8;$$

$$EI_{zz}\frac{dy}{dx}=\frac{66.667x^4}{4}-\frac{1500x^3}{3}+\frac{11250x^2}{2}-28125x+308475/8; \qquad 4.5<x<7.5$$

Calculating the displacement:

$$EI_{zz}y=\frac{1}{EI_{zz}}\left[-900\frac{x^2}{2}-1200\frac{x^3}{6}\right]+C; \qquad 0<x<3$$

$$x=0;\quad y=0;\quad C=0;\quad y(3)=-0.40299;$$

$$y(x)=\frac{1}{EI_{zz}}\left[-9900\frac{x^2}{2}+1800\frac{x^3}{6}+13500x\right]+C; \qquad 3<x<4.5$$

$$x=3;\quad y(3)=\frac{1}{EI_{zz}}\left[-9900\times\frac{3^2}{2}+1800\frac{3^3}{6}+13500\times3\right]=-0.40299;\quad C=-0.5757;$$

$$y(x)=\frac{1}{EI_{zz}}\left[-9900\frac{x^2}{2}+1800\frac{x^3}{6}+13500x\right]-0.5757;\quad y(4.5)=-1.0938;$$

$$y(x)=\frac{1}{EI_{zz}}\left[\frac{66.667x^5}{20}-\frac{1500x^4}{12}+\frac{11250x^3}{6}-28125\frac{x^2}{2}+\frac{308475}{8}x\right]+C; \qquad 4.5<x<7.5$$

$$x=4.5;\quad y(4.5)=\frac{1}{EI_{zz}}\left[\frac{66.667\times4.5^5}{20}-\frac{1500\times4.5^4}{12}+\frac{11250\times4.5^3}{6}-28125\frac{4.5^2}{2}+\frac{308475}{8}4.5\right]+C=-1.0938;\quad C=-1.7123;$$

$$y(x)=\frac{1}{EI_{zz}}\left[\frac{66.667x^5}{20}-\frac{1500x^4}{12}+\frac{11250x^3}{6}-28125\frac{x^2}{2}+\frac{308475}{8}x\right]-1.7123;; \qquad 4.5<x<7.5$$

The displacement of the beam is shown in Figure 7.3.18

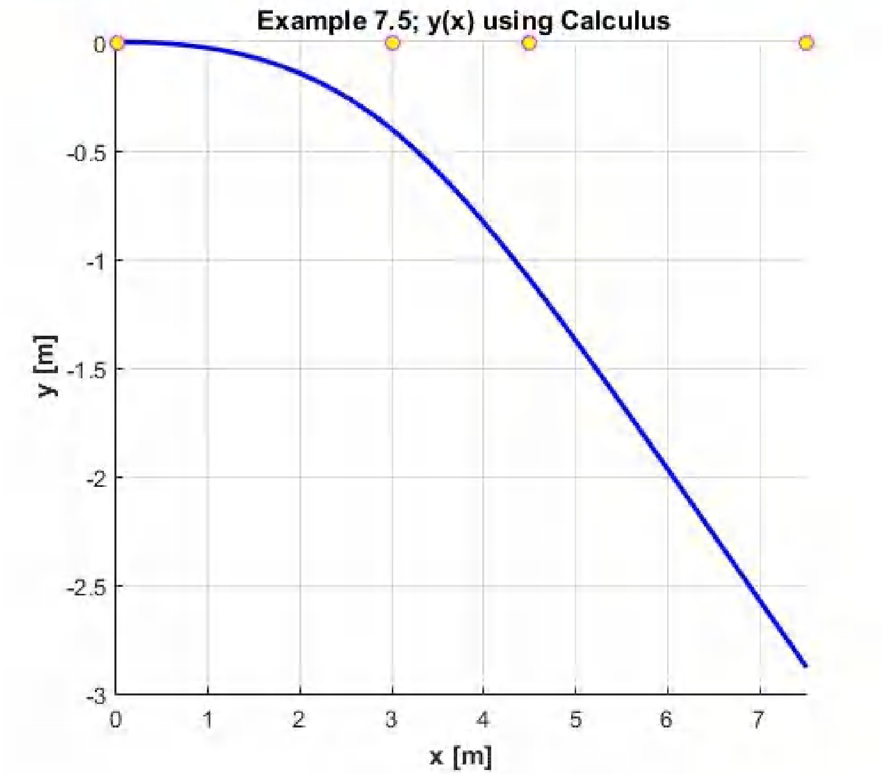


Figure 7.3.18. Beam deflection

The maximum deflection is -2.87 m. This is unacceptable. Both the modulus of elasticity and the MOI influences the deflection. The MOI has to increase by a factor of 100 for the maximum deflection to be 2.87 cm.

This completes the complete analysis of the beam deflection. Finish the problem by using MATLAB to do the complete analysis.

7.3.7 Additional Problems

Set up the following problems by hand on paper and solve them on paper and using MATLAB/Octave For each problem you must draw the FBD and work with a coordinate system. For all problems check for static indeterminacy and obtain support reactions. Obtain and plot the distributed load - w(x), shear force - V(x), and the bending moment diagrams M(x) using the engineering approach. Obtain the centroid and the MOI about the centroid. Obtain and plot the displacement curve. Choose your material and estimate the FOS.

After completing these problems, consider the problem in the previous section and solve them for the case for a rectangular cross-section of width 60 mm and height 200 mm.

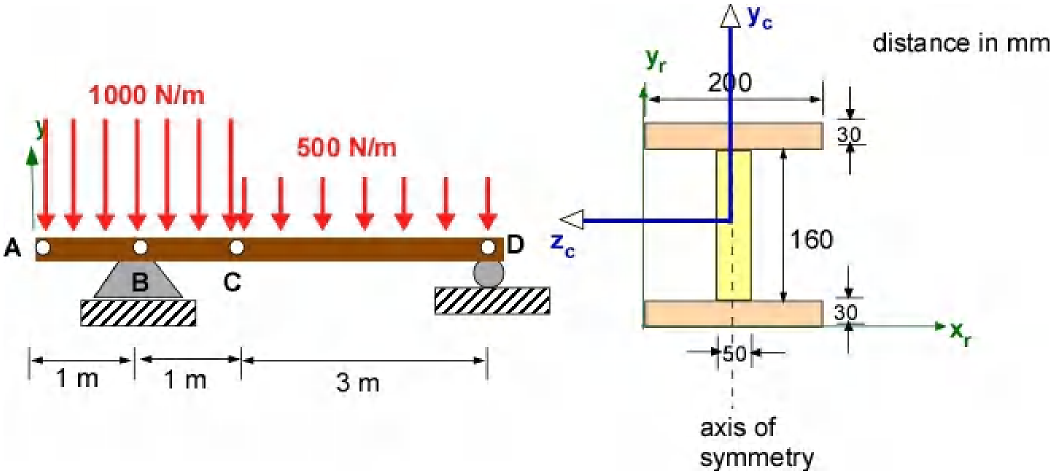
Please use the following table if necessary.

Table 7.2

Material	Aluminum	Brass	Steel	Wood
E [GPa]	70	105	200	13
Yield stress [MPa]	230	410	250	60
Ultimate Stress (tension) [MPa]	300	500	450	100

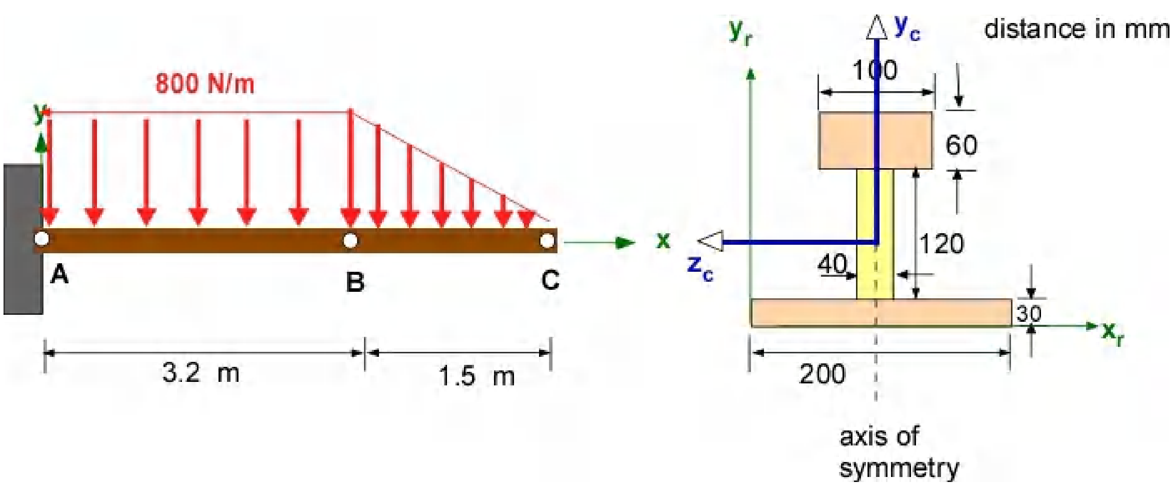
Ultimate stress (shear) [MPa]	70	200	250	10

Problem 7.3.1. Figure shown below



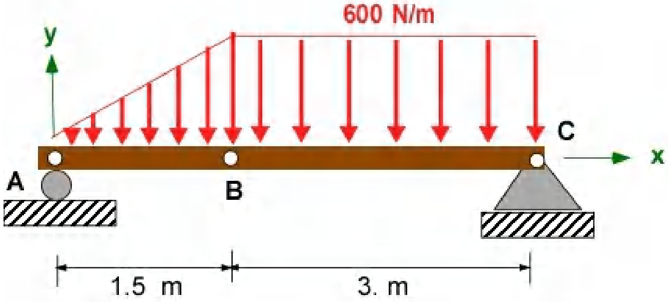
Problem 7.3.1

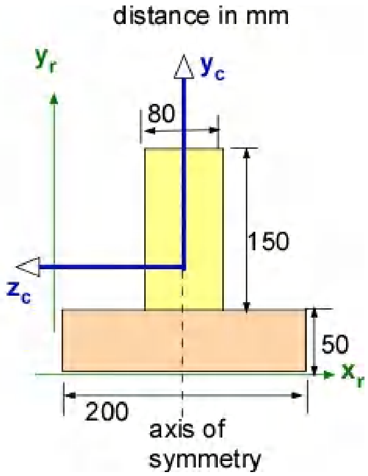
Problem 7.3.2. Figure shown below



Problem 7.3.2

Problem 7.3.3. Figure shown below


600 N/m
y
x
A
B
C
1.5 m
3. m


distance in mm
y_r
y_c
80
150
z_c
50
x_r
200
axis of
symmetry

Problem 7.3.3

7.4 SHEAR STRESS IN BEAMS

Except for beams loaded in pure bending most structural problems will have a shear force present in addition to bending. We have seen this in Examples 7.2 through Example 7.5. This must also be expected as seen in the basic equations of beam bending in Eqn. (7.1) and (7.2) repeated below. There is always a shear force distribution (also referred to as **transverse load**) along the beam that can be related to the bending moment distribution covered in the previous section.

$$\frac{dV}{dx} = -w(x); \quad V_2 - V_1 = \int_{x_1}^{x_2} -w(x)\,dx \qquad (7.1)$$

$$\frac{dM}{dx} = V(x); \quad M_2 - M_1 = \int_{x_1}^{x_2} V(x)\,dx \qquad (7.2)$$

The examples we explored had another property that we did not explicitly express. The cross-sections had an **axis of symmetry where this shear load is applied**. This produces only shear stress due to the shear force (direct shear). For a non symmetric cross-section the shear load must be applied at the shear center to avoid torsion. This is covered later. Here we are dealing with direct shear only.

In the following discussion you will contend with two kinds of distribution:

- A shear **force** distribution along the length of the beam V(x) that has appeared in the examples so far.

A shear **stress** distribution in the cross-section for each point along the length of the beam which can be expressed a τ_{xy} or τ_{xz} where the beam axis is in the x-direction. . If the shear force varies along the beam then the shear stress in the beam is actually three-dimensional. From a design perspective we will keep it two-dimension by discussing it only at particular cross-sections.

Therefore every cross-section will likely see a different shear stress distribution based on the magnitude of the shear force even if the cross-section of the beam is uniform along its length. We can examine the maximum shear force location along the beam for calculating the maximum shear stresses. However, because of the presence of normal stresses it may be more appropriate to look at the **principal stresses** to decide on **failure**. In the previous section we developed the relation between bending moment in the cross-section and normal stresses in the cross-section and deflection of beam. We can therefore expect the shear force in the cross-section to correspond to a shear stress in the cross-section. Our development of the relations between the shear force and shear stress will be similar to the discussion regarding bending moment and normal stress in the previous section.

Idealizations

1. We will keep the cross-section of the beam **uniform**. Every cross-section will carry a bending moment and shear force of a **scalar value**. From Examples 7.2 - 7.5 we understand that the shear

force will be directed along the y-axis. The sign convention also requires that the cross-section exposed on the right side of the beam will have the **positive shear force** directed along the negative y-axis.

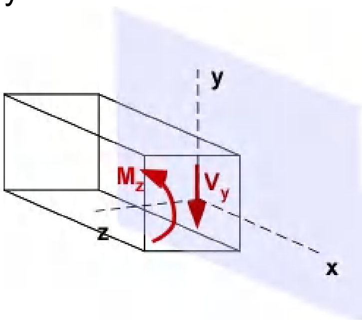


Figure: Idealization 1

2. We often develop analysis in mechanics using superposition. We will remove the bending moment from the picture since we already know how to account for it. We will retain the **centroid**, **NA**, and **MOI** from the bending calculations.

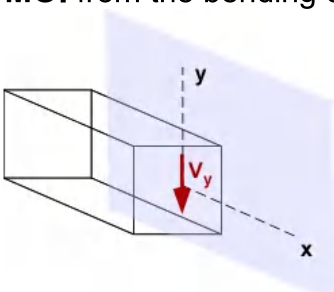


Figure: Idealization 1 - simplified for analysis

3. The shear load V in the cross-section must result in a shear stress distribution in the cross-section. We look at an elemental area **dA** in the positive quadrant of the cross-section. There should be two shear stresses (shown in the positive sense on area element dA) and they yield the relations below.

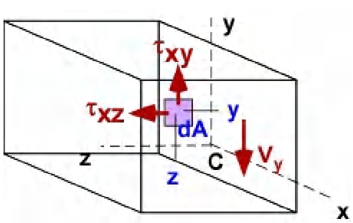


Figure : Idealization 3

$$\int_A \tau_{xy}\, dA = -V$$

$$\int_A \tau_{xz}\, dA = 0$$

The first relation suggests that the integration of the shear stress, τ_{xy}, over the cross-section, must match the applied transverse shear load V_y.

The second equation suggests that the integration of the shear stress τ_{xz} over the cross-section must be zero. It does not call for τ_{xz} to be zero everywhere.

4. Prior to developing relations between the shear force and the shear stress let us also understand the shear stress equilibrium for a typical element of volume. For example let us develop an elemental volume of the beam material using the elemental area dA. Due to equilibrium the shear stress τ_{xy} on dA is present on all four sides of the volume as shown below. Similarly the shear stress τ_{xz} is present on four sides (not shown on all sides). It is also understood that shear equilibrium at every point requires τ_{xy} is equal to τ_{yx}.

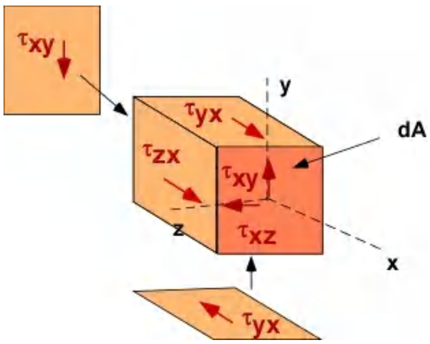


Figure: Idealization 4

7.4.1 Relation between Normal Stress and Shear Stress

To develop this idea we introduce Example 7.6 and accept that we can establish the shear force distribution, the bending moment distribution, and the normal stress distribution on every cross-section along the beam.   We have also the centroid and MOI available. We will solve this example later to we find the actual solution. At this time we do not need the exact normal stress distribution but only the recognition that this stress distribution is linear in the cross-section and the magnitudes may vary across the beam if the bending moment changes along the beam - which it should for Example 7.6 shown below. In the figure below we have drawn attention to a small portion of the beam between arbitrary locations M and N of infinitesimal length **Δx**. We will be using this portion to develop our relations in this sub-section. We will draw an exploded view of this region to develop the analysis.

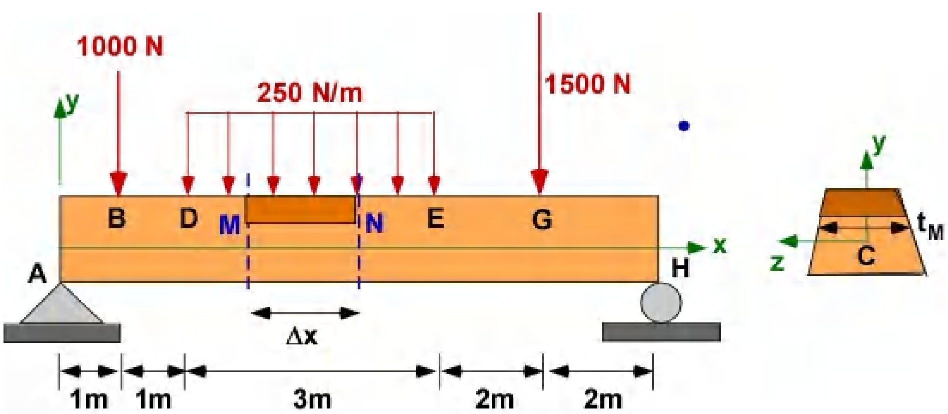


Figure 7.4.1 Example 7.6

We are going to focus on the element of the beam between M and N and draw the FBD of the piece. M' and N' are the corresponding locations at the top of the beam. The FBD of a portion of the beam is the figure on the left in Figure 7.4.2. The right figure is the portion of the beam in the cross-section. In the FBD we have taken the liberty of replacing the bending moment at M and N by the corresponding normal stress distribution over the portion of the beam. It is a truncated linear distribution measured from the centroid. It is also shown on the side view. Notice the shear force on the FBD is only a fraction of the total shear at these locations.

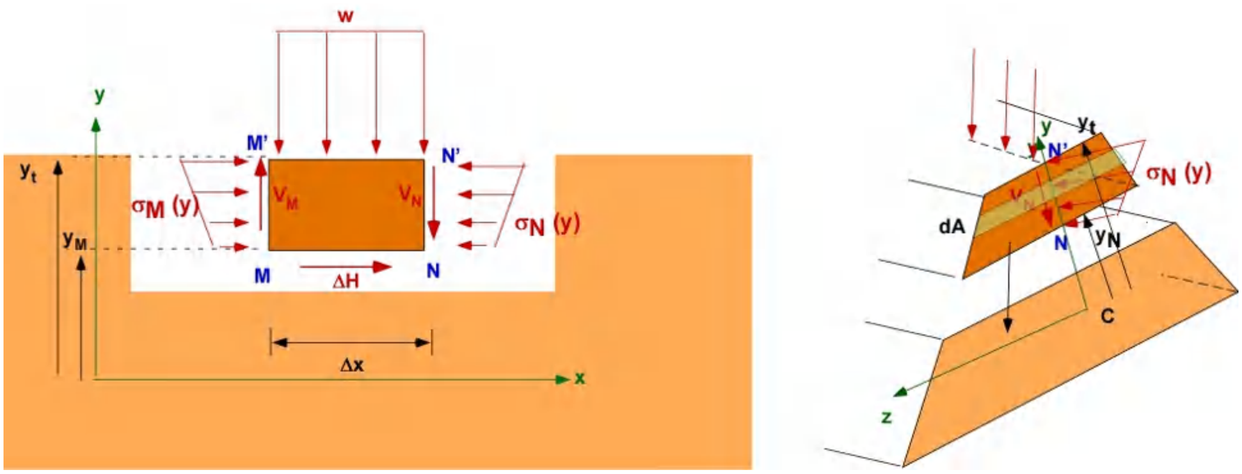


Figure 7.4.2. FBD of the beam segment between M and N

We will also invoke the traditional assumption (*in formula development*) that the stresses and shear **increase along the positive x direction**. Because the normal stress on the right **σ_N (y)** is greater than the normal stress on the left **σ_M (y),** there must be a horizontal force **ΔH** on the bottom of the segment MN for equilibrium in the x-direction. The top of the beam is assumed shear stress free. This force is a **shear force** based on its orientation to the surface on which it is acting - the bottom of the segment of the beam between M and N. This is a *horizontal shear force*. There is no horizontal force on the top as can be seen on the FBD. We will say that the area represented by the orange strip is **A_M** (or A_N). Also shown is the area **dA** for the integration. This **dA** depends on the distance **y** from the centroid. The integration will be between **y_N** and **y_t**. Here **y_t** is the location of the top surface from the centroid.

$$\sum F_x = \int_{A_M} \sigma_M(y)\,dA + \Delta H - \int_{A_M} \sigma_N(y)\,dA = 0$$

$$\Delta H = \int_{A_M} \left(\sigma_N(y) - \sigma_M(y)\right) dA = \frac{\left(M_N(x) - M_M(x)\right)}{I_{zz}} \int_{A_M} -y\,dA =$$

$$-\frac{M_N(x) - M_M(x)}{I_{zz}} Q_{y_M}$$

In the above we have substituted the normal stress with their corresponding bending moments through Eqn. (7. 11). We have also introduced a new quantity ***Qy*** to denote the first moment of the area of the cross-section defined between the point **y_M** to the top of the beam **y_t**. This moment is about the centroid (or NA).

There is a special reason we have defined the length of the beam between M and N as **Δx**. We want this to be small so that we can represent the variation in the bending moment between M and N linearly as :

$$M_N = M_M + \frac{dM}{dx}\Delta x; \quad M_N - M_M = \frac{dM}{dx}\Delta x = V(x)\,\Delta x$$

We can then introduce this into the previous expression to obtain a relation for ∆**H** in terms of **V(x).** To relate this to shear stress we first define a new entity called **shear flow** which is the **shear force per unit length**. It is usually expressed by the variable ***q***. In this discussion we can distinguish it as $\mathbf{q_x}$ since it is directed horizontally. It is evaluated at a distance $\mathbf{y_M}$ from the centroid. If we further assume that this is distributed evenly on the surface (bottom of the beam segment at MN), we can then relate it to the **horizontal shear stress**. In this case we will by divide it by the thickness of the beam at the location $\mathbf{y_M}$ which we will call $\mathbf{t_M}$. We will have to define the shear stress as τ_{yx} - as the stress is in the *x-direction* on the *y-plane*. In essence these calculations reflect an **average value** at the current location (***x*** and y_M). By making ∆**x** *(in the limit)* small these can be associated with instantaneous values. Let us finish our development:

$$\Delta H = -\frac{V(x) Q_{y_M}}{I_{zz}} \Delta x$$

$$q_x = \frac{\Delta H}{\Delta x} = -\frac{V(x) Q_{y_M}}{I_{zz}} \qquad (7.16)$$

$$\tau_{yx}(x, y_M) = \frac{\Delta H}{Area} = \frac{\Delta H}{\Delta x\, t_M} = -\frac{V(x) Q_{y_M}}{I_{zz}\, t_M} = \tau_{xy}(x, y_M) \qquad (7.17)$$

We can drop the negative sign in the expressions above with the understanding that the direction for the shear flow and the shear stress at the centroid will be the same as the shear force in the cross-section. The shear stress τ_{yx} (on the bottom) can be related to the τ_{xy} in the beam cross-section at the location y through shear equilibrium at every point. Putting it all together we collapse the points M and N to represent the location x on the beam. The average shear stress distribution in the cross-section is related to the shear load carried by the cross-section. The shear stress distribution is related to the first moment of the area of the cross-section, from the point of interest to the outside fibers of the beam, about the centroid of the cross-section. With y_1 a point between the centroid and the top and y_2 is a point between the centroid and the bottom.

$$Q_{y_t} = Q_{y_b} = 0 \quad and \quad Q_{y_C} > Q_{y_1} \quad and \quad Q_{y_C} > Q_{y_2}$$

The shear stress in the cross-section at different points is described below: The area for the first moment calculation is shown in the figures. The thickness is the bottom of the highlighted area (above centroid) and the top of the highlighted area (below centroid).

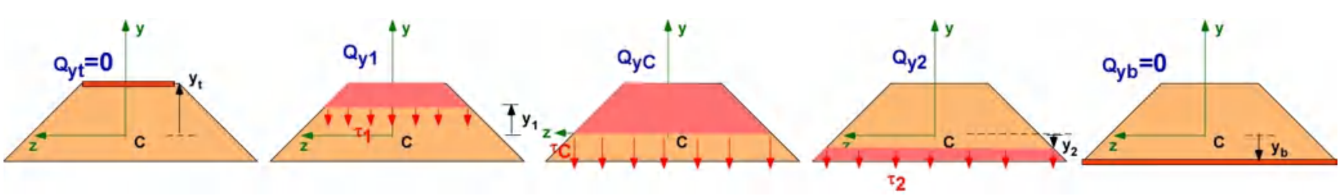


Figure 7.4.3a Shear stress distribution in the cross-section

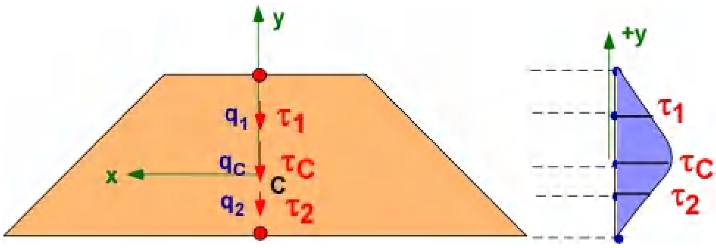


Figure 7.4.3b Another representation for the shear stress in the cross-section. (the x- coordinate should actually be z in the figure)

In Figure 7.4.3b we can represent the shear stress along the y-axis since it is the **average value** over the cross-section at each y which is shown in Figure 7.4.3a. It is computed using Eqn. (7.17) where V(x) is the same but the thickness and the first moment of area are different. The theory of elasticity suggests that the actual value varies across the thickness with the larger values at the edges. For beams with small widths compared to the depth the average value of shear stress is acceptable. Our illustration does not meet this criteria. At this stage we will not develop a finer calculation for shear stress across the width and will stick to Eqn. (7.17) for determining the shear stress. The shear stress will indeed vary over the height of the beam. While shear stress is not a vector we illustrate the magnitude and direction using arrows. We have done the same to illustrate the normal stress previously. In advanced mechanics we prefer to discuss the shear properties in the cross-section in terms of shear flow rather than the shear stress. Sometimes an analogy to fluid flow is invoked to represent the shear flow distribution. Therefore the arrows reflect the flow of shear in the cross-section. There is an equivalence between the shear flow and the shear stress as you can obtain the latter by dividing by the thickness. The shear stress is expected to be zero at the outer fibers and maximum at the centroid. This is unlike bending stress which is zero at the centroid and maximum and of different signs at the outer fibers.

The rightmost figure in Figure 7.4.3b is another description for the variation of shear stress in the cross-section. This description is **nonlinear** unlike the normal stress distribution in the cross-section which is **linear**.

Let us look at the shear stress distribution for a rectangular beam and an **I** beam which is popular in structural design. In Figure 7.4.3b we have shown the shear stress as a **maximum** at the centroid. This depends on the properties of the cross-section. While the **shear flow** will be the maximum because of the first moment of area about the centroid will be the largest, the shear stress must be divided by the thickness of the cross-section and therefore could have a minimum at a different point.

Rectangular cross-section

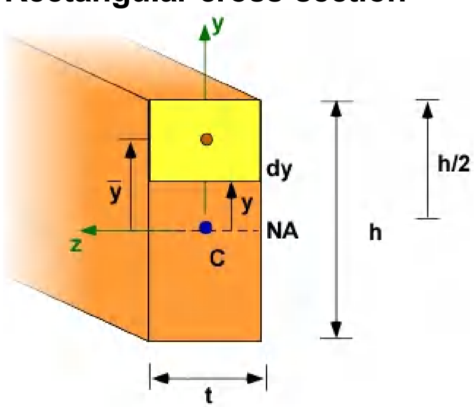


Figure 7.4.4 Rectangular cross-section

$$\tau_{xy}(y) = \frac{V(x)Q(y)}{I_{zz}t} = \frac{V(x)}{I_{z}t}\left[Area \times \bar{y}\right] = \frac{V(x)}{I_{zz}t}\left[\left(t\left(\frac{h}{2}-y\right)\right)\times 0.5\left(\frac{h}{2}+y\right)\right]$$

$$= \frac{V(x)}{I_{zz}t}\left[0.5 \times t \times \left(\frac{h^2}{4}-y^2\right)\right] = 0.5\frac{V(x)}{I_{z}}\left(\frac{h^2}{4}-y^2\right)$$

The MOI $\mathbf{I_{zz}}$ for the rectangular section is $\mathbf{th^3/12}$. We can substitute this in the expression above and recognize the area of cross-section is **th** to obtain:

$$\tau_{xy}(y) = \frac{3}{2}\frac{V(x)}{Area}\left(1-\frac{y^2}{(h/2)^2}\right) \qquad (7.18)$$

The average shear stress is distributed quadratically (parabolic) over the depth with the **maximum shear stress** at **y = 0** with a value of **1.5 [V(x)/Area].** Without the analysis of this sub-section we would have calculated an **average shear stress** in the cross-section of **V(x)/Area**. This will also be the maximum shear stress as it is constant in the cross-section. Our analysis suggest that the maximum shear stress is under estimated by 50 %.

I cross-section

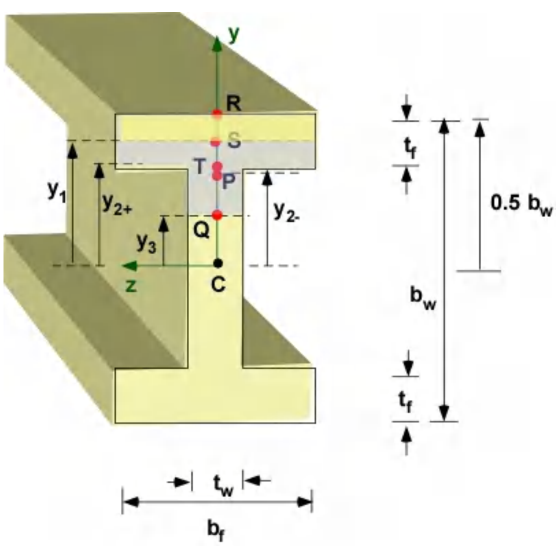


Figure 7.4.5 I-beam cross-section

To keep it simple we have a double symmetric I-beam. The centroid C is located directly at the intersection of the two axes of symmetry. The calculations for the bottom half of the cross-section of the beam will be same as the corresponding point on the top half. The I-beam has two defined regions. The flange which is the rectangle that is more wider than tall. There are usually two flanges. The single web is the the taller rectangle. The subscripts *f* and *w* are used for the flange and web respectively. We have identified several points along the y-axis to enable plotting based on relative magnitudes. Point R is the top of the beam (y = 0.5 b_w). Point S is a location within the flange (y = y_1). Point T is slightly above the intersection of the flange and the web (y = y_{2+}). Point P is just slightly below the intersection of the flange and the web (y = y_{2-}). For calculations these are the same locations which we will represent as y_2. The thickness at the two points are different. Point Q is between the centroid and the flange. Point C (y = 0) is the centroid.

We make the following observations based on the value for the corresponding first moment of area without calculations:

$$q_{xy}(y) = \frac{V(x)}{I_{zz}} Q_y; \quad \tau_{xy}(y) = \frac{q_{xy}}{t_y};$$

$$Q_C > Q_Q > Q_P = Q_T > Q_S > Q_R = 0$$

We can then plot the distribution of $\mathbf{q_{xy}(y)}$ and $\boldsymbol{\tau_{xy}(y)}$ in the cross-section as it appears in Figure 7.4.6. The distribution is quadratic between the points.

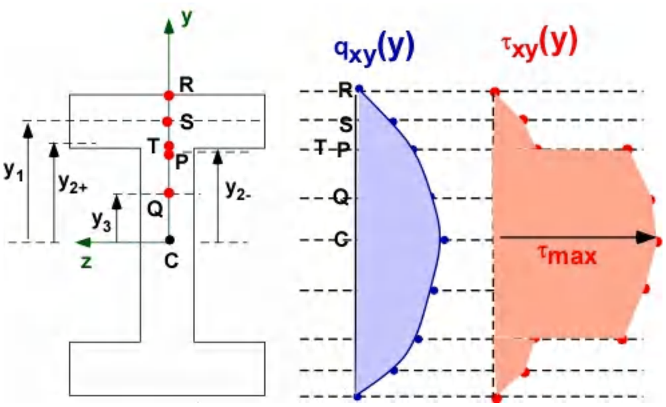


Figure 7.4.6 shear flow and stress distribution in the I-beam

As you can note the shear flow **q** is continuous while the shear stress τ_{xy} jumps. Most of the shear stress appears in the web. If the dimensions of the flange and the web are the same the maximum shear stress can be estimated as:

$$\tau_{xy}(\max) \approx \frac{V(x)}{A_{web}} \qquad (7.19)$$

For the I-beam it is possible to calculate shear flow and the corresponding stress in the z - direction also. We will return to this after working out the solution to Example 7.6.

7.4.2 Example 7.6

Example 7.6 which appeared above in the development of the analysis is restated with the cross-section defined in an exploded view in Figure 7.4.7 below. Let us assume that the beam is made of Aluminum alloy 6061-T6. This has a density of 2700 kg/m^3, Young's modulus for is 68.9GPa, a Poisson's ratio of 0.33, and a shear modulus of 26 GPa. It has a minimum tensile yield strength of 276 MPa, ultimate tensile strength of 310 MPa, shear yield strength of 140 MPa, and an ultimate shear strength of 165 MPa - *Wikipedia*.

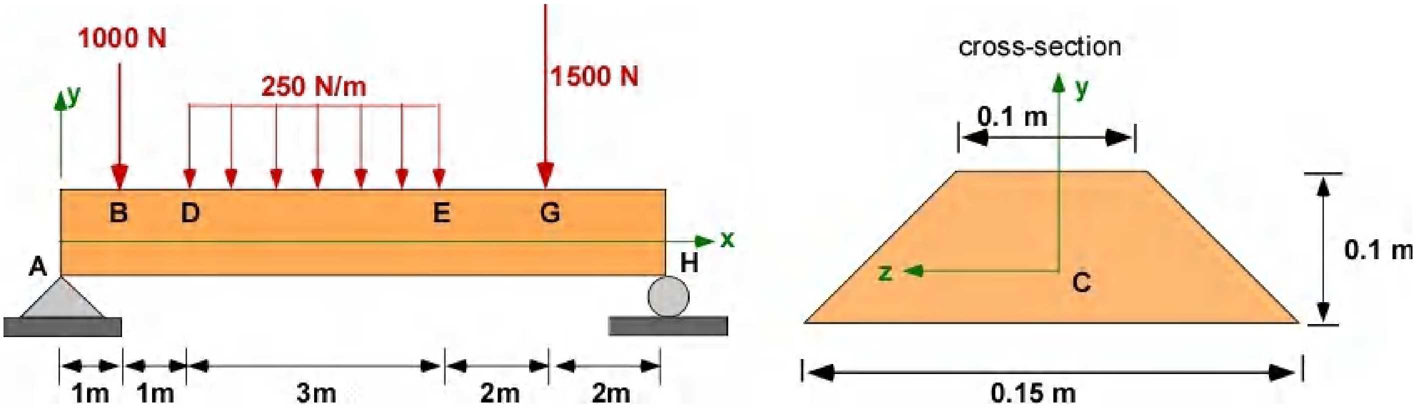


Figure 7.4.7 Example 7.6 with cross-section dimensions

Data: Beam, loading and locations are shown in the figure.
A = [0,0,0]; B = [1,0,0]; D = [2,0,0]; E = [5,0,0]; G=[7,0,0]; H = [9,0,0];
A is pin supported and H is a roller support. It is statically determinate
E = 68.9 [GPa]; G = 26 {GPa]; σ_y = 276 MPa; σ_U = 310 MPa; ν = 0.33; τ_y = 140 MPa; τ_U = 165 MPa

Find:

i. The reactions at the support.
ii. The shear force and bending moment diagram.
iii. The location of the centroid.
iv. The calculation of the moment of inertia.
v. Maximum bending stress on the beam
vi. The deflection of the beam
vii. The shear flow distribution, shear stress distribution
viii. The maximum Von Mises stress in the cross-section

Assumption: Cross-section is uniform. Ignore beam weight.

Solution:

i. The reactions at the support.

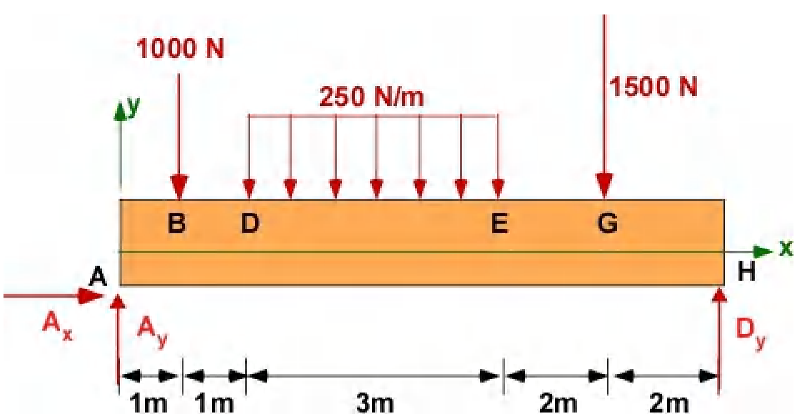


Figure 7.4.8a Example 7.6 - FBD

$$\sum F_x = 0 = A_x;$$
$$\sum F_y = 0 = A_y - 1000 - 250 \times 3 - 1500 + D_y = 0; \quad A_y = 1680.6[N];$$
$$\sum M_A = 0 = 9D_y - 1000 - 250 \times 3 \times 3.5 - 1500 \times 7; \quad D_y = 1569.4[N];$$

ii. The shear force and bending moment diagram.

Using singularity functions:

$$w(x) = -250\langle x-2\rangle^0 + 250\langle x-5\rangle^0;$$
$$V(x) = -250\langle x-2\rangle^1 + 250\langle x-5\rangle^1 + 1680.6\langle x-0\rangle^0 + \\ -1000\langle x-1\rangle^0 - 1500\langle x-7\rangle^0;$$
$$M(x) = -250\langle x-2\rangle^2/2 + 250\langle x-5\rangle^2/2 + 1680.6\langle x-0\rangle^1 + \\ -1000\langle x-1\rangle^1 - 1500\langle x-7\rangle^1;$$

An intuitive plot of the shear and bending moment diagram is shown in **Figure 7.4.8b.**

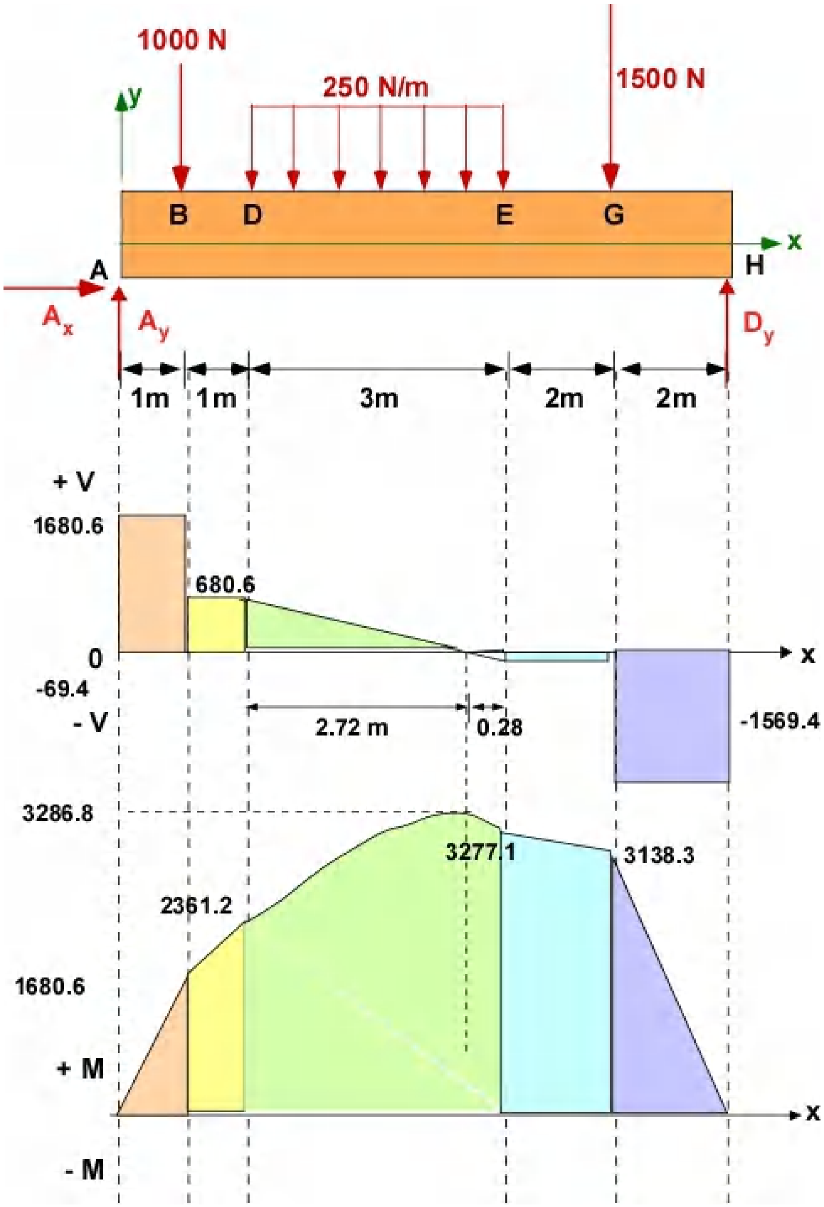


Figure 7.4.8b. Intuitive load, shear, and moment variation

iii. The location of the centroid.

This is done through integration as the width varies linearly in the cross-section. The z-centroid is on the axis of symmetry. The reference coordinates for calculating the centroid and MOI are shown in Figure 7.4.8c. Hence $z_c = 0$. Only the y coordinate of the centroid is determined. The centroid is measured from the base of the rectangle.

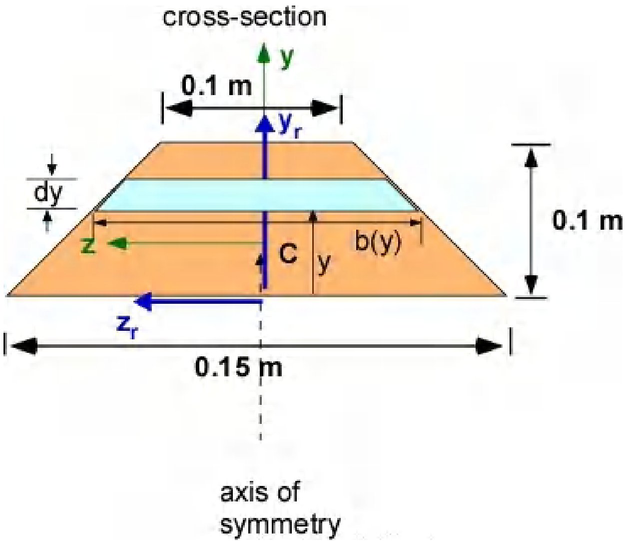


Figure 7.4.8c. Geometry for centroid and MOI calculation

$$b(y) = 015 - \frac{(0.15-0.1)}{0.1}y = 0.15 - 0.5y;$$

$$A = \int_0^{0.1} b(y)dy = \int_0^{0.1}(0.15-0.5y)dy = 0.0125\left[m^2\right];$$

$$\bar{y} = y_c = \frac{1}{A}\int_0^{0.1} yb(y)dy = \frac{1}{0.0125}\int_0^{0.1}\left(0.15y - 0.5y^2\right)dy = \frac{0.0005833}{0.0125} = 0.0467\left[m\right];$$

iv. The calculation of the moment of inertia.

Uses parallel axis theorem. The MOI at the base can be set up easily as the second moment of area using the function established above.

$$I_{z_r} = \int_0^{0.1} y^2 dA = \int_0^{0.1} y^2 b(y)dy = \int_0^{0.1} y^2(0.15-0.5y)dy = 3.75\times10^{-5}\left[m^4\right];$$

$$I_{z_r} = I_{z_c} + Ay_c^2;$$

$$I_{z_c} = 3.75\times10^{-5} - \left(0.0125\times0.0467^2\right) = 1.028\times10^{-5}\left[m^4\right];$$

v. Maximum bending stress on the beam

Bending moment is positive on the beam. The beam has compression on top and tension on the bottom along its entire length. The location and value of the maximum bending moment is shown in Figure 7.4.8b. Using these values:

Location of maximum bending moment [m] = 4.7246
Value of the maximum bending moment [Nm] = 3287.4
Maximum Tensile Stress (bottom) [MPa] = 14.93
Maximum Compressive Stress (top) [MPa] = -17.06

Design discussion: These stresses are well within the elastic limit for the material (276 MPa) suggesting very high FOS so it is not estimated. The MOI is probably high and a new design with reduced cross-section dimensions can easily handle the same loads. This could save on material costs.

vi. The deflection of the beam

Using Singularity functions with E = 68.9*10^9 and MOI determined above

$$EI\frac{d^2y}{dx^2} = M(x) = -250\langle x-2\rangle^2/2 + 250\langle x-5\rangle^2/2 + 1680.6\langle x-0\rangle^1 + $$
$$-1000\langle x-1\rangle^1 - 1500\langle x-7\rangle^1;$$

$$EI\frac{dy}{dx} = -250\langle x-2\rangle^3/2/3 + 250\langle x-5\rangle^3/2/3 + 1680.6\langle x-0\rangle^2/2 +$$
$$-1000\langle x-1\rangle^2/2 - 1500\langle x-7\rangle^2/2 + C_1$$

$$EIy(x) = -250\langle x-2\rangle^4/2/3/4 + 250\langle x-5\rangle^4/2/3/4 + 1680.6\langle x-0\rangle^3/2/3 +$$
$$-1000\langle x-1\rangle^3/2/3 - 1500\langle x-7\rangle^3/2/3 + C_1 x + C_2;$$

$$y(0) = 0 = EIy(0); \quad C_2 = 0$$

$$y(9) = 0 = EIy(9) = -\frac{250}{24}(9-2)^4 + \frac{250}{24}(9-5)^4 + \frac{1680.6}{6}(7-0)^3 +$$
$$-\frac{1000}{6}(9-1)^3 - \frac{1500}{6}(9-7)^3 + 9C_1$$

$$C_1 = -10501;$$

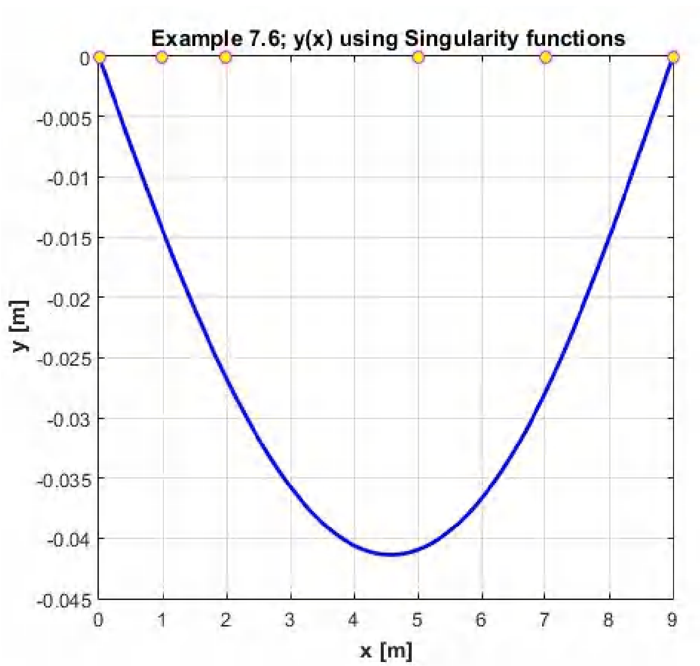


Figure 7.4.8d. Beam deflection

Design Discussion: Maximum deflection is 4 cm which should attract design attention for being high. It is possible that the beam is too long [9 m]. Another remedy is to introduce another support for the beam, maybe between D and E. This will make it statically indeterminate. We can solve the problem through the extra deflection constraint. To reduce the maximum deflection the MOI needs to increase but we have already determined that the cross-section is large for maximum bending stresses. A redesign must address the length of the beam. The mass of the beam is 304 [kg]. The weight of the beam is 2980 [N], that is almost the same as the total applied external load. Maybe the analysis should be redone to include the beam self-weight. This will introduce and additional load w(x) of about 331 N/m over the length of the beam. Such exploration of the design is easy if the problem is solved through MATLAB/Octave

vii. The shear flow distribution, shear stress distribution

The maximum shear force and location can be obtained from Figure 7.4.8b or from the singularity function.The absolute maximum is between x = 0 and x = 1 and is 1680.6 [N]. The section between x = 7 and x = 9 has almost the same magnitude and should be in consideration for calculating the Von Mises stress.

The shear flow and stress is based on the first moment of area measured with respect to the centroid. We will need the variation of the width with respect to the centroid. Let us define ***yy*** as the variable measured from the centroid that corresponds to the value of ***y*** measured from the base of the rectangle as shown in Figure 7.4.8e.. This is not the most glamorous way but it avoids another alphabet and another subscript.

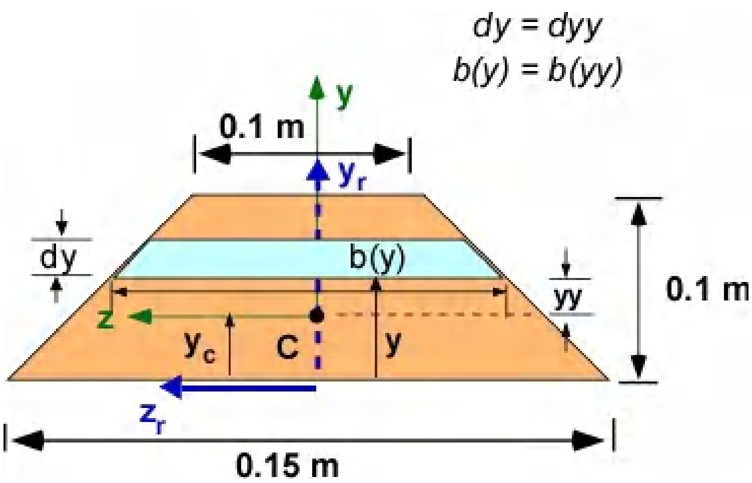


Figure 7.4.8e. Q calculation for shear flow

$$y = yy + y_c$$

$$b(yy) = 0.15 - 0.5(yy + y_c) = 0.1267 - 0.5yy;$$

$$Q(yy) = \int_{yy}^{y_t} yy\,dA = \int_{yy}^{y_t} yy\left[b(yy) \times dyy\right] = 0.1667yy^3 - 0.06333y^2 + 0.000155;$$

$$q(yy) = \frac{V_{max} Q(yy)}{I_{z_c}} = 2.72533 \times 10^7 yy^3 - 1.03562 \times 10^7 yy^2 + 25323.0;$$

$$\tau_{xy}(yy) = \frac{q(yy)}{b(yy)} = \frac{-(1.0 \times (2.72533 \times 10^7 yy^3 - 1.03562 \times 10^7 yy^2 + 25323.0))}{(0.5yy\ \text{-}\ 0.126666)}$$

We can obtain similar functions for shear flow and stress below the centroid. If *yy* is allowed to be negative below the centroid we can describe the shear flow and shear stress using the same functions. The best way to confirm and understand this variation is to plot this as a function of y measured from the centroid (yy). These are average stresses.

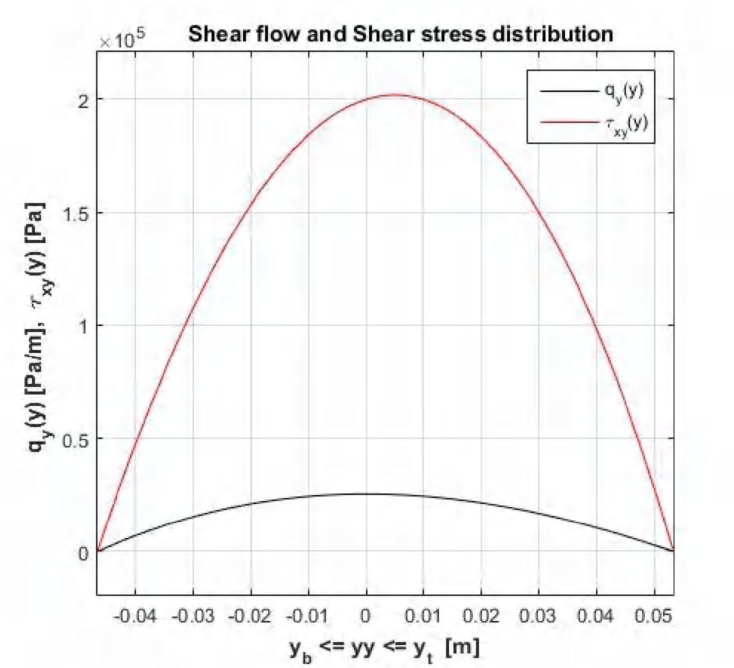


Figure 7.4.8f. Shear flow and shear stress variation in the cross-section

Design Discussion: Figure 7.4.8f is the shear flow and shear stress variation in the cross-section for the maximum shear force of 1680.6 [N]. Note that the maximum shear flow is at the centroid (y = 0) which is expected. The maximum shear stress appears to be at y = 0.005 [m] above the centroid. The maximum stress is well within the shear yield limit with a large FOS. The dimensions of the cross-section can probably be decreased saving material cost.

viii. The maximum Von Mises stress in the cross-section and FOS

The following ideas are important in this section:

- The cross-section is uniform along the beam
- Bending moment and shear force vary along the beam

- Maximum shear force and maximum bending moment may not occur at the same point along beam length
- Unless normal stress or shear stress dominate the maximum Von Mises stress calculation should include both bending and shear stresses along beam length. Possibly point G in Figure 7.4.8b
- The maximum bending moment will produce maximum normal stress in the cross-section
- In the cross-section the maximum normal stress appears on the top and bottom fibers of the beam and the maximum tensile and compressive stress may not be the same.
- The bending stress at the centroid is zero
- Shear force varies along the beam and the maximum shear stress will appear at the location of the maximum shear force
- Shear stress varies in the cross-section with the maximum close to the centroid and zero shear stress at the top and bottom of beam
- In the cross-section maximum Von Mises stress is likely between the centroid and the top and bottom

There are infinite points on the beam to calculate the Von Mises stress. To keep our efforts modest we must apply the criteria at a few competing points on the beam. These points are located on the cross-section but may be at any section along the beam. The following analysis is based on your past experience, intuition and experience. The problem is compounded when the bending moment and the shear force are at a maximum along different locations on the beam - like in this example.

Reviewing Figure 7.4.8b., we can select the points B and G for further exploration. If you want to chose only one then point G is the more likely candidate because of the relative values of the bending moment and shear force. The sign of the loads/moments do not matter it is the magnitude. One strategy to identify this point on the length of the beam by looking for the maximum of the sum of the absolute values of bending moment and shear along the length of the beam, Figure 7.4.8b. Note you are adding terms with different units. We are interested in the magnitude. A simple addition suggests that point G has a larger value because of the larger bending moment.

From Figure 7.4.8b:
Point for Exploring Von Mises [m] : 7
Bending Moment for calculating normal stresses [Nm] = 3138.3
Shear Force for calculating normal stresses [N] = -1569.4

This is the point G on the beam. We will be performing the Von Mises calculation for the beam at the location G (or x = 7 m). The positive bending moment will cause the top fiber of the beam in to be in compression and the bottom fiber to be in tension. From our previous analysis in item (v) we will be applying the criteria near the top of the beam.

Now we have to choose a point in the cross-section where will calculate the principal stresses. The top of the beam has the maximum normal stress. The maximum shear stress is around the centroid where the normal stress is zero. This distribution will not matter if one type of stress dominates the other. In our example the maximum normal stress is 17 MPa while the maximum shear stress is 2 MPa. We can effectively ignore the shear stress and choose to examine the top fiber for failure (at the point A where there is the maximum bending moment). Since there is only normal stress on the fiber this is effectively the primary principal stress while the other one is zero.

We will develop the analysis for the case where both stress are present in similar magnitudes by looking at the scaled variation of the stresses in the cross-section. We will scale the stresses by dividing by the maximum values and identify the point (we will call it Von Mises point) where the sum of the absolute value is the greatest and use the information to calculate the principal stresses.

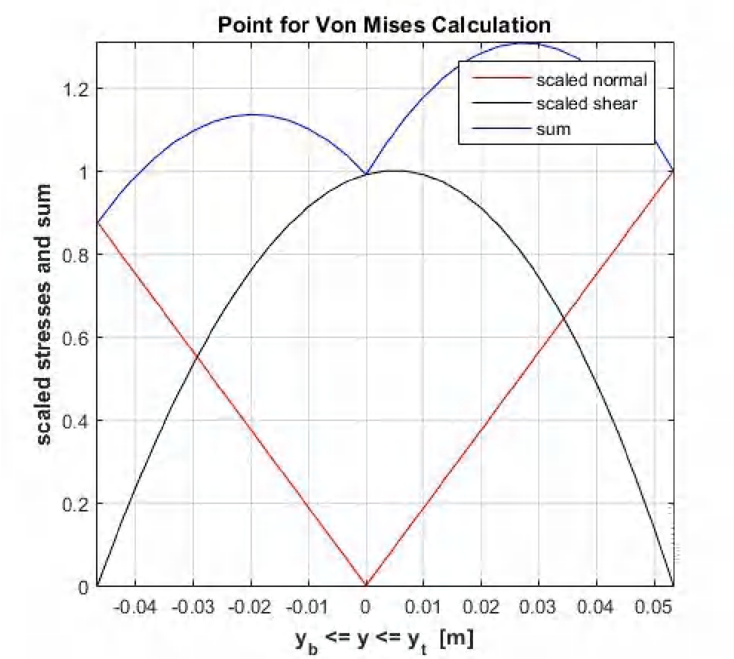


Figure 7.4.8g. Von Mises point in the cross-section

Point for calculation of stresses (from centroid) [m] = 0.026667
Maximum tensile stress in cross-section [Pa] = -8.14 e+06
Maximum shear stress in cross-section [Pa] = 1.55 e+05

Principal stresses:
σ_a [Pa] : 2936.7
σ_b [Pa] : -8.15 e+06
τ_{max} [Pa] : 4.07e+06
Direction of principal axis (deg): 88.9
Yield Strength of Material [Pa] : 276000000
σ_{vm} : Von Mises stress at this location [Pa] : 8.15 e+06
Factor of Safety (FOS) : 33

As expected this is a large value suggesting significant over design in the structure.

This was a long analysis . The calculations were incremental and followed a simple sequence based on introduction of the topics. A lot of the above values were estimated through MATLAB. In the following section we develop the code implementing the anlysis. The same code can be used for different loads, different locations, different materials, adjusting cross-section value geometry without changing shapes - all of which are extremely useful in beam design.

7.4.3 Example 7.6 through MATLAB

We will start with the FBD in Figure 7.4.9a. The code is broken into various sections for clarity below and following the sequence above.. It should be appended in one file.

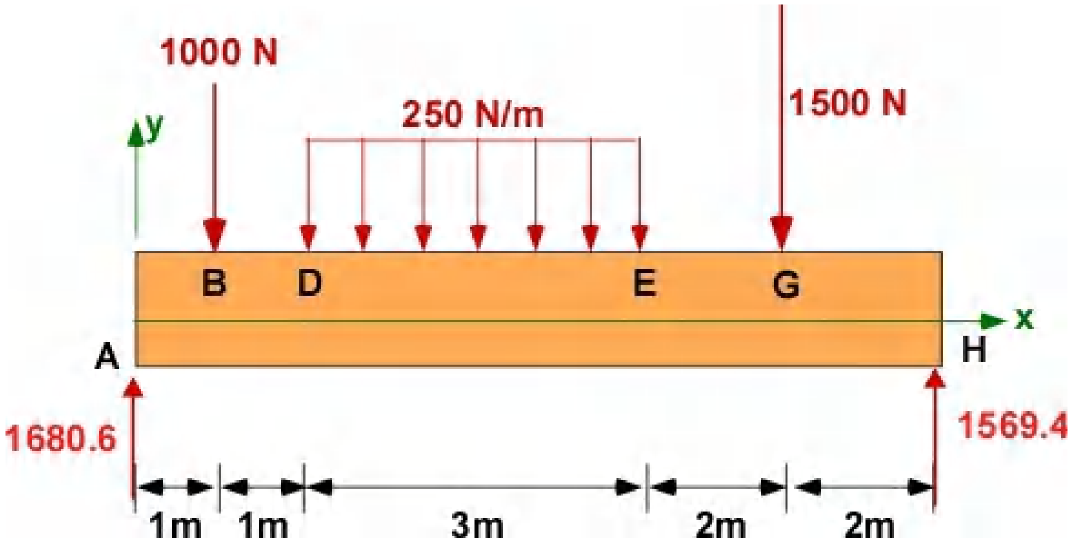


Figure 7.4.9a. FBD - Example 7.6

The MATLAB code is incremental and the code in the editor is followed by the results in the Command window, and sometimes in aa Figure window. The code should be assembled into a single file for subsequent use on different examples.

Find:

i. The reactions at the support.

```
% Essential Foundations in Mechanics
% P. Venkataraman, June 2016
%  Example 7-6
% Beams 2D - Reactions, Shear and bending diagram (singularity)
% centroid , moment of inertia, normal stress, deflection
% shear stress, Von Mises
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all, digits(5)
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 7.6 - Beam Bending with shear - Complete Analysis\n')
fprintf('----------------------------------------------------------\n')

%% Data
% The constants
%% The constants
w1 = -250;
Fb = -1000;  Fg = -1500;
xb = 1; xd = 2; xe = 5; xg = 7; xh = 9; xde = 0.5*(xd+xe);
% beam location
A= [0,0,0]; B = [xb,0,0];  D = [xd,0,0]; E = [xe,0,0]; DE = [xde,0,0];
G = [xg,0,0];  H = [xh,0,0];

fprintf('Coordinates')
fprintf('\n----------------------')
fprintf('\nPoint A [m]  : '),disp(A)
fprintf('Point B [m]  : '),disp(B)
fprintf('Point D [m]  : '),disp(D)
fprintf('Point E [m]  : '),disp(E)
```

```
fprintf('Point G [m]   : '),disp(G)
fprintf('Point H [m]   : '),disp(H)

fprintf('\nLoads')
fprintf('\n----------------------')
fprintf('\nLoad at B [N]       = '),disp(Fb)
fprintf('\nLoad at G [N]       = '),disp(Fg)
fprintf('w(x)[N/m]          = '),disp(w1)

%% (i) reactions
syms Ax Ay Hy x real
FA = [Ax,Ay,0]; FB = [0,Fb,0];  FG = [0,Fg,0];  FH = [0,Hy,0];
Fde = int(w1,x,D(1),E(1));
FDE = [0,Fde,0];
SumF = FA+FB+FDE+FG+FH  % sum of forces

rAB = B-A;  rAG = G -A;  rAH = H - A; rADE = DE-A;
SumMA = cross(rAB,FB) + cross(rAG,FG) + cross(rAH,FH)+ ...
    cross(rADE,FDE)% sum of moments about A

fprintf('\nEquilibrium - Beam\n')
fprintf('---------------------------')
fprintf('\nSumF : \n'),disp(vpa(SumF',5))
fprintf('SumMA: \n'),disp(vpa(SumMA',5))

sol1 = solve(SumF(1),SumF(2),SumMA(3));
Ax = sol1.Ax;
Ay = sol1.Ay;
Hy = sol1.Hy;
FA = subs(FA);  FH = subs(FH);
fprintf('Ax [N] = '),disp(double(Ax))
fprintf('Ay [N] = '),disp(double(Ay))
fprintf('By [N] = '),disp(double(Hy))
fprintf('----------------------------------------------------\n')
```

In Command Window:

```
Example 7.6 - Beam Bending with shear - Complete Analysis
-------------------------------------------------------------
Coordinates
----------------------
Point A [m]  :      0      0      0
Point B [m]  :      1      0      0
Point D [m]  :      2      0      0
Point E [m]  :      5      0      0
Point G [m]  :      7      0      0
Point H [m]  :      9      0      0

Loads
----------------------
Load at B [N]      =        -1000

Load at G [N]      =        -1500
w(x)[N/m]          =   -250
SumF =
```

```
[ Ax, Ay + Hy - 3250, 0]
SumMA =
[ 0, 0, 9*Hy - 14125]

Equilibrium - Beam
----------------------------
SumF :
               Ax
 Ay + Hy - 3250.0
                0
SumMA:
                0
                0
 9.0*Hy - 14125.0
Ax [N] =      0
Ay [N] =        1680.6
By [N] =        1569.4
```

ii. The shear force and bending moment diagram.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% (ii)  w(x), V(x) and M(x) - Singularity functions
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

fprintf('V(x) and M(x) in Sections by singularity function\n')
fprintf('-----------------------------------------\n')
xpoints = [0,xb,xd,xe,xg,xh];
ypoints = [0,0,0,0,0,0];

% Define an an0nymous function (used to be called inline function)
% n is expected to be greater than or equal to zero
singfun = @(x,a,n) (x-a).^n.*(x >= a);
%
% assemble the functions
% this is not symbolic implementation
% use very close points for x so you cannot see the jump
npx = 501;  % numbr of points in x
x = linspace(0,xh,npx);  % x is no longer symbolic
w = w1*singfun(x,xd,0) - w1*singfun(x,xe,0);
V = w1*singfun(x,xd,1)/1 - w1*singfun(x,xe,1)/1 + ...
     Ay*singfun(x,0,0)+ Fb*singfun(x,xb,0) + Fg*singfun(x,xg,0);

M = w1*singfun(x,xd,2)/1/2 - w1*singfun(x,xe,2)/1/2 + ...
     Ay*singfun(x,0,1)/1+ Fb*singfun(x,xb,1)/1 + ...
     Fg*singfun(x,xg,1)/1 ;



% plot the functions
figure
set(gcf,'Position',[25,50,450,400], ...
    'Color','w');
plot(x,w,'r-','LineWidth',2)
hold on
plot(x,V,'b-','LineWidth',2)
plot(x,M,'g-','LineWidth',2)
xlabel('\bfx [m]')
```

```
ylabel('\bf w [N/m], V [N], M [Nm]')
grid
legend('w(x)','V(x)','M(x)','Location','Best')
title('\bf Example 7.6 using Singularity functions')
 % points on the beam where functoions change
plot(xpoints,ypoints,'mo','MarkerFaceColor','y')
hold off

fprintf('---------------------------------------------------\n'
```

```
   -----------------------------------------------------------------
   Shear and Moment Distribution in Sections by singularity function
   ---------------------------------------------
```

In Figure Window

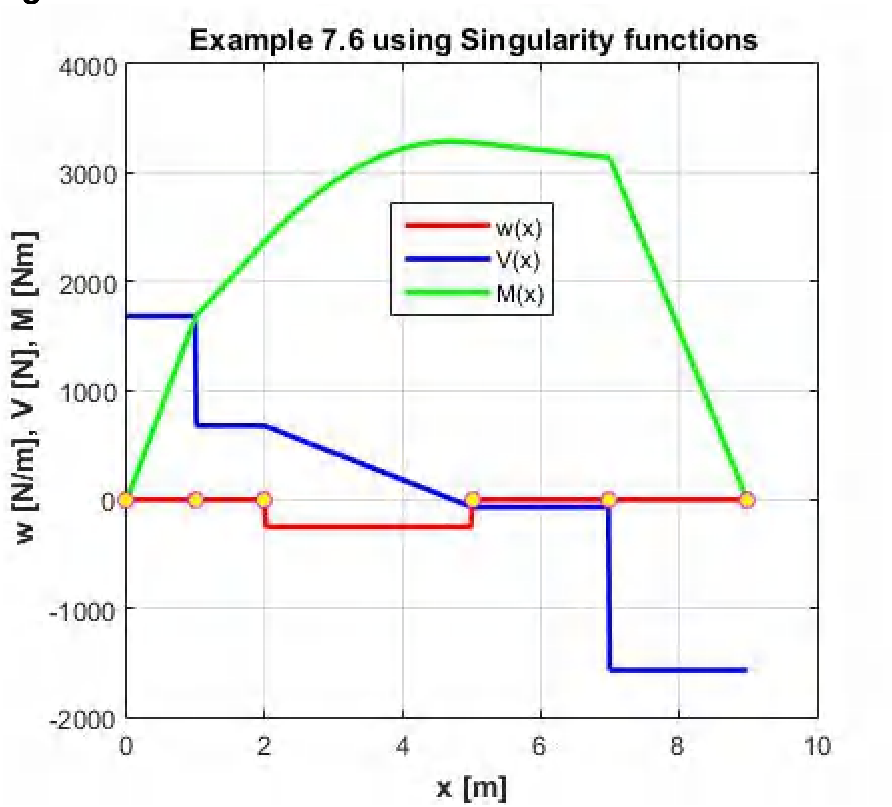


Figure 7.4.9b. Load, shear and bending moment diagram- Example 7.6

iii. The location of the centroid.
Note that x and y are used instead of z and y

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% (iii) Centroid
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Centroid by Integration \n')
fprintf('--------------------------------------------\n')
top = 0.1; base = 0.15; depth = 0.1;  % in meters
syms y a b % thickness varies from base
t1 = base; y1 = 0;
t2 = top; y2 = depth;
by = a*y + b;
eq1 = t1 - a*y1 - b;
eq2 = t2 - a*y2 - b;
```

```
sol2 = solve(eq1,eq2);
a = sol2.a;  b = sol2.b;  by = subs(by);
fprintf('Equation for thickness (measured from base): b(y) = '), ...
    disp(by)

area = int(by,y,y1,y2);  % area of the crosssection
xc = 0.5*base;  % location of x- centroid from left bottom corner
yA = int(by*y,y,y1,y2) ; % y centroid from base
yc = yA/area;
yc = double(yc);

fprintf('Area of cross-section [m^2]      = '),disp(double(area))
fprintf('y*A [m]          = '),disp(double(yA))
fprintf('y-centroid location from base [m]        = '), ...
    disp(double(yc))
fprintf('--------------------------------------------------\n')

%% draw cross-section and locate centroid
% define the corner locations
diff = base - top;
xcor = [0, base,(base-0.5*diff),(base-0.5*diff-top),0];
ycor = [0,0,depth,depth,0];

figure
set(gcf,'Position',[25,50,500,450], ...
    'Color','w');
patch(xcor,ycor,'g')
hold on
line([xc,xc-0.1],[yc,yc],'Color','r','LineWidth',2)
plot(xc-0.1,yc,'<')
text(xc-0.1,yc-0.004,'z_c')

line([xc,xc],[yc,yc+0.1],'Color','r','LineWidth',2)
plot(xc,yc+0.1,'^')
text(xc+0.004,yc+0.1,'y_c')

text(xc+0.02,yc,'C')
axis square
title('Centroid of cross-section - Example 7.4')
hold off
```

In the Command Window

```
-----------------------------------------------------------------
Centroid by Integration
-------------------------------------------
Equation for thickness (y mesaured from base): b(y) = 3/20 - y/2
Area of cross-section [m^2]      =         0.0125
y*A [m]          =    0.00058333
y-centroid location from base [m]        =      0.046667
```

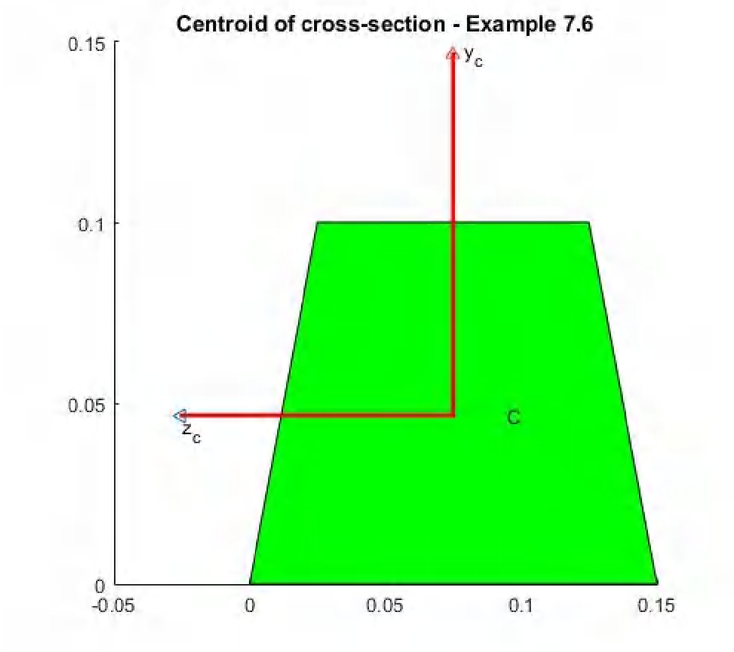


Figure 7.4.9c. Centroid - Example 7.6

iv. The calculation of the moment of inertia.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% (iv) MOI
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('MOI by integration \n')
fprintf('--------------------------------------------\n')
Iz_base = int(by*y*y,y1,y2);  % MOI about base
Izz = Iz_base - (yc^2)*area;  % MOI about centroid - parallel axis
fprintf('MOI at the base   [m^4]                   = '), ...
    disp(double(Iz_base))
fprintf('MOI at the centroid [m^4]                 = '), ...
    disp(double(Izz))


fprintf('----------------------------------------------------\n')
```

In the Command Window

```
----------------------------------------------------------------
MOI by integration
--------------------------------------------
MOI at the base   [m^4]                 =      3.75e-05
MOI at the centroid [m^4]               =    1.0277e-05
----------------------------------------------------------------
```

v. Maximum bending stress on the beam

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% (v) Maximum bending stress
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[Mmax, loc] = max(M);
yt = depth-yc;    % distance from centroid to top
yb = -yc;   % distance from centroid to bottom
sigt = double(-Mmax*yt/Izz);
```

```
sigb = double(-Mmax*yb/Izz);
fprintf('Location of Maximum bending moment [m] = '), ...
    disp(loc*xh/501)
fprintf('Maximum bending moment [Nm] = '), ...
    disp(double(Mmax))
fprintf('Maximum Tensile Stress (bottom) [MPa]  = '),...
    disp(sigb/1000000);
fprintf('Maximum Compressive Stress (top) [MPa] = '), ...
    disp(sigt/1000000);


fprintf('-----------------------------------------------------\n')
```

In the Command Window

```
----------------------------------------------------------------
Maximum Bending Moment
--------------------------------------------
Location of Maximum bending moment [m] =        4.7246
Maximum bending moment [Nm] =        3287.4
Maximum Tensile Stress (bottom) [MPa]  =        14.927
Maximum Compressive Stress (top) [MPa] =        -17.06
----------------------------------------------------------------
```

vi. The deflection of the beam

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% (vi) Beam deflection
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Beam Deflection\n')
fprintf('---------------------------\n')

EL = 68.9e09;
syms C1 C2 xx
dydx = w1*singfun(x,xd,3)/1/2/3 - w1*singfun(x,xe,3)/1/2/3 + ...
     Ay*singfun(x,0,2)/1/2+ Fb*singfun(x,xb,2)/1/2 + ...
     Fg*singfun(x,xg,2)/1/2 + C1;
yx = w1*singfun(x,xd,4)/1/2/3/4 - w1*singfun(x,xe,4)/1/2/3/4 + ...
     Ay*singfun(x,0,3)/1/2/3+ Fb*singfun(x,xb,3)/1/2/3 + ...
     Fg*singfun(x,xg,3)/1/2/3 + C1*x + C2;
eq1 = yx(1);
eq2 = yx(end);
sol = solve(eq1,eq2);
C1 = sol.C1;  C2 = sol.C2;yx = double(subs(yx));
fprintf('Integration constant - C1 = '),disp(double(C1))
fprintf('Integration constant - C2 = '),disp(double(C2))

figure
set(gcf,'Position',[25,50,500,450], ...
    'Color','w');
yx = yx/EL/Izz;
plot(x,yx,'b-','LineWidth',2)
hold on
xlabel('\bfx [m]')
ylabel('\bf y [m]')
grid

title('\bf Example 7.6; y(x) using Singularity functions')
```

```
 % points on the beam where functoions change
plot(xpoints,ypoints,'mo','MarkerFaceColor','y')
hold off
fprintf('Modulus of Elasticity [Gpa] = '),disp(EL)
fprintf('--------------------------------------------------\n')
```

In the Command Window

```
Beam Deflection
---------------------------
Integration constant - C1 =       -10501
Integration constant - C2 =      0
Modulus of Elasticity [Gpa] =      6.89e+10
----------------------------------------------------------------
```

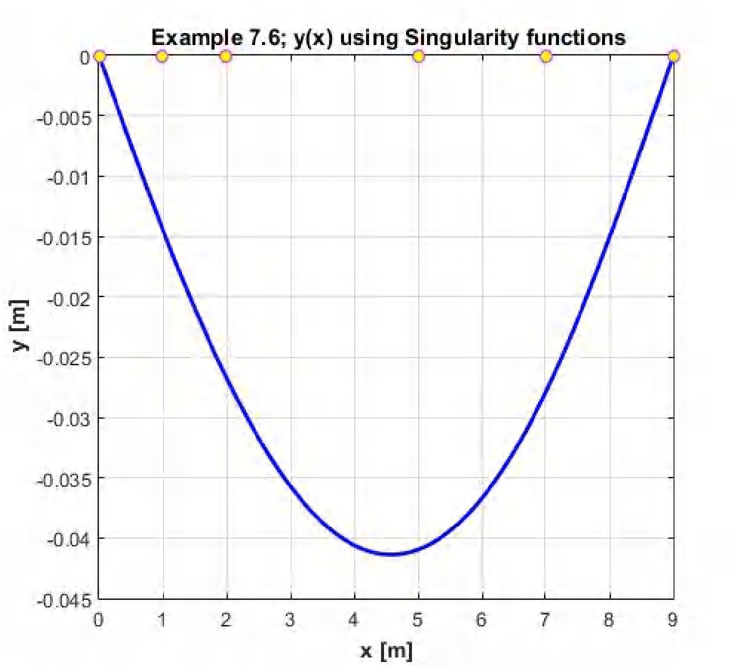


Figure 7.4.9d. Beam Deflection - Example 7.6

vii. The shear flow distribution, shear stress distribution

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% vii - shear flow and shear stress distribution
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Shear flow and stress \n')
fprintf('---------------------------\n')
[Vmax, locv] = max(abs(V));
fprintf('Location of Maximum Shear force [m] is at '),...
    disp(locv)
fprintf('Maximum Shear Force [N]                  = '),...
    disp(double(Vmax))

%%  Qy is a function of y measured from centroid
syms yy real
byy = subs(by,y,yy+yc);
fprintf('Thickness varies as y from centroid :'),disp(vpa(byy,6))

Qy_t = int(byy*yy,yy,yy,yt); % first moment of area from yy to top edge
```

```
fprintf('Q(y) - above centroid :   '),disp(vpa(Qy_t,6))

qy_t = Vmax*Qy_t/Izz;  % shear flow from centroid
tauxy_t = qy_t/byy;   % shear stress from centroid
fprintf('q(y) - :   '),disp(vpa(qy_t,6))
fprintf('tauxy(y) - :   '),disp(vpa(tauxy_t,6))
% variation of shear stress in cross-section
figure
set(gcf,'Position',[25,500,500,450], ...
    'Color','w');

hp1 = ezplot(qy_t,[yb,yt]);
set(hp1,'Color','k')
hold on
hp2 = ezplot(tauxy_t,[yb,yt]);
set(hp2,'Color','r')
hold off
grid
title('       \bfShear flow and Shear stress distribution ')
xlabel('\bfy_b <= yy <= y_t  [m]')
ylabel('\bfq_y(y) [Pa/m],  \tau_{xy}(y) [Pa] ')
legend('q_y(y)',  '\tau_{xy}(y)')
fprintf('---------------------------------------------------\n')
```

In the Command Window

```
Shear flow and stress
---------------------------
Location of Maximum Shear force [m] is at      1
Maximum Shear Force [N]                  =        1680.6
Thickness varies as y from centroid :0.126666 - 0.5*yy
Q(y) - above centroid :   0.166667*yy^3 - 0.0633332*yy^2 + 0.000154862
q(y) - :   2.72533e7*yy^3 - 1.03562e7*yy^2 + 25323.0
tauxy(y) - :   -(1.0*(2.72533e7*yy^3 - 1.03562e7*yy^2 +
25323.0))/(0.5*yy - 0.126666)
----------------------------------------------------------------
```

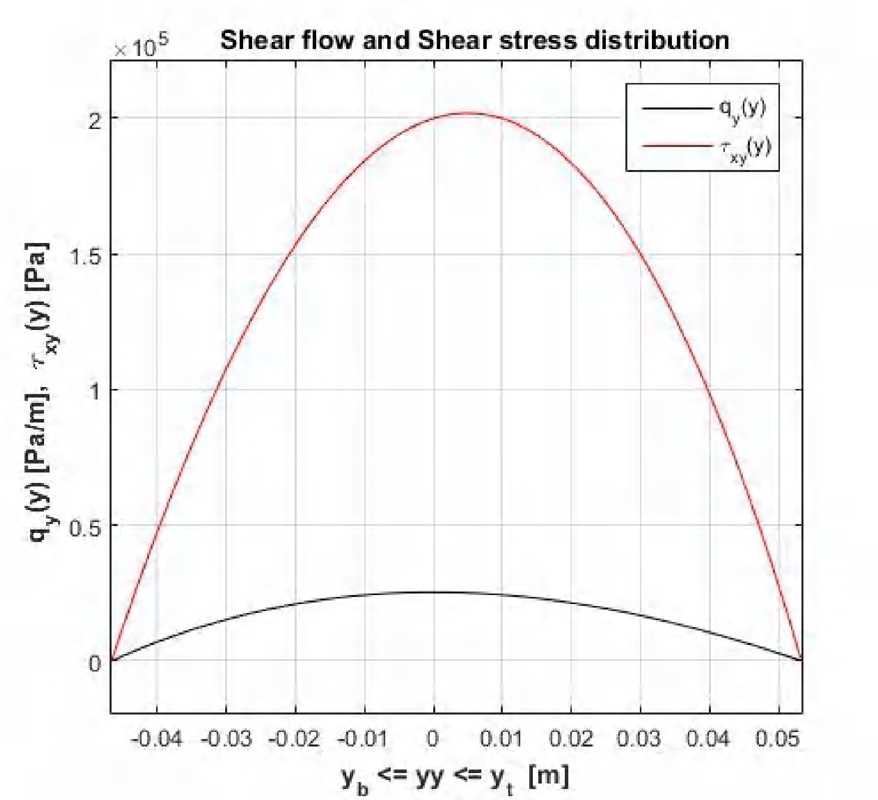


Figure 7.4.9e. Shear flow and shear stress - Example 7.6

viii. The maximum Von Mises stress in the cross-section

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% viii calculating the Von Mises criteria
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Von Mises Stress \n')
fprintf('---------------------------\n')
[VMx, locVM] = max(abs(V) + abs(M));
fprintf('Point for Exploring Von Mises [m] : '), ...
   disp((locVM-1)*xh/(npx-1))

fprintf('Sum of Bending and shear [m] : '),disp(VMx)
% Mvm = M(locVM);
% Vvm = V(locVM);
Mvm = 3138.3;   Vvm = -1594.4;

fprintf('M for calculating normal stresses [Nm] = '), ...
    disp(double(Mvm))
fprintf('V for calculating shear stresses [N]     = '), ...
    disp(double(Vvm))

% determining the point for calculation of von mises
ypts = linspace(yb,yt,31);  % points in cross-section

% determine the scaled normal stress distribution (numeric)
sigtVM = double(-Mvm*yt/Izz);
sigbVM = double(-Mvm*yb/Izz);
sigMaxVM = max(abs(sigtVM),abs(sigbVM));
sigVM = (-Mvm*(yy)/Izz);  % normal stress as a function of y
```

```
sc_sig = abs(sigVM)/sigMaxVM;
sc_sig_pts = subs(sc_sig,yy,ypts);

% calculate scaled shears stress in the cross-section
qy_VM = Vvm*Qy_t/Izz;  % shear flow from centroid
tauxyVM = abs(qy_VM/byy);    % shear stress from centroid
tauxy_pts = subs(tauxyVM,yy,ypts);
tauxyMx = max(tauxy_pts);  %
sc_tauxy = tauxy_pts/tauxyMx;

% sum of the stresses
sum_st = sc_sig_pts+sc_tauxy;
[maxsum, VMloc] = max(sum_st);
yloc = ypts(VMloc);
sigval = double(-Mvm*yloc/Izz);
tauval = double(subs(tauxyVM,yy,yloc));

% figure
figure
set(gcf,'Position',[25,500,500,450], ...
    'Color','w');

hp1 = plot(ypts,sc_sig_pts,'r-');
% set(hp1,'Color','k')
hold on
hp2 = plot(ypts,sc_tauxy,'k-');
% set(hp2,'Color','r')
hp3 = plot(ypts,sum_st,'b-');
hold off
title('\bfPoint for Von Mises Calculation')
xlabel('\bfy_b <= y <= y_t  [m]')
ylabel('\bfscaled stresses and sum')
grid
legend('scaled normal','scaled shear','sum')
axis tight

% calculate principal stress
fprintf('\nPoint calculation of stresses (from centroid)[m] = '), ...
    disp(yloc)
fprintf('Maximum tensile stress in cross-section [Pa]   = '), ...
    disp(sigval)
fprintf('Maximum shear stress in cross-section [Pa]   = '), ...
    disp(tauval)

%% Calculate Principal Stress
sigy = 0;
sigav = 0.5*(sigval + sigy);
R = sqrt((0.5*(sigval-sigy))^2 + tauval^2);
siga = sigav+R;   sigb = sigav-R;
thtp = 0.5*atan2(2*tauval,(sigval-sigy));
fprintf('\n-------------------\n')
fprintf('Principal stresses \n')
fprintf('--------------------\n')
fprintf('siga   [Pa] :'),disp(siga)
fprintf('sigb   [Pa] :'),disp(sigb)
fprintf('taumax [Pa] :'),disp(R)
```

```
fprintf('Direction of principal axis (deg):'),disp(thtp*180/pi)
yieldsig = 276e06;
VMStress = sqrt((siga^2 - siga*sigb + sigb^2));
FOS =  yieldsig/VMStress;
fprintf('Yield Strength of Material    [Pa]          :'),disp(yieldsig)
fprintf('Von Mises stress at this location    [Pa] :'),disp(VMStress)
fprintf('Factor of Safety (FOS)                      :'),disp(FOS)
```

In the Command Window

```
Von Mises Stress
---------------------------
Point for Exploring Von Mises [m] :         7.002
Sum of Bending and shear [m] : 1191955047002197280 5/2533274790395904
Bending Moment for calculating normal stresses [Nm] =        3138.3
Shear Force for calculating shear stresses [N]     =       -1594.4

Point for calculation of stresses (from centroid)[m] =      0.026666
Maximum tensile stress in cross-section [Pa]    =     -8142800
Maximum shear stress in cross-section [Pa]    =       154666

-------------------
Principal stresses
-------------------
siga   [Pa] :       2936.7
sigb   [Pa] :  -8.1457e+06
taumax [Pa] :   4.0743e+06
Direction of principal axis (deg):       88.912
Yield Strength of Material    [Pa]       :   276000000
Von Mises stress at this location    [Pa] :   8.1472e+06
Factor of Safety (FOS)                   :       33.877
```

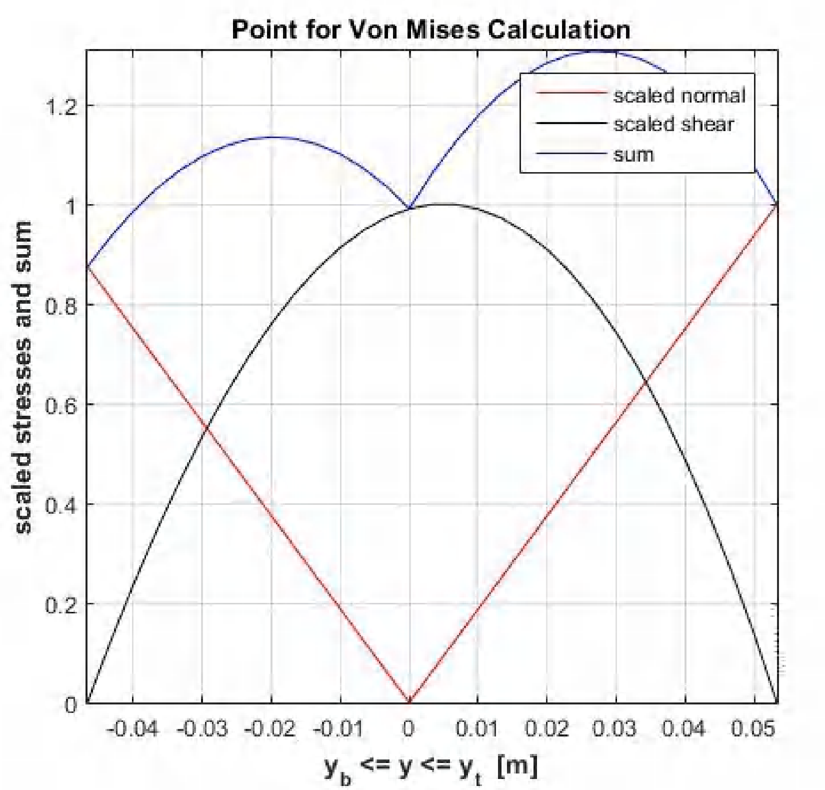


Figure 7.4.9f. Determine Von Mises point for calculating stress - Example 7.6

Execution in Octave

The code is same as in the MATLAB except for the highlighted changes. The code is run as a composite file. It was quite frustrating to debug and correct for the errors. First the code takes much longer to run. There were two main errors that had to be corrected:

1. All vector substitutions were replaced by contiguous single value substitutions
2. All information for the plot must be converted to double as the information still continues to remain rational and triggers plotting error. It appears to be a good practice to use the double function on any variable that had some connection to symbolic variable upstream anywhere before plotting

In Octave Editor

```
clc, clear, format compact, format shortg, close all,
warning off
pkg load symbolic;

fprintf('\nSumF : \n'),disp(vpa(SumF,5))
fprintf('SumMA: \n'),disp(vpa(SumMA,5))

FA = subs(FA, Ax);
FA = subs(FA, Ay);
FH = subs(FH,Hy);

w = double(w);
V = double(V);
M = double(M);

%by = subs(by);
by = subs(by,'a',a);
by = subs(by,'b',b);

yc = double(yc)

Izz = double(Izz);

C1 = double(sol.C1);  C2 = double(sol.C2);

%yx = double(subs(yx));
yx = subs(yx,'C1',C1);
yx = subs(yx,'C2',C2);
yx = double(yx);

sc_sig_pts = double(sc_sig_pts);
sc_tauxy = double(sc_tauxy);
sum_st = double(sum_st);
```

In the Octave Command Window

The entire output in the command window is included below. The code is run as a composite file

```
Example 7.6 - Beam Bending with shear - Complete Analysis
------------------------------------------------------------
Coordinates
```

```
----------------------
Point A [m]  :   0  0  0
Point B [m]  :   1  0  0
Point D [m]  :   2  0  0
Point E [m]  :   5  0  0
Point G [m]  :   7  0  0
Point H [m]  :   9  0  0

Loads
----------------------
Load at B [N]     = -1000
Load at G [N]     = -1500
w(x)[N/m]         = -250

Equilibrium - Beam
---------------------------
SumF :
  [Ax  Ay + Hy - 3250.0  0]
SumMA:
  [0  0  9.0*Hy - 14125.0]
Ax [N] = 0
Ay [N] = 1680.6
By [N] = 1569.4
----------------------------------------------------------------
(ii) Load, Shear,Moment Distribution by singularity function
------------------------------------------------------------------
----------------------------------------------------------------
(iii) Centroid by Integration
----------------------------------------------
Equation for thickness (y measaured from base): b(y) =
     y   3
  - - + --
    2   20
yc = 0.046667
Area of cross-section [m^2]     = 0.0125
y*A [m]        = 0.00058333
y-centroid location from base [m]        = 0.046667
----------------------------------------------------------------
(iv) MOI by integration
----------------------------------------------
MOI at the base  [m^4]                   = 3.75e-05
MOI at the centroid [m^4]                = 1.0278e-05
----------------------------------------------------------------
(v) Maximum Bending
----------------------------------------------
Location of Maximum bending moment [m] = 4.7246
Maximum bending moment [Nm] = 3287.4
Maximum Tensile Stress (bottom) [MPa]  = 14.927
Maximum Compressive Stress (top) [MPa] = -17.059

----------------------------------------------------------------
 (vi)Beam Deflection
---------------------------
Integration constant - C1 = -10501
Integration constant - C2 = 0
Modulus of Elasticity [Gpa] = 6.89e+10
```

```
-------------------------------------------------------------------
 (vii) Shear flow and stress
----------------------------
Location of Maximum Shear force [m] is at 1
Maximum Shear Force [N]                  = 1680.6
Thickness varies as y from centroid :  -0.5*yy + 0.126667
Q(y) - above centroid :                 3              2
  0.166667*yy  - 0.0633333*yy  + 0.000154864
q(y) - :                    3                2
  2.72523e+7*yy  - 1.03559e+7*yy  + 25322.4
tauxy(y) - :             /          3              2           \
  97297.3*\280.093*yy  - 106.435*yy  + 0.260258/
  ------------------------------------------------
                -0.5*yy + 0.126667
-------------------------------------------------------------------
 (viii)Von Mises Stress
----------------------------
Point for Exploring Von Mises [m] : 7.002
Sum of Bending and shear [m] : 4705.2
Bending Moment for calculating normal stresses [Nm] = 3138.3
Shear Force for calculating shear stresses [N]     = -1594.4

Point for calculation of stresses (from centroid)[m] = 0.026667
Maximum tensile stress in cross-section [Pa]   = -8.1426e+06
Maximum shear stress in cross-section [Pa]   = 1.5466e+05

-------------------
Principal stresses
-------------------
siga   [Pa] :2936.4
sigb   [Pa] :-8.1456e+06
taumax [Pa] :4.0742e+06
Direction of principal axis (deg):88.912
Yield Strength of Material   [Pa]        :2.76e+08
Von Mises stress at this location   [Pa] :8.147e+06
Factor of Safety (FOS)                   :33.877
```

In the Octave Figure Window
All the plot are the same. The last one is included below:

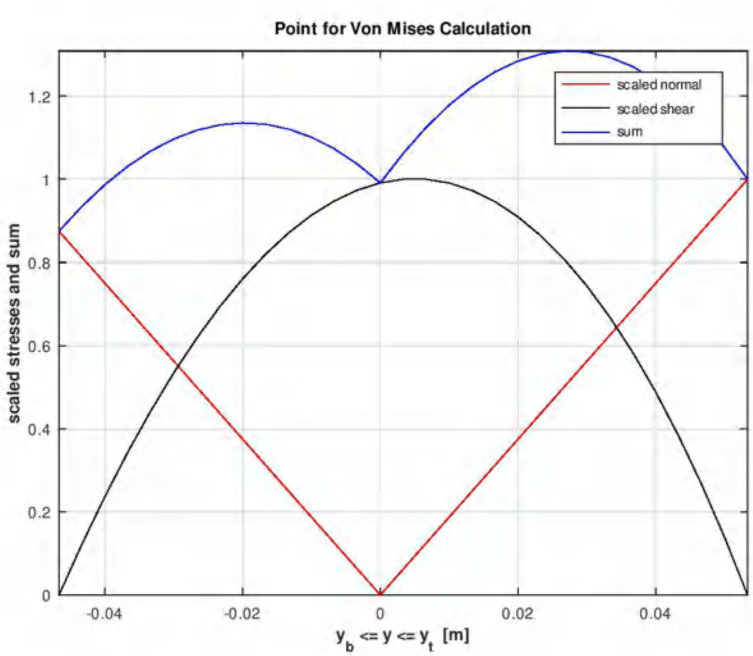


Figure 7.4.9g. Determine Von Mises point for calculating stress - Example 7.6 - OCTAVE

7.4.4 Example 7.7: Shear flow and Stress Calculation for Example 7.5

Example 7.5 defined in Figure 7.3.12 has a π shaped cross-section. We start with the known solution of Example 7.5 in Section 7.3.6 . Let us assume that the beam is made of Aluminum Alloy. It has a density of 2700 kg/m^3 , Young's modulus of 70 GPa , a Poisson's ratio of 0.32, and a shear modulus of 26 GPa. It has an yield strength of 95 MPa and ultimate tensile strength of 110 MPa in tension. The shear yield strength is 55 MPa and the the ultimate shear strength is 70 MPa.

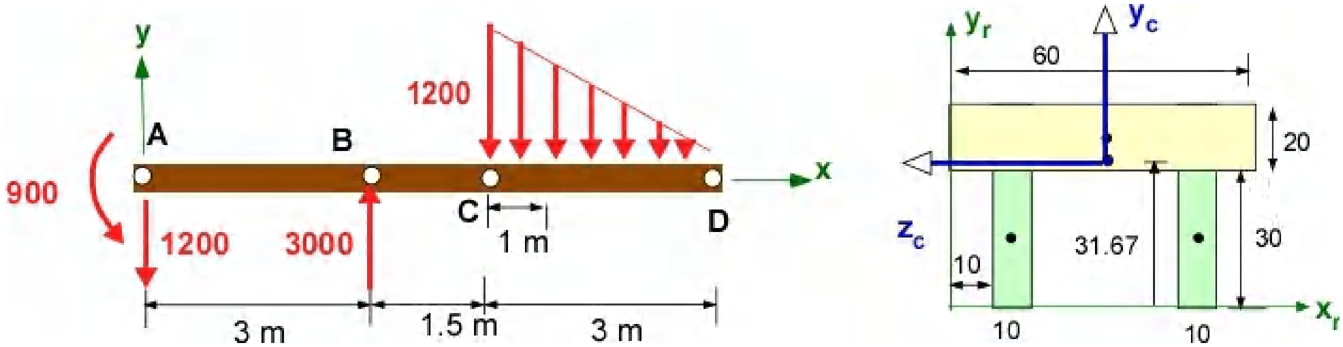


Figure 7.4.10a. FBD and Cross-section - Example 7.6

Data: Beam, loading and reactions are shown in the figure. A = [0,0,0]; B = [1,0,0]; C = [3,0,0]; D = [5,0,0]; F_B = -3000 N; w = -500 N/m.
Cantilevered at A -Statically Determinate.
E = 70 GPa; **G** = 26 GPa; σ_y = 95 MPa; σ_U = 110 MPa τ_y = 55 MPa; τ_U = 70 MPa
I_z =3.35e^{-7} m^4

Find:

i. The shear flow distribution, shear stress distribution
ii. The maximum Von Mises stress in the cross-section

Assumption: Cross-section has an axis of symmetry. Cross-section is same across the beam.

Solution for centroid, MOI, bending stresses available from solution to Example 7.5.
Below is the load, shear, and bending moment diagram

Solution:

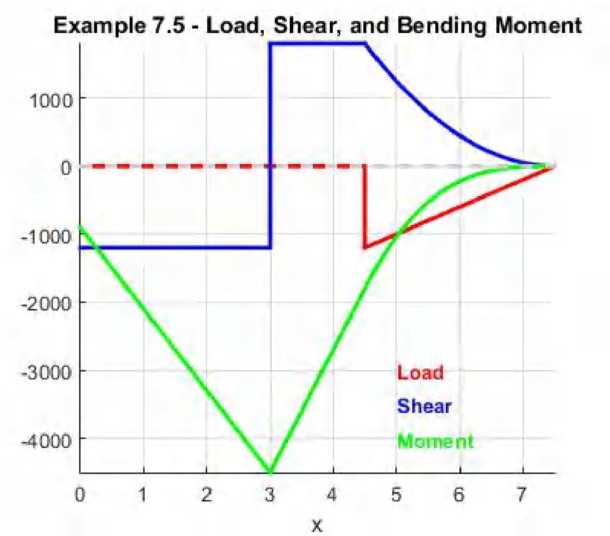


Figure 7.4.10b. Load, shear and bending moment diagram - Example 7.7

The bending moment is maximum at x = 3 m (point B) with a value of -4500 Nm.The shear force is also a maximum at this point at a value of 1800 N. Therefor the cross-section at B will have the maximum normal and shear stresses. This is the point along the beam length for design.

i. The shear flow distribution, shear stress distribution

We will use composite areas to calculate the first moment of area. We will establish some points for the calculation of shear stress in the cross-section. In Figure 7.4.10c these points are identified as a(top) , b (10 mm below top), c (centroid), d (30 mm from bottom),e (15 mm from bottom), and f (bottom).

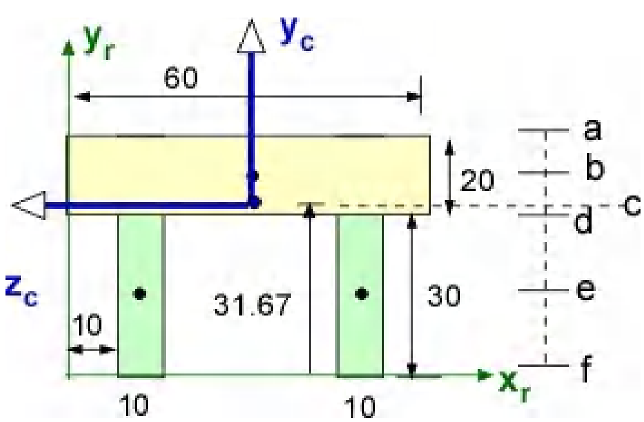


Figure 7.4.10c. Cross-section - Example 7.6

For example to calculate the shear flow and shear stress at y_b which is located 10 mm below the top surface we will reference Figure 7.4.10d

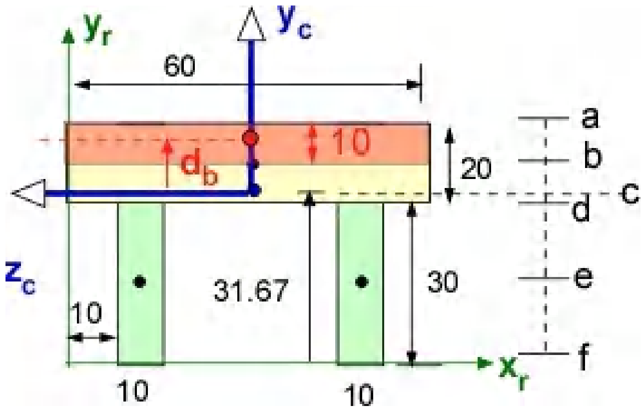


Figure 7.4.10d. First moment of area $\mathbf{Q_b}$

$$Q_b = Area \times d_b = 60 \times 10 \times (45 - 31.67) = 7998\, mm^3 = 7998 \times 10^{-9}\, m^3;$$

$$q_b = \frac{1800 \times 7998 \times 10^{-9}}{3.35 \times 10^{-7}} = 42974 \frac{N}{m};$$

$$\tau_{xy}(b) = \frac{q_b}{w_b} = \frac{42974}{60/1000} = 7.16 \times 10^5 \frac{N}{m^2};$$

The shear stress is the average shear stress at the location b.

In addition we know that the shear flow and shear stress is zero at location a and f

You should calculate:

$$Q_c = 1.008 \times 10^{-5}\, m^3; \quad q_c = 54159 \frac{N}{m}; \quad \tau_{xy}(c) = 9.03 \times 10^5 \frac{N}{m^2};$$

$$Q_d = 5.001 \times 10^{-6}\, m^3; \quad q_d = 26871 \frac{N}{m}; \quad \tau_{xy}(d) = 2.69 \times 10^6 \frac{N}{m^2};$$

$$Q_e = 3.625 \times 10^{-6}\, m^3; \quad q_e = 19480 \frac{N}{m}; \quad \tau_{xy}(e) = 1.95 \times 10^6 \frac{N}{m^2};$$

You will see that the shear flow is maximum at the centroid but the shear stress is largest at point d.

ii. The maximum Von Mises stress in the cross-section

The maximum bending/normal and shear stress will be at x = 3 m (point B). We have calculated the shear stress above. Let us borrow the maximum normal stresses from Example 7.5:

$$\sigma_t(\max) = 4500(50-31.67)/(3.35\times10^{-7}\times1000) = 246.3[MPa];$$

$$\sigma_c(\max) = 4500(31.67-0)/(3.35\times10^{-7}\times1000) = -425.4[MPa];$$

$$FOS_Y = \frac{95}{425.4} = 0.22;$$

$$FOS_U = \frac{110}{425.4} = 0.26;$$

The normal stress values suggests that the beam must be redesigned since the FOS is quite low. The maximum normal stress is about 424.5 MPa while the maximum shear stress is around 2.69 MPa.

We can ignore the shear stress and calculate the Von Mises stresses based on the value of the principal stress of $\sigma_{\mathbf{a}}$ = 425 MPa and $\sigma_{\mathbf{b}}$ = 0 for $\sigma_{\mathbf{vm}}$ = 425 MPa.

7.4.5 Additional Problems

Set up the following problems by hand on paper and solve them on paper and using MATLAB/Octave. These problems appeared in Section 7.3. Extend them to include shear flow and shear stress calculations. If the FOS based on bending is reasonable then calculate the maximum Von Mises stress for the problems..

After completing these problems, consider the problem in Section 7.2 and solve them for the case for a rectangular cross-section of width 60 mm and height 200 mm.

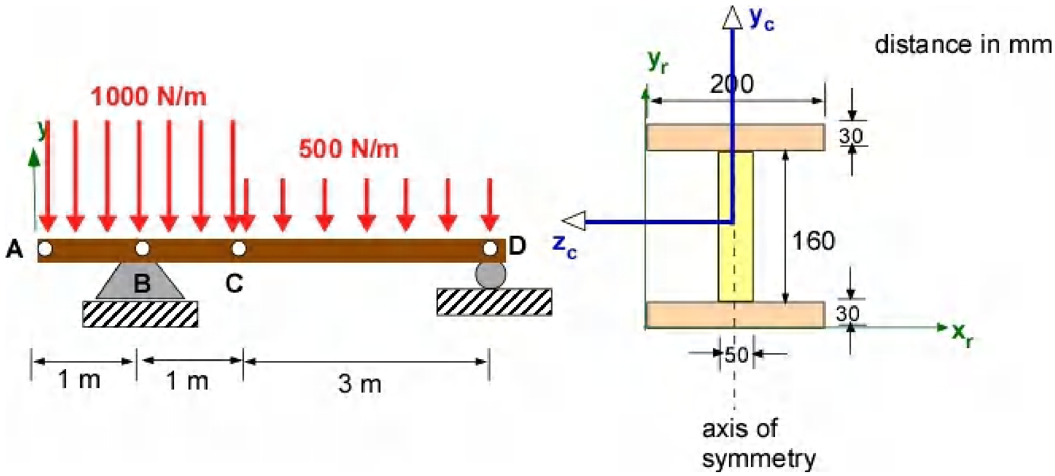
Please use the following table if necessary.

Table 7.2

Material	**Aluminum**	**Brass**	**Steel**	**Wood**
E [GPa]	70	105	200	13
Yield stress [MPa]	230	410	250	60
Ultimate Stress (tension) [MPa]	300	500	450	100
Ultimate stress (shear) [MPa]	70	200	250	10

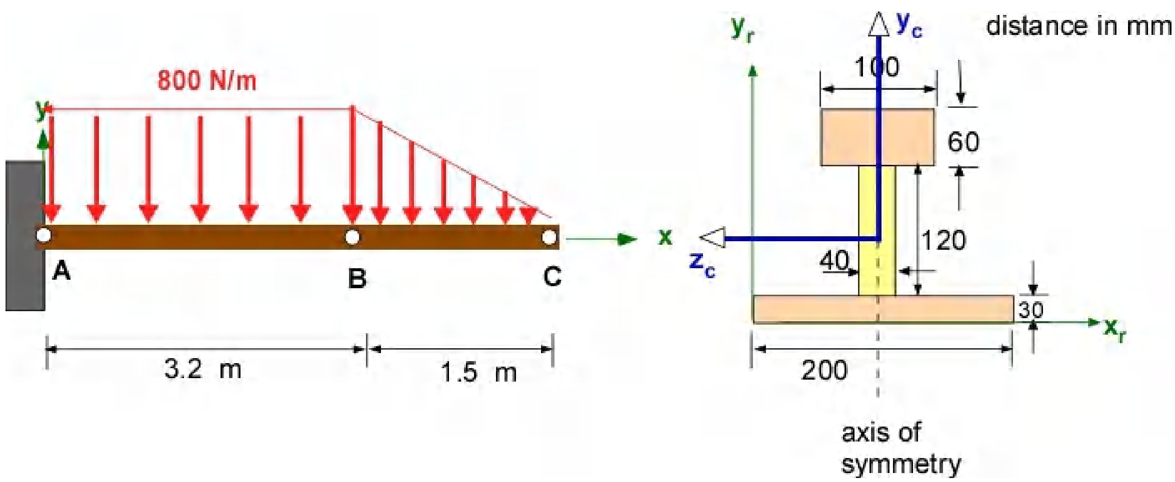
(The problems are the same but just renumbered)

Problem 7.4.1. Figure shown below



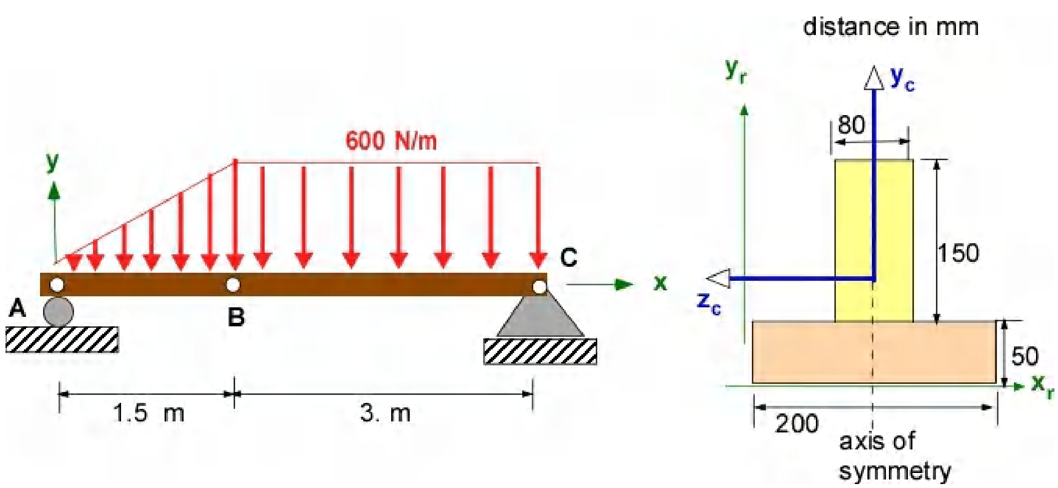
Problem 7.4.1

Problem 7.4.2. Figure shown below



Problem 7.4.2

Problem 7.4.3. Figure shown below



Problem 7.4.3

7.5 SHEAR FLOW IN THIN WALLED MEMBERS

You will find thin walled structures everywhere you care to look. This is because they use less material while avoiding failure. One industry that relies on these structures is the aircraft industry. They are motivated by minimum mass designs so that they can increase their payload capacity. They are also subject to rigorous failure standards. Figure 7.5.1 shows the skeleton of a business aircraft.

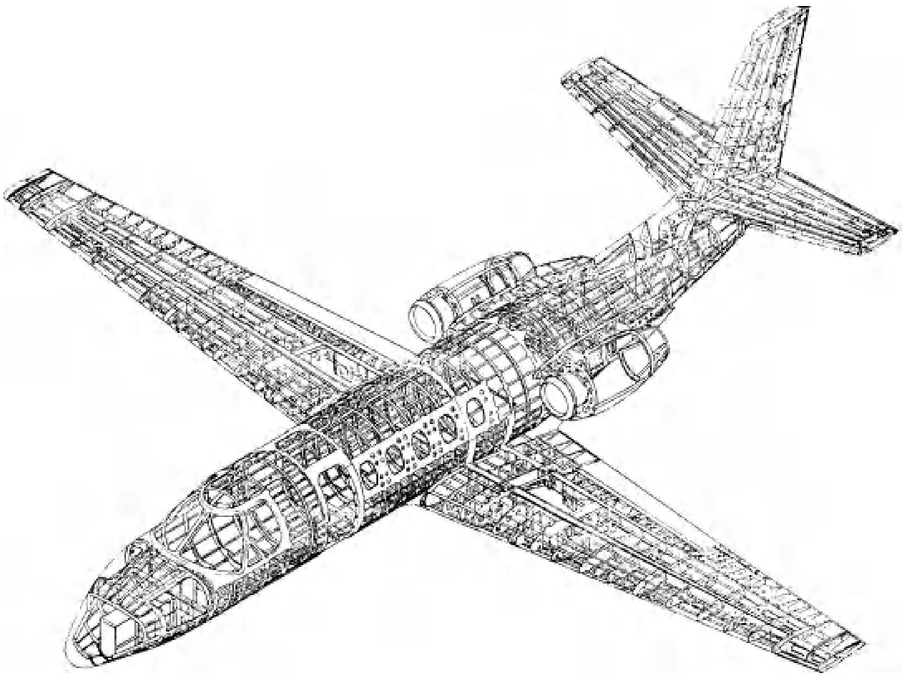


Figure 7.5.1 Aircraft Structure (Image: Stanford University)

This and other aircraft structures are composed of shell like structures. There are long beams (long structures) that have thin cross-sections that can be open or closed. They are usually classified as the alphabet shape they represent. Figure 7.5.2 are some of the cross-sections they represent. Note that except for one there is a vertical axis of symmetry.

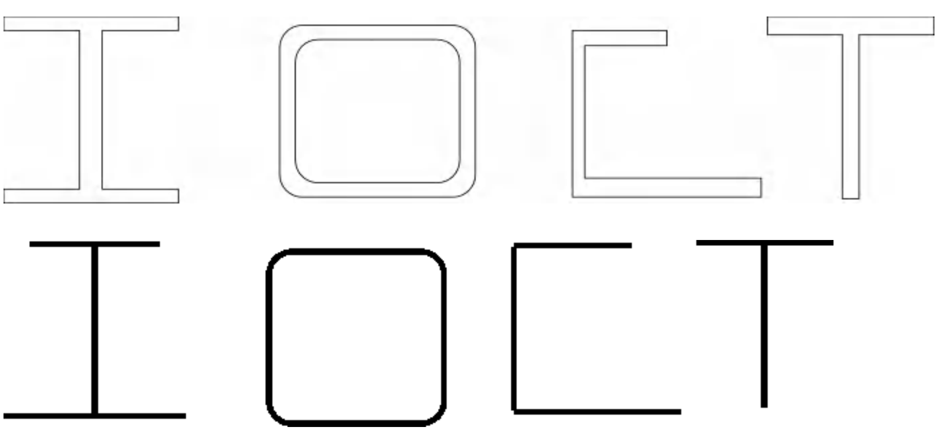


Figure 7.5.2 Thin walled beam cross-sections (with alternate representations)

The thin walls suggest that the shear flow and the shear stress at any point in the cross-section can be replaced by an average value. For that reason we can also represent the cross-section as a stick figure with same or different thickness for each section. From the previous section the shear stress and shear flow is calculated as (see Eqn. 7.16 and 7.17):

$$\tau_{ave} = \tau = \frac{VQ}{I_c t}; \quad and\ /\ or \quad q_{ave} = q = \frac{VQ}{I_c}; \qquad (7.20)$$

Here **V** is the shear load in the cross-section along the *vertical axis of symmetry*, **Q** is an appropriate first moment of area about the centroid, $\mathbf{I_c}$ is the moment of inertia of the cross-section about the centroid, and ***t*** is the thickness of the cross-section at the point of shear computation. **V** and $\mathbf{I_c}$ should not change as we calculate τ at different points in the cross-section. It is also popular to report the values as **q** - shear flow. It tends to flow around the cross-sections with a definite starting and ending point, often with a value of zero for a force loaded structure. For a vertical downward shear force in the cross-section we illustrate the choice of area for the calculation of **Q** and the nature and direction of τ in the cross-section at that point. Depending on how you decide to section the cross-section you may see more than one τ at the section. In this case you will have to apportion the calculated value from Equation 7.20 among them.   The **I** beam is among the most popular cross-section for resisting bending and you will see it used everywhere. It also resists shear load in the cross-section. In Figure 7.5.3 we we represent the shear stress around the beam in various sections by the arrow. Also note the subscripts of the shear stress as it is computed in the cross-section.

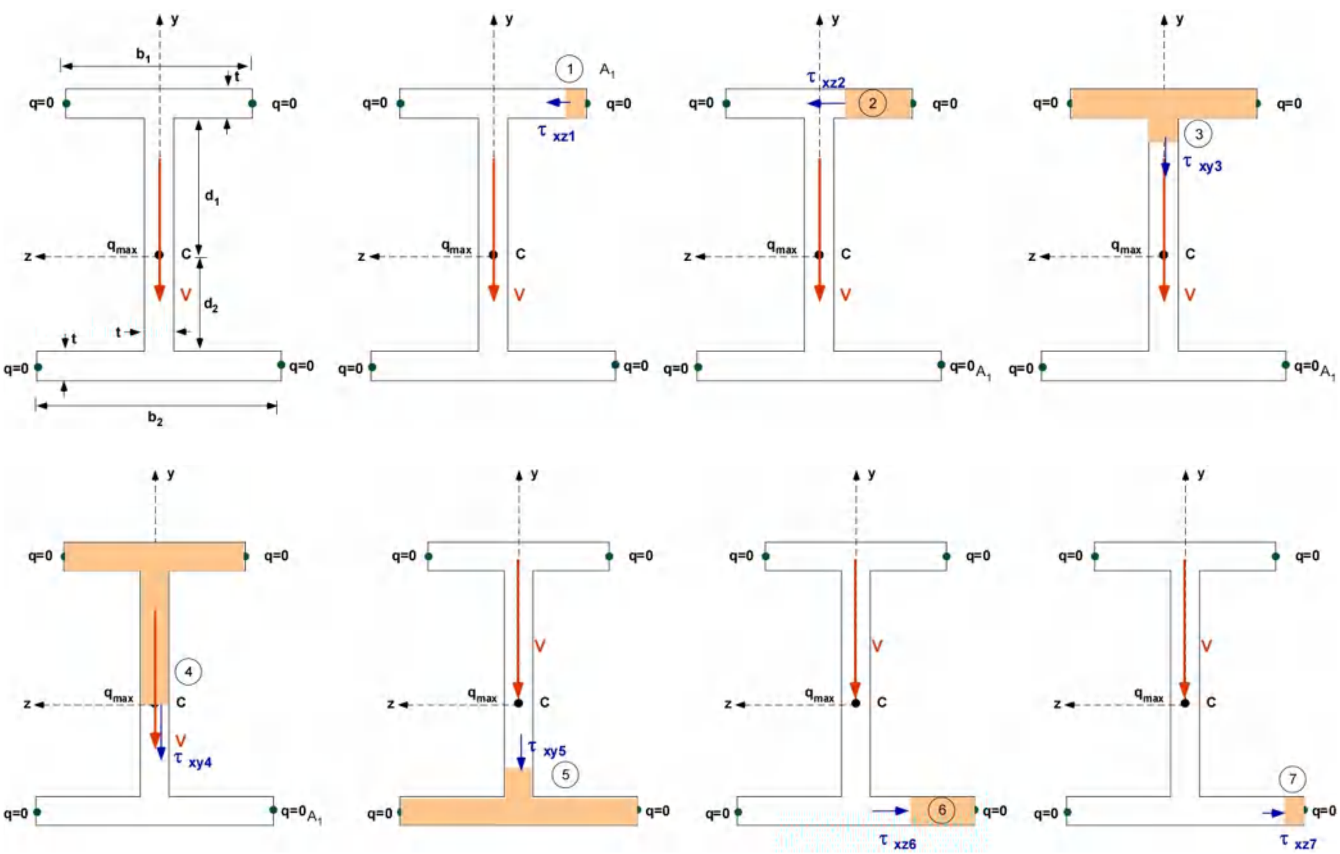


Fig. 7.5.3 Calculation of shear stress in the cross-section of I beam

The corresponding shear flow distribution is shown in Figure 7.5.4.

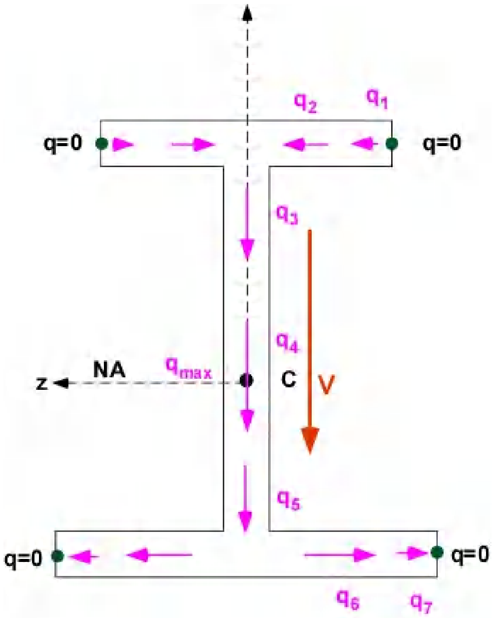


Figure 7.5.4 The shear flow distribution for the I beam

Another beam is the box beam. The shear stress and shear flow around it is illustrated in Figure 7.5.5. The distribution and values are symmetrical.

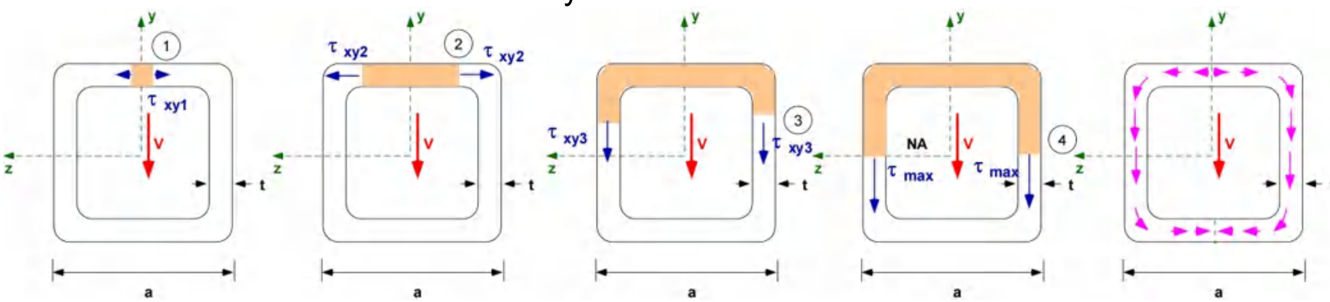


Figure 7.5.5 Shear stress and shear flow around a symmetrical box beam.

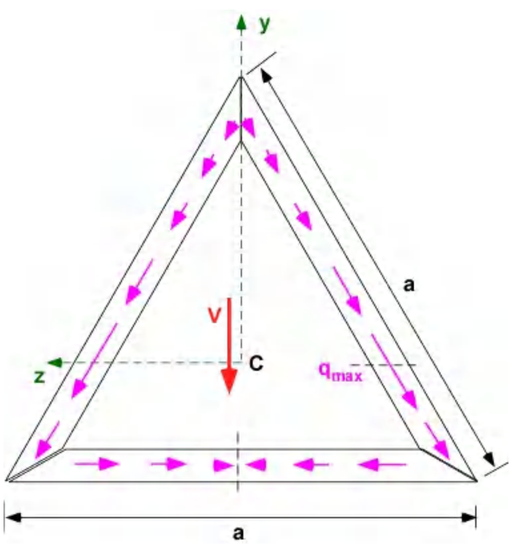


Figure 7.5.6 Shear flow around a triangle beam

An important consideration in this discussion is that the distribution of shear flow and shear stress due to the shear load (V) only, the load must be applied at the ***shear center***. A load not applied at the shear center will cause the cross-section to twist and therefore cause additional shear stress. In the examples in this section the shear center is at the centroid so it is not an issue, but for open beams this is an assumption that must be made.

7.5.1 Example 7.8

The extruded hat beam made of Aluminum alloy AL 7075-T6 is used to to support a maximum shear load of 500 N (downward) in the cross-section. Calculate the factor of safety base on the maximum shear stress. Obtain the shear flow distribution around the beam at the locations identified.

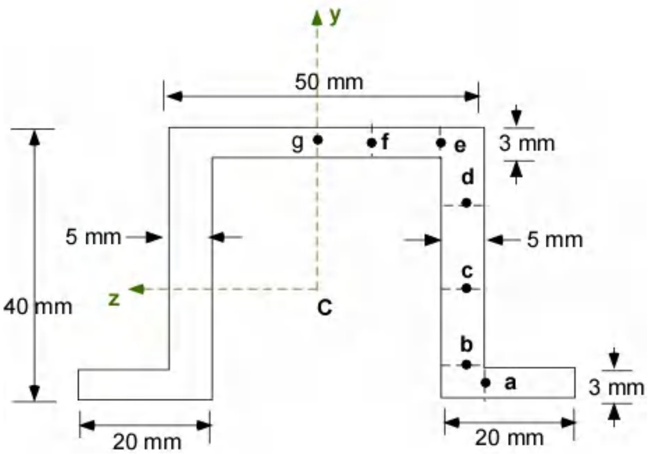


Figure 7.5.7 Example 7.8

Data: Cross-section is shown in the figure.
Material AL 7075-T6 alloy
σ_y = 503 MPa (tensile yield strength); **E** = 71.7 GPa (modulus of elasticity); ν = 0.33 (Poisson's ratio); **G** = 26.9 GPa (shear modulus); and τ_y = 331 MPa (shear yield strength).
V = - 500 N

Find:

i. The location of the centroid.
ii. The calculation of the moment of inertia.
iii. The shear flow distribution and shear stress distribution.
iv. Maximum shear stress in the cross-section.
v. The factor of safety due to shear

Assumption: Constant cross-section

Solution:

i. Location of Centroid

The cross-section is symmetric and therefor we need only the y location of the centroid from the base.

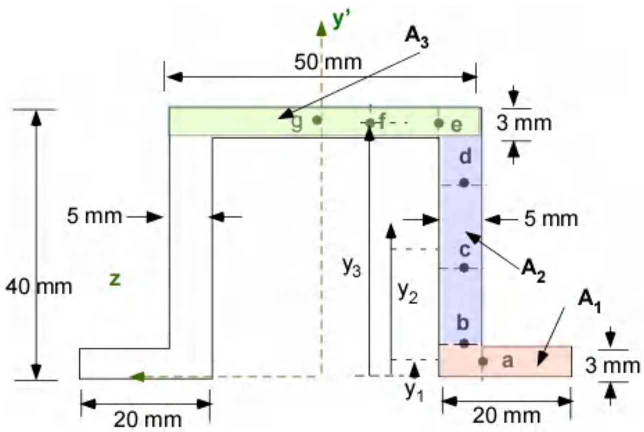


Figure 7.5.8a. Centroid calculation

$$y_C = \frac{2y_1A_1 + 2y_2A_2 + y_3A_3}{2A_1 + 2A_2 + A_3}$$

y_C = 0.0209 [m]

ii. Moment of Inertia about Centroid

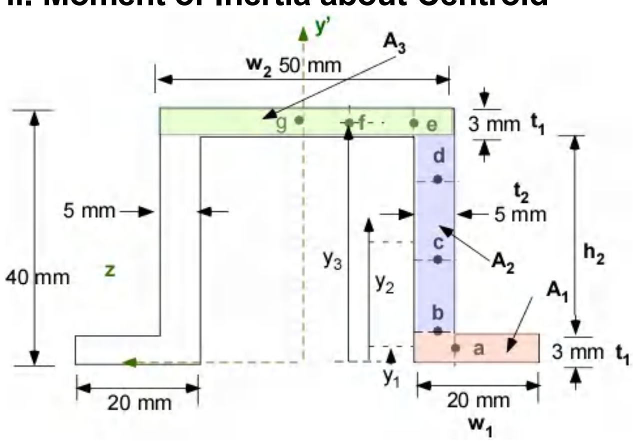


Figure 7.5.8b. MOI calculation

We can calculate the MOI about the centroid exactly or through thin wall assumptions where the cubed terms of thickness are ignored with respect to the other terms. You will notice that there is no significant difference.

$$I_{zC,exact} = 2\left(\frac{1}{12}t_1^3w_1 + A_1(y_C - y_1)^2\right) + 2\left(\frac{1}{12}h_2^3t_2 + A_2(y_C - y_2)^2\right) + \left(\frac{1}{12}t_1^3w_2 + A_3(y_C - y_3)^2\right)$$

$$= 1.2486\text{e-}07\left[m^4\right]$$

$$I_{zC,thinwall} = 2\left(A_1(y_C - y_1)^2\right) + 2\left(\frac{1}{12}h_2^3t_2 + A_2(y_C - y_2)^2\right) + \left(A_3(y_C - y_3)^2\right)$$

$$= 1.2466\text{e-}07\left[m^4\right]$$

In this calculation we used the centroid form the actual geometry. To be consistent we should recalculate the centroid based on thin wall assumption. Will it be different?

iii. The shear flow and shear stress distribution (at selected points)
Figure 7.5.8c illustrates the area for a few of the points (f, c, and a)

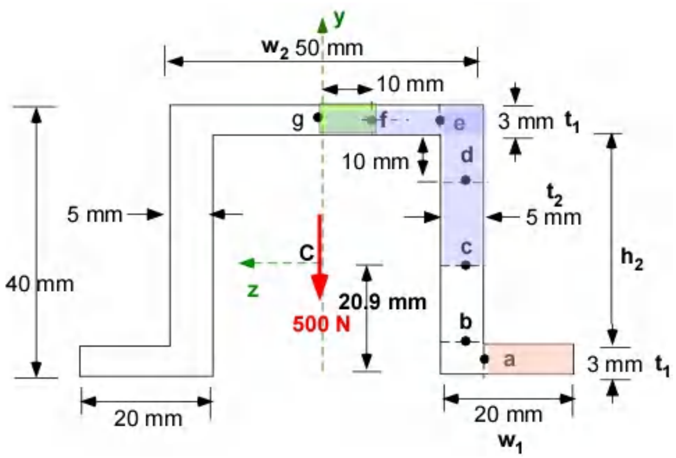


Figure 7.5.8c Areas for some shear flow calculation

The shear will be zero at g and at the ends of the bottom legs. It will be downward at c.

Point g: $\mathbf{q_g} = \tau_g = 0$

Point f: $A_f = (10/1000)*(3/1000) = 3.00e\text{-}05\ [m^2]$; $y_f = ((40-20.9) - 1.5)/1000 = 1.76e\text{-}02\ [m]$;
$Q_f = A_f*y_f = 5.28e\text{-}07\ [m^3]$; $\mathbf{q_f} = (500)*Q_{f)}/I_{zC} = 2.11e{+}03$ [Pa-m]; $\tau_f = q_f/\mathbf{t}_f = 7.04e{+}05$ [Pa]

Point e: $A_e = (20/1000)*(3/1000) = 6.00e\text{-}05\ [m^2]$; $y_e = ((40-20.9) - 1.5)/1000 = 1.76e\text{-}02\ [m]$;
$Q_e = A_f*y_f = 1.06e\text{-}06\ [m^3]$; $\mathbf{q_e} = (500)*Q_e/I_{zC} = 4.23e{+}03$ [Pa-m]; $\tau_e = q_e/\mathbf{t_e} = 1.41e{+}06$ [Pa]

Point d: Ad1 = (25/1000)*(3/1000)= 7.50e-05 [m2]; yd1= ((40-20.9) - 1.5)/1000 = 1.76e-02 [m] ;
Ad2 = (10/1000)*(5/1000) = 5.00e-05 [m2]; yd2= ((40-20.9) - 3 - 5)/1000 = 1.11e-02 [m] ;
$\mathbf{Q_d}$ = Ad1*yd1+ Ad2*yd2 = 1.87e-06 [m3]; $\mathbf{q_d} = (500)*Q_{d})/IzC = 7.50e{+}03$ [Pa-m];
$\tau_d = \mathbf{q_d}/\mathbf{t_d} = 1.50e{+}06$ [Pa]

Point c: Ac1 = (25/1000)*(3/1000) = 7.50e-05 [m2]; yc1= ((40-20.9) - 1.5)/1000 = 1.76e-02 [m] ;
Ac2 = ((40-20.91)/1000)-(3/1000))*(5/1000)= 8.05e-05 [m2];
yc2= ((40-20.9) - 3)/2000 = 8.05e-03 [m] ;
$\mathbf{Q_c}$ = Ad1*yd1+ Ad2*yd2 = 1.97e-06 [m3]; $\mathbf{q_c} = (500)*Q_{c})/IzC = 7.87e{+}03$ [Pa-m];
$\tau_c = \mathbf{q_c}/\mathbf{t_c} = 1.57e{+}06$ [Pa]

Point f: $A_b = (20/1000)*(3/1000) = 6.00e\text{-}05\ [m^2]$; $y_b = ((20.9-3/2)/1000 = 1.94e\text{-}02\ [m]$;
$Q_b = A_b*y_b = 1.16e\text{-}06\ [m^3]$; $\mathbf{q_b} = (500)*Q_b/I_{zC} = 4.66e{+}03$ [Pa-m]; $\tau_b = q_b/\mathbf{t}_b = 9.33e{+}05$ [Pa]

Point f: $A_a = (15/1000)*(3/1000) = 4.5e\text{-}05\ [m^2]$; $y_a = ((20.9-3/2)/1000 = 1.94e\text{-}02\ [m]$;
$Q_a = A_a*y_a = 8.73e\text{-}07\ [m^3]$; $\mathbf{q_a} = (500)*Q_a/I_{zC} = 3.50e{+}03$ [Pa-m]; $\tau_a = q_a/\mathbf{t}_a = 1.17e{+}06$ [Pa]

The shear flow distribution is shown in Figure 7.5.8d.

iv. Maximum shear stress in the cross-section.
The maximum shear flow is at the centroid as expected. It is interesting to note that the shear stress

at point **a** is greater than the shear stress at point **b** primarily due to the change in the section thickness, though the shear flow is lower as expected.

v. The factor of safety due to shear

The FOS in shear is therefore the shear yield divided by the maximum shear stress which is 210. This suggests that the structure is capable of carrying a lot more load than 500 N

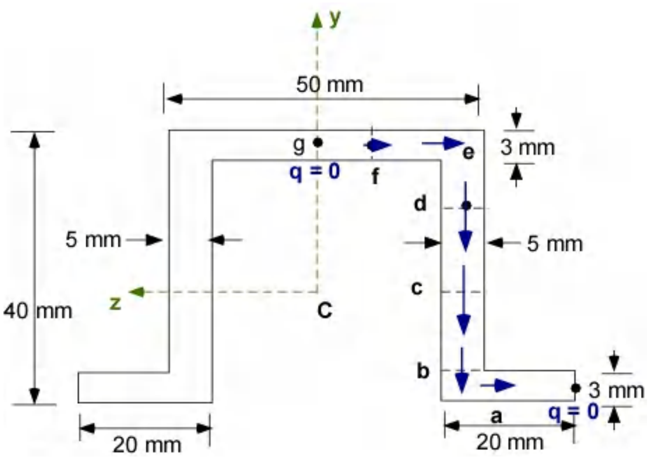


Figure 7.5.8d Shear flow distribution

Design Discussion:

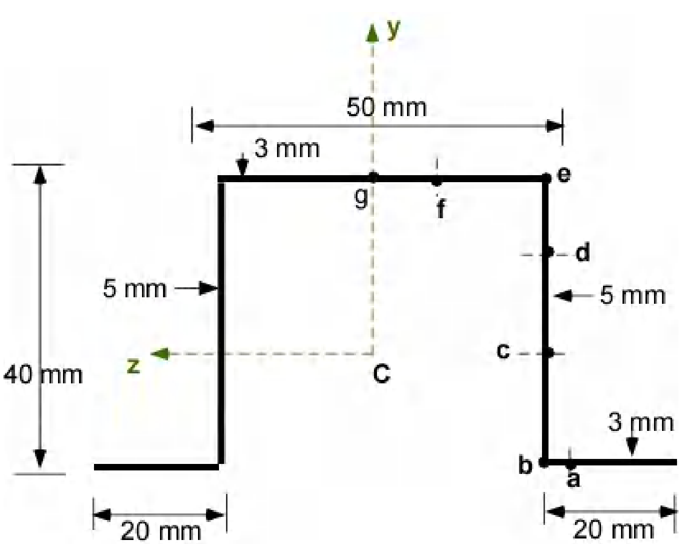
This section did not involve any new informtion

7.5.2 Additional Problems

Set up the following problems by hand on paper and solve them on paper and using MATLAB/Octave. For each problem you must work with a coordinate system. Obtain the centroid and the MOI about the centroid.

Problem 7.5.1

Solve Example 7.5 using thin wall beam in all your calculations. The material is AL 7075-T6 alloy as σ_y = 503 MPa (tensile yield strength); **E** = 71.7 GPa (modulus of elasticity); ν = 0.33 (Poisson's ratio); **G** = 26.9 GPa (shear modulus); and τ_y = 331 MPa (shear yield strength). The vertical load is 500 N. The beam is shown below. You will have to recalculate the centroid which might change the MOI.

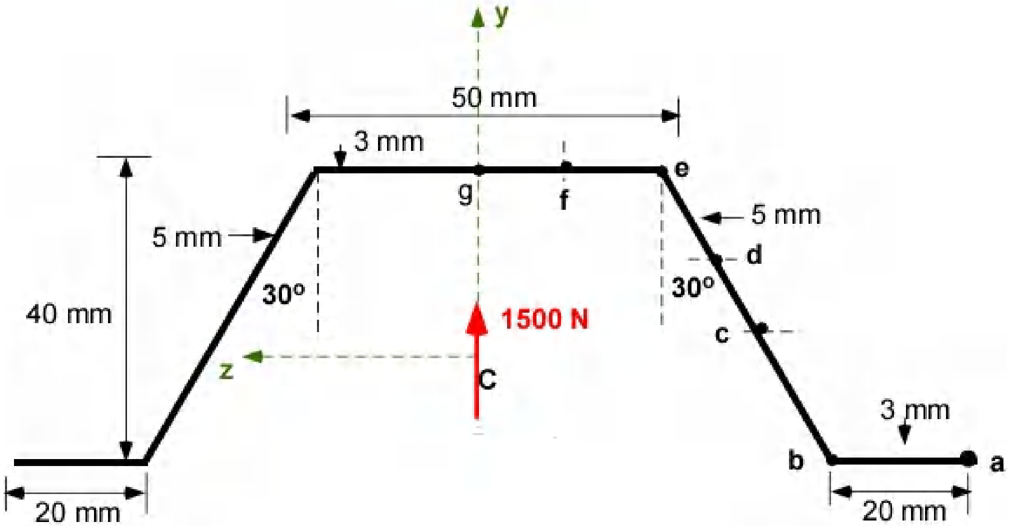


Problem 7.5.1

Solution: τ_f = 7.64e+05 [Pa]; τ_e = 1.15e+06 [Pa]; τ_d= 1.71e+06 [Pa]; $\tau_c = q_c/t_c$= 1.88e+06 [Pa]; $\tau_b = q_b/t_b$ =1.00e+06 [Pa]; $\tau_a = q_a/t_a$ =1.26e+06 [Pa]

Problem 7.5.2

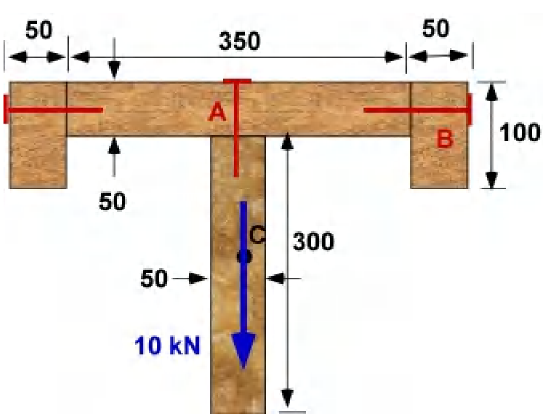
The material is Carbon composite with a shear strength of 50 MPa. The vertical load is 1500 N. The cross-section (thin walled is shown). Find the FOS based on investigating the shear at the indicated locations. You can find reference to MOI for inclined cross-section in Section 2.9.



Problem 7.5.2

Problem 7.5.3

A **T** cross-section is built up using wooden beams and nails as shown. The dimensions are in millimeters. A vertical shear load of 10 kN is applied to the cross-section in the centroidal plane. The nails are used handle the shear force at the junctions of the beam. The nails at A are spaced every 20 mm while those at B at 50 mm. Calculate the shear force handled by the nails at A and B. The shear force will be the shear flow times the nail spacing.



Problem7.5.3

7.6 ECCENTRIC LOADING

Until now the problems in this chapter considered only shear and bending loads in the cross-sections. We have ignored axial loads in the cross-sections. If the axial loads were applied at the centroid we can assume that these loads will create a direct normal stress in the cross-section. If in addition there is a bending moment in the cross-section this will create additional linear normal stress distribution in the cross-section. Finally these stress distribution will be superimposed leading to an asymmetric distribution. Superposition is the simplest way to handle the multiple loads. You are also adding their effect ignoring the fact that you will be adding a normal load to a bent member. The superposition assumes that the stresses due to the loads each act independently. An important assumption for this analysis is that the deformations are small and they can be linearly accounted for. It usually works satisfactorily for **symmetric** members. Symmetric members have one or two axis of symmetry so that the products of inertia vanish and this makes it easier to apply superposition.

There are several occasions when the axial load in the cross-section is not at the centroid. In these instances we can create an equivalent loading by moving the axial load to the centroid and adding a bending moment to the cross-section. Figure 7.6.1 illustrates this idea for the beam of rectangular cross-section.. Consider that the stress in section CC of the of the material carrying a simple axial load.P is required. In this case we are simply transferring the load from section B to section C. Since we are moving a force away from its line of action we need to provide a moment to accompany the transfer of the force. The normal stresses due to each load is shown and then added to get the final stress distribution due to the combined loading.

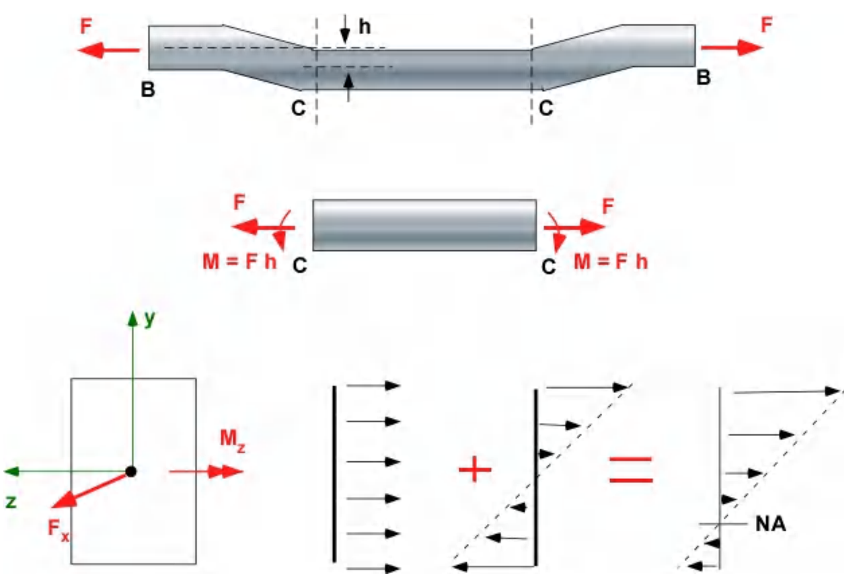


Figure 7.6.1 Example of eccentric loading

The superposition of the stresses for this example leads to the normal stress distribution where the stresses add on the top and subtract below the centroid. There is also another effect. The neutral axis (NA) is no longer at the centroid. These can be calculated using relations we have established further. A is the area of cross-section and $\mathbf{I_z}$ is the MOI about the centroidal axis.

$$\sigma_x = \frac{F_x}{A} + \frac{M_z y}{I_z}$$

$$\sigma_x = 0 \Rightarrow y = -\frac{I_z F_x}{A M_z} \qquad (7.21)$$

The neutral axis (NA) is the line through the origin and given by the second of the equation in 7.21

7.6.1 Example 7.9 - Moment Along Two Axis

Consider a symmetrical beam of 'I' cross-section that is subject to a bending moment M that is not originally directed along the coordinate axis in Fig. 7.6.2. Consider a bending moment of 100 kNm with an α of 60 degrees.

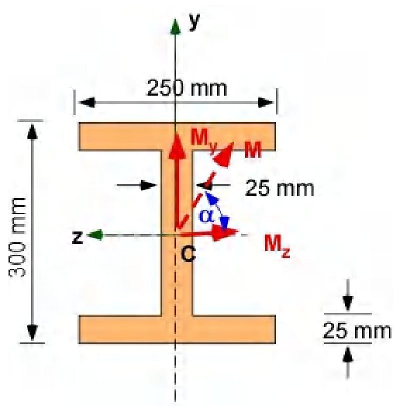


Figure 7.6.2 Example 7.9

In this case we solve the problem by taking the component of the bending moment along the coordinate axis. Hence M_z = M*cos(α) = -50000 [Nm] and M_y = M*sin(α) = 86603 [Nm]. The moment M_z is in the negative direction. It will create a tension on the material in the positive quadrant.

Data: Cross-section and dimensions are shown in the figure.
M = 100000 Nm; α = 60 degrees.

Find:

i. The moments of inertia.
ii. The normal stress distribution equation
iii. The neutral axis.
iv. The location and value of maximum stress

Assumption: Cross-section has two axis of symmetry. Cross-section is same across the beam. As there are two planes of symmetry the centroid is located by inspection at the intersection of the planes of symmetry.

(i) Moment of Inertia

$$I_z=\frac{1}{12}(25)(250)^3+2\left[\frac{1}{12}(250)(25)^3+(25)(250)(150-12.5)^2\right]=0.0002693\,m^4$$

$$I_y=2\frac{1}{12}(25)(250)^3+\frac{1}{12}(300-50)(25)^3=6.54\times10^{-5}\,m^4$$

(ii) Normal stress distribution

The stress-relations are written for a (+y) and (+z) locations - that is the positive quadrant:

The normal stress due to M_z is $\sigma_x(y,z)=-\dfrac{M_z y}{I_z}=\dfrac{50000*y}{2.69\times10^{-4}}$

The normal stress due to M_y is $\sigma_x(y,z)=\dfrac{M_y z}{I_y}=\dfrac{86603*z}{6.54\times10^{-5}}$

The total normal stress in the cross-section will be the superposition of these two distribution:

$$\sigma_x(y,z)=\frac{M_z y}{I_z}+\frac{M_y z}{I_y}=\frac{50000*y}{2.69\times10^{-4}}+\frac{86603*z}{6.54\times10^{-5}}=1.86\times10^{8}y+1.32\times10^{9}z$$

(iii) The Neutral Axis

$$\sigma_x(y,z)=1.86\times10^{8}y+1.32\times10^{9}z=0$$

$$y=-7.135z$$

This line can be drawn in the cross-section. It is now the new NA. It passes through the current centroid.

(iv) Calculating the location and value of maximum stresses

$$\sigma_x(y,z)=1.86\times10^{8}y+1.32\times10^{9}z;$$

$$\sigma_{max}=1.86\times10^{8}y_{max}+1.32\times10^{9}z_{max};$$

$$\sigma_{max}(0.150,0.125)=193.3\,MPa;$$

The cross-section is symmetrical and therefore the maximum and minimum stresses will have the same magnitude and opposite sign.The maximum positive normal stresses (tension) is at the maximum positive value of y and z at the left end of the top flange. The maximum compressive stress is the right end and at the bottom of the bottom flange.

Normal Stress Distribution over the Cross-section

There is normal stress distribution over the web and the flange. They can be calculated using the values of y and z along these surfaces. The stress along the NA is zero. The stress in the flanges will be zero where the NA axis intersects. The distribution is not symmetric. However the stress in the web is symmetric since the NA passes through the centroid. This is captured in Figure 7.6.3 below.

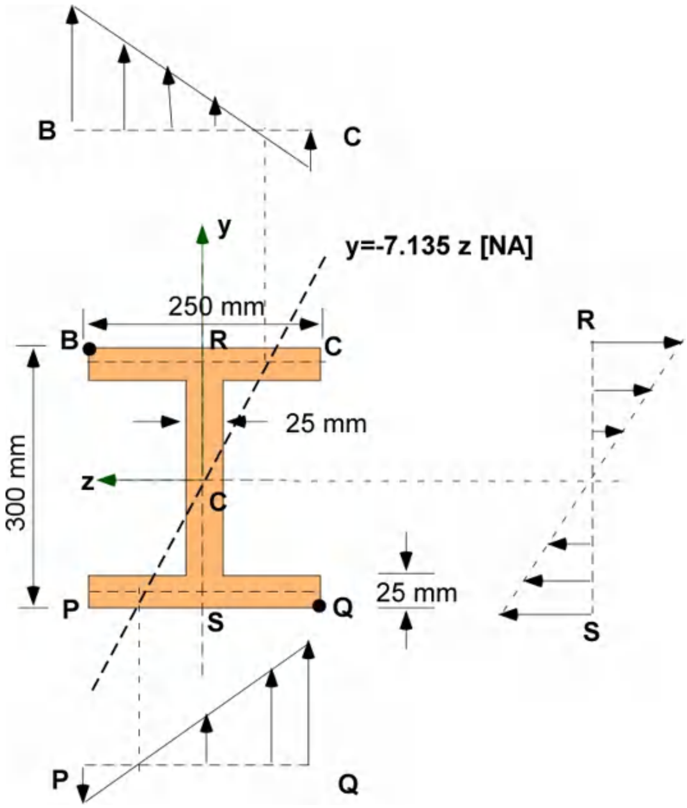


Figure 7.6.3 Stress distribution over cross-section

Design Discussion:

- Until this example we only had bending moment along the z- axis only. Our derivations of the various relations was based on bending moment along the z-axis with the axis of the beam along the x coordinate.
- In this example we encountered a bending moment along the y-direction and we set up the relations without any derivations. We just used the previous relations with different subscripts for the coordinates.
- The sign of the bending stresses corresponding to M_y was positive (unlike the stresses corresponding to moment M_z)
- This sign can be intuitive based on the actual moment direction along coordinate axis.
- A positive M_y will cause location at +z to be in tension and locations at -z to be in compression
- A positive M_z will cause location at +y to be in compression and locations at -y to be in tension

7.6.2 Example 7.10 - The General Eccentric Load

Consider a cross-section with an axial load that is not at the centroid as shown in Figure 7.6.4. The force F is 3000 N. The cross-section is square with a hole. Find (i) the stress at point B and (ii) the equation for the NA in the cross-section.

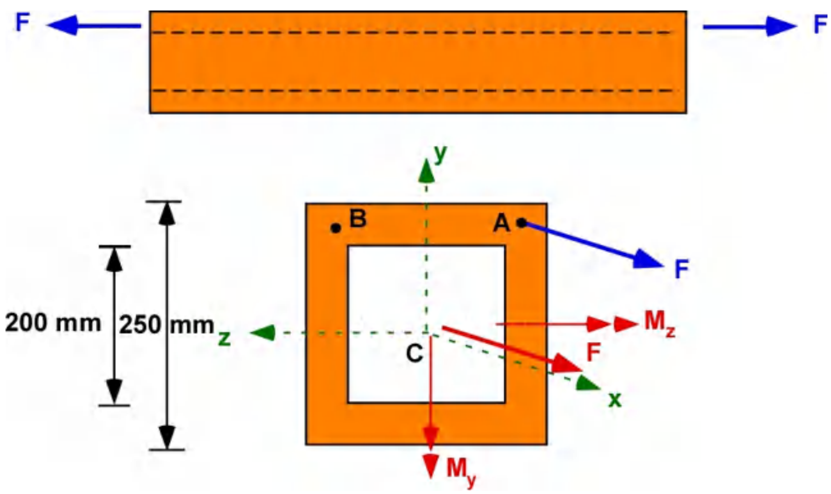


Figure 7.6.4 Example 7.10

Data: Load, cross-section is square and dimensions are shown in Figure 7.6.4. Note that all the values required for analysis can be represented in the cross-section.

Find:

i. Load at the centroid C . The eccentric load at A is transferred to C (red arrows). There will be two bending moment components and a normal force as shown.
ii. The moments of inertia
iii. Equation for the normal stress distribution
iv. Calculate the stress at point B
v. Calculate the equation to the NA

Assumption: Cross-section has an two axis of symmetry. Cross-section is same across the beam. As there are two planes of symmetry the centroid is located by inspection at the intersection of the planes of symmetry.

Solution Using MATLAB:
The code is split into the various incremental calculations in sequence. These should be appended in a file for usefulness.

In the Editor:

```
% Essential Foundations in Mechanics
% P. Venkataraman, June 2017
%  Example 7-10
% Eccentric load and bending
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 7.10 - General Eccentric Load\n')
fprintf('--------------------------------------------------------\n')

%% Data
F = 3000; % N
wo = 250/1000;  do = 250/1000;  wi =200/1000;  di = 200/1000;
fprintf(' Data\n')
fprintf('---------------------------\n')
fprintf('F [N] = '),disp(F)
fprintf('do [m]=  '),disp(do)
```

```
fprintf('wo  [m]=   '),disp(wo)
fprintf('di  [m]=   '),disp(di)
fprintf('wi  [m]=   '),disp(wi)
```

In the Command Window:

```
Example 7.10 - General Eccentric Load
------------------------------------------------------------
 Data
---------------------------
F [N] =         3000
do  [m]=          0.25
wo  [m]=          0.25
di  [m]=           0.2
wi  [m]=           0.2
```

i. Load at the centroid C . The eccentric load at A is transferred to C (red arrows). There will be two bending moment components and a normal force as shown.

In the Editor:

```
%% i. Transferring load to the centroid
L = 0.5*(wo/2 + wi/2);
Mz = -F*L;   My = -F*L;
fprintf('---------------------------\n')
fprintf('(i)  Loads at the Centroid\n')
fprintf('---------------------------\n')
fprintf('F   [N] = '),disp(F)
fprintf('L - moment arm [m] ='),disp(L)
fprintf('Mz [Nm]  = '),disp(Mz)
fprintf('My [Nm]=  '),disp(My)
```

In the Command Window:

```
  ---------------------------
  (i)  Loads at the Centroid
  ---------------------------
  F  [N] =         3000
  L - moment arm [m] =       0.1125
  Mz [Nm] =        -337.5
  My [Nm]=         -337.5
```

ii. The moments of inertia

In the Editor:

```
%% ii  Moment of inertia and area
fprintf('---------------------------\n')
fprintf('(ii)  Moments of Inertia\n')
fprintf('---------------------------\n')
Iz = wo*do^3/12 - wi*di^3/12;
Iy = Iz;
A = wo*do-wi*di;
fprintf('Cross-section area [m^2] '),disp(A)
fprintf('Iz about C [m^4] '),disp(Iz)
fprintf('Iy about C [m^4] '),disp(Iy)
```

In the Command Window:

```
  ---------------------------
  (ii)  Moments of Inertia
```

```
---------------------------
Cross-section area [m^2]        0.0225
Iz about C [m^4]     0.00019219
Iy about C [m^4]     0.00019219
```

iii. Equation for the normal stress distribution

In the Editor:

```
%% iii  - Normal stress distribution
fprintf('---------------------------\n')
fprintf('(iii)  Normal Stress Distribution\n')
fprintf('---------------------------\n')
syms y z
sigx_f = F/A;
sigx_mz = vpa(-Mz*y/Iz,6);
sigx_my = vpa(My*z/Iy,6);
fprintf('sigx_f='),disp(sigx_f)
fprintf('sigx_mz='),disp(sigx_mz)
fprintf('sigx_my='),disp(sigx_my)
sigx = vpa(sigx_f + sigx_mz + sigx_my,6);
sigx = vpa(sigx,6);
fprintf('sigx(y, z) = '),disp(sigx)
```

In the Command Window:

```
---------------------------
(iii)  Normal Stress Distribution
---------------------------
sigx_f=   1.3333e+05
sigx_mz=1756100.0*y
sigx_my=-1756100.0*z
sigx(y, z) = 1756100.0*y - 1756100.0*z + 133333.0
```

iv. Calculate the stress at point B

In the Editor:

```
%% iv - Stress at B
fprintf('---------------------------\n')
fprintf('(iv)  Stress at B\n')
fprintf('---------------------------\n')
yB = L;  zB = L;
sigxmax = vpa(subs(sigx,[y,z],[yB,zB]),4);
fprintf('[yB, zB] [m] = '),disp([yB, zB])
fprintf('sigx(B) [Pa] ='),disp(sigxmax)
```

In the Command Window:

```
---------------------------
(iv)  Stress at B
---------------------------
[yB, zB] [m] =         0.1125        0.1125
sigx(B) [Pa] =1.333e5
```

v. Calculate the equation to the NA

In the Editor:

```
%% v - locating the NA
fprintf('---------------------------\n')
fprintf('(v)  The new NA \n')
fprintf('---------------------------\n')
```

```
ysol = vpa(solve(sigx == 0),6);
fprintf('Equation for NA : y = '),disp(ysol)
```

In the Command Window:

```
---------------------------
(v)  The new NA
---------------------------
Equation for NA : y = z - 0.0759259
```

Execution in Octave
The code is same as in the MATLAB except for the highlighted changes.
Variable **do** is a system variable and is replaced by **do1** everywhere using find and replace

In Octave Editor

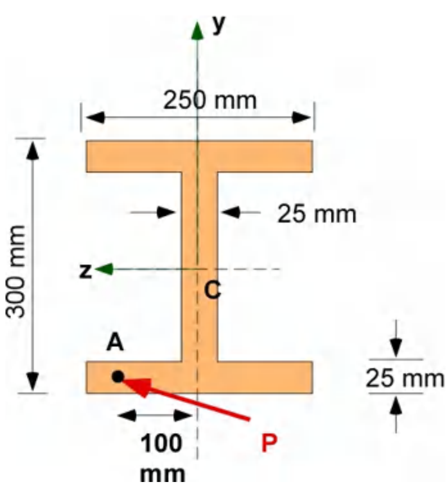
```
pkg load symbolic;

wo = 250/1000;   do1 = 250/1000;  wi =200/1000;  di = 200/1000;
```

The results are the same as that produced by MATLAB and is not included

7.6.3 Additional Problems

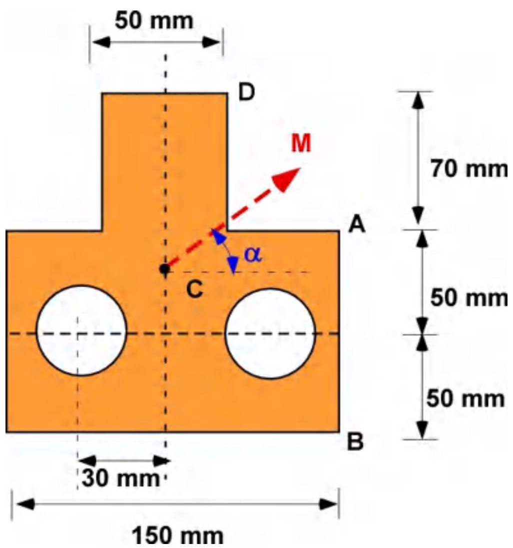
Set up the following problems by hand on paper and solve them on paper and using MATLAB/Octave. For each problem you must draw the load in the cross-section at the centroid and work with a coordinate system.

Problem 7.6.1
A horizontal load P is applied at the middle f the flange and at a distance of 100 mm from the centroidal axis as shown. The maximum compressive stress in the cross-section should not exceed 250 MPa with a FOS of 2.5. (a) What is the maximum load P that can be applied? (b) Find the equation of the NA.



Problem 7.6.1

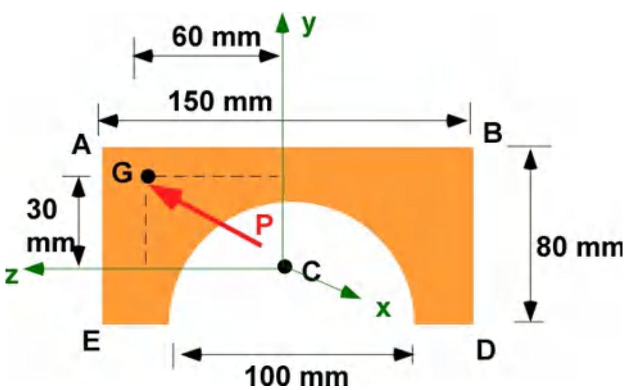
Problem 7.6.2
A bending moment M = 10000 Nm is applied at an angle of α = 30 degrees at the centroid of the symmetric cross-section shown in Figure 7.6.6. The two circular cutout have a radius of 25 mm and are symmetrically located. (a) Obtain the location of the centroid. (b) Calculate the stress at the point A. (c) Calculate the stress at the point B. (d) Calculate the stress at the point D.



Problem 7.6.2

Problem 7.6.3

A bar of rectangular cross-section with a semicircular cutout is subjected to a compressive axial load P at the point G. The material is Bronze. What is the maximum load P that can be applied with a FOS of 3.



Problem 7.6.3

7.7 UNSYMMETRIC BENDING

The analysis of beams until now involved cross-sections that had one or two axis of symmetry. The principal assumption that was part of the development, but not explicitly stated, was that the bending itself was symmetrical and the bending moment was applied in the plane of symmetry. A simple illustration is shown in Figure 7.7.1. There is a single axis of symmetry. In the cross-section on the right the double arrow representing the bending moment M_z is actually applied in the plane of symmetry as shown. The bending deflection is also symmetrical to the plane of symmetry. In addition the neutral axis (NA) lies in the plane of symmetry.

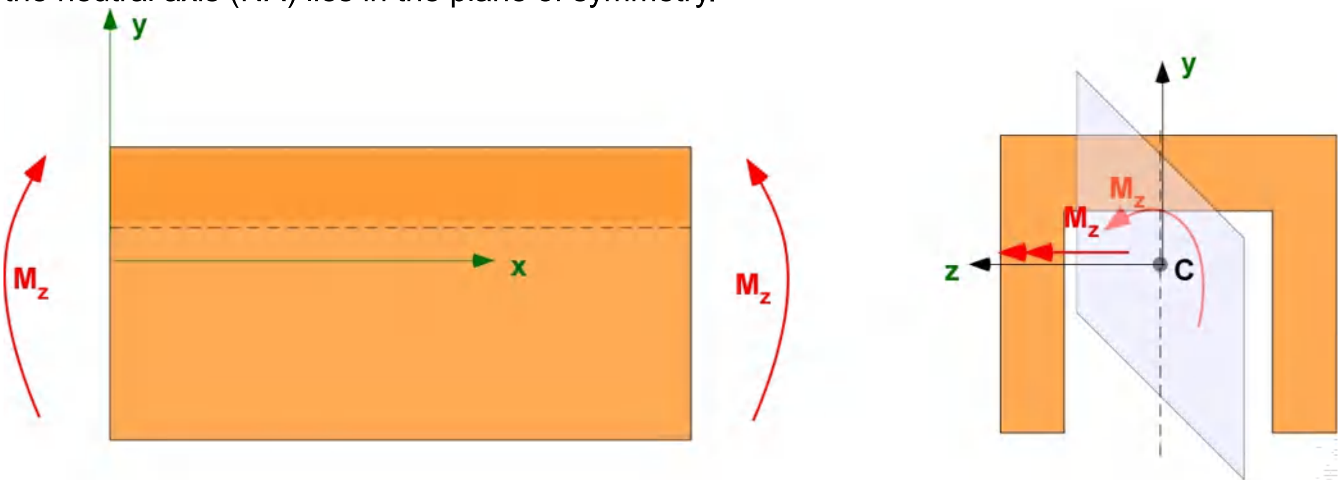


Figure 7.7.1 Symmetrical bending - illustration

Unsymmetric bending takes place when:

- The bending moment is not applied in the plane of symmetry. This was covered in the previous section eccentric bending.
- There is no axis of symmetry in the cross-section. This is the focus in this section.

7.7.1 Centroid and MOI of a Non Symmetric Cross-section

If there is no axis of symmetry in the cross-section then we have an important additional element that has to feature in the discussion - the product of inertia - I_{yz}. This term is usually zero if there is at least one axis of symmetry. This was a feature in the problems in eccentric bending, and in fact all of the problems considered so far. This is the first time we have a need to evaluate $\mathbf{I_{yz}}$ since Section 2.8. Let us review the general formula for locating the centroid and calculating the various MOI for the cross-section. In most sections.

Most text books develop relations with the cross-section described by the coordinates x and y. In our previous problems we have used x to be the axial coordinate, positive to the right. The cross-section is therefore described by y-z coordinates with z to the left. We will stick to this system for our discussion and hence we will have have new relations. Figure 7.7.2 describes the beam and the cross-section. Coordinate ***Oy'z'*** is a reference coordinate system to establish the centroid. ***Cxyz*** is the centroidal coordinate system. The elemental area is ***dA***.

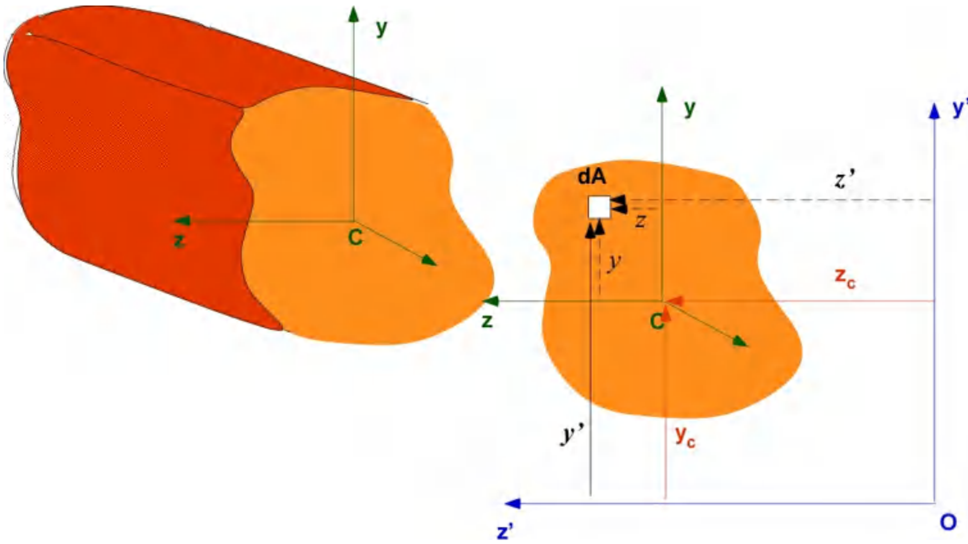


Figure 7.7.2 The beam and the cross-section

$$y_c = \frac{\int_A y' dA}{\int_A dA} = \frac{\sum_i y_i' A_i}{\sum_i A_i}; \qquad z_c = \frac{\int_A z' dA}{\int_A dA} = \frac{\sum_i z_i' A_i}{\sum_i A_i}; \quad (7.22)$$

The summation is useful for composite areas. Note that repeated double subscripts can be replaced by a single one for convenience. The various MOI about the centroid is calculated through

$$\begin{aligned} I_y &= I_{yy} = \int_A y^2\, dA; \quad I_z = I_{zz} = \int_A z^2\, dA; \quad I_{yz} = \int_A y\, z\, dA; \\ I_y &= I_{yy} = \sum_i y_i^2 A_i; \quad I_z = I_{zz} = \sum_i z_i^2 A_i; \quad I_{yz} = \sum_i y_i z_i A_i; \end{aligned} \quad (7.23)$$

The parallel axis theorem is useful to find MOI about any other axes:

$$\begin{aligned} A &= \sum_i A_i \\ I_{y'} &= I_{y'y'} = I_y + A z_c^2; \quad I_{z'} = I_{z'z'} = I_z + A y_c^2; \quad I_{y'z'} = I_{yz} + A y_c z_c; \end{aligned} \quad (7.24)$$

7.7.2 Principal Axis of Inertia

The concept of principle axis of inertia (POI) is very much like the concept of principal stresses. For principal stresses we can rotate the coordinate system until we can find a plane determined by the new coordinate axes where the ***shear stress goes to zero***. Along these coordinates there are only normal stresses - called **principal stresses** and they are the maximum and minimum possible stresses at the point.

For principal axes of inertia the coordinate system is rotated until the ***product of inertia goes to zero***. If there is an axis of symmetry then these are naturally the principal axis of inertia since there the product of inertia is zero. Consider the non-symmetric cross-section we used previously in Figure 7.7.3.

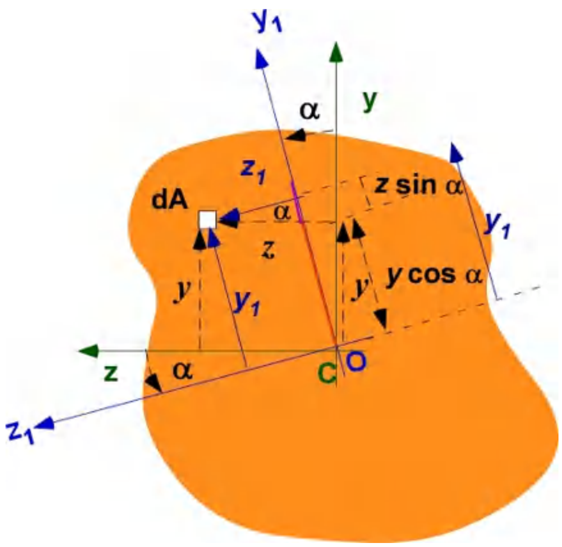


Figure 7.7.3 Cross-section with axis of rotation

In Figure 7.7.3 the centroidal axis ***Cyz*** is rotated by angle α to the coordinate system ***Cx₁y₁***. The element area **dA** is located at (y, z) with respect to the coordinate system ***Cyz***. It's location in the coordinate system ***Cx₁y₁*** is (x_1, y_1). This is also shown in Figure 7.7.3. Through geometry it can be established

$$y_1 = y\cos\alpha + z\sin\alpha \qquad (7.25a)$$

It is also possible to derive

$$z_1 = z\cos\alpha - y\sin\alpha \qquad (7.25b)$$

Calculating the MOI about the new axis ***Cx₁y₁*** axis using the formula in Equations (7.22) and (7.25)

$$\begin{aligned}
I_{y_1} &= \int_A z_1^2\,dA = \int_A \left(z\cos\alpha - y\sin\alpha\right)^2 dA = \\
&\quad \int_A z^2\cos^2\alpha\,dA + \int_A y^2\sin^2\alpha\,dA - 2\int_A yz\sin\alpha\cos\alpha\,dA \\
I_{y_1} &= \cos^2\alpha\int_A z^2\,dA + \sin^2\alpha\int_A y^2\,dA - 2\sin\alpha\cos\alpha\int_A yz\,dA \\
I_{y_1} &= I_y\cos^2\alpha + I_z\sin^2\alpha - I_{yz}\sin 2\alpha
\end{aligned}$$

$$\begin{aligned}
I_{z_1} &= \int_A y_1^2\,dA = \int_A \left(y\cos\alpha + z\sin\alpha\right)^2 dA = \\
&\quad \int_A y^2\cos^2\alpha\,dA + \int_A z^2\sin^2\alpha\,dA + 2\int_A yz\sin\alpha\cos\alpha\,dA \\
I_{z_1} &= \cos^2\alpha\int_A y^2\,dA + \sin^2\alpha\int_A z^2\,dA + 2\sin\alpha\cos\alpha\int_A yz\,dA \\
I_{z_1} &= I_z\cos^2\alpha + I_y\sin^2\alpha + I_{yz}\sin 2\alpha
\end{aligned}$$

$$I_{y_1z_1} = \int_A y_1 z_1\, dA = \int_A (y\cos\alpha + z\sin\alpha)(z\cos\alpha - y\sin\alpha)\, dA$$
$$= \int_A yz\cos^2\alpha\, dA - \int_A y^2 \sin\alpha\cos\alpha\, dA +$$
$$\int_A z^2 \sin\alpha\cos\alpha\, dA - \int_A yz\sin^2\alpha\, dA$$
$$I_{y_1z_1} = I_{yz}\cos^2\alpha - I_z \frac{\sin 2\alpha}{2} + I_y \frac{\sin 2\alpha}{2} - I_{yz}\sin^2\alpha$$
$$I_{y_1z_1} = I_{yz}\left(\cos^2\alpha - \sin^2\alpha\right) + \frac{\sin 2\alpha}{2}\left(I_y - I_z\right)$$

The equations for transforming the MOI is assembled from the above derivation as:

$$I_{y_1} = I_y\cos^2\alpha + I_z\sin^2\alpha - I_{yz}\sin 2\alpha$$
$$I_{z_1} = I_z\cos^2\alpha + I_y\sin^2\alpha + I_{yz}\sin 2\alpha \qquad (7.26)$$
$$I_{y_1z_1} = I_{yz}\left(\cos^2\alpha - \sin^2\alpha\right) + \frac{\sin 2\alpha}{2}\left(I_y - I_z\right)$$

The following trigonometric identities are also useful:

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}; \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}; \quad \cos^2\alpha - \sin^2\alpha = \cos 2\alpha \qquad (7.27)$$

The product of inertia is important and can be used to define the principal axis of inertia (POI)

$$I_{y_1z_1} = I_{yz}\cos 2\alpha + \frac{\sin 2\alpha}{2}\left(I_y - I_z\right)$$
$$\textit{For principal axis } I_{y_1z_1} = 0; \quad \textit{then} \qquad (7.28)$$
$$\tan 2\alpha_p = \frac{\sin 2\alpha_p}{\cos 2\alpha_p} = 2\frac{I_{yz}}{\left(I_z - I_y\right)}$$

Where α_p is the rotation from the **centroidal axis** to the **principal axis.**

Principal Moments of Inertia (POI)
The rotation from the centroidal axes to the principal axes is given by the angle α_p. This information is used in the equations (7.28) to calculate the values of the MOI along these principal axes - Cy_pz_p shown in Figure 7.7.4

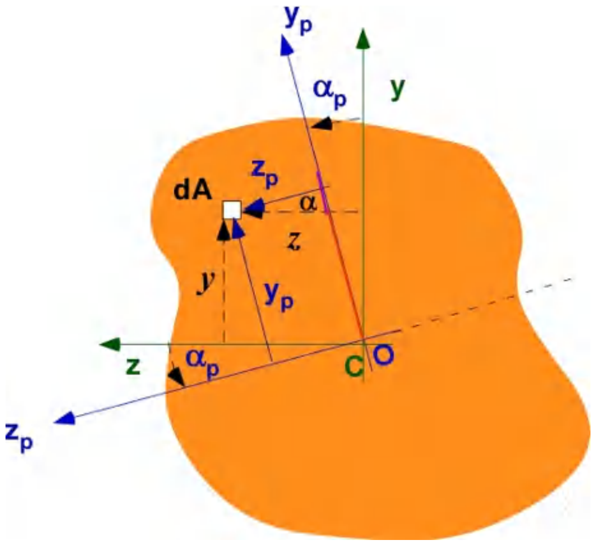


Figure 7.7.4 Principal moment of inertia axes

Prior to calculating the principal moment of inertia a few useful identities will be useful. They are obtained by processing the equation (7.28) for the angle needed to determine the principal axes. This is illustrated to for the ***sin 2***α below. The details of other derivations are left as an exercise while the results are included here.

$$\sin 2\alpha = \frac{I_{yz}}{0.5\left(I_z - I_y\right)}\cos 2\alpha;$$

$$\left(\sin 2\alpha\right)^2 = \frac{I^2_{yz}}{\left[0.5\left(I_z - I_y\right)\right]^2}\left(\cos 2\alpha\right)^2 = 1 - \left(\cos 2\alpha\right)^2;$$

$$\left(\cos 2\alpha\right)^2 = \frac{1}{1 + \dfrac{I^2_{yz}}{\left[0.5\left(I_z - I_y\right)\right]^2}} = \frac{\left[0.5\left(I_z - I_y\right)\right]^2}{\left[0.5\left(I_z - I_y\right)\right]^2 + I^2_{yz}} \qquad (7.29)$$

$$\cos 2\alpha = \frac{0.5\left(I_z - I_y\right)}{\sqrt{\left[0.5\left(I_z - I_y\right)\right]^2 + I^2_{yz}}}; \quad \sin 2\alpha = \frac{I_{yz}}{\sqrt{\left[0.5\left(I_z - I_y\right)\right]^2 + I^2_{yz}}}$$

Remember the α involved in the equations are actually α_p. The results in equations (7.26), (7.27) and (7.29) are combined to develop the equations for calculating the principal axis of inertia

$$I_{y_p} = I_y \frac{1+\cos 2\alpha_p}{2} + I_z \frac{1-\cos 2\alpha_p}{2} - I_{yz} \sin 2\alpha_p$$

$$= \frac{I_y + I_z}{2} + \frac{I_y - I_z}{2} \left\{ \frac{0.5(I_z - I_y)}{\sqrt{[0.5(I_z - I_y)]^2 + I_{yz}^2}} \right\} - I_{yz} \left\{ \frac{I_{yz}}{\sqrt{[0.5(I_z - I_y)]^2 + I_{yz}^2}} \right\}$$

$$= \frac{I_y + I_z}{2} - \left\{ \frac{[0.5(I_z - I_y)]^2 + I_{yz}^2}{\sqrt{[0.5(I_z - I_y)]^2 + I_{yz}^2}} \right\} = \frac{I_y + I_z}{2} - \sqrt{[0.5(I_z - I_y)]^2 + I_{yz}^2}$$

(7.30a)

$$I_{z_p} = \frac{I_y + I_z}{2} + \sqrt{[0.5(I_z - I_y)]^2 + I_{yz}^2} \qquad (7.30b)$$

The product of inertia $\mathbf{I_{ypzp}}$ is zero for the principal axes of inertia.

7.7.3 Example 7.11

Calculate the principal moments of inertia for the 'L' shaped cross-section shown.

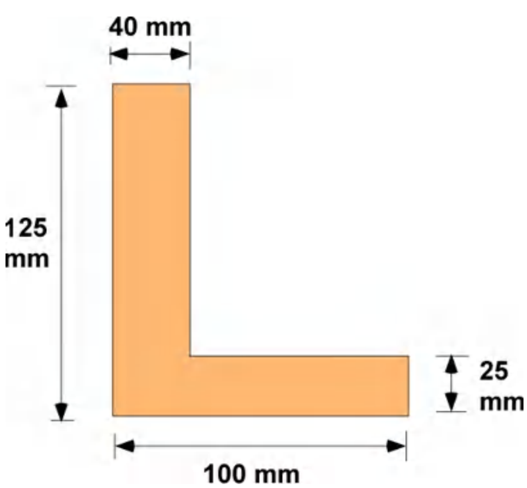


Figure 7.7.5 Example 7.11

Data: The cross-section and dimension are shown

Find:
The solution is determined through the following steps:

(i) Calculate the centroid
(ii) Calculate the MOI about the centroidal axes
(iii) Calculate the POI and determine the rotation of the principal axes from the centroidal axes.

Assumption: This is a simple cross-section with no axis of symmetry.

Solution:

Step(i): The cross-section can be neatly sectioned into two rectangles and these composite areas can be used for the calculation of the centroid and MOI. Figure 7.7.6 defines the various elements of the composite area. The initial axes for reference is the $x_r y_r$ coordinate system. The two areas are identified and their centroid are easily recognized.

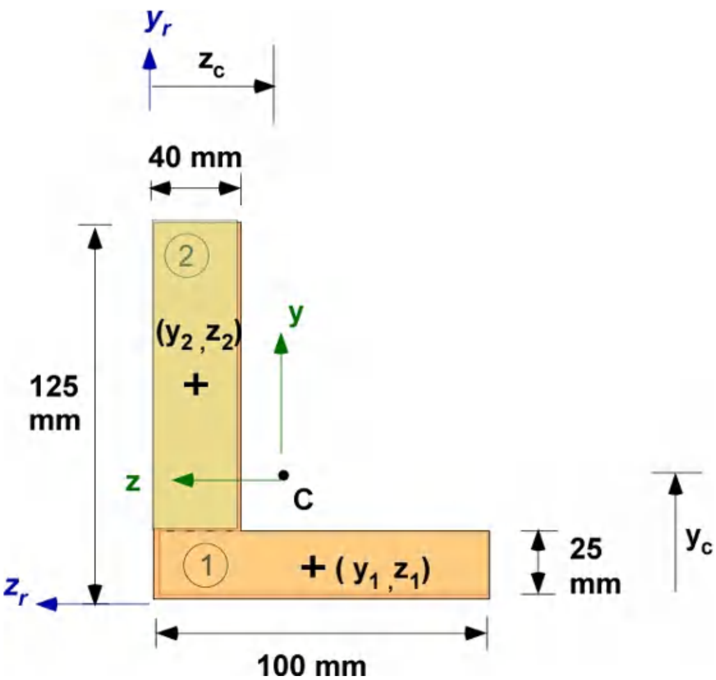


Figure 7.7.6 Composite areas for Example 7.11

$$y_c = \frac{y_1 A_1 + y_2 A_2}{A_1 + A_2} = \frac{0.0125(.1 \times 0.025) + 0.075(0.04 \times 0.1)}{(.1 \times 0.025) + (0.04 \times 0.1)} = 0.05096\,m$$

$$z_c = \frac{z_1 A_1 + z_2 A_2}{A_1 + A_2} = \frac{-0.05(0.1 \times 0.025) - 0.02\,(0.04 \times 0.1)}{(0.1 \times 0.025) + (0.04 \times 0.1)} = -0.03154\,m$$

Step(ii): MOI about centroid

$$I_y = \frac{1}{12} 0.1^3 \times 0.025 + 0.0025 \times (0.05 - 0.03154)^2 + \frac{1}{12} 0.04^3 \times 0.1 + 0.004 \times (0.02 - 0.03154)^2 = 4.0 \times 10^{-6}\,m^4$$

$$I_z = \frac{1}{12} 0.1 \times 0.025^3 + 0.0025 \times (0.0125 - 0.05096)^2 + \frac{1}{12} 0.04 \times 0.1^3 + 0.004 \times (0.075 - 0.05096)^2 = 9.47 \times 10^{-6}\,m^4$$

$$I_{yz} = A_1(y_1 - y_c)(z_1 - z_c) + A_2(y_2 - y_c)(z_2 - z_c)$$
$$= 0.0025(0.0125 - 0.05096)(-0.05 + 0.03154) +$$
$$0.004(0.075 - 0.05096)(-0.02 + 0.03154) = 2.88 \times 10^{-6}\, m^4$$

Step(iii) Calculate the MOI along principal axis. They are identified as $\boldsymbol{I_{max}}$ and $\boldsymbol{I_{min}}$

$$I_{avg} = 0.5(I_y + I_z) = 6.737 \times 10^{-6};\quad I_{dif} = 0.5(I_z - I_y) = 2.735 \times 10^{-6};\quad R = \sqrt{I_{dif}^2 + I_{yz}^2} = 3.975 \times 10^{-6}$$
$$I_{max} = I_{avg} + R = 1.071 \times 10^{-5}\left[m^4\right];$$
$$I_{min} = I_{avg} - R = 2.761 \times 10^{-6}\left[m^4\right];$$
$$\alpha = \frac{1}{2}\tan^{-1}\left[\frac{2I_{yz}}{(I_z - I_y)}\right] = \frac{1}{2}\tan^{-1}\left[\frac{2 \times 2.884 \times 10^{-6}}{(9.473 \times 10^{-6} - 4 \times 10^{-6})}\right] = 23.26[\deg]$$

7.7.4 Bending of Unsymmetric Sections

The bending calculation are very similar to the case of pure bending in Section 7.3. The list of of assumptions in this development include:

- The are no axes of symmetry in the cross-section
- The coordinate axes are drawn at the centroid
- The general bending moment in the cross-section can be resolved along the centroidal axis
- The bending moment is positive and drawn along the axis
- There is no axial force in the cross-section contributing to the normal stress. The normal stress is entirely due to bending.
- Plain sections must remain plane. This requires a neutral axis (NA) in the cross-section where the stress is zero. The radius of curvature for the prismatic beam is ρ.
- The orientation of the NA is unknown and is inclined at an angle of α to the centroidal axis.
- The stress is within the elastic limit

The NA axis passes through the centroid as derived below and is drawn so in Figure 7.7.7

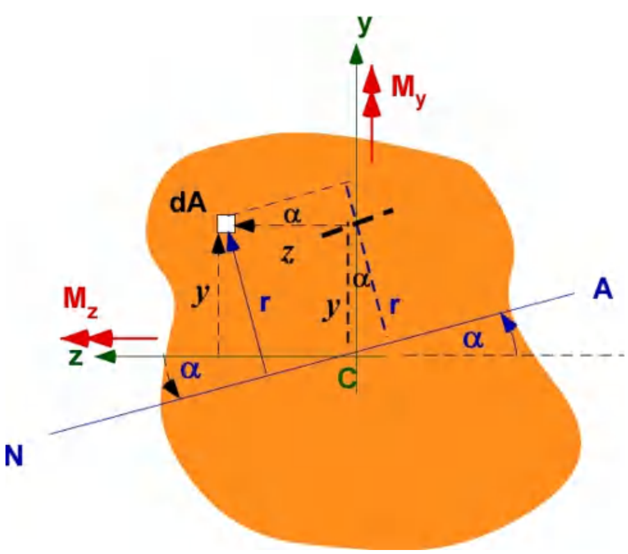


Figure 7.7.7 Unsymmetric bending

The following derivations are involved but we will use the final results in the analysis for unsymmetric bending.
At any point in the cross-section represented by the differential area **dA**, either at the location ***(y, z)*** or at a distance ***r*** from the NA. Borrowing from pure bending

$$\sigma_x(y,z) = E\varepsilon_x(y,z) = E\frac{r}{\rho}$$

As there is no axial force in the cross-section, the NA must pass through the centroid

$$\int_A \sigma_x\, dA = 0 = \int_A E\varepsilon_x\, dA = \frac{E}{\rho}\int_A r\, dA = 0$$

But form Figure 7.7.7 $\qquad r = y\cos\alpha + z\sin\alpha$

The applied bending moment can be related to the normal stress distribution $\sigma_x(y, z)$ in the cross-section and is positive in the positive x-direction (coming out of the plane)

$$M_y = \int_A \sigma_x\, z\, dA; \quad M_z = -\int_A \sigma_x\, y\, dA;$$

$$M_y = \frac{E}{\rho}\int_A (y\cos\alpha + z\sin\alpha)\, z\, dA; \quad -M_z = \frac{E}{\rho}\int_A (y\cos\alpha + z\sin\alpha)\, y\, dA;$$

$$M_y = \frac{E}{\rho}\left[\cos\alpha\, I_{yz} + I_y\sin\alpha\right]; \quad -M_z = \frac{E}{\rho}\left[\cos\alpha\, I_z + I_{yz}\sin\alpha\right]$$

For known applied bending moments $\mathbf{M_y}$ and $\mathbf{M_z}$ we have two unknowns ρ and α

It is easier to process these equations further in matrix form. The bending moment equations can be organized as a matrix equation:

$$\begin{bmatrix} M_y \\ -M_z \end{bmatrix} = \frac{E}{\rho}\begin{bmatrix} I_{yz} & I_y \\ I_z & I_{yz} \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

$$\frac{E}{\rho}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} I_{yz} & I_y \\ I_z & I_{yz} \end{bmatrix}^{-1}\begin{bmatrix} M_y \\ -M_z \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{E}{\rho}\cos\alpha \\ \dfrac{E}{\rho}\sin\alpha \end{bmatrix} = \frac{1}{\left[I_y I_z - I_{yz}^2\right]}\begin{bmatrix} -I_{yz} & I_y \\ I_z & -I_{yz} \end{bmatrix}\begin{bmatrix} M_y \\ -M_z \end{bmatrix}$$

The inverse can be calculated to yield the following solution

$$\begin{bmatrix} \frac{E}{\rho}\cos\alpha \\ \frac{E}{\rho}\sin\alpha \end{bmatrix} = \frac{1}{I_y I_z - I_{yz}^2}\begin{bmatrix} -I_{yz}M_y & I_y(-M_z) \\ I_z M_y & -I_{yz}(-M_z) \end{bmatrix}$$

It is now possible to calculate the normal stress distribution

$$\begin{aligned} \sigma_x(y,z) &= y\frac{E}{\rho}\cos\alpha + z\frac{E}{\rho}\sin\alpha \\ &= \frac{1}{\left[I_y I_z - I_{yz}^2\right]}\left\{-y\left[I_{yz}M_y + I_y M_z\right] + z\left[I_z M_y + I_{yz}M_z\right]\right\} \qquad (7.31) \\ &= \frac{1}{\left[I_y I_z - I_{yz}^2\right]}\left\{M_y\left[-yI_{yz} + zI_z\right] + M_z\left[-yI_y + zI_{yz}\right]\right\} \end{aligned}$$

Design Note: The normal stress distribution is also valid for the symmetrical bending by setting I_{yz} = 0. It is also valid for moment along a single axis as in earlier bending problems.
Eqn. (7.31) is a general relation valid for all types of bending discussed so far

The neutral axis (NA) can be obtained from (7.31) by requiring the normal stress to be zero. This should yield the equation of line of the NA as

$$\frac{y}{z}\Big|_{NA} = \frac{\left[I_z M_y + I_{yz}M_z\right]}{\left[I_{yz}M_y + I_y M_z\right]} = \tan\alpha \qquad (7.32)$$

The development of the analysis was a little challenging. However the final results can be applied as we are familiar with calculating MOI.

7.7.5 Example 7.12

Example 7.12 is the extension of Example 7.11 by applying bending moment to the cross-section at the centroid. This allows us to explore the stress calculation using Eqn. (7.31) and calculating the NA using Eqn. (7.32). $\mathbf{M_y}$ is 3000 Nm and $\mathbf{M_z}$ is -5000 Nm as shown in Figure 7.7.8.

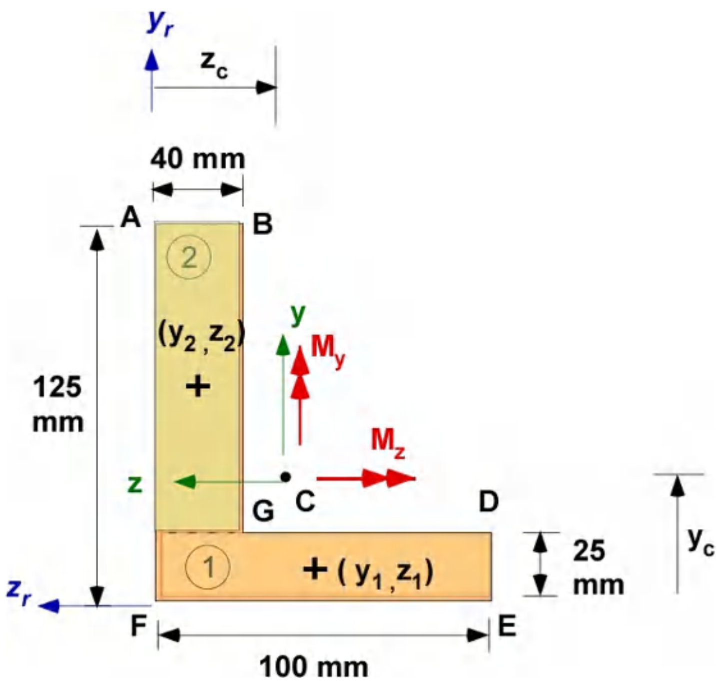


Figure 7.7.8 Example 7.12

Data: Cross-section is shown in Figure 7.7.8. M_y = 3000 Nm; M_z = - 5000 Nm

Assumption: None. Results borrowed for some calculations from Example 7.11

Find:

(i) The centroid
(ii) The MOI about the centroidal axes
(iii) The normal stress distribution across the cross-section
(iv) The NA for the cross-section and loading.

Solution:

(i) The centroid

$$y_c = 0.05096\,m;$$
$$z_c = -0.03154\,m;$$

(ii) The MOI about the centroidal axes

$$I_{y_r} = 4.0\times10^{-6}\,m^4;$$
$$I_{z_r} = 9.47\times10^{-6}\,m^4;$$
$$I_{y_r z_r} = 2.88\times10^{-6}\,m^4;$$

(iii) The normal stress distribution across the cross-section

$$I_y I_z - I_{yz}^2 = \left(4\times10^{-6}\right)\left(9.47\times10^{-6}\right)-\left(2.88\times10^{-6}\right)^2 = 29.58\times10^{-12}$$

$$\sigma_x\left(y,z\right)=\frac{1}{\left[I_y I_z - I_{yz}^2\right]}\left\{-y\left[I_{yz}M_y + I_y M_z\right]+z\left[I_z M_y + I_{yz}M_z\right]\right\}$$

$$\sigma_x\left(y,z\right)=\frac{10^{-6}}{29.58\times10^{-12}}\left\{-y\left[2.88\times3000+4\times\left(-5000\right)\right]+z\left[9.47\times3000+2.88\times\left(-5\right.\right.\right.$$

$$=\frac{1}{29.58\times10^{-6}}\left\{11360\,y+14010\,z\right\}=\left[384.04\,y+473.63\,z\right]\times10^6\left[Pa\right]$$

The stress at the various points in the cross-section are:

$$\sigma_x\left(A\right)=\sigma_x\left(0.074,0.032\right)=384.04*0.074+473.63*0.032=43.3\left[MPa\right]$$
$$\sigma_x\left(B\right)=\sigma_x\left(0.074,-0.008\right)=384.04*0.074-473.63*0.008=24.4\left[MPa\right]$$
$$\sigma_x\left(D\right)=\sigma_x\left(-0.026,-0.068\right)=-384.04*0.026-473.63*0.068=-42.4\left[MPa\right]$$
$$\sigma_x\left(E\right)=\sigma_x\left(-0.051,-0.068\right)=-384.04*0.051-473.63*0.068=-51.9\left[MPa\right]$$
$$\sigma_x\left(F\right)=\sigma_x\left(-0.051,0.032\right)=-384.04*0.051+473.63*0.032=-4.63\left[MPa\right]$$

The maximum compressive stress is at the point E and the maximum tensile stress is at point A.

(iv) The NA for the cross-section and loading

$$NA: y=\frac{\left[I_z M_y + I_{yz}M_z\right]}{\left[I_{yz}M_y + I_y M_z\right]}\times z$$

$$y=\frac{\left(9.47\times10^{-6}*3000-2.88\times10^{-6}*5000\right)}{\left(2.88\times10^{-6}*3000-4.00\times10^{-6}*5000\right)}=-1.23\,z$$

Or α = -51 degrees. That is a clockwise rotation about the centroid

7.7.6 Example 7.13

This example uses a another simple non symmetric cross-section that is frequently used in the design of “stringers” in aircraft wing-design. In addition the thickness small enough to invoke thin-wall assumptions. The example is shown in Figure 7.7.9. The objective is to calculate the stress in the cross-section at the corners of the cross-section at points A and B, and to calculate the inclination of the NA. The solution is determined symbolically.

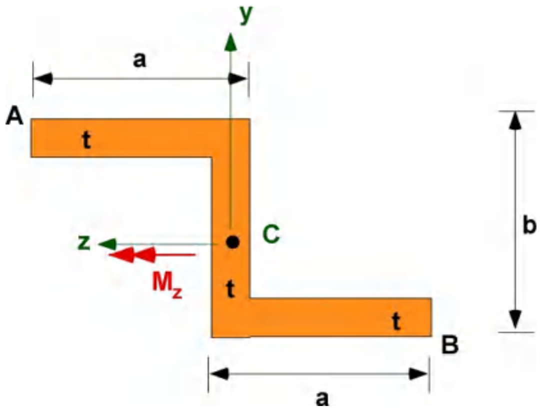


Figure 7.7.9 Example 7.13

Data: Cross-section is shown in Figure 7.7.9. M_z is the only bending moment. The solution of the problem will be symbolic.

Assumption: The cross-section is non-symmetric. However the centroid can be identified by inspection of the geometry because of equal areas and first moment of areas around special lines. The intersection of these lines is the centroid C.
Thin-walled section is assumed. Thickness t is the same for all parts of the cross-section.

Find:

(i) The centroid
(ii) The MOI about the centroidal axes
(iii) The normal stress distribution across the cross-section
(iv) The NA for the cross-section and loading.

(i) The centroid:

It is possible to establish the centroid by inspection because of equal areas. It is at a distance of b/2 from the bottom and the center of the vertical section.

(ii) The MOI about the centroidal axes

The thin walled beam assumptions should slightly over estimate the the value of the MOI. While the centroid was established by inspection the product of inertia is not zero. In terms of the symbols ***a***, ***b***, and ***t*** the various MOI should evaluate to :

$$I_{yy} = I_y = 2\left[\frac{1}{12}ta^3 + (ta)\left(\frac{a}{2}\right)^2\right] = 0.667\,a^3 t$$

$$I_{zz} = I_z = \frac{1}{12}tb^3 + 2\left[(ta)\left(\frac{b}{2}\right)^2\right] = 0.083\,b^3 t + 0.5\,a\,t\,b^2$$

$$I_{yz} = at\left(\frac{b}{2}\right)\left(\frac{a}{2}\right) + at\left(-\frac{b}{2}\right)\left(-\frac{a}{2}\right) = 0.5a^2\,b\,t$$

(iii) The normal stress distribution across the cross-section:

Using Eqn. (7.31) and noting that M_y is zero

$$\sigma_x(y,z) = \frac{1}{\left[I_y I_z - I_{yz}^2\right]}\left\{M_z\left[-y I_y + z I_{yz}\right]\right\}$$

$$I_y I_z - I_{yz}^2 = \left\{\left(0.667 a^3 t\right)\left(0.083 t b^3 + 0.5 a t b^2\right) - \left(0.5 a^2 b t\right)^2\right\}$$

$$\sigma_x(y,z) = \frac{M_z\left[-y \times \left(0.667 a^3 t\right) + z\left(0.5 a^2 b t\right)\right]}{\left\{\left(0.667 a^3 t\right)\left(0.083 t b^3 + 0.5 a t b^2\right) - \left(0.5 a^2 b t\right)\right\}}$$

The maximum tensile stress will be when z is maximum positive and y is maximum negative. The minimum tensile stress will be when y is maximum positive and z is maximum negative. Points A and B are not the locations of maximum stress.

$$\sigma_A(b/2,a) = \frac{M_z\left[-b/2 \times \left(0.667 a^3 t\right) + a\left(0.5 a^2 b t\right)\right]}{\left\{\left(0.667 a^3 t\right)\left(0.083 t b^3 + 0.5 a t b^2\right) - \left(0.5 a^2 b t\right)\right\}}$$

$$\sigma_B(-b/2,-a) = \frac{M_z\left[b/2 \times \left(0.667 a^3 t\right) - a\left(0.5 a^2 b t\right)\right]}{\left\{\left(0.667 a^3 t\right)\left(0.083 t b^3 + 0.5 a t b^2\right) - \left(0.5 a^2 b t\right)\right\}}$$

(iv) The NA for the cross-section and loading

Applying Eqn. (7.32)

$$\frac{y}{z} = \frac{I_{yz}}{I_y} = \frac{0.5 a^2 b t}{0.667 a^3 t} = 0.75\frac{b}{a}$$

Design Discussion: In this problem you have all the quantities as a function of parameters. This is a multi-variable function. For example

$$\sigma_A(a,b,t,M_z) = \frac{M_z\left[-0.5b\left(0.667 a^3 t\right) + a\left(0.5 a^2 b t\right)^2\right]}{\left(0.667 a^3 t\right)\left(0.083 t b^3 + 0.5 a t b^2\right) - \left(0.5 a^2 b t\right)^2};$$

Now you set up design and optimization problems with explicit functions

Solution using MATLAB

In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, June 2017
%  Example 7-13
```

```
% unsymmetric bending - symbolic
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 7.13 - Unsymmetric Bending - symbolic \n')
fprintf('---------------------------------------------------------\n')

%% Data
syms a b t
type = [1,2,1];  % 1 = horizontal, 2 = vertical
wd = [a,t,a]; % x-length
ht = [t,b,t]; %  y-length
ze = [a/2,0,-a/2];  % x-centroid  % left leftmost, bottom
ye = [b/2,0,-b/2];  % y-centroid

%% (i) centroid
fprintf('---------------------------\n')
fprintf('(i)  The Centroid\n')
fprintf('---------------------------\n')
area = wd.*ht;
zc = sum(ze.*area)/sum(area);
yc = sum(ye.*area)/sum(area);
fprintf('z-centroid : '),disp(vpa(zc,4))
fprintf('y-centroid : '),disp(vpa(yc,4))

%% (ii) MOI
fprintf('\n---------------------------\n')
fprintf('(ii)  MOI\n')
fprintf('---------------------------\n')

%% MOI Calculations
Iz = 0; Iy = 0; Iyz = 0;
for i = 1:length(type)
    switch type(i)
        case 1
            Iz = Iz + area(i)*(ye(i)-yc)^2;
            Iy = Iy + (1/12)*ht(i)*wd(i)^3 + area(i)*(ze(i)-zc)^2;
            Iyz = Iyz + area(i)*(ze(i)-zc)*(ye(i)-yc);
        case 2
            Iz = Iz + (1/12)*wd(i)*ht(i)^3 + area(i)*(ye(i)-yc)^2;
            Iy =  Iy + area(i)*(ze(i)-zc)^2;
            Iyz = Iyz +  area(i)*(ze(i)-zc)*(ye(i)-yc);
    end
end
Iz = vpa(Iz,4);
Iy = vpa(Iy,4);
Iyz = vpa(Iyz,4);
fprintf('Izz    : '),disp(Iz)
fprintf('Iyy    : '),disp(Iy)
fprintf('Iyz    : '),disp(Iyz)

%% (iii) Stress distribution
fprintf('\n---------------------------\n')
fprintf('(iii) Stress distribution\n')
fprintf('---------------------------\n')
```

```
syms y z Mz My
den = vpa(Iz*Iy - Iyz^2,4);
My = 0;   %  for this example - remove if  My is present in problem
sigx = (-y*(Iyz*My +Iy*Mz) + z*(Iz*My + Iyz*Mz))/den;
fprintf('\nMy {Nm] = '),disp(My)
fprintf('Mz {Nm] = '),disp(Mz)
fprintf('denominator      = '),disp(den)
fprintf('sigx (y, z) [Pa] = \n'),disp(vpa(sigx,3))

%% stress at the various points
ya = b/2;  za = a;
yb = -b/2; zb = -a;

sigA = vpa(subs(sigx,[y,z],[ya,za]),3);
fprintf('\nPoint A: sigx = '),disp(sigA)
sigB = vpa(subs(sigx,[y,z],[yb,zb]),3);
fprintf('Point B: sigx = '),disp(sigB)

%% (iv) Neutral Axis
fprintf('\n----------------------------\n')
fprintf('(iv) Equation for NA\n')
fprintf('----------------------------\n')
% Neutral axis
NArat = (Iz*My + Iyz*Mz)/(Iyz*My+Iy*Mz);
eqnNA = vpa(NArat*z,3);
fprintf('Equation for the NA : y = '),disp(eqnNA)
```

Output in Command Window

```
Example 7.13 - Unsymmetric Bending - symbolic
------------------------------------------------------------
---------------------------
 Data/ Parameters
---------------------------
No. of rectangles :      3
Width : [ a, t, a]
Height : [ t, b, t]
y-centroid: [ b/2, 0, -b/2]
z-centroid: [ a/2, 0, -a/2]
---------------------------
(i)  The Centroid
---------------------------
z-centroid : 0.0
y-centroid : 0.0

---------------------------
(ii)  MOI
---------------------------
Izz    : 0.08333*t*b^3 + 0.5*a*t*b^2
Iyy    : 0.6667*a^3*t
Iyz    : 0.5*a^2*b*t

---------------------------
(iii) Stress distribution
---------------------------
```

```
My {Nm] =       0
Mz {Nm] = Mz
denominator      = 0.6667*a^3*t*(0.08333*t*b^3 + 0.5*a*t*b^2) -
0.25*a^4*b^2*t^2
sigx (y, z) [Pa] =
-(1.0*(0.667*Mz*t*y*a^3 - 0.5*Mz*b*t*z*a^2))/(0.667*a^3*t*(0.0833*t*b^3 +
0.5*a*t*b^2) - 0.25*a^4*b^2*t^2)

Point A: sigx = (0.167*Mz*a^3*b*t)/(0.667*a^3*t*(0.0833*t*b^3 +
0.5*a*t*b^2) - 0.25*a^4*b^2*t^2)
Point B: sigx = -(0.167*Mz*a^3*b*t)/(0.667*a^3*t*(0.0833*t*b^3 +
0.5*a*t*b^2) - 0.25*a^4*b^2*t^2)

---------------------------
(iv) Equation for NA
---------------------------
Equation for the NA : y = (0.75*b*z)/a
```

Please compare the results to that obtained earlier. You should spot at least one transcribing error.

Execution in Octave
The code is same as in the MATLAB except for the highlighted changes. Most of it is formatting

In Octave Editor

```
pkg load symbolic;

fprintf('Izz    : \n'),disp(Iz)
fprintf('Iyy    : \n'),disp(Iy)
fprintf('Iyz    : \n'),disp(Iyz)

fprintf('\ndenominator     = \n'),disp(den)
fprintf('\nsigx (y, z) [Pa] = \n'),disp(vpa(sigx,3))

fprintf('\nPoint A: \nsigx = '),disp(sigA)
fprintf('\nPoint B: \nsigx = '),disp(sigB)

fprintf('Equation for the NA : \ny = '),disp(eqnNA)
```

The results are the same as that produced by MATLAB and is not included

A Design Problem: Consider the design of the cross-section in Example 7.13. There are three dimensions to fix a, b, and t. These are the essential unknowns for the design. To make a thin wall assumption let us consider that a = 10 t and b = 12 t. This has reduced the problem to a single unknown t. Choose a bending load. This reduces the normal stress distribution in terms of t alone. Now choose a material and a FOS. This determines the largest stress in the cross-section. We can then solve for t based on the maximum stress to be experienced by the cross-section.

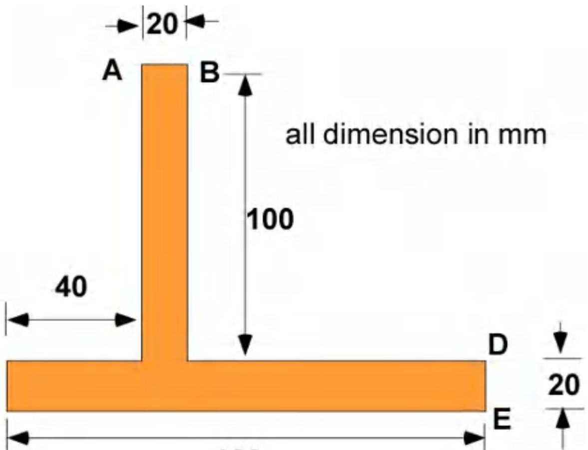
Another possibility is that using off the shelf cross-section with a known t the maximum bending load can be estimated. These are the advantages of a functional representation in terms of design parameters (symbolic approach to the problem).

7.7.7 Additional Problems.

Please consider thin-wall assumptions if appropriate.

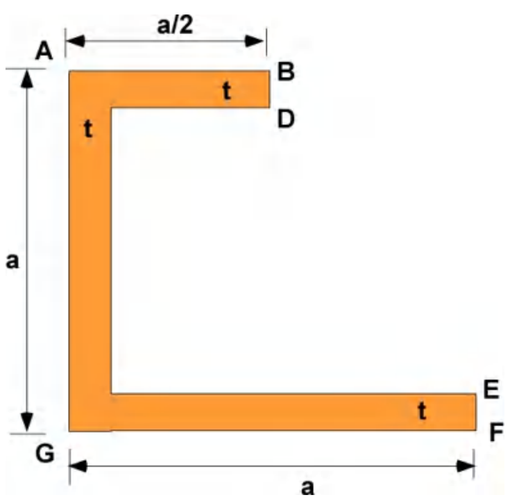
For each problem:
(i) Calculate the centroid
(ii) Calculate the MOI about the centroidal axes
(iii) Calculate the normal stress distribution across the cross-section and establish the stress at points on the cross-section based on loads
(iv) Use/Calculate FOS for given material
(v) Solve on paper and using MATLAB/Octave

Problem 7.7.1: Consider a bending load of My = - 5000 Nm and Mz = - 6000 Nm applied at the centroid of an non symmetric **T** section. Calculate the FOS if the material is made of Al 2024 alloy.



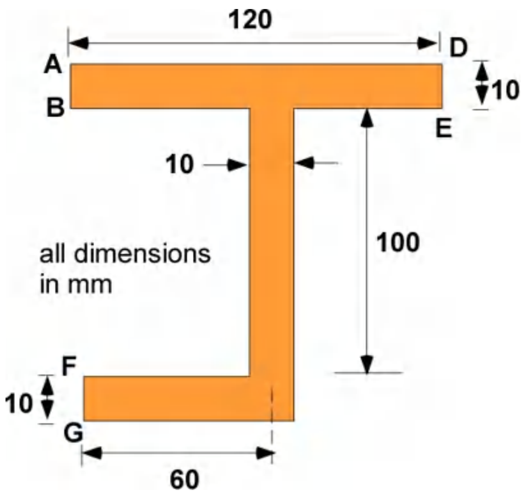
Problem 7.7.1

Problem 7.7.2: Consider a bending load of My = 5000 Nm and Mz = - 6000 Nm applied at the centroid of the non symmetric **C** section. Design the cross-section if the material is Al 2024 and the FOS is 2.5.



Problem 7.7.2

Problem 7.7.3: Consider a bending load of My = 5000 Nm and Mz = 6000 Nm applied at the centroid of the non symmetric **J** section. Calculate the stresses at the corners.



Problem 7.7.3

7.8 SHEAR IN UNSYMMETRIC CROSS-SECTIONS

Shear stress and shear flow in unsymmetrical cross-sections are interesting when they apply to thin walled structures rather than solid cross-sections. In this section we will extend the information in Section 7.5 to thin-walled sections that lack of symmetry in the cross-section. Two important changes to the calculation of shear stress will take effect. The first one is the inclusion of the product of inertia in the calculations. This is not surprising. The second effect is the introduction of a new concept of the **shear center**. Previously for symmetric sections it was accepted practice to apply shear loads at the centroid and that was useful and valid. For non-symmetric sections applying shear loads at the centroid will cause a twist in the cross-section. To avoid the twist in the cross-section the shear loads must be applied at the shear center. This is also problematic. In actual cases of shear load on unsymmetric cross-section how do you stage the load to be applied at the shear center. This is very difficult. Rather you might just apply the shear load at the centroid and include the calculation of the twist of the cross-section. Such calculations are indeed complex.

The development of the relations for calculating shear stress and the shear center are more suited for an advanced course and the details are avoided in this section. One can find the development in advanced books of structural mechanics particularly in an aerospace program. The nature of calculations is similar to the previous sections on shear flow and unsymmetrical bending and it should be easy to deploy them. In that spirit the expressions for calculations of shear stress and shear flow is developed by extensions of the previous concepts using reasonable assumptions.

7.8.1 Review of Shear Flow in Symmetric Beams

The calculation in Section 7.5 used the formula (7.20) shown below.

$$\tau_{ave} = \tau = \frac{VQ}{I_c t}; \quad and\ /\ or \quad q_{ave} = q = \frac{VQ}{I_c}; \qquad (7.20)$$

These calculations were applied to symmetric sections. Figure 7.8.1 extracts the important concepts from the section. It shows a symmetric **I** cross-section with applied shear load V_y and V_z applied independently - also regarded as superposition. Section 7.5 established that the shear flow and the average shear stress in the cross-section will vary with the maximum at the centroid going to zero at the free ends. This shear flow distribution ***q*** is illustrated in the figure through blue arrows. The integration of this shear flow in the cross-section must equal the applied shear force. The direction of the maximum shear flow is the same as the applied load at the centroid. This is true for the case of the load in the y-direction in the figure. For the load in the z-direction the maximum shear flow is at the mid-flange. The web is ineffective for the shear load in the z-direction since Q is zero.

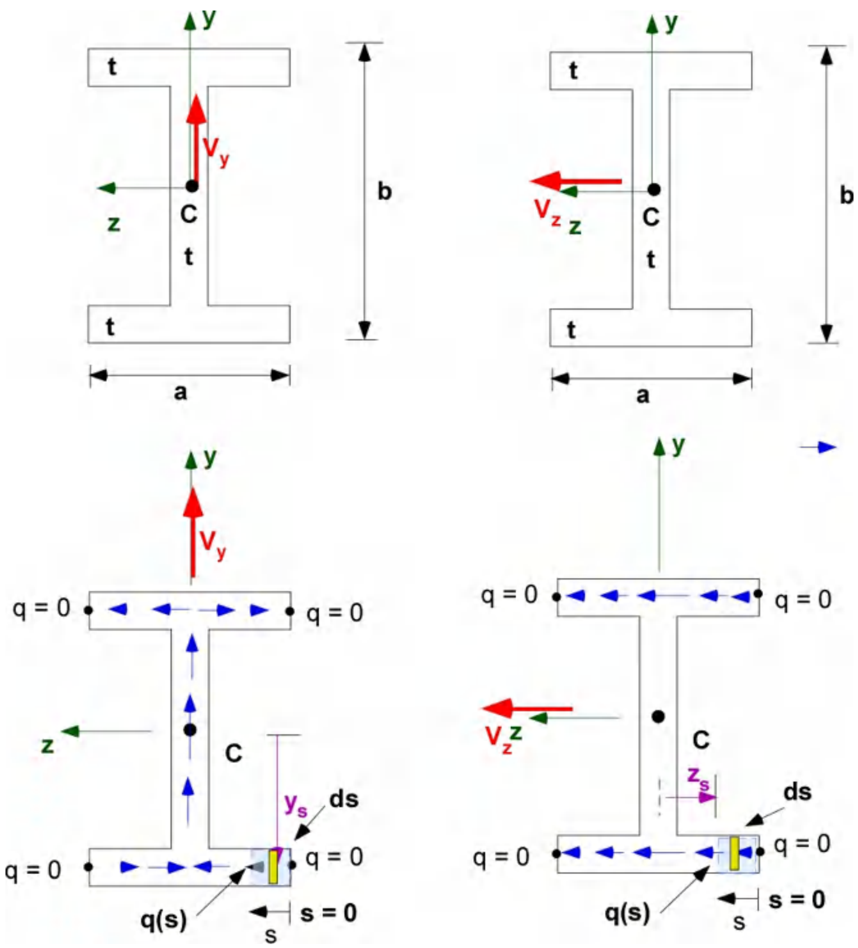


Figure 7.8.1 Shear flow distribution in symmetric **I** beam

In Figure 7.8.1 this varying shear flow distribution is represented by ***q(s)*** at ***s*** where ***s*** is measured from a point of zero shear flow. The calculation of Q in equation (7.20) is dependent on the value of ***s*** and therefore can be interpreted as ***Q(s)***. In Section 7.5 the cross-section was of constant thickness and the distance for calculating the first moment of area was at the centroid of the area. The objective was to calculate the average value over a finite region rather than the computation of the shear flow at a point. To make the formulas accommodate varying thickness and curved thin-walled sections a generalized formula that includes integration of these variable quantities will be necessary. Please note that the shear flow for the load V_y is both in z and y directions depending on the point in the cross-section. For the load V_z it is only in the z direction as shown in Figure 7.8.1. For the symmetric section we can surmise that:

For the load V_y - the shear flow is represented by $q_1(s)$:

$$q_1(s)=\frac{V_y Q(s)}{I_z}=\frac{V_y}{I_z}\int_0^s -y_s\,dA=-\frac{V_y}{I_z}\int_0^s y_s\,t\,ds=-\frac{V_y}{I_z}\int_0^s y(s)t(s)\,ds$$

For the load V_z - the shear flow is represented by $q_2(s)$:

$$q_1(s)=\frac{V_z Q(s)}{I_y}=\frac{V_z}{I_y}\int_0^s -z_s\,dA=-\frac{V_z}{I_y}\int_0^s z_s\,t\,ds=-\frac{V_z}{I_y}\int_0^s z(s)t(s)\,ds$$

For this choice of **s** both the shear flow add to provide through superposition and the equation (7.20) is now equation (7.33)

$$q(s)=q_1(s)+q_2(s)=-\frac{V_y}{I_z}\int_0^s y(s)t(s)\,ds-\frac{V_z}{I_y}\int_0^s z(s)t(s)\,ds \qquad (7.33)$$

The extension to **non-symmetric** sections is made by modifying equation (7.33) to include product of inertia in a similar manner to the case of unsymmetric bending. It is included without proof as:

$$\begin{aligned} q(s)&=q_1(s)+q_2(s)=-\frac{V_y}{I_z}\int_0^s y(s)t(s)\,ds-\frac{V_z}{I_y}\int_0^s z(s)t(s)\,ds \\ &=-\left[\frac{V_y I_y - V_z I_{yz}}{I_y I_z - I_{yz}^2}\right]\int_0^s y(s)t(s)\,ds-\left[\frac{V_z I_z - V_y I_{yz}}{I_y I_z - I_{yz}^2}\right]\int_0^s z(s)t(s)\,ds \end{aligned} \qquad (7.34)$$

There are two reasons that the equation (7.34) is satisfactory:

- Equation (7.34) reverts to equation (7.33) for symmetric sections
- The product of inertia serves to couple the shear loads. This is similar to its coupling the bending loads for calculating the normal stresses.

This will be the limit of our discussion regarding shear flow in non-symmetric cross-sections. We will look at the shear center after considering an example for calculating shear flow. We will attempt to use the new relations in the following example.

7.8.2 Example 7.14

This is same as Example 7.13 but with a shear load instead of a bending moment. The shear load is in the y-direction as shown. The cross-section is a thin walled section shown on the left in Figure 7.8.2. The shear flow variation q(s) across the section needs to be determined. Before attempting a solution it is important to have an expectation of the solution. The previous sections should suggest the shear flow distribution represented in the right part of the figure. The shear flow at A and B should be zero while at C it is a maximum in the same direction as the applied load. We will set up the problem to apply equation (7.34) in the three distinct limbs of the problem. The problem is solved symbolically. You can expect large symbolic expressions, but on the other hand, you can now design the cross-section for any combination of values for a, b, and t.

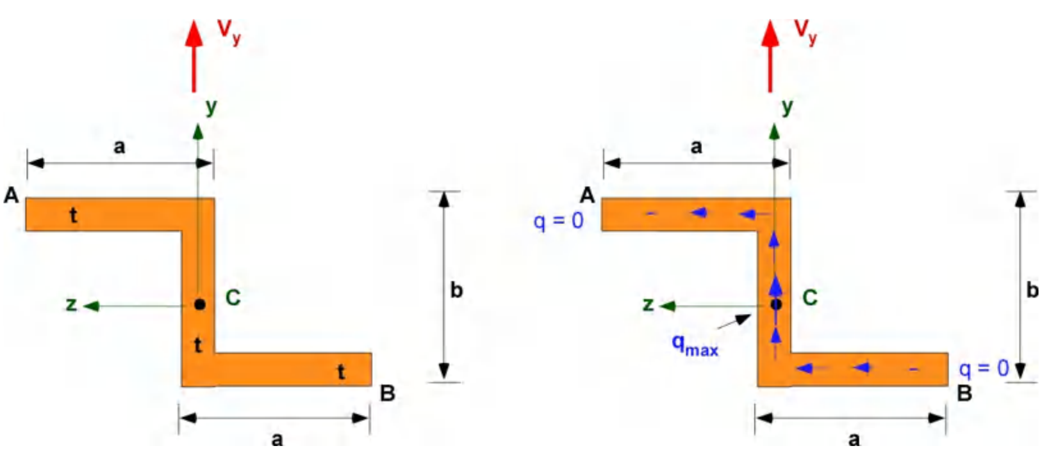


Figure 7.8.2 Example 7.10

Figure 7.8.2 Example 7.10

Data: Cross-section is shown in Figure 7.8.2. V_y is the only shear load. The solution is obtained as symbolic expressions.

Assumption: The cross-section is non-symmetric. However the centroid can be identified by inspection of the geometry because of equal areas and first moment of areas around special lines. The intersection of these lines is the centroid C.
Thin-walled section is assumed. Thickness **t** is the same for all parts of the cross-section.

Solution:
The steps for solving the problem are:

(i) Calculate the centroid
(ii) Calculate the MOI about the centroidal axes
(iii) Calculate the shear flow distribution across the cross-section.

Step(i):
It is possible to establish the centroid by inspection because of equal areas. It is at a distance of b/2 from the bottom and the center of the vertical section.

Step (ii): The thin walled beam assumptions should slightly over estimate the the value of the MOI. While the centroid was established by inspection the product of inertia is not zero. In terms of the symbols a, b, and t the various MOI should evaluate to (see Example 7.13)

$$I_{yy} = I_y = 0.667\, a^3 t$$
$$I_{zz} = I_z = 0.083 b^3\, t + 0.5\, a t b^2$$
$$I_{yz} = 0.5 a^2\, b t$$

Step (iii): Equation (7.34) is applied in the three segments of Example 7.14 below. These will affect only the **integration** terms. The terms multiplying them are the same in the three segments. We can preprocess some of the relations as:

$$I_y I_z - I_{yz}^2 = 0.667\, a^3\, t\left(0.083 t b^3 + 0.5\, a t b^2\right) - 0.25\, a^4\, b^2 t^2$$

$$-\frac{\left[V_y I_y - V_z I_{yz}\right]}{\left[I_y I_z - I_{yz}^2\right]} = \frac{-0.667\, V_y\, a^3\, t}{0.667\, a^3\, t\left(0.083 t b^3 + 0.5\, a t b^2\right) - 0.25\, a^4\, b^2 t^2}$$

$$-\frac{\left[V_z I_z - V_y I_{yz}\right]}{\left[I_y I_z - I_{yz}^2\right]} = \frac{0.5 V_y\, a^2\, b t}{0.667\, a^3\, t\left(0.083 t b^3 + 0.5\, a t b^2\right) - 0.25\, a^4\, b^2 t^2}$$

The cross-section is redrawn in Figure 7.8.3a with more more details:

Segment BD:

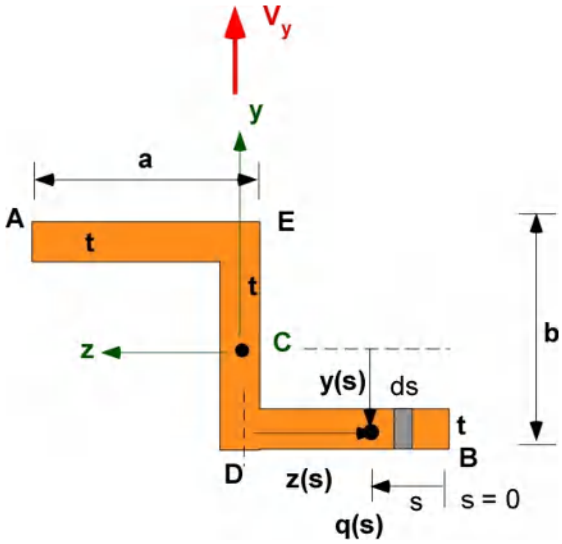


Figure 7.8.3a Segment BD

The integration is with respect to ***s***. The first task is to obtain y(s) and z(s) for the segment BD. In general we will see a linear relationship in the variable **s**.

$$s=0;\quad y=-b/2;\quad z=-a$$
$$s=a;\quad y=-b/2;\quad z=0$$
$$linear\ equation:\quad y(s)=c\times s+d:\quad substitute\ y,s$$
$$-b/2=0+d;\quad -b/2=a\times s+d;\quad Therefore:\quad y(s)=-b/2$$

$$linear\ equation:\quad z(s)=c\times s+d:\quad substitute\ z,s$$
$$-a=0+d;\quad 0=c\times a+d;\quad Therefore:\quad z(s)=s-a$$

The integrals are :

$$\int_0^s y(s)t\,ds=\int_0^s(-b/2)t\,ds=-\frac{b}{2}ts$$
$$\int_0^s z(s)t\,ds=\int_0^s t(s-a)\,ds=t\left(\frac{s^2}{2}-as\right)$$

The shear flow in the segment is :

$$q(s)=\frac{0.333V_y a^3 bst^2-0.25V_y a^2 bst^2(2a-s)}{0.667a^3 t(0.0833tb^3+0.5atb^2)-0.25a^4b^2t^2}$$

$$q(B)=0$$

$$q(D)=\frac{0.0833V_y a^4 b t^2}{0.667a^3 t(0.0833tb^3+0.5atb^2)-0.25a^4 b^2 t^2)}$$

Segment DE:

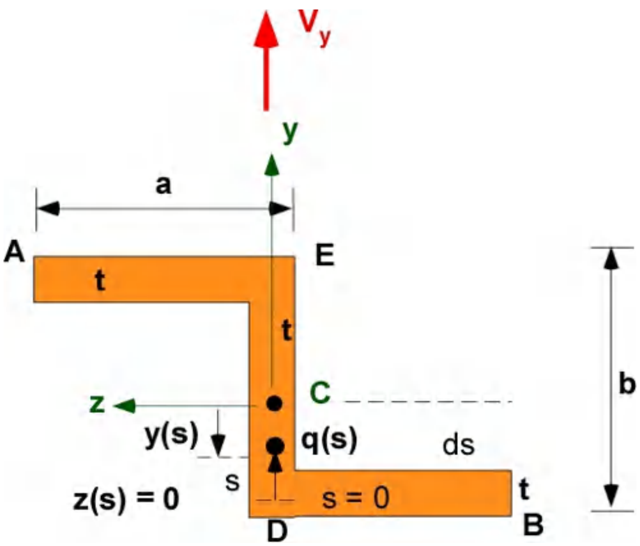


Figure 7.8.3b Segment DE

In this segment ***s*** starts at zero at D instead of continuing from the previous segment. This will require a modification to the calculation of the shear flow by adding the value of the shear flow at D to the calculations for the segment similar to above.

$$q(s)_{DE}=q(D)+q(s)$$

Identify the linear relation y(s) and z(s) :

$$s=0;\quad y=-b/2;\quad z=0$$

$$s=b;\quad y=+b/2;\quad z=0$$

$$linear\ equation:\quad y(s)=c\times s+d:\quad substitute\ y,s$$

$$-b/2=0+d;\quad b/2=c\times b+d;\quad Therefore:\quad y(s)=s-b/2$$

$$linear\ equation:\quad z(s)=c\times s+d:\quad substitute\ z,s$$

$$0=0+d;\quad 0=c\times b+d;\quad Therefore:\quad z(s)=0$$

The integrals:

$$\int_0^s y(s)\,t\,ds = \int_0^s (s - b/2)\,t\,ds = t\frac{s}{2}(s-b)$$

$$\int_0^s z(s)\,t\,ds = \int_0^s t0\,ds = 0$$

The shear flow distribution q(s) along the segment DE is :

$$q(s) = \frac{(0.0833 V_y a^4 b t^2)}{(0.667 a^3 t(0.0833 t b^3 + 0.5 a t b^2) - 0.25 a^4 b^2 t^2)} + \frac{(0.333 V_y a^3 s t^2 (b-s))}{(0.667 a^3 t(0.0833 t b^3 + 0.5 a t b^2) - 0.25 a^4 b^2 t^2)}$$

The first term is the q(D). The shear flow at D, C, and E are:

$$q(D) = \frac{0.0833 V_y a^4 b t^2}{(0.667 a^3 t(0.0833 t b^3 + 0.5 a t b^2) - 0.25 a^4 b^2 t^2)}$$

$$q(C) = \frac{0.0833 V_y a^3 b t^2 (a+b)}{(0.667 a^3 t(0.0833 t b^3 + 0.5 a t b^2) - 0.25 a^4 b^2 t^2)} +$$

$$q(E) = \frac{0.0833 V_y a^4 b t^2}{(0.667 a^3 t(0.0833 t b^3 + 0.5 a t b^2) - 0.25 a^4 b^2 t^2)}$$

Segment EA:

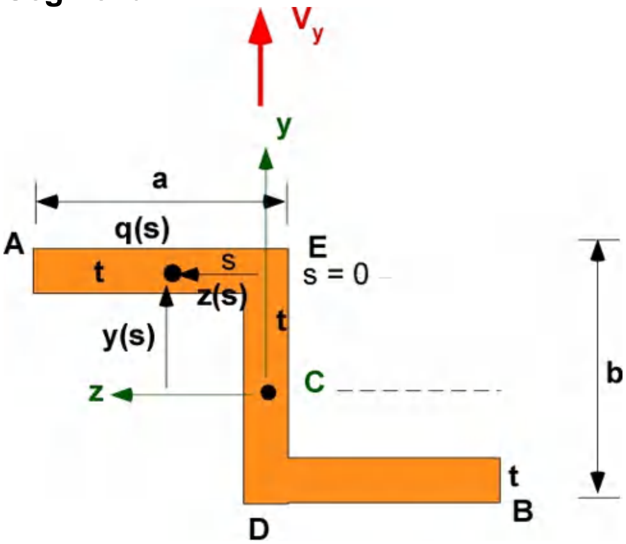


Figure 7.8.3c Segment EA

By symmetry the shear distribution in this segment must mirror the segment BD with shear flow going to zero at A. Our calculations should verify this. The shear flow starts with s = 0 at E. Therefor we must add the value of q(E) to the distribution in the segment:

$$q(s)_{EA} = q(E) + q(s)$$

Identify the linear relations y(s) and z(s):

$$s = 0; \quad y = b/2; \quad z = 0$$
$$s = a; \quad y = b/2; \quad z = a$$
$$linear\ equation: \quad y(s) = c \times s + d: \quad substitute\ y, s$$
$$b/2 = 0 + d; \quad b/2 = c \times a + d; \quad Therefore: \quad y(s) = b/2$$

$$linear\ equation: \quad z(s) = c \times s + d: \quad substitute\ z, s$$
$$0 = 0 + d; \quad a = c \times a + d; \quad Therefore: \quad z(s) = s$$

The integrals:

$$\int_0^s y(s)\,t\,ds = \int_0^s \frac{b}{2} t\,ds = t\frac{b}{2}s$$
$$\int_0^s z(s)\,t\,ds = \int_0^s ts\,ds = t\frac{s^2}{2}$$

The shear flow distribution q(s) in segment EA is:

$$q(s) = \frac{0.0833 V_y\, a^4\, bt^2}{(0.667\, a^3\, t(0.0833\, t\, b^3 + 0.5\, a\, t\, b^2) - 0.25\, a^4\, b^2\, t^2)} + \frac{0.25\, V_y\, a^2\, b\, t^2\, s^2}{(0.667\, a^3\, t(0.0833\, t\, b^3 + 0.5\, a\, t\, b^2) - 0.25\, a^4\, b^2\, t^2)} - \frac{0.333 V_y\, a^3\, b\, t^2\, s}{(0.667\, a^3\, t(0.0833\, t\, b^3 + 0.5\, a\, t\, b^2) - 0.25\, a^4\, b^2\, t^2)}$$

The shear flow at points E and A:

$$q(E) = \frac{0.0833 V_y\, a^4\, bt^2}{(0.667\, a^3\, t(0.0833\, t\, b^3 + 0.5\, a\, t\, b^2) - 0.25\, a^4\, b^2\, t^2)}$$
$$q(A) = 0$$

Note:

The solution is more sophisticated than the one in Section 7.5 as the solution defines the shear flow in the segments as a continuous distribution in terms of the variable ***s*** - which can also be curvilinear. The calculations in Section 7.5 are the same but they are used to estimate shear flow at discrete points. In engineering this is more useful as it possible to identify approximate locations for the maximum shear flow and shear stress without needing a continuous variation. These calculations are for open thin-walled sections. In genera it is useful to start the calculations at s = 0 where q(s) = 0.

7.8.3 Shear Center

The shear center is the point in the cross-section where the applied shear load will not cause a twist in the cross-section. Reversing the argument the shear center is also the center for twist when a torsional load is applied in the cross-section. Practically it is difficult to locate the shear load at the shear center for an unsymmetric section. Consider the example of the regular and the rotated C section with a shear load in the y- direction. The load is applied at the centroid.

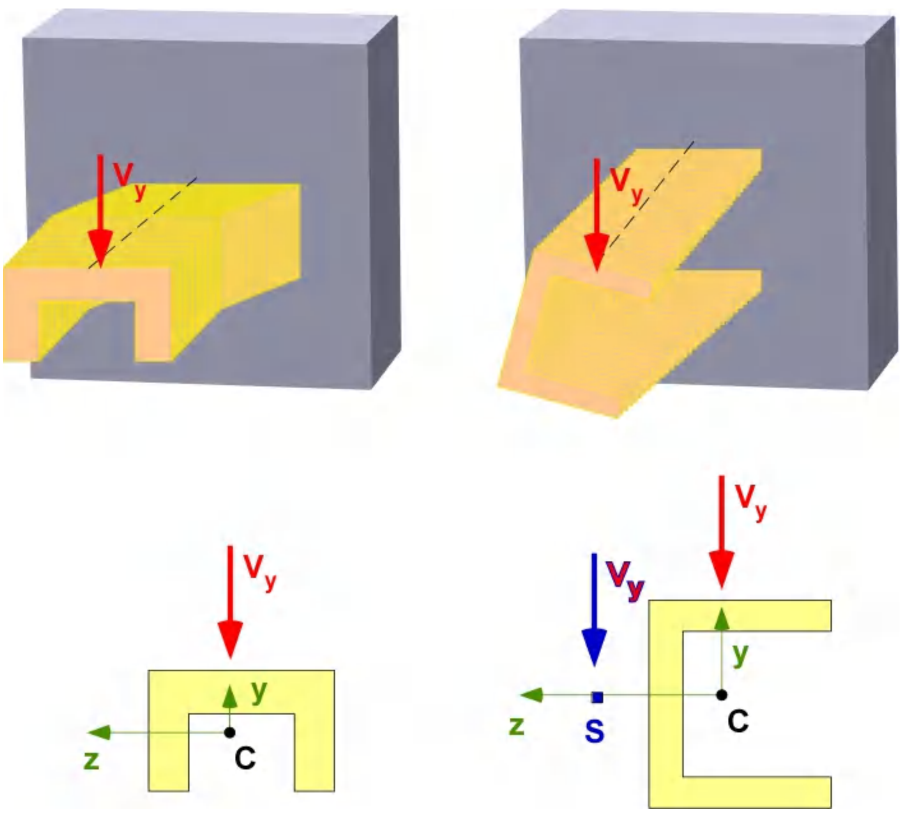


Figure 7.8.4 The shear center

For the beam on the left the shear load will just cause bending. For the beam on the right the shear load will cause a twist along with bending. To avoid the twist the shear load must be applied at the shear center **S**. The cross-section has one axis of symmetry. The twisting action occurs if the load does not pass through the axis of symmetry.

Location of the Shear Center: The shear center is located at a point where the applied shear load produces the same moment about any point in the cross-section as the moment produced by the shear force distribution in the cross-section. . This shear force distribution is realted to the shear flow distribution. In practice:

- If there is an axis of symmetry the shear center will lie on this axis
- If the beam has two sections and they intersect then the shear center must lie at the

intersection as in Figure 7.8.5. The moment due to the shear force in the cross-section about this point is zero and therefore the external shear load must be placed here.

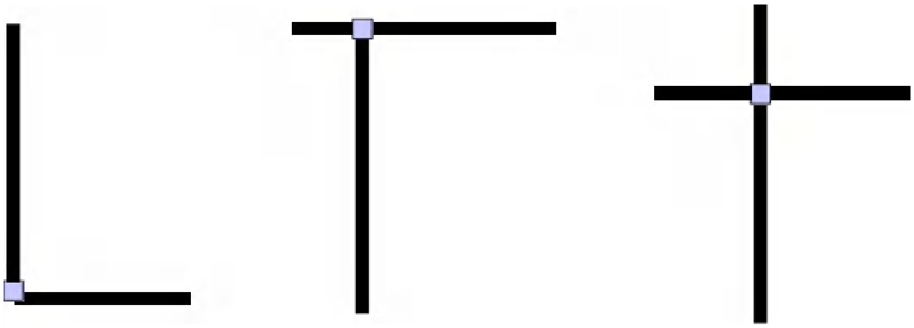


Figure 7.8.5 Shear center at intersection

Calculating shear center: Consider the simple **C** section used in previous illustration detailed in Figure 7.8.6. The point D on the cross-section is a convenient point for calculation of the moment. The moment by the applied shear load is **V_y *z_{sc}**. The shear force in sections DE and BD will not produce a moment about D since the moment arm is zero. The shear distribution in BA is the only one that will provide a moment at D.

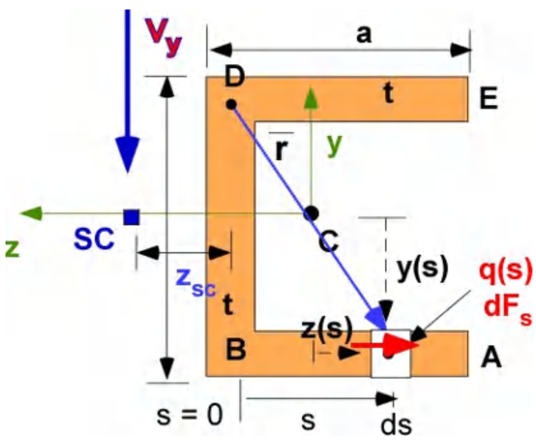


Figure 7.8.6 Calculating shear center

$$V_y z_{sc} = \int_0^a \bar{r} \times d\bar{F}_s = \int_0^a \bar{r} \times \left[q(s) t ds \left(-\hat{k}\right) \right] \quad (7.35)$$

The shear flow is distribution q(s) is available in Eq. (7.34). Eq. (7.35) can then be solved for **z_{sc}**.

7.8.4 Example 7.15

This example requires the determination of the shear flow and the calculation of the shear center. Parts of the cross-section are varying in thickness. The size of the section is changed to keep in non-symmetric. The applied vertical shear load is 6000 N in the direction shown. Assume the load is applied at the shear center.

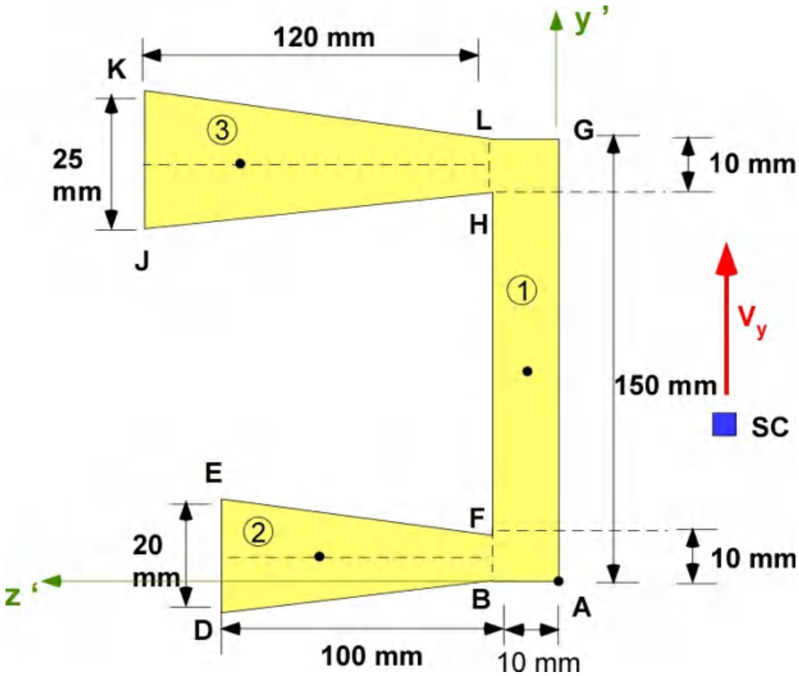


Figure 7.8.7 Example 7.15

Data: Cross-section is shown in Figure 7.8.7. V_y is the only shear load applied. The cross-section is recognized in three segments.

Assumption: The cross-section is non-symmetric. The centroid must be identified first. Thickness varies in the flanges.

Solution:
The steps for solving the problem are:
(i) Calculate the centroid
(ii) Calculate the MOI about the centroidal axes
(iii) Calculate the shear flow distribution across the cross-section.
(iv) Locate the shear center for the cross-section

Step (i) The Centroid
The two areas on the left are identified as isosceles trapezoid. The expressions for the centroid and MOI for this type of geometry can be derived or looked up. The values here are obtain from (www.efunda.com). You should be able to easily verify it. This will be used to identify the centroid and MOI in Example 7.15.

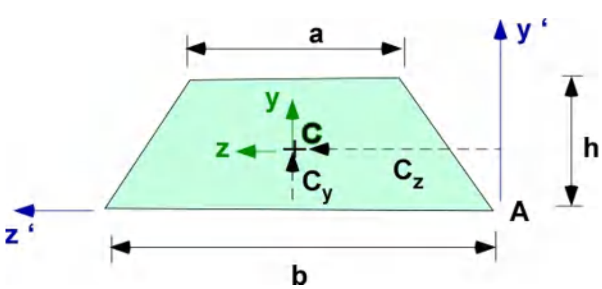


Figure 7.8.8 Isosceles Trapezoid

$$C_y = \frac{h}{3}\left(\frac{2a+b}{a+b}\right);\quad C_x = \frac{b}{2};\quad A = \frac{h}{2}(a+b)$$

From the same reference the MOI are

$$I_{yc} = \frac{h(a+b)(a^2+b^2)}{48};\quad I_{xc} = \frac{h^3(a^2+4ab+b^2)}{36(a+b)}$$

For the isosceles segment 2:
(Note the actual trapezoid is rotated compared to the formula above)
b = 10; a = 20; h = 100

$Area[mm^2]$:   1500

$C_x(from\,base)[mm]$:   55.56

$C_y(from\,base)[mm]$:   5

$I_x[mm^4]$:   31250

$I_{yc}[mm^4]$: 1.204×10^6

The isosceles segment 3:
(Note the actual trapezoid is rotated compared to the formula above)

b = 10; a = 25; h = 120

$Area[mm^2]$:   2100

$C_x(from\,base)[mm]$:   68.57

$C_y(from\,base)[mm]$:   5

$I_x[mm^4]$:   63438

$I_{yc}[mm^4]$: 2.366×10^6

The vertical segment values can be easily inferred.
h = 10; b = 150;

$$Area\left[mm^2\right]:\quad 1500$$

$$C_z\left(from\,base\right)\left[mm\right]:\quad 5$$

$$C_y\left(from\,base\right)\left[mm\right]:\quad 75$$

$$I_{zc}\left[mm^4\right]:\quad 2812500$$

$$I_{yc}\left[mm^4\right]:\quad 12500$$

The centroid of the cross-section is (please verify these values):

$$area = [A_1; A_2; A_3];$$

$$z_e = \left[C_{z1}; C_{z2}+10; C_{z3}+10\right];$$

$$y_e = \left[C_{y1}; C_{y2}+10; C_{y3}+140\right];$$

$$z_c = \frac{sum\left(z_e^{\,T} area\right)}{sum\left(area\right)} = 53.1\left[mm\right];$$

$$y_c = \frac{sum\left(y_e^{\,T} area\right)}{sum\left(area\right)} = 83.24\left[mm\right];$$

Step (ii):
The MOI about the centroid is calculated using the parallel axis theorem without thin-walled assumptions (please verify these values) :

$$I_{zz} = \left(I_{zc1} + A_1\times\left[y_{e_1} - y_c\right]^2\right) + \left(I_{zc2} + A_2\times\left[y_{e_2} - y_c\right]^2\right) +$$
$$= \left(I_{zc3} + A_3\times\left[y_{e_3} - y_c\right]^2\right) = 2.02\times10^7\left[mm^4\right]$$

$$I_{yy} = \left(I_{yc1} + A_1\times\left[z_{e_1} - z_c\right]^2\right) + \left(I_{yc2} + A_2\times\left[z_{e_2} - z_c\right]^2\right) +$$
$$= \left(I_{yc3} + A_3\times\left[z_{e_3} - z_c\right]^2\right) = 8.65\times10^6\left[mm^4\right]$$

$$I_{yz} = A_1\times\left[z_{e_1} - z_c\right]\left[y_{e_1} - y_c\right] + A_2\times\left[z_{e_2} - z_c\right]\left[y_{e_2} - y_c\right] +$$
$$A_3\times\left[z_{e3} - z_c\right]\left[y_{e3} - y_c\right] = 2.44\times10^6\left[mm^4\right]$$

Step (iii):

The shear flow is calculated assuming the shear force Vy is at the shear center using:

$$q(s)=q_1(s)+q_2(s)=-\left[\frac{V_y I_y - V_z I_{yz}}{I_y I_z - I_{yz}^2}\right]\int_s^0 y(s)t(s)ds$$
$$-\left[\frac{V_z I_z - V_y I_{yz}}{I_y I_z - I_{yz}^2}\right]\int_s^0 z(s)t(s)ds$$

The coefficients of the integral are the same in all segments (please verify these values) :

$$-\left[\frac{V_y I_y - V_z I_{yz}}{I_y I_z - I_{yz}^2}\right] = -3.07\times10^{-4}$$
$$-\left[\frac{V_z I_z - V_y I_{yz}}{I_y I_z - I_{yz}^2}\right] = 8.66\times10^{-5}$$

Segment DB: Establish the linear relations:

$$s=0;\quad t=20;\quad y=-78.23;\quad z=56.9$$
$$s=100;\quad t=10;\quad y=-78.23;\quad z-43.1$$
$$y(s)=-78.2$$
$$z(s)=56.9-s$$
$$t(s)=20-0.1s$$

The shear flow $q_1(s)$ and $q_2(s)$ and the values at D and B are (please verify these expressions):

$$q_1(s) = -0.0012\,s(s - 400.0)$$
$$q_2(s)=2.83\times10^{-8}\,s\,(102.0s^2-3.93\times10^4\,s+3.48\times10^6)$$
$$q_D = 0$$
$$q_B = 37.7\left[\frac{N}{mm}\right]$$

Segment BL: Vertical Segment (between the centers of the isosceles trapezoid) -BL (not exactly at B or L)

$$s = 0;\quad t = 10;\quad y = -78.23;\quad z = -48.1$$

$$s = 140;\quad t = 10;\quad y = 61.76;\quad z - 48.1$$

$$y(s) = s - 78.2$$

$$z(s) = -48.1$$

$$t(s) = 10$$

The shear flow value $\mathbf{q_B}$ from the previous segment must be added to the shear flows $q_1(s)$ and $q_2(s)$ in this segment (please verify these expressions). The values at B and L are:

$$q_1(s) = -9.04 \times 10^{-5}\, s\,(17.0s - 2666.0)$$

$$q_2(s) = -0.0417\, s$$

$$q_B = 37.7\left[N/mm\right]$$

$$q_L = 35.4\left[N/mm\right]$$

Segment LK: The Segment LK should have a shear flow of 35.4 at L and go to zero at K. You should be able to apply the analysis to solve for this segment to complete this section of the solution.

Step (iv): The following ideas may be useful in establishing the shear center - see Figure 7.8.9.

- By taking the moment of the shear force about the intersection of 1 and 3 (point M) we only need the shear force distribution in the segment DB. The shear force distribution in segments BL and LK will have no moment about point M.

- With shear load $\mathbf{V_y}$ we can only calculate the z-location of the shear center. We will need to apply the $\mathbf{V_z}$ to calculate the y-offset of the shear center. The magnitude of the shear load is not important. Positive loads $\mathbf{V_y}$ and $\mathbf{V_z}$ are applied.

The figure for the example is then edited appropriately in Figure 7.8.9

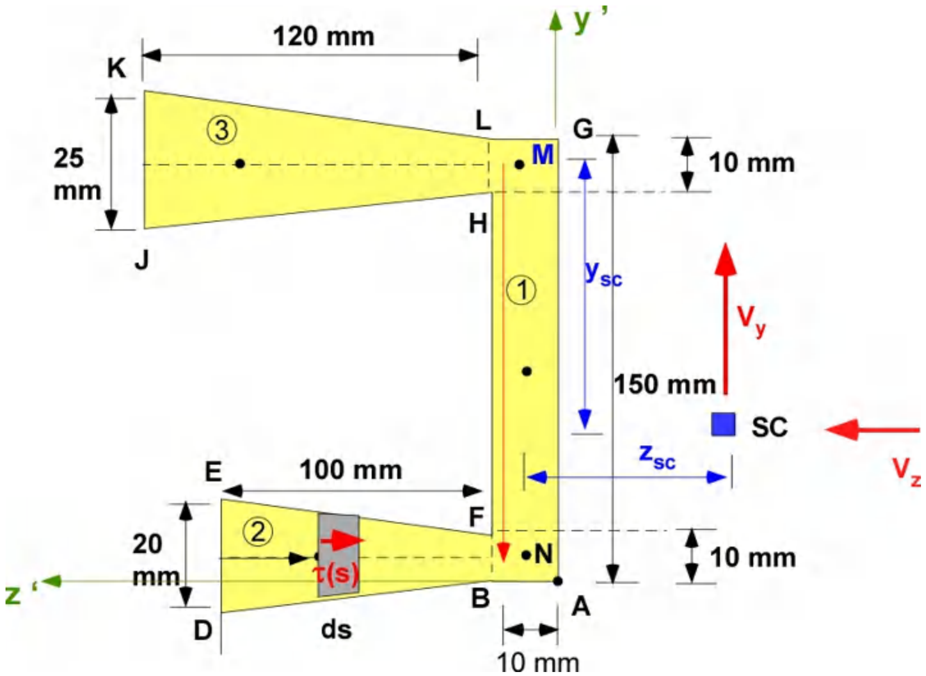


Figure 7.8.9 Establishing the shear center

Calculation of z_{sc}:

$$q(s) = q_1(s) + q_2(s) = 4.72\times10^{-12} V_y\, s(102.0s^2 - 3.93\times10^4 s + 3.48\times10^6) - 2.0\times10^{-7} V_y\, s(s - 400.0)$$

$$dF = \tau(s)\, t(s)\, ds = q(s)\, ds$$

$$dM = y_{MN}\, dF = 140\, dF$$

$$M = \int_{100}^{0} dM = V_y\, z_{sc} = 51.3 V_y$$

$$z_{sc} = 51.3\,[mm]$$

Calculation of y_{sc}: Note : The shear force distribution is calculated to the right in segment DB. The applied shear force is to the left (see Figure 7.8.9). The moment at M will be opposite in sign and this is incorporated in the calculations below.

$$q(s) = q_1(s) + q_2(s) = 3.91\times10^{-11} V_z\, s(102.0s^2 - 3.93\times10^4 s + 3.48\times10^6) - 5.65\times10^{-8} V_z\, s(s - 400.0)$$

$$dF = \tau(s)\, t(s)\, ds = q(s)\, ds$$

$$dM = y_{MN}\, dF = 140\, dF$$

$$M = \int_{100}^{0} dM = -V_z\, y_{sc} = -50.7 V_z$$

$$y_{sc} = 50.7\,[mm]$$

7.8.5 Example 7.15 Using MATLAB

The cross-section is the same as in Figure 7.8.9 reused below with local axes definition The preliminary description is also imported.

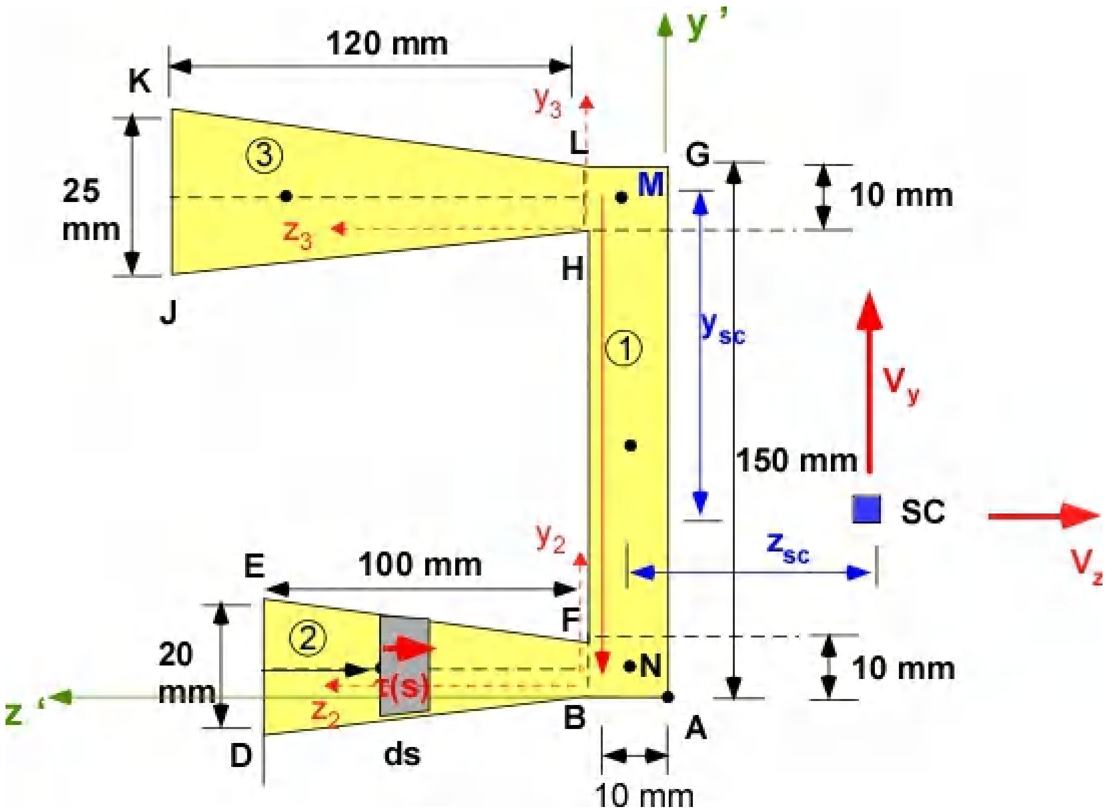


Figure 7.8.10 Establishing the shear center

Data: Cross-section is shown in Figure 7.8.10. V_y is the only shear load applied. The cross-section is recognized in three segments.

Assumption: The cross-section is non-symmetric. The centroid must be identified first. Thickness varies in the flanges.

Solution:
The steps for solving the problem are:

(i) Calculate the centroid
(ii) Calculate the MOI about the centroidal axes
(iii) Calculate the shear flow distribution across the cross-section.
(iv) Locate the shear center for the cross-section

MATLAB Code:

- The following ideas are implemented:
- The code calculates the shear center.
- The shear flow calculation for segment DB is only used to calculate the shear center.
- The shear flow calculation steps are the same but must be performed for V_y alone and V_z alone (superposition).

- Shear flow calculation for segment BL is also included for V_y only.
- Shear flow calculation in LK is left as an exercise.

The code is broken up into a sequence of calculations below: Remember to consolidate the code in a single file.

(i) Calculate the centroid

```
% Essential Foundations in Mechanics
% P. Venkataraman, June 2017
%  Example 7-15
% unsymmetric shear -
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 7.15 - Unsymmetric Shear \n')
fprintf('----------------------------------------------------------\n')
%{
The steps for solving the problem are:
(i) Calculate the centroid
(ii) Calculate the MOI about the centroidal axes
(iii) Calculate the shear flow distribution across the cross-section.
(iv) Locate the shear center for the cross-section
%}
%% Data
fprintf('------------------------------------------\n')
fprintf(' Preprocessing - CG, MOI of isoceles trapezoid  \n')
fprintf('----------------------------------------------\n')
syms a b h

% formula from handbook
Cz = h*(2*a + b)/(a +b)/3;
Cy = b/2;
A = h*(a + b)/2;
Izc = h*(a+b)*(a^2 + b^2)/48;
Iyc = h^3*(a^2 + 4*a*b + b^2)/36/(a+b);

fprintf('Cy  = '),disp(Cy)
fprintf('Cz  = '),disp(Cz)
fprintf('A  = '),disp(A)
fprintf('Iyc = '),disp(Iyc)
fprintf('Izc = '),disp(Izc)

%% (i) centroid
fprintf('--------------------------\n')
fprintf('(i)  The Centroid\n')
fprintf('--------------------------\n')
fprintf('Local CG and MOI for Segment 3\n')
fprintf('---------------------------------------------\n')
a3 = 25;  b3 = 10; h3 = 120;
A3 = double(subs(A,[a,b,h],[a3,b3,h3]));
Cz3 = double(subs(Cz,[a,b,h],[a3,b3,h3]));
Cy3 = double(subs(Cy,[a,b,h],[a3,b3,h3]));
Izc3 = double(subs(Izc,[a,b,h],[a3,b3,h3]));
```

```
Iyc3 = double(subs(Iyc,[a,b,h],[a3,b3,h3]));
fprintf('Area Segment 3 [mm^2]       : '),disp(A3)
fprintf('Cz from Base for segment 3 : '),disp(Cy3)
fprintf('Cy at Base for segment 3   : '),disp(Cz3)
fprintf('MOIz centroid segment 3    : '),disp(Izc3)
fprintf('MOIy centroid segment 3    : '),disp(Iyc3)

% segment 2
fprintf('---------------------------\n')
fprintf('Local CG and MOI for Segment 2\n')
fprintf('----------------------------------------------------\n')
a2 = 20;  b2 = 10; h2 = 100;
A2 = double(subs(A,[a,b,h],[a2,b2,h2]));
Cz2 = double(subs(Cz,[a,b,h],[a2,b2,h2]));
Cy2 = double(subs(Cy,[a,b,h],[a2,b2,h2]));
Izc2 = double(subs(Izc,[a,b,h],[a2,b2,h2]));
Iyc2 = double(subs(Iyc,[a,b,h],[a2,b2,h2]));
fprintf('Area Segment 2 [mm^2]       : '),disp(A2)
fprintf('Cz from Base for segment 2 : '),disp(Cz2)
fprintf('Cy at Base for segment 2   : '),disp(Cy2)
fprintf('MOIz centroid segment 2    : '),disp(Izc2)
fprintf('MOIy centroid segment 2    : '),disp(Iyc2)

fprintf('---------------------------\n')
fprintf('Local CG and MOI for Segment 1\n\n')
fprintf('----------------------------------------------------\n')
A1 = 10*150;
Cz1 = 5;
Cy1 = 75;
Izc1 = 10*150^3/12;
Iyc1 = 150*10^3/12;
fprintf('Area Segment 1 [mm^2]       : '),disp(A1)
fprintf('Cz from Base for segment 1 : '),disp(Cz1)
fprintf('Cy at Base for segment 1   : '),disp(Cy1)
fprintf('MOIz centroid segment 1    : '),disp(Izc1)
fprintf('MOIy centroid segment 1    : '),disp(Iyc1)

fprintf('---------------------------\n')
fprintf('Cross-section CG and MOI about CG\n')
fprintf('----------------------------------------------------\n')

% element information
area = [A1,A2,A3]; % segment areas
ze = [Cz1,Cz2+10,Cz3+10];  % from reference
ye = [Cy1,Cy2,140+Cy3];  % from reference
Iyc = [Iyc1, Iyc2, Iyc3];
Izc = [Izc1, Izc2, Izc3];

% centroid
zc = sum(ze.*area)/sum(area);
yc = sum(ye.*area)/sum(area);
fprintf('z-centroid from A [mm]: '),disp(vpa(zc,4))
fprintf('y-centroid from A [mm] : '),disp(vpa(yc,4))
```

(ii) Calculate the MOI about the centroidal axes

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% (ii) Calculating the MOI
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('---------------------------\n')
fprintf('(ii) MOI \n')
fprintf('---------------------------\n')
Iz = Izc + area.*(ye - yc).^2;
Iy = Iyc + area.*(ze - zc).^2;
Iyzc = area.*(ye -yc).*(ze - zc);
Izz = sum(Iz);  Iyy = sum(Iy) ;  Iyz = sum(Iyzc);
fprintf('Izz [mm^4]   : '),disp(Izz)
fprintf('Iyy [mm^4]   : '),disp(Iyy)
fprintf('Iyz [mm^4]   : '),disp(Iyz)
```

(iii) Calculate the shear flow distribution across the cross-section.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% (iii) Calculating the shear flow in cross-section
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('------------------------------------------------\n')
fprintf('(iii) Calculation of shear flow in segments \n')
fprintf('------------------------------------------------------------\n')

%% Calculating the shear flow in segment DB
% and the calculation of shear center Zsc
Vz = 0;  Vy = 6000;
den = Izz*Iyy - Iyz^2;
num1 = -(Vy*Iyy - Vz*Iyz);
num2 = -(Vz*Izz - Vy*Iyz);

fprintf('Vy [N] = '),disp(Vy)
fprintf('Vz [N] = '),disp(Vz)
fprintf('-(Vy*Iy-Vz*Iyz) ='),disp(num1)
fprintf('-(Vz*Iz-Vv*Iyz) ='),disp(num2)
fprintf('Izz*Iyy - Iyz^2  ='),disp(den)
coef1 = num1/den;  coef2 = num2/den;
fprintf('coef1 ='),disp(coef1)
fprintf('coef2 ='),disp(coef2)


fprintf('---------------------------\n')
fprintf('Calculation q(s) in DB \n')
fprintf('------------------------\n')
syms y z s t
% calculating linear functions
syms c d
ydb = -(yc-5); zd = 110 - zc;  zb = 10-zc;
td = 20;  tb = 10;
fprintf('s, t, y, z at D = '), disp([0,td,ydb,zd])
fprintf('s, t, y, z at B = '), disp([100,tb,ydb,zb])
eqy = y-c*s - d;
eq(1) = subs(eqy,[y,s],[ydb,0]);
eq(2) = subs(eqy,[y,s],[ydb,100]);
soly = solve(eq,c,d);
c = soly.c;
d = soly.d;
```

```
ys = c*s +d;
fprintf('Equation for y(s) = '),disp(vpa(ys,3))

clear c d
syms c d
eqz = z-c*s - d;
eq(1) = subs(eqz,[z,s],[zd,0]);
eq(2) = subs(eqz,[z,s],[zb,100]);
solz = solve(eq,c,d);
c = solz.c;
d = solz.d;
zs = c*s+d;
fprintf('Equation for z(s) = '),disp(vpa(zs,3))

clear c d
syms c d
eqz = t-c*s - d;
eq(1) = subs(eqz,[t,s],[td,0]);
eq(2) = subs(eqz,[t,s],[tb,100]);
solt = solve(eq,c,d);
c = solt.c;
d = solt.d;
ts = c*s+d;
fprintf('Equation for t(s) = '),disp(vpa(ts,3))

% shear flow calculation
qs1 = coef1*int(ys*ts,s,0,s);
qs2 = coef2*int(zs*ts,s,0,s);
qs = qs1 + qs2;
fprintf('q1(s) = '),disp(vpa(qs1,3))
fprintf('q2(s) = '),disp(vpa(qs2,3))
fprintf('q(s) = '),disp(vpa(qs,3))

qD = subs(qs,s,0);
qB = subs(qs,s,100);
fprintf('qD [N/mm] = '),disp(double(qD))
fprintf('qB [N/mm]= '),disp(double(qB))

fprintf('----------------------------\n')
fprintf('Calculation q(s) in BL \n')
fprintf('-------------------------\n')
clear c d zs ys ts
syms c d
yb = -(yc-5); zb = 5- zc;  tb = 10; sb = 0;
yl = 145-yc; zl = 5- zc;  tl = 10;  sl = 140;
fprintf('s, t, y, z at B = '), disp([sb, tb, yb, zb])
fprintf('s, t, y, z at L = '), disp([sl, tl, yl, zl])
eqy = y-c*s - d;
eq(1) = subs(eqy,[y,s],[yb, sb]);
eq(2) = subs(eqy,[y,s],[yl, sl]);
soly = solve(eq,c,d);
c = soly.c;
d = soly.d;
ys = c*s +d;
fprintf('Equation for y(s) = '),disp(vpa(ys,3))
```

```
syms c d
eqz = z-c*s - d;
eq(1) = subs(eqz,[z,s],[zb, sb]);
eq(2) = subs(eqz,[z,s],[zl, sl]);
solz = solve(eq,c,d);
c = solz.c;
d = solz.d;
zs = c*s+d;
fprintf('Equation for z(s) = '),disp(vpa(zs,3))

syms c d
eqz = t-c*s - d;
eq(1) = subs(eqz,[t,s],[tb, sb]);
eq(2) = subs(eqz,[t,s],[tl, sl]);
solt = solve(eq,c,d);
c = solt.c;
d = solt.d;
ts = c*s+d;
fprintf('Equation for t(s) = '),disp(vpa(ts,3))

% shear flow calculation
qs11 = coef1*int(ys*ts,s,0,s);
qs21 = coef2*int(zs*ts,s,0,s);
qss = qs11+qs21;
fprintf('q1(s) = '),disp(vpa(qs11,3))
fprintf('q2(s) = '),disp(vpa(qs21,3))
fprintf('q(s) = '),disp(vpa(qss,3))
%
qB1 = subs(qs11,s,sb) + subs(qs21,s,sb);
qBB  = qB + qB1;
qL1 = subs(qs11,s,sl) + subs(qs21,s,sl);
qLL = qB + qL1;
fprintf('qB1 = '),disp(vpa(qB1,3))
fprintf('qB(B) = '),disp(vpa(qBB,3))
fprintf('qL1 = '),disp(vpa(qL1,3))
fprintf('qL(L) = '),disp(vpa(qLL,3))
%
```

(iv) Locate the shear center for the cross-section

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% (iv) Calculating the shear center
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('------------------------------------------------\n')
fprintf('(iv) Calculation of shear centers \n')
fprintf('    Taking moments at point M  \n')
fprintf('------------------------------------------------------\n')
fprintf('----------------------------------------------------\n')
fprintf('Calculation of Zsc - q(s) in DB  due to Vy only\n')
fprintf('----------------------------------------------------\n')
ymn = 140;
dM = ymn*qs;
M = int(dM,s,0,100);
fprintf('Moment about M of distributed shear force : '),disp(vpa(M,3))
syms Zsc
solz = double(solve(Vy*Zsc - M));
```

```
fprintf('Shear center Zsc [mm] : '),disp(vpa(solz,3))

fprintf('--------------------------------------------------\n')
fprintf('Calculation of Ysc : q(s) in DB due to Vz only  \n')
fprintf('---------------------------------------------------\n')

syms y z s Vy Vz t
den = Izz*Iyy - Iyz^2;
Vy = 0;
num1 = -(Vy*Iyy - Vz*Iyz);
num2 = -(Vz*Izz - Vy*Iyz);
fprintf('Vy [N] = '),disp(Vy)
fprintf('Vz [N] = '),disp(Vz)
fprintf('-(Vy*Iy-Vz*Iyz) ='),disp(vpa(num1,3))
fprintf('-(Vz*Iz-Vv*Iyz) ='),disp(vpa(num2,3))
fprintf('Izz*Iyy - Iyz^2  ='),disp(vpa(den,3))
coef1 = num1/den;  coef2 = num2/den;
fprintf('coef1 ='),disp(vpa(coef1,3))
fprintf('coef2 ='),disp(vpa(coef2,3))
fprintf('-------------------\n')
fprintf('Segment DB: \n')
fprintf('----------------\n')

syms c d
ydb = -(yc-5); zd = 110 - zc;  zb = 10-zc;
td = 20;  tb = 10;
fprintf('s, t, y, z at D = '), disp([0,td,ydb,zd])
fprintf('s, t, y, z at B = '), disp([100,tb,ydb,zb])
eqy = y-c*s - d;
eq(1) = subs(eqy,[y,s],[ydb,0]);
eq(2) = subs(eqy,[y,s],[ydb,100]);
soly = solve(eq,c,d);
c = soly.c;
d = soly.d;
ys = c*s +d;
fprintf('Equation for y(s) = '),disp(vpa(ys,3))

syms c d
eqz = z-c*s - d;
eq(1) = subs(eqz,[z,s],[zd,0]);
eq(2) = subs(eqz,[z,s],[zb,100]);
solz = solve(eq,c,d);
c = solz.c;
d = solz.d;
zs = c*s+d;
fprintf('Equation for z(s) = '),disp(vpa(zs,3))

syms c d
eqz = t-c*s - d;
eq(1) = subs(eqz,[t,s],[td,0]);
eq(2) = subs(eqz,[t,s],[tb,100]);
solt = solve(eq,c,d);
c = solt.c;
d = solt.d;
ts = c*s+d;
fprintf('Equation for t(s) = '),disp(vpa(ts,3))
```

```
%
qs1 = coef1*int(ys*ts,s,0,s);
qs2 = coef2*int(zs*ts,s,0,s);
qs = qs1 + qs2;
fprintf('q(s) = '),disp(vpa(qs,3))
ymn = 140;
dM = ymn*qs;
M = int(dM,s,0,100);
fprintf('Moment about M of distributed shear force : '),disp(vpa(M,3))
syms Ysc
soly = double(solve(-Vz*Ysc - M));
fprintf('Shear center Ysc [mm] : '),disp(vpa(soly,3))
```

In the Command Window:

```
Example 7.15 - Unsymmetric Shear
-----------------------------------------------------------
------------------------------------------
 Preprocessing - CG, MOI of isoceles trapezoid
----------------------------------------------
Cy  = b/2
Cz  = (h*(2*a + b))/(3*(a + b))
A  = (h*(a + b))/2
Iyc = (h^3*(a^2 + 4*a*b + b^2))/(36*(a + b))
Izc = (h*(a + b)*(a^2 + b^2))/48
---------------------------
(i)  The Centroid
---------------------------
Local CG and MOI for Segment 3
---------------------------------------------
Area Segment 3 [mm^2]      :         2100
Cz from Base for segment 3 :      5
Cy at Base for segment 3   :        68.571
MOIz centroid segment 3    :         63438
MOIy centroid segment 3    :    2.3657e+06
---------------------------
Local CG and MOI for Segment 2
---------------------------------------------
Area Segment 2 [mm^2]      :         1500
Cz from Base for segment 2 :        55.556
Cy at Base for segment 2   :      5
MOIz centroid segment 2    :        31250
MOIy centroid segment 2    :    1.2037e+06
---------------------------
Local CG and MOI for Segment 1

---------------------------------------------
Area Segment 1 [mm^2]      :         1500
Cz from Base for segment 1 :      5
Cy at Base for segment 1   :     75
MOIz centroid segment 1    :      2812500
MOIy centroid segment 1    :        12500
---------------------------
Cross-section CG and MOI about CG
---------------------------------------------
```

```
z-centroid from A [mm]: 53.1
y-centroid from A [mm] : 83.24
---------------------------
(ii) MOI
---------------------------
Izz [mm^4]   :    2.0201e+07
Iyy [mm^4]   :    8.6475e+06
Iyz [mm^4]   :    2.4363e+06
-------------------------------------------------
(iii) Calculation of shear flow in segments
-------------------------------------------------------
Vy [N] =         6000
Vz [N] =      0
-(Vy*Iy-Vz*Iyz) =  -5.1885e+10
-(Vz*Iz-Vv*Iyz) =   1.4618e+10
Izz*Iyy - Iyz^2  =   1.6876e+14
coef1 =  -0.00030746
coef2 =    8.662e-05
---------------------------
Calculation q(s) in DB
------------------------
s, t, y, z at D =            0           20      -78.235       56.895
s, t, y, z at B =          100           10      -78.235      -43.105
Equation for y(s) = -78.2
Equation for z(s) = 56.9 - 1.0*s
Equation for t(s) = 20.0 - 0.1*s
q1(s) = -0.0012*s*(s - 400.0)
q2(s) = 2.83e-8*s*(102.0*s^2 - 3.93e4*s + 3.48e6)
q(s) = 2.83e-8*s*(102.0*s^2 - 3.93e4*s + 3.48e6) - 0.0012*s*(s - 400.0)
qD [N/mm] =      0
qB [N/mm]=        37.699
---------------------------
Calculation q(s) in BL
------------------------
s, t, y, z at B =            0           10      -78.235      -48.105
s, t, y, z at L =          140           10       61.765      -48.105
Equation for y(s) = s - 78.2
Equation for z(s) = -48.1
Equation for t(s) = 10.0
q1(s) = -9.04e-5*s*(17.0*s - 2666.0)
q2(s) = -0.0417*s
q(s) = - 0.0417*s - 9.04e-5*s*(17.0*s - 2666.0)
qB1 = 0.0
qB(B) = 37.7
qL1 = -2.29
qL(L) = 35.4
-------------------------------------------------
(iv) Calculation of shear centers
    Taking moments at point M
-------------------------------------------------------
-----------------------------------------------------
Calculation of Zsc - q(s) in DB  due to Vy only
-----------------------------------------------------
Moment about M of distributed shear force : 3.08e5
Shear center Zsc [mm] : 51.3
--------------------------------------------------
```

```
Calculation of Ysc : q(s) in DB due to Vz only
---------------------------------------------------
Vy [N] =      0
Vz [N] = Vz
-(Vy*Iy-Vz*Iyz) =2.44e6*Vz
-(Vz*Iz-Vv*Iyz) =-2.02e7*Vz
Izz*Iyy - Iyz^2  =1.69e14
coef1 =1.44e-8*Vz
coef2 =-1.2e-7*Vz
-------------------
Segment DB:
----------------
s, t, y, z at D =             0           20      -78.235       56.895
s, t, y, z at B =           100           10      -78.235      -43.105
Equation for y(s) = -78.2
Equation for z(s) = 56.9 - 1.0*s
Equation for t(s) = 20.0 - 0.1*s
q(s) = 5.65e-8*Vz*s*(s - 400.0) - 3.91e-11*Vz*s*(102.0*s^2 - 3.93e4*s +
3.48e6)
Moment about M of distributed shear force : -50.7*Vz
Shear center Ysc [mm] : 50.7
```

Execution in OCTAVE

The code is same as in the MATLAB except for the highlighted changes. Most of it is formatting for the better appearance in the Command Window.. The code works for most of the calculations and the changes are identified through color.

In Octave Editor

The code is same as in the MATLAB except for the highlighted changes. Most of it is formatting for the better appearance in the Command Window.

```
% OCTAVE must be able to reach python and sympy - for example
% >> setenv python C:\Users\venka\Anaconda3\python.exe
% OTHERWISE
pkg load symbolic;  % if python path is already set up

fprintf('Cy  = \n'),disp(Cy)
fprintf('Cz  = \n'),disp(Cz)
fprintf('A  = \n'),disp(A)
fprintf('Iyc = \n'),disp(Iyc)
fprintf('Izc = \n'),disp(Izc)

fprintf('q1(s) =\n '),disp(vpa(qs1,3))
fprintf('q2(s)  = \n'),disp(vpa(qs2,3))
fprintf('q(s)  = \n'),disp(vpa(qs,3))


clear num1 num2 coef1 coef2 den
syms y z s Vy Vz t num1 num2
den = Izz*Iyy - Iyz^2;
Vy = 0;
num1 = -(Vy*Iyy - Vz*Iyz);
num2 = -(Vz*Izz - Vy*Iyz);
fprintf('Vy [N] = '),disp(Vy)
```

```
fprintf('Vz [N] = '),disp(Vz)
fprintf('-(Vy*Iy-Vz*Iyz) ='),disp(vpa(num1,6))
fprintf('-(Vz*Iz-Vv*Iyz) ='),disp(vpa(num2,6))
fprintf('Izz*Iyy - Iyz^2  ='),disp(vpa(den,6))

den= sym(den);
num1
num2
coef1 = (num1/den)
coef2 = (num2/den)
fprintf('coef1 ='),disp(vpa(coef1,3))
fprintf('coef2 ='),disp(vpa(coef2,3))

fprintf('q(s) = \n'),disp(vpa(qs,3))
```

In Octave Command Window

There is calculation error which I cannot debug in step (iv) . For some reason the calculation of variables coef1 and coef2 is corrupted. It is happening during a simple division. I have tried to correct this by clearing the variables but it was not useful. I am hoping one of you will help resolve this error.

```
Example 7.15 - Unsymmetric Shear
-------------------------------------------------------------
-------------------------------------------
 Preprocessing - CG, MOI of isoceles trapezoid
-----------------------------------------------
Cy  =
  b
  -
  2
Cz  =
  h*(2*a + b)
  -----------
   3*(a + b)
A  =
  h*(a + b)
  ---------
      2
Iyc =
   3 / 2            2\
  h *\a  + 4*a*b + b /
  --------------------
       36*(a + b)
Izc =
            / 2    2\
  h*(a + b)*\a  + b /
  -------------------
           48
---------------------------
(i)  The Centroid
---------------------------
Local CG and MOI for Segment 3
-----------------------------------------------
Area Segment 3 [mm^2]      : 2100
Cz from Base for segment 3 : 5
Cy at Base for segment 3   : 68.571
```

```
MOIz centroid segment 3    : 63438
MOIy centroid segment 3    : 2.3657e+06
---------------------------
Local CG and MOI for Segment 2
----------------------------------------------
Area Segment 2 [mm^2]      : 1500
Cz from Base for segment 2 : 55.556
Cy at Base for segment 2   : 5
MOIz centroid segment 2    : 31250
MOIy centroid segment 2    : 1.2037e+06
---------------------------
Local CG and MOI for Segment 1

----------------------------------------------
Area Segment 1 [mm^2]      : 1500
Cz from Base for segment 1 : 5
Cy at Base for segment 1   : 75
MOIz centroid segment 1    : 2.8125e+06
MOIy centroid segment 1    : 12500
---------------------------
Cross-section CG and MOI about CG
----------------------------------------------
z-centroid from A [mm]:   53.10
y-centroid from A [mm] :   83.24
---------------------------
(ii) MOI
---------------------------
Izz [mm^4]   : 2.0201e+07
Iyy [mm^4]   : 8.6475e+06
Iyz [mm^4]   : 2.4363e+06
-------------------------------------------------
(iii) Calculation of shear flow in segments
-------------------------------------------------------
Vy [N] = 6000
Vz [N] = 0
-(Vy*Iy-Vz*Iyz) =-5.1885e+10
-(Vz*Iz-Vv*Iyz) =1.4618e+10
Izz*Iyy - Iyz^2  =1.6876e+14
coef1 =-0.00030746
coef2 =8.662e-05
---------------------------
Calculation q(s) in DB
------------------------
s, t, y, z at D =   0  20  -78.235  56.895
s, t, y, z at B =   100  10  -78.235  -43.105
Equation for y(s) =   -78.2
Equation for z(s) =   -s + 56.9
Equation for t(s) =   -0.1*s + 20.0
q1(s) =
             2
  - 0.0012*s  + 0.481*s
q2(s) =
           3            2
  2.89e-6*s  - 0.00111*s  + 0.0986*s
q(s) =
           3            2
```

```
  2.89e-6*s  - 0.00232*s  + 0.58*s
qD [N/mm] = 0
qB [N/mm]= 37.699
---------------------------
Calculation q(s) in BL
-----------------------
s, t, y, z at B =   0  10  -78.235  -48.105
s, t, y, z at L =   140  10  61.765  -48.105
Equation for y(s) =   s - 78.2
Equation for z(s) =   -48.1
Equation for t(s) =   10.0
q1(s) =              2
  - 0.00154*s  + 0.241*s
q2(s) =   -0.0417*s
q(s) =              2
  - 0.00154*s  + 0.199*s
qB1 =   0
qB(B) =   37.7
qL1 =   -2.29
qL(L) =   35.4
-------------------------------------------------
(iv) Calculation of shear centers
    Taking moments at point M
-------------------------------------------------------
-----------------------------------------------------
Calculation of Zsc - q(s) in DB  due to Vy only
-----------------------------------------------------
Moment about M of distributed shear force :   3.08e+5
Shear center Zsc [mm] :   51.3
--------------------------------------------------
Calculation of Ysc : q(s) in DB due to Vz only
---------------------------------------------------
Vy [N] = 0
Vz [N] =   Vz
-(Vy*Iy-Vz*Iyz) =  2.43627e+6*Vz
-(Vz*Iz-Vv*Iyz) =  -2.02013e+7*Vz
Izz*Iyy - Iyz^2  =  1.68756e+14
num1 = (sym)
       _____________
  12*\/ 41218288105 *Vz   This is correct
num2 = (sym)
     _________________
  -\/ 408092729644585 *Vz
den = 1.6876e+14
coef1 = (sym)
       ______________________________
  12*\/ 77586042023404250460838750 15 *Vz
  ------------------------------------        This is wrong
           131762457669353940 1      This is a simple division num1/den
coef2 = (sym)
     __________________________________
  -\/ 768161443070939040844457748276 55 *Vz
  -------------------------------------------
             131762457669353940 1
coef1 =  0.000802*Vz
coef2 =  -0.00665*Vz        This is wrong
```

```
--------------------
Segment DB:
----------------
s, t, y, z at D =   0  20  -78.235  56.895
s, t, y, z at B =   100  10  -78.235  -43.105
Equation for y(s) =   -78.2
Equation for z(s) =   -s + 56.9
Equation for t(s) =   -0.1*s + 20.0

q(s) =
              /      2            \             /          3          2
  0.000802*Vz*\3.91*s  - 1.56e+3*s/ - 0.00665*Vz*\0.0333*s  - 12.8*s  +
1.14e+3*

   \
  s/  Not sure what this is?
Moment about M of distributed shear force :   -2.82e+6*Vz
soly =
  0
  2.8194e+06

Shear center Ysc [mm] :   [   0   ]
  [       ]
  [2.82e+6]   This is wrong
```

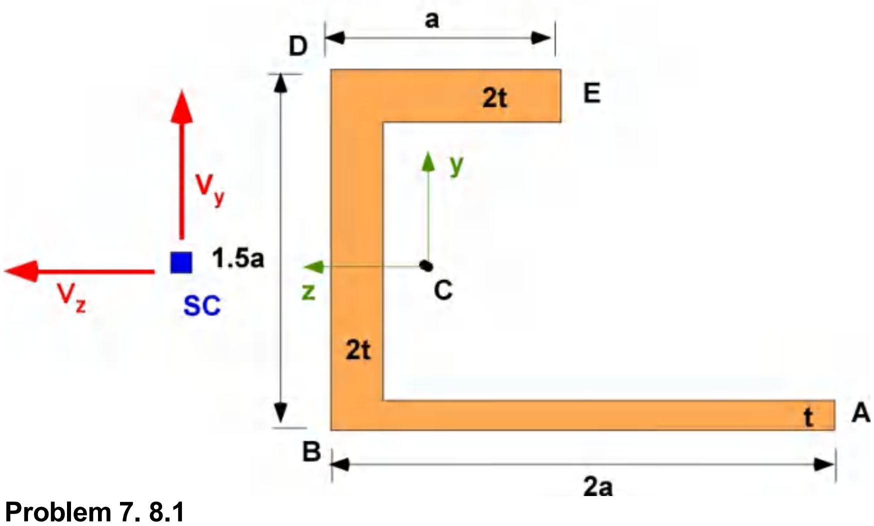
7.8.6 Additional Problems

Please consider thin-wall assumptions if appropriate. Solve on paper and MATLAB/Octave

The steps for solving the problem are:

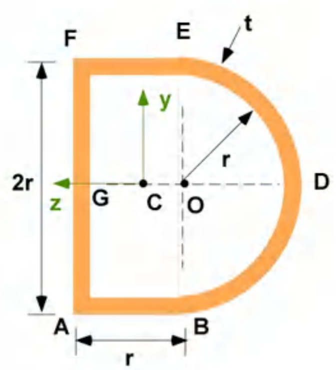
(i) Calculate the values symbolically
(ii) Establish the centroid
(iii) Calculate the MOI about the centroidal axes
(iv) Calculate the shear flow distribution across the cross-section and establish values at the lettered points on the figure.
(v) Calculate the shear center of the cross-section
(vi) Assume your own set of numerical values for the constants and evaluate items (ii) through (v) above
(vii) Determine maximum load you can apply to your cross-section for your choice in (vi) and aluminum alloy

Problem 7.8.1: The cross-section and loading is indicated in the figure Problem 7.8.1



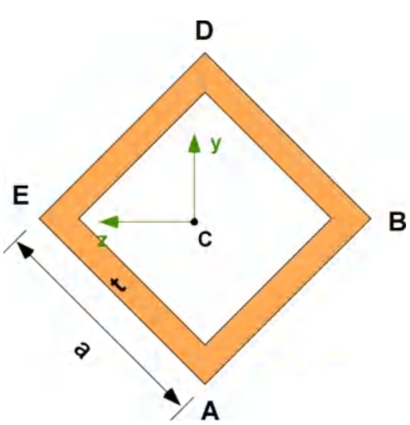
Problem 7. 8.1

Problem 7.8.2: The cross-section is a D-section. This problem will require imagination and originality . Assume the load $\mathbf{V_y}$ is applied at the shear center. Where is the shear center?



Problem 7.8.2

Problem 7.8.3: This should be simple since it is a symmetric section. This exercise should allow you to calculate shear flow in inclined sections.. Assume a shear load $\mathbf{V_y}$ at the shear center. It is a diamond section.



Problem 7.8.3

7.9 BENDING OF BEAMS MADE FROM MULTIPLE MATERIALS

The discussion on bending previously focused on a single material. The analysis of the beam made of a **single material** is the starting point for the analysis of the beams made of **multiple materials**. The corner stone of the analysis is that a beam made of multiple material must deflect the same way as a beam of a single material in pure bending since it must handle the bending moment while remaining an integrated structure. However the mathematical relations are adjusted to incorporate multiple materials. Figure 7.9.1 summarizes the bending analysis of a beam of single material.

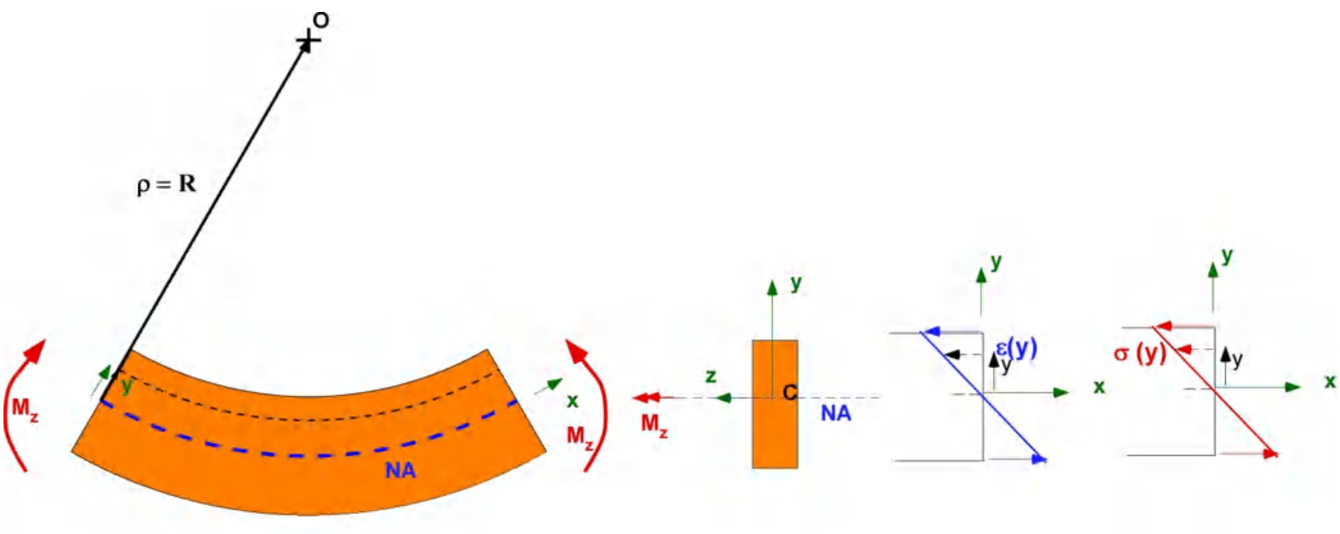


Figure 7.9.1 Pure bending of a beam of a single material.

The bending deflection is an arc of a circle. There is a neutral axis (NA) which experiences zero strain. There is a linear distribution of strain along the y-axis with compressive strain above the NA and tensile strains below. This depends on the direction of the applied bending moment. This translates to a corresponding linear distribution of stress since they are designed to remain below the elastic limit. The NA lies on the axis of symmetry. The analysis is made through the relations in

$$\varepsilon(y) = \frac{-y}{R}; \quad \sigma(y) = E\varepsilon(y); \quad \frac{1}{R} = \frac{M_z}{EI_z}; \quad \sigma(y) = \frac{M_z(-y)}{I_z}; \quad (7.36)$$

Figure 7.9.2 describes a beam of two materials, marked as 1 and 2, with different properties. The cross-section is kept simple with both materials having the same area. The bending deflection is an arc of a circle. There is a neutral axis (NA) which experiences zero strain. There is a linear distribution of strain along the y-axis with compressive strain above the NA and tensile strains below.

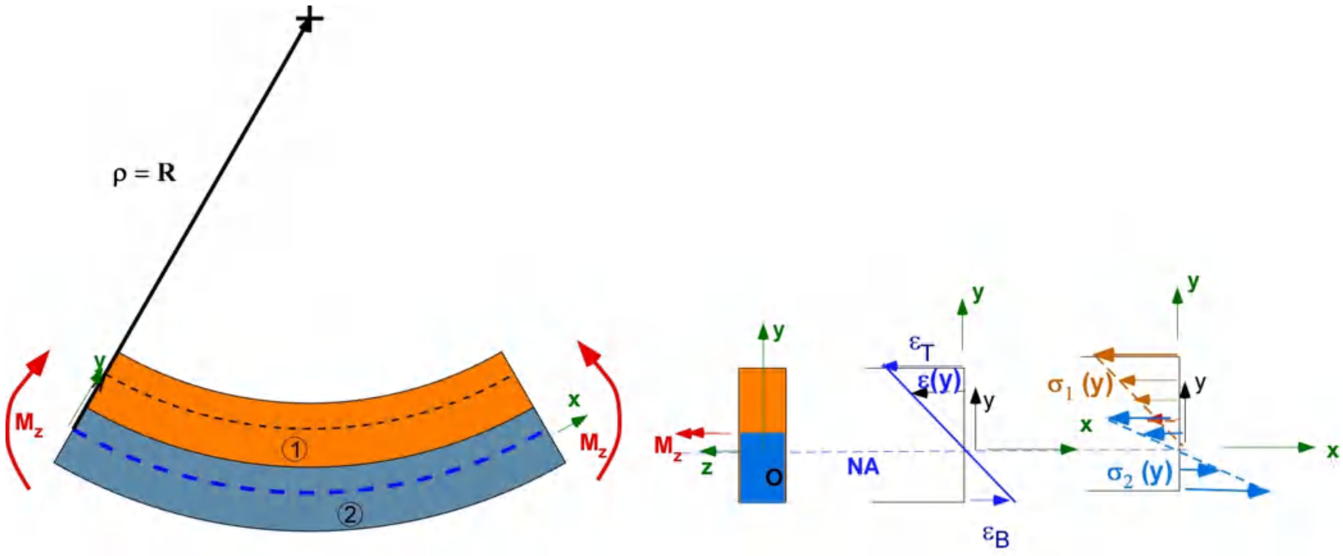


Figure 7.9.2 Pure bending of a beam made of two materials.

The location of the neutral axis NA is unknown. The properties of the two materials should influence its determination as that is the only difference between Figure 7.9.1 and 7.9.2. The stress distribution is linear with a different slope for each material. There is a jump in the stresses at the interface of the two materials even if the strain must be continuous. This section includes simple introductory information on this topic.

7.9.1 Determining the Neutral Axis

Consider the case that the modulus of elasticity for material 2 is greater than the modulus of elasticity for material 1.

$$\frac{E_2}{E_1} = n > 1$$

Consider an elemental area dA in the cross-section. Consider the force on this area in both materials.

$$dF_1 = \sigma_1 \, dA = E_1 \, \varepsilon_1 \, dA = E_1 \left(\frac{-y}{R} \right) dA$$

$$dF_2 = \sigma_2 \, dA = E_2 \, \varepsilon_2 \, dA = E_2 \left(\frac{-y}{R} \right) dA = (n \, E_1) \left(\frac{-y}{R} \right) dA = E_1 \left(\frac{-y}{R} \right) (n \, dA)$$

The force on material 2 for the elemental area dA is the same as the force on material 1 if it the area is scaled by the factor n. The cross-section then can be reconfigured to be made of material 1 alone. This is done by replacing material 2 of area A_2 by material 1 with an area of **n A_2**. The area must be scaled in a direction parallel to the NA in the cross-section. In this case it will be parallel to the z-axis. This is illustrated in Figure 7.9.3. With this change in the cross-section the new **centroid** is determined along with the moment of inertia about this centroid, $\mathbf{I_z}$. The centroid locates the NA. The stress is based on **y** measured from this neutral axis. If y lies in material 1 the stress is the based on the equation in Figure 7.9.3. If y lies in material 2 then we multiply the corresponding stress by 'n'.

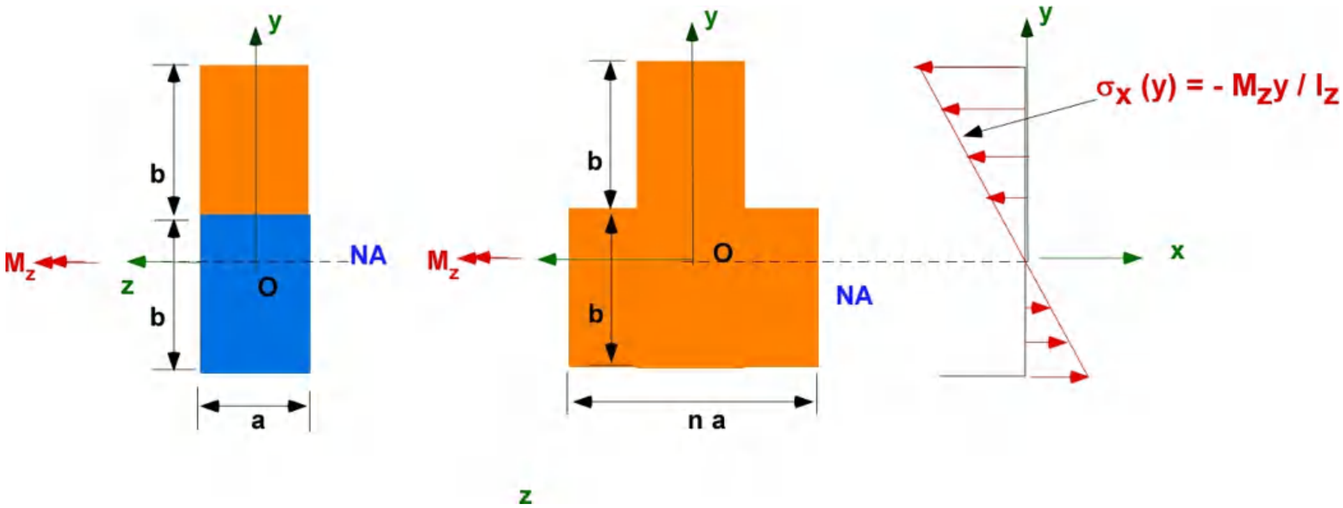


Figure 7.9.3 Equivalent beam cross-section

Therefore the relations for the beam with two materials:

$$\frac{1}{R} = \frac{M_z}{E_1 I_z} \tag{7.37}$$

$$\sigma_1(y) = -\frac{M_z y}{I_z}; \quad \sigma_2(y) = -\frac{n M_z y}{I_z};$$

7.9.2 Example 7.16

In this example we will work through a problem similar to the illustration in Section 7.9.1. The example should be simple. The cross-section is made up of two materials side by side. Material 1 is brass with a modulus of elasticity of 105 GPa. Material 2 is steel modulus of elasticity 200 GPa. The cross-section of the two materials are the same. The bending moment M_z is 5000 Nm. What is the maximum stress in brass and steel?

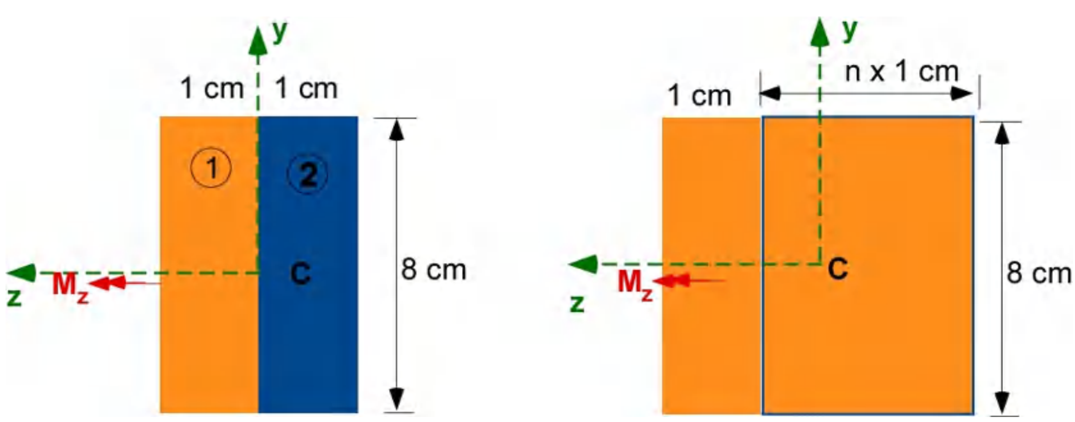


Figure 7.9.4 Example 7.16 - original and modified to be of a single material

Data: Cross-section is shown in Figure 7.9.4 It is made of two materials. Both have the same area and dimensions.
M_z = 5000 Nm. E_1 = 105 GPa; E_2 = 200 GPa

Assumption: Multi material bending

Solution: Steps for Solution

(a) Calculate the ration of the modulus - n
(b) Adjust the cross-section to be made of one material (preferably the lower value of E)
(c) Identify the centroid and MOI
(d) Determine the maximum distance to the outer fiber to calculate maximum stresses

Step (a): $\frac{E_2}{E_1} = 1.9048$

Step (b): $w_{steel} = 0.01905\,[m];\quad b = 0.02905\,[m]$

Step (c): Centroid does not change in y direction

$$I_{z} = 1.2394 \times 10^{-6}\left[m^4\right]$$

Step (d): d_{max} = 0.04 m
Stress (max) in brass [MPa] = 161.37
Stress (max) in steel [MPa] = 307.38

You should now be able to create your own MATLAB code for this problem.

7.9.3 Additional Problems

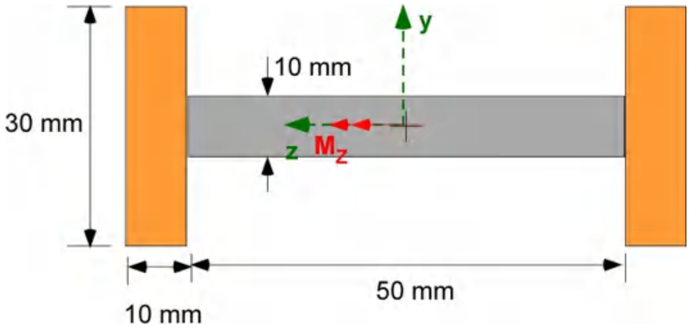
For the problems below you can vary the combination of materials for the same geometry (assuming they can be bonded as shown). Consider that the gray material has a higher modulus of elasticity than the orange material.

Table 7.2

Material	Aluminum	Brass	Steel	Wood
E [GPa]	70	105	200	13
Yield stress [MPa]	230	410	250	60
Ultimate Stress (tension) [MPa]	300	500	450	100
Ultimate stress (shear) [MPa]	70	200	250	10

Problem 7.9.1

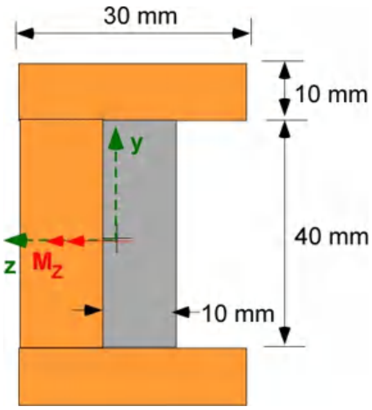
For any combination of two materials calculate the maximum bending moment M_z that can be applied.



Problem 7.9.1

Problem 7.9.2

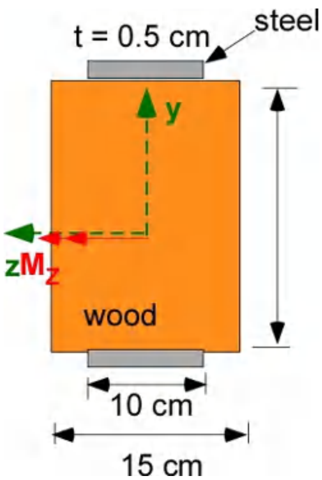
For any combination of two materials calculate the maximum bending moment M_z that can be applied.



Problem 7.9.2

Problem 7.9.3

Calculate the maximum bending moment M_z that can be applied. Reinforcing the wooden cross-section using a steel plate.



Problem 7.9.3

7.10 PLASTIC DEFORMATION IN BENDING

The criteria for design until now required that the stresses in the material be within the elastic limit so that when the load is removed the material recovers its original shape. Another way this criteria was assessed was that the defections were small so that strains were small and therefore stress remained within the elastic limit. The materials used always had a linear stress strain behavior. This linearity was important in making the calculations easy. This was an important idealization. This was true for most metallic structural materials. Today researchers are inventing new material where linear behavior may not be standard, especially with micro- and nano- materials. Foam is a structural material today. Sandwich construction and laminated construction are very prevalent in design. Another new category is meta materials - elements made from composites that include metals or plastics - engineered to have properties that are not natural from our exposure of the course so far. Sometimes plastic deformation is necessary to absorb energy.

Plastic deformation is best described by working hard on the chewing gum and then pulling it when held between your teeth. You should notice that the gum stretches even when you do not increase the force. This is unlike elastic deformation where the displacement depends on the force. Plastic deformation is the increase in displacement without any corresponding increase in the force that causes the displacement. This can be translated equivalently to stress strain. Plastic strain can occur without increase in stress. Once the force is removed some of the displacement is still present. This is called residual strain or permanent set. This means that the object has lengthened and therefore changed shape. The design ideas till now did not permit this change. Permitting the change will account for energy absorption. Here are some important considerations for design:

Mild steel is durable. It can handle cyclic loading - loading and unloading cycles
Rubber is elastic but does not have a linear stress strain curve
Kevlar is tough. It can withstand impact
Glass is brittle. The stress strain curve is linear but it breaks suddenly at the end

In general plastic deformation is a nonlinear phenomena and that should give us pause because nonlinear problems are difficult to solve. One approach to make the analysis tractable is to use idealized stress-strain curves. Until this section the material behavior was idealized as "ideally elastic". Figure 7.10.1 indicates some idealized stress-strain behavior. Figure 7.10.2 illustrates the real and the idealized behavior. This section only includes some introductory discussion on this topic.

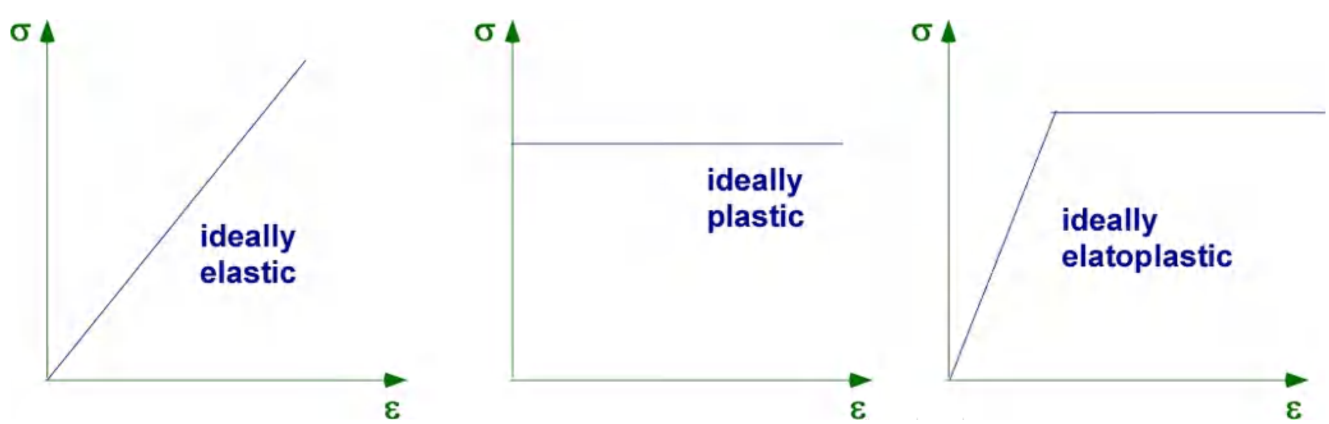


Figure 7.10.1 Idealized stress-strain behavior

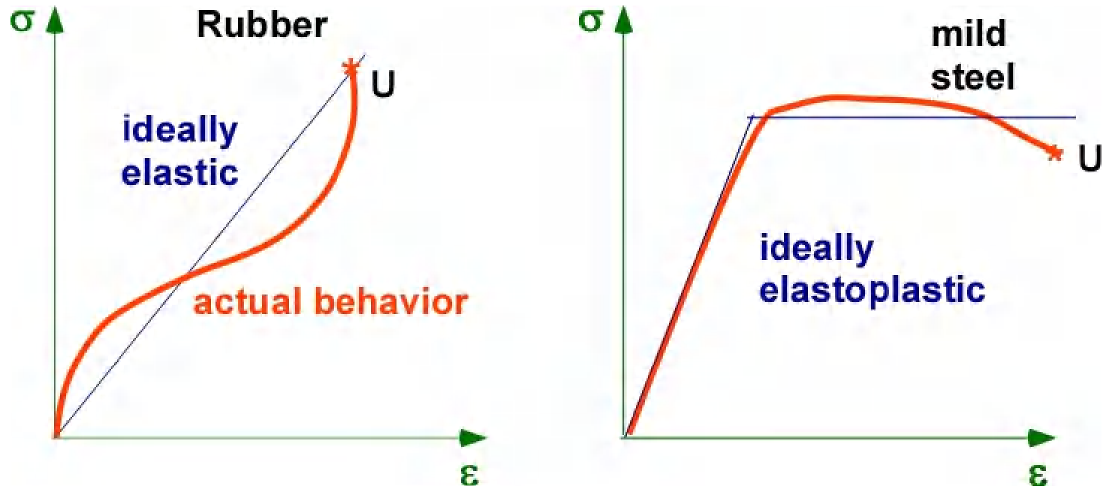


Figure 7.10.2 Real and idealized behavior

7.10.1 Nonlinear Stress-Strain in Pure Bending

The description of pure bending described in an earlier section is reproduced in Figure 7.10.3. The bending deformation is the strain distribution ε(y) and this is available through the relations in (7.8). At this stage there were no assumptions made on the material property except that bending was an arc of a circle. The elasticity or the plasticity of the material were not part of the discussion. The maximum stresses were at the top and the bottom with the distance being measured from the neutral axis(NA).

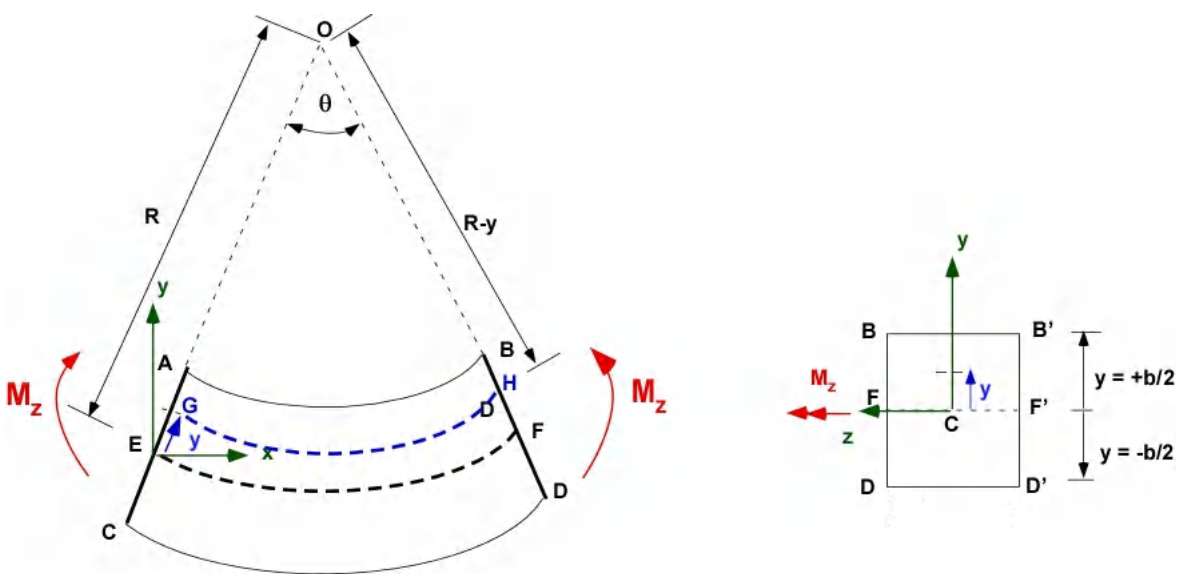


Figure 7.10.3 Pure bending

$$\delta = GH - EF = (R-y)\theta - R\theta = -y\theta$$

$$\varepsilon(y) = \frac{\delta}{L} = \frac{-y\theta}{R\theta} = \frac{-y}{R} \qquad (7.8)$$

$$\varepsilon_B = \frac{-b}{2R}; \quad \varepsilon_D = \frac{+b}{2R}$$

It is fair to conclude that the strain distribution is still linear for plastic deformation - since material property is not involved. However for design the maximum strain or the strain at the top/bottom must be specified.

$$\varepsilon_x(y) = -\frac{y}{b/2}\varepsilon_B = -\frac{y}{b/2}\varepsilon_m \quad (7.38)$$

What does not work anymore is that the NA passes through the centroid of the section. This will be true for a special case where the cross-section is symmetrical along both axis and the stress-strain behavior is the same for tension and compression. For the stresses to be symmetrical the NA will coincide with the axis of symmetry

For the general case the location of the NA is by trial and error dictated by the satisfaction of equilibrium using the actual stress strain behavior as indicated in Figures 7.10.4 and relating it to the applied moment. It is helpful if the stress strain behavior can be defined through an analytical function.

$$\sigma_x = f(\varepsilon_x); \quad M_z = \int_A -y\sigma_x \, dA$$

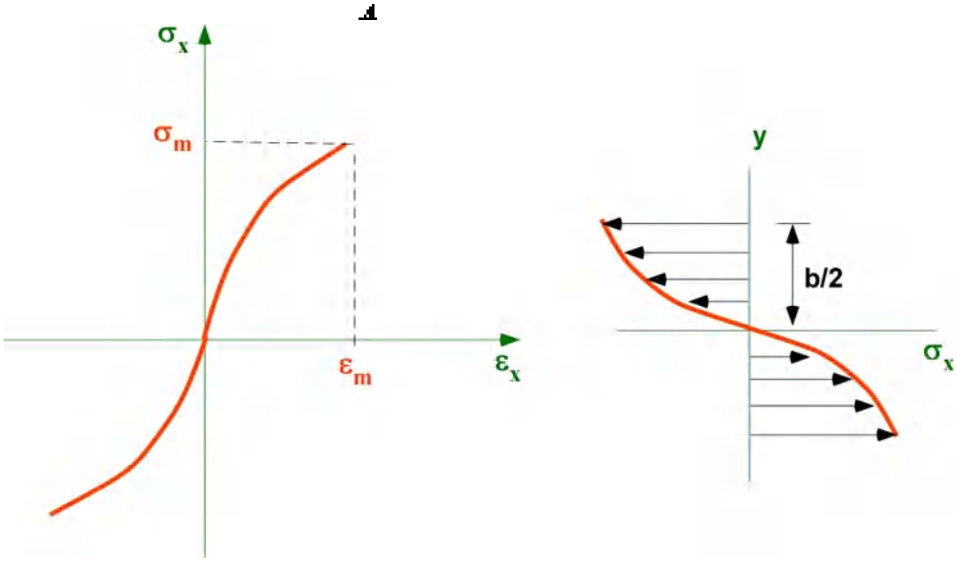


Figure 7.10.4 Nonlinear stress behavior and distribution in cross-section

7.10.2 Ideal Elastoplastic Material

Actual plastic behavior requires actual plastic properties that is not guaranteed because of differences in loading, manufacturing process, and material properties. A standard approach is to use ideal material behavior to develop design information. A common analysis of plastic behavior is based on the ideal elastoplastic material. Pure bending of the beam made of this material is likely to involve one of the four following scenarios which are shown in Figure 7.10.5.

(1) Maximum stress in the cross-section is less than the yield stress (σ_Y)

(2) Maximum stress in the cross-section is the yield stress. This was the design consideration for elastic behavior of materials.

(3) The outer areas of the cross-section are plastic while the material close to the NA is elastic.

(4) The entire cross-section is plastic.

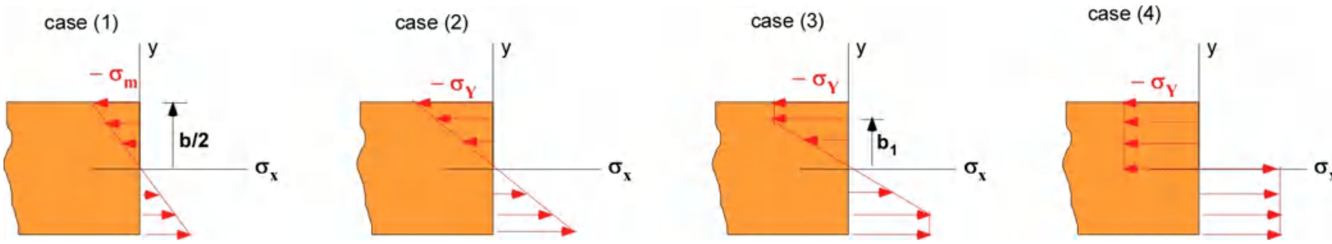


Figure 7.10.5 Possible stress distribution for elastoplastic behavior

In Figure 7.10.5 the linear variation of stress is the elastic region. The constant stress region is the plastic region. Cases (1) and (2) are elastic. For case (1) the maximum stress is less than the elastic limit. Case (3) is partly elastic and partly plastic. Case (4) is completely plastic.. The bending moment for the corresponding stress distribution can be obtained if the geometry of the cross-section is known. For simplicity the cross-section is rectangular with height b and width a. The area is **A** = ab. The moment of inertia **I** is $ab^3/12$. The general relation between the applied bending moment and the stress distribution can be calculated through

$$M = -2\int_{b_1}^{0} y\left(-\frac{\sigma_Y}{b_1}y\right)a\,dy - 2\int_{b/2}^{b_1} y\left(-\sigma_Y\right)a\,dy$$

This relation is most useful for case (3). For other cases you need only one of the terms. In the limit the plastic moment (**M_P**) is about **1.5 M_Y** for this geometry. **M_Y** is the maximum elastic moment - case (2).

7.10.3 Example 7.17

A simple inverse problem is to specify the moment in the cross-section and discover the extant of the elastic and the plastic regions. A beam of rectangular cross-section of height 100 mm and width 40 mm is subject to a bending moment of 20000 Nm. The E for the material is 70 GPa and the yield stress (σ_Y) is 250 MPa. Calculate the extant of the elastic region if any.

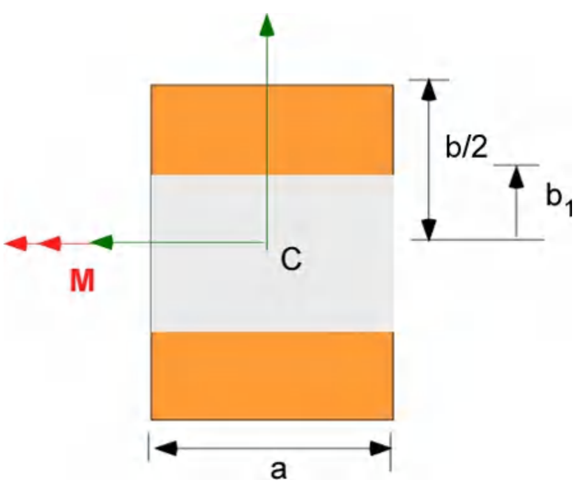


Figure 7.10.6 Example 7.17

Data: Cross-section is shown in Figure 7.10.6 The cross-section is rectangular.
M_z = 20000 Nm. E = 70 GPa; σ_Y = 250 MPa

Assumption: elastoplastic behavior

Solution:

The steps for solving the problem are:

(i) Calculate elastic Moment M_Y

(ii) Calculate plastic Moment M_P

(iii) If given M is between M_Y and M_P there is elastoplastic deformation. Calculate using the integral relation above.

Step (i) The maximum elastic moment MY can be established from

$$M_Y = \frac{\sigma_Y I}{(b/2)} = 16667 [Nm]$$

Step (ii) For this geometry the plastic moment is

$$M_P = 1.5 M_Y = 25000 [Nm]$$

Step (iii) The given moment is between the elastic and plastic region suggesting elastoplastic behavior. We can solve for $\mathbf{b_1}$ form solving the integral

$$20000 = -2\int_0^{b_1} y\left(-\frac{250\times 10^6}{b_1} y\right) 0.04\, dy - 2\int_{b_1}^{b} y\left(-250\times 10^6\right) 0.04\, dy$$

$$b_1 = 0.0387 [m]$$

These are large values for the moment. To decrease the moments a FOS should be assumed

7.10.4 Cross-section with a single axis of symmetry.

The discussion of plastic deformation so far has involved a cross-section with two axes of symmetry The stress distribution had the same magnitude at the points which were at the same distance with respect to the centroid, though of opposite signs. The symmetry allowed us to consider elastic and plastic regions in the cross-section. For the cross-section showed with a single axis of symmetry in Figure 7.10.7 the bottom surface will yield first. As the moment is increased the plastic region will spread from the bottom while the top region may remain elastic. This asymmetric behavior is difficult to constrain through the ideally elastic - ideally plastic model for the material. Once some of the material is in the plastic region further deformation is does not require additional stress.

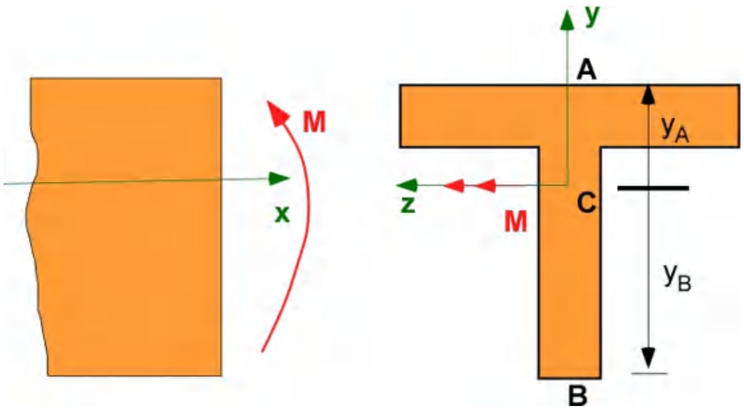


Figure 7.10.7 Cross-section with a single axis of symmetry

$$\sigma_B = \frac{M\, y_B}{I} > \sigma_A = \frac{M\, y_A}{I}$$

The question is whether the region between C and A can be plastic? This is beyond the scope of our discussion. Nevertheless for design information it is possible to analyze the cross-section if the entire cross-section is completely plastic with constant yield stress. This is considered in a problem below.

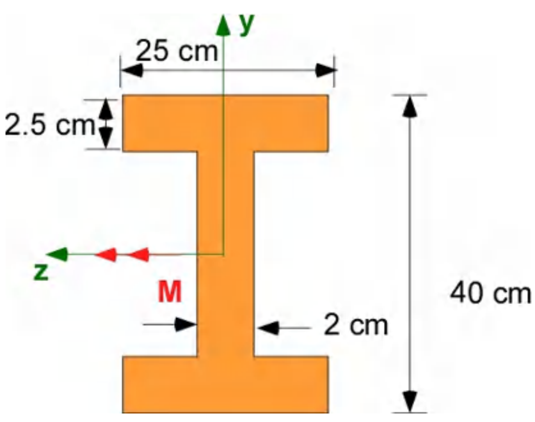
7.10.5 Additional Problems

Consider the material has an elastoplastic behavior in all problems below.

Problem 7.10.1

The material (mild steel) of the I beam shown below is idealized to behave as an elastoplastic material. The beam is symmetric along both axis. Calculate the bending moment when

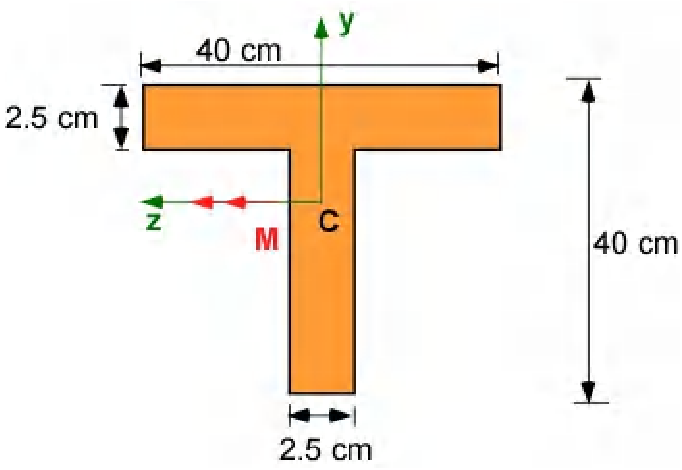
(a) The outer fibers of the beam are at the yield stress;
(b) The flange is fully plastic but the web is elastic
(c) The cross-section is plastic



Problem 7.10.1

Problem 7.10.2

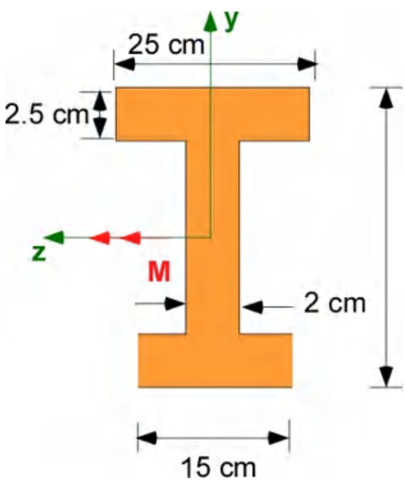
Consider the 'T' cross-section shown below. Choose a material for the beam. (a) Calculate the applied bending moment when the yield stress first appears in the cross-section. (b) Calculate the bending moment when the entire cross-section is experiencing the yield stress.



Problem 7.10.2

Problem 7.10.3

Consider the ‘T’ cross-section shown below. Choose a material for the beam. (a) Calculate the applied bending moment when the yield stress first appears in the cross-section. (b) Calculate the bending moment when the entire cross-section is experiencing the yield stress.



Problem 7.10.3

8. TORSION IN SHAFTS

Torsion was introduced in Section 3.10. While the section was introductory it did encompass many useful ideas and relations that are the basis of most problems. It is worthwhile revisiting the definitions and becoming familiar with the relations. Torsional design likes a circular cross-section. From statics, the torque (T) is due to the integration of moment due to the distribution of the shear stress from its location from the center (r) as shown below.

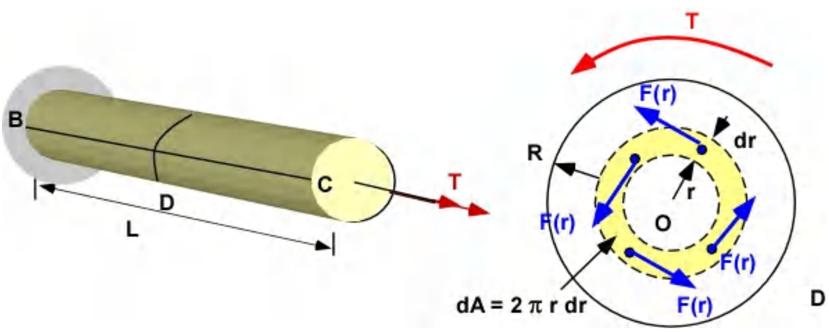


Figure 8.1 The mechanics of torsion

In this illustration the shear stress is only a function of its radial location from the center. While not shown you can see that it is not helpful to have subscripts on the shear stress.

$$F(r) = \tau(r)\,dA$$
$$dT(r) = r \times F(r) = r\,F(r) = \tau(r)\,r\,dA \qquad (8.1)$$
$$T = \int_0^R dT(r) = \int_0^R \tau(r)\,r\,dA$$

These relations are the same in in Section 3.10. The torsional deformation or the twist (ϕ) of the section is defined through

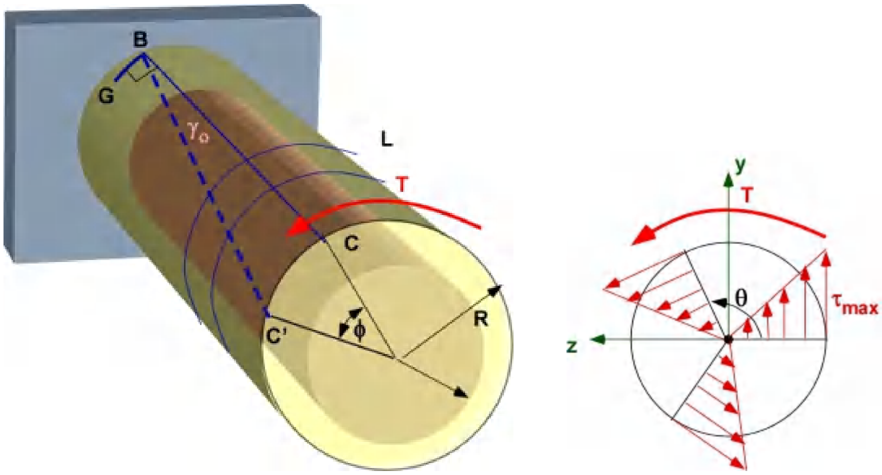


Figure 8.2 Torsional elastic deformation

Note in this illustration the end at B has no twist deformation because of the fixed boundary condition. Nevertheless the relations can be considered relative between the ends.

$$\tan \gamma_o \simeq \gamma_o = \frac{CC'}{L} = \frac{R\phi}{L}$$
$$\tan \gamma \simeq \gamma = \frac{FF'}{L} = \frac{r\phi}{L} \qquad (8.2)$$
$$\gamma = \gamma_o \frac{r}{R}$$

Since the shear stress is assumed linearly distributed in the cross-section with a maximum at the outside diameter (τ_{max}).

$$\tau(r) = G\gamma = \frac{G\gamma_o}{R} r = \tau_o \frac{r}{R}; \quad \tau_o = \tau_{max}$$
$$T = \int_0^R \tau_{max}\, r\, r\, dA = \frac{\tau_{max}}{R}\int_0^R r^2 dA = \frac{\tau_{max} J}{R}; \quad J = \int_0^R r^2 dA \qquad (8.3)$$

G is the modulus of rigidity of the material, J is the polar moment of inertia, and γ is the shear strain. Elastic deformation is also assumed. Finally

$$\tau_{max} = \frac{TR}{J} = G\gamma_{max} = G\frac{R\phi}{L}$$

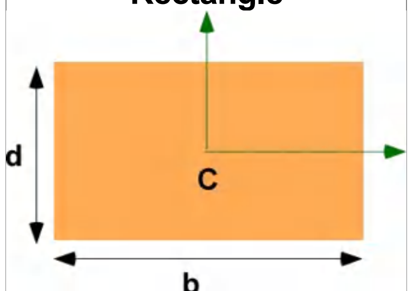
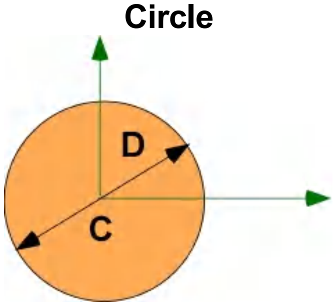
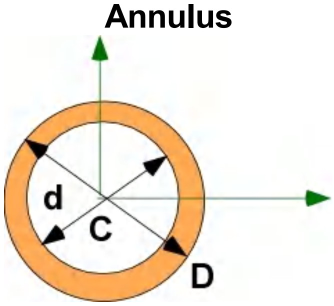
$$\phi = \frac{TL}{GJ}; \quad \textit{is not a function of } r \qquad (8.4)$$

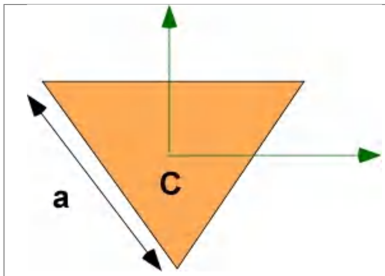
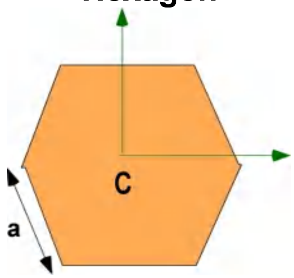
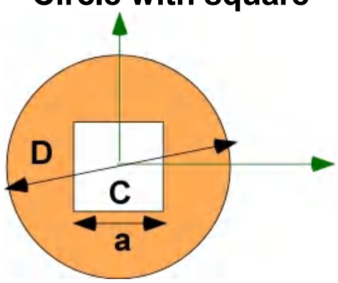
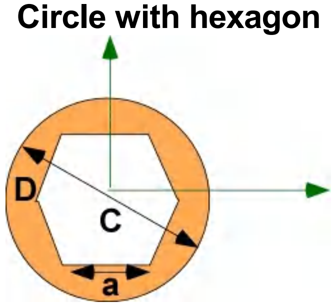
$$\gamma_{max} = \frac{R\phi}{L}$$

$$\tau(r) = \frac{Tr}{J} = G\gamma(r)$$

Eqn. (8.1) - (8.4) is the set of relations for elastic one-dimensional torsion. It is dependent on geometry and the corresponding polar moment of inertia. Some relations for J for regular geometry are included in Table 8.1.

Table 8.1 Polar Moment of Inertia for Some Sections

Section	**Polar MOI (J)**
Rectangle 	$\frac{b\,d\left(b^2+d^2\right)}{12}$
Circle 	$\frac{\pi D^4}{32}$
Annulus 	$\frac{\pi}{32}\left(D^4-d^4\right)$
Triangle	

<table>
<tr>
<td>
</td>
<td>$$\frac{\sqrt{3}}{48}a^4$$</td>
</tr>
<tr>
<td>Hexagon

</td>
<td>$$\frac{5\sqrt{3}}{8}a^4$$</td>
</tr>
<tr>
<td>Circle with square

</td>
<td>$$\frac{\pi D^4}{32}-\frac{a^4}{6}$$</td>
</tr>
<tr>
<td>Circle with hexagon

</td>
<td>$$\frac{\pi D^4}{32}-\frac{5\sqrt{3}}{8}a^4$$</td>
</tr>
</table>

8.1 EQUILIBRIUM IN TORSION

The introductory information on torsion presented in this book was focused on the stress and displacement due to an applied torque on an object. This determines the torsional response of the object for the design. But there are simpler problems to consider before the discussion of design. These problems include equilibrium of the body in torsion as well torque transmission that allow us to determine torque distribution in the mechanism. These torques are the applied loads that must be carried without failure. Since the relations are simple these topics are introduced through examples.

Torsional loading on a single shaft is a one-dimensional problem. The load is a torque and the shaft must be in equilibrium throughout. Example 8.1.1 provides a simple example of a stepped shaft in equilibrium.

8.1.1 Example 8.1

The shaft AD is subject to a torque T at the end A. It is held fixed at end B. What is the torque at the end D? And at a point C along the shaft.

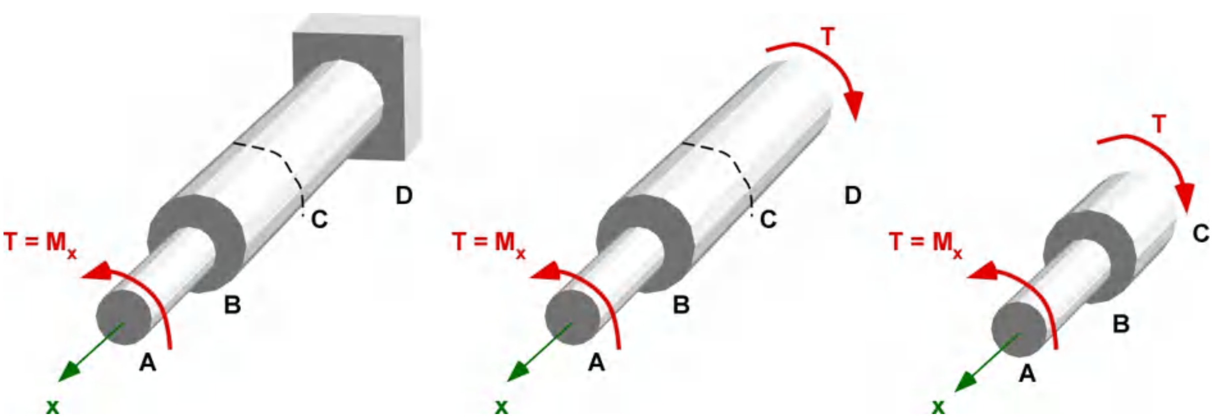


Figure 8.1.1a Example 8.1

Solution:
Data: As shown

Find: (a) Torque at D. (b) Torque at an arbitrary point C.

Assumption: One-dimensional problem. Applied torque T along x (also M_x)

Analysis:

(a) Draw FBD of AD. This is the figure in the center.

For equilibrium : $\sum M_x = 0$

Therefore torque at D is equal and opposite to torque at A.
This is true even if the cross-section has changed

(b) Draw FBD AC. This is the figure on the right. For equilibrium

$$\sum M_x = 0$$

Therefore torque at C is equal and opposite to torque at A.

The same torque exists along every section of the shaft. The stress distributions in the cross-section will be different in AB and BD

Figure 8.1.1b is the exact representation of Figure 8.1.1a with the torsional moment represented by double arrows. With this representation the one-D torsional equilibrium equation is similar to the force balance equation for equilibrium. This representation is often handy and will also be used in further discussion.

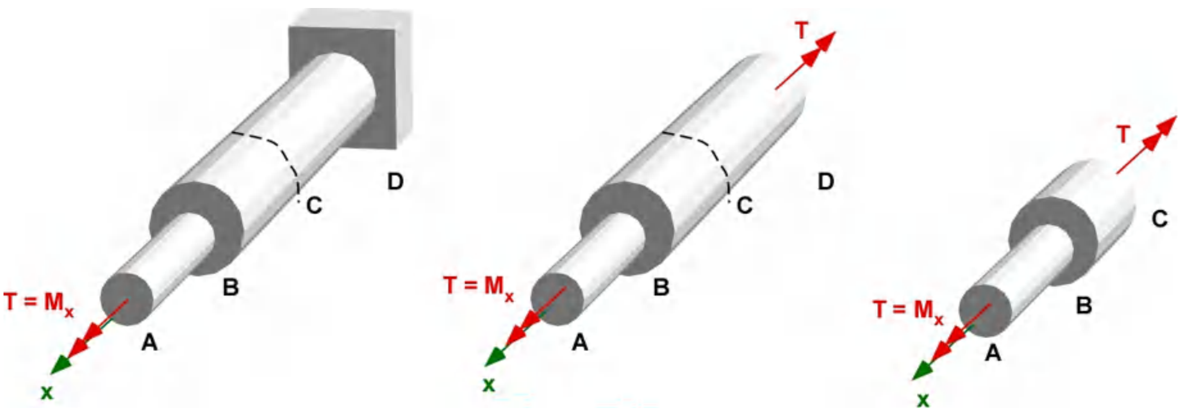


Figure 8.1.1b.1b An equivalent representation of Example 8.1

8.1.2 Example 8.2

In this example we have a single axis that is subject to different torsional moments along its length through a pulley system. The end D holds the shaft rigid or is a the fixed end. The pulleys are a mechanism to provide the torsional moment. The shaft between them are the design elements and are analyzed for stress and deflection. The shafts are solid. The material is Aluminum 6061 - T6 alloy for the shafts. The shear yield stress (τ_y) is 130 [MPa] and the modulus of rigidity (**G**) is 28 [GPa].The information for the problem is available in Table 8.2. The figure describing the problem is in Figure 8.1.2a and the FBD of the shafts between the pulleys is in Figure 8.1.2b. Calculate the torque carried by each shaft, the maximum stresses, the twist in the shafts, and the FOS.

Table 8.2 Information for Example 8.2

Pulley	Torque [Nm]	Diameter of shaft [mm]	Length of shaft [m]
A	50	15	0.6
B	100	20	0.5
C	75	25	0.7

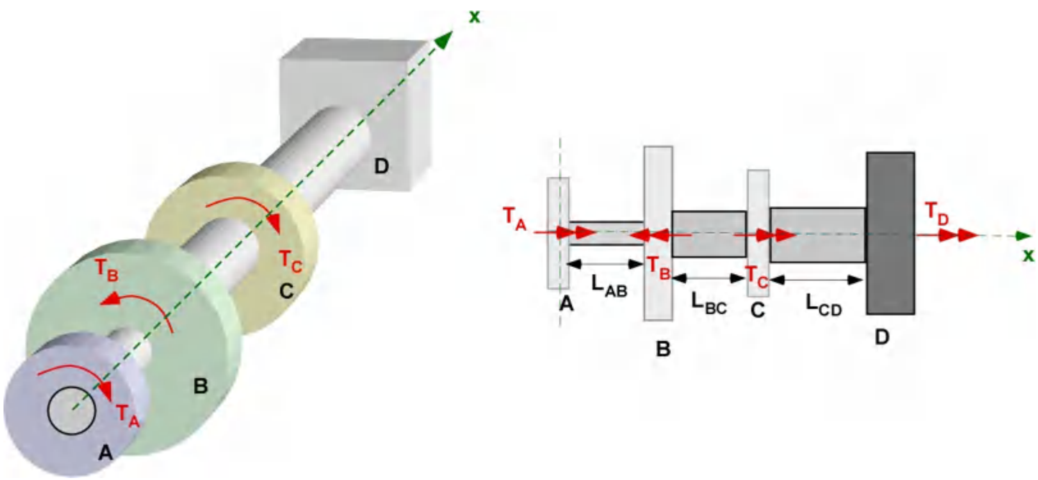


Figure 8.1.2a Example 8.2 (Equivalent representations)

The FBD for the analysis is in Figure 8.1.2b. All torsion is along x - direction and therefore a 1 dimensional problem in torsion.

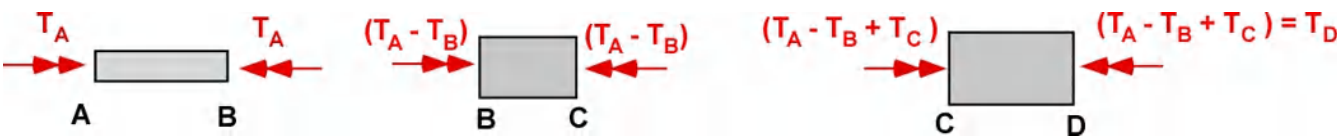


Figure 8.1.2b FBD for the shafts

Solution:
Data: All data is available in Table 8.2. The FBD in Figure 8.1.2b are for the shafts between the pulleys. Positive torque implies torque in the positive x-direction.

Assumption: The shaft is in equilibrium

Solution: (i) The torque in the shafts:

$T_{AB} = T_A = 50$ [Nm]
$T_{BC} = 50 - 100 = - 50$ [Nm]
$T_{CD} = 50 - 100 + 75 = 25$ [Nm]

(ii) The maximum stress in the shafts are: $\dfrac{TR}{J}$

Shaft AB : J(AB) = 4.97×10^{-9} [m^4]; τ_{max} (AB) = 75.451 [MPa];
Shaft BC : J(BC) = 1.57×10^{-8} [m^4]; τ_{max} (BC) = 31.831 [MPa];
Shaft CD : J(CD) = 3.83×10^{-8} [m^4]; τ_{max} (CD) = 8.148 [MPa];

(iii) The general relation for twist between ends A and B is :

$$\phi_B = \phi_A + \frac{T_{AB} L_{AB}}{G_{AB} J_{AB}}$$

For this example we can start at the end D since we know the twist there must be zero.

$$\phi_D = 0$$

$$\phi_C = \phi_D + \frac{T_{CD}\, L_{CD}}{G\, J_{CD}} = 0.108[rad]$$

$$\phi_B = \phi_C + \frac{T_{BC}\, L_{BC}}{G\, JBC} = 0.051[rad]$$

$$\phi_A = \phi_B + \frac{T_{AB}\, L_{AB}}{G\, J_{AB}} = 0.266[rad]$$

(iv) Factor of Safety : $FOS = \dfrac{\tau_y}{\tau_{max}}$

FOS (AB) = 1.72
FOS(BC) = 4.08
FOS(CD) = 15.95

FOS (shaft) = 1.72

8.1.3 Torque Transmission and Equilibrium - Example 8.3

Another set of problems where equilibrium is involved is in torque transmission through gears. Two gears in contact apply equal and opposite forces at the point of contact. This also scales the torque transmitted. In this example, Figure 8.1.3, a torque is applied at the free end A of the shaft AB. This shaft is attached to shaft EC through the gears B and C. The end E of the shaft EC is fixed.

Figure 8.1.3 illustrates the problem through three representative figures.

- The problem is described through the figure in part (a).
- The FBD of the shafts and the torque transmission is illustrated through the figure in part (b).
- The gear effect is represented through the figure in part (c).
- The shafts are supported by bearings which handle the force but transmit the torque without resistance. Therefore the bearings resist the force F_D and the corresponding bending moments. These are ignored since the focus is on torsion.
- The shaft is continuous through the bearings. The contacts are elastic and there is no loss or slip in the gears.

In this problem we will calculate the torque carried by each shaft, the maximum stresses, the twist in the shafts. The applied torque $\mathbf{T_A}$ is 300 [Nm]. The material is ASTM-A709 Grade 345 Steel for the shafts. The shear yield stress (τ_y) is 145 [MPa] and the modulus of rigidity (**G**) is77.2 [GPa].The geometric values are given in Table 8.3.

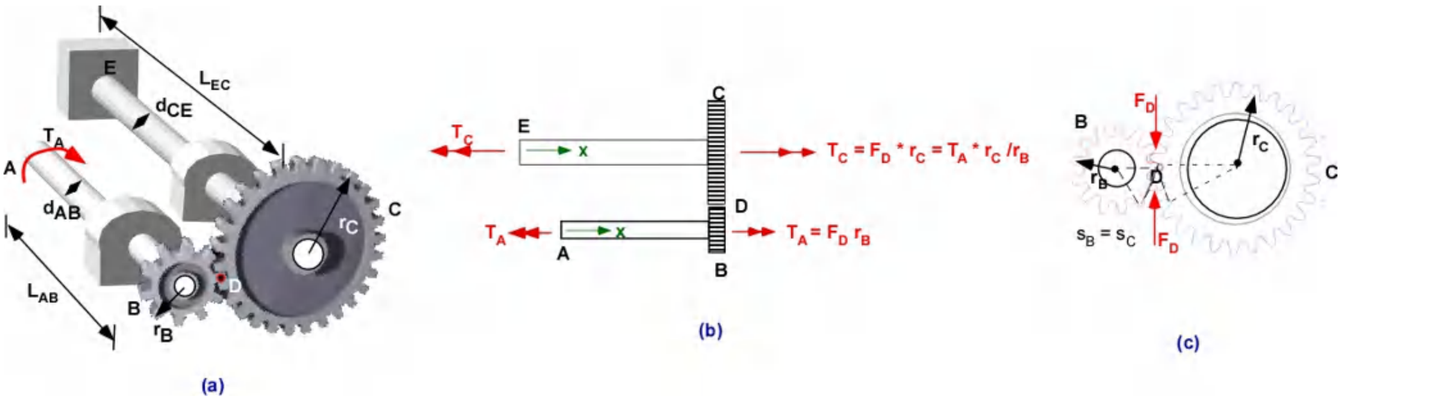


Figure 8.1.3 Example 8.3

Table 8.3. Example 8.3

Shaft	Gear radius [mm]	Diameter of shaft [mm]	Length of shaft [m]
AB	25	40	0.6
EC	75	60	0.8

Solution:
Data: Geometric data is available in Table 8.3. The applied torque T_A = 100 [Nm].

Assumption: The shafts are in equilibrium

Solution: (i) Equilibrium: Both the shafts carry a single torque.

For shaft AB the applied torque is 100 [Nm] in the negative -x direction. For equilibrium the torque at B must be the same in the +x direction.

This torque is generated by the force F_D acting the point of contact (which is assumed to be at the radius of the gear B) applied by the gear C.

This is shown in part (c)

$$T_A = F_D\, r_B$$
$$F_D = \frac{T_A}{r_B} = \frac{300}{25/1000} = 12000[N] \qquad (8.5)$$

The gear at C experiences the equal and opposite force F_D (due to gear B)and generates a torque T_C at the end C of the shaft EC.

$$T_C = F_D\, r_C = 12000\frac{75}{1000} = 900[Nm]$$

The gearing scaled the torque in the ratio of the radius (or teeth). Since $\mathbf{r_C > r_B}$, Torque in EC is amplified and is in the same direction.

The torque in EC at C is in the same direction as the torque at the end B in shaft AB.

At the end E the magnitude of the torque is the same as T_C but oppositely directed due to equilibrium.

(ii) The maximum stress in the shafts and FOS:

$$\tau_{max} = \frac{TR}{J}$$

Shaft AB : J(AB) = 2.513 x10^{-7} [m^4]; τ_{max} (AB) = 23.87 [MPa];
Shaft EC : J(EC) = 1.272x10^{-6} [m^4]; τ_{max} (EC) = 21.22 [MPa]

FOS (AB) = 6.07
FOS(BC) = 6.83

FOS (design) = 6.07

(iii) The relative twist in the shafts: Since end E is fixed the twist there is zero. The end C of shaft EC will twist counter clockwise due to T_C.

$$\begin{aligned}
\phi_C &= \phi_E + \frac{T_C L_{EC}}{GJ_{EC}} = 0.007331[rad] \\
s &= r_C\,\phi_C = r_B\,\phi_B \quad (no\ slip) \\
\phi_B &= \phi_C\,\frac{r_C}{r_B} = 0.022[rad] : clockwise \\
\phi_A &= \phi_B + \frac{T_B L_{AB}}{GJ_{AB}} = 0.0312[rad]
\end{aligned} \tag{8.6}$$

The analysis is performed in MATLAB below.

Solution Using MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, September 2017
% Example 8.3- 1 D Torsion
% Gearing
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 8.3 Gearing\n')
fprintf('---------------------------\n')
%% Data
Ta = 300;  % Torque in [Nm]
dab = 40/1000;  dec  = 60/1000;    % diameter in [m]
Jab = pi*dab^4/32; Jec = pi*dec^4/32;    % m^4
Lab = 0.6;  Lec = 0.8;  % in [m] =
rb = 25/1000;  rc = 75/1000;  % [m]
G = 77.2e09; tauy = 145e6;

fprintf('Data')
fprintf('\n-----------------------')
fprintf('\ndab [m]  : '),disp(dab)
fprintf('dec [m]  : '),disp(dec)
```

```
fprintf('Lab [m]   : '),disp(Lab)
fprintf('Lec[m]   : '),disp(Lec)
fprintf('rb [m]   : '),disp(rb)
fprintf('rc [m]   : '),disp(rc)
fprintf('\nTa [Nm] = '),disp(Ta)
fprintf('G [Pa] = '),disp(G)
fprintf('tauy [Pa] = '),disp(tauy)

fprintf('\nJab [m4] = '),disp(Jab)
fprintf('Jec [m4] = '),disp(Jec)

%% (i)  Torsional Equilibrium
fprintf('\n---------------------------------------\n')
fprintf('(i)  Torsional Equilibrium with Gears \n')
fprintf('------------------------------------------\n')

Fd = Ta/rb;
Tc = rc*Fd;
fprintf('Tc [Nm] = '),disp(Tc)
fprintf('Fd [N]  = '),disp(Fd)

%% (ii)  Maximum stress
fprintf('\n---------------------------------\n')
fprintf('(ii) Maximum stress \n')
fprintf('---------------------------------\n')
taum_ab = abs(Ta*dab/2/Jab);
taum_ec = abs(Tc*dec/2/Jec);
fprintf('tau_max AB [MPa] = '),disp(taum_ab/1e6)
fprintf('tau_max EC [MPa] = '),disp(taum_ec/1e6)

% Factor of Safety
fprintf('\nFOS - shaft AB = '),disp(tauy/taum_ab)
fprintf('FOS - shaft EC = '),disp(tauy/taum_ec)

FOSmin = min([tauy/taum_ab,tauy/taum_ec,]);
fprintf('FOS - Design    = '),disp(FOSmin)

%% (iii)  Twist - of the shafts
fprintf('\n---------------------------------\n')
fprintf('(iii) Shaft Twist \n')
fprintf('---------------------------------\n')

phi_e = 0;
phi_c = phi_e + (Tc*Lec/(G*Jec));
phi_bc = phi_c*rc/rb;
phi_a = phi_bc + (Ta*Lab/(G*Jab));
fprintf('Twist angle at e[rad] = '),disp(phi_e)
fprintf('Twist angle at c[rad] = '),disp(phi_c)
fprintf('Twist angle at b[rad] = '),disp(phi_bc)
fprintf('Twist angle at a[rad] = '),disp(phi_a)
```

In Command Window

```
Example 8.3 Gearing
--------------------------
Data
```

```
----------------------
dab [m]  :         0.04
dec [m]  :         0.06
Lab [m]  :          0.6
Lec[m]  :          0.8
rb [m]  :        0.025
rc [m]  :        0.075

Ta [Nm] =    300
G [Pa] =      7.72e+10
tauy [Pa] =    145000000

Jab [m4] =    2.5133e-07
Jec [m4] =    1.2723e-06

---------------------------------------
(i)  Torsional Equilibrium with Gears
------------------------------------------
Tc [Nm] =    900
Fd [N]  =       12000

---------------------------------
(ii) Maximum stress
---------------------------------
tau_max AB [MPa] =        23.873
tau_max EC [MPa] =        21.221

FOS - shaft AB =        6.0737
FOS - shaft EC =         6.833
FOS - Design    =        6.0737

---------------------------------
(iii) Shaft Twist
---------------------------------
Twist angle at e[rad] =      0
Twist angle at c[rad] =     0.0073301
Twist angle at b[rad] =       0.02199
Twist angle at a[rad] =      0.031267
```

Execution in Octave

There is no change in the code.

The results in the Command Window are also the same

8.1.4 Statically Indeterminate Torsion Example - Example 8.4

A statically indeterminate torsion problem would indicate that you cannot determine the solution to the problem through the equations of statics alone. In such a case the additional equations are developed by considering displacements and other constraints if any. In this example a single shaft is acted on by a torque T at a point on the shaft but both the ends are fixed. The problem and the FBD are described in Figure 8.1.4

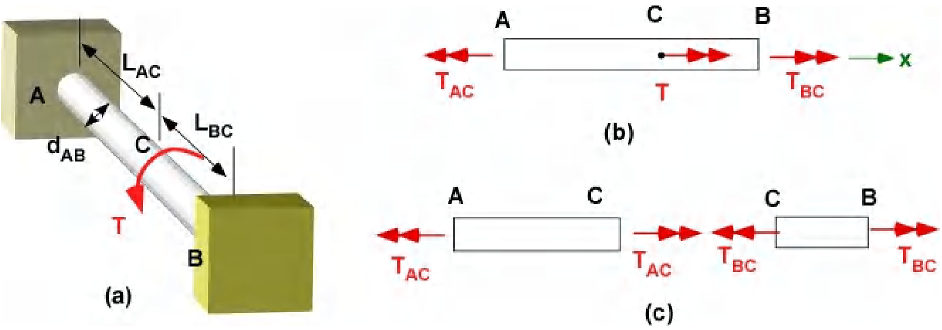


Figure 8.1.4 Example 8.4

Solution:
Equilibrium:

$$\sum M_x = 0 = -T_{AC} + T + T_{BC}$$

$$\phi_A = \phi_B = 0 = \phi_A + \phi_{C:A} + \phi_{B:C} = \frac{T_{AC}\,L_{AC}}{J_{AC}\,G_{AC}} + \frac{T_{BC}\,L_{BC}}{J_{BC}\,G_{BC}} = 0$$

Since the geometry and material for the segments are the same: $T_{AC} = -T_{BC}\frac{L_{BC}}{L_{AC}}$

This solves for the torque in the shafts. The twist can be solved using the values for the torques.

Solution: using values: T = 500 Nm; d_{AB} = 50 mm; L_{AC} = 0.6 m; L_{BC} = 0.4 m

$$T_{AC} = -T_{BC}\frac{0.4}{0.6}$$

$$T_{BC}\frac{2}{3} + 500 + T_{BC} = 0; \quad T_{BC} = -300\,Nm;$$

$$T_{AC} = 200\,Nm$$

Solution in

8.1.5 Additional Problems

Please rework the following problems for each of the materials or a combination of them when there are two or more shafts involved in the problem given below in the table.

Table 8.4 (also Table 7.9.1)

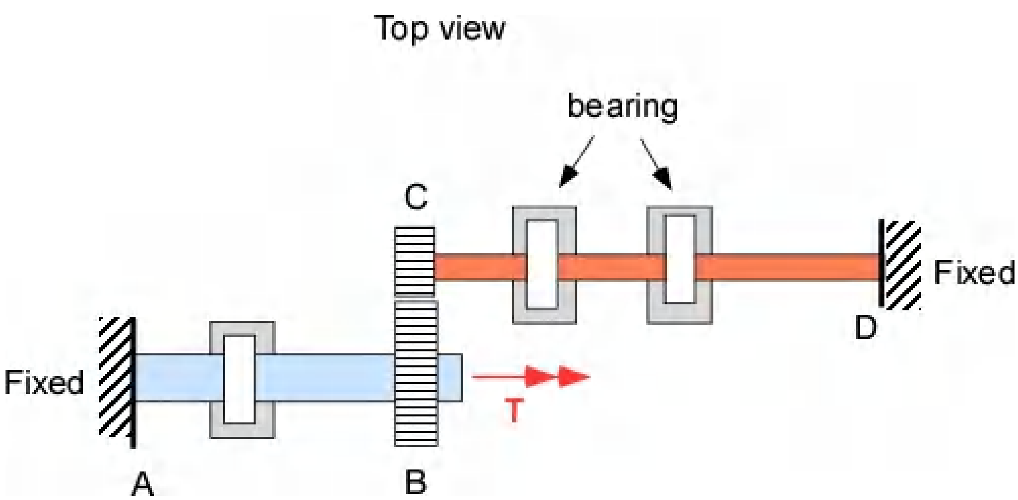
Material	Aluminum	Brass	Steel	Wood
G [GPa]	77	39	77	0.7
Yield Stress [MPa] (shear)	55	250	145	7.6

Problem 8.1.1 (design problem)
A 1.5 m shaft is subject to torque of 8 kN-m. The relative twist between the ends should not exceed

2.5 degrees. The maximum factor of safety is 3.0. Find the minimum diameter of the shaft. Please make sure to use a FBD before working out the solution.

Problem 8.1.2

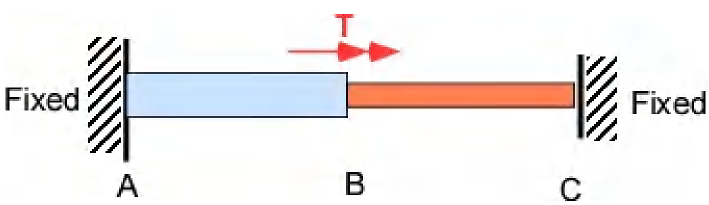
Shafts AB and CD are connected through gears B and C. The end A and D of the shafts are fixed. The shafts are made of different materials. They are supported on bearings transmit the torque freely. The length of shaft AB is 250 mm and its diameter is 50 mm. The length of shaft CD is 500 mm and its diameter is 40 mm. The radius of gear B is 90 mm. The radius of gear C is 50 mm. The applied Torque is 5 kN-m. Calculate (a) the maximum stress in the shafts and the minimum factor of safety. (b) the deflection of the gear B. (c) the deflection of the gear C.



Problem 8.1.2

Problem 8.1.3

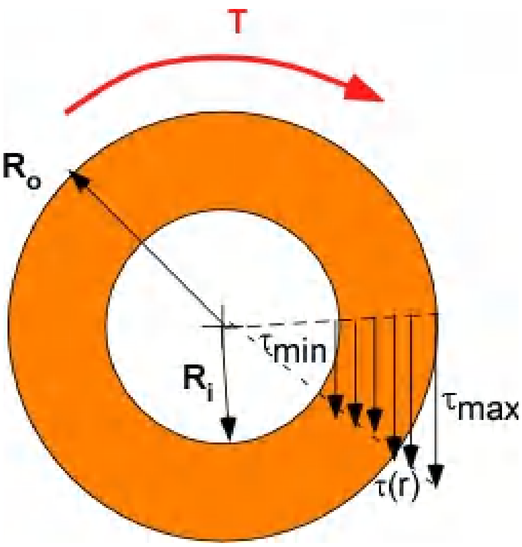
Shafts AB and BC are bonded at B and held fixed at their other ends as shown in the figure. They are made of different materials. The length of shaft AB is 200 mm and its diameter is 40 mm. The length of shaft BC is 300 mm and its diameter is 60 mm. The torque applied 1500 Nm. Calculate the maximum shear stress in each of the shafts. (b) Calculate the deflection in each shaft.



Problem 8.1.3

8.2 HOLLOW SHAFTS

Section 8.1 applies to shafts be they solid or hollow. In every example we could have chosen a shaft with an annular cross-section and the only change is the computation of the polar moment of inertia. This is true until the thickness is small compared to the diameter and this creates a simpler model for the torque response. This is later discussed as thin walled hollow shafts. For reasonable thickness of the shaft and for stresses within the elastic limit the shear stress is still linearly distributed as a function of the radius but the minimum stress is not zero.



$$\tau(r) = \frac{T\,r}{J}; \quad \tau_{\max} = \frac{T\,R_o}{J}; \quad \tau_{\min} = \frac{T\,R_i}{J};$$

$$J = \frac{\pi}{2}\left(R_o^4 - R_i^4\right)$$

$$\gamma = \frac{r\,\phi}{L}; \quad \gamma_{\max} = \frac{\tau_{\max}}{G}; \quad \gamma_{\min} = \frac{\tau_{\min}}{G};$$

$$\phi = \frac{T\,L}{G\,J}$$

(8.7)

Figure 8.2.1 Hollow shaft and relations

8.2.1 Example 8.5

This example deals with the design of a hollow shaft. For a shaft in pure torsion the state of stress at the outer surface is examined. The Mohr's circle is simple since the principal stresses are zero. For any other orientation of the planes the surface is likely to see a combination of normal stress and shear stress, For a rotation of 45 degrees the principal stresses will have the same magnitude as the shear stress. Consider L = 1 m; R_i = 0.04 m and R_o is 0.055 m. Material is brass (σ_y = 410 MPa, τ_y = 250 MPa, E=105 GPa, G = 39 GPa, ρ = 8910 kg/m^3). Find (a) the maximum torque for FOS of 2.5 in yield; (b) the minimum shearing stress for this load; (c) the principal principal stress for this loading for a state of stress at point a; (d) the state of stress at the point b on the outer diameter; (e) maximum twist in the cross-section.

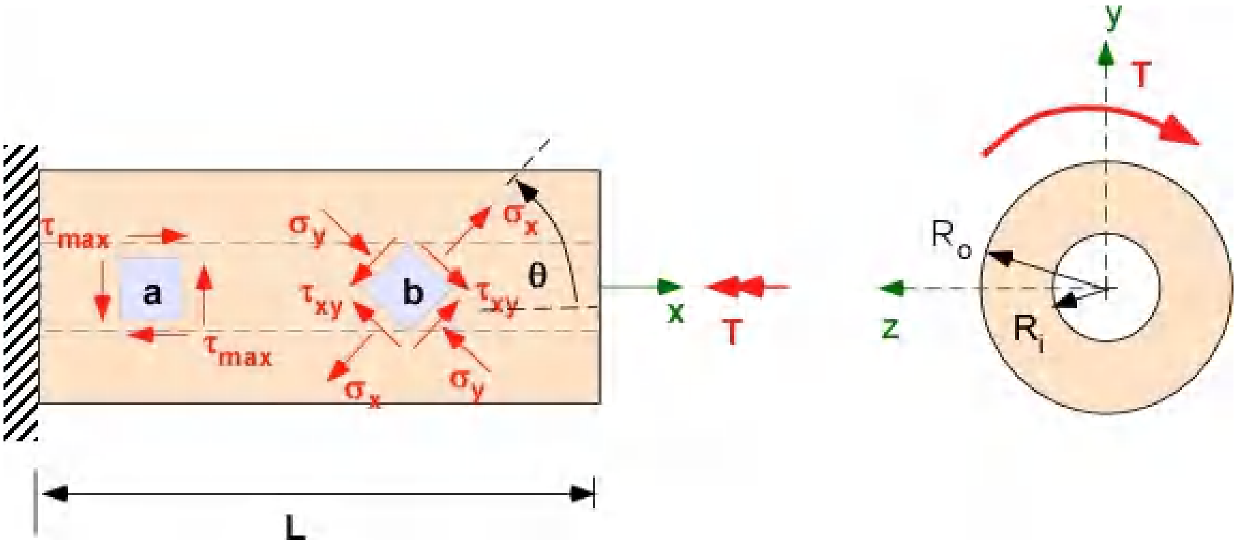


Figure 8.2.2 Example 8.5

Solution:

Data: L = 1 m; R_i = 0.04 m and R_o is 0.055 m

Material: σ_y = 410 MPa, τ_y = 250 MPa, E=105 GPa, G = 39 GPa, ρ = 8910 kg/m^3

FOS = 2.5

Assumption: FOS based on yield.

Solution: (a) Equilibrium: The allowable shear stress is based on the maximum shear stress in the cross-section - which Iis based on the outside diameter.

$$\tau_{all} = \frac{\tau_y}{FOS} = 100[MPa]$$

$$J = \frac{\pi}{32}\left(d_o^4 - d_i^4\right) = 1.035\times10^{-5}\left[m^4\right]$$

$$T_{max} = \frac{J\,\tau_{all}}{0.5\,d_o} = 18.82[kN-m]$$

(b) $$\tau_{min} = \frac{d_i}{d_o}\tau_{all} = 72.73[MPa]$$

(c) $$\sigma_1 = \sigma_2 = 0$$

(d) $$\sigma_1 = \tau_{all} = 100[MPa]$$

(e) $$\phi = \frac{T_{max}\,L}{G\,J} = 0.066[rad]$$

8.2.2 Example 8.6 - Statically Indeterminate Torsion

The circular shaft AB, in Figure 8.2.3a, between fixed supports is solid between AC and hollow between CB. A torque T is applied at C.
(a) Find the reactions at the support. A and B; (b) if the material is the same, what is the maximum stress in the shaft; (c) what is angle of twist at C. This is the same as Example 8.4 except we have a hollow part of the shaft.

Solve for any combination of material in Table 8.4 and geometry and load in Table 8.5

Table 8.4 Material and shear properties

Material	Aluminum	Brass	Steel	Wood
G [GPa]	26	39	77	0.7
Yield Stress [MPa] (shear)	55	250	145	7.6

Table 8.5 Information for Example 8.6

Case	Torque [Nm]	Diameter d_o [mm]	Diameter d_i [mm]	L_1 [m]	L_2 [m]
(i)	50	15	10	0.6	0.6
(ii)	100	30	24	0.5	0.7
(iii)	75	25	20	0.7	0.5

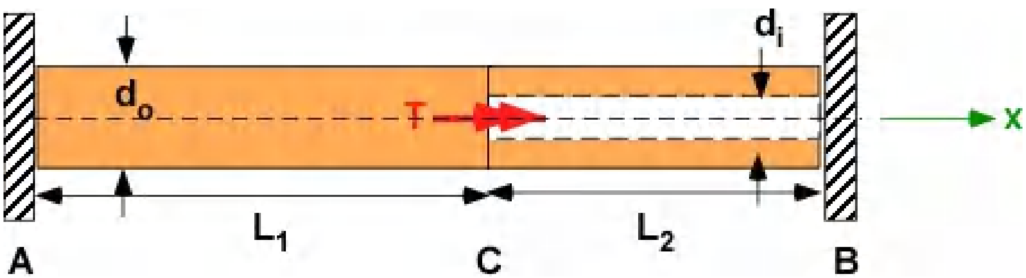


Figure 8.2.3a. Example 8.6

Solution:
Data: Given T, L_1, L_2, G, J_1, J_2, T_A, T_B

Assumption: None

Solution: Part (a) This is statically indeterminate. Applying moment equilibrium in x-direction

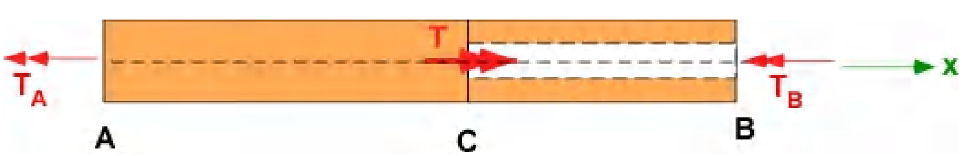


Figure 8.2.3b. FBD of shaft AB

$$\sum M_x = 0 = -T_A - T_B + T \qquad (a)$$

The second equation must come from the twist. The angle of twist is zero at A and B. Using the FBD of part AC and part CB and applying equation (b)

$$\phi_B = 0 = \phi_{C:A} + \phi_{B:C} \qquad (b)$$

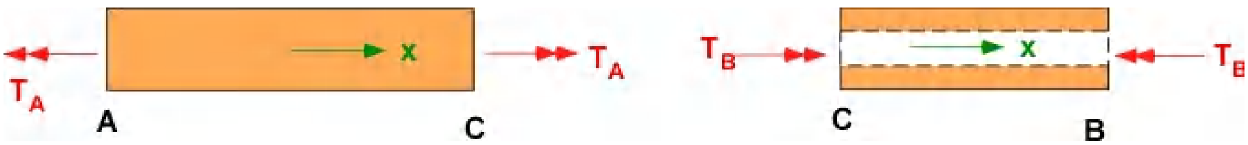


Figure 8.2.3c. FBD of parts of shaft

$$0 = \frac{T_A L_1}{G J_1} - \frac{T_B L_2}{G J_2}; \quad T_A = T_B \frac{L_2}{L_1} \frac{J_1}{J_2};$$

$$substituting\, in\, (a)$$

$$T_B = \frac{T}{\left[1 + \frac{L_2}{L_1} \frac{J_1}{J_2}\right]}; \qquad (c)$$

Part (b)

$$\tau_{max}(AC) = \frac{T_A d_o / 2}{J_1}; \quad \tau_{max}(CB) = \frac{T_B d_o / 2}{J_2}$$

Part (c)

$$\phi_C = \phi_{C:A} = \frac{T_A L_1}{G J_1}$$

For each material and each case we can calculate all of the information through programming. The MATLAB code is more sophisticated and uses programming instead of direct calculation. It calculates all values for all materials, for all geometry and publishes the information in a table and also creates a **.csv** file of values. The units are SI.

Solution Using MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, Feb 2018
% Example 8.6  1D Torsion
% Statically indeterminate with parly hollow shaft
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 8.6\n')
fprintf('-------------------------\n')
fprintf('Hollow Shaft in Statically Indeterminate Torsion\n')
fprintf('-----------------------------------------------\n')
%% Data
mat = [26,55;39,250;77,145;0.7,7.6]; % matrix of G and tauy
```

```
material = {'Aluminum','Brass','Steel','Wood'}; % material
geom = [50,15,10,0.6,0.6;  % three sets of geometry
        100,30,24,0.7,0.5;
        75,25,20,0.7,0.5];
[m1,n1] = size(mat);  % getting the size from the matrix
[m2,n2] = size(geom);  % getting the size from the matrix

k = 1;
for i = 1:m1
    for j = 1:m2
        % btain the specific material and geometry
        Mat = material(i);
        G = mat(i,1)*10^09;
        tau = mat(i,2)*10^6;
        T = geom(j,1);
        do = geom(j,2)/1000;
        di = geom(j,3)/1000;
        L1 = geom(j,4);
        L2 = geom(j,5);
        J1 = pi*do^4/32;
        J2 = pi*(do^4 - di^4);
        TB = T/(1+(L2*J1/L1/J2));
        TA = T - TB;
        taumAC = TA*do/2/J1;
        taumCB = TB*do/2/J2;
        phi = TA*L1/G/J1;

        % store the values in a struct variable for printing and saving
        st86(k).Mat = Mat;
        st86(k).G = G;
        st86(k).tau = tau;
        st86(k).T = T;
        st86(k).do =do;
        st86(k).di = di;
        st86(k).L1 = L1;
        st86(k).L2 = L2;
        st86(k).J1 = J1;
        st86(k).J2 = J2;
        st86(k).TB = TB;
        st86(k).TA = TA;
        st86(k).taumAC = taumAC;
        st86(k).taumCB = taumCB;
        st86(k).phi = phi;

        % increment the index
         k = k+1;
    end
end
fprintf('The Table of Results\n')
fprintf('---------------------\n')
T = struct2table(st86);  % create a table from a structure
% column headings are the index to the struct variable
disp(T)  % show it on the screen
 % write the table as a csv file (can open in Excel)
 % in the same directory as the code
writetable(T,'Ex8_6.csv')
```

In the Command Window

Output is not shown because the table is long. It should appear in the command window. The units are SI.

In addition a “csv” file that can be opened by Microsoft EXCEL is created in the same directory.

We could have split the information in two tables to be able to include it here. This is left to the student.

Execution in Octave

There is no change in the code except for the ones highlighted below. The Octave program does not support the **struct2table** or the **writetable** MATLAB commands. Instead you can use the formatted print command to print to the Command window. Those commands are commented in the Octave code.

The **csvwrite** command in Octave appears to work with numeric information only

In Octave Editor

```
##  Octave does not have a struct2table command
## so we will print the information using fprintf

##T = struct2table(st86);  % create a table from a structure
##% column headings are the index to the struct variable
##disp(T)  % show it on the screen
## % write the table as a csv file (can open in Excel)
## % in the same directory as the code
##writetable(T,'Ex8_6.csv') T


fprintf('\nMaterial       G [Pa]  Torque [Nm]  L1[mm]   taumax(AC)[Pa]
taumax(CB)[Pa] phi(rad)\n')
k = 1;
for i = 1:m1
    for j = 1:m2
      st86str = char(st86(k).Mat);
      fprintf('\n%10s    %5.2e    %5.1f     %5.2f    %5.2e          %5.2e
%5.2e', ...
      st86str,st86(k).G,st86(k).T,st86(k).L1,st86(k).taumAC,st86(k).taumCB
,st86(k).phi);
      k = k + 1;

    end
end
fprintf('\n\n')
```

In Octave Command Window

(Only a subset of information is printed. You can print all or another set)

```
Example 8.6
--------------------------
Hollow Shaft in Statically Indeterminate Torsion
-------------------------------------------------
The Table of Results
---------------------
```

Material	G [Pa]	Torque [Nm]	L1[mm]	taumax(AC)[Pa]	taumax(CB)[Pa]	phi(rad)
Aluminum	2.60e+10	50.0	0.60	2.83e+06	2.83e+06	8.70e-03
Aluminum	2.60e+10	100.0	0.70	6.87e+05	9.62e+05	1.23e-03
Aluminum	2.60e+10	75.0	0.70	8.91e+05	1.25e+06	1.92e-03
Brass	3.90e+10	50.0	0.60	2.83e+06	2.83e+06	5.80e-03
Brass	3.90e+10	100.0	0.70	6.87e+05	9.62e+05	8.22e-04
Brass	3.90e+10	75.0	0.70	8.91e+05	1.25e+06	1.28e-03
Steel	7.70e+10	50.0	0.60	2.83e+06	2.83e+06	2.94e-03
Steel	7.70e+10	100.0	0.70	6.87e+05	9.62e+05	4.16e-04
Steel	7.70e+10	75.0	0.70	8.91e+05	1.25e+06	6.48e-04
Wood	7.00e+08	50.0	0.60	2.83e+06	2.83e+06	3.23e-01
Wood	7.00e+08	100.0	0.70	6.87e+05	9.62e+05	4.58e-02
Wood	7.00e+08	75.0	0.70	8.91e+05	1.25e+06	7.12e-02

8.2.3 Thin-Walled Circular Shafts

This is an annular shaft where the outer radius and the inner radius differ by a small value. This difference is the wall thickness. For such a design there is a very small difference between the maximum and the minimum shear stress and therefore it can be replaced by a constant value on the wall. Consider a thin walled circular cylinder with inner radius R and a thickness t. The average shear stress τ_a can be calculated using the average radius R_a and is shown in Figure 8.2.4

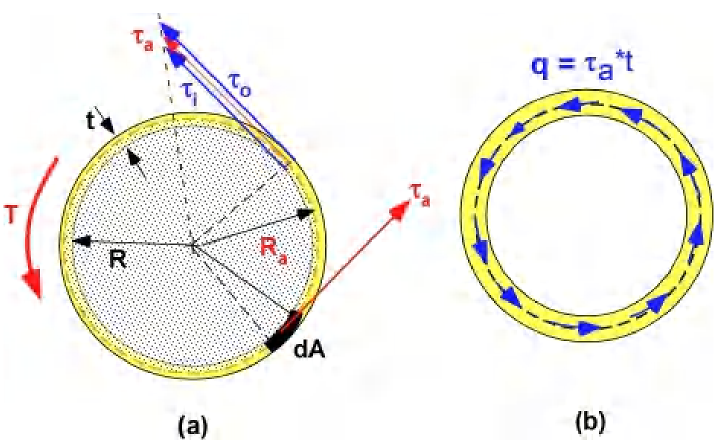


Figure 8.2.4 Thin walled circular shaft

In Figure 8.2.4a, the relations between the Torque T and the shear stress can be developed as follow:

$$dA = R_a t\, d\theta$$
$$dF = \tau_a dA = \tau_a R_a t\, d\theta$$
$$dT = R_a dF = \tau_a t R_a^2 d\theta$$
$$T = \int_{2\pi}^{0} dT = \int_{2\pi}^{0} \tau_a t R_a^2 d\theta = \tau_a t R_a^2 \int_{2\pi}^{0} d\theta = 2\pi \tau_a t R_a^2 = 2\tau_a t A_e \qquad (8.8)$$
$$\tau_a = \frac{T}{2 t A_e};$$

A_e: This is the average area or the enclosed area by the average radius. If the wall is thin this can be approximated by the inside area.

The last equation in the set (8.8) can be related to the shear flow "**q**" which appeared in Chapter 7. It is popular in design discussions where thin walls are involved. This is shown in Figure 8.2.4b.

$$q = \tau_a t = \frac{T}{2A_e} \quad (8.9)$$

Example 8.7
A thin walled tube of **material 1** and a solid circular shaft of **material 2** are connected to a fixed support at the end A and a rigid end plate at the end B. The length of the shafts is **L**. A torque T is applied to the end plate as shown in Figure 8.2.5. The outer diameter of the shaft made of material 1 is $\mathbf{d_1}$ and the thickness is t. The diameter of the solid shaft is $\mathbf{d_2}$. Given the material properties (**G** and σ_y) calculate the maximum torque that can be applied for a factor of safety of 2.5. Consider the two cases in Table 8.6. This is a design problem.

Table 8.6 Material and shear properties

Case	Diameter d_1 [mm]	Diameter d_2 [mm]	t [mm]	L [m]	Material 1	Material 2
(i)	100	50	10	0.75	steel	aluminum
(ii)	100	50	10	0.75	aluminum	steel

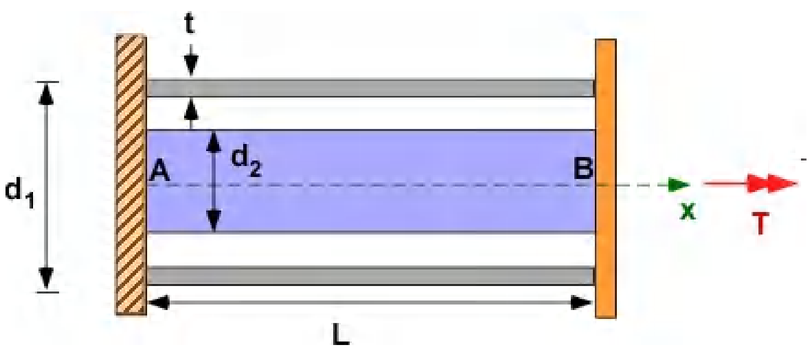


Figure 8.2.5a Example 8.7

Data: Table 8.6 preserves the geometry but flips the material. This provides an instinct on the torque carrying capacity of mixed material structures in torsion.

Assumption: This is a statically indeterminate problem as the applied torque is shared between the two shafts. Since the two objects are rigidly connected at the ends, the angle of twist must be the same for both materials.

Solution: Use Table 8.4 for material properties

The dimensions are same for both cases . Therefore

$$J_1 = \frac{\pi}{32}\left[d_1^4 - (d_1 - 2t)^4\right] = 5.7962\text{e-}06\left[m^4\right]$$

$$J_2 = \frac{\pi}{32}\left[d_2^4\right] = 6.1359\text{e-}07\left[m^4\right]$$

$$L = 0.75[m]$$

$$FOS = 2.5$$

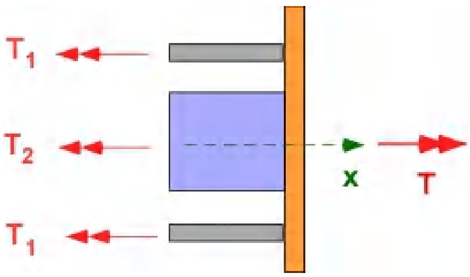


Figure 8.2.5b Statically indeterminate problem

$$T = T_1 + T_2 \qquad (a)$$

$$\phi_1 = \frac{T_1 L}{J_1 G_1} = \phi_2 = \frac{T_2 L}{J_2 G_2} \qquad (b)$$

For maximum torque then either

$$T_{1,\max} = \frac{(\tau_{1y} / FOS) J_1}{(d_1/2)} \quad OR \quad T_{2,\max} = \frac{(\tau_{2y} / FOS) J_2}{(d_2/2)} \qquad (c)$$

For each case the maximum torque on each material is calculated and the stress limit on the other material is checked to make sure it is less than the yield stress to provide valid loading. Once either $T_{1,max}$ or $T_{2,max}$ are determined then the other torque is obtained using equation (b). If the stress in the other material is below the yield stress then the sum of the torques is the maximum torque that can be applied on the design. If both calculations suggest a valid design then the maximum torque is reported.

This was programmed in MATLAB with the results below in some detail. You can calculate using the equations above and confirm the results. Better still you should be able to develop the code.

In the Command window:

```
-----------------------
Example 8.7
---------------
Hollow Shaft in Statically Indeterminate Torsion
---------------------------------------------------
J1 [m^4] =    5.7962e-06
```

```
J2 [m^4] =    6.1359e-07
L    [m] =          0.75
FOS      =           2.5

Case :       1
----------------
Case : Material 1 is at yield stress
Material 1 : Steel
Torque T1 [Nm] =        6723.6
Torque T2 [Nm] =        260.65
Material 2 : Aluminum
Max. shear in material 2 [MPa] =        10.62
Yield shear in material 2 [MPa] =     22
Shear stress in material 2 below yield
Total Torque [Nm] =        6984.3
Twist in Material 1 [rad]      0.012254
Twist in Material 2 [rad]      0.012254

Case : Material 2 is at yield stress
Material 2 : Aluminum
Torque T1 [Nm] =         13929
Torque T2 [Nm] =        539.96
Material 1 : Steel
Max. shear in material 1 [MPa] =       120.15
Yield shear in material 1 [MPa] =     58
Consider Shear stress in material 1 for design

Case :       2
----------------
Case : Material 1 is at yield stress
Material 1 : Aluminum
Torque T1 [Nm] =        2550.3
Torque T2 [Nm] =        737.25
Material 2 : Steel
Max. shear in material 2 [MPa] =       30.038
Yield shear in material 2 [MPa] =     58
Shear stress in material 2 below yield
Total Torque [Nm] =        3287.6
Twist in Material 1 [rad]      0.012692
Twist in Material 2 [rad]      0.012692

Case : Material 2 is at yield stress
Material 2 : Steel
Torque T1 [Nm] =        4924.4
Torque T2 [Nm] =        1423.5
Material 1 : Aluminum
Max. shear in material 1 [MPa] =       42.479
Yield shear in material 1 [MPa] =     22
Consider Shear stress in material 1 for design
```

The previous solution did not use thin wall assumptions. Let us see how the solution changes using thin walled relations with

$$T = \tau_a(2At)$$

```
----------------------
Example 8.7
---------------
Hollow Shaft in Thin Wall Relation
-------------------------------------------------
J1 [m^4] =    5.7962e-06
J2 [m^4] =    6.1359e-07
L    [m] =          0.75
FOS      =           2.5

Case :      1
----------------
Case : Material 1 is at yield stress
Material 1 : Steel
Torque T1 [Nm] =         7379.6
Torque T2 [Nm] =         286.08
Material 2 : Aluminum
Max. shear in material 2 [MPa] =        11.656
Yield shear in material 2 [MPa] =     22
Shear stress in material 2 below yield
Total Torque [Nm] =        7665.7
Twist in Material 1 [rad]      0.013449
Twist in Material 2 [rad]      0.013449

Case : Material 2 is at yield stress
Material 2 : Aluminum
Torque T1 [Nm] =          13929
Torque T2 [Nm] =         539.96
Material 1 : Steel
Max. shear in material 1 [MPa] =        120.15
Yield shear in material 1 [MPa] =     58
Consider Shear stress in material 1 for design

Case :      2
----------------
Case : Material 1 is at yield stress
Material 1 : Aluminum
Torque T1 [Nm] =         2799.2
Torque T2 [Nm] =         809.18
Material 2 : Steel
Max. shear in material 2 [MPa] =        32.969
Yield shear in material 2 [MPa] =     58
Shear stress in material 2 below yield
Total Torque [Nm] =        3608.3
Twist in Material 1 [rad]      0.013931
Twist in Material 2 [rad]      0.013931

Case : Material 2 is at yield stress
Material 2 : Steel
Torque T1 [Nm] =         4924.4
Torque T2 [Nm] =         1423.5
Material 1 : Aluminum
Max. shear in material 1 [MPa] =        42.479
Yield shear in material 1 [MPa] =     22
Consider Shear stress in material 1 for design
```

The maximum torque is slightly overestimated using the thin wall relations. This can be analytically justified.

8.2.4 Thin Walled Non-circular Shafts

Thin walls are a principle design feature for light weight structures specially in aerospace design. They usually provide conduit for shear flow or handling shear stress in the structures due to torsion or shear load or both. There is a direct extension of circular thin wall relations to non circular thin wall cross-sections. One important feature of the analysis is that the wall thickness allows for a constant shear flow in the cross-section. This is sufficiently accurate until a thickness of about 20% of the smaller diameter, or the smallest dimension in the cross-section for non circular shapes. By extension the shear flow at any point in the cross-section is the shear flow divided by the thickness. The thickness does not necessarily have to be constant. Examples of structures with thin walled cross-section are shown in Figure 8.2.6.

Figure 8.2.6 Thin walled cross-sections

In the examples above all the thin walled sections are closed sections. They are also classified as a single cell or a multiple cell structure. The fuselage is an example of a multiple cell structure. Each of the cells will have a different shear flow. However they would be reconciled using an analogy of fluid flow or current flow where the flows obey a conservation principle. In this book we will only consider a single cell and closed sections. The open thin wall cross-sections behave very similarly with reduced torque resisting capacity.

To relate the shear flow and applied torque consider Figure 8.2.7 where a torque is applied to a thin walled shaft of an arbitrary closed cross-section.

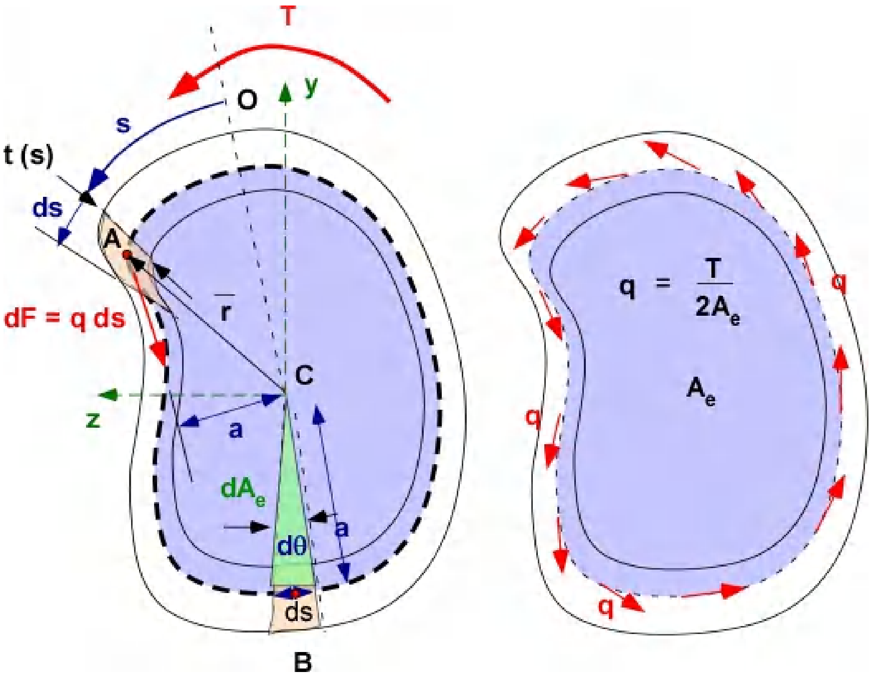


Figure 8.2.7 Torque and shear flow

The development of relations is usually set up using the positive quadrant of the coordinate system - point A. The applied torque will be the same as the torque resisted by the shear flow.

$$dT = \left|\bar{r} \times d\bar{F}\right| = a(q\,ds)$$

The same information is easy to follow at the point B.

$$ds = a\,d\theta$$
$$dT = q\,a(a\,d\theta) = q\,a^2\,d\theta$$
$$dA_e = \frac{1}{2}(a\,d\theta)a; \quad a^2\,d\theta = 2\,dA_e \qquad (8.10)$$
$$T = \int dT = \int_{A_e} 2q\,dA_e = 2qA_e$$

This is the same expression for hollow circular shaft in Eqn. (8.8). Calculating the enclosed area requires more information. Additional relations for the stress and twist can be obtained as follows:

$$\tau(s) = \frac{q}{t(s)} = \frac{T}{2tA_e}; \quad \tau_{max} = \frac{T}{2t_{min}A_e} \qquad (8.11)$$

The change in twist is usually developed in an advanced course. The development is omitted but the relations are due to Bredt-Batho are set up as

$$\frac{d\phi}{dx} = \frac{q}{2GA_e}\oint\frac{ds}{t} = \frac{T}{4GA_e^2}\oint\frac{ds}{t}$$ (8.12)

$$\phi = \frac{\tau L p}{2A_e G}; \quad t: \text{constant}$$

By comparing with the relation for circular shaft in the last of Equation (8.7) it can be deduced that

$$J = \frac{4A_e^2}{\oint\frac{ds}{t}} = \frac{4tA_e^2}{p}; \quad p: perimeter$$

Example 8.8
Compute the shear flow and stress in a rectangular tube with thin wall shown in Figure 8.2.8. Choose the material of the shaft if the maximum torque required to be carried is 1200 Nm with a factor of safety of 2. Limit your material choices to Table 8.4

Table 8.4 Material and shear properties

Material	Aluminum	Brass	Steel	Wood
G [GPa]	26	39	77	0.7
Yield Stress [MPa] (shear)	55	250	145	7.6

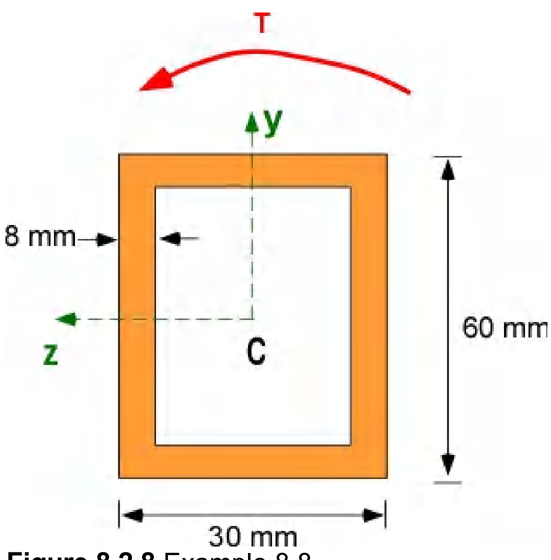


Figure 8.2.8 Example 8.8

Data: FOS = 2. Dimensions are available in the figure. T = 1200 Nm.

Assumption: Use thin wall relations.This example involves straight forward application of the results established in this section. Also note given the torque T and the geometry the shear flow and the shear stress is calculated. The problem is completely determinate. How will the choice of the material come into the picture? What if more than one material satisfies the stresses?

Solution: Use Table 8.4 for material properties

$$A_e = (30-8)\times(60-8)\times 10^{-6} = 0.001144\left[m^2\right]$$

$$q = \frac{T}{2A_e} = \frac{1200}{0.001144} = 5.245\times 10^5\left[N/m\right]$$

$$\tau = \frac{q}{t} = \frac{5.245\times 10^5}{0.008} = 65.66\left[MPa\right]$$

$$\tau_Y(req) = \tau \times FOS = 65.66\times 2 = 131.12\left[MPa\right]$$

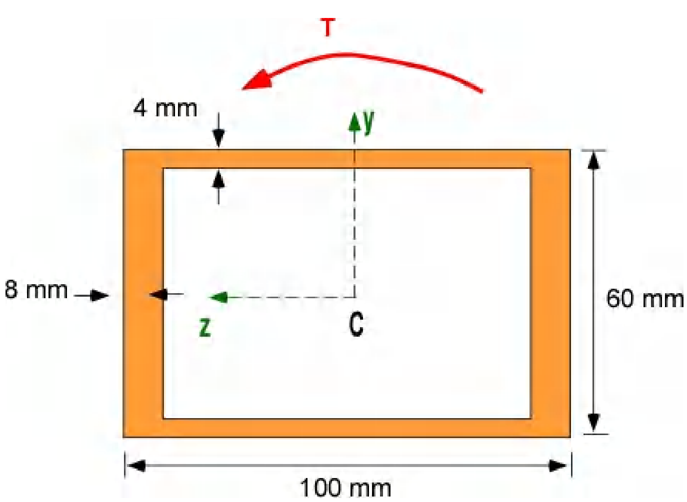
From Table 8.4 either brass or steel will serve the purpose. This allows additional choice based on cost or weight or corrosion based on access to information. Hence it can become a multi-objective design problem.

8.2.5 Additional Problems.

Consider solving these problems using a different material or applied torque.

Problem 8.2.1

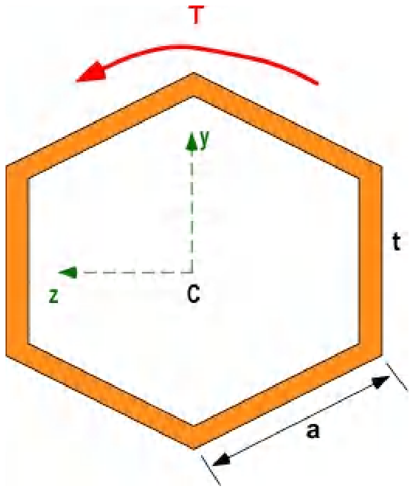
Consider the rectangular aluminum (Figure: Problem 8.2.1) cross-section with two different wall thickness but still symmetrical. For a torque of 1500 Nm calculate the maximum FOS for the design. Use thin walled assumption. What would be the result if the torque was increased or decreased. Is there a direct relationship?



Problem 8.2.1

Problem 8.2.2

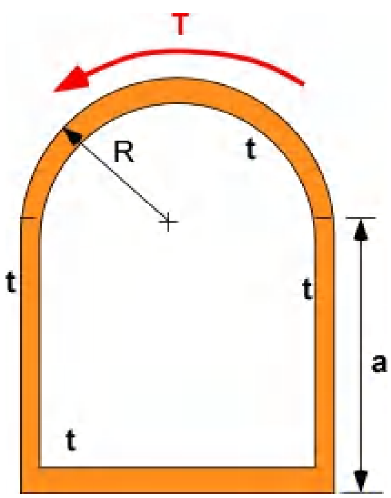
Consider the thin walled regular hexagon of side **a** and thickness **t**. The length of the shaft is 0.75 m. The material is brass. Calculate the dimensions of the hexagon for a torque of 150 Nm and a factor of safety of 2.5. The maximum twist angle over the length of the shaft is limited to 0.5 degrees. Note: sharp corners are likely to see stress concentrations. It is ignored in this problem.



Problem 8.2.2

Problem 8.2.3

Consider the thin walled cross-section in the figure below. The material is aluminum and the applied torque is 3000 Nm. Use MATLAB to compute the shear flow, shear stress, and the twist for a 0.5 m length of the section for a range of values of the various dimensions in the problem. Consider 5 values of R between 50 and 100 mm. Consider 7 values of a between 75 and 150 mm. Consider 10 values of t between 5 and 14 mm.



Problem 8.2.3

8.3 ADDITIONAL TOPICS IN TORSION

In this section a grab bag of problems that include plastic torsion, design of transmission shafts, and combination of bending and torsion are discussed. The sections before involved design problems where shear stresses were below or at the shear yield strength. This allowed the computation of the factor safety, a valuable piece of design information. A once in a lifetime event can load the structure beyond the yield limit and this would cause the material to behave plastically. Sometimes material properties are not as uniform as expected which makes the material behavior deviate from the linear stress-strain relationship. All the derivations, particularly the deflections in this chapter assume the validity of Hooke's law in shear. This is a linear relationship between shear stress and shear strain. A brittle material will have a nonlinear material behavior. Incorporating nonlinear behavior is both difficult and requires an analytical or numerical functional dependence making our computations cumbersome and beyond the scope of this text. A limited discussion is possible by considering an elastoplastic material that was featured in bending. Some discussion here can also be used for bending even if has not been been discussed there.

8.3.1 Stressing Beyond Elastic Limit

A typical shear stress (τ) and shear strain (γ) relation is shown in Figure 8.3.1a. The region for the validity of the Hooke's law in shear is the straight line region shown as a blue line. The slope of this line if the modulus of rigidity or the shear modulus **G**. The limit of this region is the the yield strength (τ_y). As you increase the load beyond this limit the stress strain behavior is nonlinear. The maximum shear stress is the ultimate shear stress or strength (τ_U). The last point F is the fracture stress (τ_f) when the material has snapped in shear. The nonlinear region is difficult to use for design for the behavior must be described through a nonlinear relationship.

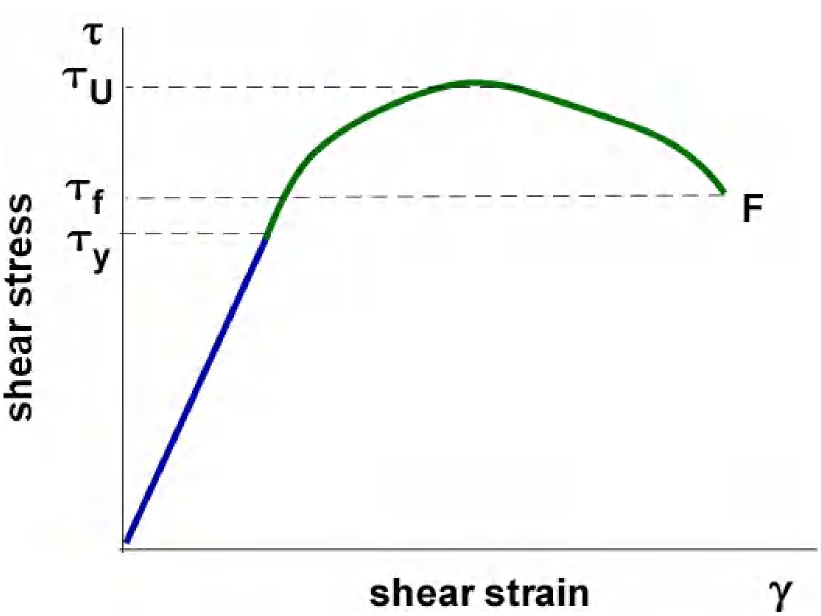


Figure 8.3.1a Typical shear stress - shear strain behavior

In many cases the application of the load and the subsequent development of stress is cyclic. In most designs the structure is also unloaded when the force is no longer exerted on the structure. This causes a cycling of the load and requires the stress must return back to zero. If the applied stress was in the linear region, both the stress and strain return to zero. If an event causes the stress to exceed the linear region, the unloading of the material is parallel to the elastic region but will suffer a deformation when the stress becomes zero. This is a permanent strain in the structure leading to a permanent deformation.

First consider the load applied until the limit A in Figure 8.3.1b. As load increases the stress and strain increase linearly. As the load is removed the stress is brought to zero along the straight line shown in blue. This is the elastic region and the structure is undeformed at the end of the load - unload cycle.

In the second case the applied shear load makes the stress exceed the elastic limit and stops at B. During the unloading when the stress slowly returns to zero the unloading occurs along a line parallel to the blue line but starting at B. illustrated by the red arrows.When the stress is zero the point on the stress strain cure is O_1. The strain here is not zero. There is a resident strain - γ_1. This is a permanent shear deformation in the structure even if there is no stress.

The next time the structure is loaded it will progress along the line O_1B. If the stress now exceeds the stress at B it will continue along the original stress - strain curve until the point C. When unloaded it will descend parallel to the elastic limit with a resident strain - γ_2. It is possible that these changes may change the original material by hardening. That is for an advanced course.

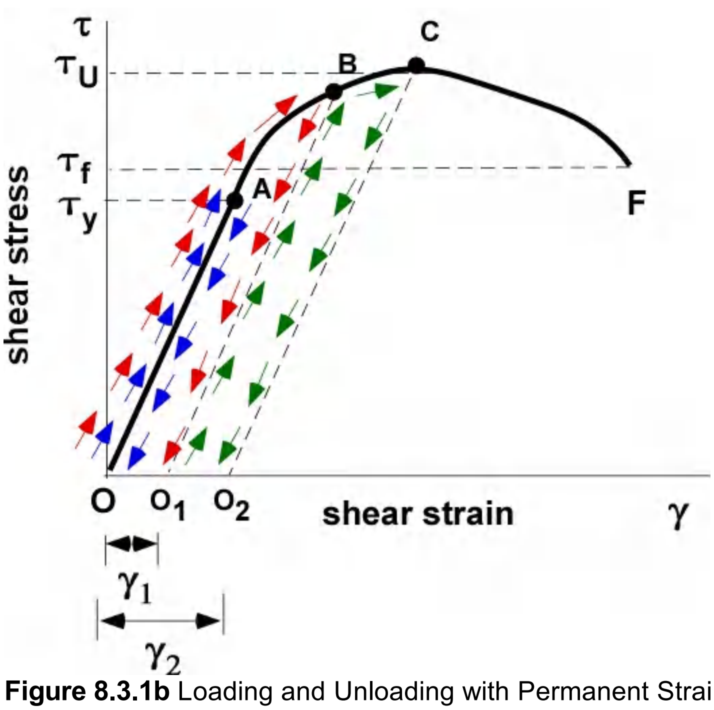


Figure 8.3.1b Loading and Unloading with Permanent Strain

These stress - strain loading and unloading cases describe plastic deformation. This is the deformation beyond the elastic range. In the stress - strain curve in in Figure 8.3.1 note that after the point C there is an increase in strain with a decrease in stress - another characteristic of plastic behavior. To use the stress - strain curve until fracture it in necessary to model the curve in three regions. The elastic region between OA. The nonlinear increase in strain with stress between BC. The nonlinear increase in strain with decrease in stress between CF.

8.3.2 Idealization of Plastic Behavior

The early approaches to avoid nonlinear behavior was to linearize the segments creating a piecewise stress-strain model for the material. The left figure in Figure 8.3.2 is modeled as three regions. This simplifies the calculations even though the stress-strain behavior underestimates the stress. The right figure in Figure 8.3.2 illustrates an even simpler model of the plastic behavior. This is the ideal elastic and the ideal plastic behavior, also called elastoplastic behavior. This was introduced during discussion on bending.

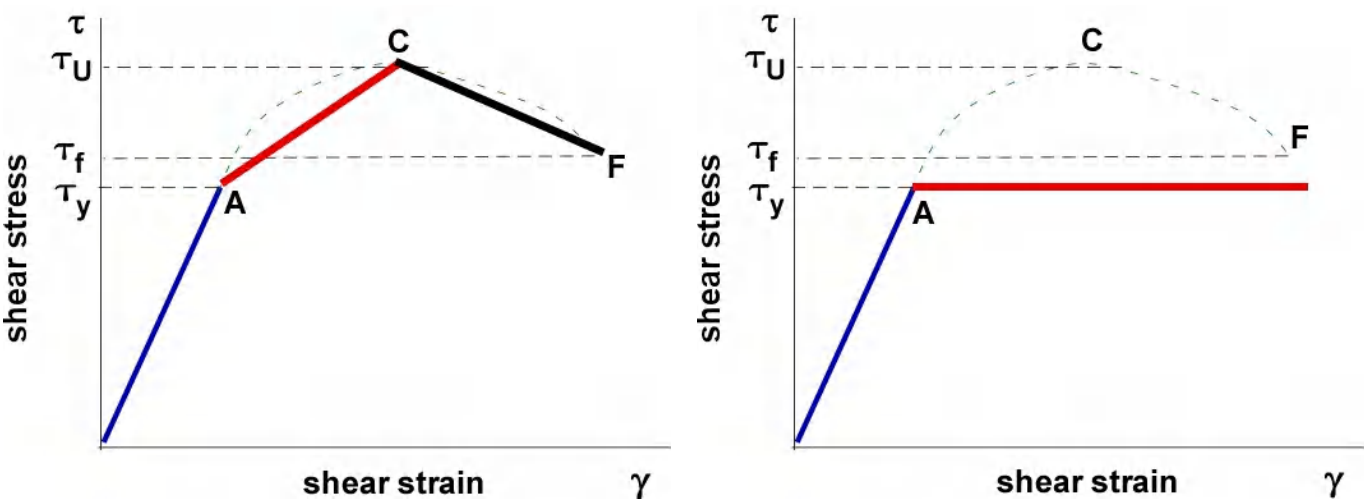


Figure 8.3.2 Idealization of the stress-strain behavior

An elastoplastic material is a simple model for the nonlinear behavior using piecewise linear segments. The first segments is the standard linear behavior captured by the Hooke's law. After the elastic limit the stress-strain curve becomes horizontal until fracture. Stressing beyond the elastic limit and removing the load can cause a permanent strain in the structure. Consider a simple solid shaft , having the ideal elastoplastic behavior and subject to increased torque. The shear stress distribution at any radial line is shown in Figure 8.3.3.

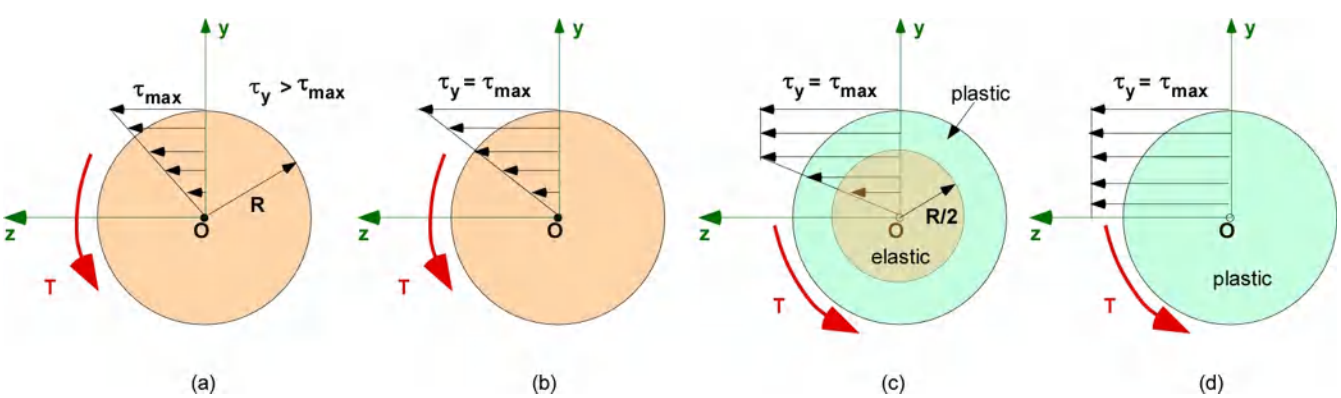


Figure 8.3.3 Elastoplastic behavior

The type of material response to a simple torsion loading is shown in the four different figures in Figure 8.3.3.

- In Figure 8.3.3 (a) the maximum shear stress in the material is less than the shear yield strength everywhere.
- In Figure 8.3.3 (b) the maximum shear stress in the material is at the shear yield strength.
- In Figure 8.3.3 (c) part of the material exhibits elastic behavior and part of the material is in the plastic region.
- In Figure 8.3.3 (d) the entire cross-section behaves plastically.

Example 8.9
Calculate the applied torque T for each of the torsional behavior in Figure 8.3.3.

Data: Use generic symbols shown in Figure 8.3.3.

Assumption: The relation between torque T and the shear stress distribution $\tau(r)$ was established earlier in the development. The moment due to the shear force distribution in the cross-section must be the same as the applied torque.

Solution: The case for the linear elastic behavior is considered below through the Figure 8.3.4. The remaining cases are inferred without a figure. These relations are valid for a solid circular shaft.

Case (a):

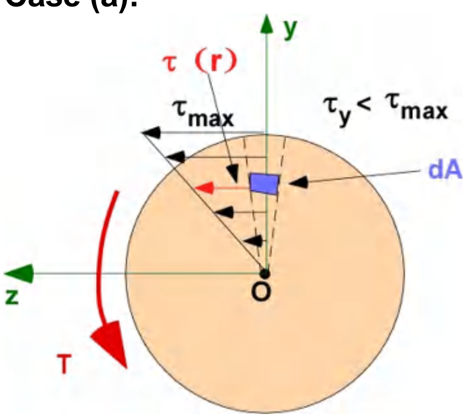


Figure 8.3.4. Example 8.9

The relation between **T** and τ_{max} **:**

$$\tau(r) = \tau_{max}\frac{r}{R} = \frac{\tau_{max}}{R}r; \quad dA = dr(r\,d\theta);$$

$$dF = \tau(r)dA = \frac{\tau_{max}}{R}r^2 dr\,d\theta;$$

$$T_a = \int_{2\pi}^{0}\int_{R}^{0} r\,dF = \int_{2\pi}^{0}\int_{R}^{0}\frac{\tau_{max}}{R}r^3 dr\,d\theta = \int_{R}^{0}2\pi\frac{\tau_{max}}{R}r^3 dr$$

$$= 2\pi\frac{\tau_{max}}{R}\int_{R}^{0}r^3 dr = \frac{\tau_{max}}{R}\left(\frac{\pi}{2}R^4\right) = \frac{\tau_{max}}{R}J$$

Case (b): $\tau_{max} = \tau_y$

$$T_b = \frac{\tau_y}{R}J:$$ **Maximum Elastic Torque**

Case (c) : Elastoplastic behavior: Elastic region : R/2

$$dF_e = \tau(r)dA = \frac{\tau_y}{R/2}r^2 dr\, d\theta; \quad dF_p = \tau_y\, dA = \tau_y\, r dr\, d\theta;$$

$$T_c = \int_0^{2\pi}\int_0^{R/2} r\, dF_e + \int_0^{2\pi}\int_{R/2}^{R} r\, dF_p$$

$$T_c = 2\pi\frac{\tau_y}{R/2}\int_0^{R/2} r^3 dr + 2\pi\tau_y\int_{R/2}^{R} r^2 dr = 2\pi\frac{\tau_y}{R/2}\frac{(R/2)^4}{4} + \frac{2\pi\tau_y}{3}\left[R^3 - (R/2)^3\right] = \tau_y\,\pi R^3\left(\frac{62}{96}\right)$$

Case (d):

$$T_d = 2\pi\tau_y\int_0^R r^2 dr = \frac{2\pi}{3}\tau_y R^3 = \frac{4}{3}T_b$$ **: Maximum Plastic Torque**

8.3.3 Plastic Deformation in Torsion

In Example 8.9 the calculation of elastoplastic deformation when the shaft has both elastic and plastic region present is interesting. For Case (c) in the example the torque was calculated for a prescribed elastic region (R/2). Here we consider the inverse problem where the size of the elastic region indicated by $\mathbf{r_Y}$ for a known large angle of twist (ϕ) that has caused elastoplastic deformation needs to be determined. It will be possible to calculate the associated torque and then look at the torque versus angle of twist variation.

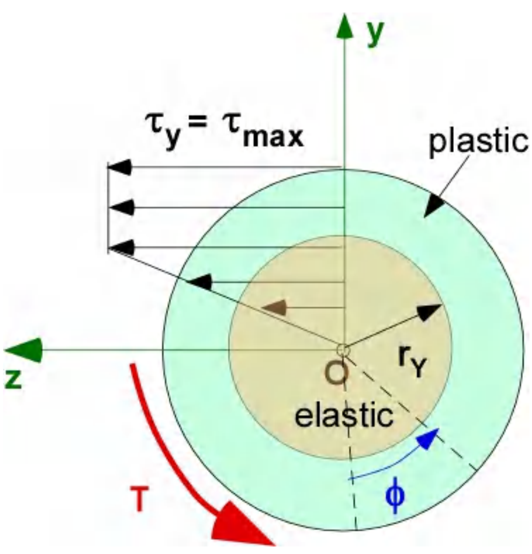


Figure 8.3.5 Elastoplastic deformation

The stress versus strain curve is still linear for the elastic region and is given by Equation 3.28. If the twist ϕ causes elastoplastic behavior then the extant of the elastic region over a length L of the rod can be obtained as:

$$r_Y = \frac{L\,\gamma_Y}{\phi}; \quad \tau_Y = G\,\gamma_Y;$$

If ϕ_Y is the twist at maximum elastic torque T_Y (T_b above), then

$$R = \frac{L\,\gamma_Y}{\phi_Y}; \quad \frac{r_Y}{R} = \frac{\phi_Y}{\phi};$$

Revisiting Case (c) with r_y instead of R/2 and working through the integral, using the relation above, and collecting terms the relation between applied torque T, maximum elastic torque T_Y, maximum elastic twist ϕ_Y and the current twist ϕ, can be determined (for $\phi > \phi_Y$) as:

$$T = \frac{4}{3} T_Y \left(1 - \frac{1}{4} \left(\frac{\phi_Y}{\phi} \right)^3 \right) \qquad (8.13)$$

The non dimensional plot of applied torque versus the twist for the range of elastoplastic behavior for a solid circular shaft can be seen in Figure 8.3.6. It is the information developed in Equation 8.7. The elastic range is linear. The plastic range is nonlinear. The torque ratio plateaus at a value of 4/3. The is a significant change in twist for small changes in torque.

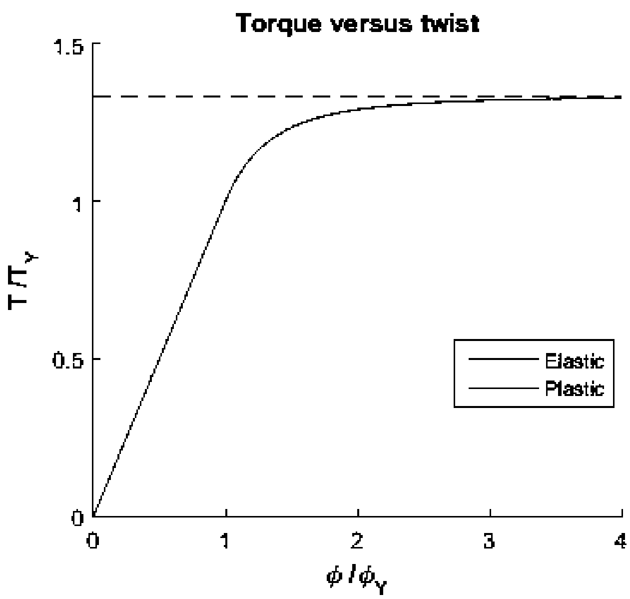


Figure 8.3.6 Elastoplastic torque and twist for solid circular shaft

Example 8.10

Find (a) the maximum elastic torque, (b) the maximum plastic torque, and (c) the torque versus twist curve in pure torsion for an annular shaft made of Aluminum Alloy 2014-T6 shown in Figure 8.3.7. The properties alloy are: density = 2800 kg/m^3; ultimate strength in shear is 275 MPa; shear yield is 230 MPa; modulus of rigidity is 27 GPa. The length of the shaft is 1.2 m.

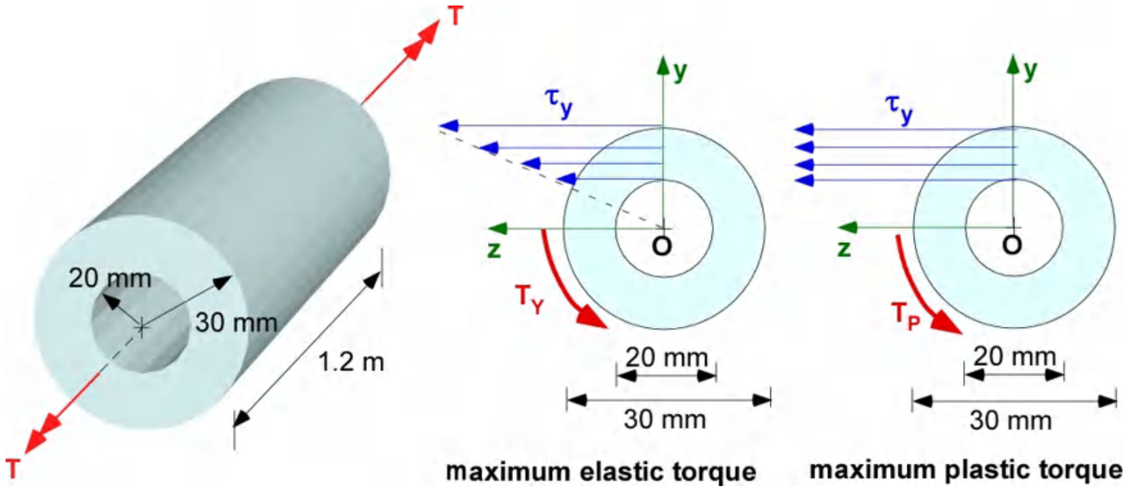


Figure 8.3.7 Example 8.10

Data: R_1 = 0.02 m; R_2 = 0.03 m; L = 1.2 m; τ_y = 230 MPa; τ_u = 270 MPa; G = 27 GPa

Assumption: For maximum elastic torque the stress distribution is linear across the section with the maximum at the shear yield strength. For maximum plastic torque the stress distribution is constant at the value of the shear yield strength.

Solution: The calculation of the maximum elastic and plastic torque are the same as Cases (b) and (c) in Example 8.9. This is implemented in MATLAB below:

Solution Using MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, June 2018
% Section 8.3.3 - Example 8.10
% Annular cylindrical shaft
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 8.10\n')
fprintf('--------------------------\n')
fprintf('Elastoplastic deformation of annular shaft\n')
fprintf('----------------------------------------------\n')
%% Data
r1 = 0.02;  r2 = 0.03;  L = 1.2;  tauy = 30e06;  G = 27e09;
J = 0.5*pi*(r2^4 - r1^4);

% maximum elastic torque
Ty = tauy*J/r2;
gamy = tauy/G;
phiy = gamy*L/r2;

% maximum plastic torque
syms r
int(r^2,r,r1,r2);
Tp = 2*pi*tauy*int(r^2,r,r1,r2);
phip = gamy*L/r1;
```

```
% print
fprintf('inner radius    [m] :'),disp(r1)
fprintf('outer radius    [m] :'),disp(r2)
fprintf('Length          [m] :'),disp(L)
fprintf('Yield strength  [MPa]:'),disp(tauy/1e06)
fprintf('Shear modulus   [GPa]:'),disp(G/1e09)
fprintf('MOI J -         [m^4]:'),disp(J)

fprintf('----------------------\n')
fprintf('Maximum Elastic Torque\n')
fprintf('-----------------------\n')
fprintf('T_Y (torque)    [Nm] :'),disp(Ty)
fprintf('yield strain    [rad]:'),disp(gamy)
fprintf('Elastic twist  -[rad]:'),disp(phiy)

fprintf('----------------------\n')
fprintf('Maximum Plastic Torque\n')
fprintf('-----------------------\n')
fprintf('T_P (torque)    [Nm] :'),disp(double(Tp))
fprintf('Plastic twist  -[rad]:'),disp(phip)

%% twist curve
fg1 = figure;
set(fg1,'Position',[50,50,400,350])
% below maximum elastic torque
syms rp tm
T1 = 2*pi*tm*int(r^3,r,r1,r2)/r2;
phi1 = T1*L/G/J;
tr1 = T1/Ty;    % normalize torque
phir1 = phi1/phiy;  % normalized twist

% between maximum elastic and plastic torque
T2 = 2*pi*tauy*int(r^3,r,r1,rp)/rp + ...
    2*pi*tauy*int(r^2,r,rp,r2);
phi2 = tauy*L/rp/G;  % based of yield strain at the inbetween radius
tr2 = T2/Ty;
phir2 = phi2/phiy;  % normalized twist

%plot
ezplot(phir1,tr1,[0,tauy]);
hold on
ezplot(phir2,tr2,[r1,r2]);

hold off
grid
xlabel('\bf\phi /\phi_Y')
ylabel('\bfT /T_Y')
title('\bfTorque versus twist')
legend('Elastic','Plastic','Location','Best')
box off
```

In the Command Window:

```
----------------------
Example 8.10
```

```
----------------------------------------------
Elastoplastic deformation of annulaThe setr shaft
-----------------------------------------------
inner radius    [m] :         0.02
outer radius    [m] :         0.03
Length          [m] :          1.2
Yield strength  [MPa]:    30
Shear modulus   [GPa]:    27
MOI J -         [m^4]:    1.021e-06
----------------------
Maximum Elastic Torque
-----------------------
T_Y (torque)    [Nm] :          1021
yield strain    [rad]:    0.0011111
Elastic twist  -[rad]:     0.044444
----------------------
Maximum Plastic Torque
-----------------------
T_P (torque)    [Nm] :        1193.8
Plastic twist  -[rad]:     0.066667
```

In the Figure Window

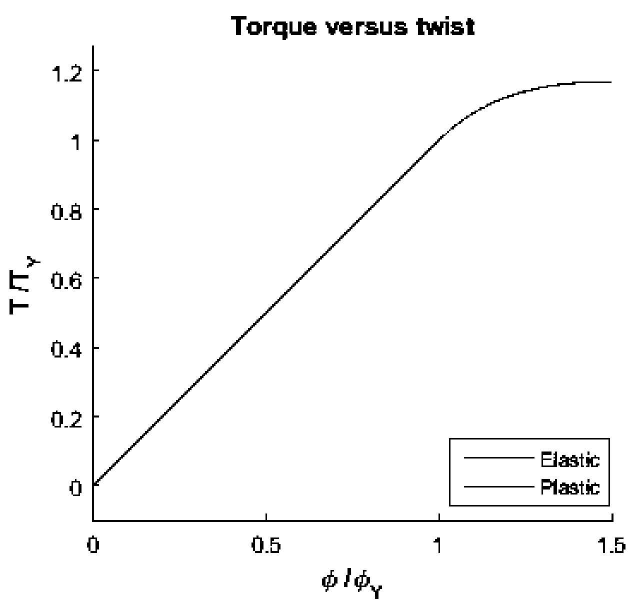


Figure 8.3.8 Normalized torque versus twist for Example 8.10

The maximum normalized torque levels of at 1.1692 beyond the range of twist angle plotted.

8.3.4 Residual Stresses

Residual stress and strain are present when the material is stressed beyond the elastic limit and then unstressed. Using Example 8.9 for illustration and incorporating the idea from Section 8.3.1, consider the applied torque (T_A) at 1.2 the maximum elastic torque (T_A/T_Y = 1.2) and then brought to zero. Figure 8.3.9 illustrates the response of the material in several ways.

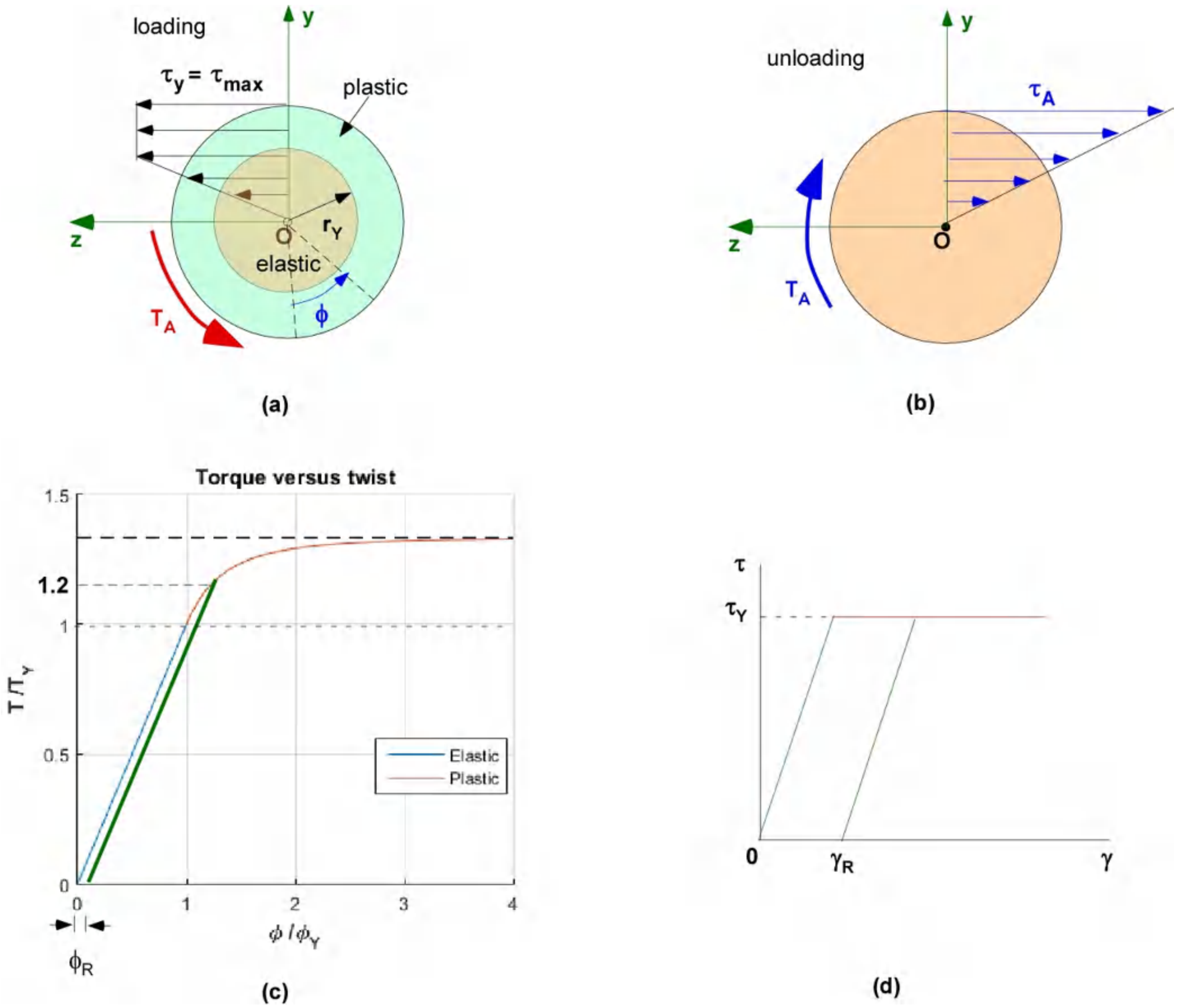


Figure 8.3.9 Residual stress and twist

1. Figure 8.3.9(a) shows the initial loading where the stresses are elastoplastic. Using equation (8.7) the ratio of the actual twist can be obtained corresponding to the applied torque T_A ratio. The extent of the elastic region can also be calculated using the relations above. The set of values for the specific applied torque T_A is:

$$1.2 = (4/3)\left[1 - \frac{1}{4}\left(\frac{\phi_Y}{\phi}\right)^3\right]; \quad \frac{\phi_Y}{\phi} = 0.737; \quad \phi_A = \frac{\phi_Y}{0.737} = 1.357\,\phi_Y$$

$$r_Y = 0.737R; \quad \phi_Y = \frac{\tau_Y L}{GR}; \quad T_Y = \frac{\tau_Y J}{R}; \quad T_A = 1.2T_Y$$

2. Figure 8.3.9(b) is the unloading of the applied torque. Since this process is linear and parallel to the original elastic line a new applied stress τ_A which will be larger than the yield strength τ_Y.

$$\tau_A = \frac{T_A R}{J} = \frac{1.2T_Y R}{J};$$

3. Figure 8.3.9(c) illustrates that the unloading is along the green line establishing a residual twist of ϕ_R.

$$\phi_R = \phi_A - \frac{\tau_A L}{GR} = \phi_A - 1.2\phi_Y = 0.157\,\phi_Y;$$

4. Figure 8.3.9(d) is same information as Figure 8.3.9(c) on the elastoplastic stress strain curve.

The residual stresses is obtained by adding the stress distribution in Figure 8.3.9(b) and Figure 8.3.9(a). This is shown in Figure 8.3.10.

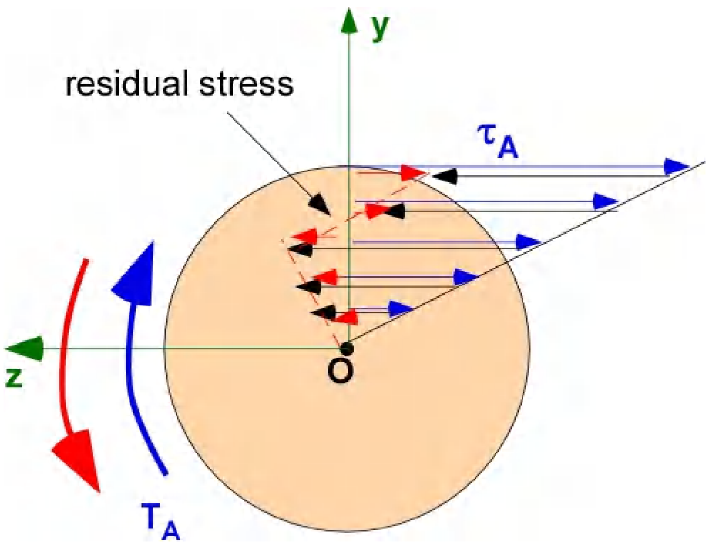


Figure 8.3.10 Residual stress is the shaft

8.3.5 Transmission Shafts

Shafts and torsion are the primary mechanism that generates and delivers mechanical power in many of the devices that people use everyday. Transmission shafts and gears are the structures that make this possible. The design must meet the shear yield criteria laid out in the previous sections. Two new variables appear in the discussion of transmission shafts. They are power (**P**) and the angular speed of rotation (Ω or ω). The former is the speed expressed in revolutions per minute and the latter is the speed expressed in radians per second.

The additional relations are :

$$P = T\omega \qquad (8.14)$$

$$\omega = \frac{2\pi\Omega}{60}$$

Example 8.11

A Ford Mustang GT outputs 360 hp (horsepower) at 6350 rpm (revolutions per minute). The transmission shaft is a steel tube with an outside diameter of 60 mm. Find the thickness of the tube for a factor of safety of 2.5. The shear yield strength is 150 MPa and the modulus of rigidity is 75 GPa.

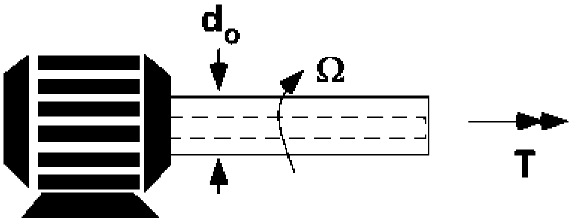


Figure 8.3.11 Example 8.11

Data: P = 360 hp = 268.56 kW; Ω = 6350 rpm = 664.97 rad/s; FOS = 2.5; τ_y = 150 MPa; d_o = 60 mm; R_2 = 30 mm

Find: $t = R_2 - R_1$

Assumption: Use the relations for maximum elastic torque.

Solution: The calculations are straight forward

$$T = \frac{268560}{664.97} = 403.87\left[Nm\right]$$

$$\tau_{all} = \frac{150\times10^6}{2.5} = 60\times10^6\left[Pa\right]$$

$$J = \frac{\pi}{2}\left[\left(\frac{30}{1000}\right)^4 - R_1^4\right] = \frac{TR_2}{\tau_{all}} = 2.0193\times10^{-7}\left[m^4\right]$$

$$R_1 = 0.0287; \quad t = 0.0013\left[m\right] = 1.3\left[mm\right]$$

The angular speed has a very important effect on the thickness. Consider working through the problem with different values to get an idea for the design of such shafts. For example an angular speed of 635 rpm rpm will require a thickness of 11.4 mm.

8.3.6 Eccentric Loading Resulting in Bending and Torsion

This topic was visited in bending where an eccentric pair of forces also produces bending. A simple case of multiple loading occurs when a force is offset from the center of the cross-section of a long shaft. In Figure 8.3.12a force F is offset from the center of a shaft of an annular cross-section of length L. In figure 8.3.12b we see the loads developed at the cross-section of a point A along the length. There is shear force, torque and a bending moment developed in the cross-section. Every point in the cross-section will see both normal and shear stress. From a design perspective it will be appropriate to evaluate the design based on a failure criteria because of the presence of normal and shear stresses.

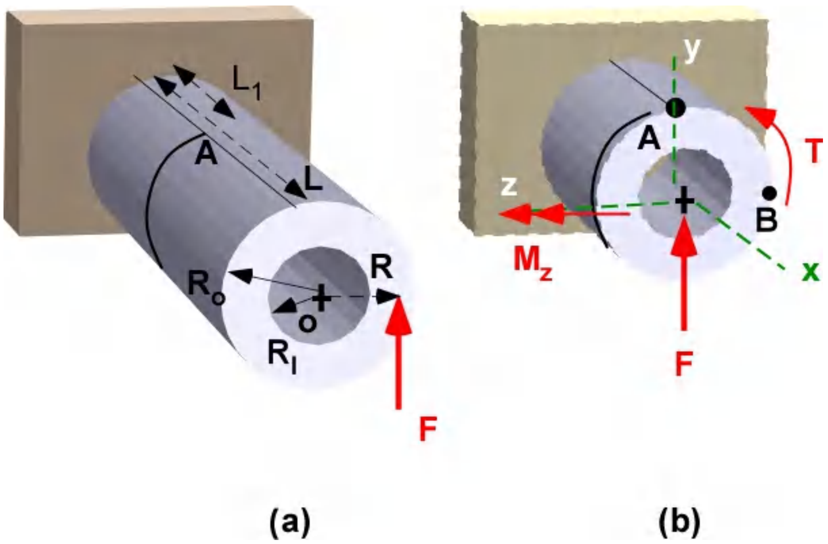


Figure 8.3.12 Eccentric loading

The loads in the cross-section at A, apart from F, are obtained by moving a force parallel to its line of action. They are

$$T = R\,F;$$

$$M_z = F\left(L - L_1\right)$$

Example 8.12 provides an opportunity of following through with the calculations.

Example 8.12
An annular steel shaft. Figure 8.3.12a is subject to an eccentric load of 2500 N at a radius of 56 mm. The outside diameter of the shaft is 60 mm and the inside diameter is 50 mm. The shear yield strength is 150 MPa and the yield strength in tension is 260 MPa. The modulus of rigidity is 75 GPa while the modulus of elasticity is 190 GPa. The length of the shaft is 0.8 m. The cross-section at A is located at 0.1m from the fixed end. Find the Von Mises stress at the point A on the outside surface of the shaft and calculate the factor of safety.

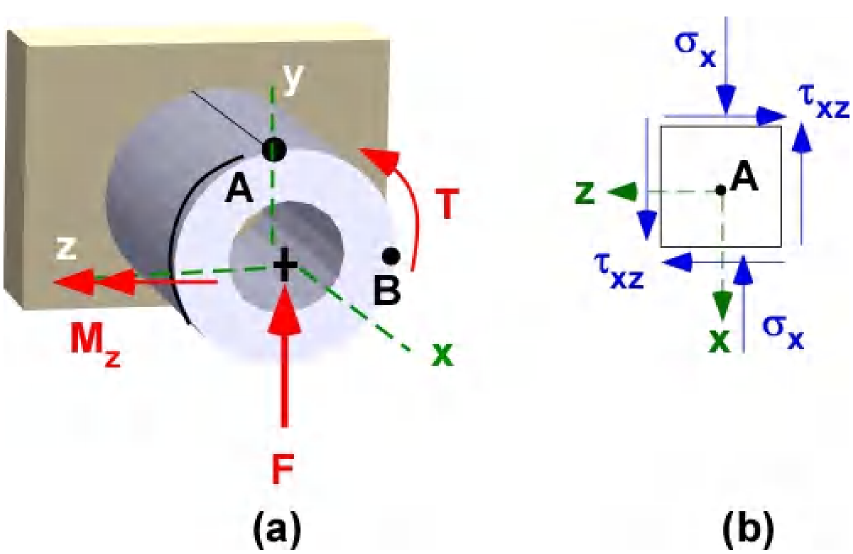


Figure 8.3.13 Example 8.12

Data: F = 2500 N; R_o = 60 mm; R_i = 50 mm; R = 56 mm; L = 0.8 m; L_1 = 0.1 m; σ_Y = 190 GPa;

Find: Von Mises stress at A and FOS

Assumption: For the point A the loads will create a state of stress indicated in Figure 8.3.13b. First calculate the normal and shear stresses at point A. Then calculate the principal stress and the corresponding Von Mises stress at point A.

Solution: The calculations are direct. It is implemented in MATLAB below

Solution Using MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, June 2018
% Section 8.3.6 - Example 8.12
% Annular cylindrical shaft
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 8.12\n')
fprintf('---------------------------\n')
fprintf('Eccentric Loading - bending and torsion\n')
fprintf('-----------------------------------------------\n')
%% Data
ro = 60/1000;  ri = 50/1000;  r = 56/1000;
L = 0.8;  L1 = 0.1;
tauy = 150e06;  G = 75e09;
sigy = 260e6;  E = 190e9;
J = 0.5*pi*(ro^4-ri^4);
Iz = J/2;
fprintf('outer radius  [m] ='),disp(ro)
fprintf('inner radius  [m] ='),disp(ri)
fprintf('moment radius [m] ='),disp(r)
fprintf('Polar MOI [m^4]   ='),disp(J)
fprintf('Iz -  MOI [m^4]   ='),disp(Iz)
fprintf('\nMaterial steel \n')
fprintf('Yield Strength [MPa] ='),disp(sigy/1e6)
%% Loads
F = 8000;
T = F*r;
Mz = F*(L-L1);
fprintf('\nForce(F)      [N] ='),disp(F)
fprintf('Torque(T)     [Nm] ='),disp(T)
fprintf('Bending Mom.  [Nm] ='),disp(Mz)

%% Stresses
tauA = T*ro/J;
sigA = -Mz*ro/Iz;
fprintf('\nStress at A')
fprintf('\n--------------\n')
fprintf('No shear stress due to F\n')
fprintf('Shear stress due to T [MPa]   = '),disp(tauA/1e6)
fprintf('Normal stress due to Mz [MPa] = '),disp(sigA/1e6)
```

```
%% Principal stresses
avgsig = sigA/2;
R = sqrt((sigA/2)^2 + tauA^2);
sig1 = avgsig + R;
sig2 = avgsig - R;
fprintf('\nPrincipal stress 1 [MPa]    = '),disp(sig1/1e6)
fprintf('Principal stress 2 [MPa]    = '),disp(sig2/1e6)
%% Von Mises stress and FOS
sigvm = sqrt(sig1^2 -sig1*sig2 + sig2^2);
FOS = sigy/sigvm;
fprintf('Von Mises stress at A [MPa]    = '),disp(sigvm/1e6)
fprintf('Factor of Safety - FOS         = '),disp(FOS)
```

In the Command Window

```
Example 8.12
--------------------------
Eccentric Loading - bending and torsion
----------------------------------------------
outer radius  [m] =         0.06
inner radius  [m] =         0.05
moment radius [m] =        0.056
Polar MOI [m^4]   =    1.054e-05
Iz -  MOI [m^4]   =     5.27e-06

Material steel
Yield Strength [MPa] =   260

Force(F)      [N] =        8000
Torque(T)     [Nm] =   448
Bending Mom.  [Nm] =         5600

Stress at A
--------------
No shear stress due to F
Shear stress due to T [MPa]    =        2.5503
Normal stress due to Mz [MPa] =       -63.757

Principal stress 1 [MPa]    =       0.10185
Principal stress 2 [MPa]    =       -63.859
Von Mises stress at A [MPa]    =         63.91
Factor of Safety - FOS         =        4.0682
```

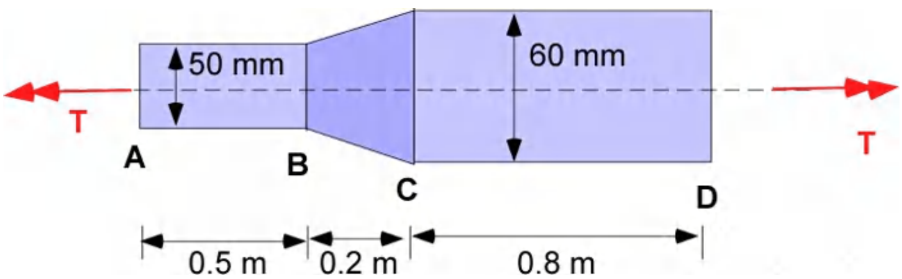
Execution in OCTAVE
There is no change in the code.

The results in the Command Window are also the same

8.3.6 Additional Problems

Problem 8.3.1
A tapered solid circular steel shaft is subject to a torque 7500 Nm. The shaft is assumed to be elastoplastic with the shear yield strength at 150 MPa and the modulus of rigidity of 77 GPa. (a) Describe the stress distribution in the region AB; (b) Describe the stress distribution in the region CD;

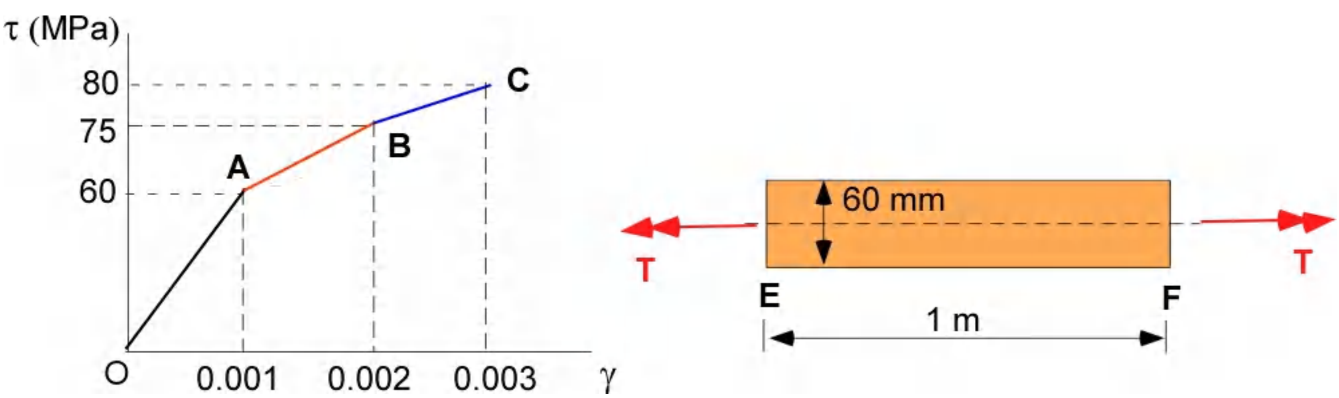
(c) What is the largest torque that can be applied to the shaft?



Problem 8.3.1

Problem 8.3.2

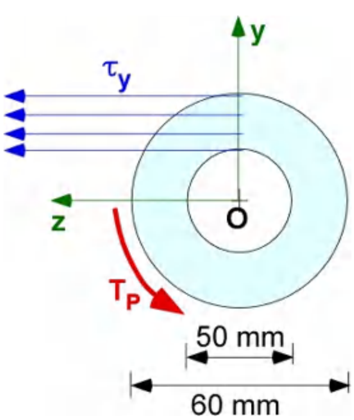
A solid circular shaft carrying a torque over a meter of length has the accompanying stress-strain curve. (a) What is the torque if the maximum shearing stress in the shaft is 78 MPa. (b) What is the angle of twist over the length of the shaft for this torque.



Problem 8.3.2

Problem 8.3.3

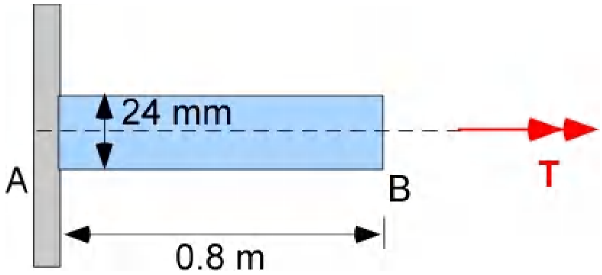
The annular elastoplastic aluminum shaft, shear yield strength of 230 MPa and modulus of rigidity of 27 GPa, is subject to a maximum plastic torque and then slowly unloaded. (a) Calculate the maximum torque applied. (b) Calculate the residual shearing stress in the shaft.



Problem 8.3.3

Problem 8.3.4

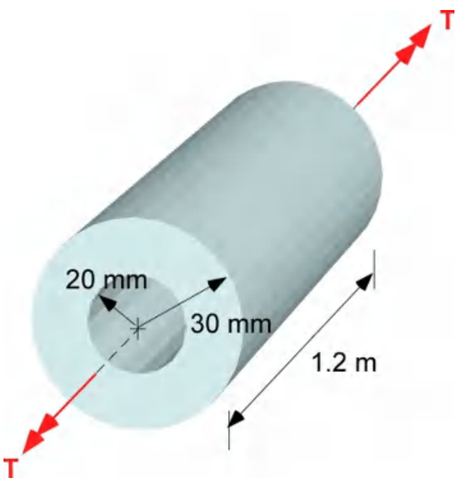
The elastoplastic solid steel shaft, with shear yield strength at 150 MPa and the modulus of rigidity of 77 GPa, is twisted through 5.5 degrees and then unloaded. (a) Calculate the residual shearing stress in the shaft. (b) Calculate the residual twist in the shaft.



Problem 8.3.4

Problem 8.3.5

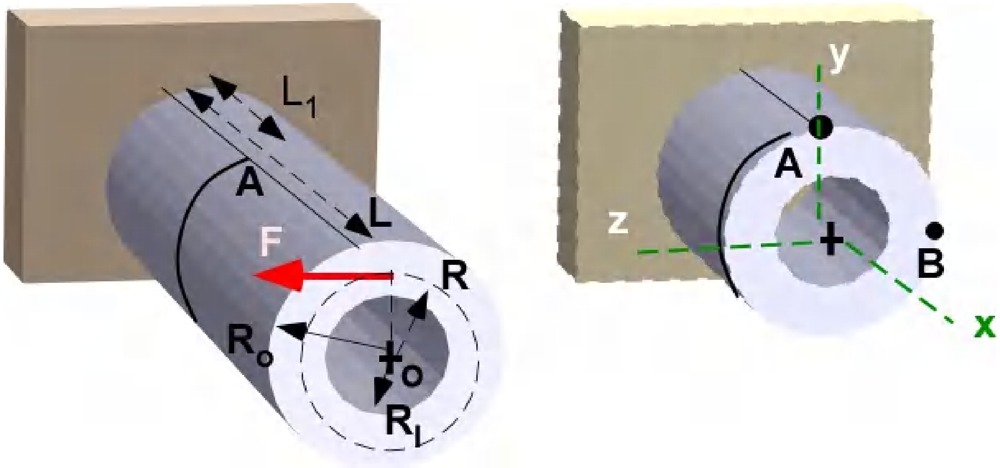
A hollow shaft with an yield strength of 260 MPa and a modulus of rigidity of 75 GPa rotates at 320 rpm. The angle of twist for the shaft is measured as 3.5 degrees. (a) What is the power transmitted by the shaft? (b) What is the FOS for this operation?



Problem 8.3.5

Problem 8.3.6

This is same as Example 8.12 but the applied eccentric force has a different orientation. The eccentric load is 4000 N at a radius of 55 mm.The rest of the information is same. The steel shaft is annular. The outside diameter of the shaft is 60 mm and the inside diameter is 50 mm. The shear yield strength is 150 MPa and the yield strength in tension is 260 MPa. The modulus of rigidity is 75 GPa while the modulus of elasticity is 190 GPa. The length of the shaft is 0.8 m. The cross-section at A is located at 0.1m from the fixed end. Points A and B ore on the outside diameter and located as shown. Find the Von Mises stress at the point A and B on the shaft and calculate the factor of safety. Draw the loads in the cross-section. Draw the state of stress at A and B.



Problem 8.3.6

9. BUCKLING

(This section first appeared in Section 3.12)

Buckling is another mode of failure. This happens with tall or long structures that are carrying a compressive load.

The simple deflection of the structure causes a bending load due to the eccentricity of the load because of the deflection. Very often this is also termed as column buckling. Figure 9.1 (also Figure 3.16.1) illustrates buckling of columns in testing performed at NIST (National Institute of Standards) in the NIST Building and Fire Research Laboratory, particularly the column identified as Col 79. This column is bent under the application of a compressive load. Note Col 80 also bends but has a different shape. Both of the columns are deflected sideways.

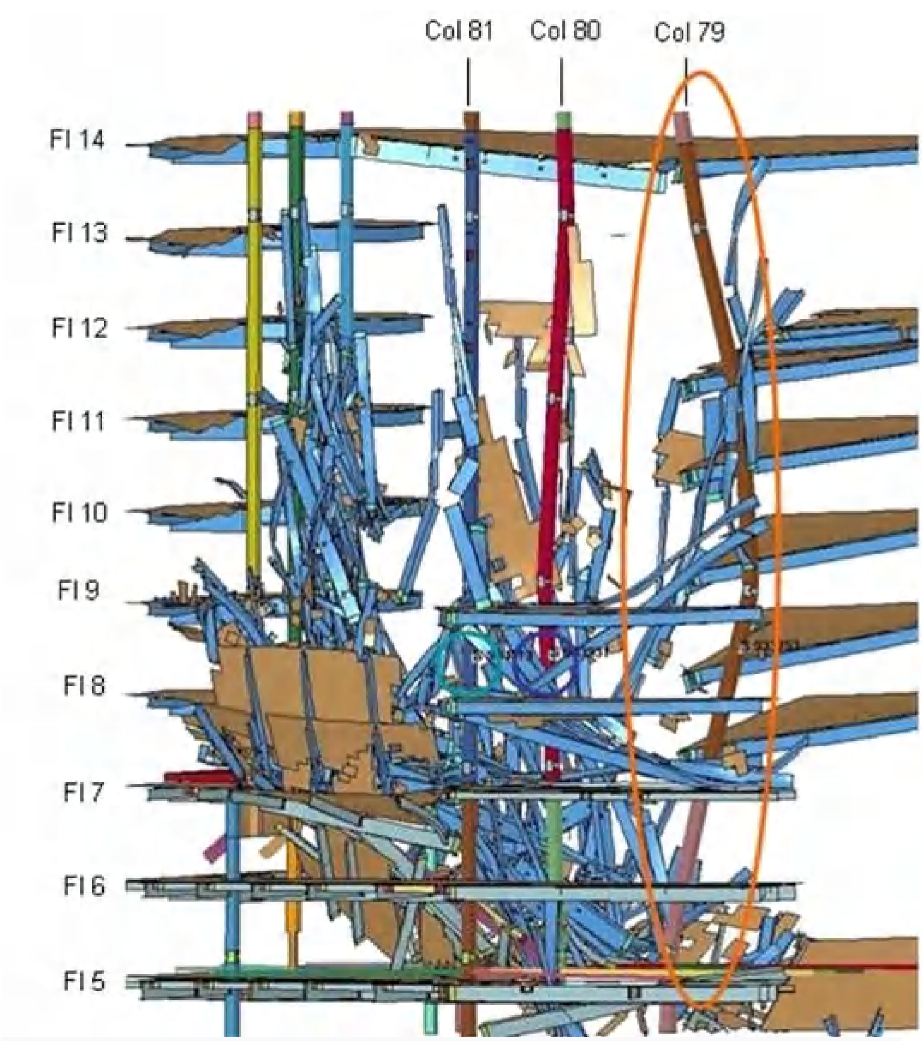


Figure 9.1 Column buckling (NIST Building and Fire Research Laboratory).

Buckling can occur at a load well below the direct compressive stress that lead to compressive failure. For this reason bucking is considered a *structural* ***instability***. For a structure carrying a load the simple idea of instability is that if the load exceeds a certain value it will no longer be able to support the load and may fail catastrophically. Instability is bad in all structural designs because you cannot guarantee the safety or performance of the structure. Apart from the failure criteria that was presented earlier it is important that the design will not experience the *critical buckling stress* or the *critical buckling* load. If loading increases beyond the critical buckling load the structure will respond unpredictably leading to the loss of any capacity to carry the load.

Nevertheless *buckling* is a special case of loading. The structure is long and is subject to compression. Buckling can be related to the length of the column as can be derived for the basic model of buckling. Compressive loads can develop on a structure due to thermal strains. An example of this is the gap between between adjacent sections of railway tracks. If the outside temperature exceeds the design maximum the tracks will expand and press against each other creating enough compressive load to detach from the foundation, Figure 9.2 (also Figure 3.16.2). There are other types of buckling like multiple column buckling (bicycle wheels), plate buckling (extension of column buckling), surface buckling (creation of potholes in winter). They will always cause failure if they occur.


Figure 9.2 Buckling of tracks (Wikipedia)

Column buckling relations are obtained from the model of buckling that is described by differential equations and their solutions. The model itself should be easy to develop based on eccentric loading discussed elsewhere. The solutions are borrowed directly and their interpretation are taken for granted.

9.1 COLUMN BUCKLING - PIN CONNECTED ENDS

(This section first appeared in Section 3.16)

The simplest model for column buckling is the Euler formula for columns having pin connections at the ends. The buckling shape or deformation is influenced by the type of end connections as shown in Figure 9.1.1 (Figure 3.16.3). To keep this material independent we will use new figure numbers during the development even if they have appeared before in Section 3.16.

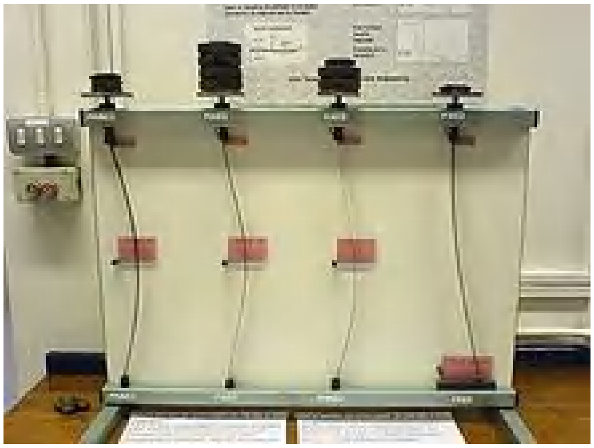


Figure 9.1.1 Types of buckling deformation (Wikipedia)

Since buckling deformation is a transverse deformation the applied load must be able travel so the model has the compressive load mounted on a rail as shown in Figure 9.1.2.

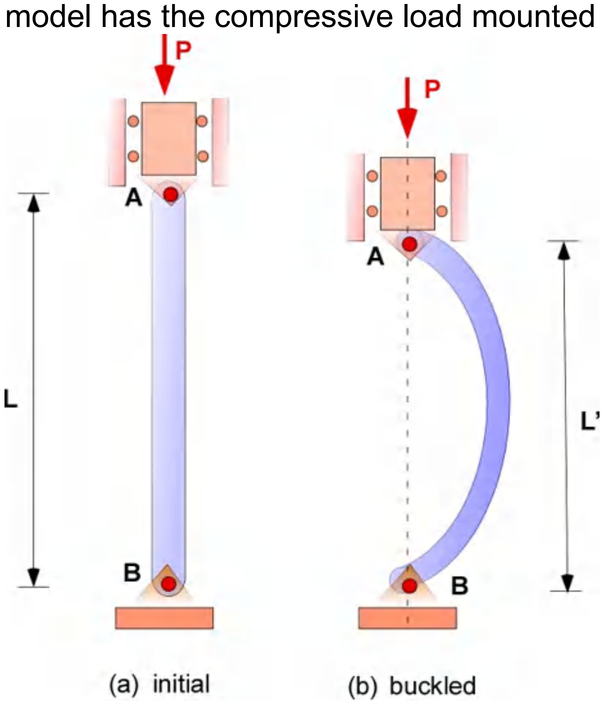


Figure 9.1.2 Illustration of pinned column buckling

The Euler buckling model is developed using a free body diagram of the part of the buckling deformation shown in Figure 9.1.2 b. In the model the pin supports cannot deflect. These conditions are translated to the boundary conditions on the differential equations describing the model. Small deflections are also assumed so that the relations from pure bending are incorporated into the development. One anomaly is that the original length L of the column and the deflected length L' are considered the same even though they are visually different - even in the experimental setups above. It is necessary for the completeness of mathematical model since two boundary conditions are required. The model development is shown in Figure 9.1.3.

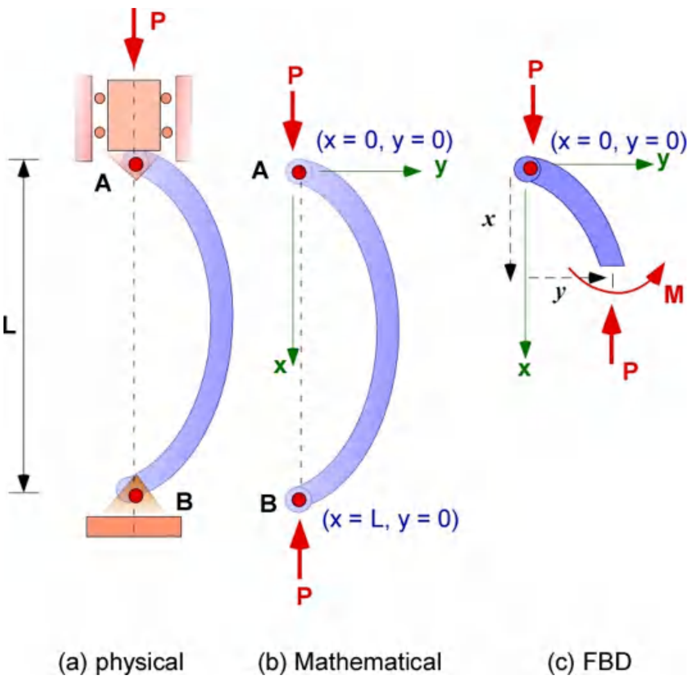


Figure 9.1.3 Euler buckling model

First apply the equations of equilibrium:

$$\sum F_x = 0 = P - P$$

$$\sum M_z = 0 = M + Py$$

Now replace the bending moment M through its relationship to the radius of curvature in pure bending (3.33 or 7.13) as:

$$M = EI\frac{d^2y}{dx^2} = -Py; \quad OR$$

$$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = 0; \quad OR \qquad (9.1)$$

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0; \quad \lambda^2 = \frac{P}{EI}$$

In the above equation ***x*** is considered the *independent* variable and ***y*** is the *dependent* variable. The solution for the deflection being sought is y as a function of x or ***y(x)***. The final equation is called a *differential equation* since it contains *derivatives* and there is an *equal to* sign in the expression. It is usually expressed by grouping terms in the dependent variable on the *left side* of the equal sign. The beam deflection equation in Chapter 7 was also a differential equation but it could be solved by direct integration. Here it is not so. Some additional concepts are necessary for the solution to this equation.

This book will likely be used in the first year of the engineering curriculum. It is likely that the student may not yet be formally introduced to differential equations from the core math curriculum. Rather than avoid this topic a very brief and to the point ideas and discussions are introduced in the following.

The equation is considered *homogeneous* if there is no terms on the right of the equal to sign. The *order* of the equation is based on the highest derivative in the expression. If the powers of the terms involving the independent variable is *one in the terms on the left side* then it is considered *linear.* If the coefficients of the terms on the left are constant it is considered a *constant coefficient* differential equation.

The differential equation in (9.1) is a homogeneous, second order, linear, differential equation with constant coefficients. To emphasize, it is linear because the power of the second derivative is one and the power of y is one.The general solution for this equations is:

$$y(x) = C\sin\lambda x + D\cos\lambda x$$

C and D are two constants that must be determined to describe the solution - the buckling deformation. The differential equation in (9.1) by itself is incomplete. A solution can only be found if the boundary conditions are given. A second order equation must be accompanied by two boundary conditions and these are used to determine the constants C and D. The natural boundary conditions for the pinned support at A (x = 0) and B (x = L) are that the displacement y is zero at these locations.

At x = 0 :

$$y(0) = D = 0$$

At x = L:

$$y(L) = C\sin\lambda L = 0$$

If C = 0 then there is no buckling (simple compression). If C is not zero - there is lateral displacement - then the ***sine*** term must be zero.Therefore

$$\sin(\lambda L) = 0 \quad OR \quad \lambda L = \pm n\pi \quad OR \quad \lambda^2 L^2 = n\pi^2$$

$$\lambda^2 = \frac{n\pi^2}{L^2}; \quad P = EI\lambda^2 = \frac{n\pi^2 EI}{L^2}; \qquad (9.2)$$

$$y(x) = C\sin\left(\frac{\pm n\pi x}{L}\right)$$

The smallest value of P, the critical buckling load, corresponds to n = 1 and I therefore

$$P_{cr} = \frac{\pi^2 EI}{L^2};$$
$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL^2} \qquad (9.3)$$

The relations above are the Euler's formula for buckling (for pinned ends). The critical buckling stress is also obtained above. If the MOI can be expressed using the cross-sectional area and the radius of gyration (k) - in most handbooks for common cross-sections then

$$I = Ak^2;$$
$$\sigma_{cr} = \frac{\pi^2 E}{(L/k)^2} \qquad (9.4)$$

The ratio ***L/k*** is called the ***slenderness ratio*** of the column. Since buckling is a response to compressive load the structural design considerations is a combined response to simple compressive stress and critical buckling stress. The latter is usually lower but kicks in at a higher value of the slenderness ratio. For structural steel with E = 200 GPa and σ_y = 250 MPa the design limits for compressive loads are seen in Figure 9.1.4. The buckling stresses above the yield strength can be ignored since the material will fail in simple compression. The critical slenderness ratio for the plot is 88.86. This requires the critical buckling stress to be the same value as the yield stress. This is the maximum buckling stress. An increase in slenderness ration will decrease the critical buckling stress.

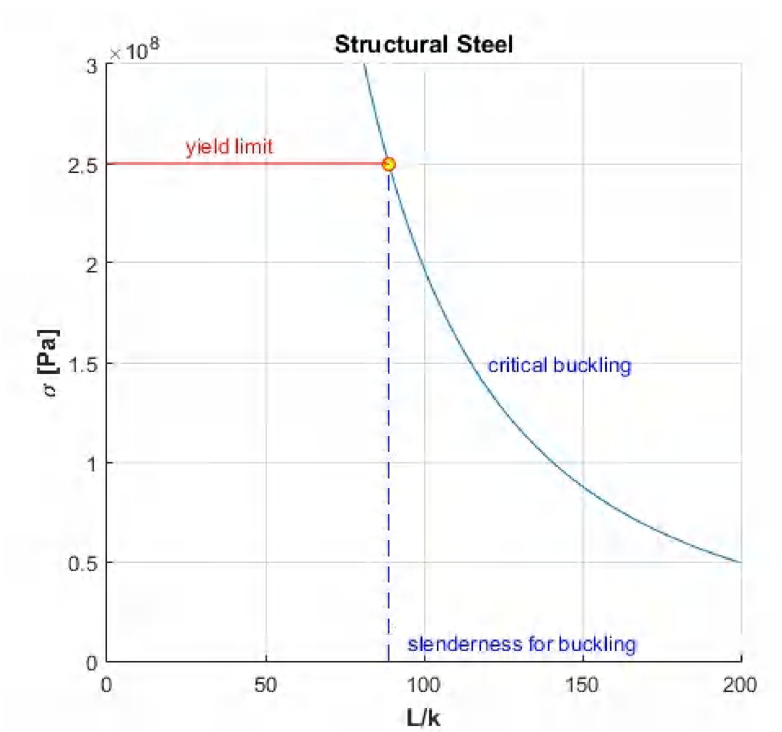


Figure 9.1.4 Column design for compressive stress

Buckling Deflection

When the column has buckled its displacement , or the elastic curve, is given by

$$y(x) = C\sin\left(\frac{\pm n\pi x}{L}\right)$$

This is a sine plot between x = 0 and x = L which is half a wave length. We still have to determine C. We have used the boundary conditions to establish the value of the critical buckling load instead of C. The solution will remain indeterminate unless additional information is available. The value of C is also the maximum displacement. From a design point of view we wish to stay below the critical stress and prevent the initiation of buckling. All of these relations assume that everything is neat and centered. A small force on the side of the column while carrying a load may be able to buckle the column below the critical load.. Hence the idea of ***instability***.

9.1.1 Example 9.1

A 6 m column made of steel , E = 200 GPa, is subject to a compressive load. Calculate the critical buckling load and stress if the cross=section is solid with an outside diameter of 100 mm.

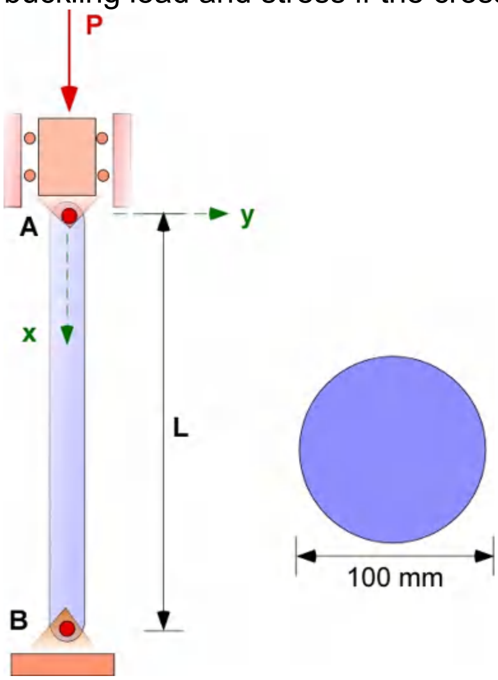


Figure 9.1.5 Example 9.1

Data: L = 6 m; r = 50 mm; E = 200 GPa; (σ_Y = 250 MPa);

Find: Critical buckling load and stress

Assumption: The formulas are available in Eqn. 9.3

Solution: The calculations are direct.

$$A = \pi\left(r^2\right) = 0.00785\left[m^2\right]$$

$$I = \frac{\pi}{4}\left(r^4\right) = 4.908\times10^{-6}\left[m^4\right]$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 269.15\left[kN\right]$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = 34.27\left[MPa\right]$$

$$k = \sqrt{\frac{I}{A}} = 0.025\left[m\right]$$

$$\frac{L}{k} = 240$$

$$P_{comp} = \sigma_y A = 1963.5\left[kN\right]$$

The last calculation is the compressive load required to cause direct compressible yield. The column would start buckling long before this value.

What is the factor of safety for this design. This is not a simple question in light of previous discussion of FOS. The column has buckled and this is an instability and this must be prevented We can then ask a different design question.

What must be the length of the column to just avoid the critical buckling load?

$$\left(\frac{L}{k}\right)_{max} = \sqrt{\frac{\pi^2 E}{\sigma_y}} = 88.86$$

$$L_m = k\left(\frac{L}{k}\right)_{max} = 2.22\left[m\right]$$

The solid column with the same outside diameter performs poorly compared to the hollow column with the same outside diameter (see Example 3.16.1). Note that columns can be shorter than L_m to avoid buckling.

9.1.2 Example 9.2

Timber columns are often used in building construction. Consider a 3 m column of square cross-section with made from Douglas Fir. An aside here is that properties of wood in compression depends on the direction of force in relation to the direction of grain, and depending if the wood id wet or dry. For example

Table 9.1 Douglas Fir - Maximum Compressive Stress (MPa)

Perpendicular to grain		Parallel to grain	
Wet	**Dry**	**Wet**	**Dry**
2.88	4.31	9.38	11.7

Consider E = 13 GPa and σ_y = 12 MPa. The FOS is 2.5 for the critical buckling load. What should be the minimum dimension of the area of cross-section to support a 200 kN load?

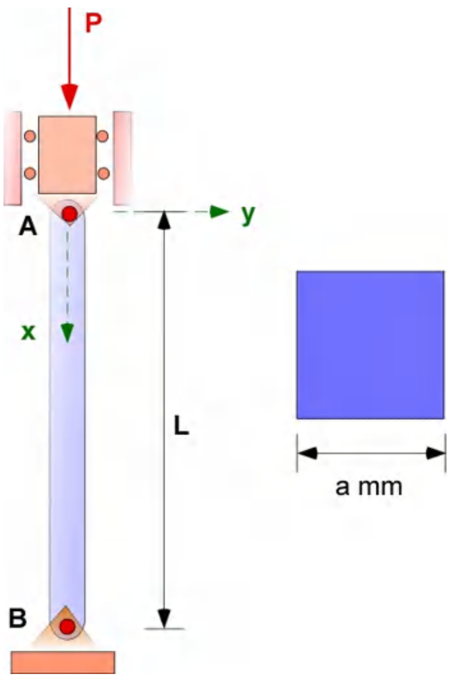


Figure 9.1.6 Example 9.2

Data: L = 3 m; E = 13 GPa; σ_Y = 12 MPa, FOS = 2.5; P = 200 kN.

Find: Calculate dimension a

Assumption: Calculate a and verify that the compressive stresses are below yield strength

Solution: The calculations are direct.

Solution Using MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, August 2018
% Section 9.1.2 - Example 9.2
% Buckling and Design
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 9.2\n')
fprintf('---------------------------\n')
fprintf('Design for Buckling\n')
fprintf('---------------------------------------------\n')
```

```
%% Data
syms a real
E = 13e09;  sigy = 12e06;  L = 3;  P = 200000;  FOS = 2.5;
Pa = FOS*P;
A = a^2;
I = a^4/12;
sigcr = Pa/A;
Eq1 = Pa - pi^2*E*I/L^2;
a = double(max(solve(Eq1)));
A = double(subs(A));
I = double(subs(I));
sigcr = double(subs(sigcr));

% confirmation of calculation
Pcr1 = pi^2*E*I/L^2;

% sigcr = Pcr/A;
k = sqrt(I/A);
Sr = L/k;
sigc = P/A;
fprintf('Material Wood\n')
fprintf('E    [GPa] = '),disp(E/1e09)
fprintf('sigy [MPa] = '),disp(sigy/1e06)
fprintf('L      [m] = '),disp(L)
fprintf('a      [m] = '),disp(a)
fprintf('A    [m^2] = '),disp(A)
fprintf('I    [m^4] = '),disp(I)
fprintf('k      [m] = '),disp(k)
fprintf('L/k        = '),disp(Sr)
fprintf('\nActual Load [kN]        = '),disp(P/1000)
fprintf('Design Load  (FOS) [kN] = '),disp(Pa/1000)
fprintf('Max. Critical Load (check)[kN] = '),disp(Pcr1/1000)
fprintf('sigcr(buckling) [MPa] = '),disp(sigcr/1e6)
fprintf('sigc (compression) [MPa] = '),disp(sigc/1e6)
```

In Command Window

```
Example 9.2
--------------------------
Design for Buckling
----------------------------------------------
Material Wood
E    [GPa] =     13
sigy [MPa] =     12
L      [m] =      3
a      [m] =       0.14323
A    [m^2] =      0.020515
I    [m^4] =    3.5073e-05
k      [m] =      0.041347
L/k        =        72.556

Actual Load [kN]        =    200
Design Load  (FOS) [kN] =    500
Max. Critical Load (check)[kN] =    500
sigcr(buckling) [MPa] =        24.372
sigc (compression) [MPa] =        9.7489
```

In this problem the size determined by the critical buckling load is valid because the compressive stress is less than yield. If the direct compressive stress was over the yield strength then the size would be determined by direct compressive stress (P/A). Note the buckling stress is higher than yield strength since the point is on the curve in the region above the yield strength (see Figure 9.1.4) and this region is ignored.

Execution in Octave

There is some change in the code.

For symbolic calculation we must use **`sym(pi)`** otherwise the results are different.

The code must be adjusted for scalar substitution

In Octave Editor

```
Eq1 = Pa - sym(pi)^2*E*I/L^2;

a1 = double(max(solve(Eq1)));
%A = double(subs(A));
A = double(subs(A,a,a1));
%I = double(subs(I));
I = double(subs(I,a,a1));
%sigcr = double(subs(sigcr));
sigcr = double(subs(sigcr,a,a1));
```

In Octave Command Window

The results are exactly the same

9.1.3 Additional Problems

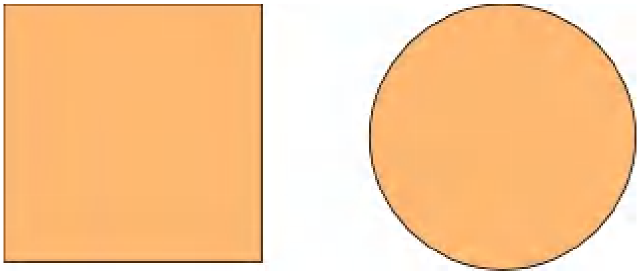
Use the following table for your calculations

Table 9.2

Material	Aluminum	Brass	Steel	Wood
E [GPa]	70	105	200	13
Allowable Stress [MPa]	100	160	250	25
G [GPa]	26	39	77	0.7

Problem 9.1.1

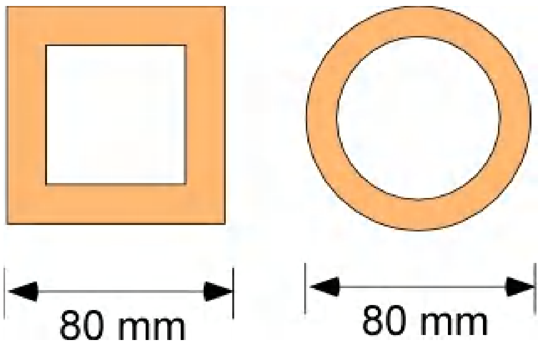
A column of length 3-m is designed using the cross-sections (solid square and circle) shown in Figure Problem 9.1.1 For each of the material listed in Table 9.2 identify the maximum cross-section dimension for both of the cross-sections for the maximum critical buckling stress. Use a FOS of 2. Will you use this dimension for the actual design?



Problem 9.1.1

Problem 9.1.2

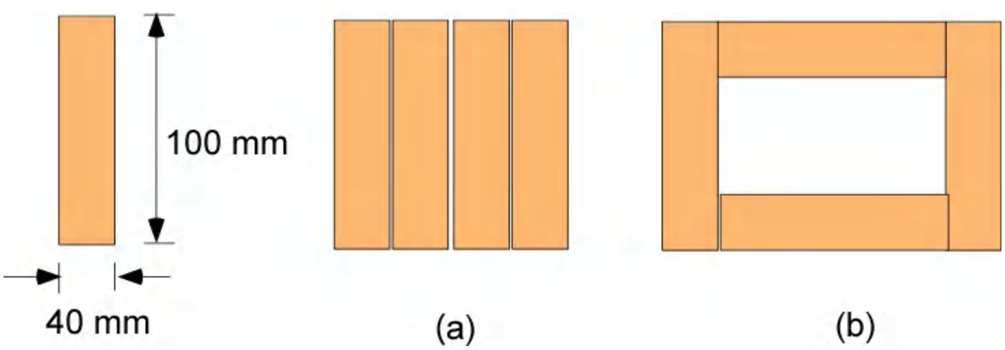
A column of length 3-m is designed using the cross-sections (annular square and circle) shown in Figure Problem 9.1.2 For each of the material listed in Table 9.2 identify the constant wall thickness for both of the cross-sections for the maximum critical buckling stress. Use a FOS of 2. Will you use this dimension for the actual design?



Problem 9.1.2

Problem 9.1.3

A column of length 3-m whose cross-section is glued together using four rectangles of base size 40 mm x 100 mm. Calculate the critical buckling load for each of the two cross-sections and for each of the material listed in Table 9.2. Calculate the critical buckling stress in each case and the FOS.



Problem 9.1.3

9.2 EULER BUCKLING WITH OTHER END CONDITIONS

Section 9.1 was the model for column buckling with two pin connections at the ends. In this section we look at a few other end conditions. These are shown in Figure 9.2.1. Part (a) is the pin connected ends already discussed but is included to illustrate the various end conditions.

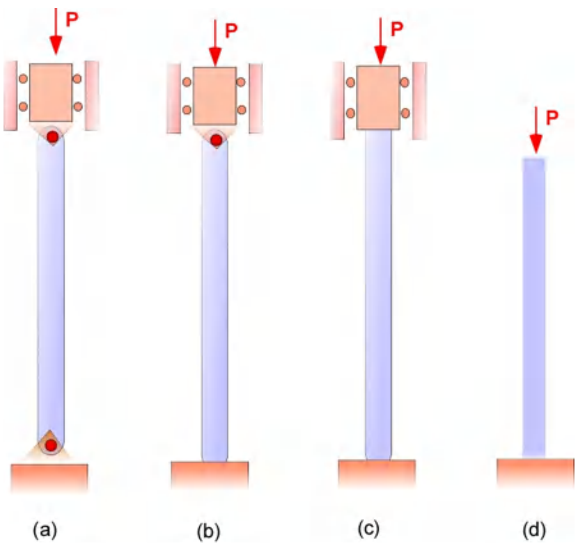


Figure 9.2.1 Column with different en conditions.

Part (b) of Figure 9.2.1 illustrates one end fixed and the other pinned. In part (c) we have both ends fixed. Part (d) is the case where one end is fixed and one end is free. In all of the cases above the column still bends sideways but the fixed ends cause the displacement and the slope of the column to be zero, much like the cantilevered beam in bending. This has to be incorporated into our previous model for the deflection through the boundary conditions.

9.2.1 Column with One End Fixed and One End Pinned

The loading is illustrated in part (b) of Figure 9.2.1 which is imported as part (a) in Figure 9.2.2. The FBD of the entire deflected column is shown in Figure 9.2.2, part (b). From previous discussion on bending with the fixed end there must be a resisting moment M_B on the FBD at the fixed end. The pin at end A cannot support a moment. This moment M_B now requires a shear force at the pin end which must have an equal and opposite component in the fixed end. The boundary conditions of this model including the zero slope of the deflection at the fixed end are also indicated on the figure. There are three boundary conditions. The FBD of a part of the column to obtain the mathematical model is shown in Figure 9.2.2, part(c).

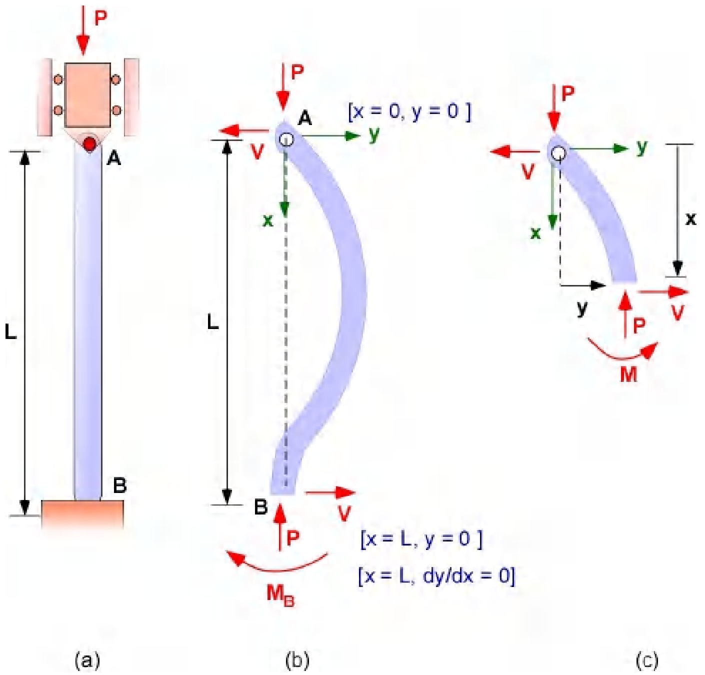


Figure 9.2.2 Column with fixed and pinned ends

The development of the mathematical model starts with the static equilibrium applied to Figure 9.2.2 part(c):

$$\sum F_x = P - P = 0; \quad (satisfied)$$
$$\sum F_y = -V + V = 0; \quad (satisfied)$$
$$\sum M_A = Vx + Py + M = 0;$$

The first two equations are identically satisfied. There is only one equation to identify unknowns in the problem. If this is a second order linear differential equation then we will have two unknown constants. Hence the system is statically indeterminate. However there are extra boundary conditions that may assist in determining a solution. Substituting for bending moment M

$$M = EI\frac{d^2y}{dx^2}$$

The differential equation describing the deflection (also the elastic curve) and the boundary conditions for this model are:

$$\frac{d^2y}{dx^2}+\left(\frac{P}{EI}\right)y=-\left(\frac{V}{EI}\right)x;$$

$$\frac{d^2y}{dx^2}+\lambda^2y=-\left(\frac{V}{EI}\right)x; \quad \lambda^2=\frac{P}{EI} \tag{9.5}$$

Boundary conditions :

$$y(0)=0;$$

$$y(L)=0; \quad \frac{dy}{dx}(L)=0;$$

This is a ***non-homogeneous***, *second order, linear, constant coefficient ,differential equations.* This is also a *two-point boundary value problem* as the boundary conditions are at the two ends. These problems are also called *boundary-value problems*. The problem requires two boundary conditions but there are three available and this decreases the degree of static indeterminacy by one.

Solution of the Differential Equation:
The solution for these problems are taught in the first course in differential equations and is usually through the addition of the homogeneous solution and a particular solution.

$$y(x)=y_h(x)+y_p(x)$$

The particular solution for this case is:

$$y_p(x)=-\frac{V}{\lambda^2 EI}x=-\frac{V}{P}x;$$

The homogeneous solution is the same as in the previous section and is written as:

$$y_h(x)=C\sin(\lambda x)+D\cos(\lambda x)$$

The total solution and the first derivative is:

$$y(x)=C\sin(\lambda x)+D\cos(\lambda x)-\frac{V}{P}x;$$

$$\frac{dy}{dx}(x)=C\lambda\cos(\lambda x)-D\lambda\sin(\lambda x)-\frac{V}{P}$$

The constants C and D are determined using the boundary conditions

$$y(0) = C(0) + D - 0; \quad D = 0;$$

$$y(L) = C\sin(\lambda L) - \frac{V}{P}L = 0; \quad C\sin(\lambda L) = \frac{V}{P}L$$

$$\frac{dy}{dx}(L) = C\lambda\cos(\lambda L) - \frac{V}{P} = 0; \quad C\lambda\cos(\lambda L) = \frac{V}{P}$$

Combing the last two equations above the solution is the solution to this equation:

$$\tan(\lambda L) = \lambda L \qquad (9.6)$$

This must be solved numerically. A graphic solution using MATLAB is shown in Figure 9.2.3

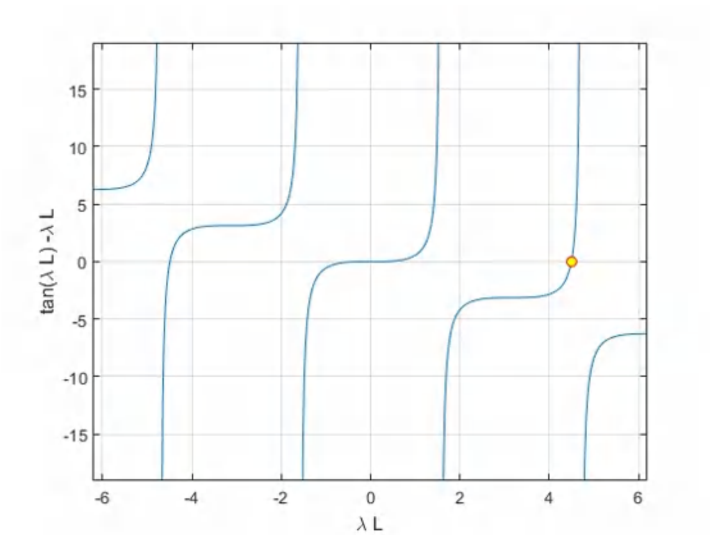


Figure 9.2.3 Solution for Equation (9.5)

The smallest value for λL is between 4.492 and 4.493 to third decimal place (zooming) or 4.4925.

The critical buckling load is for a column compressed between a fixed end and a free end is :

$$P_{cr} = \frac{(4.4925)^2 EI}{L^2} = \frac{20.183\,EI}{L^2} \qquad (9.7)$$

Once again note that this is dependent on the geometry of the problem and directly related to the material property. Unlike earlier chapters there is no applied load that is required for the design.

Effective Length of Column (L_e) : The effective length of the column is an artificial definition. It is the length of the column if the critical buckling load was calculated using Eqn. (9.3) - that is the buckling load corresponding to pin supported ends. To calculate this we equate Eqn. (9.3) to Eqn. (9.6)

$$\frac{\pi^2 E I}{L_e^2} = \frac{20.183\, E I}{L^2}; \qquad (9.8)$$

$$L_e = 0.699\, L \cong 0.7\, L$$

The critical buckling stress is :

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{20.183 E I}{A L^2} = \frac{20.183 E}{(L/k)^2}; \quad I = k^2 A \qquad (9.9)$$

Compare this to the critical bucking stress for pin supported ends.

Example 9.3
A annular tube of length L is subject to buckling.The tube is embedded at the bottom and connected at the top so that it can pivot along one of the vertical planes - essentially a pin connection. The outside diameter is d and the wall thickness is t. FOS is the factor of safety. Explore the design problem.

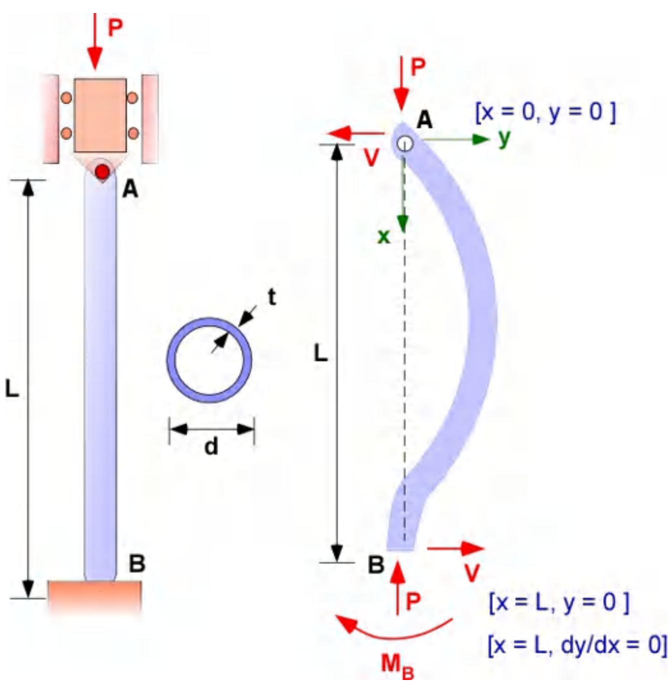


Figure 9.2.4 Example 9.3

Data: diameter of the cross-section = d; wall thickness = t; length of column = L;
Factor of safety = FOS; modulus of elasticity = E; yield strength = σ_y;

Find: The relation between the design variables d, t, and L

Assumption: Assume buckling with one end fixed and the other pin supported

Solution: The calculations are straightforward and leads to more variables than equations. Traditionally the material and the FOS is given for the design.
If steel is the material then E = 200 GPa and σ_y = 250 MPa. Assume FOS = 2.5

In this case we need to determine d, t, and L using one equation - Eq. (9.9)

$$I = \frac{\pi}{64}\left[d^4 - (d-2t)^4\right] = \frac{\pi}{64}\left[d^2 - (d-2t)^2\right]\left[d^2 + (d-2t)^2\right]$$

$$A = \frac{\pi}{4}\left[d^2 - (d-2t)^2\right]$$

$$\frac{I}{A} = \frac{\left[d^2 + (d-2t)^2\right]}{16}$$

$$\sigma_{cr} = \frac{20.183\,E}{L^2}\frac{I}{A} = \frac{20.183\,E}{L^2}\frac{\left[d^2 + (d-2t)^2\right]}{16} = \frac{\sigma_y}{FOS}$$

This leads to the following equation:

$$(25000000*(4*L^2 - 20183*d^2 + 40366*d*t - 40366*t^2))/L^2 = 0 \quad OR$$

$$(4*L^2 - 20183*d^2 + 40366*d*t - 40366*t^2) = 0$$

$$L^2 - \frac{20183}{4}\left(d^2 - 2d\,t + 2t^2\right) = 0$$

This reduces to one equation and three variables. If **t** is much smaller than **d** then we can simplify this further as:

$$L^2 = 5045.27\left(d^2 - 2d\,t\right)$$

One way to solve this equation is to assume a value of length of the column **(L)** and varying the thickness **(t)** find the value of the diameter **(d)**. The following plot is generated in MATLAB.

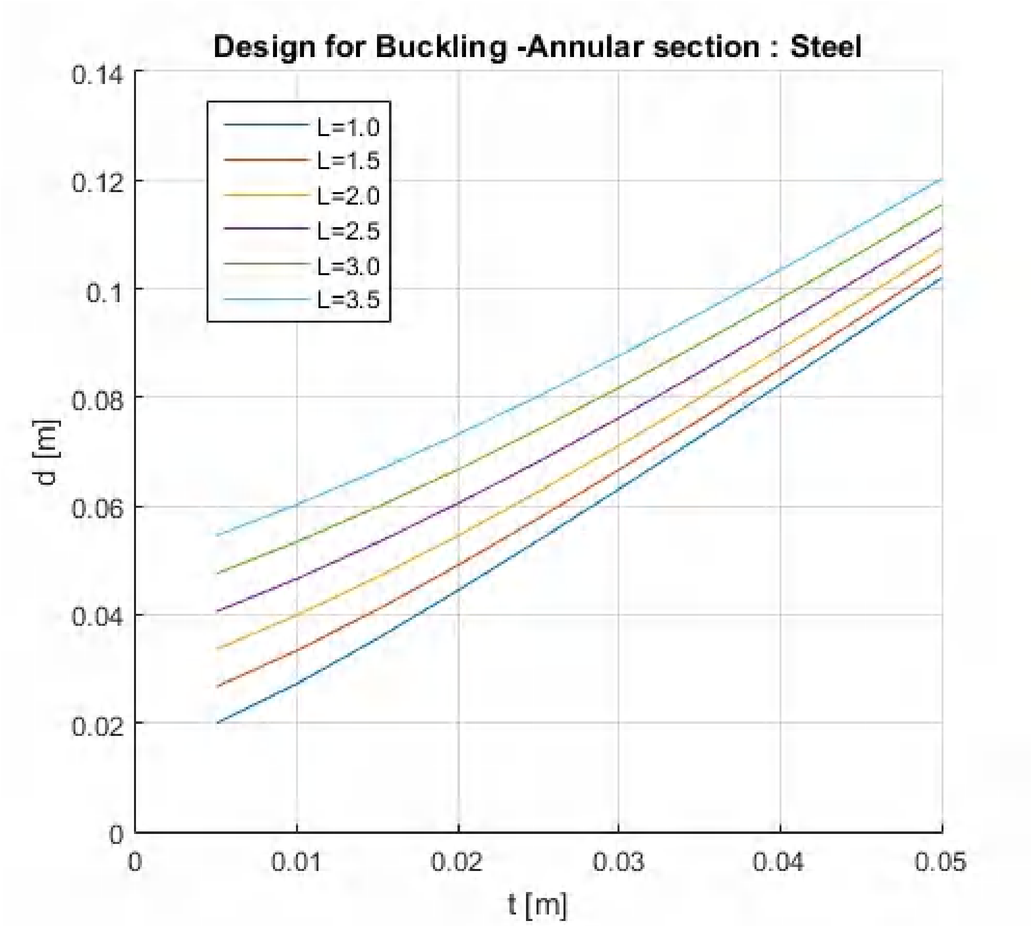


Figure 9.2.5 Design solution for Example 9.3

9.2.2 Column with Both End Fixed

The loading is illustrated in Figure 9.2.6 part (a). The FBD of the column under load with the boundary conditions is shown in Figure 9.2.6(b) and the FBD of the column over a length x to assist in the development of the model is shown in Figure 9.2.6(c).

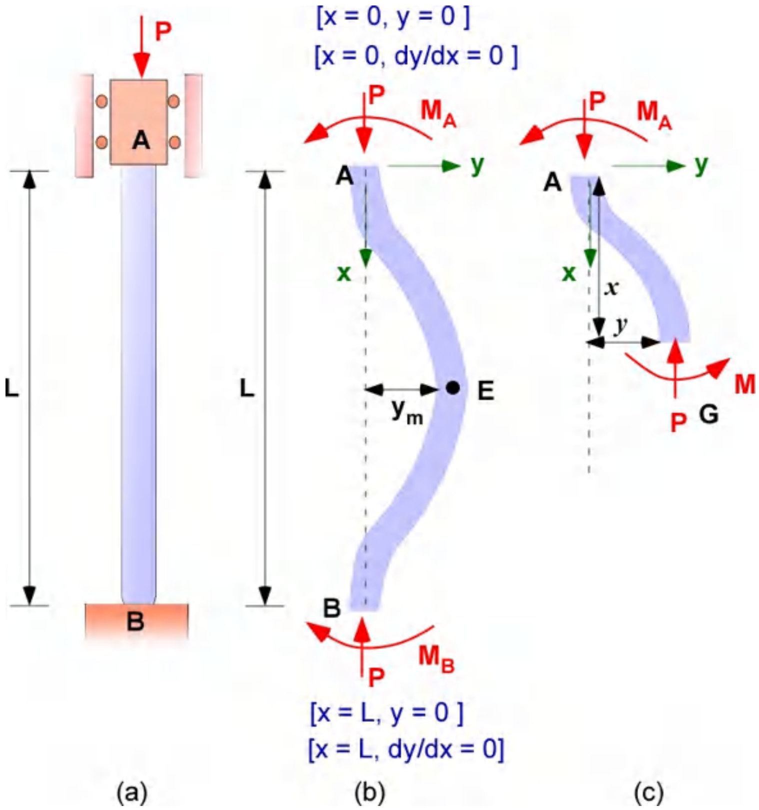


Figure 9.2.6 Column buckling with fixed end conditions

In the FBD of the column in Figure 9.2.6 (b), the column is in static equilibrium and the boundary conditions are reflected in the deformation which is symmetric. There is no shear force necessary at the ends A and B for equilibrium unlike Section 9.2.1. Consider a section of the beam as in Figure 9.2.6 (c). There is no shear force necessary for equilibrium again. This suggests that there is no shear force in the beam over the length.

The mathematical model starts with the FBD of the the segment AG of the beam of length x and the displacement y. There are two equations of equilibrium.

$$\sum F_x = 0 = P - P$$

$$\sum M_{AG} = 0 = M_A + P\,y + M(y)$$

Substituting for M(y) and collecting terms in y on the left

$$EI\frac{d^2y}{dx^2} + Py = -M_A$$

$$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = \left(-\frac{M_A}{EI}\right) \qquad (9.10)$$

$$\frac{d^2y}{dx^2} + \lambda^2 y = \left(-\frac{M_A}{EI}\right); \quad \lambda^2 = \frac{P}{EI}$$

This is a non-homogeneous second order constant coefficient boundary value problem with the following boundary conditions:

$$x = 0; \quad y = 0;$$
$$x = 0; \quad \frac{dy}{dx} = 0;$$
$$x = L; \quad y = 0; \qquad (9.11)$$
$$x = L; \quad \frac{dy}{dx} = 0;$$

The effect of the boundary conditions on the deformation of the column is indicated in Figure 9.2.6 part (b).

Solution of the Boundary Value Problem:
The solution the addition of the homogeneous solution and a particular solution.

$$y(x) = y_h(x) + y_p(x)$$

The particular solution is constant

$$y_p(x) = -\frac{M_A}{P};$$

The homogeneous solution is the same as in the previous sub-section and is written as:

$$y_h(x) = C\sin(\lambda x) + D\cos(\lambda x)$$

The total solution and the first derivative is:

$$y(x) = C\sin(\lambda x) + D\cos(\lambda x) - \frac{M_A}{P};$$
$$\frac{dy}{dx}(x) = C\lambda\cos(\lambda x) - D\lambda\sin(\lambda x)$$

The constants C and D are determined using the boundary conditions

$$y(0) = 0 = C(0) + D - \frac{M_A}{P}; \quad D = \frac{M_A}{P};$$
$$\frac{dy}{dx}(0) = 0 = C\lambda = 0; \quad C = 0$$
$$y(L) = 0 = D\cos(\lambda L) - \frac{M_A}{P} = 0;$$
$$\cos(\lambda L) = 1 \quad OR \quad \lambda L = \pm 2n\pi \tag{9.12}$$

The principal value is :

$$\lambda L = 2\pi; \quad \lambda = \frac{2\pi}{L}; \quad \lambda^2 = \frac{P}{EI} = \frac{4\pi^2}{L^2}$$
$$P_{cr} = \frac{4\pi^2 EI}{L^2} \tag{9.13}$$

Effective Length of Column (L_e) : The effective length of the column for a column fixed at both ends is the critical buckling load calculated using the expression in Eqn. (9.3) - that is the buckling load corresponding to pin supported ends. To calculate this we equate Eqn. (9.3) to Eq. (9.12)

$$\frac{\pi^2 EI}{L_e^2} = \frac{4\pi^2 EI}{L^2};$$
$$L_e = \frac{L}{2} = 0.5L \tag{9.14}$$

The critical buckling stress is:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{4\pi^2 EI}{AL^2} = \frac{4\pi^2 E}{(L/k)^2}; \quad k: radius\ of\ gyration \tag{9.15}$$

Example 9.4
A column of square cross-section of length a, wall thickness t, and length L is subject to buckling.The column is fixed at both ends . The material is aluminum and the FOS is 2.5. Explore the connection between the variables a, t and L. This example is similar to Example 9.3 but with different end fixity and material.

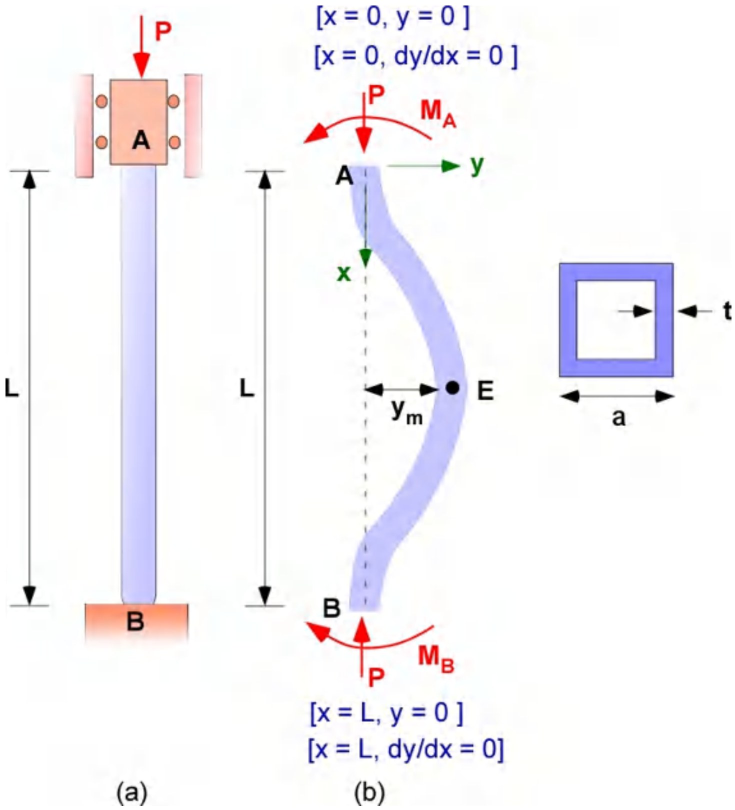


Figure 9.2.7 Example 9.4

Data: dimension of wall side = a; wall thickness = t; length of column = L;
Factor of safety = FOS; modulus of elasticity = E; yield strength = σ_y;

Find: The relation between design variables a, t, and L for critical buckling of a column with fixed ends.

Assumption: Assume buckling with both ends fixed - Eq, (9.14)

Solution: The calculations are straightforward and leads to more variables than equations. Traditionally the material and the FOS is given for the design.
If aluminum is the material then E = 70 GPa and σ_y = 100 MPa. Assume FOS = 2.5

$$I = \frac{a^4 - (a-2t)^4}{12} = \frac{\left[a^2 - (a-2t)^2\right]\left[a^2 + (a-2t)^2\right]}{12};$$

$$A = \left[a^2 - (a-2t)^2\right];$$

$$\frac{I}{A} = \frac{\left[a^2 + (a-2t)^2\right]}{12}$$

$$\sigma_{cr} = \frac{4\pi^2 E}{L^2}\left(\frac{I}{A}\right) = \frac{4\pi^2 E}{L^2}\frac{\left[a^2 + (a-2t)^2\right]}{12}$$

$$\frac{4\pi^2 E}{L^2}\frac{\left[a^2 + (a-2t)^2\right]}{12} = \frac{\sigma_y}{FOS} \quad (solve)$$

For a choice of L and t the value for a can be obtained. This is solved through MATLAB and results in the following figure

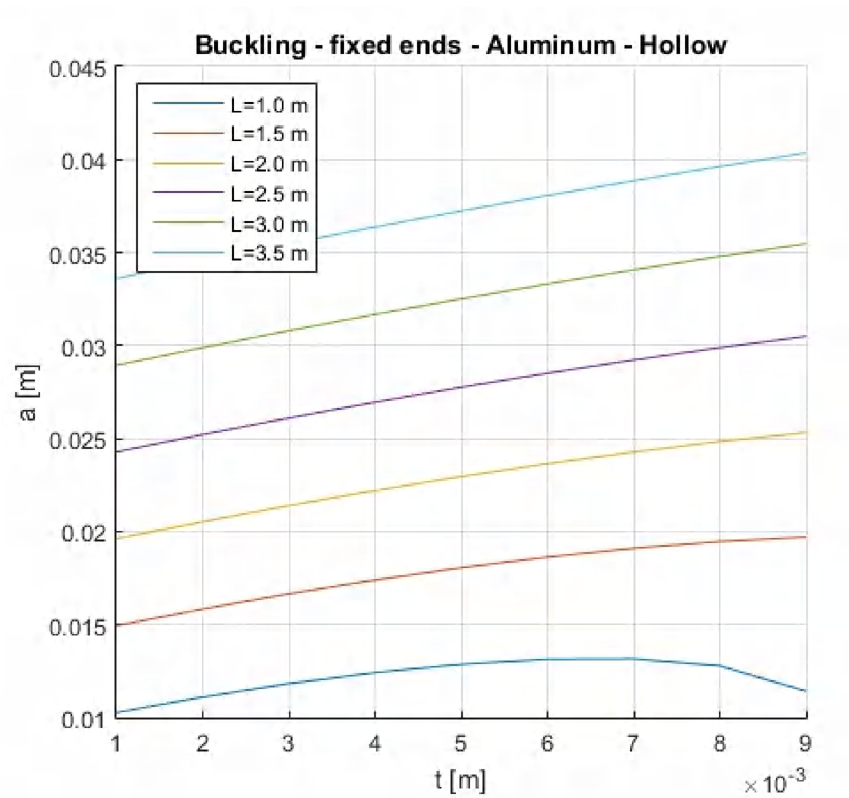


Figure 9.2.8 Example 9.4

Figure 9.2.8 is a powerful design guide for both ends fixed, hollow aluminum square cross-section column in buckling. These are information typically found in handbooks.

9.2.3 Column with One End Fixed and One End Free

The loading is illustrated in Figure 9.2.9 part (a). The FBD of the column under load with the boundary conditions is shown in Figure 9.2.9(b) and the FBD of the column over a length x to assist in the development of the model is shown in Figure 9.2.9(c).

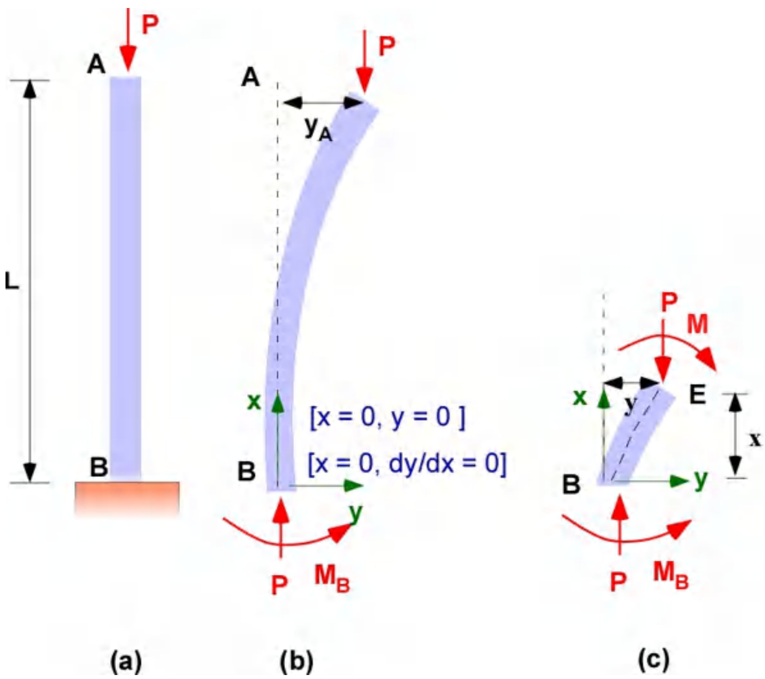


Figure 9.2.9 Column with one end fixed and one end free

The fixed end conditions at B will require that the deflection and the slope there to be zero (Figure 9.2.9 part(b)). End A is free to deflect and is expected to have a maximum deflection in this model. The load at A is only P. Therefore the model does not require a shear force along the column. Since boundary conditions are available at end B we use the point B for the origin of the coordinate system. The static equilibrium of the column is:

$$\sum F_x = 0 = P - P;$$
$$\sum M_z = 0 = -M_B + P\,y_A; \quad y_A = \frac{M_B}{P}; \qquad (9.16)$$

The model is generated using the part BE of the column of length x. The FBD of BE is shown in Figure 9.2.9 part (c). The equilibrium of BE leads to:

$$\sum F_x = 0 = P - P$$
$$\sum M_{Bz} = 0 = -M_B + P\,y + M(y)$$

Substituting for M(y) in terms of its elastic equivalent:

$$EI\frac{d^2y}{dx^2}+Py=M_B$$

$$\frac{d^2y}{dx^2}+\left(\frac{P}{EI}\right)y=\left(\frac{M_B}{EI}\right) \tag{9.17}$$

$$\frac{d^2y}{dx^2}+\lambda^2 y=\left(\frac{M_B}{EI}\right); \quad \lambda^2=\frac{P}{EI}$$

This is a non-homogeneous second order constant coefficient boundary value problem with the following boundary conditions:

$$\begin{aligned} x=0; \quad & y=0; \\ x=0; \quad & dy/dx=0; \end{aligned} \tag{9.18}$$

Solution of the Boundary Value Problem:
The solution the addition of the homogeneous solution and a particular solution.

$$y(x)=y_h(x)+y_p(x)$$

The particular solution is constant

$$y_p(x)=\frac{M_B}{P};$$

The homogeneous solution is the same as in the previous sub-section and is written as:

$$y_h(x)=C\sin(\lambda x)+D\cos(\lambda x)$$

The total solution and the first derivative is:

$$y(x)=C\sin(\lambda x)+D\cos(\lambda x)+\frac{M_B}{P};$$

$$\frac{dy}{dx}(x)=C\lambda\cos(\lambda x)-D\lambda\sin(\lambda x)$$

The constants C and D are determined using the boundary conditions

$$y(0)=0=C(0)+D+\frac{M_B}{P}; \quad D=-\frac{M_B}{P};$$

$$\frac{dy}{dx}(0)=0=C\lambda=0; \quad C=0$$

D and M_B are both unknowns. We will need additional information to obtain the critical bucking load for this case. We will use the fact that the deflection is maximum at point A.

$$y(x) = D\cos(\lambda x) + \frac{M_B}{P} = \frac{M_B}{P}\left(1 - \cos(\lambda x)\right)$$

$$y_A = y(L) = \frac{M_B}{P}\left(1 - \cos(\lambda L)\right) = \frac{M_B}{P};$$

$$\cos(\lambda L) = 0; \quad \lambda L = \frac{\pi}{2}; \quad \lambda = \frac{\pi}{2L} = \sqrt{\frac{P}{EI}};$$

$$P_{cr} = \frac{\pi^2 EI}{4L^2}; \qquad (9.19)$$

Effective Length of Column (L_e) : The effective length of the column for a column fixed at both ends is the critical buckling load calculated using the expression in Eq. (9.3) - that is the buckling load corresponding to pin supported ends. To calculate this we equate Eqn. (9.3) to Eqn. (9.13)

$$\frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{4L^2}; \qquad (9.20)$$

$$L_e = 2L$$

The critical buckling stress is:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{4AL^2} = \frac{\pi^2 E}{4(L/k)^2}; \quad k: radius\ of\ gyration \qquad (9.21)$$

Example 9.5

This example compares the critical load for a given column and material for the same length, area of cross-section and inertia. It is a straight forward application of the relations and is used here for insight. The material is aluminum. The length is 3 m. The area of cross-section is 0.6×10^{-3} [m^2]. The MOI is 0.75×10^{-7} [m^4].

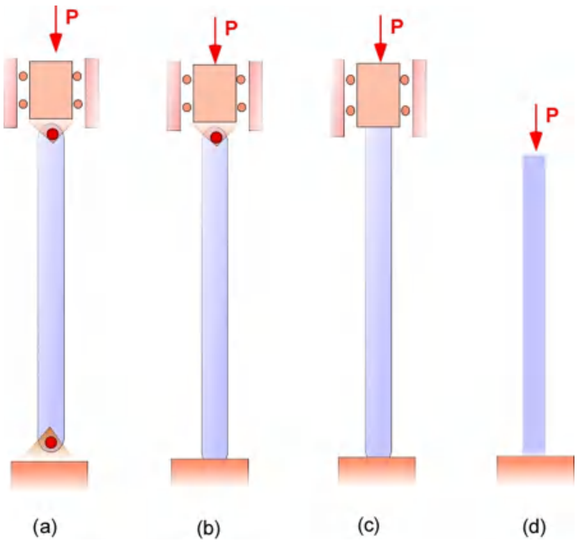


Figure 9.2.10 Example 9.5

Data: L = 3 [m]; A = 0.6x10^{-3} [m^2]; I = 0.75x10^{-7} [m^4]; E = E = 70 [GPa] ; σ_y = 100 [MPa].

Find: The critical buckling load for various end supports

Assumption: None.

Solution: The calculations are straightforward. It is implemented in MATLAB including a bar chart for comparing the bar chart for the various cases.

Solution Using MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, August 2018
% Section 9.2.3 - Example 9.5
% Buckling and Design
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 9.5\n')
fprintf('-------------------------------------\n')
fprintf('Critical Buckling- all end conditions\n')
fprintf('--------------------------------------\n')
%% Data
E = 70e09;   sigy = 100e6; FOS = 2.5;
fprintf('Material : Aluminum\n')
fprintf('E    [GPa] = '),disp(E/1e09)
fprintf('sigy [MPa] = '),disp(sigy/1e06)
% fprintf('FOS        = '),disp(FOS)
L = 3;
I = 0.75e-7;
A = 0.6e-3;
fprintf('L [m]   :'),disp(L)
fprintf('A [m^2] :'),disp(A)
fprintf('I [m^4] :'),disp(I)
```

```
con = pi^2*E*I/L/L;
Pcr = con*[1,(20.183/pi^2),4,(1/4)];
fprintf('\nPcr (a) [N] = '),disp(Pcr(1))
fprintf('Pcr (b) [N] = '),disp(Pcr(2))
fprintf('Pcr (c) [N] = '),disp(Pcr(3))
fprintf('Pcr (d) [N] = '),disp(Pcr(4))

%% create design plot
h1 = figure;
set(h1,'Position',[20 50 400 350], ...
    'Units','normalized', ...
    'Color',[1,1,1], ...
    'Name','Compressive Design', ...
    'NumberTitle','on', ...
    'Colormap',jet, ...
    'DoubleBuffer','on', ...
    'Pointer','arrow');

bar(Pcr,'FaceColor',[0 .5 .5],'EdgeColor',[0 .9 .9],'LineWidth',1.5)
set(gca,'xticklabel',{'(a)','(b)','(c)','(d)'});
YTickLabel = get(gca,'YTick');
set(gca,'yticklabel',num2str(YTickLabel'));
title('Critical Load - various end supports')
grid
```

In the Command Window

```
Example 9.5
--------------------------------------
Critical Buckling- all end conditions
---------------------------------------
Material : Aluminum
E    [GPa] =     70
sigy [MPa] =    100
L [m]   :     3
A [m^2] :       0.0006
I [m^4] :      7.5e-08

Pcr (a) [N] =        5757.3
Pcr (b) [N] =         11773
Pcr (c) [N] =         23029
Pcr (d) [N] =        1439.3
```

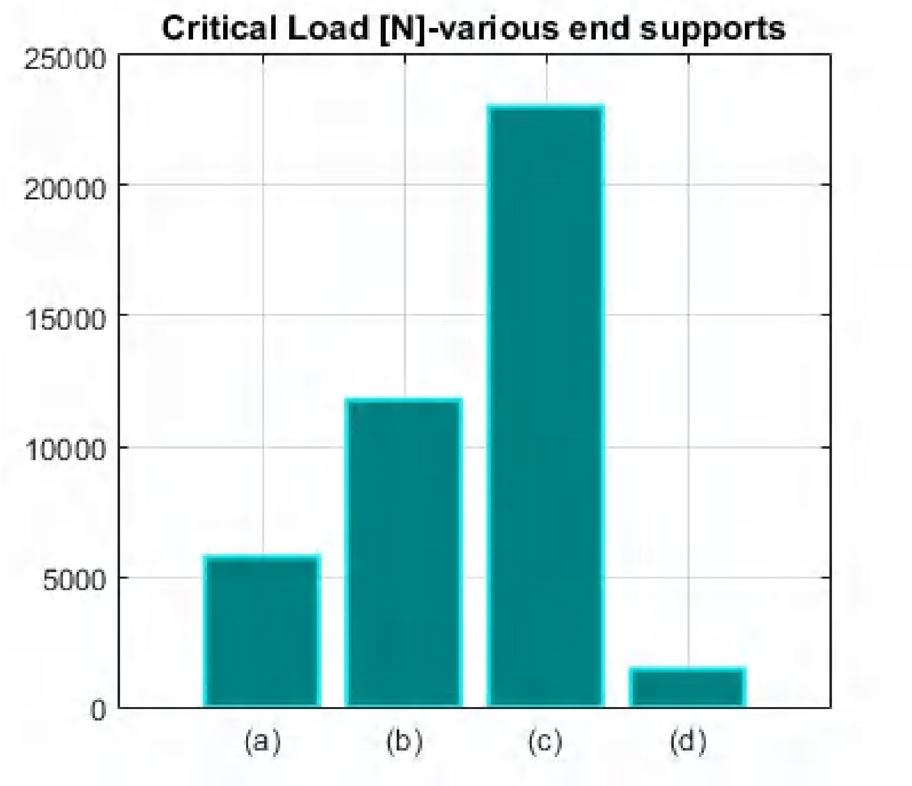


Figure 9.2.11 Solution: Example 9.5

Figure 9.2.11 indicates that case (d), one end fixed and one end free is able to carry the least load. Two pin connected ends, case (a) is better than case (d). The largest load compressive load is handled when the two ends are fixed.

Execution in Octave
There is no change in the code.

In Octave Command Window
The results in the Command Window are also the same.

In Octave Figure Window
The Figure is the same

9.2.4 Example 9.6

Consider a column of length L with one end fixed and one end free with a rectangular cross-section defined by the sides a and b, see Figure 9.2.12. What is the critical buckling load for this design?

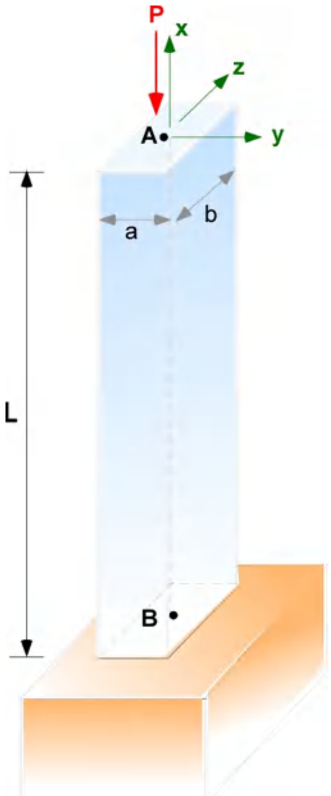


Figure 9.2.12. Example 9.6

Data: Length = L; Area = a x b; Modulus of Elasticity E ; yield strength = σ_y.

Find: The critical buckling load for fixed end B and free end A

Assumption: None.

Solution: The three dimensional figure is for an important reason. The previous model for bending assumed that the bending moment was directed along the z-axis only. While it is possible to design this as a constraint for the pin supported and fixed ends, the free end is unconstrained to bend in any direction. In effect in the simplest case two critical load can be calculated for this example:

$$P_{cr_1} = \frac{\pi^2 E I_z}{4L^2} = \frac{\pi^2 E\left[\frac{ba^3}{12}\right]}{4L^2} = \frac{\pi^2 E\, b\, a^3}{48L^2};$$

$$P_{cr_2} = \frac{\pi^2 E I_y}{4L^2} = \frac{\pi^2 E\left[\frac{ab^3}{12}\right]}{4L^2} = \frac{\pi^2 E\, a\, b^3}{48L^2};$$

If b > a then the second critical load is greater than the first and therefore the design will be limited by the first critical buckling load. This is based on the assumption that there are only two possible directions for bending.

Consider the case that the cross-section is symmetric, for example a circular cross-section. This means that the bending can occur along any axis normal to the x-direction. This buckling is unpredictable even if the critical buckling load is known. This is a true instability. In this case the column should be designed with a constraint to make it bend along a certain direction to control the failure.

9.2.5 Summary of Various Type of End Constraints

Four types of end restraints on a column with compressive load was examined in previous sections. Instead of different expression for the critical load a single formula with the effective length is useful. Accordingly the critical buckling load is given by:

$$P_{cr} = \frac{\pi^2 EI}{L_e^2};$$

The values for the effective length are given in Table 9.3

Table 9.3 Effective length of column

Both ends pinned	One fixed end and one pinned	Both ends fixed	One fixed and one free
$L_e = L$	$L_e = 0.7\ L$	$L_e = 0.5\ L$	$L_e = 2\ L$

9.2.6 Additional Problems

Use the following table for your calculations if needed. Solve on paper and using MATLAB/Octave

Table 9.2

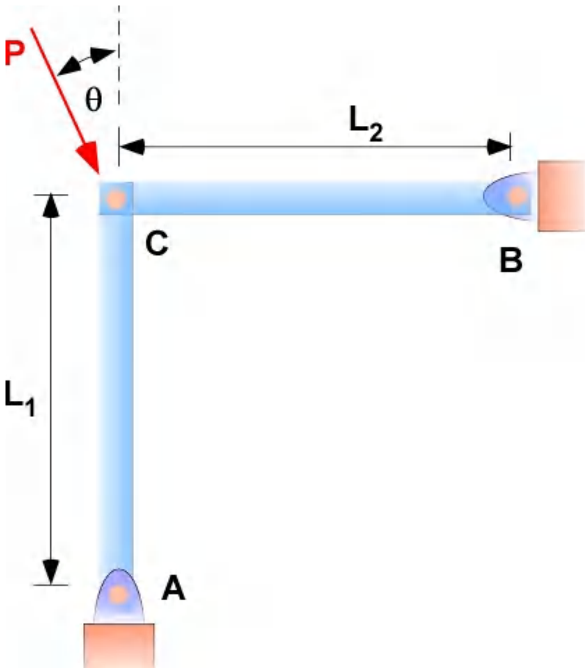
Material	**Aluminum**	**Brass**	**Steel**	**Wood**
E [GPa]	70	105	200	13
Allowable Stress [MPa]	100	160	250	25
G [GPa]	26	39	77	0.7

Problem 9.2.1

The arrangement in Problem 9.2.1 is pin supported at A, B and C. The sections AC and BC are pipes having the same dimensions. Identify the diameter and the wall thickness for a FOS of 3.0. Choose any material from Table 9.2. Consider the cases in Table 9.4. Report the corresponding value of load and the stress.

Table 9.4 Problem 9.2.1

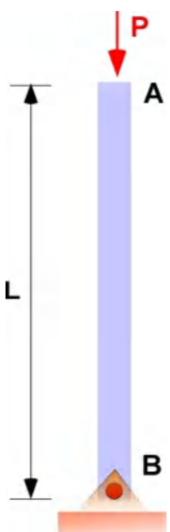
θ**(deg)**	L_1 **[m]**	L_2 **[m]**
30	3.0	4.0
45	3.0	3.0
15	4.0	3.0



Problem 9.2.1

Problem 9.2.2

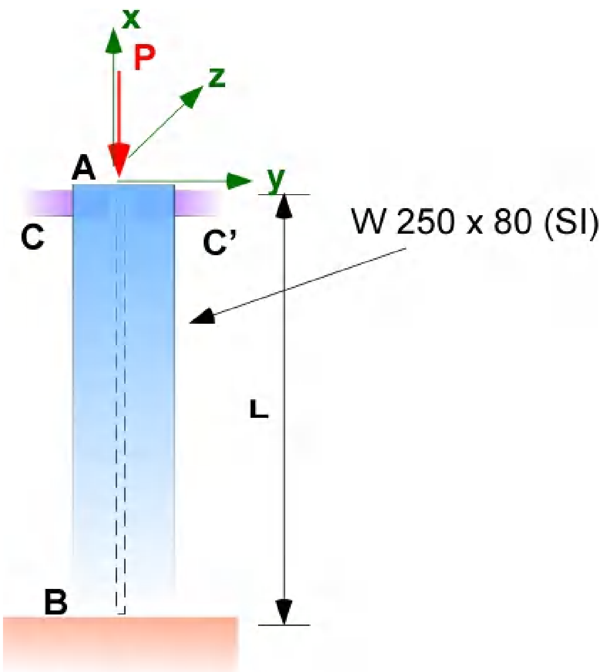
Column buckling was examined for the four cases shown in Figure 9.2.1 and the corresponding critical buckling load was determined using a mathematical model. Develop a model and examine the critical buckling load for the case there is a pin supported end and a free end as shown in Problem 9.2.2.



Problem 9.2.2

Problem 9.2.3

Column AB is a wide flange rolled steel shape (W 250 x 80 SI) with properties in Table 9.2. The pins C and C' prevent deflection in the x-y plane only. The factor of safety is 2.5. The load P is 70 kN. What should be the length of the column?



Problem 9.2.3

9.3 COLUMNS WITH ECCENTRIC LOADING

The derivation and examples in the previous sections had the compressive load through the centroid of the cross-section. This was not explicitly stated. The formulas for the various end supports are valid for a centric load only. Structural loads at the centroid are a convenient assumption since the analysis is simpler. In a real situation it is difficult to guarantee that the load will pass through the centroid unless it is enforced through design. Designing for a centric load is expensive and may involve in additional structures that are not necessary for load bearing. Loads that do not pass through the centroid are termed as eccentric load. In previous chapters the eccentric loading was replaced by a centric load and an moment. The response of the structure to eccentric load in previous sections was typically the superimposition of the effect of the centric load and response of the structure to the additional moment that were supposed to act independently of each other.

For the case of column design which involves elastic bending and differential equation superposition is avoided because the response of the structure is not linear. However the model includes the bending moment as an additional bending moment on the structure that will function as a forcing function for the boundary value problem. This will provide a new solution to the differential equation.This may change the expression used for computing the critical load ($\boldsymbol{P_{cr}}$). In addition it may motivate the column to buckle in a certain direction. Figure 9.3.1 illustrates the application of the load at a distance of *e* from the centroid (eccentricity). The traditional reduction of the eccentric load to a centric load with accompanying moment is shown in the figure.

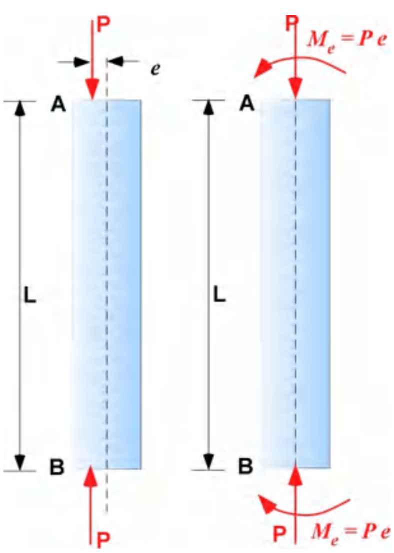


Figure 9.3.1 Handling eccentricity

9.3.1 Eccentric Load with Pin Like Support

This is the same problem described in Section 9.1 with an additional bending moment at the ends. It is difficult to justify a pin support because the pins cannot transmit a moment. It can be considered to be pinned with a bending moment applied to the bars directly. Therefore we start with column under load and then use the FBD of the portion AE of the column to set up the mathematical model and determine the solution.

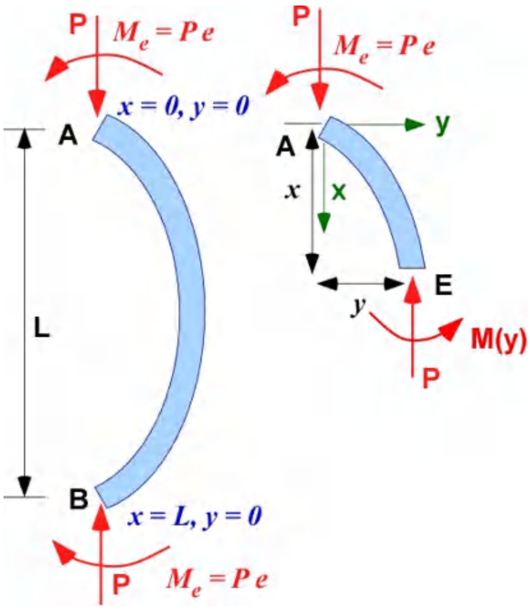


Figure 9.3.2 Model for column with eccentric load

The analysis for this model requires the solution to a differential equations that is a common feature for column buckling. The difference may be in the forcing function on the right hand side of the differential equation that may lead to a different solution. The first step is applying the equations of equilibrium.

$$\sum F_x = 0 = P - P$$
$$\sum M_z = 0 = M_e + M(y) + Py$$

We replace M(y) by the elastic equivalent and collect terms in the variable ***y*** on the left to reorganize the moment equations as:

$$M(y) = EI\frac{d^2y}{dx^2} = -Py - M_e = -Py - Pe;$$
$$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = -\left(\frac{P}{EI}\right)e; \qquad (9.22)$$
$$\frac{d^2y}{dx^2} + \lambda^2 y = -\lambda^2 e; \quad \lambda^2 = \frac{P}{EI}$$

The boundary conditions for this example are:

$$x = 0; \quad y = 0;$$
$$x = L; \quad y = 0; \qquad (9.23)$$

This is a ***non-homogeneous***, *second order, linear, constant coefficient, boundary value problem.* The term on the right of the equal sign is the f*orcing function*. The solution is the combination of the homogeneous solution and a particular solution with two constants to be determined using the boundary condition.

$$y(x) = y_h(x) + y_p(x)$$

The homogeneous solution is the same as in other examples. The particular solution has the same form as the forcing function and in this case is

$$y_p(x) = -e;$$

$$y_h(x) = C\sin(\lambda x) + D\cos(\lambda x)$$

The complete solution is therefore

$$y(x) = C\sin(\lambda x) + D\cos(\lambda x) - e;$$

Applying the boundary conditions:

$$y(0) = 0 = D - e; \quad D = e;$$

$$y(L) = 0 = C\sin(\lambda L) + e\cos(\lambda L) - e;$$

$$C\sin(\lambda L) = e\left[1 - \cos(\lambda L)\right] = e\left[2\sin^2\left(\frac{\lambda L}{2}\right)\right]$$

$$2C\sin\left(\frac{\lambda L}{2}\right)\cos\left(\frac{\lambda L}{2}\right) = e\left[2\sin^2\left(\frac{\lambda L}{2}\right)\right]$$

$$C = e\tan\left(\frac{\lambda L}{2}\right)$$

$$y(x) = e\tan\left(\frac{\lambda L}{2}\right)\sin(\lambda x) + e\cos(\lambda x) - e$$

This can be written as:

$$y = e\left[\tan\left(\frac{\lambda L}{2}\right)\sin(\lambda x) + \cos(\lambda x) - 1\right] \qquad (9.24)$$

At this time we have not resolved the value of λ that will give us the critical load P_{cr}. We need another condition and that will be associated with the maximum deflection. The principal deflection is symmetric and therefore the maximum deflection will be at x = L/2.

$$y_{max} = e\left[\tan\left(\frac{\lambda L}{2}\right)\sin\left(\frac{\lambda L}{2}\right) + \cos\left(\frac{\lambda L}{2}\right) - 1\right];$$

$$= e\left[\frac{\sin^2\left(\frac{\lambda L}{2}\right) + \cos^2\left(\frac{\lambda L}{2}\right)}{\cos\left(\frac{\lambda L}{2}\right)} - 1\right];$$

$$= e\left[\sec\left(\frac{\lambda L}{2}\right) - 1\right]$$

This still does not determine the critical load. The maximum deflection must remain finite and therefore the angle of the secant must be less than $\pi/2$. At the limit this is:

$$\frac{\lambda L}{2} = \frac{\pi}{2}; \quad P_{cr} = \frac{\pi^2 EI}{L^2}; \qquad (9.25)$$

This is the same result for a centric load with pinned ends. If the critical load for the centric load is $\mathbf{P_{cr}}$ then we can re-write the expression for the eccentric load P as

$$y_{max}(P) = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]; \qquad (9.26)$$

The computation of maximum stress requires the FBD of one-half the column where the bending moment is the maximum:

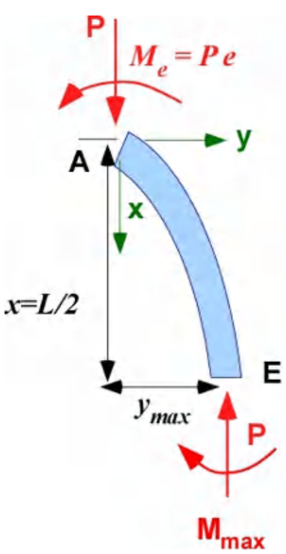


Figure 9.3.3 Maximum bending moment

$$\sum M_{cd} = 0 = Pe + P\,y_{max} - M_{max};$$

$$M_{max} = Pe + P\,y_{max};$$

The maximum normal stress for design is based on M_{max} and includes the direct compression due to P as well as the stress due to maximum bending. This requires the property of the cross-section. If c is the maximum distance from the neutral axis (NA) then the normal stress can be calculated as:

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max}\,c}{I};$$

$$= \frac{P}{A} + \frac{(Pe + P\,y_{max})c}{I}; \quad I = Ak^2;$$

$$= \frac{P}{A}\left[1 + \frac{(e + y_{max})c}{k^2}\right];$$

Substituting for y_{max} from Eqn. (9.26) the maximum normal stress for a load P can be expressed in terms of the stress corresponding to the critical buckling load $\mathbf{P_{cr}}$ as:

$$\sigma_{max}(P) = \frac{P}{A}\left[1 + \frac{ec}{k^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]; \qquad (9.27)$$

Equation (9.27) was derived with zero deflection at both ends. This equation is also valid for other cases of end supports that have zero deflection at the ends. The value of $\mathbf{P_{cr}}$ is adjusted according to those end conditions with centric load. This is also referred as the *Secant Formula*. For the design of these columns the FOS is usually applied on the load rather than the stress. Remember e and c are distances.

Example 9.7

An aluminum tube of length L = 0.7 m is subject to a compressive load of 15 kN applied with an eccentricity e = 5 mm along the y-axis as shown in Figure 9.3.4. The load must be applied on a welded plate. The diameter of the tube is d =35 mm with a wall thickness of t = 3 mm. Determine the maximum stress in the column and the deflection of the end for a FOS of 2.5. Use the material property from Table 9.2.

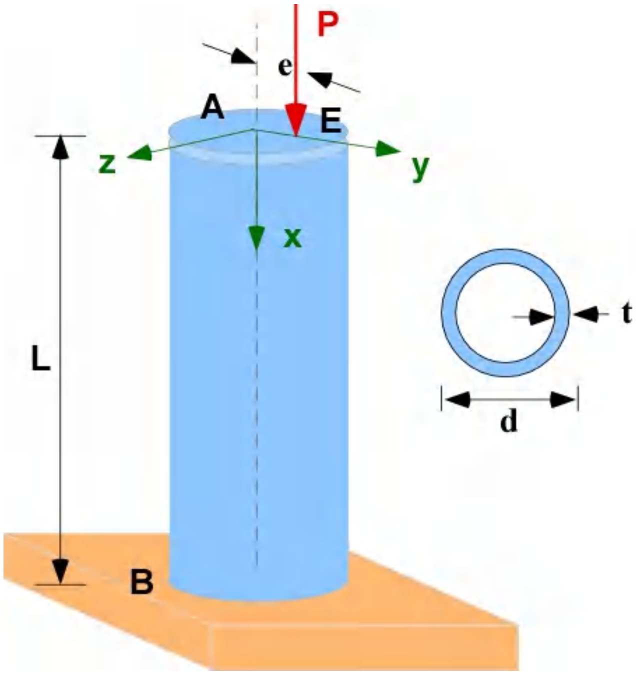


Figure 9.3.4 Example 9.7

Data: L = 0.7 m; d = 35 mm; t = 3 mm; e = 5 mm; FOS= 2.5E = 70 GPa ; σ_y = 100 MPa;

Find: (a)The maximum stress in the column, and (b) the maximum deflection.

Assumption: The column is fixed at B and free at A

Solution: We can directly use the formulas derived in the section.

Solution Using MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, August 2018
% Section 9.3.1 - Example 9.7
% Buckling and Design
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 9.7\n')
fprintf('-------------------------------------\n')
fprintf('Critical Buckling- eccentric loading\n')
fprintf('---------------------------------------\n')
%% Data
E = 70e09;   sigy = 100e6; FOS = 2.5;
L = 0.7;  d = 35/1000;  t = 3/1000;   e = 5/1000;
econ = 4; % type of end constraints
fprintf('Material : Aluminum\n')
fprintf('E    [GPa] = '),disp(E/1e09)
fprintf('sigy [MPa] = '),disp(sigy/1e06)
fprintf('FOS        = '),disp(FOS)
fprintf('\nGeometry  :\n')
fprintf('--------------\n')
fprintf('L [m]   :'),disp(L)
```

```
fprintf('d [m]   :'),disp(d)
fprintf('t [m]   :'),disp(t)
fprintf('e [m]   :'),disp(e)

fprintf('\nGeometric Properties  :\n')
fprintf('--------------------------\n')
A = pi*(d^2 - (d-2*t)^2)/4;
I = pi*(d^4 - (d-2*t)^4)/64;
k = sqrt(I/A);
c = d/2;
fprintf('A [m^2] :'),disp(A)
fprintf('I [m^4] :'),disp(I)
fprintf('k [m]   :'),disp(k)
fprintf('c [m]   :'),disp(c)

fprintf('\nEnd Constraints  :\n')
fprintf('--------------------------\n')
switch econ
    case 1
        fprintf('Both ends pinned\n')
        Le = L;
        fprintf ('Le = L = '),disp(Le);
    case 2
        fprintf('One Fixed One Pinned\n')
        Le = 0.7*L;
        fprintf ('Le = 0.7L = '),disp(Le);
    case 3
        fprintf('Both ends Fixedd\n')
        Le = 0.5*L;
        fprintf ('Le = 0.5L = '),disp(Le);
    case 4
        fprintf('One Fixed One Free\n')
        Le = 2*L;
        fprintf ('Le = 2L = '),disp(Le);
end
fprintf('--------------------------\n')
%% Critical load

Pcr = pi^2*E*I/Le^2;
Pmax = Pcr/FOS;
sigcen = Pmax/A;
sigmax = (Pmax/A)*(1 + (e*c/k^2)*sec(pi*sqrt(Pmax/Pcr)/2));
ymax = e*(sec(pi*sqrt(Pmax/Pcr)/2)-1);
fprintf('Pcr (centric)[N]  :'),disp(Pcr)
fprintf('Pmax (allow) [N]  :'),disp(Pmax)
fprintf('Normal stress (centric) (allow) [MPa]  :'),disp(sigcen/1e6)
fprintf('Normal stress (max) (eccentric) [MPa]  :'),disp(sigmax/1e6)
if sigmax < sigy
    fprintf('\tMax normal stress is less than yield stress\n')
else
    fprintf('\tMax normal stress is greater than yield stress\n')
    fprintf('\tRedesign for safety\n')
end

fprintf('y(max) (eccentric) [m]  :'),disp(ymax)
```

In the Command Window

```
Example 9.7
--------------------------------------
Critical Buckling- eccentric loading
---------------------------------------
Material : Aluminum
E    [GPa] =     70
sigy [MPa] =    100
FOS        =           2.5

Geometry  :
--------------
L [m]   :          0.7
d [m]   :        0.035
t [m]   :        0.003
e [m]   :        0.005

Geometric Properties  :
--------------------------
A [m^2] :   0.00030159
I [m^4] :   3.8943e-08
k [m]   :   0.011363
c [m]   :       0.0175

End Constraints  :
--------------------------
One Fixed One Free
Le = 2L =          1.4
--------------------------
Pcr (centric)[N]  :        13727
Pmax (allow) [N]  :       5490.8
Normal stress (centric) (allow) [MPa]  :       18.206
Normal stress (max) (eccentric) [MPa]  :        40.81
           Max normal stress is less than yield stress
y(max) (eccentric) [m]  :     0.004161
```

The code must be adjusted for different type of cross-sections.

Execution in Octave
There is no change in the code.

In Octave Command Window
The results in the Command Window are also the same.

9.3.2 Reconciling displacement of the loaded column

The derivation for the critical load and maximum displacement due to eccentric load began with pin supported ends. The final formula for the critical load and displacement applied to other end conditions as well with the understanding that the corresponding critical load used in the expression must match the one derived for the specific pair of end conditions. This can also be accommodated by using the corresponding *effective length* (L_e) in the formula. The pinned or the fixed end conditions match the boundary conditions used in the deriving the maximum displacement. However, in Example 9.7 we used the displacement formula for the free end conditions, which does not meet the boundary conditions. This can be justified visually in Figure 9.3.5 by using the effective length of the column.

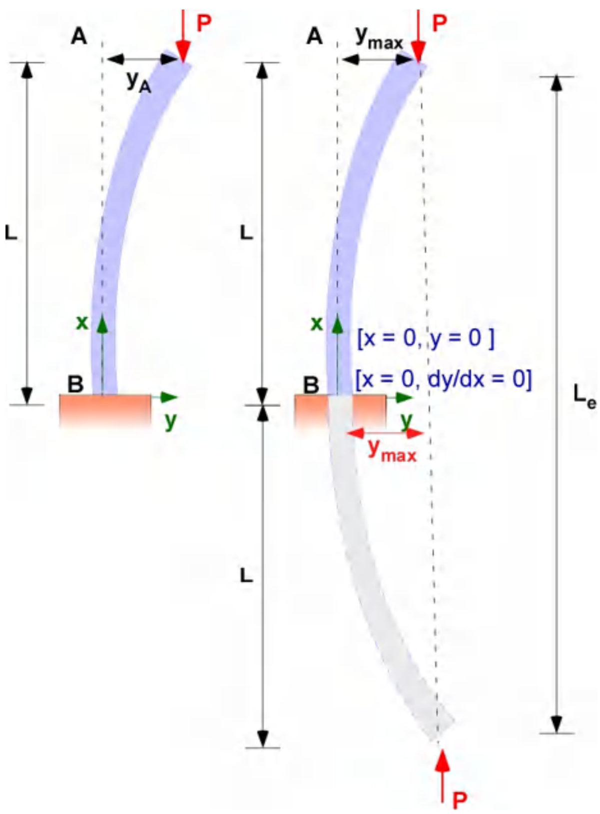


Figure 9.3.5 Displacement of column with free end

In Figure 9.3.5 the effective length is twice the actual length of the column for one fixed and one free end as derived in Section 9.2. This is represented by mirroring the displacement of the column with respect to the fixed end. This makes it appear that the maximum displacement is at the middle of the effective column which also appear to simulate pinned columns with a compressive load. The deflection at A is the same as the maximum displacement at the center of the effective column. As the effective length is used in the calculations we can therefore use the same expression to determine the maximum displacement of the free end.

9.3.3 Relation between Load and Geometry for Column Buckling

Equation (9.27) can be used to develop a relation between load and geometry for column buckling of a given material. This should be very useful for design. Given the geometry of the cross-section and the eccentricity the allowable stress can be determined. It is possible to generate this information over a range of values and present it as a design chart for a given material.

$$\sigma_{max}(P) = \frac{P}{A}\left[1 + \frac{ec}{k^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right];$$

$$\sigma_Y = \frac{P}{A}\left[1 + \frac{ec}{k^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{\frac{\pi^2 EI}{L_e^2}}}\right)\right];$$

$$\sigma_Y = \frac{P}{A}\left[1 + \left(\frac{ec}{k^2}\right)\sec\left(\frac{1}{2\sqrt{E}}\sqrt{\frac{P}{\frac{Ak^2}{L_e^2}}}\right)\right]; \qquad (9.28)$$

$$\sigma_Y = \frac{P}{A}\left[1 + \left(\frac{ec}{k^2}\right)\sec\left(\frac{1}{2\sqrt{E}}\left(\frac{L_e}{k}\right)\sqrt{\frac{P}{A}}\right)\right];$$

In the last equation in Eqn. (9.28) , for a given material (σ_Y, E), for a set of geometric parameters (e^*c/k^2) and (L_e/k), we can identify the maximum load through the parameter (P/A). We can use MATLAB to generate the graphical relation between these values.

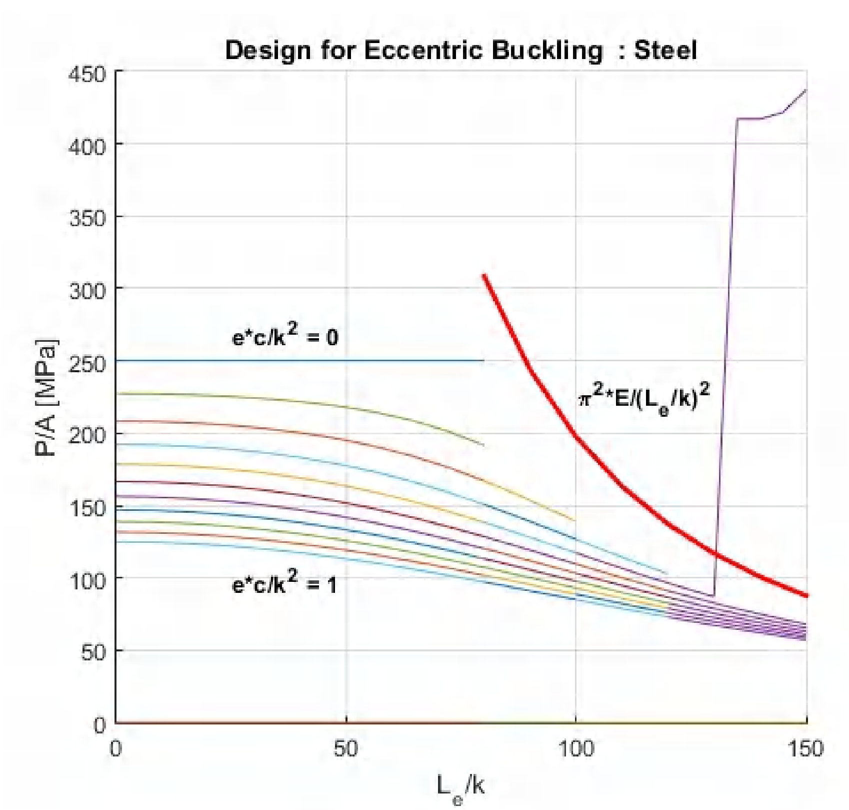


Figure 9.3.6 Eccentric buckling - steel

Figure 9.3.6 is the plot of the last relation in Eqn. (9.28). It is for steel with E = 200 GPa and σ_Y = 200 MPa. For a pair of values of e^*c/k^2 and L_e/k the value of P/A is obtained. The value of P/A is attempted symbolically but most of the time there is a warning that symbolic solution is not available and a numerical solution is obtained instead. The values for e^*c/k^2 varied between 0 and 1 in steps of 0.1. The curve is singular at certain values of L_e/k in this approach. . The singularity for e^*c/k^2 = 0.4 is illustrated for those who wish to reproduce the curve.Therefore the curves in Figure 9.3.6 are generated for different ranges of L_e/k to avoid the singularity. This can be gleamed by the change in color along the same curve. The generic critical stress is also plotted to indicate the limit. It is possible to avoid the singularity and cover the entire range by obtaining a numerical solution instead of attempting a symbolic one. This is not done here as it requires knowledge of numerical techniques - a third year course.

9.3.4 Additional Problems

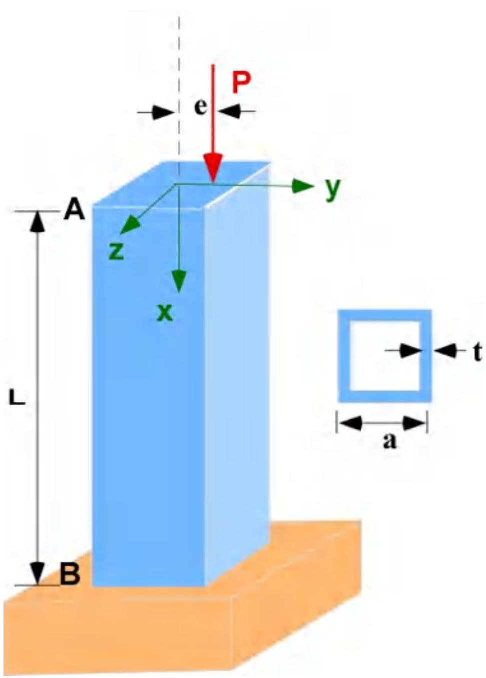
Use the following table for your calculations if needed. Solve on paper and using MATLAB/Octave

Table 9.2

Material	**Aluminum**	**Brass**	**Steel**	**Wood**
E [GPa]	70	105	200	13
Allowable Stress [MPa]	100	160	250	25
G [GPa]	26	39	77	0.7

Problem 9.3.1

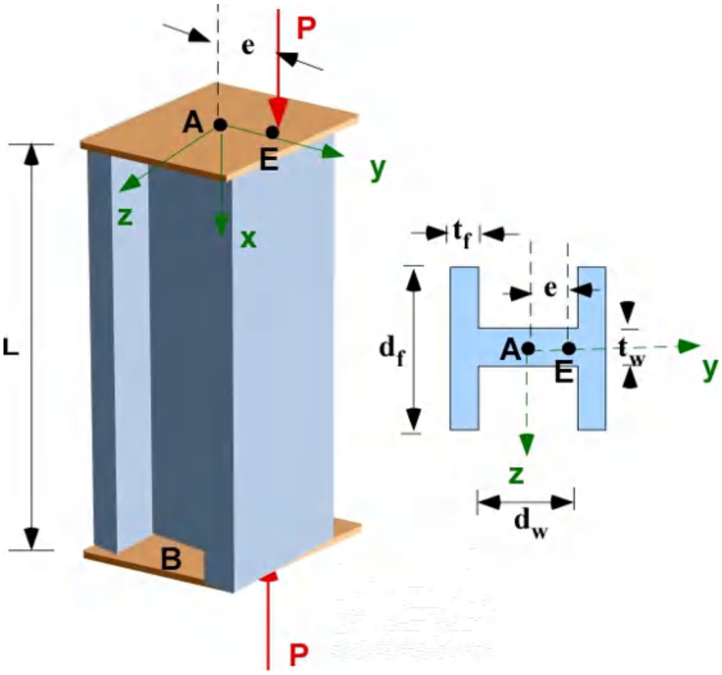
A steel square tube of length L = 0.7 m is subject to a compressive load of 25 kN applied with an eccentricity e = 5 mm along the y-axis as shown. The load must be applied on a welded plate. The side of the tube is a =50 mm with a wall thickness of t = 4 mm. Determine the maximum stress in the column and the deflection of the end for a FOS of 2.5. Use the material property from Table 9.2.



Problem 9.3.1

Problem 9.3.2

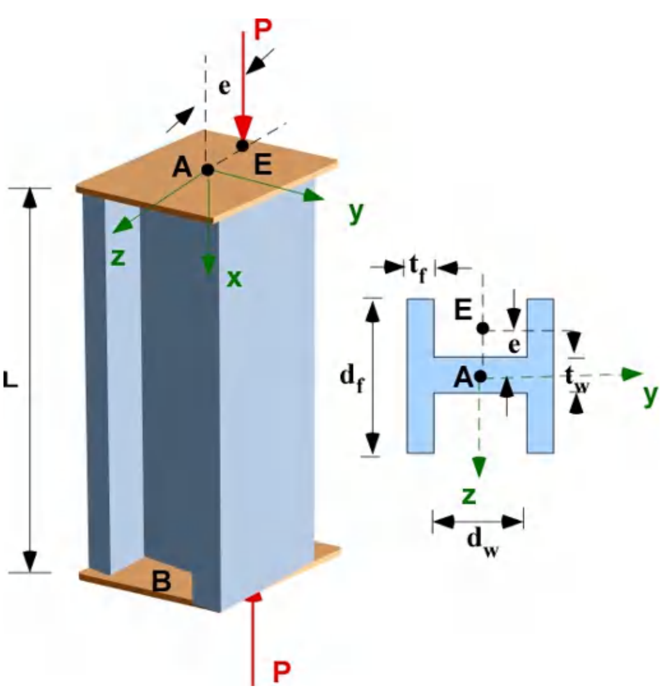
An aluminum I beam of length L = 1.2 m is subject to a compressive load of 15 kN applied with an eccentricity e = 10 mm along the y-axis as shown. The load must be applied on a welded plate integral with the beam. The I beam has flange width (d_w) of 125 mm, flange thickness (t_f) of 12 mm, a beam depth (d_w) of 180 mm, and web thickness (t_w) of 10 mm. Determine the (a) maximum stress in the column and (b) the maximum deflection of the column for a FOS of 3.0. Use the material property from Table 9.2.



Problem 9.3.2

Problem 9.3.3

A steel I beam of length L =3.0 m is subject to a compressive load of 50 kN applied with an eccentricity e = 20 mm along the z-axis as shown. The load must be applied on a welded plate integral with the beam. The I beam has flange width (d_w) of 100 mm, flange thickness (t_f) of 8 mm, a beam depth (d_w) of 300 mm, and web thickness (t_w) of 6 mm. Determine the (a) maximum stress in the column and (b) the maximum deflection of the column for a FOS of 2.5. Use the material property from Table 9.2.



Problem 9.3.3

9.4 BUCKLING OF THIN WALLED COLUMNS

Thin walled structural elements are important in aerospace engineering and other areas of engineering because of minimum mass designs. Many structural elements like stringers and longerons, which are basically long columns, provide stiffening of thin web structures, and are susceptible to failure by bucking. In this section we discuss these columns that carry a compressive load and also have open cross-sections like 'I' sections, 'C' sections or 'L' sections. Examples of closed circular and rectangular sections have appeared in previous sections in this chapter.

One reason for this section is that the buckling of open thin sections is very different from the ideas in the previous section. Thin walled open sections may not buckle in bending as predicted by Euler theory. They could twist instead. They can twist and bend simultaneously causing warping. This is termed as flexural-torsional buckling. The derivation is usually advanced and is beyond the scope of the ideas we have dealt in this book. In the previous sections we dealt with a single second order differential equation. The simultaneous presence of bending and twisting sets up three differential equations. Two of them relate to bending in perpendicular directions while the third is a differential equation that relates to twisting. The torsion equation is a fourth order differential equation. In special cases these equations are uncoupled and provide three solutions for the critical buckling load. The solution is beyond the scope of this book but the results are presented and used with examples.

9.4.1 Bending-Torsional Bucking of Thin Walled Columns

Figure 9.4.1(a) is a thin-walled open cross-section column subject to a compressive load. Figure 9.4.1(b) is an element of column reacting to the compressive load by twisting and bending along perpendicular axis.

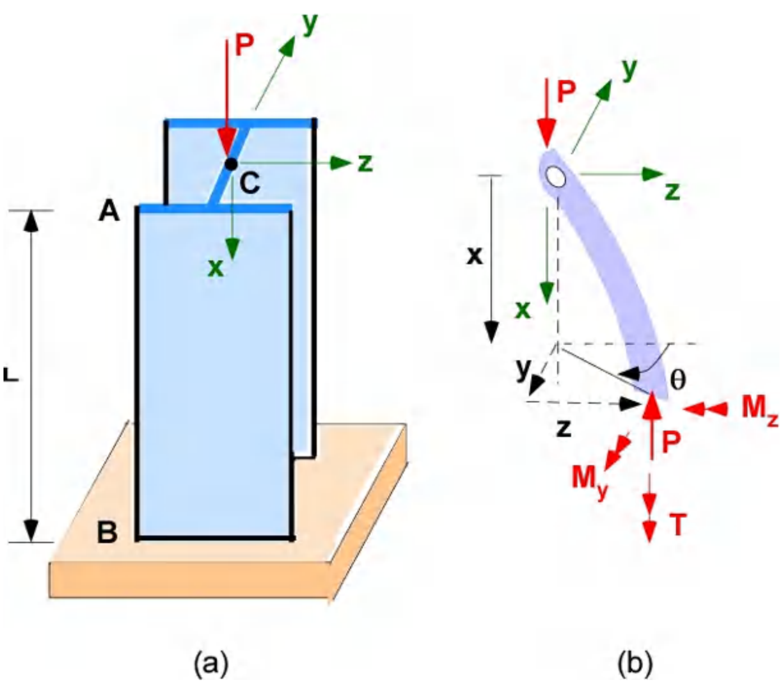


Figure 9.4.1 Bending-Torsional Buckling

There is no deflection at the ends A and B along the y- z axis. There is also no rotation about the x-axis at A and B. However the ends are free to rotate about the y and z axis and are also free to warp. This translates to the following boundary conditions:

$$\begin{aligned} x=0;\quad & y=0;\quad z=0;\quad \theta=0;\quad M_y=0;\quad M_z=0; \\ x=L;\quad & y=0;\quad z=0;\quad \theta=0;\quad M_y=0;\quad M_z=0; \end{aligned} \tag{9.29}$$

This leads to three values for the critical load :

$$\begin{aligned} & P_{cr}(y)=\frac{\pi^2 E I_{yy}}{L^2};\quad P_{cr}(z)=\frac{\pi^2 E I_{zz}}{L^2}; \\ & P_{cr}(\theta)=\frac{A}{I_c}\left(GJ+\frac{\pi^2 E\Gamma}{L^2}\right);\quad \Gamma=\text{torsion-bending constant} \end{aligned} \tag{9.30}$$

The new constant Γ is purely a function of the geometry of the cross-section and is related to the torsion with the bending of the flanges. Needless to say the use of Eqn. (9.30) is conditioned on the availability of the relation to calculate this new constant. $\mathbf{I_c}$ is the polar moment of the cross-section about the centroid. We will examine only symmetric cross-sections in this section so that we can ignore shear center calculations and its effects.

For a cross-section with two flanges with thickness t_1 and t_2, flange width b_1 and b_2, located a distance h apart, Figure 9.4.2, one recommended formulation for the calculation of Γ is :

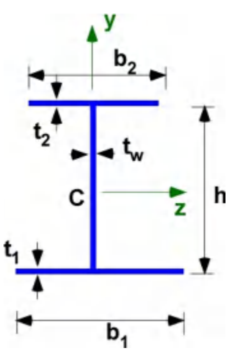


Figure 9.4.2 Geometry definition

$$\Gamma=\frac{h^2\, t_1\, t_2\, b_1^3\, b_2^3}{12\left(t_1 b_1^3+t_2 b_2^3\right)}; \tag{9.31}$$

The torsional constant J for thin-walled open section beams is also a new concept for this section as we have only encountered closed wall sections before. We use the formula without discussion but it can be related to shear flow discussions earlier.

$$J=\sum_i \frac{t_i s_i^3}{3}=\frac{t_1 b_1^3+t_2 b_2^3+t_w h^3}{3} \tag{9.32}$$

9.4.2 Example 9.8

A 2.5 m aluminum column with a thin walled symmetric I beam cross-section, see Figure 9.4.2, is subject to a compressive load at its pinned ends. Calculate the buckling load and mode of buckling. Use Table 9.2 for properties of aluminum.

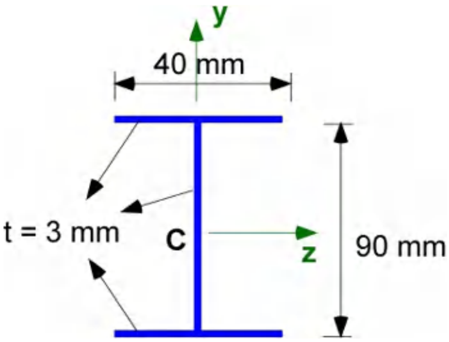


Figure 9.4.3 Example 9.8

Data: L = 2.5 m; $b_1 = b_2 = 40$ mm; $t_1 = t_2 = t_w = 3$ mm; h = 90 mm; E = 70 GPa ; $\sigma_y = 100$ MPa; G = 26 GPa;

Find: (a)The critical buckling load, and (b) the buckling mode

Assumption: The ends of the column are pinned. Note the column is symmetric along both axis and therefor the shear center is at the centroid.

Solution: We can directly use the formulas in Eqn. (9.29) and makes it simple to use MATLAB for routine calculation

Solution MATLAB
In the Editor

```
% Essential Foundations in Mechanics
% P. Venkataraman, August 2018
% Section 9.4.2 - Example 9.8
% Buckling - Thin Walled Section
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clc, clear, format compact, format shortg, close all,
warning off
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fprintf('Example 9.8\n')
fprintf('---------------------------------------\n')
fprintf('Critical Buckling- Thin walled Section\n')
fprintf('----------------------------------------\n')
%% Data
E = 70e09;   sigy = 100e6; G =26e9;
L = 2.5;  b1 = 40/100;  b2 = 40/1000;  h = 90/1000;
t1 = 3/1000;  t2 = 3/1000;  tw = 3/1000;
econ = 1;  % type of end constraints
fprintf('Material : Aluminum\n')
fprintf('E    [GPa] = '),disp(E/1e09)
fprintf('G    [GPa] = '),disp(G/1e09)
fprintf('sigy [MPa] = '),disp(sigy/1e06)

fprintf('\nGeometry  :\n')
fprintf('--------------\n')
```

```
fprintf('Cross-section :'),disp('symmetric I - beam')
fprintf('L [m]    :'),disp(L)
fprintf('h [m]    :'),disp(h)
fprintf('t1 = t2 = [m]    :'),disp(t1)
fprintf('b1 = b2 = [m]    :'),disp(b1)

fprintf('\nGeometric Properties  :\n')
fprintf('--------------------------\n')
A = t1*(2*b1 + h);
Iz = 2*t1*b1*(h/2)^2 + h^3*t1/12;
Iy = 2*t1*b1^3/12;
Ic = Iy + Iz;
J = (b1*t1^3 + b2*t2^3 + h*tw^3)/3;
Gam = h^2*t1*t2*b1^3*b2^3/(12*(t1*b1^3+t2*b2^3));
fprintf('A [m^2] :'),disp(A)
fprintf('Iz [m^4] :'),disp(Iz)
fprintf('Iy [m^4] :'),disp(Iy)
fprintf('Ic [m^4]   :'),disp(Ic)
fprintf('J  [m^4]   :'),disp(J)
fprintf('Gam  [m^6]   :'),disp(Gam)
fprintf('--------------------------\n')
%% Critical load
Pcr_y = pi^2*E*Iy/L^2;
Pcr_z = pi^2*E*Iz/L^2;
Pcr_t = (A/Ic)*(G*J + pi^2*E*Gam/L^2);

fprintf('Pcr - y [kN]  :'),disp(Pcr_y/1000)
fprintf('Pcr - z [kN]  :'),disp(Pcr_z/1000)
fprintf('Pcr - t [kN]  :'),disp(Pcr_t/1000)

[Pmin,i] = min([Pcr_y,Pcr_z ,Pcr_t]);
fprintf('\n*************************************')
if i == 1
    fprintf('\nFailure likely in bending in y')
elseif i == 2
    fprintf('\nFailure likely in bending in z')
else
    fprintf('\nFailure likely in torsion')
end
fprintf('\n*************************************')
```

In the Command Window

```
Example 9.8
---------------------------------------
Critical Buckling- Thin walled Section
----------------------------------------
Material : Aluminum
E     [GPa] =     70
G     [GPa] =     26
sigy [MPa] =    100

Geometry  :
--------------
Cross-section :symmetric I - beam
L [m]    :          2.5
```

```
h [m]    :         0.09
t1 = t2 = [m]    :        0.003
b1 = b2 = [m]    :          0.4

Geometric Properties  :
--------------------------
A [m^2] :      0.00267
Iz [m^4] :   5.0422e-06
Iy [m^4] :      3.2e-05
Ic [m^4]   :   3.7042e-05
J  [m^4]   :     4.77e-09
Gam  [m^6]   :   1.2947e-10
--------------------------
Pcr - y [kN]  :       3537.3
Pcr - z [kN]  :       557.37
Pcr - t [kN]  :       9.9709

*************************************
Failure likely in torsion
*************************************
```

Execution in Octave
There is no change in the code.

In Octave Command Window
The results in the Command Window are also the same.

The formulas in this section is limited to doubly symmetric open sections with two flanges. Single axis symmetry or unsymmetric section will need the calculation of the centroid, shear center and further adjustment of the above formulas. Nevertheless, these calculations will always provide a rough estimate for the critical load if used without any corrections.

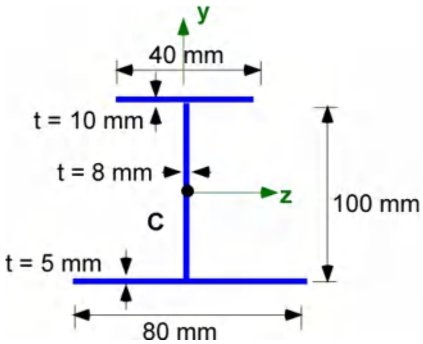
9.4.3 Additional Problems

Use the following table for your calculations if needed. You will have to calculate the centroid and MOI in some cases. Ignore shear center calculations. If shear center is not at the centroid then the calculations are approximate. Solve on paper and using MATLAB/Octave

Table 9.2

Material	**Aluminum**	**Brass**	**Steel**	**Wood**
E [GPa]	70	105	200	13
Allowable Stress [MPa]	100	160	250	25
G [GPa]	26	39	77	0.7

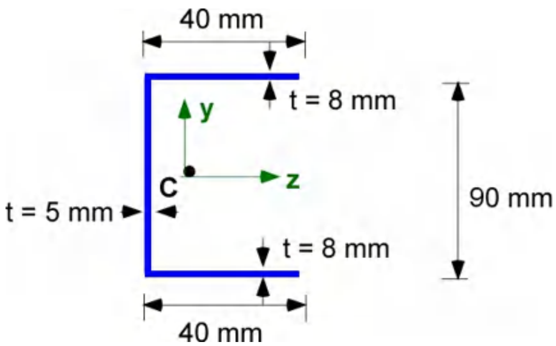
Problem 9.4.1
A 3 m steel column with a thin walled I beam cross-section, see figure Problem 9.4.1, is subject to a compressive load at its pinned ends. Calculate the buckling load and mode of buckling. Use Table 9.2 for properties of steel.



Problem 9.4.1

Problem 9.4.2

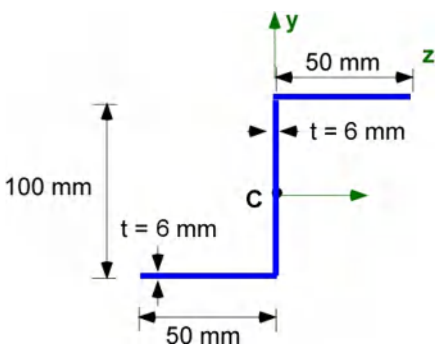
A 2 m aluminum column with a thin walled C beam cross-section, see figure Problem 9.4.2, is subject to a compressive load at its pinned ends. Calculate the buckling load and mode of buckling. Use Table 9.2 for properties of aluminum.



Problem 9.4.2

Problem 9.4.3

A 1.5 m brass column with a thin walled Z beam cross-section, see figure Problem 9.4.3, is subject to a compressive load at its pinned ends. Calculate the buckling load and mode of buckling. Use Table 9.2 for properties of brass.



Problem 9.4.3